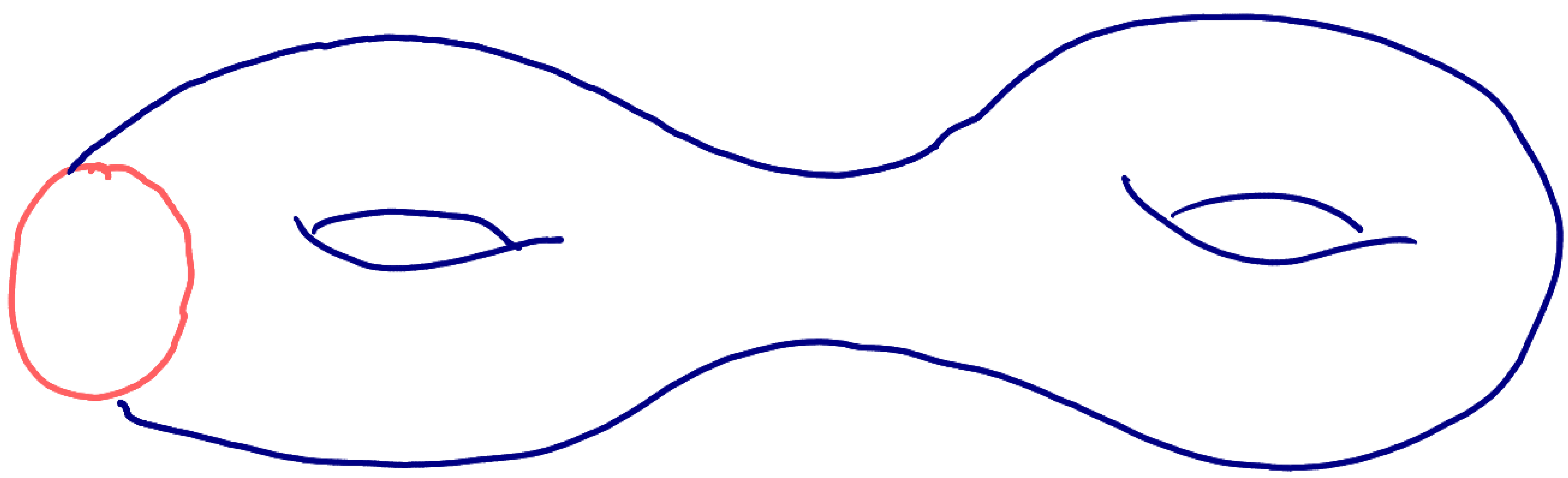
Mapping class group

We denote by $\Sigma_{g,1}$ the surface of genus g with one boundary component

$\Sigma_{2,1}$



$\mathrm{Diff}_\partial(\Sigma_{g,1})$ = topological group of diffeomorphisms of $\Sigma_{g,1}$ fixing a neighbourhood of the boundary

Fact: $$\mathrm{Diff}_\partial(\Sigma_{g,1}) \xrightarrow{\simeq} \pi_0(\mathrm{Diff}_\partial(\Sigma_{g,1}))$$

i.e. all components are contractible. $\qquad \pi_0(\mathrm{Diff}_\partial(\Sigma_{g,1})) = \Gamma_{g,1}$ $\uparrow$ group

Definition: $\Gamma_{g,1} = \pi_0(\mathrm{Diff}_\partial(\Sigma_{g,1}))$ is the <u>mapping class group</u>.

→ All the homotopical information is in the group $\Gamma_{g,1}$.

Examples

* $g=0$: (Alexander's trick) $\Gamma_{0,1} = *$

In D^2, any diffeomorphism (homeomorphism) fixing the boundary is homotopic to the identity.

* $\Gamma_{g,0} = SL_2(\mathbb{Z})$ with $\phi \in \mathrm{Diff}(S^1 \times S^1)$ determined by $\phi_*: H_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z} \cong H_1(S^1 \times S^1)$

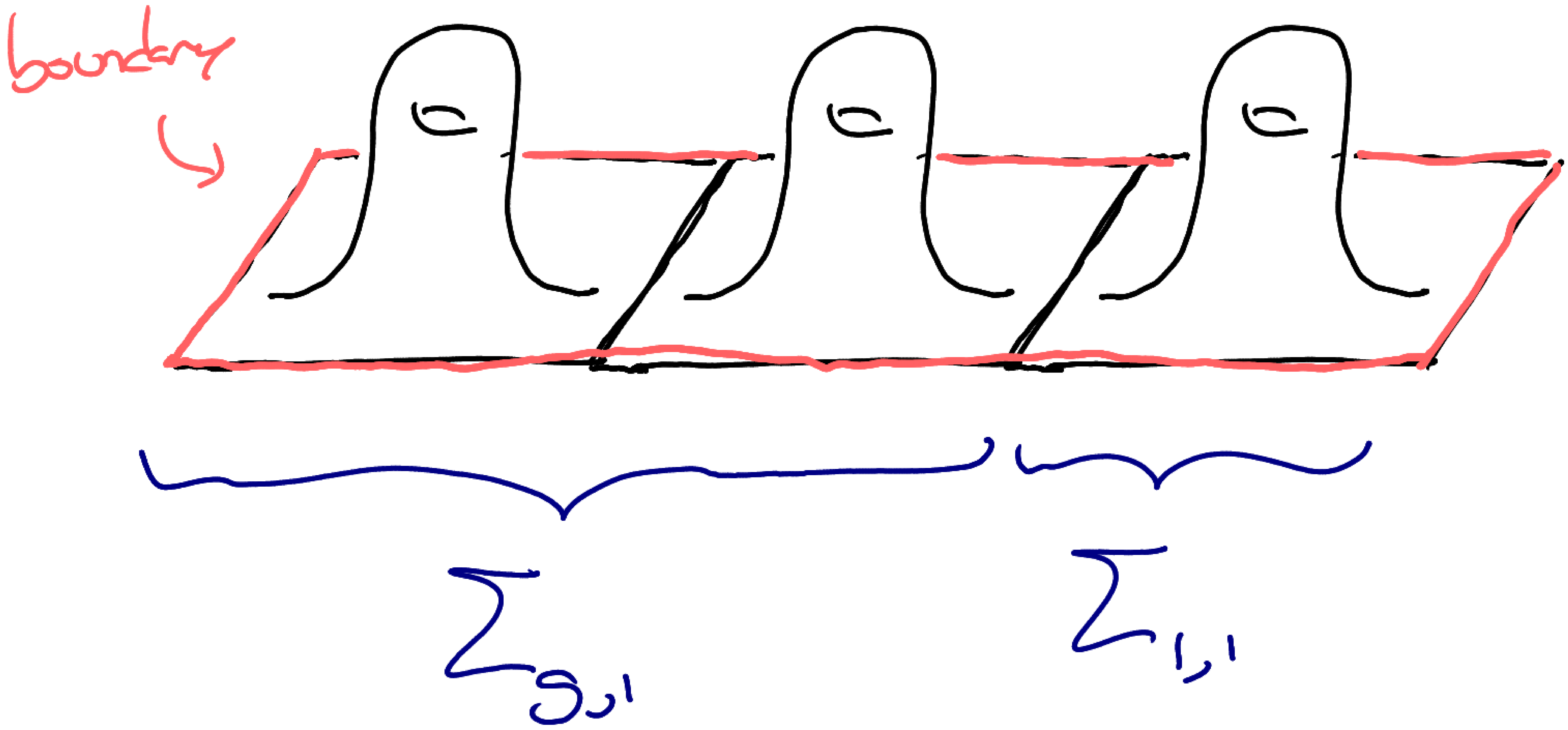
We want to study the homology of the classifying space $B\Gamma_{g,1}$. There is a bijection

$[X, B\Gamma_{g,1}] \xleftrightarrow{\cong}$ classes of manifold bundles over X with fiber $\Sigma_{g,1}$ which trivialise on the boundary.

surface bundle

$H^*(B\Gamma_{g,1}) \rightsquigarrow$ characteristic classes of surf. bundles

We represent $\Sigma_{g,1}$ as copies of



Given $f \in \mathrm{Diff}_\partial(\Sigma_{g,1})$ we consider it extended by the identity on the $(g+1)$-cube $\Rightarrow$ gives an element in $\mathrm{Diff}_\partial(\Sigma_{g+1,1})$. This defines a group homomorphism $\Gamma_{g,1} \to \Gamma_{g+1,1}$, which induces the

<u>stabilisation map</u> $\sigma : B\Gamma_{g,1} \longrightarrow B\Gamma_{g+1,1}$

Theorem [Harer '85] (Homological stability)

$$\sigma_* : H_d(B\Gamma_{g-1,1};\mathbb{Z}) \to H_d(B\Gamma_{g,1};\mathbb{Z})$$

is a surjection for $d \leq \frac{2g-1}{3}$ and an iso for $d \leq \frac{2g-4}{3}$

→ In the 'stable range' the homology groups are well-understood:

* $\operatorname{colim}_{g\to\infty} H_d(B\Gamma_{g,1};\mathbb{Z}) \cong H_d(\Omega_0^\infty MTSO(2);\mathbb{Z})$

* Rationally the algebra structure is understood: the stable cohomology is the free-graded commutative algebra on the Miller-Morita-Mumford classes.

Homology groups outside the stable range?

In the stable range $H_d(B\Gamma_{g,1}, B\Gamma_{g+1,1};\mathbb{Z}) = 0$. In the <u>metastable range</u> $\neq 0$ but they are isomorphic

<u>Theorem</u> (Secondary homological stability)

$$\varphi_*: H_{d-2}(B\Gamma_{g-3,1}, B\Gamma_{g-4,1};\mathbb{Z}) \to H_d(B\Gamma_{g,1}, B\Gamma_{g-2,1};\mathbb{Z})$$

surj for $d \leq \frac{3g-1}{4}$ and iso for $d \leq \frac{3g-5}{4}$

<u>Question</u>: $\operatorname{colim}_{k\to\infty} H_{d+2k}(B\Gamma_{g+3k,1}, B\Gamma_{g+3k-1,1};\mathbb{Z})$

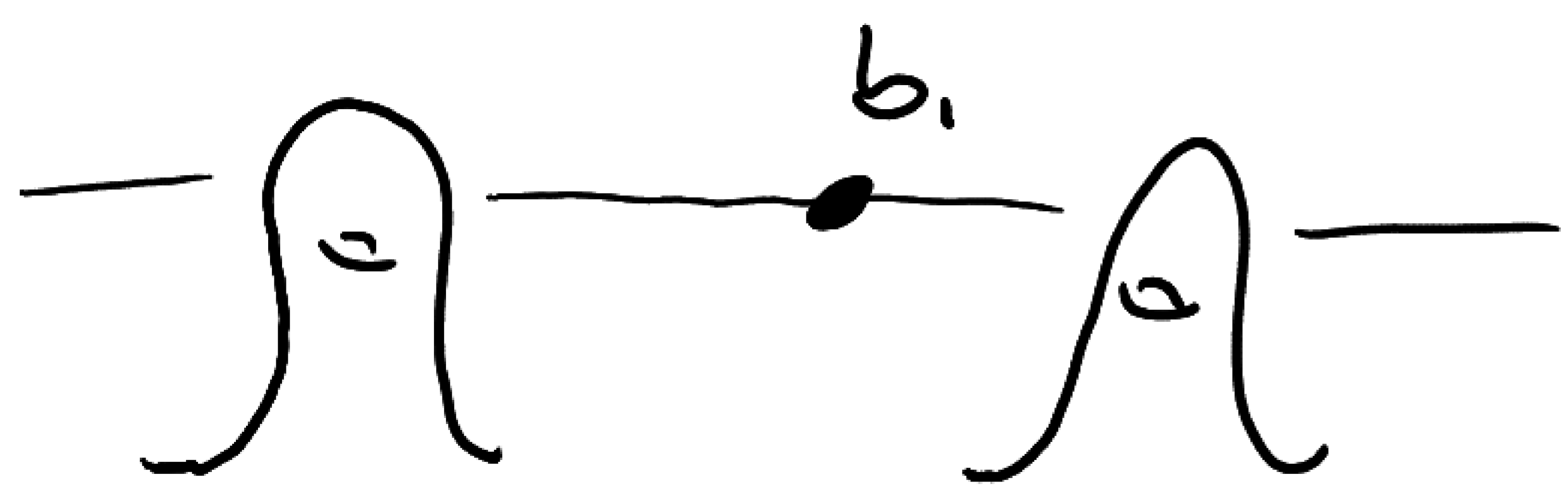
Arc complexes

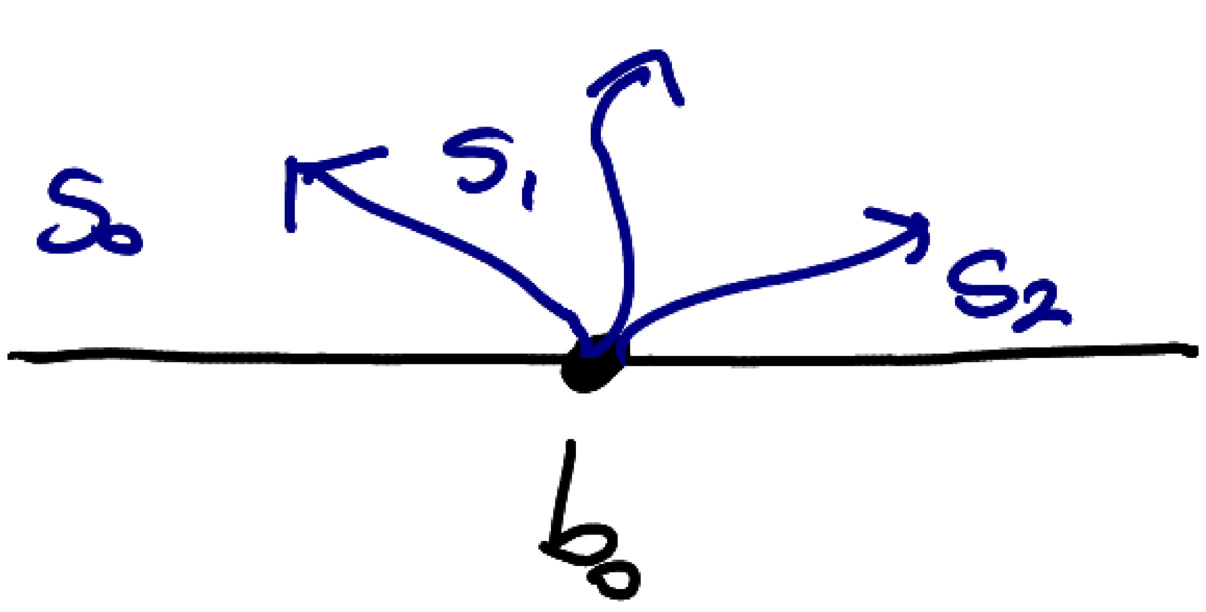
Fix two points b_0, b_1 in $\partial \Sigma_{g,1}$.

Consider a list of paths $s_0, \dots s_p$ in $\Sigma_{g,1}$ from b_0 to b_1. We impose 3 conditions:

i) $s_0, \dots s_p$ are disjoint (except for the endpoints)

ii) At b_0, the curves are ordered clockwise

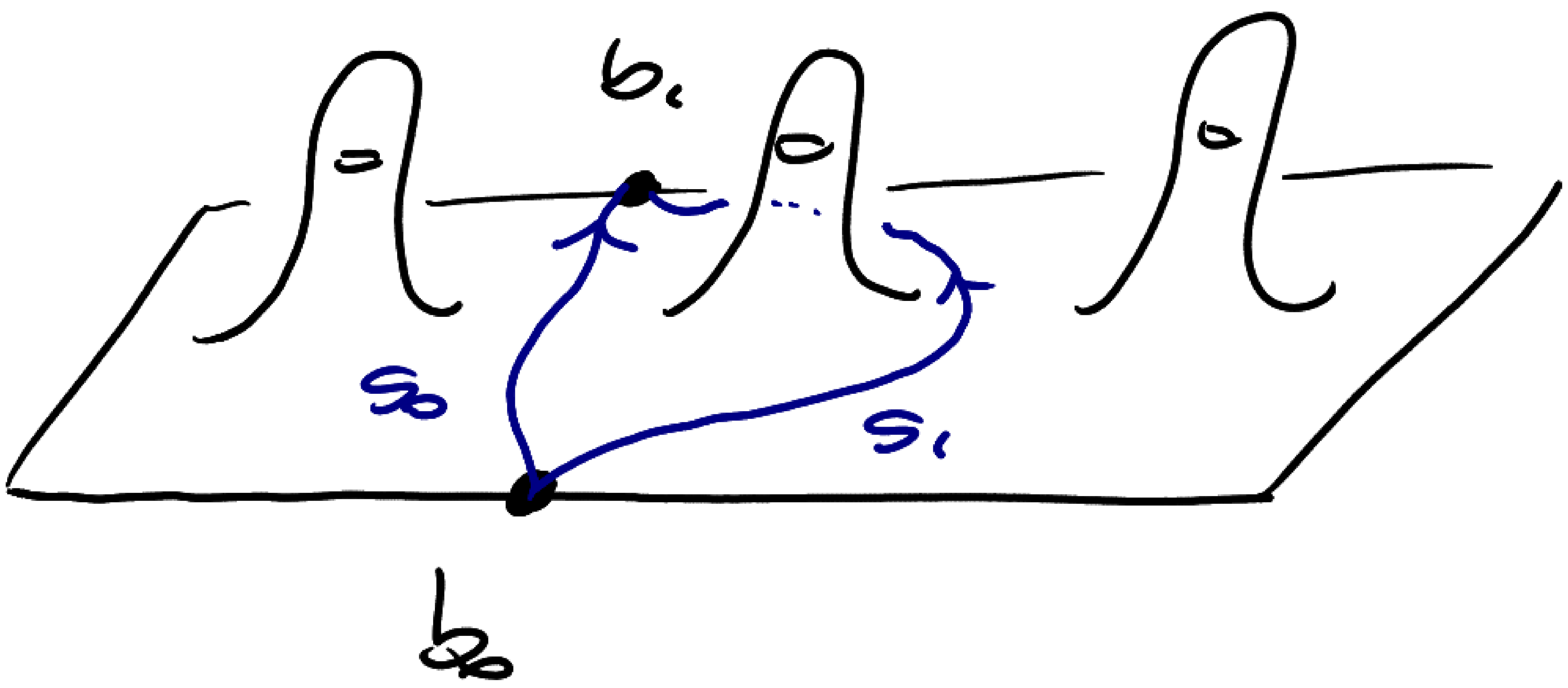




iii) Each curve splits $\Sigma_{g,1}$ into two subsurfaces

The curves splits $\Sigma_{g,1}$ into $(p+2)$ subsurfaces.

All these regions are required to have positive genus.

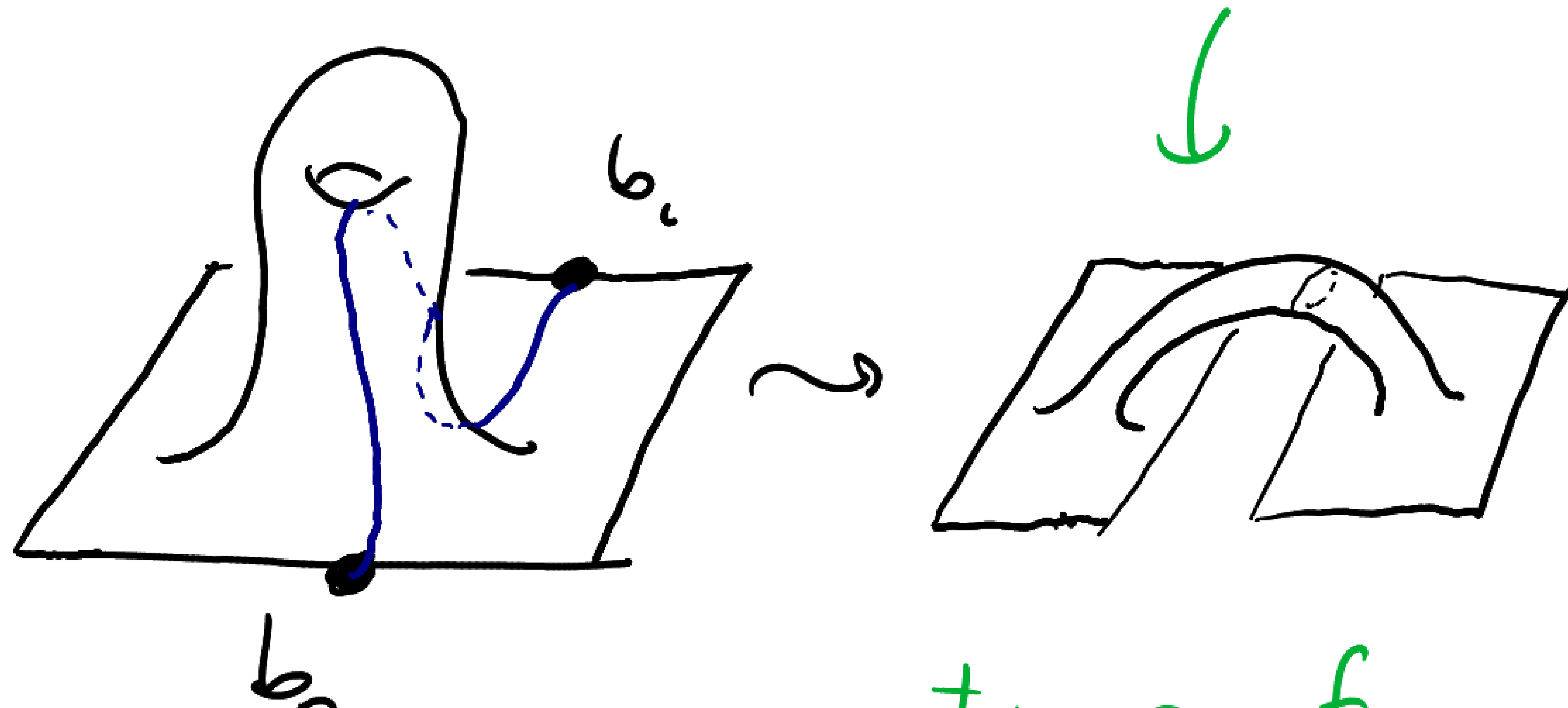


Condition iii) limits the number of curves:

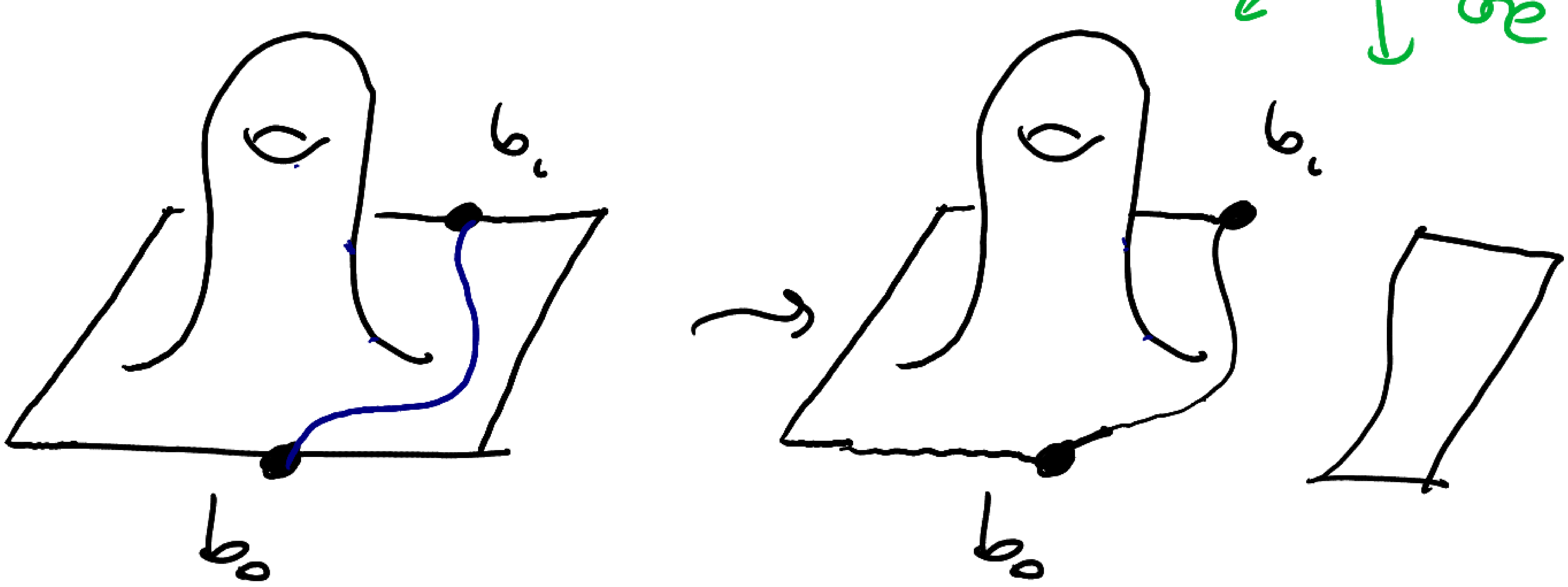
In $\Sigma_{g,1}$ the largest number of curves is $g-1$

Example

only one surface



two surfaces but one of genus 0



Not valid situations!

* The space of arcs in a given isotopy class is contractible. Also true for tuples of arcs disjoints.

$\Rightarrow$ The condition does not depend on the choice of representatives.

<u>Definition</u>: $S(\Sigma_{g,1}, b_0, b_1)_\bullet = S_\bullet$ is a semi-simp. set where

* $S(\Sigma_{g,1}, b_0, b_1)_p = \{([s_0], \dots [s_p])$ p+1 classes of paths satisfying i), ii), iii) $\}$

* The i-th face d_i forgets $[s_i]$.

• $S_\bullet$ is $(g-2)$ dimensional.

<u>Theorem</u>: $|S_\bullet|$ is $(g-3)$-connected.

Both facts imply that $|S_\bullet| \simeq \bigvee_I S^{g-2}$

Low-degree homology of $\Gamma_{g,1}$

	$\Gamma_{0,1}$	$\Gamma_{1,1}$	$\Gamma_{2,1}$	$\Gamma_{3,1}$	$\Gamma_{4,1}$
H_2	0	0	$\mathbb{Z}/2 \xhookrightarrow{\sigma}$	$\mathbb{Z}/2 \oplus \mathbb{Z}\langle\lambda\rangle \xrightarrow{\sigma}$	$\mathbb{Z}\langle\sigma\tau\rangle$
H_1	$\mathbb{Z}$	$\mathbb{Z}\langle\tau\rangle$	$\mathbb{Z}/10\langle\sigma\tau\rangle$	0	$0 \cdots$

Furthermore

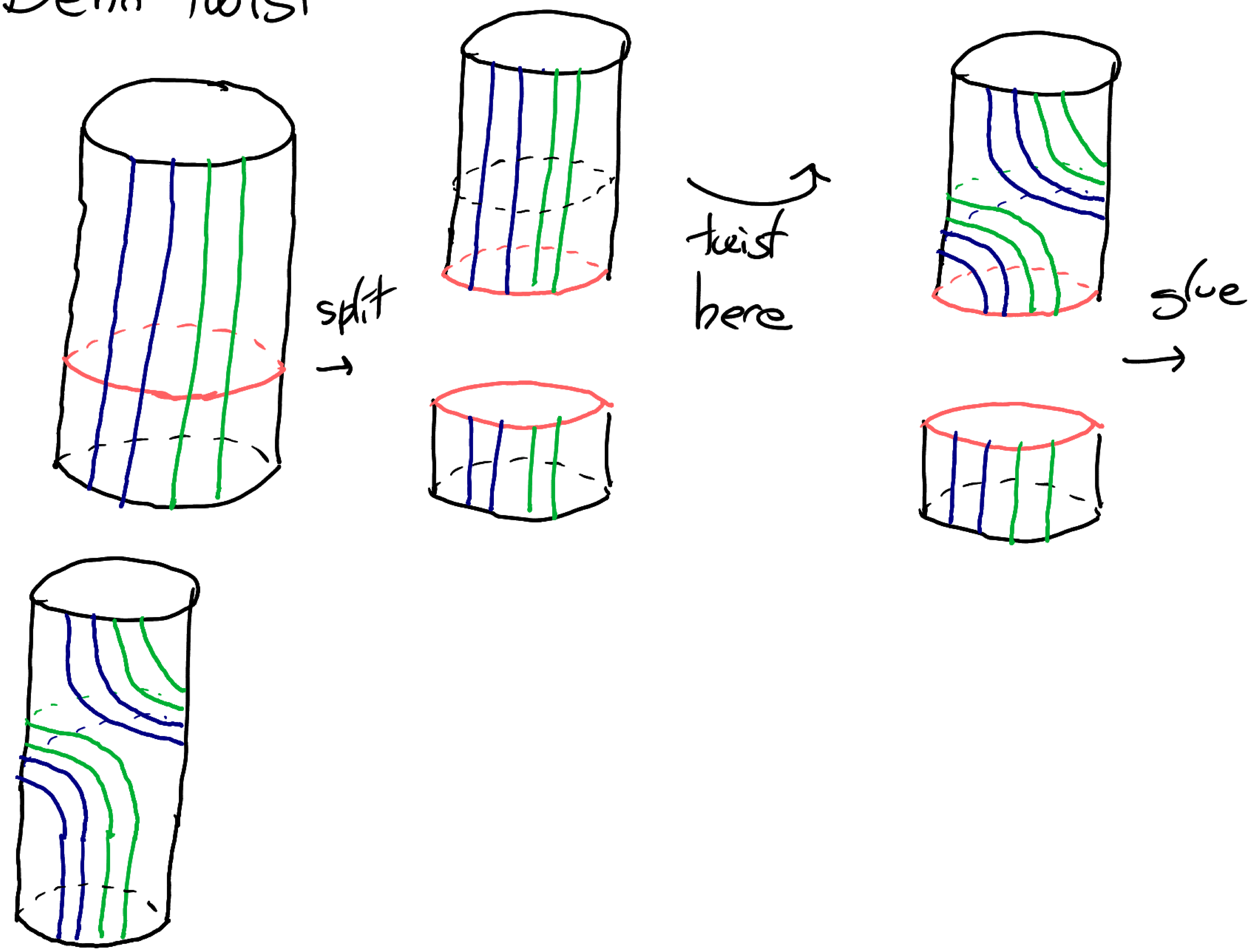
$$0 \to \frac{H_2(\Gamma_{3,1};\mathbb{Z})}{\mathrm{im}(\sigma)} \to H_2(\Gamma_{3,1},\Gamma_{2,1};\mathbb{Z}) \to H_1(\Gamma_{2,1};\mathbb{Z}) \to 0$$

isomorphic to

$$0 \to \mathbb{Z}\langle\lambda\rangle \xrightarrow{\lambda \mapsto 10\mu} \mathbb{Z}\langle\mu\rangle \xrightarrow{\mu \mapsto \sigma\tau} \mathbb{Z}/10\langle\sigma\tau\rangle \to 0$$

Presentations of mapping class groups

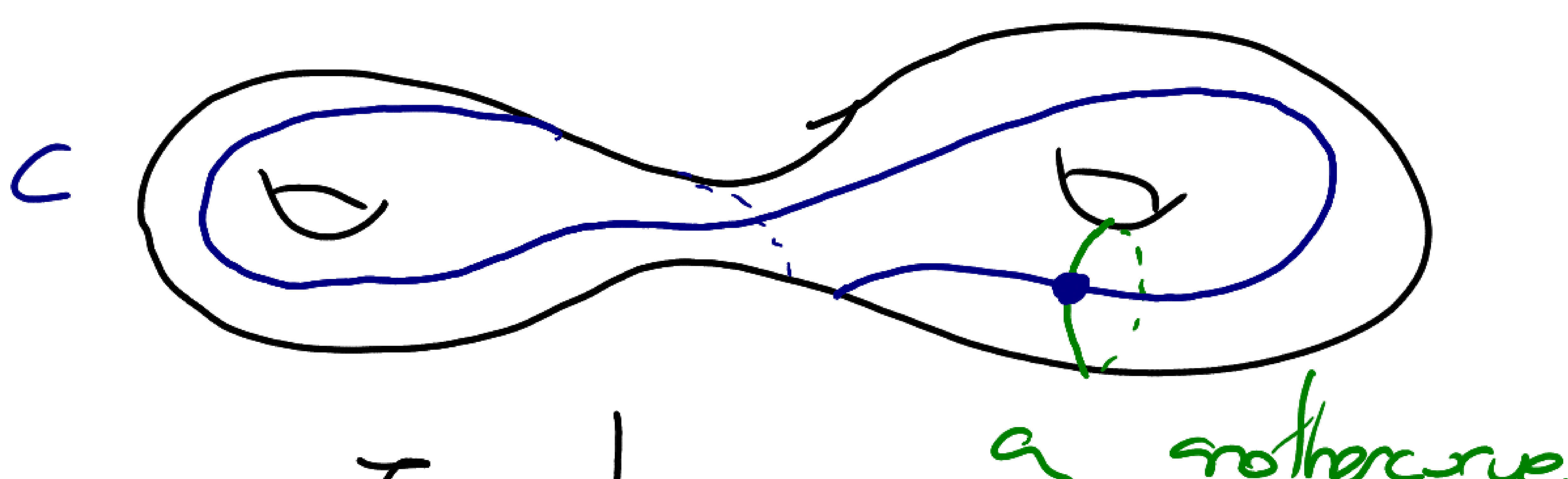
- Dehn twist



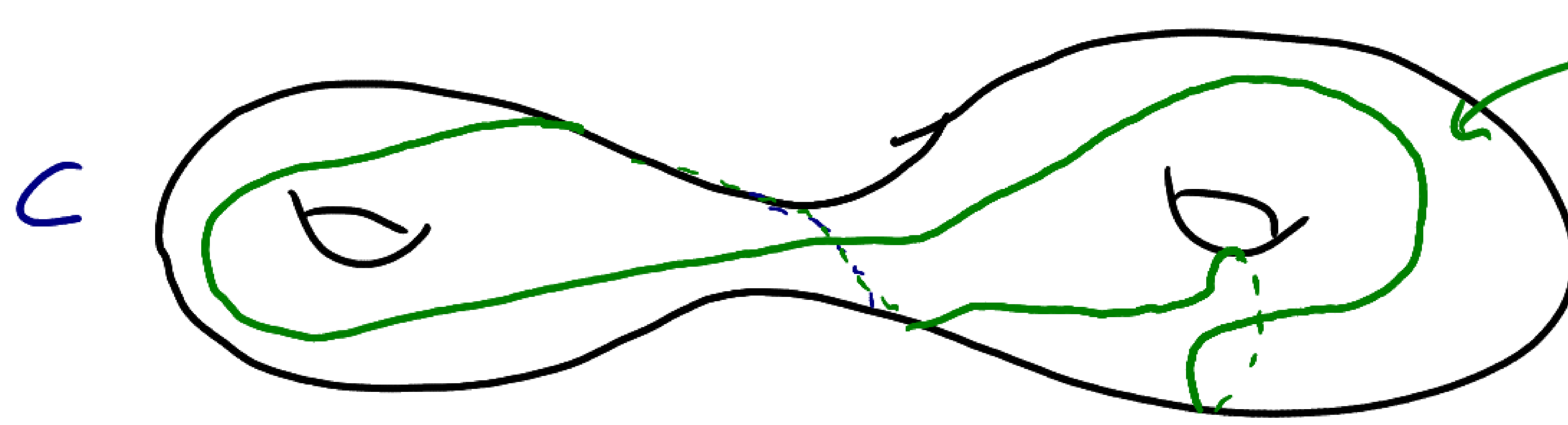
- Given a closed curve in an oriented surface Σ, it has an (oriented) neighbourhood diffeomorphic to the cylinder:

Perform the Dehn twist on the annulus to obtain a diffeomorphism τ_c

* τ_c depends only on the isotopy class of c.
* $\phi \circ \tau_c \circ \phi^{-1} = \tau_{\phi(c)}$ for ϕ diffeomorphism



$\tau_c \downarrow$



Lemma: The stabiliser of any isotopy class (with fixed end points) of an arc $[a]$ for the $\Gamma(\Sigma)$-action is the subgroup $\Gamma(\Sigma \setminus \text{neighbourhood}(a)) \leq \Gamma(\Sigma)$

Algorithm: We want to prove ϕ is trivial.

1) Show $\phi(a)$ isotopic to $a \Rightarrow \phi \simeq \phi'$ in the stabiliser of $[a]$

2) Show ϕ is trivial on $\Sigma \setminus \text{neighbourhood}(a)$ which is a simpler surface $\Rightarrow$ go to step 1.

3) When the genus is $0 \Rightarrow \Gamma_{0,1} = 0 \Rightarrow \phi$ trivial.

We use this algorithm to prove relations in

$\Gamma_{g,1}$

* Braid relation (disjoint)

If a, b closed curves with $a \cap b = \emptyset$, then

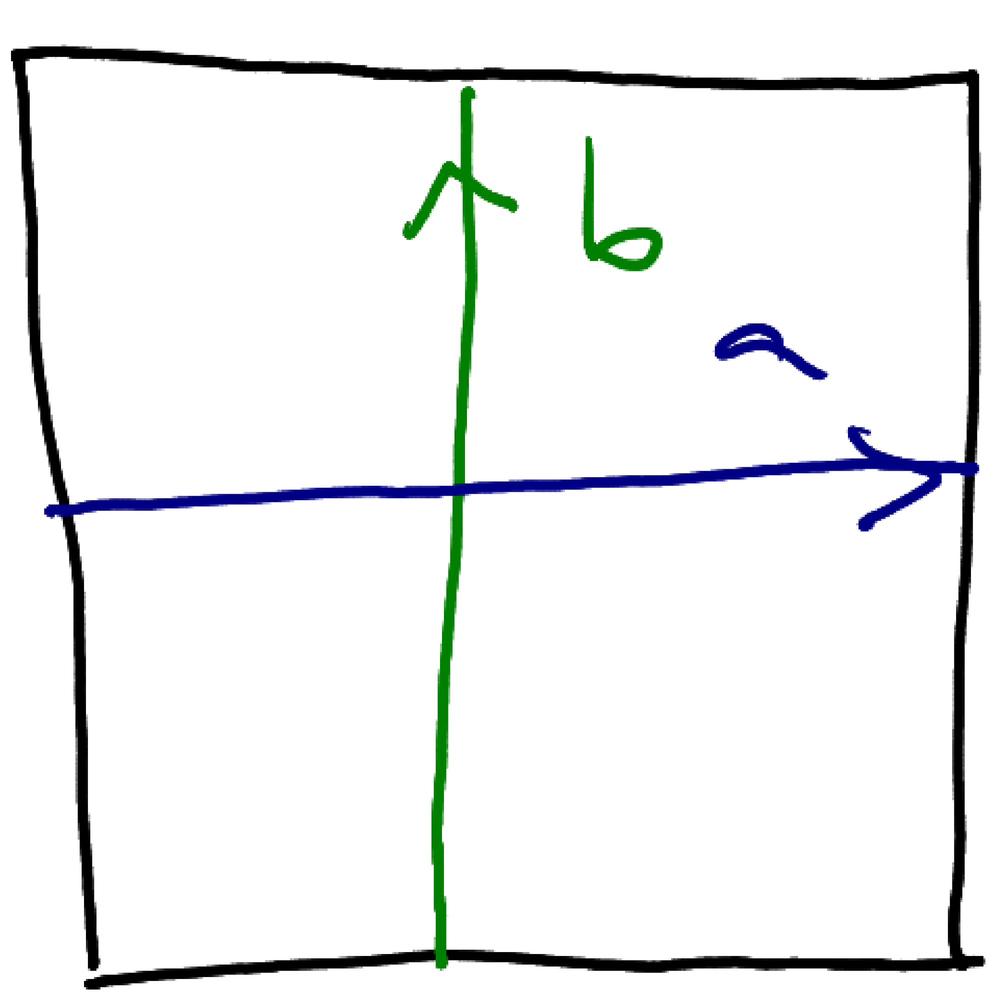
$$[\tau_a, \tau_b] = 0.$$

Proof: Clearly $\phi = [\tau_a, \tau_b]$ fix a and $b \Rightarrow$ we restrict ϕ to $\Sigma \setminus (\text{neigh}(a) \cup \text{neigh}(b))$ but ϕ is the identity here $\Rightarrow \phi = e$.

* Braid relation (one point intersection)

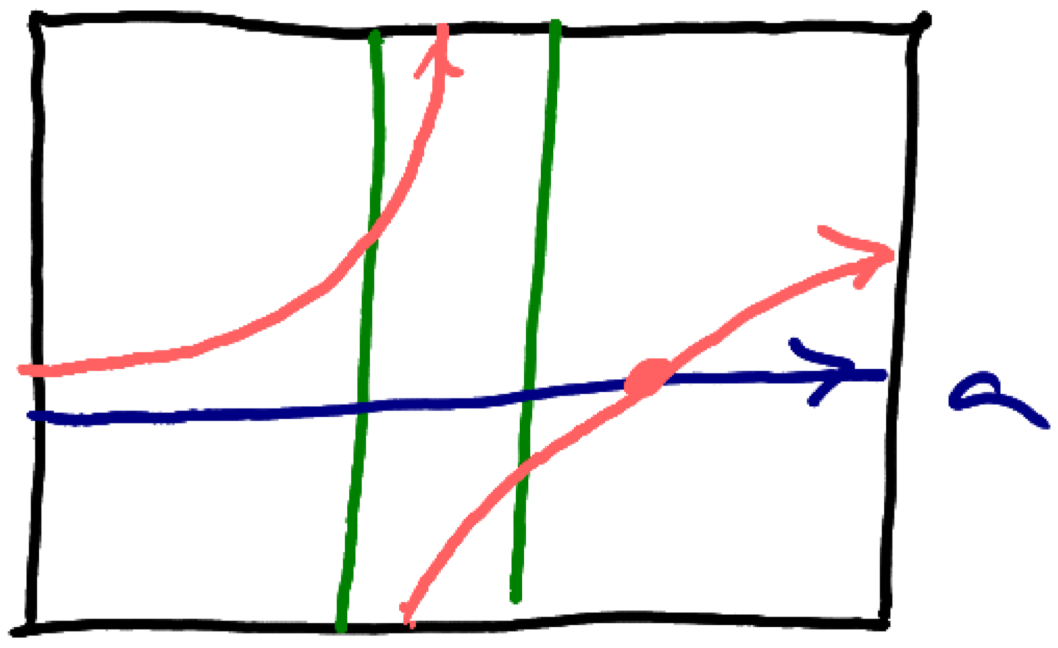
If $a \cap b = *$, then $\tau_a \tau_b \tau_a = \tau_b \tau_a \tau_b$

Proof:

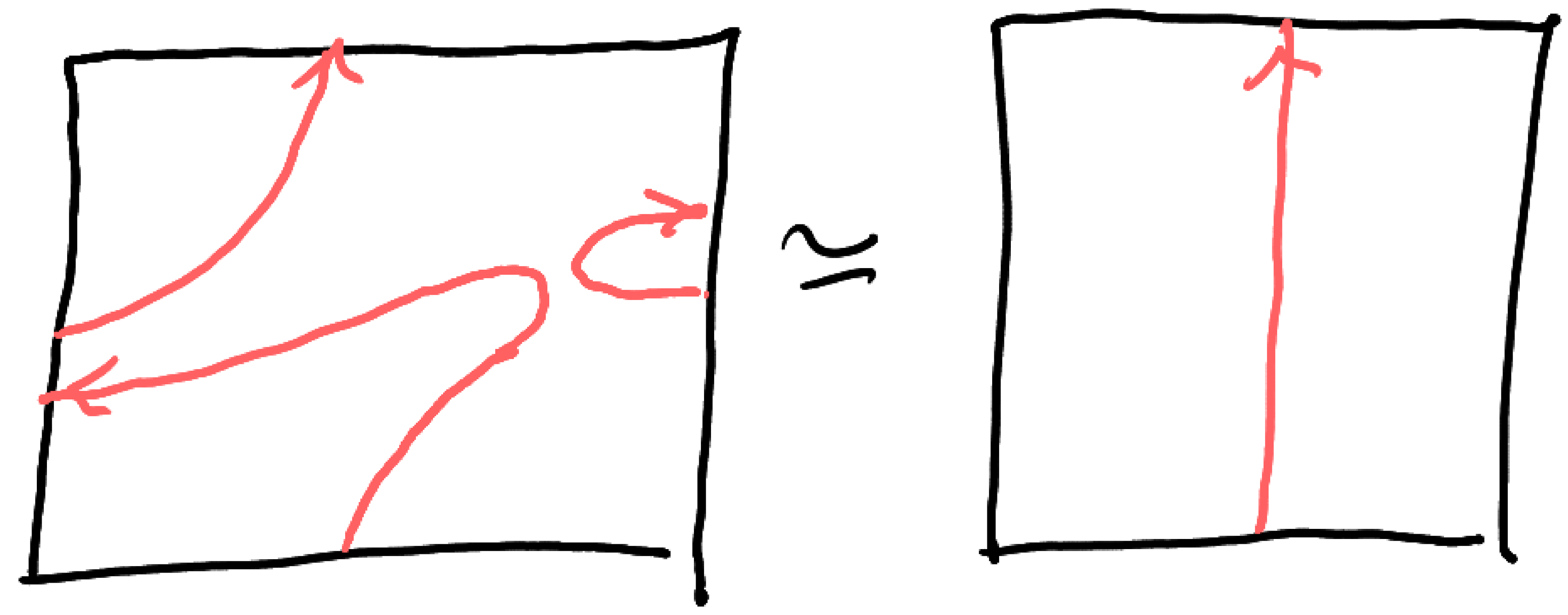


Claim: $\tau_a^{-1} \tau_b (a) \overset{(*)}{=} b$

$\tau_b(a)$ $\qquad\qquad\qquad$ $\tau_a^{-1}\tau_b(a) = b$



$\tau_b(a)$



Then $\tau_b^{-1}\tau_a\tau_b^{-1}\tau_a^{-1}\tau_b\tau_a^{-1}(a) =$

$$= \tau_b^{-1}\tau_a\tau_b^{-1}\tau_a^{-1}\tau_b(a) \overset{(*)}{=}$$

$$= \tau_b^{-1}\tau_a\tau_b^{-1}(b) = \tau_b^{-1}\tau_a(b) \overset{(*)}{=} a$$

and $\tau_b^{-1}\tau_a\tau_b^{-1}\tau_a^{-1}\tau_b\tau_a^{-1}(b^{-1}) \overset{(*)}{=}$

$$= \tau_b^{-1}\tau_a\tau_b^{-1}\tau_a^{-1}(a^{-1}) = \tau_b^{-1}\tau_a\tau_{b^{-1}}(a^{-1}) =$$

$$\overset{(*)}{=} \tau_b^{-1}(b^{-1}) = b^{-1}$$

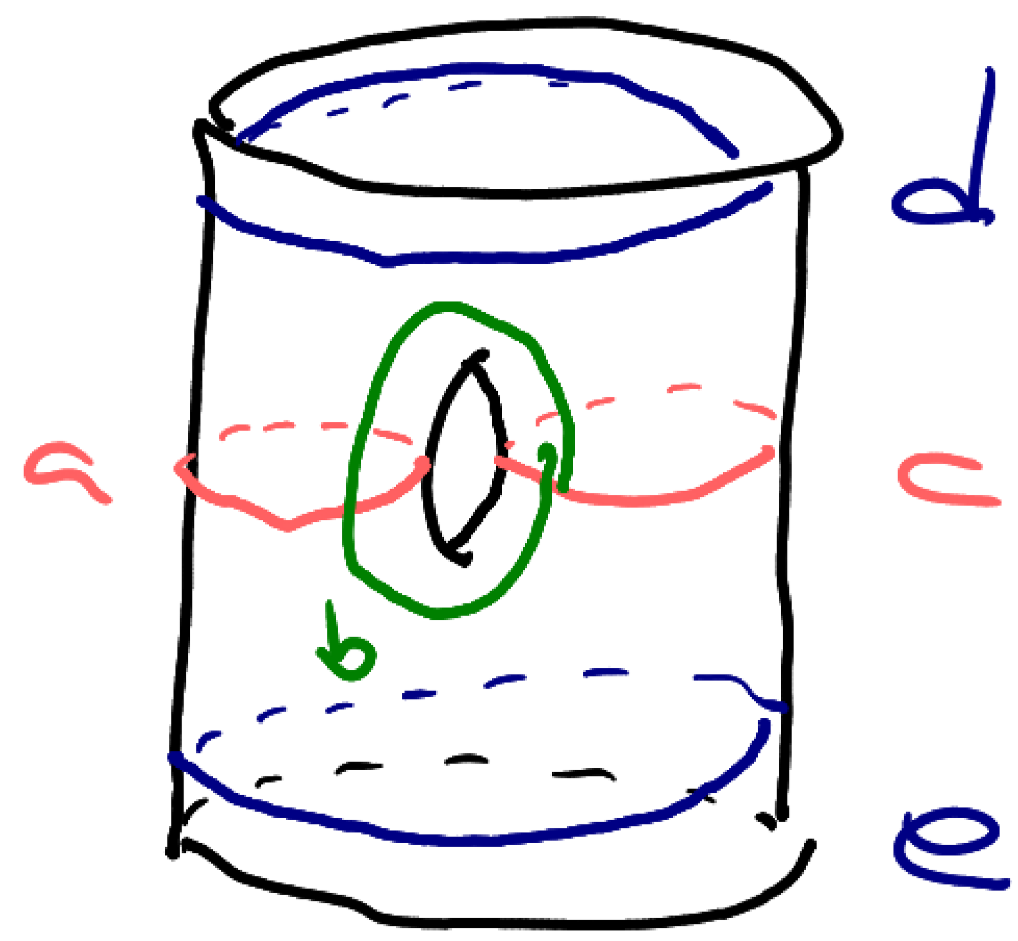
And in $\Sigma_{g,1} \setminus$ neigh$(a) \cup$ neigh(b) is the identity

$\Rightarrow$ $\tau_b^{-1}\tau_a\tau_b^{-1}\tau_a^{-1}\tau_b\tau_a^{-1} \simeq \mathrm{id} \Rightarrow \tau_a\tau_b\tau_a = \tau_b\tau_a\tau_b$

We have other types of relations.

* Two-holed torus relation

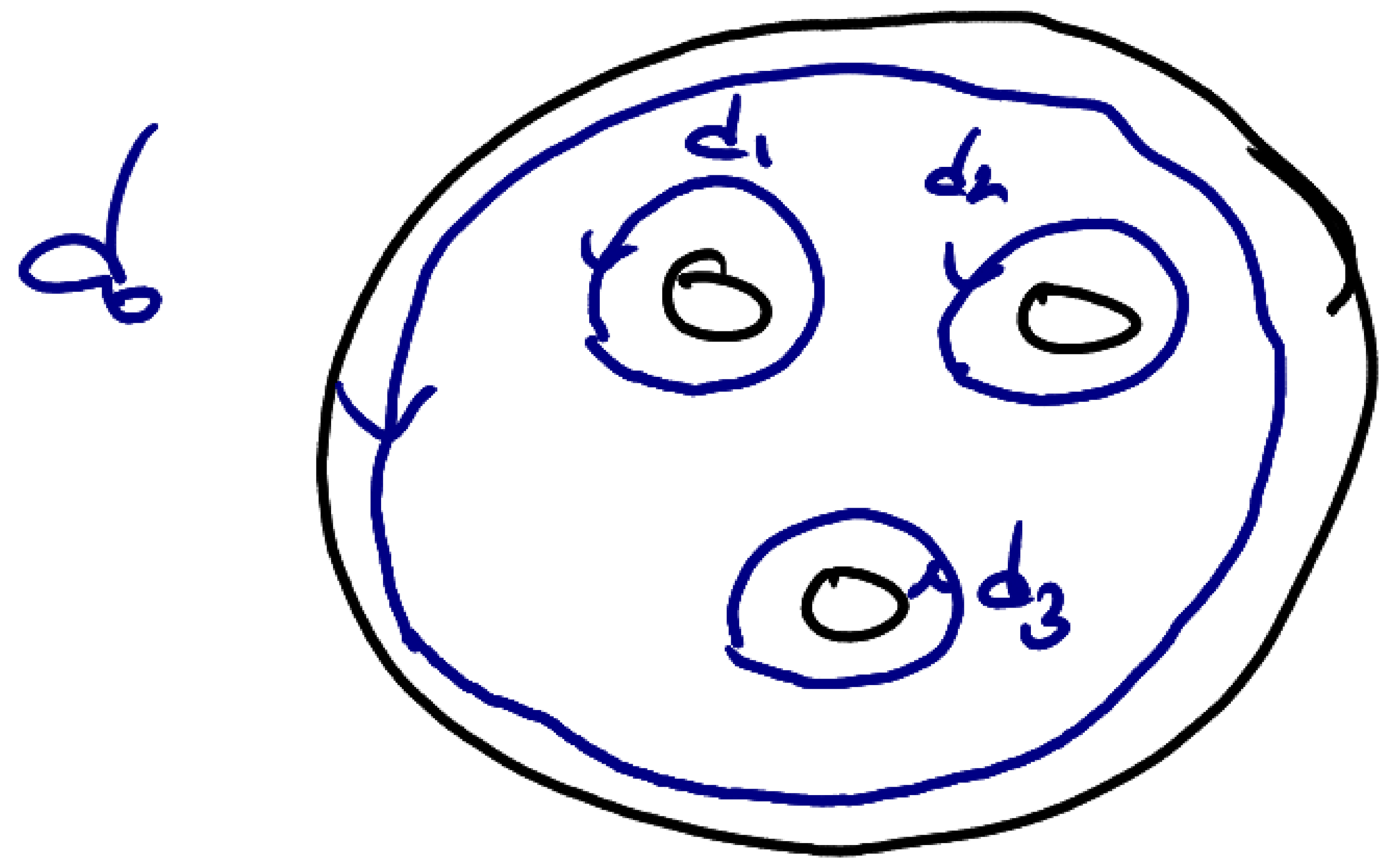
In $\Sigma_{1,2}$

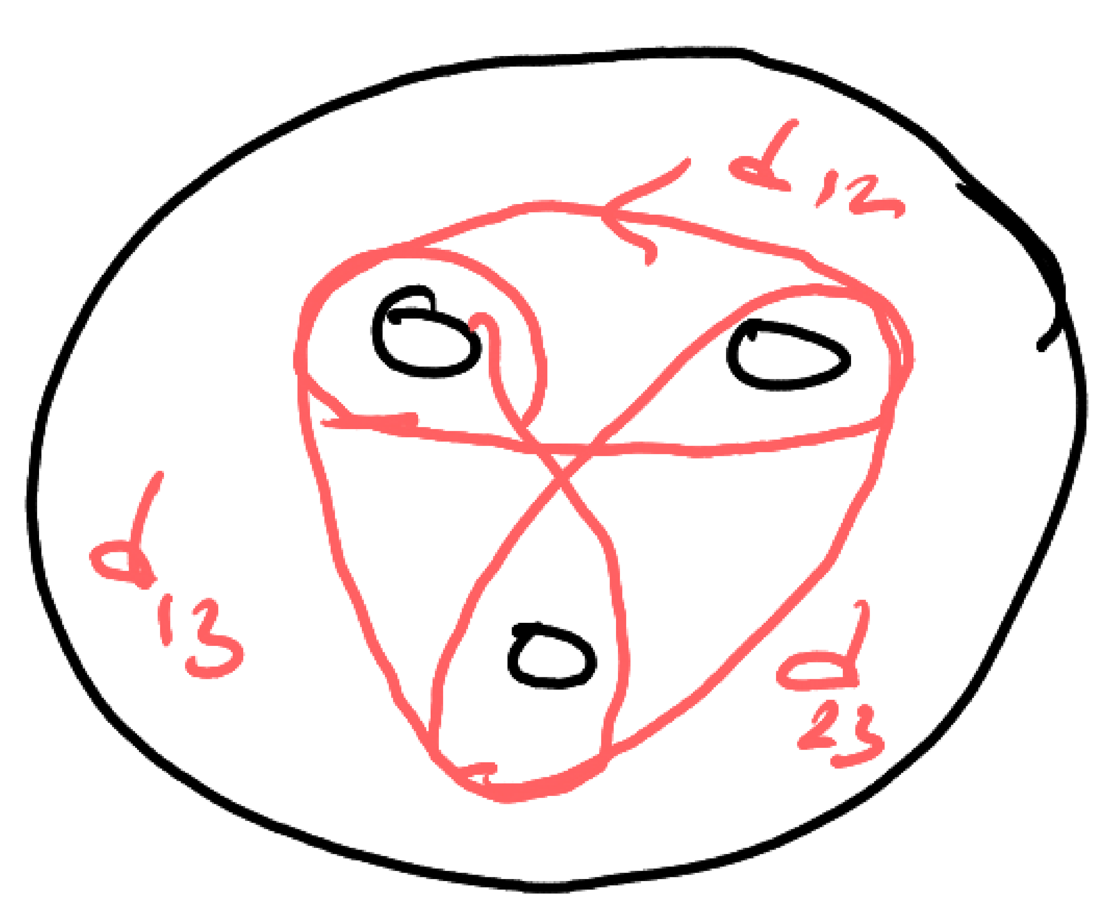


$$(\tau_a \tau_b \tau_c)^4 = \tau_d \tau_e$$

* Lantern relation

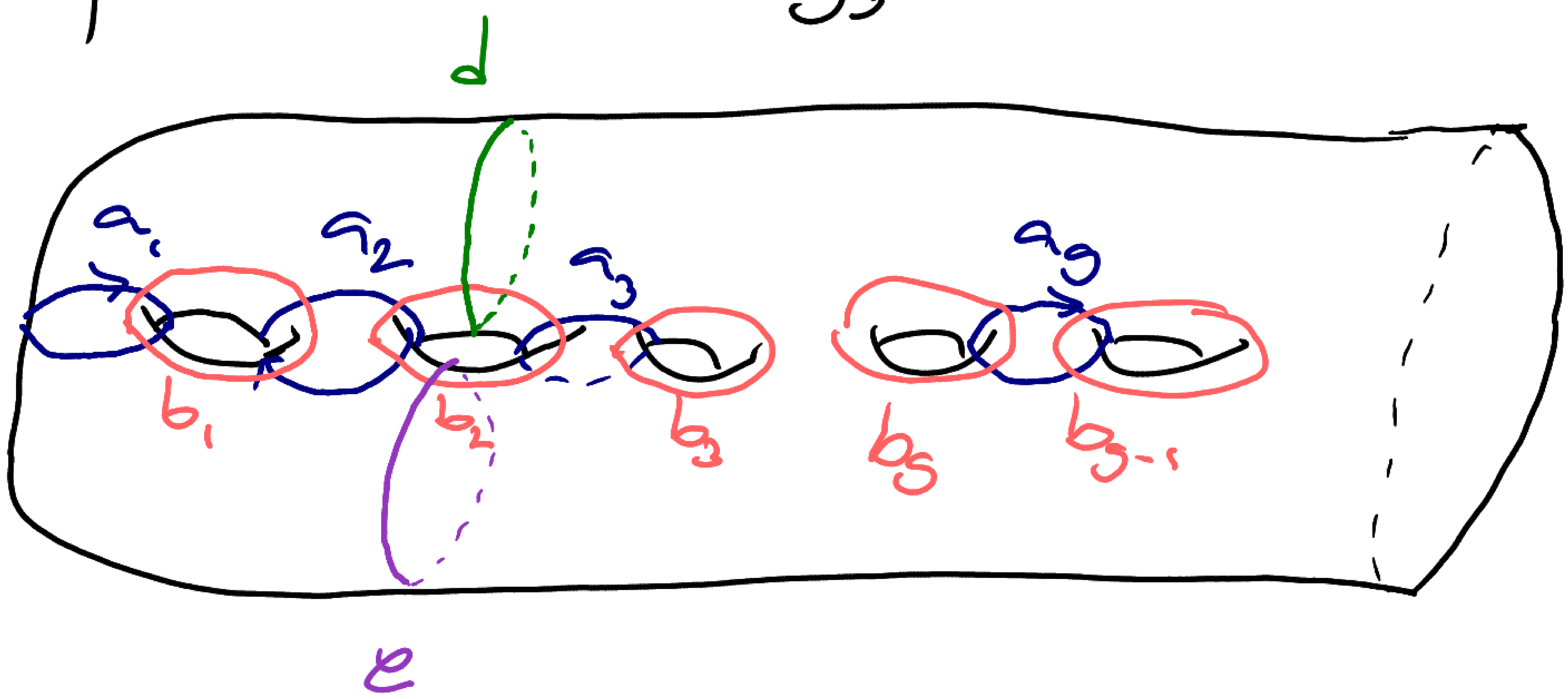
In $\Sigma_{0,4}$ = sphere with 4 boundary components





$$\tau_{d_0} \tau_{d_1} \tau_{d_2} \tau_{d_3} = \tau_{d_{12}} \tau_{d_{13}} \tau_{d_{23}}$$

A presentation of $\Gamma_{g,1}$



<u>Theorem</u>: $\tau_{a_1}, \ldots, \tau_{a_g}, \tau_{b_1}, \ldots, \tau_{b_g}, \tau_d$ are generators of $\Gamma_{g,1}$. The relations are

i) Braid relation (depending on if they are disjoint)

ii) Two-holed torus relation: $(\tau_{a_1}\tau_{b_1}\tau_{a_2})^4 = \tau_d \tau_e$
(if $g \geq 2$)

not a generator!

iii) Lantern relation involving a_1, a_2, a_3

Example: $\Gamma_{0,1} = 0$ Braid group on 3 strands

$$\Gamma_{0,2} = \langle a, b \mid aba = bab \rangle = Br_3$$

- Computing $H_1(\Gamma_{g,1})$

$$H_1(\Gamma_{g,1}) = Ab(\Gamma_{g,1})$$

→ The braid relation, when abelianized gives

$$xyx = yxy \Rightarrow x \equiv y.$$

Since we can connect each 'generating curve' to any other by a chain of intersecting curves

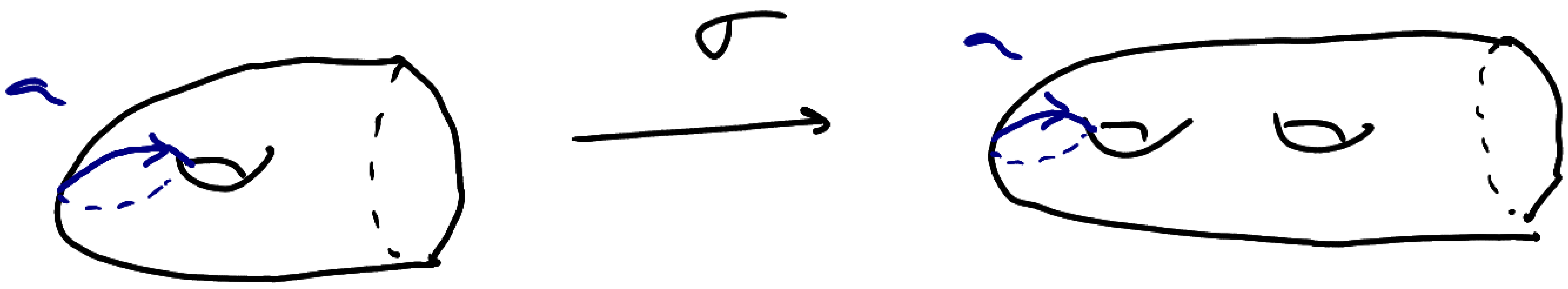
$$\Rightarrow \tau := \tau_c = \tau_{c'}$$

For $g=1$, no other relation: $H_1(\Gamma_{1,1}) = \mathbb{Z}\langle \tau \rangle$

For $g=2$, ii) says $\tau^{10} = \tau^2$: $H_1(\Gamma_{2,1}) = \mathbb{Z}/_{10}\langle \tau \rangle$

For $g \geq 3$, iii) says $\tau^4 \equiv \tau^3$: $H_1(\Gamma_{g,1}) = 0$

Moreover $\sigma : \mathbb{Z} \to \mathbb{Z}/10$.
$\tau \mapsto \tau$



- Computing $H_2(\Gamma_{g,1})$

For $g=1$, we have $\Gamma_{g,1} = Br_3$, then

$$B\Gamma_{1,1} \simeq (S^1 \vee S^1) \underset{\tau_a\tau_b\tau_a\tau_b^{-1}\tau_a^{-1}\tau_b^{-1}}{\cup} D^2 \cup \{\text{cells of dim} \geq 3\}$$

so its cellular chains are of the form

$$\mathbb{Z} \leftarrow \mathbb{Z}\oplus\mathbb{Z} \xleftarrow{(1,-1)} \mathbb{Z} \leftarrow ? \leftarrow ? \leftarrow \cdots$$

So we conclude

$$H_2(\Gamma_{1,1}) = 0$$

For $g \geq 2$, it can be used the Hopf's formula

$$G = F/R \, ; \quad H_2(G;\mathbb{Z}) = \frac{[F,F]\cap R}{[F,R]}$$

[Korkmaz-Stipsicz '03]

$$\begin{tikzcd}
H_2(\Gamma_{2,1};\mathbb{Z}) \arrow[r, phantom, "\cong"] & \mathbb{Z}/2 \arrow[d, "\sigma_*"', "\cong"] & & \\
H_2(\Gamma_{3,1};\mathbb{Z}) \arrow[r, phantom, "\cong"] & \mathbb{Z}/2 \oplus \mathbb{Z} \arrow[d, "\cong"', "\sigma_*"] & & \\
H_2(\Gamma_{4,1};\mathbb{Z}) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{Z} \arrow[d, "\cong"', "\sigma_*"] & & \\
H_2(\Gamma_{g,1};\mathbb{Z}) \arrow[r, phantom, "\cong"] & \mathbb{Z} & \forall g \geq 4 &
\end{tikzcd}$$

We want to identify the generators.

Given $\phi \in \Sigma_{g,1}$ it induces an isomorphism on $H_1(\Sigma_{g,1};\mathbb{Z})$. It respects the intersection form (which is skew-symmetric and non singular). Therefore there is a group homomorphism

$$\Gamma_{g,1} \longrightarrow Sp_{2g}(\mathbb{Z}).$$

topological group

Then consider

$$B\Gamma_{g,1} \longrightarrow BSp_{2g}(\mathbb{Z}) \longrightarrow BSp_{2g}(\mathbb{R}) \xleftarrow{\simeq} BU(g)$$

$U(g)$ maximal compact subgroup

→ Take the Chern class in $H^2(BU(g))$ and pullback to $\lambda_1 \in H^2(B\Gamma_{g,1})$ λ_1 is called Hodge class

<u>Facts</u>: λ_1 generates $H^2(B\Gamma_{g,1}) \cong \mathbb{Z}$ for $g \geq 3$

Then $\lambda \in H_2(B\Gamma_{g,1})$ (dual to λ_1) generates the free part of $H_2(B\Gamma_{2,1}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ and the whole $H_2(B\Gamma_{g,1}) \cong \mathbb{Z}$

if $g \geq 3$.

$\to$ This Hodge class is natural with respect to σ.