Hurewicz theorem and cellular algebras

<u>Theorem</u> (Hurewicz):

(X, A) topological spaces pair

$$h_*: \pi_k(X, A) \longrightarrow H_k(X, A)$$

If $n \geq 2$, $\pi_k(X, A) = 0$ for $k < n$, then

$H_k(X, A) = 0$ for $k < n$ and h_n is surjective.

<u>Theorem</u> (Brantner - Mathew)

$\mathbb{k}$,

$$\mathrm{SCR}_{\mathbb{k}}^{*, \mathrm{compl}, \mathrm{Nt}} \rightleftarrows \mathrm{Mod}_{\mathbb{k}}^{1\text{-}\mathrm{conn}}$$

$$(A \to \mathbb{k}) \longrightarrow L_{\mathbb{k}/A}[-1]$$

this adjunction is comonadic.

S, G = groupoid, $\oplus$, $\mathcal{C} = S^G$, $\mathcal{O}$ operad in $\mathcal{C}$

If $\mathcal{O} \longrightarrow \mathbb{k}[\mathbb{1}]$

$f: R \to S$ $\mathcal{O}$-algebras in $\mathcal{C}$

$H_*(S, R; \mathbb{k})$ $H_*^G(S, R; \mathbb{k})$

Slogan: for good $f: R \to S$ there is a relative cellular structure $R \to \operatorname{colim} sk(f) \xrightarrow{\simeq} S$

<u>Theorem</u> S = (pt) semistable left cpl

$\mathcal{O}$ = augmented h 0-conn Σ-cof

G = Artinian groupoid (ex $\mathbb{N}^{dist}$)

$$\mathcal{O}(1) = \begin{cases} 1)\ \varepsilon: H_{*,0}(\mathcal{O}(1); \mathbb{k}) \xrightarrow{\simeq} \mathbb{k}[1] \\ 2)\ \ker(\varepsilon) \text{ is nilpotent and } \oplus \text{ inv} \cong \mathbb{1}_G \end{cases}$$

$R \xrightarrow{f} S \in \mathrm{Alg}_{\mathcal{O}}(S^G)$ both h-$\mathcal{O}$-connected

If i) R, S are "reduced" or ii) $\mathcal{O} \simeq \mathcal{O}(1)$

$c: G \to [-\infty, \infty]_{\geq}$ such that

$$H^G_{g,d}(S, R; \mathbb{k}) = 0 \text{ for } d < c(g)$$

Then $\exists$ CW-structure

$$f: R \to \operatorname{colim} sk(f) \xrightarrow{\sim} S$$

$sk(f)$ has no (g,d)-cell for $d \leq c(g)$

§1. Connectivity

Category $[-\infty, \infty]_{\geq}$: obj: $\{-\infty, \infty\} \cup \mathbb{R}$; $\mathrm{Morph}(x,y) = \begin{cases} * & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$

$+$ is a symmetric monoidal structure $\infty + (-\infty) = \infty$

$c: G \to [-\infty, \infty]_{\geq}$ abstract connectivity

Day convolution: $c * c'$

$$c*c'(g) = \inf \{ |c(a) + c'(b)| \mid \exists\, a \otimes b \to g \}$$

<u>Definition</u>: $f: X \to Y$ in $S^G = \mathcal{C}$ is

<u>homological-c-con</u> if $H_{g,d}(Y, X; \mathbb{k}) = 0$ for $d < c(g)$

<u>Example</u>: $G = \mathbb{Z}^{\text{discrete}}$, $S = \mathrm{Mod}_R$

$c: n \longmapsto n$

Belison t-structure

<u>Lemma</u>: 1) $X, X' \in \mathcal{C} = S^G$ cofibrant are h-c, c'-connected then

$X \otimes X'$ is h-$(c*c')$-connected

2) $f: X'' \to X'$ is h-c_f-connected

X is h-c-connected

Then $X \otimes X'' \longrightarrow X \otimes X'$ is $h - c + c_f$ - connect.

<u>Proof</u>: Künneth, Spect. seq., $\mathrm{Tor}\left(H_{*,d}(X;k), H_{*,d'}(X';k)\right)$

$\Rightarrow \mathrm{Tor}\left(H_{*,d}(X \otimes X'; k)\right)$

<u>Corollary</u>: $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$

$f \otimes g : X \otimes Y \longrightarrow X' \otimes Y'$ is

$$h - \max\left\{ \begin{array}{l} \min\{c_X + c_g,\ c_f + c_{Y'}\} \\ \min\{c_f + c_Y,\ c_{X'} + c_g\} \end{array} \right\} \text{-connected}$$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\mathrm{id} \otimes g"] \arrow[d, "f \otimes \mathrm{id}"'] & X \otimes Y' \arrow[d] \\
X' \otimes Y \arrow[r] & X' \otimes Y'
\end{tikzcd}$$

$\min\{\ldots\}$

$\min\{c_f + c_Y,\ c_{X'} + c_g\}$

Corollary: $f: X \to Y$, $f^{\otimes n}: X^{\otimes n} \to Y^{\otimes n}$ is

h-$\min \{ c_X^{*a} * c_f * c_Y^{*b}, \ a+b=n-1 \}$

Proposition: $\mathcal{C} = \mathcal{S}^G$, G is non-unitary, Σ-cofibrant, h-0-connected

$R, S \in Alg_G(\mathcal{C})$ both h-c-connected,

$R \xrightarrow{f} S$ is h-c_f-connected, $c * c \geq c$

$$\begin{tikzcd}
(U^G R)_+ \arrow[r] \arrow[d] & Q^G_{G(1)} R \arrow[d] \\
(U^G S)_+ \arrow[r] & Q^G_{G(1)} S
\end{tikzcd}$$

is h-$(1+\delta)$-connected, $\delta = \min \{ c * c_f, c_f * c \}$

$$H_{g,d}(S,R) \longrightarrow H_{g,d}(Q^G_{G(1)} S,\ Q^G_{G(1)} R)$$

epimorphism for $d < 1 + \delta(g)$ and
isomorphism for $d < \delta(g)$

Proof:

$$\begin{array}{ccccc}
\|\mathrm{Dec}^G_{G(1)}(R) & \to & \|\mathrm{Bar}(G,G,R)\|_+ & \longrightarrow & \coprod Q^G_{G(1)} R \\
\downarrow & & \downarrow & & \downarrow \\
\|\mathrm{Dec}^G_{G(1)}(S) & \to & \|\mathrm{Bar}(O,O,S)\|_+ & \longrightarrow & \coprod Q^G_{G(1)} S \\
\downarrow & & \downarrow & & \downarrow \\
Z & \longrightarrow & X & \longrightarrow & Y
\end{array}$$

Z is h-δ-connected.

In the bar construction (decomposable) we have

$$\underbrace{G^{\geq 2} \circ G^{\circ(p-1)}}_{P} \circ R = \coprod_{n \geq 2} \left(P(n) \otimes R^{\otimes n}\right)_{S_p}$$

<u>Corollary</u>: $\mathcal{O}$ non-unitary h-0-connected Σ-cofibrant operad, G = Artin groupoid

$f: S \to R$ in $Alg_{\mathcal{O}}(\mathcal{C})$, S, R cofibrant, reduced $h.0$-connected, such that

$$H_{g',d'}(S,R) = 0 \text{ whenever } (\omega(g'), d') < (\omega(g), d)$$

then $H_{g,i}(S,R) \longrightarrow H_{g,i}(Q^{G}_{G(1)} S, Q^{G}_{G(1)} R)$

is isomorphism for $i \leq d$, surjection for $i = d+1$.

<u>Proposition</u>: $R, S \in Alg_{\mathcal{O}}(\mathcal{C})$ $\quad \mathcal{C} = S^{G}$

h-c-connected, $c * c \geq c$.

If $f: R \to S$ is h-c_f-connected

$$\begin{tikzcd}
(U^{G}R)_{+} \arrow[r] \arrow[d] & Q^{G}_{G(1)} R \arrow[d] \\
(U^{G}S)_{+} \arrow[r] & Q^{G}_{G(1)} S
\end{tikzcd}$$

is $H(\min\{c_f * c, c * c_f\})$ - cocartesian

<u>Definition</u>: $(G, \oplus, \mathbb{1})$ <u>Artinian</u>

$n \leq m \Leftrightarrow n \longrightarrow m$

$\mathcal{B}$ reduced if $H_{g,0}(\mathcal{B})=0$ for g $\oplus$-invertible

<u>Proof</u>: Consider $\begin{cases} c(g)=1 & \text{if } g \text{ is } \oplus\text{-invertible} \\ c(g)=0 & \text{else} \end{cases}$

$c*c \geq 0$.

f is c_f-connected

$$(c_f * c)(g) = \inf_{a \oplus b \to g} c_f(a) + c(b)$$

Two cases:

i) $\omega(b)=1$, $\omega(a) < \omega(g)$

$H_{a,i}(S,\mathcal{B})=0$ for $i \leq d$.

$c_f(a) \geq d+1 \Rightarrow c_f(a) + c(b) \geq d+1$

ii) $\omega(b)=0 \Rightarrow b$ invertible, $c(b) \geq 1$, $\omega(a)=\omega(g)$

$c_f(a) + c(b) \geq d+1$.

§3. Horewicz (II)

$$A := \mathcal{O}(1) \xrightarrow{\varepsilon} \mathbb{1}_{\mathcal{C}}$$

$$a = H_0(A; \Bbbk)$$

$$Q^{\mathcal{O}(1)}(cB) = B /\!\!/^{h}_{A}$$

<u>Lemma</u>: If it is h-0-connected, $f: B \to S$ such that

$$H_{*,d'}(S, B; \Bbbk) = 0 \quad \text{for } d' < d$$

$$\Bbbk[\mathbb{1}] \otimes_a H_*(S, B; \Bbbk) \longrightarrow H_*^{\mathcal{O}(1)}(S, B; \Bbbk)$$

is isomorphism for $i \leq d$, surjective for $i = d+1$.

§4. Whitehead theorem

$f: B \to S$, if $\mathbb{L}Q^{\mathcal{O}}(f) \simeq 0$ then f is a h-equivalence.

<u>Lemma</u>: G is Artinian and

i) $\varepsilon : a \longrightarrow \Bbbk[\mathbb{1}]$ is an isomorphism.

ii) $\ker(\varepsilon)$ is nilpotent and all $\oplus$-invertible

$g \simeq \mathbb{1}_G$

If M is an a-mod such that

$$\Bbbk[\mathbb{1}] \underset{a}{\otimes} M \simeq 0 \text{ for } g \in G \text{ if } \omega(g) \leq r$$

then $M(g) = 0$ for g such that $\omega(g) \leq r$

<u>Proposition</u>: $\mathcal{O}$ = augmented non-unitary

h - $\mathcal{O}$-connected Σ-cofibrant operad

G as in lemma.

$f : B \to S$ h-$\mathcal{O}$-connected algebras such that

$$H^{\mathcal{O}}_{*,d'}(S, B) = 0 \text{ for } d' \leq d$$

If either $\begin{cases} \text{i) } \mathcal{B}, \mathcal{S} \text{ are reduced} \\ \text{ii) } \mathcal{G} \simeq \mathcal{O}(1) \end{cases}$

then $H_{*,d}(\mathcal{S}, \mathcal{B}) = 0$ for $d' \leq d$ as well

§ 5. CW-approximation

$\mathcal{S}$ pointed and semi stable

$\mathcal{S}_* \times \mathcal{S} \longrightarrow \mathcal{S}$ $\rightsquigarrow \pi_*(X)$ for $X \in \mathcal{S}$

$\{\text{weak-equivalences in } \mathcal{S}\} \Leftrightarrow \left\{ \begin{matrix} \text{maps that are} \\ \text{bijective on } \pi_* \end{matrix} \right\}$

Theorem: $\mathcal{G}$ such that $\mathcal{G}_0$ has Nakayama?

$f: \mathcal{B} \to \mathcal{S}$ between h-0-connected algebras such that

either $\begin{cases} \text{i) } \mathcal{B}, \mathcal{S} \text{ reduced} \\ \text{ii) } \mathcal{G} \simeq \mathcal{O}(1) \end{cases}$

$c: G \longrightarrow [-\infty, \infty]$ such that $H^{\mathcal{G}}_{g,d}(\mathcal{S}, \mathcal{B}; \Bbbk) = 0$

for $d < c(g)$

then $\exists$ a relative CW-structure

$f : B \to \operatorname{colim} sk(A) \to S$ such that $sk(f)$ has

no (g,d)-cell for $d < c(g)$ $\quad Z_e = \operatorname{colim} sk_e(f)$

$$B \simeq sk_{-1}(f) \to sk_0(f) \to \cdots \xrightarrow{h_e} sk_e(f) \to O_* S$$

a) $H^G_{*,d}(S, \operatorname{colim} sk_v(f)) = 0$ for $d \leq v$

b) $h : sk_{i-1}(f) \to sk_i(f)$

only happens on dim v, and g such that $c(g) \leq e$

$\to$ If $e = \varepsilon - 1$ is done

$$H^G_{*,d}(S, Z_{\varepsilon-1}) = 0 \text{ for } d \leq \varepsilon - 1 \Rightarrow$$

$$H_{*,d}(S, Z_{\varepsilon-1}) = 0 \text{ for } d \leq \varepsilon - 1$$

<u>Claim</u>: $\pi_{*,\varepsilon}(S, Z_{\varepsilon-1}) \to H^G_{*,\varepsilon}(S, Z_\varepsilon ; \Bbbk)$ surjective

Proof:

$$\begin{array}{c}
\pi_{*,\varepsilon}(S, Z_{\varepsilon-1}) \cong H_{*,\varepsilon}(S, Z_{\varepsilon-1}) \\
\Big\downarrow \; {\scriptstyle d \leq \varepsilon - 1,\ \text{surj for } d = \varepsilon} \\
H_{*,d}\left(Q^G_{G(1)} S, Q^G_{G(1)} Z_{\varepsilon-1}; \Bbbk\right) \\
\Big\downarrow \\
\Bbbk[\mathbb{1}] \underset{a}{\otimes} \, ? \\
\Big\downarrow \\
H^G_{*,\varepsilon}(S, Z_{\varepsilon-1}; \Bbbk)
\end{array}$$

$\square$