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Tuesday 21 January 2025 14:03

Hurewicz Thm & cellular algebras

Thm (Hurewicz):

(X, A) nice top pair

$$h_k : \pi_k(X, A) \longrightarrow H_k(X, A)$$

If $\forall n \geq 2$ $\pi_k(X, A) = 0$ for $k < n$

then $H_k(X, A) = 0$ for $k < n$ and h_n is surjective.

Thm (Brantner–Mathew):

$\mathbb{K}$ $A \to \mathbb{K} \to \mathbb{L}_{\mathbb{K}/A}[-1]$

$$SCR_{\mathbb{K}}^{*, \text{cpl Noetherian}} \underset{\longleftarrow}{\overset{\text{cot}}{\longrightarrow}} \mathrm{Mod}_{\mathbb{K}}^{1\text{-conn}}$$

then adjunction is comonadic.

Let $\mathcal{S}$ category of spaces, G = gpd $\oplus$

$\mathcal{C} = \mathcal{S}^G$ $\mathcal{O}$ opd here

If $\mathcal{O} \longrightarrow \mathbb{K}[\mathbb{1}]$

$f : R \to S$ $\mathcal{O}$-alg in $\mathcal{C}$

$H_*(S; R; \mathbb{K})$ $H^{\mathcal{O}}(S; R; \mathbb{K})$

$$\mathbb{K}[\mathbb{1}] \circ^{\mathbb{L}}_{\mathcal{O}} R \longrightarrow \mathbb{K}[\mathbb{1}] \circ^{\mathbb{L}}_{\mathcal{O}} S \longrightarrow \mathrm{cof}$$

Slogan: For good $f: R \to S$ there is a relative cellular str

$R \to \operatorname{colim} sk(f) \xrightarrow{\sim} S$

Thm $\mathcal{S} = (pt)$ admissible left cpl

$\mathcal{O} :=$ Aug h. O-Comm Σ-cof

G = Artinian grpd (ex $\mathbb{N}^{disc}$)

$\mathcal{O}(1) = \begin{cases} ① \ \varepsilon: H_{*,0}(\mathcal{O}(1); \mathbb{K}) \xrightarrow{\sim} \mathbb{K}[\#] \\ ② \ \ker(\varepsilon) \text{ is nilpotent } \& \ \oplus \text{ invertible} \Rightarrow \simeq \mathbb{1}_G \end{cases}$

$R \xrightarrow{f} S \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{S}^G)$ both h-O-comm

if either

i) R, S are "reduced"

ii) $\mathcal{O} = \mathcal{O}(1)$

$G \to [-\infty, \infty]_{\geq}$ s.t.

$H^{\mathcal{O}}_{g,d}(S, R; \mathbb{K}) = 0$ for $d \leq c(g)$

Then $\exists$ CW-str

$f: R \to \operatorname{colim} sk(f) \xrightarrow{\sim} S$

$sk(f)$ has no (g,d)-cells for $d \leq c(g)$

§1 Connectivity

$[-\infty, \infty]_{\mathbb{Z}}$

$$\mathrm{Mor}(x,y) = \begin{cases} * & \text{if } x \geq y \\ \emptyset & \text{else} \end{cases}$$

Sym monoidal: $\infty + (-\infty) = \infty$.

$C : G \longrightarrow [-\infty, \infty]_{\mathbb{Z}}$ abstract connectivity

Day convolution: $C \circledast C$

$C \circledast C(g) = \inf \{ C(a) + C(b) \mid \exists\, a \otimes b \to g \}$

Df: $f : X \to Y$ in $\mathcal{S}^G$

is h-c-conn if $\pi_{g,d}(Y, X; k) = 0$ for $d \leq c(g)$.

homologically

Ex $G = \mathbb{Z}^{\text{discrete}}$

$\mathcal{S} = \mathrm{Mod}_k$ $\quad c : n \mapsto n$

Beilinson t-str

lemma ① $X, X' \in \mathcal{C} = \mathcal{S}^G$ cof

h-c, c' - connected

Then $X \otimes X'$ is h-$(c \circledast c')$-conn

② $f : X'' \to X'$ h-c_f-conn

X h-C-conn

then $X \otimes X'' \to X \otimes X'$ is h-$c \circledast c_f$-conn

Pf: Künneth spectral seq. ☒

Cor $X \xrightarrow{f} X'$

$Y \xrightarrow{g} Y'$

$$Y \xrightarrow{g} Y'$$

$f \otimes g : X \otimes Y \longrightarrow X' \otimes Y'$ is h-$\max\{\min\{c_X * c_g \,;\, c_f \otimes c_{Y'}\},$

$\min\{c_f * c_Y,\ c_{X'} * c_g\}\}$ -comm

Pf:

$$\begin{tikzcd}
X \otimes Y \arrow[r] \arrow[d] & X \otimes Y' \arrow[d] \\
X' \otimes Y \arrow[r] & X' \otimes Y'
\end{tikzcd}$$

Cor: $f : X \to Y$, c_X, c_Y, c_f

$f^{\otimes n} : X^{\otimes n} \longrightarrow Y^{\otimes n}$ is h-$\min\{c_X^{*a} * c_f \otimes c_Y^{b},\ a+b = n-1\}$ -comm

$\mathcal{C} = \mathcal{S}^G$ σ is a non-unitary h-σ-comm Σ-cof operad

Prop $R, S \in Alg_\sigma(\mathcal{C})$ both h-c-comm and $R \xrightarrow{h} S$ h-c_f-comm

$$\begin{tikzcd}
(U^\sigma R)_+ \arrow[r] \arrow[d] & Q^\sigma_{\sigma(1)} R \arrow[d] \\
(U^\sigma S)_+ \arrow[r] & Q^\sigma_{\sigma(1)} S
\end{tikzcd}$$

is h-$(1+\delta)$-cocartesian where $\delta = \min\{c * c_f,\ c_f * c\}$

$\Downarrow$

$Hg_d(S, R) \longrightarrow Hg_d(Q^\sigma_{\sigma(1)} S,\ Q^\sigma_{\sigma(1)} S)$ is epimorphism for $d \leq 1+\delta(g)$

isomorphism for $d \leq \delta(g)$

Proof: