Hurewicz theorem and cellular algebras

Theorem (Hurewicz):

(X, A) topological spaces pair

$$h_*: \pi_k(X, A) \longrightarrow H_k(X, A)$$

If $n \geq 2$, $\pi_k(X, A) = 0$ for $k < n$, then

$H_k(X, A) = 0$ for $k < n$ and h_n is surjective.

Theorem (Brantner-Mathew)

$\Bbbk$,

$$\mathrm{SCR}_{\Bbbk}^{*,\mathrm{cpl},\mathrm{Art}} \rightleftarrows \mathrm{Mod}_{\Bbbk}^{1\text{-conn}}$$

$$(A \to \Bbbk) \longrightarrow L_{\Bbbk/A}[-1]$$

this adjunction is comonadic.

S, G = groupoid, $\oplus$, $\mathcal{C} = S^G$, $\mathcal{O}$ operad in $\mathcal{C}$

If $\mathcal{O} \longrightarrow \Bbbk[\mathbb{1}]$

$f: R \to S$ $\mathcal{O}$-algebras in $\mathcal{C}$

$H_*(S, R; \Bbbk)$ $\quad$ $H_*^{\mathcal{O}}(S, R; \Bbbk)$

Slogan: for good $f: R \to S$ there is a relative cellular structure $R \to \operatorname{colim} sk(f) \xrightarrow{\simeq} S$

Theorem S = (pt) semistable left cpl

$\mathcal{O}$ = augmented $\wedge$ 0-conn Σ-cof

G = Artinian groupoid (ex $\mathbb{N}^{dist}$)

$$\mathcal{O}(1) = \begin{cases} 1) \ \varepsilon: H_{*,0}(\mathcal{O}(1); \Bbbk) \xrightarrow{\simeq} \Bbbk[1] \\ 2) \ \ker(\varepsilon) \text{ is nilpotent and } \oplus \text{ inv} \cong \mathbb{1}_G \end{cases}$$

$R \xrightarrow{f} S \in \mathrm{Alg}_{\mathcal{O}}(S^G)$ both h-$\mathcal{O}$-connected

If i) R, S are "reduced" or ii) $\mathcal{O} \simeq \mathcal{O}(1)$

$c : G \to [-\infty, \infty]_{\geq}$ such that

$$H^G_{g,d}(S, R; \Bbbk) = 0 \text{ for } d < c(g)$$

Then $\exists$ CW-structure

$$f : R \to \operatorname{colim} sk(f) \xrightarrow{\sim} S$$

$sk(f)$ has no (g,d)-cell for $d \leq c(g)$

§1. Connectivity

Category $[-\infty, \infty]_{\geq}$ — obj: $\{-\infty, \infty\} \cup \mathbb{R}$; $\mathrm{Morph}(x,y) = \begin{cases} * & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$

$+$ is a symmetric monoidal structure $\infty + (-\infty) = \infty$

$c : G \to [-\infty, \infty]_{\geq}$ <u>abstract connectivity</u>

Day convolution: $c * c'$

$$c * c'(g) = \inf \{ |c(a) + c'(b)| \mid \exists\, a \otimes b \to g \}$$

Definition: $f: X \to Y$ in $\mathcal{S}^G = \mathcal{C}$ is

homological-c-con if $H_{g,d}(Y, X; \Bbbk) = 0$ for $d < c(g)$

Example: $G = \mathbb{Z}^{\text{discrete}}$, $\mathcal{S} = \mathrm{Mod}_R$

$c: n \longmapsto n$

Belison t-structure

Lemma: 1) $X, X' \in \mathcal{C} = \mathcal{S}^G$ cofibrant are

h-c, c'-connected then

$X \otimes X'$ is h-$(c * c')$-connected

2) $f: X'' \to X'$ is h-c_f-connected

X is h-c-connected

Then $X \otimes X'' \longrightarrow X \otimes X'$ is $h - c + c_f$ - connect.

Proof: Künneth, Spect. seq., $\mathrm{Tor}\left(H_{*,d}(X;\Bbbk), H_{*,d'}(X';\Bbbk)\right)$

$\Rightarrow \mathrm{Tor}\left(H_{*,d}(X \otimes X'; \Bbbk)\right)$

Corollary: $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$

$f \otimes g : X \otimes Y \longrightarrow X' \otimes Y'$ is

$$h - \max\left\{ \begin{matrix} \min\{c_X + c_g,\ c_f + c_{Y'}\} \\ \min\{c_f + c_Y,\ c_{X'} + c_g\} \end{matrix} \right\} \text{-connected}$$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\mathrm{id} \otimes g"] \arrow[d, "f \otimes \mathrm{id}"'] & X \otimes Y' \arrow[d] \\
X' \otimes Y \arrow[r] & X' \otimes Y'
\end{tikzcd}$$

$\min\{\ldots\}$

$\min\{c_f + c_Y,\ c_{X'} + c_g\}$

<u>Corollary</u>: $f: X \to Y$, $f^{\otimes n}: X^{\otimes n} \to Y^{\otimes n}$ is

$$h\text{-}\min\left\{ c_X^{*a} * c_f * c_Y^{*b},\ a+b=n-1 \right\}$$

<u>Proposition</u>: $\mathcal{C} = S^G$, G is non-unitary, Σ-cofibrant, h-0-connected

$R, S \in Alg_G(\mathcal{C})$ both h-c-connected,

$R \xrightarrow{f} S$ is h-c_f-connected, $c * c \geq c$

$$\begin{tikzcd}
(U^G R)_+ \arrow[r] \arrow[d] & Q^G_{G(1)} R \arrow[d] \\
(U^G S)_+ \arrow[r] & Q^G_{G(1)} S
\end{tikzcd}$$

is h-$(1+\delta)$-connected, $\delta = \min\{ c * c_f, c_f * c \}$

$$H_{g,d}(S,R) \longrightarrow H_{g,d}(Q^G_{G(1)} S, Q^G_{G(1)} R)$$

epimorphism for $d < 1 + \delta(g)$ and
isomorphism for $d < \delta(g)$

Proof:

$$\begin{tikzcd}
\| \mathrm{Dec}^G_{G(1)}(R) \| \arrow[r] \arrow[d] & \| \mathrm{Bar}(G,G,R) \|_+ \arrow[r] \arrow[d] & \coprod Q^G_{G(1)} R \arrow[d] \\
\| \mathrm{Dec}^G_{G(1)}(S) \| \arrow[r] \arrow[d] & \| \mathrm{Bar}(G,G,S) \|_+ \arrow[r] \arrow[d] & \coprod Q^G_{G(1)} S \arrow[d] \\
Z \arrow[r] & X \arrow[r] & Y
\end{tikzcd}$$

Z is h-δ-connected.

In the bar construction (decomposable) we have

$$\underbrace{G^{\geq 2} \circ G^{\circ(p-1)}}_{P} \circ R = \coprod_{n \geq 2} \left(P(n) \otimes R^{\otimes n} \right)_{S_p}$$