

Hurewicz theorem and cellular algebras

Theorem (Hurewicz):

(X, A) topological spaces pair

$$h_*: \pi_k(X, A) \longrightarrow H_k(X, A)$$

If $n \geq 2$, $\pi_k(X, A) = 0$ for $k < n$, then

$H_k(X, A) = 0$ for $k < n$ and h_n is surjective.

Theorem (Brantner - Mathew)

$$|k|, \quad \text{SCR}_{|k}^{*, \text{compl}, Nt} \rightleftarrows \text{Mod}_{|k}^{1\text{-Conn}}$$

$$(A \rightarrow |k|) \longrightarrow \bigcup_{|k/A} [-1]$$

this adjunction is comonadic.

$S, G = \text{groupoid}, \oplus, \mathcal{L} = S^G, G \text{ operad in } \mathcal{L}$

If $G \longrightarrow \mathbb{k}[1]$

$f: R \rightarrow S$ G -algebras in $\mathcal{L}$

$$H_*(S, R; \mathbb{k}) \quad H_*^G(S, R; \mathbb{k})$$

Slogan: for good $f: R \rightarrow S$ there is a relative cellular structure $R \rightarrow \text{colim } \text{sk}(f) \xrightarrow{\sim} S$

Theorem $S = (\text{pt})$ semistable left $\mathcal{O}$

$G = \text{augmented h O-conn } \Sigma\text{-coet}$

$G = \text{Artinian groupoid (ex } \mathbb{N}^{\text{dist}})$

$$G(1) = \begin{cases} 1) \varepsilon: H_{*,0}(G(1); \mathbb{k}) \xrightarrow{\sim} \mathbb{k}[1] \\ 2) \ker(\varepsilon) \text{ is nilpotent and } \oplus \text{ inu} \cong 1_G \end{cases}$$

$R \xrightarrow{f} S \in \text{Alg}_G(S^G)$ both h- $\mathcal{O}$ -connected

If i) R, S are "reduced" or ii) $\mathcal{O} \simeq \mathcal{O}(1)$

$c: G \rightarrow [-\infty, \infty]_{\geq}$ such that

$$H_{\mathcal{O}, d}^G(S, R; k) = 0 \text{ for } d < c(S)$$

Then $\exists$ CW-structure

$$f: R \rightarrow \text{colim sh}(f) \xrightarrow{\sim} S$$

$\text{sh}(f)$ has no $(\mathcal{O}, d)$ -cell for $d \leq c(S)$

§1. Connectivity

Category $[-\infty, \infty]_{\geq}$ $\begin{cases} \text{obj: } \{-\infty, \infty\} \cup \mathbb{R} \\ \text{Morph}(x, y) = \begin{cases} * & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases} \end{cases}$

$+$ is a symmetric monoidal structure $\infty + (-\infty) = \infty$

$c: G \rightarrow [-\infty, \infty]_{\geq}$ abstract connectivity

Day convolution: $c * c'$

$$c * c'(g) = \inf \{ |c(a) + c'(b)| \mid \exists a \otimes b \rightarrow g \}$$

Definition: $f: X \rightarrow Y$ in $S^G = \mathcal{C}$ is

homological-c-con if $H_{\mathbb{Z}_d}(Y, X; k) = 0$ for $d < c(g)$

Example: $G = \mathbb{Z}^{\text{discrete}}$, $S = \text{Mod}_R$

$$c: n \mapsto n$$

Beilinson t-structure

Lemma: 1) $X, X' \in \mathcal{C} = S^G$ cofibrant are

h - c, c' -connected then

$X \otimes X'$ is h - $(c * c')$ -connected

2) $f: X'' \rightarrow X'$ is h - c_f -connected

X is h - c -connected

Then $X \otimes X'' \rightarrow X \otimes X'$ is $h - c + c_f - \text{connect.}$

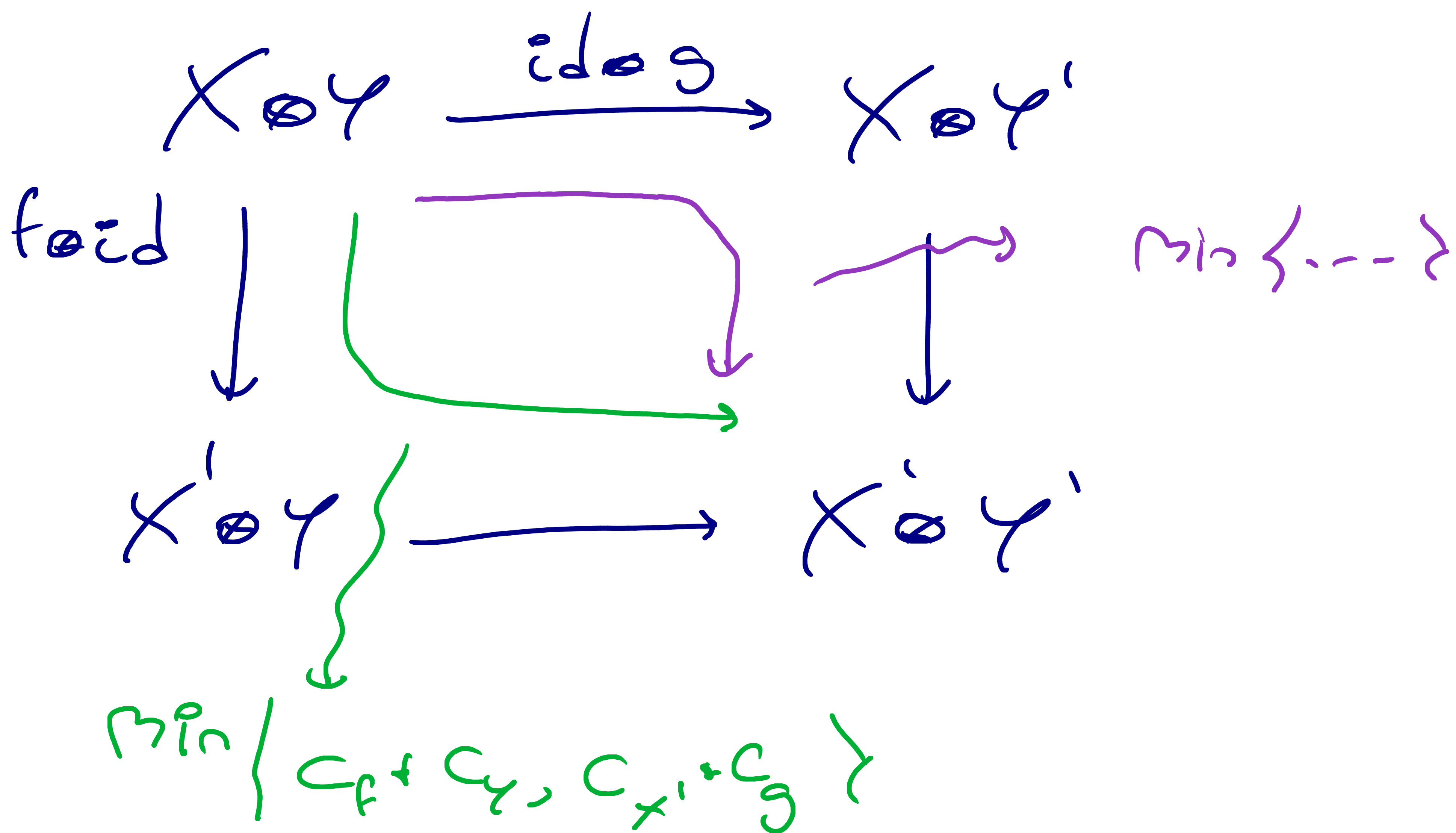
Proof: Künneth, Spect. seq., $\text{Tor}(H_{*,d}(X;k), H_{*,d'}(X';k))$

$$\Rightarrow \text{Tor}(H_{*,d}(X \otimes X'; k))$$

Corollary: $X \xrightarrow{f} X', Y \xrightarrow{g} Y'$

$f \otimes g: X \otimes Y \rightarrow X' \otimes Y'$ is

$$h - \max \left\{ \begin{array}{l} \min \{ c_X + c_g, c_f + c_{Y'} \} \\ \min \{ c_f + c_Y, c_{X'} + c_g \} \end{array} \right\} - \text{connected}$$



Corollary: $f: X \rightarrow Y$, $f^{\otimes n}: X^{\otimes n} \rightarrow Y^{\otimes n}$ is

$$h\text{-min} \left\{ c_X^{*a} * c_f * c_Y^{*b}, a+b=n-1 \right\}$$

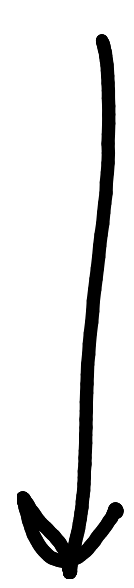
Proposition: $\mathcal{C} = S^G$, G is non-unitary,

Σ -cofibrant, h -0-connected

$R, S \in \text{Alg}_G(\mathcal{C})$ both h -c-connected,

$R \xrightarrow{f} S$ is h - c_f -connected, $c * c \geq c$

$$(U^G R)_+ \longrightarrow Q_{G(n)}^G R$$



$$(U^G S)_+ \longrightarrow Q_{G(n)}^G S$$

is h -($1+\delta$)-connected, $\delta = \min \{ c * c_f, c_f * c \}$

$$H_{g,d}(S, R) \longrightarrow H_{g,d}(Q_{G^{(1)}}^G, S, Q_{G^{(1)}}^G, R)$$

epimorphism for $d < 1 + \delta(g)$ and

isomorphism for $d < \delta(g)$

Proof:

$$\begin{array}{ccccc}
 \mathbb{L} \text{Dec}_{G^{(1)}}^G(R) \rightarrow \|\text{Bar}(G, G, R)\|_+ & \longrightarrow & \mathbb{L} Q_{G^{(1)}}^G R & & \\
 \downarrow & & \downarrow & & \\
 \mathbb{L} \text{Dec}_{G^{(1)}}^G(S) \rightarrow \|\text{Bar}(G, G, S)\|_+ & \longrightarrow & \mathbb{L} Q_{G^{(1)}}^G S & & \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbb{Z} & \longrightarrow & X & \longrightarrow & \mathbb{Z}
 \end{array}$$

$\mathbb{Z}$ is h - δ -connected.

In the bar construction (decomposable) we have

$$\underbrace{G^{\geq 2}}_P \circ G^{o(p-1)} \circ R = \bigsqcup_{n \geq 2} \left(P_n \otimes R^{\otimes n} \right)_{S_p}$$