Hurewicz thm & cellular algs

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Thm (Hurewicz)

(X, A) top. space pair

$h_*: \pi_*(X, A) \longrightarrow H_k(X, A)$

If $n \geq 2$, $\pi_k(X, A) = 0$ for $k < n$, then

$H_k(X, A) = 0$ for $k < n$ and h_n is surjective.

Thm (Brantner-Mathew)

$\Bbbk$ $\quad (A \to \Bbbk) \longmapsto \mathbb{L}_{\Bbbk/A}[-1]$

$$SCR_{\Bbbk}^{*, cpl\, Nt} \rightleftarrows \mathrm{Mod}_{\Bbbk}^{1\text{-conn}}$$

this adjunction is comonadic.

$\mathcal{S}$, $G = $ gpd $\quad \oplus$

$\mathcal{C} = \mathcal{S}^G$, $\quad \mathcal{O}$ operad here (non unitary $\mathcal{O}(0) = \phi$)

If $\mathcal{O} \longrightarrow \Bbbk[1]$

$f: R \longrightarrow S$ $\quad \mathcal{O}$-alg. in $\mathcal{C}$

$H_*(S, R, \Bbbk)$ $\quad H_*^{\mathcal{O}}(S, R; \Bbbk)$

$\Bbbk[1] \circ_G^{\mathbb{L}} R \longrightarrow \Bbbk[1] \circ_G^{\mathbb{L}} S \longrightarrow$ Cofiber

Slogan: for good $f: R \to S$, there is a relative cellular
struct. : $R \longrightarrow \operatorname{colim} sk(f) \xrightarrow{\sim} S$.

Thm: $\mathcal{S}$ = (pt) semistable left cpl $\circlearrowleft \Bbbk$
$\pi_* = H(-,\Bbbk)$

$\mathcal{G}$ = aug. h. 0-connected Σ-cof.

G = Artinian gpd (ex.: $\mathbb{N}^{dist}$)

$$\mathcal{G}(1) = \begin{cases} ① \ \varepsilon : H_{*,0}(\mathcal{G}(1); \Bbbk) \xrightarrow{\simeq} \Bbbk[n] \\ ② \ \ker(\varepsilon) \text{ is nilpotent \& } \oplus\text{invertible} \iff \cong \mathbb{1}_G \end{cases}$$

$R \to S \in Alg_{\mathcal{G}}(\mathcal{S}^G)$ both h-0-conn.

If i) R, S are "reduced"

$\boxed{\text{OR}}$

ii) $\mathcal{G} \simeq \mathcal{G}(1)$

$c : G \longrightarrow [-\infty, \infty]_{\geq}$ s.t. $H^{\mathcal{G}}_{g,d}(S, R; \Bbbk) = 0$

for $d < c(g)$

Then $\exists$ CW str. $f: R \longrightarrow \operatorname{colim} sk(f) \xrightarrow{\sim} S$

$sk(f)$ has no (g,d)-cell for $d < c(g)$.

§1. Connectivity

$[-\infty, \infty]_{\geq}$

obj: $[-\infty, \infty] \cup \mathbb{R}$

$$Mor(x,y) = \begin{cases} * & \text{if } x \geq y \\ \emptyset & \text{else} \end{cases}$$

$+$ is a sym. mon. str.

$\infty + (-\infty) = \infty$.

$c : G \longrightarrow [-\infty, \infty]_{\geq}$ **abstract connectivity**

Day convolution : $c * c'$

$$c * c'(g) = \inf\{ c(a) + c'(b) \mid \exists\, a \otimes b \to g \}$$

Def: $f : X \to Y$ in $\mathcal{S}^G$ is $\underline{h\text{-}c\text{-conn.}}$ if

$H_{g,d}(Y, X; k) = 0$ for $d < c(g)$.

Ex.: $G = \mathbb{Z}^{disc}$, $\mathcal{S} = Mod_k$, $c : n \mapsto n$

Belinson struct.

Lemma: ① $X, X' \in \mathcal{C} = \mathcal{S}^G$ cof.

$h\text{-}c, c'\text{-conn}$.

$X \otimes X'$ is $h\text{-}(c * c')\text{-conn}$.

② $f : X'' \to X'$ is $h\text{-}c_f\text{-conn}$.

X is $h\text{-}c\text{-conn}$.

Then $X \otimes X'' \longrightarrow X \otimes X'$ is $h\text{-}c * c_f\text{-conn}$.

Pf: Künneth spectral seq.

$$\mathrm{Tor}(H_{*,d}(X; k), H_{*,d'}(X'; k)) \Rightarrow H_{*,d}(X \otimes X'; k) \ldots$$

Cor: $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$,

$X \otimes Y \xrightarrow{f \otimes g} X' \otimes Y'$ is $h\text{-}\max\{\min\{c_X * c_g, c_f * c_{Y'}\},$ $\min\{c_f * c_Y, c_{X'} * c_g\}\}\text{-conn}$.

Pf:
$$\begin{tikzcd}
X \otimes Y \arrow[r, "id \otimes g"] \arrow[d, "f \otimes id"'] & X \otimes Y' \arrow[d] \\
X' \otimes Y \arrow[r] & X' \otimes Y'
\end{tikzcd}$$

$$\max\left\{ \begin{matrix} \min\{c_f * c_Y, c_{X'} * c_g\} \\ \min\{ \ldots \} \end{matrix} \right\}$$

∎

Cor: $f : X \to Y$, c_X, c_Y, c_f.

$f^{\otimes n} : X^{\otimes n} \to Y^{\otimes n}$ is h-$\min\{ c_X^{*a} * c_f * c_Y^{*b} \mid a+b = n-1 \}$-con.

h-0-conn

Prop.: $R, S \in \mathrm{Alg}_G(\mathcal{E})$ both h-c-conn.

$R \xrightarrow{f} S$ h-c_f-conn.

$c * c \geq c$

$$\begin{array}{ccc}
(U^G R)_+ & \longrightarrow & Q^G_{G(1)} R \\
\downarrow & & \downarrow \\
(U^G S)_+ & \longrightarrow & Q^G_{G(1)} S
\end{array}$$

is h-$(1+\delta)$-conn. with $\delta = \min\{ c * c_f, c_f * c \}$

$H_{g,d}(R, S) \longrightarrow H_{g,d}(Q^G_{G(1)} R, Q^G_{G(1)} S)$

epimorphism for $d < 1 + \delta(g)$

isomorphism for $d < \delta(g)$