Homological stability for braided monoidal groupoid

We consider $(G, \oplus, \mathbb{1}_G)$ a braided monoidal groupoid, i.e. $G_x \oplus G_y \xrightarrow[\beta_{x,y}]{\cong} G_y \oplus G_y$ is a braid.

We denote $G_x = \hom_G(x,x)$, which is a group.

Moreover we have a braided monoidal functor

$$r : (G, \oplus, \mathbb{1}_G) \longrightarrow (\mathbb{N}, +, 0).$$

We assume 3 simplyfing hypothesis

i) $r(x) = 0 \iff x \cong \mathbb{1}_G$

ii) $G_{\mathbb{1}_G} = *$

iii) $- \oplus - : G_x \oplus G_y \longrightarrow G_{x \oplus y}$ injective.

Then we can construct a classifying space

$$BG \simeq \coprod_{[x]\in \pi_0(G)} BG_x$$

but this is not, a priori, a unital (graded) E_2-algebra.

- We consider the following main example

<u>Example</u>: Let $G = MCG$ the mapping class groupoid

$$Obj(G) = \mathbb{N} = \{0, 1, \dots\}$$

$$G_n = \Gamma_{n,1} = \pi_0(\mathrm{Diff}_\partial(\Sigma_{n,1}))$$

The monoidal structure is given as follows:

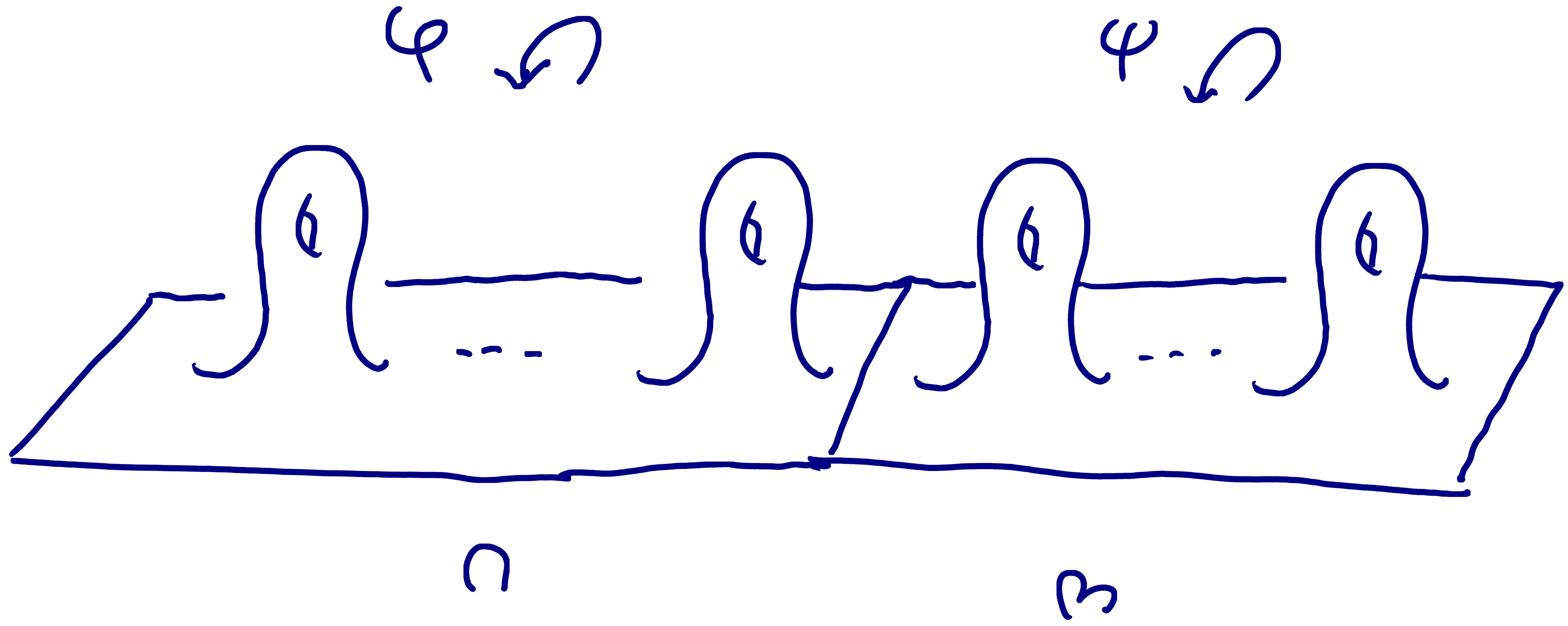
$$n \oplus m = n+m$$

and given $[\varphi] \in \mathrm{Diff}_\partial(\Sigma_{n,1})$

$$[\psi] \in \mathrm{Diff}_\partial(\Sigma_{m,1})$$

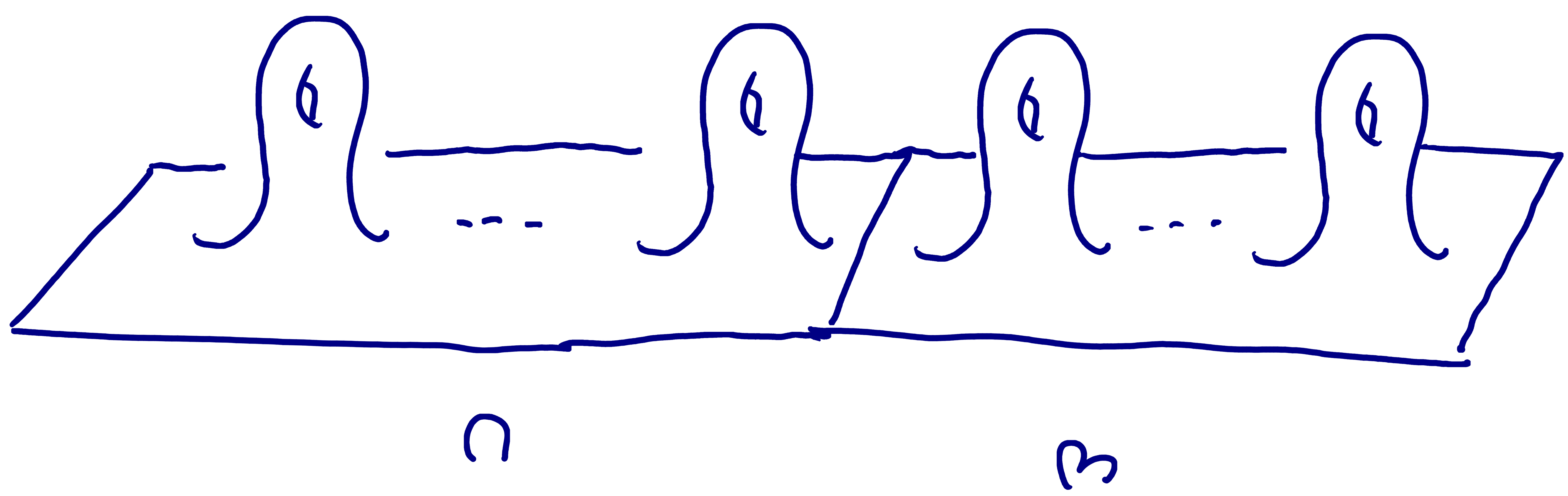
Consider

$\varphi \oplus \varphi$:

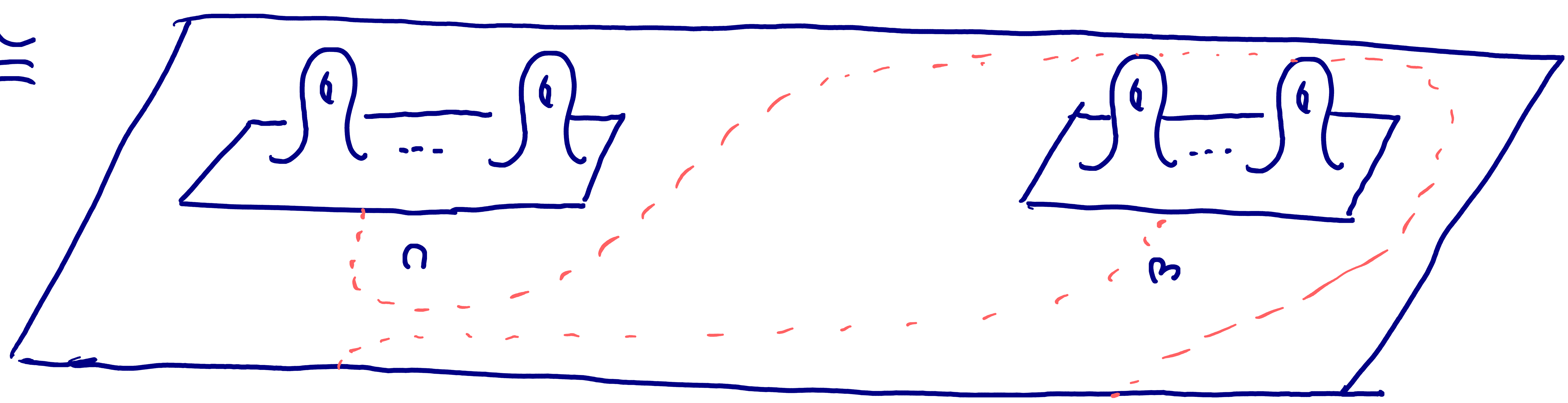


The braided monoidal structure is given by

$n \oplus m =$

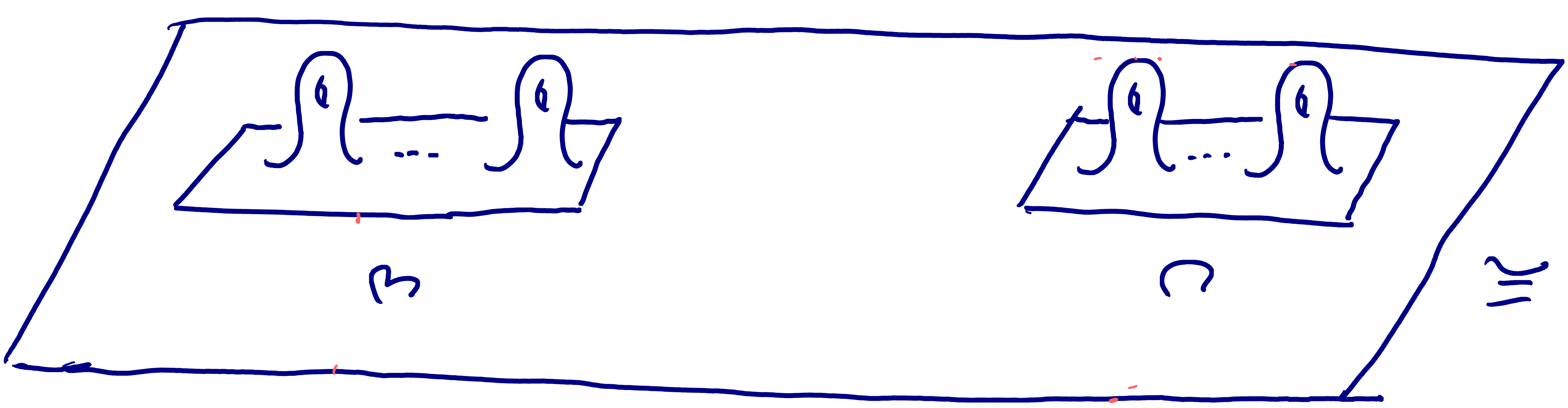


$\cong$



Perform Dehn twist

$\downarrow \cong$



$\cong m \oplus n$

$r: MCG \longrightarrow \mathbb{N}$ is just the identity on objects

$n \longmapsto n$

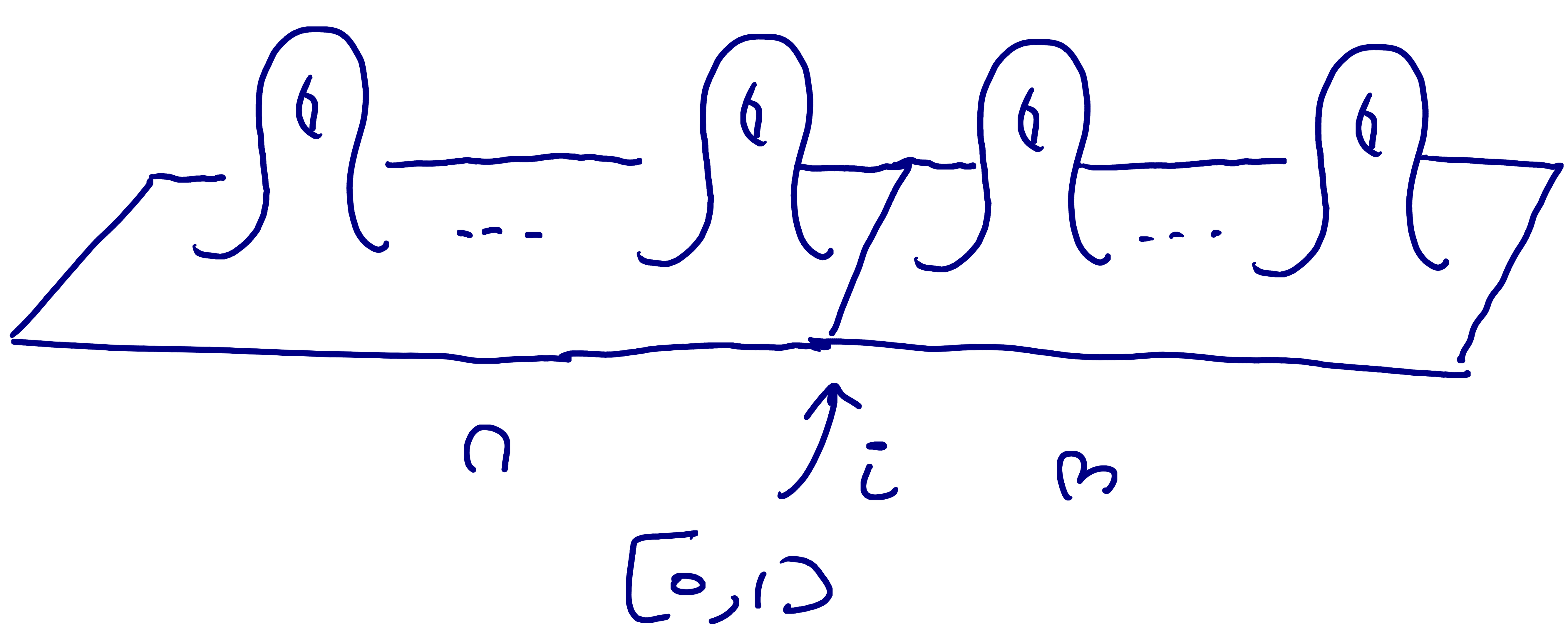
The three conditions hold:

i) $r(n) = 0 \Leftrightarrow n = 0$

ii) $G_0 = \Gamma_{0,1} = *$

iii) <u>Lemma</u>: $- \oplus - : \Gamma_{n,1} \times \Gamma_{m,1} \longrightarrow \Gamma_{n+m,1}$ is injective

<u>Proof</u> (sketch) Let $i : [0,1) \to \Sigma_{n+m,1}$



$$\mathrm{Diff}_\partial(\Sigma_{n+m,1}) \xrightarrow{i^*} \mathrm{Emb}_\partial([0,1), \Sigma_{n+m,1})$$

$$\varphi \longmapsto \varphi \circ i$$

is a Serre fibration (!)

whose fiber is $\mathrm{Diff}_\partial(\Sigma_{g,1}) \times \mathrm{Diff}_\partial(\Sigma_{h,1})$.

Since $\mathrm{Emb}_\partial([0,1], \Sigma_{g+h+1}) \simeq \coprod *$ (!)

By a long exact sequence argument

$$\pi_1(\coprod *) \to \Gamma_{g,1} \times \Gamma_{h,1} \hookrightarrow \Gamma_{g+h,1} \to \pi_0(\coprod *)$$

$\pi_1(\coprod *) = 0$

The goal is to endow BG with a unital $\mathbb{N}$-graded E_2-algebra structure.

We work in the category $sSet^G$ which, by the Day convolution, is again a braided monoidal categ.

We can define the E_2-monad in this category

$$E_2(-): sSet^G \to sSet^G$$

and we can talk about E_2-algebras.

Consider $* \in sSet^G$ defined as

$$*(x) = \begin{cases} \emptyset & \text{if } x \cong 1\!\!1_G \\ * & \text{otherwise.} \end{cases}$$
$x \in G$

→ $*$ is an algebra over any operad (its endomorphism operad is the terminal operad).

In particular over E_2.

However it is not cofibrant in $Alg_{E_2}(sSet^G)$ (Not even in $sSet^G$, except if $G_* = *$).

Thank to the theory developed, we can take a cellular approximation $T \xrightarrow{\simeq} *$ in $Alg_{E_2}(sSet)$.

We have a diagram

$$\begin{tikzcd} G \arrow[r, "T"] \arrow[d, "r"'] & sSet \\ \mathbb{N} \arrow[ur, dashed, "r_* T"'] & \end{tikzcd}$$

We can perform the left Kan extension to obtain

$$r_* T = R \in Alg_{E_2}(sSet^{\mathbb{N}})$$

which is again cellular.

<u>Remark</u>: Let's compute it explicitly in the case $G = MCG$

$$R(n) = \operatorname{colim}_{r \downarrow const_n} T(n) = \operatorname{colim}_{G_n} T(n) \simeq T(n)/G_n \simeq BG_n$$

since $T(n) \simeq *$ and G_n acts freely : T is cofibrant in $Alg_{E_2}(sSet^G) \Rightarrow$ cofibrant in $sSet^G$.

In general $\mathcal{B}(n) \simeq \coprod_{\substack{[x] \in \pi_0(\mathcal{G}) \\ r(x) = n}} BG_x$

Conclusion: $\mathcal{B}$ is (graded) homotopy equivalent to $B\mathcal{G}$ and $\mathcal{B}$ is a E_2-algebra.

Remark: If $\mathcal{G}$ is symmetric rather than braided, we obtain an E_∞-algebra.

- Homological stability of $\mathcal{B}$.

From now on, we suppose $\mathrm{Obj}(\mathcal{G}) = \mathbb{N}$.

To compute the $\Bbbk$-homology we linearize $\mathcal{B}$

$$[-]_{\Bbbk} : \mathrm{sSet} \to \mathrm{sMod}_{\Bbbk}$$

$$\mathcal{B} \longmapsto \mathcal{B}_{\Bbbk} \in \mathrm{Alg}_{E_2}(\mathrm{sMod}_{\Bbbk}^{\mathbb{N}})$$

Consider also the unitalisation $\mathcal{R}^+_{\Bbbk}$.

Since $\mathcal{R}^+(1) \simeq BG(1) = BG_1$ is connected

pick any point to define

$$\sigma : \Bbbk\left[\begin{matrix}\deg = 0\\ \text{genus} = 1\end{matrix}\right] \otimes \mathcal{R}^+_{\Bbbk} \xrightarrow{\text{pt} \otimes \text{id}} \mathcal{R}^+_{\Bbbk} \otimes \mathcal{R}^+_{\Bbbk} \xrightarrow{\bullet} \mathcal{R}^+_{\Bbbk}$$

$S^{1,0}$

operation as E_2-algebra

to obtain a map

$$\sigma : \mathcal{R}^+_{\Bbbk}(\bullet - 1) \to \mathcal{R}^+_{\Bbbk}(\bullet)$$

In our example, we see that this is the map induced by

$$\Gamma_{n-1,1} \longrightarrow \Gamma_{1,1} \times \Gamma_{n-1,n} \xrightarrow{\oplus} \Gamma_{n,1}$$

$$\varphi \longmapsto (\mathrm{id}, \varphi)$$

i.e. the stabilization map.

We take the homotopy cofiber $\mathcal{B}^+_{\Bbbk}/\sigma$ in $s\mathrm{Mod}_{\Bbbk}^{\mathbb{N}}$.

Then $H_{n,d}(\mathcal{B}^+_{\Bbbk}/\sigma;\Bbbk) \cong H_d(\mathcal{B}(n),\mathcal{B}(n-1);\Bbbk)$

$\cong H_d(G_n, G_{n-1};\Bbbk)$.

Therefore the goal is to find (n,d) where the homology of $\mathcal{B}^+_{\Bbbk}/\sigma$ vanishes.

- Computing the derived indecomposables of $\mathcal{B}^+$

Recall (from the last talk): $\widetilde{B}^{E_k}(-) \simeq S^k \wedge Q_{\mathbb{L}}(-)$ (as functors). Therefore

$$\widetilde{B}^{E_1}(\mathcal{B}^+) \simeq S^1 \wedge Q^{E_1}_{\mathbb{L}}(\mathcal{B}) \Rightarrow$$

canonical augmentation

$$\Rightarrow B^{E_1}(\mathcal{B}^+) \simeq S^0 \vee (S^1 \wedge Q^{E_1}_{\mathbb{L}}(\mathcal{B}))$$

$*$ Since $r_*: sSet^G \to sSet^{\mathbb{N}}$ is symmetric monoidal and preserves colimits

left Kan extension "quotient by G_x"

then $B^{E_1}(B^+) = B^{E_1}(r_* T^+) = r_* B^{E_1}(T^+)$

We define a new object: a semi-simplicial set to compare it with $B^{E_1}(T^+)$.

<u>Definition</u>: Given $x \in G$, we define the semi-simplicial set $S_\bullet^{E_1}(x)$, called E_1-splitting complex as

$$S_p^{E_1}(x) = \operatorname*{colim}_{\substack{x_0, \dots x_{p+1} \\ r(x_i) > 0}} \hom_G(x_0 \oplus \dots \oplus x_{p+1}, x)$$

and where the faces are induced by "merging x_i's" induced by $G_{x_i} \times G_{x_{i+1}} \longrightarrow G_{x_i \oplus x_{i+1}}$

Example: $G = MCG$

In our case, we obtain a more explicit form

$$S_p^{\epsilon_1}(n) = \coprod_{\substack{n_0+\dots+n_{p+1}=n \\ n_i>0}} \frac{G_n}{G_{n_0}\times G_{n_1}\times\dots\times G_{n_{p+1}}}$$

considered as subgroup

Since $Obj(G)=\mathbb{N}$. The faces are induced by

$$G_{n_i}\times G_{n_{i+1}} \hookrightarrow G_{n_i+n_{i+1}} \Rightarrow$$

$$\Rightarrow \frac{G}{\dots\times G_{n_i}\times G_{n_{i+1}}\times\dots} \xrightarrow{d_i} G / \dots \times G_{n_i+n_{i+1}}\times\dots$$

Remark: $S_p^{\epsilon_1}(x)=\emptyset$ if $p>r(x)-2$

Since $r(x_i)>0 \Rightarrow r(x_0\oplus\dots\oplus x_{p+1}) \geq p+2 \Rightarrow$ if $r(x)<p+2 \Rightarrow \nexists\, x_0,\dots,x_{p+1}$ such that $x_0\oplus\dots\oplus x_{p+1}=x$.

Theorem: fix $x \in G$. There is a G_x-equivariant homotopy equivalence

thick realization

$$B^{E_1}(T^+)(x) \simeq \Sigma^2 \| S^{G}_{\bullet}(x) \|$$

Proof (sketch):

Recall the bar construction:

tensor product in $sSet_*^G$ (Day convolution)

$$B_p^{E_1}(T^+) = P_p \times (T^+)^{\otimes p}$$

$$B_p^{E_1}(T^+)(x) = \operatorname*{colim}_{x_1, \dots, x_p \in G} P_{p+} \wedge \hom_G(x_1 \oplus \cdots \oplus x_p, x)_+ \wedge \underbrace{T^+(x_1)}_{\simeq *} \wedge \cdots \wedge \underbrace{T^+(x_p)}_{\simeq *}$$

$$\simeq \operatorname*{colim}_{x_1, \dots, x_p \in G} \hom_G(x_1 \oplus \cdots \oplus x_p, x)_+$$

P_p contractible

Since $G_{x_1} \oplus \cdots \oplus G_{x_p} \to G_{x_p}$ injective this last term is a discrete set.

Moreover this discrete set agrees with $S_{p-2}^{\epsilon_1(x)}$..

It can be proved that, effectively (*)

$$B^{\epsilon_1}(T^+)(x) \simeq \Sigma^2 \| S_\bullet^{\epsilon_1(x)} \|$$ and

by construction it is G_x-equivariant.

(*) The faces are constructed similarly

<u>Corollary</u>: $S^1 \wedge Q_{\amalg}^{\epsilon_1}(\mathcal{B})(n) \simeq \bigvee_{\substack{[x]\in\pi_0(G) \\ r(x)=n}} \Sigma^2 \| S_\bullet^{\epsilon_1(x)} \| /\!/ G_x$ ← homotopy orbits in $sSet_*$

for $n > 0$.

<u>Proof</u>: $S^0 \vee (S^1 \wedge Q_{\amalg}^{\epsilon_1}(\mathcal{B})) \simeq B^{\epsilon_1}(\mathcal{B}^+)$

$$\Rightarrow S^1 \wedge Q_{\amalg}^{\epsilon_1}(\mathcal{B})(n) \simeq B^{\epsilon_1}(\mathcal{B}^+)(n) =$$

$$= B^{\epsilon_1}(r_* \mathcal{B}^+)(n) = r_* B^{\epsilon_1}(\mathcal{B}^+)(n) \simeq$$

$$\simeq \bigvee_{\substack{[x]\in\pi_0(G) \\ r(x)=n}} \Sigma^2 \| S_\bullet^{\epsilon_1(x)} \| /\!/ G_x$$

$d>0$ "equivariant homology"

In particular,

(*) $H^{\epsilon_1}_{n,d-1}(B) \cong \bigoplus_{\substack{[x]\in\pi_0(G)\\ r(x)=n}} H_d\left(\Sigma^2 \|S^{\epsilon_1}_\bullet(x)\| /\!/ G_x\right)$

Proposition: If $\widetilde{H}_*(\|S^{\epsilon_1}_\bullet(x)\|; \Bbbk)=0$ for $*<r(x)-2$, then $H_{n,d}(B;\Bbbk)=0$ for $d<n-1$

Proof: Recall that $\widetilde{H}_*(\|S^{\epsilon_1}_\bullet(x)\|;\Bbbk)=0$ for $*>r(x)-2$

$\Rightarrow$ its cohomology is concentrated at $*=r(x)$

$$H_{r(x)}(\Sigma^2\|S^{\epsilon_1}_\bullet(x)\|;\Bbbk)=M$$

Steinberg module

and we consider it as a $\mathbb{Z}[G_x]$-module.

Remark: there is a spectral sequence (Cartan-Leray)

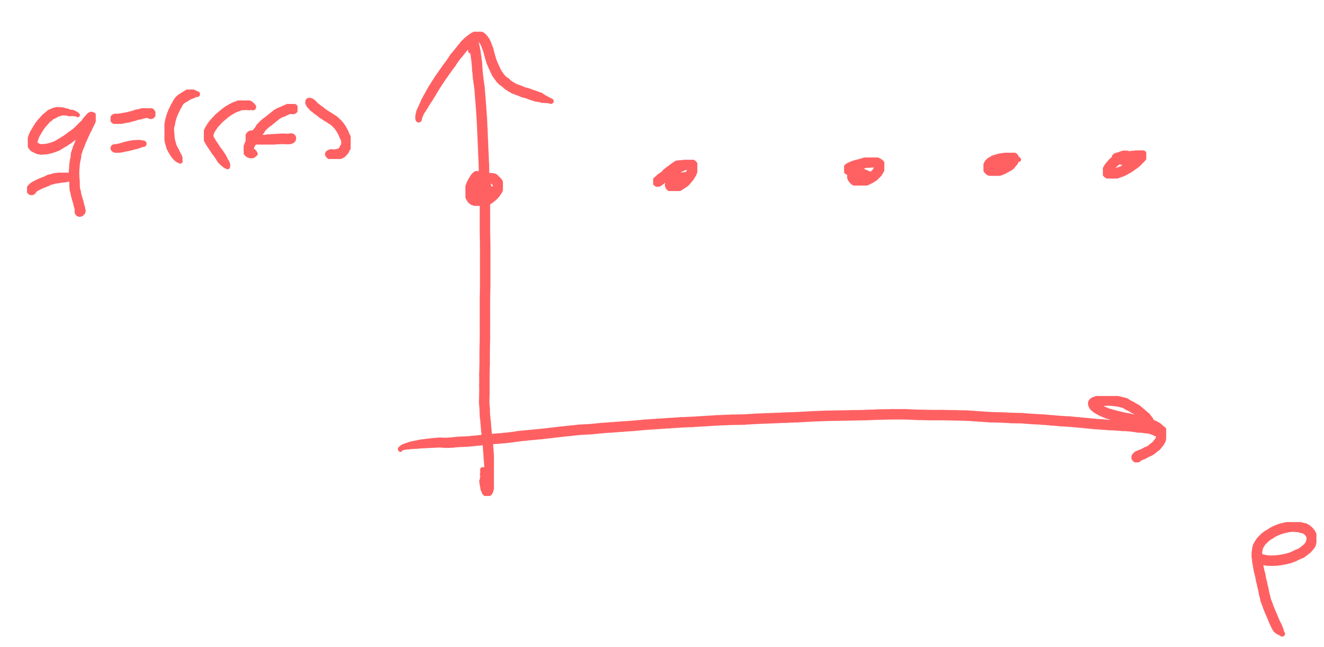
$$H_p(G;H_q(X;\Bbbk)) \Rightarrow H_{p+q}(X/\!/G;\Bbbk)$$

(!) Pointed version $H_p(G;\widetilde{H}_q(X;\Bbbk)) \Rightarrow H_{p+q}(X/\!/G;\Bbbk)$

If this is true, then

$$H_p(G_x; M) \Rightarrow H_{p+q}\left(\Sigma^2 \|S_\bullet^{E_1}(x)\| /\!/ G_x\right)$$

M concentrated at degree $r(x)+2$



Then $H_*\left(\Sigma^2 \|S_\bullet^{E_1}(x)\| / G_x\right) = 0$

for $* < r(x)$

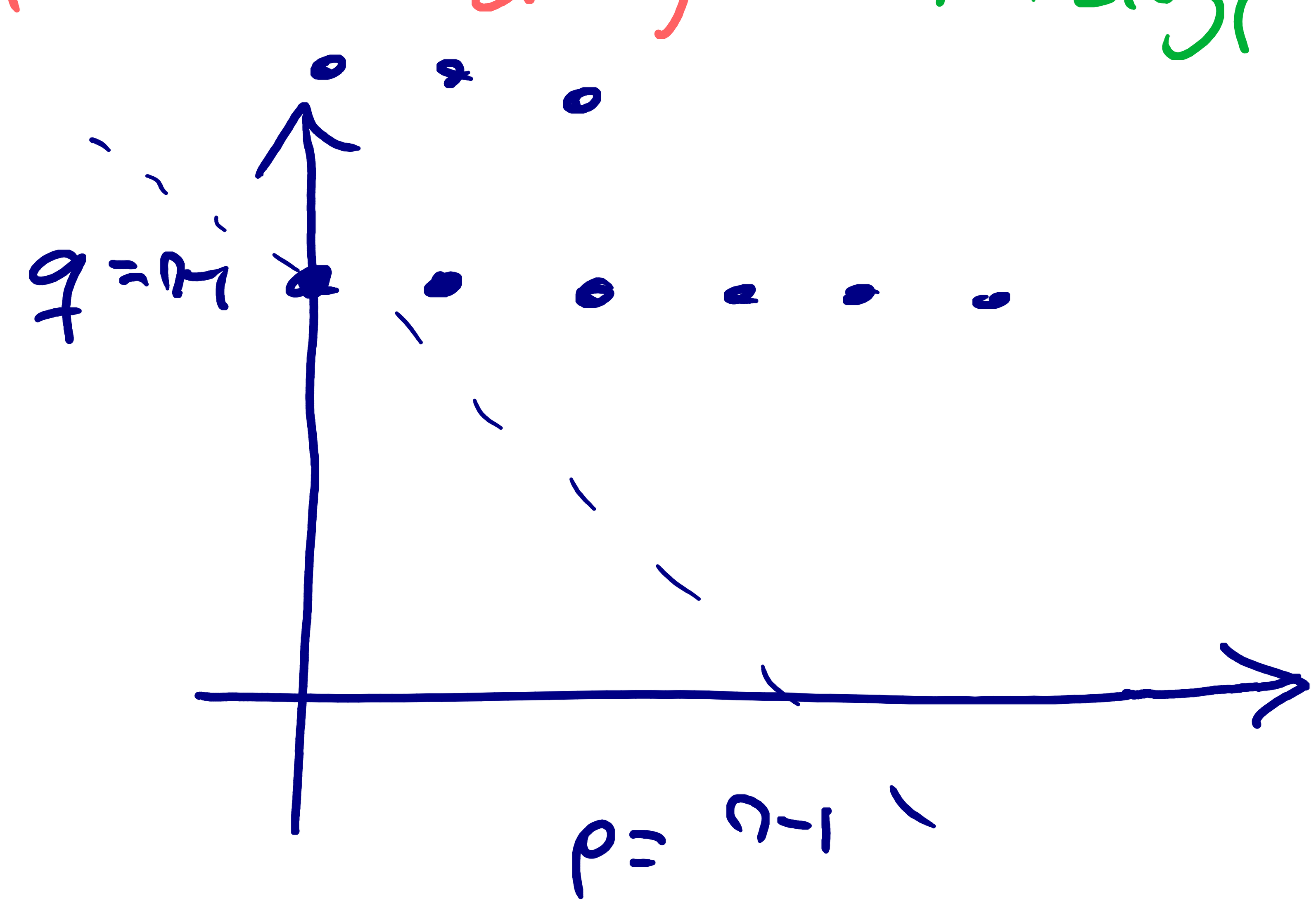
Therefore, from (*) we deduce that $H^{E_1}_{n,*}(\mathcal{R}) = 0$

for $* < n-1$.

Finally, we use the "vanishing lines" spectral sequence

$$E^1_{p,q} = H_q\Big(\underbrace{B^{E_1}(\mathcal{R})^{\otimes p}(n)}_{\text{homology starts at } *=n-1}; \Bbbk\Big) \Rightarrow H_{p+q}\left(B^{E_2}(\mathcal{R})(n); \Bbbk\right)$$

(Relative version)



$\Rightarrow H^{E_2}_{n,d}(\mathcal{R}; \Bbbk) = 0$

for $d < n-1$

<u>Remark</u> : If G symmetric, then

$$H^{E_\infty}_{n,d}(B; \mathbb{k}) = 0 \text{ for } d < n-1 \text{ also.}$$