Chapter 4

Optimal Control of Evolution Equations with Unbounded Control Operators

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Introduction to the optimal control of evolution equations with Unbounded Control Operators or with Unbounded Observation Operators

Neumann boundary control of the heat equation

Existence of optimal controls

Characterization of optimal controls

Neumann boundary control of the wave equation

Dirichlet boundary control of the heat equation

Dirichlet boundary control of the wave equation

Optimal control of evolution equations

Setting of the problem

We consider equations of the form

$$(E) \qquad y' = Ay + Bu + f, \qquad y(0) = y_0.$$

Assumptions

Y and U are Hilbert spaces.
The unbounded operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup on Y, denoted by $(e^{tA})_{t\geq 0}$.
We want to study problems for which

$$B \notin \mathcal{L}(U; Y).$$

1. To study equation (E) we look for an extension of A. We look for $\widehat{Y}$ and an unbounded operator $(\widehat{A}, D(\widehat{A}))$ on $\widehat{Y}$ for which

Y is densely embedded in $\widehat{Y}$,

$D(A)$ is densely embedded in $D(\widehat{A})$,

$Ay = \widehat{A}y$ for all $y \in D(A)$,

B belongs to $\mathcal{L}(U; \widehat{Y})$.

This kind of extension will be useful to study boundary control problems for parabolic or hyperbolic equations.

2. Extend the notion of weak solutions. Prove the existence by approximation.

We consider control problems of the form

$$(P)\ \ \text{inf}\{J(y,u)\mid u\in L^2(0,T;U),\ (y,u) \text{ obeys } (E)\}.$$

with

$$J(y,u)=\frac{1}{2}\int_0^T |Cy(t)-z_d(t)|_Z^2$$

$$+\frac{1}{2}|Dy(T)-z_T|_{Z_T}^2+\frac{1}{2}\int_0^T |u(t)|_U^2.$$

Bounded observations. $C\in\mathcal{L}(\widehat{Y};Z)$, and $D\in\mathcal{L}(\widehat{Y};Z_T)$.

If we observe the state on the boundary $\Gamma\times(0,T)$ of the domain $\Omega\times(0,T)$, C and D may be unbounded operators.

Neumann boundary control of the heat equation

The state equation

Let Ω be a bounded domain in $\mathbb{R}^N$, with a boundary Γ of class C^2. Let $T > 0$, set $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. We consider the heat equation with a Neumann boundary control

$$(HE) \qquad \begin{aligned} & \frac{\partial y}{\partial t} - \Delta y = f \quad \text{in } Q, \\ & \frac{\partial y}{\partial n} = u \quad \text{on } \Sigma, \qquad y(x,0) = y_0 \quad \text{in } \Omega. \end{aligned}$$

The function $f \in L^2(Q)$ is a given source of temperature, and the function u is a control variable. We consider the control problem

$$(P) \quad \inf\{J(y,u) \mid u \in L^2(\Sigma),\ (y,u) \text{ obeys } (HE)\},$$

where

$$J(y,u) = \frac{1}{2}\int_Q |y - y_d|^2$$
$$+\frac{1}{2}\int_\Omega |y(T) - y_d(T)|^2 + \frac{\beta}{2}\int_\Sigma u^2,$$

$\beta > 0$ and $y_d \in C([0,T]; L^2(\Omega))$.

The heat equation with a nonhomogeneous Neumann boundary condition

Recall that

$$D(A) = \left\{ \xi \in H^2(\Omega) \mid \frac{\partial \xi}{\partial n} = 0 \right\}, \quad Ay = \Delta y,$$

the operator $(A, D(A))$ is the generator of a semigroup of contractions on $L^2(\Omega)$. If $u = 0$ a weak solution of (HE) is a function $y \in L^2(0, T; L^2(\Omega))$ such that for all $\xi \in D(A)$, the mapping $t \mapsto \int_\Omega y(t)\, \xi$ belongs to $H^1(0, T)$, $\int_\Omega y(0)\, \xi = \int_\Omega y_0\, \xi$, and

$$\frac{d}{dt} \int_\Omega y(t)\xi = \int_\Omega y(t)\Delta\xi + \int_\Omega f(t)\xi.$$

If y is a regular solution of (HE) then

$$\int_\Omega \Delta y(t)\xi = \int_\Omega y(t)\Delta\xi + \int_\Gamma u(t)\xi, \qquad \forall \xi \in D(A).$$

Definition of a weak solution

Definition. A function $y \in L^2(0,T;L^2(\Omega))$ is a weak solution to equation (HE) if, for all $\xi \in D(A)$, the mapping $t \mapsto \int_\Omega y(t)\,\xi$ belongs to $H^1(0,T)$, $\int_\Omega y(0)\,\xi = \int_\Omega y_0\,\xi$, and

$$\frac{d}{dt}\int_\Omega y(t)\xi = \int_\Omega y(t)\Delta\xi + \int_\Omega f(t)\xi + \int_\Gamma u(t)\xi.$$

Theorem. Equation (HE) admits at most one weak solution in $L^2(0,T;L^2(\Omega))$.

Proof. Suppose that y_1 and y_2 are two weak solutions. Set $z = y_1 - y_2$. Then for all $\xi \in D(A)$, the mapping $t \mapsto \int_\Omega z(t)\,\xi$ belongs to $H^1(0,T)$, $\int_\Omega z(0)\,\xi = 0$, and

$$\frac{d}{dt}\int_\Omega z(t)\xi = \int_\Omega z(t)\Delta\xi.$$

From Chapter 2, we know that $z = 0$.

Approximation by regular controls

Let u be in $L^2(\Sigma)$ and let $(u_n)_n$ be a sequence in $C^1([0,T]; H^{1/2}(\Gamma))$, converging to u in $L^2(\Sigma)$. Denote by $Nu_n(t) = w_n(t)$ the solution to equation

$$-\Delta w + w = 0 \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial n} = u_n(t) \quad \text{on } \Gamma.$$

From elliptic regularity results we know that w_n belongs to $C^1([0,T]; H^2(\Omega))$. Let z_n be the solution to

$$\frac{\partial z}{\partial t} - \Delta z = f - \frac{\partial w_n}{\partial t} + \Delta w_n \quad \text{in } Q,$$
$$\frac{\partial z}{\partial n} = 0 \quad \text{on } \Sigma, \qquad z(x,0) = (y_0 - w_n(0))(x) \quad \text{in } \Omega.$$

Then $y_n = z_n + w_n$ is the solution to (HE) corresponding to (f, u_n, y_0).

Estimates on y_n

Since $(y_0 - w_n(0)) \in L^2(\Omega)$ and $f - \frac{\partial w_n}{\partial t} + \Delta w_n$ belongs to $L^2(Q)$. Thus z_n and w_n are regular enough so that y_n obeys:

$$\int_\Omega |y_n(t)|^2 + 2\int_0^t \int_\Omega |\nabla y_n|^2$$
$$= 2\int_0^t \int_\Omega f y_n + 2\int_0^t \int_\Gamma u y_n + \int_\Omega |y_0|^2 \,,$$

for every $t \in]0, T]$. We first get

$$\|y_n\|^2_{C([0,T];L^2(\Omega))} + 2\|\nabla y_n\|^2_{L^2(0,T;L^2(\Omega))}$$
$$\leq 2\|f\|_{L^2}\|y_n\|_{L^2(Q)} + 2\|u_n\|_{L^2}\|y_n\|_{L^2(\Sigma)} + \|y_0\|^2_{L^2(\Omega)}.$$

Thus with Young's inequality, we next obtain

$$\|y_n\|_{C([0,T];L^2(\Omega))} + \|y_n\|_{L^2(0,T;H^1(\Omega))}$$
$$\leq C\Big(\|f\|_{L^2(Q)} + \|u_n\|_{L^2(\Sigma)} + \|y_0\|_{L^2(\Omega)}\Big).$$

From the weak formulation we can next prove that, for every $\zeta \in D(A)$,

$$\Big\|\frac{d}{dt}\int_\Omega y_n(\cdot)\zeta\Big\|_{L^2(0,T)} \leq \|y_n\|_{L^2(Q)}\|\zeta\|_{H^2(\Omega)}$$
$$+\|f\|_{L^2(Q)}\|\zeta\|_{L^2(\Omega)} + \|u_n\|_{L^2(\Sigma)}\|\zeta\|_{L^2(\Gamma)}.$$

Let $(\zeta_j)_{j\in\mathbb{N}} \subset D(A)$ be a Hilbertian basis in $L^2(\Omega)$. Using the diagonal process, we can prove that there exist subsequence, still indexed by n to simplify the writing, and $y \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, such that

$$y_n \longrightarrow y \qquad \text{in } C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$

$$\int_\Omega y_n(\cdot)\zeta_j \longrightarrow \int_\Omega y(\cdot)\zeta_j \quad \text{in } H^1(0,T), \text{ for all } j.$$

Thus we can pass to the limit in

$$\frac{d}{dt}\int_\Omega y_n(t)\zeta_j = \int_\Omega y_n(t)\Delta\zeta_j + \int_\Omega f(t)\zeta_j + \int_\Gamma u_n(t)\zeta_j,$$

$$\int_\Omega y_n(0)\zeta_j = \int_\Omega y_0\,\zeta_j,$$

and we obtain

$$\frac{d}{dt}\int_\Omega y(t)\zeta_j = \int_\Omega y(t)\Delta\zeta_j + \int_\Omega f(t)\zeta_j + \int_\Gamma u(t)\zeta_j,$$

$$\int_\Omega y(0)\zeta_j = \int_\Omega y_0\,\zeta_j,$$

for all $j \in \mathbb{N}$. Since $(\zeta_j)_{j\in\mathbb{N}} \subset D(A)$ is a Hilbertian basis in $L^2(\Omega)$, we prove that y is a weak solution of (HE).

Theorem. For every $u \in L^2(\Sigma)$, $f \in L^2(Q)$, $y_0 \in L^2(\Omega)$, the heat equation (HE) admits a unique solution y in $L^2(0,T;L^2(\Omega))$ and

$$\|y\|_{C([0,T];L^2(\Omega))} + \|y\|_{L^2(0,T;H^1(\Omega))}$$
$$\leq C\Big(\|f\|_{L^2(Q)} + \|u\|_{L^2(\Sigma)} + \|y_0\|_{L^2(\Omega)}\Big).$$

The semigroup approach

We set $\widehat{Y} = (H^1(\Omega))'$. The norm on $(H^1(\Omega))'$ is defined by

$$y \longmapsto \|(-\Delta + I)^{-1} y\|_{H^1(\Omega)},$$

where $\xi = (-\Delta + I)^{-1}\zeta$ is the solution of

$$-\Delta\xi + \xi = \zeta \quad \text{in } \Omega, \qquad \frac{\partial \xi}{\partial n} = 0 \quad \text{on } \Gamma.$$

The associated inner product is

$$\Big(y, \zeta\Big)_{(H^1(\Omega))'} = \Big((-\Delta + I)^{-1} y, (-\Delta + I)^{-1}\zeta\Big)_{H^1(\Omega)}.$$

To define the continuous extension of A we observe that if $y \in D(A)$ we have

$$\Big(Ay, \zeta\Big)_{(H^1(\Omega))'} = \Big((-\Delta + I)^{-1}\Delta y, (-\Delta + I)^{-1}\zeta\Big)_{H^1(\Omega)}$$
$$= -\int_\Omega \nabla y \cdot \nabla(-\Delta + I)^{-1}\zeta\,.$$

Thus we define the unbounded operator $\widehat{A}$ on $(H^1(\Omega))'$ by $D(\widehat{A}) = H^1(\Omega)$, and

$$\Big(\widehat{A}y, \zeta\Big)_{(H^1(\Omega))'} = -\int_\Omega \nabla y \cdot \nabla(-\Delta + I)^{-1}\zeta$$

for every $\zeta \in (H^1(\Omega))'$, or equivalently

$$\Big\langle \widehat{A}y, z\Big\rangle_{(H^1(\Omega))', H^1(\Omega)} = -\int_\Omega \nabla y \cdot \nabla z \qquad \forall z \in H^1(\Omega).$$

Theorem. The operator $(\widehat{A}, D(\widehat{A}))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $(H^1(\Omega))'$.

Proof. The proof relies on the Hille-Yosida theorem.

$\widehat{A}$ is dissipative.

$$\Big(\widehat{A}y, y\Big)_{(H^1(\Omega))'} = -\int_\Omega \nabla y \cdot \nabla(-\Delta + I)^{-1} y$$

$$= -\int_\Omega y^2 + \int_\Omega [(-\Delta + I)^{-1} y]\, y \le 0.$$

Indeed

$$\|(-\Delta + I)^{-1} y\|_{L^2(\Omega)} \le \|y\|_{L^2(\Omega)}.$$

$\widehat{A}$ is m-dissipative. Let $\lambda > 0$. For all $f \in (H^1(\Omega))'$, the equation

$$\lambda y - \widehat{A}y = f$$

admits a unique solution in $D(\widehat{A})$.

This equation is nothing else than

$$\int_\Omega \Big(\lambda yz + \nabla y \cdot \nabla z\Big) = \Big\langle f, z \Big\rangle_{(H^1(\Omega))', H^1(\Omega)}$$

for every $z \in H^1(\Omega)$.

We want to write equation (HE) in the form

$$y' = \widehat{A}y + f + Bu, \qquad y(0) = y_0,$$

where $B \in \mathcal{L}(L^2(\Gamma); (H^1(\Omega))')$ must be identified.

As before, we first suppose that $u \in C^1([0,T]; H^{1/2}(\Gamma))$. Write y the solution to (HE) corresponding to (f, u, y_0) in the form $y = z + w$, where $w(t) = Nu(t)$. Recall that

$$\frac{\partial z}{\partial t} - \Delta z = f - w' - w \quad \text{in } Q,$$
$$\frac{\partial z}{\partial n} = 0 \quad \text{on } \Sigma, \qquad z(x,0) = (y_0 - w(0))(x) \quad \text{in } \Omega.$$

We have

$$z(t) = e^{\widehat{A}t}(y_0 - w(0)) + \int_0^t e^{\widehat{A}(t-s)}(f(s) - w'(s) + w(s))\,ds.$$

With an integration by parts we can write

$$\int_0^t e^{\widehat{A}(t-s)} w'(s)\,ds = \int_0^t \widehat{A} e^{\widehat{A}(t-s)} w(s)\,ds + w(t) - e^{\widehat{A}t} w(0).$$

Thus

$$z(t) = e^{\widehat{A}t} y_0 + \int_0^t (-\widehat{A} + I) e^{\widehat{A}(t-s)} w(s)\,ds - w(t),$$

that is

$$y(t) = e^{\widehat{A}t} y_0 + \int_0^t e^{\widehat{A}(t-s)} (-\widehat{A} + I) N u(s)\,ds.$$

Thus we can write

$$y' = \widehat{A}y + (-\widehat{A} + I)Nu.$$

We set $Bu(t) = (-\widehat{A} + I)Nu(t)$.

$$N \ : \ L^2(\Gamma) \longmapsto H^{3/2}(\Omega)$$

$$-\widehat{A} + I \ : \ H^1(\Omega) \longmapsto (H^1(\Omega))'$$

Thus $B \in \mathcal{L}(L^2(\Gamma); (H^1(\Omega))')$ and the representation of y by the above equation is still meaningful even if $u \in L^2(\Sigma)$. Accordingly y is a weak solution of the evolution equation iff

$$\frac{d}{dt}\Big(y(t), \zeta\Big)_{(H^1(\Omega))'} = \Big(y(t), \widehat{A}\zeta\Big)_{(H^1(\Omega))'} + \Big(Bu, \zeta\Big)_{(H^1(\Omega))'}.$$

Is it the same definition as above ?

From the definition of $(-\Delta + I)^{-1}w$ it follows that

$$\Big(w, \zeta\Big)_{(H^1(\Omega))'} = \int_\Omega w(-\Delta + I)^{-1}\zeta$$

Thus from the definition of $\widehat{A}$ we get

$$\Big((-\widehat{A} + I)Nu, \zeta\Big)_{(H^1(\Omega))'}$$

$$= \int_\Omega \Big(\nabla Nu \cdot \nabla(-\Delta + I)^{-1}\zeta + Nu(-\Delta + I)^{-1}\zeta\Big)$$

$$= \int_\Gamma u\,(-\Delta + I)^{-1}\zeta\,ds$$

We have

$$(Bu, \zeta)_{(H^1(\Omega))'} = \int_\Gamma u\,(-\Delta + I)^{-1}\zeta \qquad \text{for all } \zeta \in (H^1(\Omega))'.$$

We can check that

$$\Big(y,\zeta\Big)_{(H^1(\Omega))'} = \int_\Omega \Big(\nabla(-\Delta+I)^{-1}y\cdot\nabla(-\Delta+I)^{-1}\zeta$$
$$+(-\Delta+I)^{-1}y\,(-\Delta+I)^{-1}\zeta\Big)$$
$$= \int_\Omega y\,(-\Delta+I)^{-1}\zeta,$$

and

$$(y,\widehat{A}\zeta)_{(H^1(\Omega))'} = -\int_\Omega \nabla(-\Delta+I)^{-1}y\cdot\nabla\zeta$$
$$= \int_\Omega y\,\Delta(-\Delta+I)^{-1}\zeta.$$

Replacing $\xi \in D(A)$ in the first definition by $(-\Delta+I)^{-1}\zeta$, with $\zeta \in L^2(\Omega)$, we obtain the second definition.

Existence of a unique optimal control

1. Set $F(u) = J(y(u), u)$. Let $(u_n)_n$ be a minimizing sequence in $L^2(\Sigma)$, that is

$$\lim_{n\to\infty} F(u_n) = \inf_{u\in L^2(\Sigma)} F(u).$$

Let y_n the solution of (HE) corresponding to u_n, suppose that $(u_n)_n$ is bounded in $L^2(\Sigma)$, and that

$$u_n \rightharpoonup \bar{u} \qquad \text{weakly in } L^2(\Sigma).$$

2. Let $\bar{y} = y(\bar{u})$.

The operator

$$\Lambda \;:\; u \longrightarrow (y(u), y(u)(T))$$

is affine and continuous from $L^2(\Sigma)$ to $L^2(Q) \times L^2(\Omega)$.

The sequence $(y_n)_n$ converges to $\bar{y}$ for the weak topology of $L^2(Q)$, and $(y_n(T))_n$ converges to $\bar{y}(T)$ for the weak topology of $L^2(\Omega)$.

3. Using the weakly lower semicontinuity of F, we obtain

$$F(\bar{u}) \leq \mathrm{liminf}_{n\to\infty} F(u_n) = m.$$

Thus $\bar{u}$ is a solution to (P). The uniqueness follows from the strict convexity of F.

Optimality conditions

Directional Derivative

$$F'(u)v = \int_Q (y(u) - y_d)z(v)$$

$$+ \int_\Omega (y(u)(T) - y_d(T))z(v)(T) + \beta \int_\Sigma uv,$$

where $z(v)$ is the solution of

$$\frac{\partial z}{\partial t} - \Delta z = 0 \quad \text{in } Q,$$

$$\frac{\partial z}{\partial n} = v \quad \text{on } \Sigma, \quad z(x,0) = 0 \quad \text{in } \Omega.$$

Identification of $F'(u)$

We look for q such that

$$\int_Q (y(u) - y_d) z(v) + \int_\Omega [(y(u) - y_d) z(v)](T) = \int_\Sigma q\, v.$$

Let p be a regular function defined on $\overline{Q}$ and write an integration by parts between $z(v)$ and p:

$$0 = \int_Q (z_t - \Delta z) p$$

$$= \int_Q z(-p_t - \Delta p) + \int_\Omega z(T) p(T) - \int_\Sigma vp + \int_\Sigma \frac{\partial p}{\partial n} z$$

Identification with

$$\int_Q (y(u) - y_d)z + \int_\Omega [(y(u) - y_d)z](T) = \int_\Sigma q\, v.$$

We set

$$-\frac{\partial p}{\partial t} - \Delta p = y(u) - y_d \quad \text{in } Q,$$

$$\frac{\partial p}{\partial n} = 0 \quad \text{on } \Sigma, \qquad p(x, T) = (y(u) - y_d)(T) \quad \text{in } \Omega,$$

and we have

$$F'(u)v = \int_\Sigma (p + \beta u)v,$$

if the above calculation are justified.

The adjoint equation

Let $g \in L^2(Q)$, $p_T \in L^2(\Omega)$. The terminal boundary value problem

$$(AE) \qquad \begin{aligned} -\frac{\partial p}{\partial t} - \Delta p = g \quad \text{in } Q, \\ \frac{\partial p}{\partial n} = 0 \quad \text{on } \Sigma, \quad p(x,T) = p_T \quad \text{in } \Omega, \end{aligned}$$

is well posed.

$$\|p\|_{C([0,T];L^2(\Omega))} \leq C(\|g\|_{L^2(Q)} + \|p_T\|_{L^2(\Omega)}).$$

Integration by parts between z and p

Theorem. Suppose that $g \in L^2(Q)$, $p_T \in L^2(\Omega)$, and $v \in L^2(\Sigma)$. Then the solution z of equation

$$\frac{\partial z}{\partial t} - \Delta z = 0 \quad \text{in } Q, \qquad \frac{\partial z}{\partial n} = v \quad \text{on } \Sigma, \qquad z(x,0) = 0 \quad \text{in } \Omega,$$

and the solution p of (AE) satisfy the following formula

$$\int_\Sigma v\, p = \int_Q z\, g + \int_\Omega z(T) p_T.$$

Proof. We prove the IBP formula for $p_T \in H_0^1(\Omega)$, $g \in L^2(Q)$, $v \in C^1([0,T]; H^{1/2}(\Omega))$. In that case z and p belong to $L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$, and the IBP formula is satisfied. When $p_T \in L^2(\Omega)$ and $v \in L^2(\Sigma)$ we use a density argument.

Theorem. (i) If $(\bar{y}, \bar{u})$ is the solution to (P) then $\bar{u} = -\frac{1}{\beta} p|_{\Sigma}$, where p is the solution to the adjoint equation corresponding to $\bar{y}$.

(ii) Conversely, if a pair $(\tilde{y}, \tilde{p}) \in C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega))$ obeys the system

$$
\begin{aligned}
&\frac{\partial \tilde{y}}{\partial t} - \Delta \tilde{y} = f \quad \text{in } Q, \\
&\frac{\partial \tilde{y}}{\partial n} = -\frac{1}{\beta} \tilde{p} \quad \text{on } \Sigma, \quad \tilde{y}(0) = \bar{y}_0 \quad \text{in } \Omega, \\
&-\frac{\partial \tilde{p}}{\partial t} - \Delta \tilde{p} = \tilde{y} - y_d \quad \text{in } Q, \\
&\frac{\partial \tilde{p}}{\partial n} = 0 \quad \text{on } \Sigma, \quad \tilde{p}(T) = y(T) - y_d(T) \quad \text{in } \Omega,
\end{aligned}
$$

then the pair $(\tilde{y}, -\frac{1}{\beta}\tilde{p}|_{\Sigma})$ is the optimal solution to problem (P).

Proof. (i) The necessary optimality condition is already proved.

(ii) The sufficient optimality condition can be proved with the sufficient optimality condition stated in Chapter 1.

Neumann boundary control of the wave equation

The state equation

The notation Ω, Γ, T, Q, Σ, as well as the assumptions on Ω and Γ, are the ones of the previous section. We consider

$$(WE) \qquad \begin{aligned} & y'' - \Delta y = f \quad \text{in } Q, \quad \frac{\partial y}{\partial n} = u \quad \text{on } \Sigma, \\ & y(x,0) = y_0 \quad \text{and} \quad y'(x,0) = y_1 \quad \text{in } \Omega, \end{aligned}$$

with $(y_0, y_1) \in H^1 \times L^2(\Omega)$, $f \in L^2(Q)$, and $u \in L^2(\Sigma)$.

We set $D(A) = \{y_1 \in H^2(\Omega) \mid \frac{\partial y_1}{\partial n} = 0\} \times H^1(\Omega)$, $Y = H^1(\Omega) \times L^2(\Omega)$, and

$$Az = A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ \Delta z_1 - z_1 \end{pmatrix}.$$

Theorem. The operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on Y. If $f \in L^2(Q)$, $y_0 \in H^1(\Omega)$, $y_1 \in L^2(\Omega)$, and $u = 0$, equation (WE) admits a unique weak solution which belongs to $C([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$.

To study the wave equation with nonhomogeneous boundary conditions, we set $D(\widehat{A}) = H^1(\Omega) \times L^2(\Omega)$,

$\widehat{Y} = L^2(\Omega) \times (H^1(\Omega))'$, and

$$\widehat{A}z = \widehat{A}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ \tilde{A}z_1 - z_1 \end{pmatrix},$$

where $(\tilde{A}, D(\tilde{A}))$ is the unbounded operator on $(H^1(\Omega))'$ defined by

$$D(\tilde{A}) = H^1(\Omega),$$

$$\left(\tilde{A}z_1, \zeta\right)_{(H^1(\Omega))'} = -\int_\Omega \nabla z_1 \cdot \nabla(-\Delta + I)^{-1}\zeta\,.$$

Theorem. The operator $(\widehat{A}, D(\widehat{A}))$ is the infinitesimal generator of a semigroup of contractions on $\widehat{Y}$.

Now, we consider equation (WE) with a control in the Neumann boundary condition. As for the heat equation we can prove that equation (WE) may be written in the form

$$\frac{dz}{dt} = (\widehat{A} + L)z + F + Bu, \quad z(0) = z_0,$$

F, $Bu \in L^2(0,T;L^2(\Omega)) \times L^2(0,T;(H^1(\Omega))')$, $z_0 \in L^2(\Omega) \times (H^1(\Omega))'$, are defined by

$$Bu = \begin{pmatrix} 0 \\ B_2 u \end{pmatrix}, \quad L\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ z_1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix},$$

$$z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}, \quad \text{and} \quad B_2 u = (-\tilde{A} + I)Nu.$$

Theorem. For every $(f, u, y_0, y_1) \in L^2(Q) \times L^2(\Sigma) \times L^2(\Omega) \times (H^1(\Omega))'$, equation (WE) admits a unique weak solution $z(f, u, y_0, y_1) = (y(f, u, y_0, y_1), y'(f, u, y_0, y_1))$ in $C([0, T]; L^2(\Omega)) \times C([0, T]; (H^1(\Omega))')$.

The control problem

$$(P) \quad \inf\{J(y,u) \mid (y,u) \text{ obeys } (WE),\ u \in L^2(\Sigma)\},$$

the functional J is defined by

$$J(y,u) = \frac{1}{2}\int_Q |y - y_d|^2 + \frac{1}{2}\int_\Omega |y(T) - y_d(T)|^2 + \frac{\beta}{2}\int_\Sigma u^2,$$

where the function y_d belongs to $C([0,T]; L^2(\Omega))$.

Theorem. Assume that $f \in L^2(Q)$, $y_0 \in H^1(\Omega)$, $y_1 \in L^2(\Omega)$, and $y_d \in C([0,T]; L^2(\Omega))$. Problem (P) admits a unique solution $(\bar{y}, \bar{u})$.

Existence of a unique optimal control

1. Set $F(u) = J(y(u), u)$. Let $(u_n)_n$ be a minimizing sequence in $L^2(\Sigma)$, that is

$$\text{lim}_{n\to\infty} F(u_n) = \text{inf}_{u\in L^2(\Sigma)} F(u).$$

We suppose that

$$u_n \rightharpoonup \bar{u} \qquad \text{weakly in } L^2(\Sigma).$$

Let y_n the solution of (WE) corresponding to u_n, suppose that $(u_n)_n$ is bounded in $L^2(\Sigma)$, and that

$$u_n \rightharpoonup \bar{u} \qquad \text{weakly in } L^2(\Sigma).$$

Passage to the limit in the equation. Let $\bar{y} = y(\bar{u})$.

The operator

$$\Lambda \ : \ u \longrightarrow \Big(y(u), y(u)(T)\Big)$$

is affine and continuous from $L^2(\Sigma)$ to $L^2(Q) \times L^2(\Omega)$.

We conclude that problem (P) admits a unique solution $(\bar{y}, \bar{u})$.

Optimality conditions for (P)

By a classical calculation we have

$$F'(u)v = \int_Q (y(u) - y_d)z(v)$$

$$+ \int_\Omega (y(u)(T) - y_d(T))z(v)(T) + \beta \int_\Sigma uv,$$

where $z(v)$ is the solution of

$$z'' - \Delta z = 0 \quad \text{in } Q, \qquad \frac{\partial z}{\partial n} = v \quad \text{on } \Sigma,$$

$$z(x,0) = 0 \text{ and } \; z'(x,0) = 0 \quad \text{in } \Omega.$$

Identification of $F'(u)$

We look for q such that

$$\int_Q (y(u) - y_d) z(v) + \int_\Omega [(y(u) - y_d) z(v)](T) = \int_\Sigma q\, v.$$

Let p be a regular function defined on $\bar{Q}$ and write an integration by parts between $z(v)$ and p:

$$\begin{aligned} 0 &= \int_Q (z'' - \Delta z) p \\ &= \int_Q z(p'' - \Delta p) + \int_\Omega z'(T) p(T) \\ &- \int_\Omega z(T) p'(T) - \int_\Sigma v p + \int_\Sigma \frac{\partial p}{\partial n} z \end{aligned}$$

Identification with

$$\int_Q (y(u) - y_d)z + \int_\Omega [(y(u) - y_d)z](T) = \int_\Sigma q\, v.$$

We set

$$p'' - \Delta p = y(u) - y_d \quad \text{in } Q, \qquad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Sigma,$$

$$p(T) = 0 \text{ and } \; p'(T) = -(y(u) - y_d)(T) \quad \text{in } \Omega.$$

and we have

$$F'(u)v = \int_\Sigma (p + \beta u)v,$$

if the above calculation are justified.

Theorem. (i) If $(\bar{y}, \bar{u})$ is the solution to (P) then $\bar{u} = -\frac{1}{\beta} p|_{\Sigma}$, where p is the solution to the adjoint equation corresponding to $\bar{y}$:

$$p'' - \Delta p = \bar{y} - y_d \quad \text{in } Q, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Sigma,$$

$$p(T) = 0 \text{ and } \; p'(T) = -(\bar{y} - y_d)(T) \quad \text{in } \Omega.$$

(ii) Conversely, if a pair $(\tilde{y}, \tilde{p}) \in C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega))$ obeys the system

$$
\begin{aligned}
&\tilde{y}'' - \Delta \tilde{y} = f \quad \text{in } Q, \quad \frac{\partial \tilde{y}}{\partial n} = -\frac{1}{\beta}\tilde{p} \quad \text{on } \Sigma,\\
&\tilde{y}(0) = y_0, \quad \tilde{y}'(0) = y_1, \quad \text{in } \Omega,\\
&\tilde{p}'' - \Delta \tilde{p} = \tilde{y} - y_d \quad \text{in } Q, \quad \frac{\partial \tilde{p}}{\partial n} = 0 \quad \text{on } \Sigma,\\
&\tilde{p}(T) = 0, \quad \tilde{p}'(T) = -\tilde{y}(T) + y_d(T) \quad \text{in } \Omega,
\end{aligned}
$$

then the pair $(\tilde{y}, -\frac{1}{\beta}\tilde{p}|_{\Sigma})$ is the optimal solution to (P).

Dirichlet boundary control of the heat equation

The state equation

Let Ω be a bounded domain in $\mathbb{R}^N$, with a boundary Γ of class C^2. Let $T > 0$, set $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. We consider the heat equation with a Dirichlet boundary control

$$(HE) \qquad \begin{aligned} &\frac{\partial y}{\partial t} - \Delta y = f \quad \text{in } Q, \\ &y = u \quad \text{on } \Sigma, \quad y(x, 0) = y_0 \quad \text{in } \Omega. \end{aligned}$$

The function $f \in L^2(Q)$ is a given source of temperature, and the function u is a control variable. We suppose that $y_0 \in L^2(\Omega)$.

We consider the control problem

$$(P) \quad \text{inf}\{J(y,u) \mid u \in L^2(\Sigma),\ (y,u) \text{ obeys } (HE)\},$$

$$J(y,u) = \frac{1}{2}\|y(T) - y_d(T)\|^2_{H^{-1}(\Omega)} + \frac{1}{2}\int_Q |y - y_d|^2 + \frac{\beta}{2}\int_\Sigma u^2,$$

$\beta > 0$ and $y_d \in C([0,T]; L^2(\Omega))$.

Recall that

$$\|y(T) - y_d(T)\|^2_{H^{-1}(\Omega)} = \Big\langle (-\Delta)^{-1}(y(T) - y_d(T)), y(T) - y_d(T) \Big\rangle_{H^1_0(\Omega), H^{-1}(\Omega)}.$$

The heat equation with a nonhomogeneous Dirichlet boundary condition

Recall that

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \qquad Ay = \Delta y,$$

the operator $(A, D(A))$ is the generator of a semigroup of contraction on $L^2(\Omega)$. If $u = 0$ a weak solution of (HE) is a function $y \in L^2(0,T;L^2(\Omega))$ such that for all $\xi \in D(A)$, the mapping $t \mapsto \int_\Omega y(t)\,\xi$ belongs to $H^1(0,T)$, $\int_\Omega y(0)\,\xi = \int_\Omega y_0\,\xi$, and

$$\frac{d}{dt}\int_\Omega y(t)\xi = \int_\Omega y(t)\Delta\xi + \int_\Omega f(t)\xi.$$

If y is a regular solution of (HE) then

$$\int_\Omega \Delta y(t)\xi = \int_\Omega y(t)\Delta\xi - \int_\Gamma u(t)\frac{\partial\xi}{\partial n}, \qquad \forall \xi \in D(A).$$

Definition of a weak solution

Definition. A function $y \in L^2(0,T;L^2(\Omega))$ is a weak solution to equation (HE) if, for all $\xi \in D(A)$, the mapping $t \mapsto \int_\Omega y(t)\,\xi$ belongs to $H^1(0,T)$, $\int_\Omega y(0)\,\xi = \int_\Omega y_0\,\xi$, and

$$\frac{d}{dt}\int_\Omega y(t)\xi = \int_\Omega y(t)\Delta\xi + \int_\Omega f(t)\xi - \int_\Gamma u(t)\frac{\partial\xi}{\partial n}.$$

Theorem. Equation (HE) admits at most one weak solution in $L^2(0,T;L^2(\Omega))$.

Proof of uniqueness. Suppose that y_1 and y_2 are two weak solutions. Set $z = y_1 - y_2$. Then for all $\xi \in D(A)$, the mapping $t \mapsto \int_\Omega z(t)\,\xi$ belongs to $H^1(0,T)$, $\int_\Omega z(0)\,\xi = 0$, and

$$\frac{d}{dt}\int_\Omega z(t)\xi = \int_\Omega z(t)\Delta\xi.$$

From Chapter 2, we know that $z = 0$.

Approximation by regular controls

Let u be in $L^2(\Sigma)$ and let $(u_n)_n$ be a sequence in $C^1([0,T];H^{3/2}(\Gamma))$, converging to u in $L^2(\Sigma)$. Denote by $Du_n(t) = w_n(t)$ the solution to equation

$$-\Delta w = 0 \quad \text{in } \Omega, \qquad w = u_n(t) \quad \text{on } \Gamma.$$

From elliptic regularity results we know that w_n belongs to $C^1([0,T];H^2(\Omega))$. Let z_n be the solution to

$$\frac{\partial z}{\partial t} - \Delta z = f - \frac{\partial w_n}{\partial t} + \Delta w_n \quad \text{in } Q,$$
$$z = 0 \quad \text{on } \Sigma, \qquad z(x,0) = (y_0 - w_n(0))(x) \quad \text{in } \Omega.$$

Then $y_n = z_n + w_n$ is the solution to (HE) corresponding to (f, u_n, y_0).

Estimates on y_n

Since $(y_0 - w_n(0)) \in L^2(\Omega)$ and $f - \frac{\partial w_n}{\partial t} + \Delta w_n$ belongs to $L^2(Q)$, z_n and w_n are regular enough so that y_n obeys:

$$\int_\Omega y_n(t)(-\Delta)^{-1} y_n(t) + 2\int_0^t \int_\Omega |y_n|^2 - \|y_0\|^2_{H^{-1}(\Omega)}$$

$$= 2\int_0^t \int_\Omega f(-\Delta)^{-1} y_n + 2\int_0^t \int_\Gamma u_n \frac{\partial}{\partial n}\Big[(-\Delta)^{-1} y_n\Big].$$

for some $t \in]0, T]$.

We first get

$$\|y_n\|^2_{C([0,T];H^{-1}(\Omega))} + 2\|y_n\|^2_{L^2(0,T;L^2(\Omega))}$$
$$\leq \|y_0\|^2_{H^{-1}(\Omega)} + 2C\|f\|_{L^2}\|y_n\|_{L^2(Q)}$$
$$+2\|u_n\|_{L^2(\Sigma)}\Big\|\frac{\partial}{\partial n}(-\Delta)^{-1}y_n\Big\|_{L^2(\Sigma)}.$$

Observe that

$$\Big\|\frac{\partial}{\partial n}(-\Delta)^{-1}y_n\Big\|_{L^2(\Sigma)} \leq C\|y_n\|_{L^2(Q)}.$$

Thus with Young's inequality, we next obtain

$$\|y_n\|_{C([0,T];H^{-1}(\Omega))} + \|y_n\|_{L^2(0,T;L^2(\Omega))}$$
$$\leq C\Big(\|f\|_{L^2(Q)} + \|u_n\|_{L^2(\Sigma)} + \|y_0\|_{L^2(\Omega)}\Big).$$

An other estimate on y_n

From the variational formulation we can next prove that, for every $\zeta \in D(A)$,

$$\left\|\frac{d}{dt}\int_\Omega y_n(\cdot)\zeta\right\|_{L^2(0,T)} \leq \|y_n\|_{L^2(Q)}\|\zeta\|_{H^2(\Omega)}$$
$$+\|f\|_{L^2(Q)}\|\zeta\|_{L^2(\Omega)} + \|u_n\|_{L^2(\Sigma)}\|\zeta\|_{L^2(\Gamma)}.$$

Let $(\zeta_j)_{j\in\mathbb{N}} \subset D(A)$ be a Hilbertian basis in $L^2(\Omega)$. Using the diagonal process, we can prove that there exist subsequence, still indexed by n to simplify the writing, and $y \in C([0,T]; H^{-1}(\Omega)) \cap L^2(0,T; L^2(\Omega))$, such that

$$y_n \longrightarrow y \qquad \text{in } C([0,T];H^{-1}(\Omega)) \cap L^2(0,T;L^2(\Omega)),$$

$$\int_\Omega y_n(\cdot)\zeta_j \longrightarrow \int_\Omega y(\cdot)\zeta_j \quad \text{in } H^1(0,T), \text{ for all } j.$$

Thus we can pass to the limit in

$$\frac{d}{dt}\int_\Omega y_n(t)\zeta_j = \int_\Omega y_n(t)\Delta\zeta_j + \int_\Omega f(t)\zeta_j - \int_\Gamma u_n(t)\frac{\partial \zeta_j}{\partial n},$$

$$\int_\Omega y_n(0)\zeta_j = \int_\Omega y_0\,\zeta_j,$$

and we obtain

$$\frac{d}{dt}\int_\Omega y(t)\zeta_j = \int_\Omega y(t)\Delta\zeta_j + \int_\Omega f(t)\zeta_j - \int_\Gamma u(t)\frac{\partial\zeta_j}{\partial n},$$
$$\int_\Omega y(0)\zeta_j = \int_\Omega y_0\,\zeta_j,$$

for all $j \in \mathbb{N}$. Since $(\zeta_j)_{j\in\mathbb{N}} \subset D(A)$ is a Hilbertian basis in $L^2(\Omega)$, we prove that y is a weak solution of (HE).

Theorem. For every $u \in L^2(\Sigma)$, $f \in L^2(Q)$, $y_0 \in L^2(\Omega)$, the heat equation (HE) admits a unique solution y in $L^2(0,T;L^2(\Omega))$ and

$$\|y\|_{C([0,T];H^{-1}(\Omega))} + \|y\|_{L^2(0,T;L^2(\Omega))}$$
$$\leq C\Big(\|f\|_{L^2(Q)} + \|u\|_{L^2(\Sigma)} + \|y_0\|_{L^2(\Omega)}\Big).$$

The semigroup approach

We set $\widehat{Y} = (H^2(\Omega)\cap H_0^1(\Omega))'$. The norm on $(H^2(\Omega)\cap H_0^1(\Omega))'$ is defined by

$$y \longmapsto \|(-\Delta)^{-1}y\|_{L^2(\Omega)},$$

where $\xi = (-\Delta)^{-1}\zeta$ is the solution of

$$-\Delta\xi = \zeta \quad \text{in } \Omega, \quad \xi = 0 \quad \text{on } \Gamma.$$

The associated inner product is

$$\Big(y,\zeta\Big)_{(H^2(\Omega)\cap H_0^1(\Omega))'} = \Big((-\Delta)^{-1}y,(-\Delta)^{-1}\zeta\Big)_{L^2(\Omega)}.$$

To define the continuous extension of A we observe that if $y \in D(A)$ we have

$$\Big(Ay, \zeta\Big)_{(H^2 \cap H_0^1(\Omega))'} = \Big((-\Delta)^{-1}\Delta y, (-\Delta)^{-1}\zeta\Big)_{L^2(\Omega)}$$

$$= -\int_\Omega y(-\Delta)^{-1}\zeta\,.$$

Thus we define the unbounded operator $\widehat{A}$ on $(H^2(\Omega) \cap H_0^1(\Omega))'$ by $D(\widehat{A}) = L^2(\Omega)$, and

$$\Big(\widehat{A}y, \zeta\Big)_{(H^2(\Omega) \cap H_0^1(\Omega))'} = -\int_\Omega y(-\Delta)^{-1}\zeta\,.$$

Theorem. The operator $(\widehat{A}, D(\widehat{A}))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $(H^2(\Omega) \cap H_0^1(\Omega))'$.

Proof. The proof relies on the Hille-Yosida theorem.

$\widehat{A}$ is dissipative.

$$(\widehat{A}y, y)_{(H^2 \cap H_0^1(\Omega))'} = -\int_\Omega (-\Delta)^{-1} y\, y \leq 0.$$

$\widehat{A}$ is m-dissipative. Let $\lambda > 0$. For all $f \in (H^2 \cap H_0^1(\Omega))'$, the equation

$$\lambda y - \widehat{A}y = f$$

admits a unique solution in $D(\widehat{A})$.

We want to write equation (HE) in the form

$$y' = \widehat{A}y + f + Bu, \qquad y(0) = y_0,$$

where $B \in \mathcal{L}(L^2(\Gamma); (H^2 \cap H_0^1(\Omega))')$ must be identified.

As before, we first suppose that $u \in C^1([0,T]; H^{3/2}(\Gamma))$. Write y the solution to (HE) corresponding to (f, u, y_0) in the form $y = z + w$, where $w(t) = Du(t)$. Recall that

$$\frac{\partial z}{\partial t} - \Delta z = f - w' \quad \text{in } Q,$$

$$z = 0 \quad \text{on } \Sigma, \qquad z(x,0) = (y_0 - w(0))(x) \quad \text{in } \Omega.$$

We have

$$z(t) = e^{\widehat{A}t}(y_0 - w(0)) + \int_0^t e^{\widehat{A}(t-s)}(f(s) - w'(s))\,ds.$$

With an integration by parts we can write

$$\int_0^t e^{\widehat{A}(t-s)}w'(s)\,ds = \int_0^t \widehat{A}e^{\widehat{A}(t-s)}w(s)\,ds + w(t) - e^{\widehat{A}t}w(0).$$

Thus

$$z(t) = e^{\widehat{A}t}y_0 + \int_0^t (-\widehat{A})e^{\widehat{A}(t-s)}w(s)\,ds - w(t),$$

that is

$$y(t) = e^{\widehat{A}t}y_0 + \int_0^t e^{\widehat{A}(t-s)}(-\widehat{A})Du(s)\,ds.$$

Thus we can write

$$y' = \widehat{A}y + (-\widehat{A})Du.$$

We set $Bu(t) = (-\widehat{A})Du(t)$.

$$\begin{aligned} D \ &: \ L^2(\Gamma) \longmapsto H^{1/2}(\Omega) \\ -\widehat{A} \ &: \ L^2(\Omega) \longmapsto (H^2 \cap H_0^1(\Omega))' \end{aligned}$$

Thus $B \in \mathcal{L}(L^2(\Gamma); (H^2 \cap H_0^1(\Omega))')$ and the representation of y by the above equation is still meaningful even if $u \in L^2(\Sigma)$. Accordingly y is a weak solution of the evolution equation iff

$$\frac{d}{dt}(y(t), \zeta)_{\widehat{Y}} = (y(t), \widehat{A}\zeta)_{\widehat{Y}} + (Bu, \zeta)_{\widehat{Y}}.$$

Is it the same definition as above ?

From the definition of $\widehat{A}$ and with a Green formula, we get

$$
\begin{aligned}
&\Big((-\widehat{A})Du,\zeta\Big)_{(H^2\cap H_0^1(\Omega))'}\\
&=\int_\Omega Du(-\Delta)^{-1}\zeta=\int_\Omega Du(-\Delta)(-\Delta)^{-2}\zeta\\
&=\int_\Gamma u\,\frac{\partial}{\partial n}\Big[(-\Delta)^{-2}\zeta\Big]
\end{aligned}
$$

We have

$$
\Big(Bu,\zeta\Big)_{(H^2\cap H_0^1(\Omega))'}=\int_\Gamma u\,\frac{\partial}{\partial n}\Big[(-\Delta)^{-2}\zeta\Big]\qquad\forall\zeta\in\widehat{Y}.
$$

We can check that

$$\Big(y,\zeta\Big)_{(H^2\cap H_0^1(\Omega))'} = \int_\Omega (-\Delta)^{-1} y\,(-\Delta)^{-1}\zeta$$

$$= \int_\Omega y\,(-\Delta)^{-2}\zeta,$$

and

$$\Big(y,\widehat{A}\zeta\Big)_{(H^2\cap H_0^1(\Omega))'} = -\int_\Omega (-\Delta)^{-1} y\,\zeta$$

$$= \int_\Omega y\,\Delta(-\Delta)^{-2}\zeta.$$

Replacing $\xi \in D(A)$ in the first definition by $(-\Delta)^{-2}\zeta$, with $\zeta \in (H^2\cap H_0^1(\Omega))'$, we obtain the second defintion.

Remark. With the semigroup approach we obtain the existence of a solution in $C([0,T];(H^2 \cap H_0^1(\Omega))')$. With the variational method, the approximation by regular controls and the estimates we obtain the existence of a solution in $C([0,T];H^{-1}(\Omega)) \cap L^2(0,T;L^2(\Omega))$.

Existence of a unique optimal control

1. Set $F(u) = J(y(u), u)$. Let $(u_n)_n$ be a minimizing sequence in $L^2(\Sigma)$, that is

$$\lim_{n\to\infty} F(u_n) = \inf_{u\in L^2(\Sigma)} F(u).$$

Let y_n the solution of (HE) corresponding to u_n, suppose that $(u_n)_n$ is bounded in $L^2(\Sigma)$, and that

$$u_n \rightharpoonup \bar{u} \qquad \text{weakly in } L^2(\Sigma).$$

2. Let $\bar{y} = y(\bar{u})$. The operator

$$\Lambda \ : \ u \longrightarrow (y(u), y(u)(T))$$

is affine and continuous from $L^2(\Sigma)$ to $L^2(Q) \times$

$H^{-1}(\Omega)$.

The sequence $(y_n)_n$ converges to $\bar{y}$ for the weak topology of $L^2(Q)$, and $(y_n(T))_n$ converges to $\bar{y}(T)$ for the weak topology of $H^{-1}(\Omega)$.

3. Using the weakly lower semicontinuity of F, we obtain

$$F(\bar{u}) \leq \text{liminf}_{n \to \infty} F(u_n) = m.$$

Thus $\bar{u}$ is a solution to (P). The uniqueness follows from the strict convexity of F.

Optimality conditions

Directional Derivative

$$F'(u)v = \int_Q (y(u) - y_d) z(v)$$

$$+\Big(y(u)(T) - y_d(T), z(v)(T)\Big)_{H^{-1}(\Omega)} + \beta \int_{\omega\times(0,T)} uv,$$

where $z(v)$ is the solution of

$$\frac{\partial z}{\partial t} - \Delta z = 0 \quad \text{in } Q,$$

$$z = v \quad \text{on } \Sigma, \quad z(x,0) = 0 \quad \text{in } \Omega.$$

Identification of $F'(u)$

We look for q such that

$$\int_Q (y(u) - y_d) z + \Big((y(u) - y_d)(T), z(T)\Big)_{H^{-1}} = \int_\Sigma q\, v.$$

Let p be a regular function defined on $\bar{Q}$ and write an integration by parts between $z(v)$ and p:

$$-\int_\Sigma v \frac{\partial p}{\partial n} = \int_Q (z_t - \Delta z) p$$

$$= \int_Q z(-p_t - \Delta p) + \Big\langle p(T), z(T) \Big\rangle_{H_0^1, H^{-1}} - \int_\Sigma \frac{\partial z}{\partial n} p$$

Identification with

$$\int_Q (y(u)-y_d)z + \Big((y(u)-y_d)(T), z(v)(T)\Big)_{H^{-1}} = \int_\Sigma q\, v.$$

We set

$$-\frac{\partial p}{\partial t} - \Delta p = y(u) - y_d \quad \text{in } Q, \qquad p = 0 \quad \text{on } \Sigma,$$

$$p(T) = (-\Delta)^{-1}[(y(u) - y_d)(T)] \quad \text{in } \Omega,$$

and we have

$$F'(u)v = \int_\Sigma (-\frac{\partial p}{\partial n} + \beta u)v,$$

if the above calculation are justified.

The adjoint equation

Let $g \in L^2(Q)$, $p_T \in H_0^1(\Omega)$. The terminal boundary value problem

$$(AE) \qquad \begin{aligned} &-\frac{\partial p}{\partial t} - \Delta p = g \quad \text{in } Q, \\ &p = 0 \quad \text{on } \Sigma, \quad p(x,T) = p_T \quad \text{in } \Omega, \end{aligned}$$

is well posed.

$$\begin{aligned} &\|p\|_{C([0,T];H_0^1(\Omega))} + \|p\|_{L^2(0,T;H^2(\Omega))} \\ &\leq C(\|g\|_{L^2(Q)} + \|p_T\|_{L^2(\Omega)}). \end{aligned}$$

Integration by parts between z and p

Theorem. Suppose that $g \in L^2(Q)$, $p_T \in L^2(\Omega)$, and $v \in L^2(\Sigma)$. Then the solution z of equation

$$\frac{\partial z}{\partial t} - \Delta z = 0 \quad \text{in } Q, \qquad z = v \quad \text{on } \Sigma, \qquad z(0) = 0 \quad \text{in } \Omega,$$

and the solution p of (AE) satisfy the following formula

$$-\int_\Sigma v \, \frac{\partial p}{\partial n} = \int_Q z \, g + \langle z(T), p_T \rangle_{H^{-1}, H_0^1}.$$

Proof. We prove the IBP formula for $p_T \in H_0^1(\Omega)$, $g \in L^2(Q)$, $v \in C^1([0,T]; H^{3/2}(\Omega))$. If z and p belong to $L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$, and the IBPF is proved. When $v \in L^2(\Sigma)$ we use a density argument.

Theorem. (i) If $(\bar{y}, \bar{u})$ is the solution to (P) then $\bar{u} = \frac{1}{\beta}\frac{\partial p}{\partial n}$, where p is the solution to the adjoint equation corresponding to $\bar{y}$:

$$-\frac{\partial p}{\partial t} - \Delta p = \bar{y} - y_d \quad \text{in } Q,$$

$$p = 0 \quad \text{on } \Sigma, \quad p(x,T) = (-\Delta)^{-1}[(\bar{y} - y_d)(T)] \quad \text{in } \Omega,$$

(ii) Conversely, if a pair $(\tilde{y}, \tilde{p}) \in C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega))$ obeys the system

$$\frac{\partial \tilde{y}}{\partial t} - \Delta \tilde{y} = f \quad \text{in } Q,$$
$$\tilde{y} = \frac{1}{\beta}\frac{\partial \tilde{p}}{\partial n} \quad \text{on } \Sigma, \quad \tilde{y}(0) = \bar{y}_0 \quad \text{in } \Omega,$$
$$-\frac{\partial \tilde{p}}{\partial t} - \Delta \tilde{p} = \tilde{y} - y_d \quad \text{in } Q,$$
$$\tilde{p} = 0 \quad \text{on } \Sigma, \quad \tilde{p}(T) = (-\Delta)^{-1}[\tilde{y}(T) - y_d(T)] \quad \text{in } \Omega,$$

then the pair $(\tilde{y}, \frac{1}{\beta}\frac{\partial \tilde{p}}{\partial n})$ is the optimal solution to problem (P).

Dirichlet boundary control of the wave equation

The state equation

The notation Ω, Γ, T, Q, Σ, as well as the assumptions on Ω and Γ, are the ones of the previous section. We consider

$$(WE) \qquad \begin{aligned} & y'' - \Delta y = f \quad \text{in } Q, \qquad y = u \quad \text{on } \Sigma, \\ & y(x,0) = y_0 \quad \text{and} \quad y'(x,0) = y_1 \quad \text{in } \Omega, \end{aligned}$$

with $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, $f \in L^2(Q)$, and $u \in L^2(\Sigma)$.

We set $D(A) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, $Y = H_0^1(\Omega) \times L^2(\Omega)$, and

$$Az = A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ \Delta z_1 \end{pmatrix}.$$

Theorem. The operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on Y. If $f \in L^2(Q)$, $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$, and $u = 0$, equation (WE) admits a unique weak solution which belongs to $C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$.

To study the wave equation with nonhomogeneous Dirichlet boundary conditions, we set $D(\widehat{A}) = L^2(\Omega) \times$

$H^{-1}(\Omega)$, $\widehat{Y} = H^{-1}(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))'$, and

$$\widehat{A}z = \widehat{A}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ \tilde{A}z_1 \end{pmatrix},$$

where $(\tilde{A}, D(\tilde{A}))$ is the unbounded operator on $(H^2(\Omega) \cap H_0^1(\Omega))'$ defined by

$$D(\tilde{A}) = L^2(\Omega),$$

$$\left(\tilde{A}z_1, \zeta\right)_{(H^2(\Omega) \cap H_0^1(\Omega))'} = -\int_\Omega z_1(-\Delta)^{-1}\zeta\,.$$

Theorem. The operator $(\widehat{A}, D(\widehat{A}))$ is the infinitesimal generator of a semigroup of contractions on $\widehat{Y}$.

Now, we consider equation (WE) with a control in the Dirichlet boundary condition. As for the heat equation we can prove that equation (WE) may be written in the form

$$\frac{dz}{dt} = \widehat{A}z + F + Bu, \quad z(0) = z_0,$$

$F, \quad Bu \quad \in \quad L^2(0,T;L^2(\Omega)) \times L^2(0,T;(H^2(\Omega) \cap H_0^1(\Omega))')$, $z_0 \in L^2(\Omega) \times H_0^1(\Omega)$, are defined by

$$Bu = \begin{pmatrix} 0 \\ B_2u \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad \text{and} \quad z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix},$$

and

$$B_2u = -\tilde{A}Du.$$

Theorem. For every $(f, u, y_0, y_1) \in L^2(Q) \times L^2(\Sigma) \times H^{-1}(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))'$, equation (WE) admits a unique weak solution $z(f, u, y_0, y_1) = (y, y')$ in $C([0,T]; H^{-1}(\Omega)) \cap C^1([0,T]; (H^2(\Omega) \cap H_0^1(\Omega))')$.

Existence of a solution in $C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$

- Approximation by regular controls
- Definition of solutions in the sense of transposition
- Estimates on y_n by duality

New regularity results for the wave equation Let θ be the solution to

$$\theta'' - \Delta\theta = g \quad \text{in } Q, \;\; \theta = 0 \quad \text{on } \Sigma,$$

$$\theta(0) = \theta_0, \quad \theta'(0) = \theta_1 \quad \text{in } \Omega.$$

Theorem. The solution θ satisfies the following estimates

$$\|\theta\|_{C([0,T];H_0^1(\Omega))} + \|\theta\|_{C^1([0,T];L^2(\Omega))} + \left\|\frac{\partial\theta}{\partial n}\right\|_{L^2(\Sigma)}$$
$$\leq C\Big(\|\theta_0\|_{H_0^1(\Omega)} + \|\theta_1\|_{L^2(\Omega)} + \|g\|_{L^1(0,T;L^2(\Omega))}\Big).$$

Approximation by regular controls

Let u be in $L^2(\Sigma)$ and let $(u_n)_n$ be a sequence in $C_c^2(]0,T[;H^{3/2}(\Gamma))$, converging to u in $L^2(\Sigma)$. Denote by $Du_n(t) = w_n(t)$ the solution to equation

$$-\Delta w(t) = 0 \quad \text{in } \Omega, \quad w(t) = u_n(t) \quad \text{on } \Gamma.$$

From elliptic regularity results we know that w_n belongs to $C_c^2(]0,T[;H^2(\Omega))$. Let $(y_{0,n})_n$ be a sequence in $H^2(\Omega) \cap H_0^1(\Omega)$, converging to y_0 in $L^2(\Omega)$, and let $(y_{1,n})_n$ be a sequence in $H_0^1(\Omega)$, converging to y_1 in $H^{-1}(\Omega)$. Let z_n be the solution to

$$z'' - \Delta z = f - w_n'' + \Delta w_n \quad \text{in } Q, \ z = 0 \quad \text{on } \Sigma,$$

$$z(0) = y_{0,n}, \quad z'(0) = y_{1,n} \quad \text{in } \Omega.$$

Then $y_n = z_n + w_n$ is the solution to (WE) corresponding to $(f, u_n, y_{0,n}, y_{1,n})$.

Let θ be the solution to

$$\theta'' - \Delta\theta = g \quad \text{in } Q, \;\; \theta = 0 \quad \text{on } \Sigma,$$

$$\theta(T) = 0, \quad \theta'(T) = 0 \quad \text{in } \Omega,$$

where g is a given function in $L^1(0, T; L^2(\Omega))$. The functions y_n and θ are regular enough to justify integrations by parts. We obtain

$$\int_Q y_n\, g = -\int_\Sigma u_n \frac{\partial\theta}{\partial n} - \int_\Omega y_{0,n}\, \theta'(0) + \int_\Omega y_{1,n}\, \theta(0).$$

Definition of a solution in the sense of transposition

Definition. A function $y \in L^2(0,T;L^2(\Omega))$ is a solution to equation (WE) in the transposition sense if and only if

$$\int_Q y\, g$$

$$= -\int_\Sigma u\,\frac{\partial\theta}{\partial n} - \int_\Omega y_0\,\theta'(0) + \Big\langle \theta(0), y_1 \Big\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}$$

for all $g \in L^1(0,T;L^2(\Omega))$, where θ is the solution to

$$\theta'' - \Delta\theta = g \quad \text{in } Q, \;\; \theta = 0 \quad \text{on } \Sigma,$$

$$\theta(T) = 0, \quad \theta'(T) = 0 \quad \text{in } \Omega.$$

Theorem. Equation (WE) admits at most one solution in $L^2(0,T;L^2(\Omega))$ in the transposition sense.

Proof. Suppose that y_1 and y_2 are two solutions. Set $z = y_1 - y_2$. Then

$$\int_Q z\, g = 0$$

for all $g \in L^1(0,T;L^2(\Omega))$. Thus $z = 0$.

First estimate on y_n

We have

$$\|y_n\|_{L^\infty(0,T;L^2(\Omega))} = \textsf{sup}\Big\{ \int_Q y_n\, g \mid \|g\|_{L^1(0,T;L^2(\Omega))} = 1 \Big\}$$

$$\leq \|u_n\|_{L^2(\Sigma)} \Big\| \frac{\partial\theta}{\partial n} \Big\|_{L^2(\Sigma)} + \|y_{0,n}\|_{L^2(\Omega)} \|\theta'(0)\|_{L^2(\Omega)}$$

$$+ \|\theta(0)\|_{H_0^1(\Omega)} \|y_{1,n}\|_{H^{-1}(\Omega)}$$

$$\leq C\Big(\|u\|_{L^2(\Sigma)} + \|y_0\|_{L^2(\Omega)} + \|y_1\|_{H^{-1}(\Omega)} \Big).$$

Thus $(y_n)_n$ is a Cauchy sequence in $C([0,T];L^2(\Omega))$. Denote by y the limit of this sequence.

By passing to the limit in the variational formulation satisfied by y_n we prove that

$$\int_Q y\,g = -\int_\Sigma u\,\frac{\partial\theta}{\partial n} - \int_\Omega y_0\theta'(0) + \Big\langle \theta(0), y_1 \Big\rangle_{H_0^1,H^{-1}}$$

for all $g \in L^1(0,T;L^2(\Omega))$. Thus we have proved the existence of a unique solution to (WE) in $C([0,T];L^2(\Omega))$.

Second estimates on y_n

For $0 \leq \tau \leq T$, let θ_τ be the solution to

$$\theta'' - \Delta\theta = 0 \quad \text{in } Q, \;\; \theta = 0 \quad \text{on } \Sigma,$$

$$\theta(\tau) = \theta_0, \quad \theta'(\tau) = 0 \quad \text{in } \Omega.$$

We can verify that

$$\left\langle y_n'(\tau), \theta_0 \right\rangle_{H^{-1},H_0^1}$$

$$= \int_\Omega y_{1,n}\theta_\tau(0) - \int_\Omega y_{0,n}\theta_\tau'(0) - \int_\Sigma u_n \frac{\partial\theta_\tau}{\partial n}.$$

Thus

$$\|y_n'\|_{C([0,T];H^{-1})} = \text{sup}_\tau \text{sup}_{\|\theta_0\|_{H_0^1}=1} \left| \left\langle y_n'(\tau), \theta_0 \right\rangle_{H^{-1},H_0^1} \right|.$$

We have

$$\|y_n'\|_{C([0,T];H^{-1})}$$
$$\leq C\Big(\|u_n\|_{L^2(\Sigma)} + \|y_{0,n}\|_{L^2(\Omega)} + \|y_{1,n}\|_{H^{-1}(\Omega)}\Big),$$

and

$$\|y'\|_{C([0,T];H^{-1})}$$
$$\leq C\Big(\|u\|_{L^2(\Sigma)} + \|y_0\|_{L^2(\Omega)} + \|y_1\|_{H^{-1}(\Omega)}\Big).$$

The control problem

$$(P) \quad \inf\{J(y,u) \mid (y,u) \text{ obeys } (WE),\ u \in L^2(\Sigma)\},$$

the functionals J is defined by

$$J(y,u) = \frac{1}{2}\int_Q |y - y_d|^2 + \frac{1}{2}\int_\Omega |y(T) - y_d(T)|^2 + \frac{\beta}{2}\int_\Sigma u^2,$$

where the function y_d belongs to $C([0,T]; L^2(\Omega))$.

Theorem. Assume that $f \in L^2(Q)$, $y_0 \in L^2(\Omega)$, $y_1 \in H^{-1}(\Omega)$, and $y_d \in C([0,T]; L^2(\Omega))$. Problem (P) admits a unique solution $(\bar{y}, \bar{u})$.

Existence of a unique optimal control

1. Set $F(u) = J(y(u), u)$. Let $(u_n)_n$ be a minimizing sequence in $L^2(\Sigma)$, that is

$$\lim_{n\to\infty} F(u_n) = \inf_{u\in L^2(\Sigma)} F(u).$$

We suppose that

$$u_n \rightharpoonup \bar{u} \qquad \text{weakly in } L^2(\Sigma).$$

Let y_n the solution of (WE) corresponding to u_n.

Passage to the limit in the equation. Let $\bar{y} = y(\bar{u})$.

The operator

$$\Lambda \ : \ u \longrightarrow \Big(y(u), y(u)(T)\Big)$$

is affine and continuous from $L^2(\Sigma)$ to $L^2(Q) \times L^2(\Omega)$.

We conclude that problem (P) admits a unique solution $(\bar{y}, \bar{u})$.

Optimality conditions for (P)

By a classical calculation we have

$$F'(u)v = \int_Q (y(u) - y_d)z(v)$$

$$+ \int_\Omega (y(u)(T) - y_d(T))z(v)(T) + \beta \int_\Sigma uv,$$

where $z(v)$ is the solution of

$$z'' - \Delta z = 0 \quad \text{in } Q, \qquad z = v \quad \text{on } \Sigma,$$

$$z(x,0) = 0 \text{ and } \; z'(x,0) = 0 \quad \text{in } \Omega.$$

Identification of $F'(u)$

We look for q such that

$$\int_Q (y(u) - y_d) z(v) + \int_\Omega [(y(u) - y_d) z(v)](T) = \int_\Sigma q\, v.$$

Let p be a regular function defined on $\overline{Q}$ and write an integration by parts between $z(v)$ and p:

$$\begin{aligned} 0 &= \int_Q (z'' - \Delta z) p \\ &= \int_Q z(p'' - \Delta p) + \int_\Omega z'(T) p(T) \\ &- \int_\Omega z(T) p'(T) - \int_\Sigma \frac{\partial z}{\partial n} p + \int_\Sigma \frac{\partial p}{\partial n} v \end{aligned}$$

Identification with

$$\int_Q (y(u) - y_d)z + \int_\Omega [(y(u) - y_d)z](T) = \int_\Sigma q\, v.$$

We set

$$p'' - \Delta p = y(u) - y_d \quad \text{in } Q, \qquad p = 0 \quad \text{on } \Sigma,$$

$$p(x,T) = 0 \text{ and } \; p'(x,T) = -(y(u) - y_d)(T) \quad \text{in } \Omega.$$

and we have

$$F'(u)v = \int_\Sigma (-\frac{\partial p}{\partial n} + \beta u)v,$$

if the above calculation are justified.

Theorem. (i) If $(\bar{y}, \bar{u})$ is the solution to (P) then $\bar{u} = \frac{1}{\beta}\frac{\partial p}{\partial n}$, where p is the solution to the adjoint equation corresponding to $\bar{y}$:

$$p'' - \Delta p = \bar{y} - y_d \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma,$$

$$p(x,T) = 0 \text{ and } p'(x,T) = -(\bar{y} - y_d)(T) \quad \text{in } \Omega.$$

(ii) Conversely, if a pair $(\tilde{y}, \tilde{p}) \in C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega))$ obeys the system

$$\tilde{y}'' - \Delta\tilde{y} = f \quad \text{in } Q, \tilde{y} = \frac{1}{\beta}\frac{\partial\tilde{p}}{\partial n} \quad \text{on } \Sigma,$$

$$\tilde{y}(0) = y_0, \quad \tilde{y}'(0) = y_1, \quad \text{in } \Omega,$$

$$\tilde{p}'' - \Delta\tilde{p} = \tilde{y} - y_d \quad \text{in } Q, \quad \tilde{p} = 0 \quad \text{on } \Sigma,$$

$$\tilde{p}(T) = 0, \quad \tilde{p}'(T) = -\tilde{y}(T) + y_d(T) \quad \text{in } \Omega,$$

then the pair $(\tilde{y}, \frac{1}{\beta}\frac{\partial\tilde{p}}{\partial n})$ is the optimal solution to (P).

References

[1] A. Bensoussan, G. Da Prato, M. C. Delfour, S. K. Mitter, Representation and Control of Infinite Dimensional Systems, Vol. 1, Birkhäuser, 1992.

[2] S. Kesavan, Topics in Functional Analysis and Applications, Wiley-Eastern, New Delhi, 1989.

[3] V. Komornik, Exact Controllability and Stabilization, J. Wiley and Masson, Paris,1994.

[4] J.-L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer, 1971.

[5] J.-L. Lions, Contrôlabilité exacte et Stabilisation

des systèmes distribués, Vol. 1, 2, Masson, Paris 1988.

[6] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, 1983.