Chapter 3

Optimal Control of Evolution Equations
with Bounded Control Operators

Jean-Pierre Raymond
Introduction to the optimal control of evolution equations

Distributed control of the heat equation

Existence of optimal controls

Characterization of optimal controls

Distributed control of the wave equation

A general control problem

Control of a first order hyperbolic system
Optimal control of evolution equations
Setting of the problem

We consider equations of the form

\[(E) \quad y' = Ay + Bu + f, \quad y(0) = y_0.\]

Assumptions

\(Y\) and \(U\) are Hilbert spaces.
The unbounded operator \((A, D(A))\) is the infinitesimal generator of a strongly continuous semigroup on \(Y\).
This semigroup will be denoted by \((e^{tA})_{t\geq 0}\).
The operator \(B\) belongs to \(\mathcal{L}(U; Y)\).
The control problem

\[(P) \quad \inf \{ J(y, u) \mid u \in L^2(0, T; U), \ (y, u) \text{ obeys } (E) \}, \]

\[
J(y, u) = \frac{1}{2} \int_0^T |Cy(t) - zd(t)|_Z^2 + \frac{1}{2} |Dy(T) - z_T|_{Z_T}^2 \\
+ \frac{1}{2} \int_0^T |u(t)|_U^2.
\]

**Assumption**

$Z$ and $Z_T$ are Hilbert spaces.

The operator $C$ belongs to $\mathcal{L}(Y; Z)$, and the operator $D$ belongs to $\mathcal{L}(Y; Z_T)$. The function $zd$ belongs to $L^2(0, T; Z)$ and $z_T \in Z_T$. 
Optimal control
of the heat equation
The state equation

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, with a boundary $\Gamma$ of class $C^2$. Let $T > 0$, set $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. We consider the heat equation with a distributed control

$$\frac{\partial y}{\partial t} - \Delta y = f + \chi_\omega u \quad \text{in } Q,$$

$$y = 0 \quad \text{on } \Sigma, \quad y(x, 0) = y_0 \quad \text{in } \Omega.$$
The control problem

\[ (P) \quad \inf \{ J(y, u) \mid u \in L^2(\omega \times (0, T)) \}, \]

\[(y, u) \text{ obeys } (HE) \}, \]

where

\[
J(y, u) = \frac{1}{2} \int_Q |y - y_d|^2 + \frac{1}{2} \int_\Omega |y(T) - y_d(T)|^2 + \frac{\beta}{2} \int_{\omega \times (0, T)} u^2,
\]

\[\beta > 0 \text{ and } y_d \in C([0, T]; L^2(\Omega)).\]

Estimate for the state variable

\[
\|y\|_{C([0,T]; L^2(\Omega))} \leq C(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2(Q)} + \|u\|_{L^2(\omega \times (0,T))}).
\]
Existence of a unique optimal control

1. Set $F(u) = J(y(u), u)$. Let $(u_n)_n$ be a minimizing sequence in $L^2(\omega \times (0, T))$, that is

$$\lim_{n \to \infty} F(u_n) = \inf_{u \in L^2(\omega \times (0, T))} F(u).$$

Let $y_n$ the solution of $(HE)$ corresponding to $u_n$, suppose that $(u_n)_n$ is bounded in $L^2(\omega \times (0, T))$, and that

$$u_n \rightharpoonup \bar{u} \quad \text{weakly in } L^2(\omega \times (0, T)).$$
2. Let $\bar{y} = y(\bar{u})$.

The operator

$$\Lambda : u \mapsto y(u)$$

is affine and continuous from $L^2(\omega \times (0, T))$ to $L^2(Q)$, and

$$\Lambda_T : u \mapsto y(u)(T)$$

is affine and continuous from $L^2(\omega \times (0, T))$ to $L^2(\Omega)$.

The sequence $(y_n)_n$ converges to $\bar{y}$ for the weak topology of $L^2(Q)$, and $(y_n(T))_n$ converges to $\bar{y}(T)$ for the weak topology of $L^2(\Omega)$. 
3. Using the weakly lower semicontinuity of $\| \cdot \|^2_{L^2(Q)}$, $\| \cdot \|^2_{L^2(\Omega)}$, $\| \cdot \|^2_{L^2(\omega \times (0,T))}$, we obtain

$$\int_{\omega \times (0,T)} \bar{u}^2 \leq \liminf_{n \to \infty} \int_{\omega \times (0,T)} u_n^2,$$

$$\int_{Q} |\bar{y} - y_d|^2 \leq \liminf_{n \to \infty} \int_{Q} |y_n - y_d|^2,$$

and

$$\int_{\Omega} |\bar{y}(T) - y_d(T)|^2 \leq \liminf_{n \to \infty} \int_{\Omega} |y_n(T) - y_d(T)|^2.$$

Combining these results, we have

$$F(\bar{u}) \leq \liminf_{n \to \infty} F(u_n) = m.$$

Thus $\bar{u}$ is a solution to $(P)$. 
Uniqueness. Recall that the mappings

\[ u \rightarrow y(u) \quad \text{and} \quad u \rightarrow y(u)(T) \]

are affine. Thus

\[ u \rightarrow \frac{1}{2} \int_Q |y(u) - y_d|^2 + \frac{1}{2} \int_\Omega |y(u)(T) - y_d(T)|^2 \]

is convex. The mapping

\[ u \rightarrow \frac{\beta}{2} \int_Q \chi \omega u^2 \]

is strictly convex. Thus the uniqueness follows from the strict convexity of \( F \).
Optimality conditions
Derivative of the state variable

Equation satisfied by $z_\lambda = y(u + \lambda v) - y(u)$

\[
\frac{\partial z}{\partial t} - \Delta z = \lambda \chi_{\omega} v \quad \text{in } Q,
\]

\[
z = 0 \quad \text{on } \Sigma, \quad z(x, 0) = 0 \quad \text{in } \Omega.
\]

From the estimate for $(HE)$ it follows that

\[
\|z_\lambda\|_{C([0,T];L^2(\Omega))} \leq C |\lambda| \|v\|_{L^2(\omega \times (0,T))}.
\]

Thus

\[
y(u + \lambda v) \xrightarrow{C([0,T];L^2(\Omega))} y(u).
\]
\[ F'(u)v = \lim_{\lambda \searrow 0} \frac{F(u + \lambda v) - F(u)}{\lambda}. \]

By a classical calculation we have

\[
F'(u)v = \int_Q (y(u) - y_d)z(v)
+ \int_{\Omega} (y(u)(T) - y_d(T))z(v)(T) + \beta \int_{\omega \times (0,T)} uv,
\]

where \( z(v) \) is the solution of

\[
\frac{\partial z}{\partial t} - \Delta z = \chi_\omega v \quad \text{in } Q,
\]

\[ z = 0 \quad \text{on } \Sigma, \quad z(x,0) = 0 \quad \text{in } \Omega. \]
Identification of $F'(u)$

We look for $q$ such that

$$
\int_Q (y(u) - y_d) z(v) + \int_\Omega [(y(u) - y_d) z(v)](T) = \int_{\omega \times (0,T)} q v.
$$

Let $p$ be a regular function defined on $\overline{Q}$ and write an integration by parts between $z(v)$ and $p$:

$$
\int_{\omega \times (0,T)} v p = \int_Q (z_t - \Delta z)p
$$

$$
= \int_Q z(-p_t - \Delta p) + \int_\Omega z(T)p(T) - \int_\Sigma \frac{\partial z}{\partial n} p
$$
Identification with

\[ \int_Q (y(u) - y_d)z + \int_\Omega [(y(u) - y_d)z](T) = \int_{\omega \times (0, T)} q v. \]

We set

\[ -\frac{\partial p}{\partial t} - \Delta p = y(u) - y_d \quad \text{in } Q, \]

\[ p = 0 \quad \text{on } \Sigma, \quad p(x, T) = (y(u) - y_d)(T) \quad \text{in } \Omega, \]

and we have

\[ F'(u)v = \int_{\omega \times (0, T)} (p + \beta u)v, \]

if the above calculation are justified.
The adjoint equation

Let \( g \in L^2(Q), \ p_T \in L^2(\Omega) \). The terminal boundary value problem

\[
\begin{aligned}
-AE &\quad \frac{\partial p}{\partial t} - \Delta p = g & \text{in } Q, \\
p &= 0 & \text{on } \Sigma, \\
p(x, T) &= p_T & \text{in } \Omega,
\end{aligned}
\]

is well posed.

\[
\|p\|_{C([0,T];L^2(\Omega))} \leq C(\|g\|_{L^2(Q)} + \|p_T\|_{L^2(\Omega)}).
\]
Proof. A weak solution in $L^2(0, T; L^2(\Omega))$ to $(AE)$ is a function $p \in L^2(0, T; L^2(\Omega))$ such that, for all $z \in H^2 \cap H^1_0(\Omega)$, the mapping

$$t \mapsto \langle p(t), z \rangle$$

belongs to $H^1(0, T)$ and obeys

$$-\frac{d}{dt} \langle p(t), z \rangle = \langle y(t), A^* z \rangle + \langle g(t), z \rangle,$$

$$\langle p(T), z \rangle = \langle p_T, z \rangle.$$
The function \( p \) is a weak solution to \((AE')\) if and only if the function \( q \) defined by

\[
q(x, t) = p(x, T - t)
\]

is the solution to the equation

\[
\frac{\partial q}{\partial t} - \Delta q = \tilde{g} \quad \text{in } Q,
\]

\[
q = 0 \quad \text{on } \Sigma, \quad q(x, 0) = p_T \quad \text{in } \Omega,
\]

where \( \tilde{g}(x, t) = g(x, T - t) \).
Integration by parts between $z$ and $p$

**Theorem.** Suppose that $\phi \in L^2(Q)$, $g \in L^2(Q)$, and $p_T \in L^2(\Omega)$. Then the solution $z$ of equation

$$\frac{\partial z}{\partial t} - \Delta z = \phi \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x, 0) = 0 \quad \text{in } \Omega,$$

and the solution $p$ of (AE) satisfy the following formula

$$\int_Q \phi p = \int_Q zg + \int_\Omega z(T)p_T.$$
Proof. If \( p_T \in H_0^1(\Omega) \), due to a Theorem of Chapter 2, \( z \) and \( p \) belong to \( L^2(0, T; D(A)) \cap H^1(0, T; L^2(\Omega)) \). In that case, with the Green formula we have

\[
\int_\Omega -\Delta z(t)p(t) \, dx = \int_\Omega -\Delta p(t)z(t) \, dx
\]

for almost every \( t \in [0, T] \), and

\[
\int_0^T \int_\Omega \frac{\partial z}{\partial t} p = - \int_0^T \int_\Omega \frac{\partial p}{\partial t} z + \int_\Omega z(T)p_T.
\]
Thus the IBP formula is established in the case when \( p_T \in H^1_0(\Omega) \). If \( (p_{Tn})_n \) is a sequence in \( H^1_0(\Omega) \) converging to \( p_T \) in \( L^2(\Omega) \), due to the '\( C([0,T]; L^2(\Omega)) \)-estimate', \( (p_n)_n \) - where \( p_n \) is the solution to (AE) corresponding to \( p_{Tn} \) - converges to \( p \) (the solution of (AE) associated with \( p_T \)) in \( C([0,T]; L^2(\Omega)) \) when \( n \) tends to infinity. Thus, in the case when \( p_T \in L^2(\Omega) \), the IBP formula can be deduced by passing to the limit in the formula satisfied by \( p_n \).
Theorem. (i) If $(\bar{y}, \bar{u})$ is the solution to $(P)$ then

\[ \bar{u} = -\frac{1}{\beta} p|_{\omega \times (0, T)}, \]

where $p$ is the solution to the adjoint equation corresponding to $\bar{y}$:

\[
-\frac{\partial p}{\partial t} - \Delta p = \bar{y} - y_d \quad \text{in } Q, \\
p = 0 \quad \text{on } \Sigma, \quad p(x, 0) = \bar{y}(T) - y_d(T) \quad \text{in } \Omega.
\]
(ii) Conversely, if a pair \((\tilde{y}, \tilde{p}) \in C([0, T]; L^2(\Omega)) \times C([0, T]; L^2(\Omega))\) obeys the system

\[
\begin{align*}
\frac{\partial \tilde{y}}{\partial t} - \Delta \tilde{y} &= f - \frac{1}{\beta} \chi \omega \tilde{p} \quad \text{in } Q, \\
\tilde{y} &= 0 \quad \text{on } \Sigma, \quad \tilde{y}(x, 0) = \bar{y}_0 \quad \text{in } \Omega, \\
-\frac{\partial \tilde{p}}{\partial t} - \Delta \tilde{p} &= \tilde{y} - y_d \quad \text{in } Q, \\
p &= 0 \quad \text{on } \Sigma, \quad \tilde{p}(T) = \tilde{y}(T) - y_d(T) \quad \text{in } \Omega,
\end{align*}
\]

then the pair \((\tilde{y}, -\frac{1}{\beta} \tilde{p})\) is the optimal solution to problem \((P)\).

Proof. (i) The necessary optimality condition is already proved.

(ii) The sufficient optimality condition can be proved with a theorem stated in Chapter 1.
Optimal control of the wave equation
The state equation

The assumptions on $\Omega$, $\Gamma$, $\omega$, $T$, $Q$, $\Sigma$ are the ones of the previous section. We consider

\[(WE)\]

$$y'' - \Delta y = f + \chi_\omega u \quad \text{in } Q, \quad y = 0 \text{ on } \Sigma,$$

$$y(x, 0) = y_0 \quad \text{and} \quad y'(x, 0) = y_1 \quad \text{in } \Omega,$$

with $(y_0, y_1) \in H^1_0(\Omega) \times L^2(\Omega)$, $f \in L^2(Q)$, and $u \in L^2(\omega \times (0, T))$.

The operator

$$(f + \chi_\omega u, y_0, y_1) \mapsto y(f + \chi_\omega u, y_0, y_1)$$

is linear and continuous from $L^2(Q) \times H^1_0(\Omega) \times L^2(\Omega)$ into $C([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega))$. 
The family of control problems

\[
(P_i) \quad \inf \{ J_i(y, u) \mid (y, u) \text{ obeys } (WE), \ u \in L^2 \},
\]

with, for \( i = 1, \ldots, 3 \), the functionals \( J_i \) are defined by

\[
J_1(y, u) = \frac{1}{2} \int_Q |y - y_d|^2 + \frac{1}{2} \int_\Omega |y(T) - y_d(T)|^2 + \frac{\beta}{2} \int_{\omega \times (0,T)} u^2,
\]

\[
J_2(y, u) = \frac{1}{2} \int_\Omega |\nabla y(T) - \nabla y_d(T)|^2 + \frac{\beta}{2} \int_{\omega \times (0,T)} u^2,
\]

\[
J_3(y, u) = \frac{1}{2} \int_\Omega \left| y'(T) - y'_d(T) \right|^2 + \frac{\beta}{2} \int_{\omega \times (0,T)} u^2,
\]

where the function \( y_d \in C([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \).
Theorem. Assume that $f \in L^2(Q)$, $y_0 \in H^1_0(\Omega)$, $y_1 \in L^2(\Omega)$, and $y_d \in C([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega))$. For $i = 1, \ldots, 3$, problem $(P_i)$ admits a unique solution $(\bar{y}_i, \bar{u}_i)$. 
Existence of a unique optimal control

1. Set $F(u) = J(y(u), u)$. Let $(u_n)_n$ be a minimizing sequence in $L^2(\omega \times (0, T))$, that is

$$\lim_{n \to \infty} F(u_n) = \inf_{u \in L^2(\omega \times (0, T))} F(u).$$

We suppose that

$$u_n \rightharpoonup \bar{u} \quad \text{weakly in } L^2(\omega \times (0, T)).$$

Let $y_n$ the solution of $(WE)$ corresponding to $u_n$, suppose that $(u_n)_n$ is bounded in $L^2(\omega \times (0, T))$, and that

$$u_n \rightharpoonup \bar{u} \quad \text{weakly in } L^2(\omega \times (0, T)).$$
Passage to the limit in the equation.

Let \( \bar{y} = y(\bar{u}) \). The operator

\[
\Lambda : u \longrightarrow \left( y(u), y(u)(T), y(u)'(T) \right)
\]

is affine and continuous from \( L^2(\omega \times (0, T)) \) to \( L^2(Q) \times H^1_0(\Omega) \times L^2(\Omega) \).

We may conclude that, for \( i = 1, \ldots, 3 \), problem \((P_i)\) admits a unique solution \((\bar{y}_i, \bar{u}_i)\).
Optimality conditions for \((P_1)\)
\[ J_1(y, u) \]
\[ = \frac{1}{2} \int_Q |y - y_d|^2 + \frac{1}{2} \int_\Omega |y(T) - y_d(T)|^2 + \frac{\beta}{2} \int_{\omega \times (0,T)} u^2, \]

By a classical calculation we have

\[ F'(u)v = \int_Q (y(u) - y_d)z(v) \]
\[ + \int_\Omega (y(u)(T) - y_d(T))z(v)(T) + \beta \int_{\omega \times (0,T)} uv, \]

where \( z(v) \) is the solution of

\[ z'' - \Delta z = \chi_\omega v \quad \text{in} \ Q, \quad z = 0 \quad \text{on} \ \Sigma, \]
\[ z(x, 0) = 0 \quad \text{and} \quad z'(x, 0) = 0 \quad \text{in} \ \Omega. \]
**Identification of $F'(u)$**

We look for $q$ such that

$$\int_{Q} (y(u)-y_d)z(v) + \int_{\Omega} [(y(u)-y_d)z(v)](T) = \int_{\omega \times (0,T)} q \, v.$$

Let $p$ be a regular function defined on $\overline{Q}$ and write an integration by parts between $z(v)$ and $p$:

$$\int_{\omega \times (0,T)} v \, p = \int_Q (z'' - \Delta z) p$$

$$= \int_Q z(p'' - \Delta p) + \int_{\Omega} z'(T)p(T)$$

$$- \int_{\Omega} z(T)p'(T) - \int_{\Sigma} \frac{\partial z}{\partial n} p$$
Identification with
\[ \int_Q (y(u) - y_d)z + \int_\Omega [(y(u) - y_d)z](T) = \int_{\omega \times (0,T)} q v. \]

We set
\[ p'' - \Delta p = y(u) - y_d \quad \text{in} \quad Q, \quad p = 0 \quad \text{on} \quad \Sigma, \]
\[ p(x, T) = 0 \quad \text{and} \quad p'(x, T) = (y(u) - y_d)(T) \quad \text{in} \quad \Omega. \]

and we have
\[ F'(u)v = \int_{\omega \times (0,T)} (p + \beta u)v, \]

if the above calculation are justified.
**Theorem.**  (i) If \((\bar{y}, \bar{u})\) is the solution to \((P_1)\) then 
\[
\bar{u} = -\frac{1}{\beta} p|_{\omega \times (0,T)},
\]
where \(p\) is the solution to:

\[
\begin{align*}
p'' - \Delta p &= \bar{y} - y_d \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \\
p(x, T) &= 0, \quad p'(x, T) = \bar{y}(T) - y_d(T) \quad \text{in } \Omega,
\end{align*}
\]

(ii) Conversely, if \((\tilde{y}, \tilde{p}) \in (C([0, T]; L^2(\Omega)))^2\) obeys:

\[
\begin{align*}
\tilde{y}'' - \Delta \tilde{y} &= f - \frac{1}{\beta} \chi_\omega \tilde{p} \quad \text{in } Q, \quad \tilde{y} = 0 \quad \text{on } \Sigma, \\
\tilde{y}(x, 0) &= y_0, \quad \tilde{y}'(x, 0) = y_1 \quad \text{in } \Omega, \\
\tilde{p}'' - \Delta \tilde{p} &= \tilde{y} - y_d \quad \text{in } Q, \quad \tilde{p} = 0 \quad \text{on } \Sigma, \\
\tilde{p}(T) &= 0, \quad \tilde{p}'(T) = y(T) - y_d(T) \quad \text{in } \Omega,
\end{align*}
\]

then the pair \((\tilde{y}, -\frac{1}{\beta} \tilde{p})\) is the optimal solution to \((P_1)\).
Optimality conditions for $(P_2)$
Recall that

\[ J_2(y, u) = \frac{1}{2} \int_\Omega |\nabla y(T) - \nabla y_d(T)|^2 + \frac{\beta}{2} \int_{\omega \times (0,T)} u^2. \]

**Theorem.** (i) If \((\bar{y}, \bar{u})\) is the solution to \((P_2)\) then

\[ \bar{u} = -\frac{1}{\beta} p|_{\omega \times (0,T)}, \]

where \(p\) is the solution to the adjoint equation

\[ p'' - \Delta p = 0 \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \]

\[ p(T) = 0 \text{ and } p'(T) = -\Delta (\bar{y}(T) - y_d(T)) \quad \text{in } \Omega. \]

\[ (p, p') \in C([0, T]; L^2(\Omega)) \times C([0, T]; H^{-1}(\Omega)). \]
(ii) Conversely, if a pair $(\tilde{y}, \tilde{p}) \in C([0, T]; L^2(\Omega)) \times C([0, T]; L^2(\Omega))$ obeys the system

\[
\tilde{y}'' - \Delta \tilde{y} = f - \frac{1}{\beta} \chi_\omega \tilde{p} \quad \text{in } Q, \quad \tilde{y} = 0 \quad \text{on } \Sigma,
\]

\[
\tilde{y}(x, 0) = y_0, \quad \tilde{y}'(x, 0) = y_1, \quad \text{in } \Omega,
\]

\[
\tilde{p}'' - \Delta \tilde{p} = 0 \quad \text{in } Q, \quad \tilde{p} = 0 \quad \text{on } \Sigma,
\]

\[
\tilde{p}(T) = 0, \quad p'(T) = -\Delta(\tilde{y}(T) - y_d(T)) \quad \text{in } \Omega,
\]

then the pair $(\tilde{y}, -\frac{1}{\beta} \tilde{p})$ is the optimal solution to $(P_2)$. 
**Remark 1.** We set

\[ F_2(u) = J_2(y(u), u). \]

We have

\[ F_2'(u)v = \int_{\Omega} \left( \nabla y(T) - \nabla y_d(T) \right) \cdot \nabla z(T) + \beta \int_{\omega \times (0,T)} u v , \]

where \( z \) is the solution to

\[ z'' - \Delta z = \chi_\omega v \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \]

\[ z(x, 0) = 0, \quad z'(x, 0) = 0, \quad \text{in } \Omega. \]
Moreover

\[
\int_{\Omega} \left( \nabla y(T) - \nabla y_d(T) \right) \cdot \nabla z(T)
\]

\[
= \left\langle z(T), (-\Delta)(y(T) - y_d(T)) \right\rangle_{H^1_0(\Omega), H^{-1}(\Omega)}.
\]

This is why we have

\[
p'(x, T) = -\Delta(\bar{y}(T) - y_d(T))
\]

in the adjoint equation.
Remark 2. If \( \tilde{y} \in C([0, T]; H^1_0(\Omega)) \), then \( \Delta \tilde{y}(T) \) belongs to \( H^{-1}(\Omega) \). Thus the adjoint equation is stated with \( p'(T) \) in \( H^{-1}(\Omega) \). We are going to prove that the wave equation is well posed with an initial condition in \( L^2(\Omega) \times H^{-1}(\Omega) \).

Let us recall a result from chapter 2. Set \( Y = H^1_0(\Omega) \times L^2(\Omega) \) and endow \( Y \) with the inner product

\[
(u, v)_Y = \int_\Omega \nabla u_1 \cdot \nabla v_1 + \int_\Omega u_2 v_2,
\]

where \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \). Set \( D(A) = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \) and

\[
Ay = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \Delta y_1 \end{pmatrix}, \quad \text{and} \quad y_0 = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}.
\]
In chapter 2 we have proved that \((A, D(A))\) and \((-A, D(A))\) are m-dissipative in \(Y\).

Now we set \(\hat{Y} = L^2(\Omega) \times H^{-1}(\Omega)\). We equip \(\hat{Y}\) with the inner product

\[
(u, v)_{\hat{Y}} = \int_{\Omega} u_1 \cdot v_1 + \left\langle (-\Delta)^{-1} u_2, v_2 \right\rangle_{H^1_0(\Omega), H^{-1}(\Omega)},
\]

where \(u = (u_1, u_2)\) and \(v = (v_1, v_2)\). Set \(D(\hat{A}) = H^1_0(\Omega) \times L^2(\Omega)\) and

\[
\hat{A} y = \hat{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \Delta y_1 \end{pmatrix}.
\]

We can prove that \((\hat{A}, D(\hat{A}))\) and \((-\hat{A}, D(\hat{A}))\) are m-dissipative in \(\hat{Y}\).
Optimality conditions for $\left( P_3 \right)$
The functional is

\[ J_3(y, u) = \frac{1}{2} \int_\Omega \left| y'(T) - y'_d(T) \right|^2 + \frac{\beta}{2} \int_{\omega \times (0,T)} u^2. \]

**Theorem.** (i) If \((\bar{y}, \bar{u})\) is the solution to \((P_3)\) then \(\bar{u} = -\frac{1}{\beta} p|_{\omega \times (0,T)}\), where \(p\) is the solution to the adjoint

\[
\begin{align*}
p'' - \Delta p &= 0 \quad \text{in } Q, \\
p &= 0 \quad \text{on } \Sigma, \\
p(T) &= (\bar{y}' - y'_d)(T) \quad \text{and} \quad p'(T) = 0 \quad \text{in } \Omega.
\end{align*}
\]
(ii) Conversely, if a pair \((\tilde{y}, \tilde{p}) \in C([0, T]; L^2(\Omega)) \times C([0, T]; L^2(\Omega))\) obeys the system

\[
\tilde{y}'' - \Delta \tilde{y} = f - \frac{1}{\beta} \chi_\omega \tilde{p} \quad \text{in } Q, \quad \tilde{y} = 0 \quad \text{on } \Sigma,
\]

\[
\tilde{y}(x, 0) = y_0, \quad \tilde{y}'(x, 0) = y_1, \quad \text{in } \Omega,
\]

\[
\tilde{p}'' - \Delta \tilde{p} = 0 \quad \text{in } Q, \quad \tilde{p} = 0 \quad \text{on } \Sigma,
\]

\[
\tilde{p}(T) = (\tilde{y}' - y'_d)(T), \quad \tilde{p}'(T) = 0 \quad \text{in } \Omega,
\]

then the pair \((\tilde{y}, -\frac{1}{\beta} \tilde{p})\) is the optimal solution to \((P_3)\).
Optimal control of evolution equations
The state equation

\[ (SE) \quad y' = Ay + Bu + f, \quad y(0) = y_0. \]

Assumptions

\( Y \) and \( U \) are Hilbert spaces.

The unbounded operator \((A, D(A))\) is the infinitesimal generator of a strongly continuous semigroup on \( Z \).

This semigroup will be denoted by \((e^{tA})_{t \geq 0}\).

The operator \( B \) belongs to \( \mathcal{L}(U; Y) \).

The control problem

\[ (P) \quad \inf\{ J(y, u) \mid u \in L^2(0, T; U), \ (y, u) \text{ obeys } (SE) \}. \]
with

\[ J(y, u) = \frac{1}{2} \int_0^T |Cy(t) - z_d(t)|_Z^2 \]
\[ + \frac{1}{2} |Dy(T) - z_T|_{Z_T}^2 + \frac{1}{2} \int_0^T |u(t)|_U^2. \]

**Assumption**

\( Z \) and \( Z_T \) are Hilbert spaces.

The operator \( C \) belongs to \( \mathcal{L}(Y; Z) \), and the operator \( D \) belongs to \( \mathcal{L}(Y; Z_T) \). The function \( z_d \) belongs to \( L^2(0, T; Z) \) and \( z_T \in Z_T \).
Existence of a unique optimal control

If the assumptions on $B$, $C$, $D$ are satisfied. Problem $(P)$ admits a unique solution $(y, u)$.

The proof is based on the existence of a minimizing sequence $(u_n)_n$, bounded in $L^2(0, T; U)$, and on the fact that the operator

$$
\Lambda : u \longrightarrow \left( Cy(u) - z_d, Dy(u)(T) - z_T \right)
$$

is affine and continuous from $L^2(0, T; U)$ to $L^2(0, T; Z) \times Z_T$. 
Optimality conditions

The adjoint equation for \((P)\) will be of the form

\[
(AE) \quad -p' = A^*p + g, \quad p(T) = p_T.
\]

From chapter 2, we know that \((A^*, D(A^*))\) is the infinitesimal generator of a strongly continuous semigroup on \(Y'\). Thus \((AE)\) is well posed if \(p_T \in Y'\) and if \(g \in L^1(0, T; Y')\). For simplicity we identify \(Y\) and \(Y'\).

Integration by parts formula

We state an integration by parts formula between the adjoint state \(p\) and the solution \(z\) to the equation

\[
(LE) \quad z' = Az + f, \quad z(0) = 0.
\]
Theorem. For every $f \in L^2(0, T; Y)$, and every $(g, p_T) \in L^2(0, T; Y) \times Y$, the solution $z$ to equation (LE) and the solution $p$ to equation (AE) satisfy the following formula

\[
\int_0^T \left( f(t), p(t) \right)_Y \ dt = \int_0^T \left( z(t), g(t) \right)_Y \ dt + \left( z(T), p_T \right)_Y - \left( z_0, p(0) \right)_Y.
\]
Proof. Suppose that $f$ and $g$ belong to $C^1([0, T]; Y)$ and that $p_T$ belongs to $D(A^*)$. In this case we can write

$$\int_0^T (f(t), p(t))_Y dt = \int_0^T (z'(t) - Az(t), p(t))_Y dt$$

$$= \int_0^T - (z(t), p'(t))_Y dt + (z(T), p_T)_Y$$

$$- (z_0, p(0))_Y - \int_0^T (Az(t), p(t))_Y dt$$

$$= \int_0^T (z(t), g(t))_Y dt + (z(T), p_T)_Y - (z_0, p(0))_Y.$$

Thus, the IBP formula can be deduced from this case by using density arguments.
Optimality conditions

**Theorem.** If \((\bar{y}, \bar{u})\) is the solution to \((P)\) then \(\bar{u} = -B^*p\), where \(p\) is the solution to equation

\[-p' = A^*p + C^*(C\bar{y} - z_d), \quad p(T) = D^*(D\bar{y}(T) - z_T).\]

Conversely, if a pair \((\tilde{y}, \tilde{p}) \in C([0, T]; Y) \times C([0, T]; Y)\) obeys the system

\[
\begin{align*}
\tilde{y}' &= A\tilde{y} - BB^*\tilde{p} + f, \quad \tilde{y}(0) = y_0, \\
-\tilde{p}' &= A^*\tilde{p} + C^*(C\tilde{y} - z_d), \\
\tilde{p}(T) &= D^*(D\tilde{y}(T) - z_T),
\end{align*}
\]

then the pair \((\tilde{y}, -B^*\tilde{p})\) is the optimal solution to problem \((P)\).
Proof. Let $(\bar{y}, \bar{u})$ be the optimal solution to problem $(P)$. Set $F(u) = J(y(u), u)$. For every $u \in L^2(0, T; U)$, we have

$$F'(\bar{u})u = \int_0^T \left( C\bar{y}(t) - z_d, Cz(t) \right)_Z$$

$$+ \left( D\bar{y}(T) - z_T, Dz(T) \right)_{Z_T} + \int_0^T \left( \bar{u}(t), u(t) \right)_U$$

$$= \int_0^T \left( C^*(C\bar{y}(t) - z_d), z(t) \right)_Y$$

$$+ \left( D^*(D\bar{y}(T) - z_T), z(T) \right)_Y + \int_0^T \left( \bar{u}(t), u(t) \right)_U,$$

where $z$ is the solution to

$$z' = Az + Bu, \quad z(0) = 0.$$
Applying the IBP formula to \( p \) and \( z \), we obtain

\[
F'(\bar{u})u = \int_0^T (p(t), Bu(t))_Y + \int_0^T (\bar{u}(t), u(t))_U \\
= \int_0^T (B^*p(t) + \bar{u}(t), u(t))_U.
\]

The first part of the Theorem is established. The second part follows from the sufficient optimality condition stated in Chapter 1.
Exercise

Let $L > 0$ and $a$ be a function in $H^1(0, L)$ such that $0 < c_1 \leq a(x)$ for all $x \in H^1(0, L)$. Consider the equation

$$(TE) \quad y_t + ay_x = f + \chi_{(\ell_1, \ell_2)}u, \quad \text{in } (0, L) \times (0, T),$$
$$y(0, t) = 0, \quad \text{in } (0, T),$$
$$y(x, 0) = y_0, \quad \text{in } (0, L),$$

where $f \in L^2((0, L) \times (0, T))$, $\chi_{(\ell_1, \ell_2)}$ is the characteristic function of $(\ell_1, \ell_2) \subset (0, L)$, $u \in L^2((\ell_1, \ell_2) \times (0, T))$, and $y_0 \in L^2(0, L)$.

Prove that (TE) admits a unique solution in $C([0, T]; L^2(0, L))$ (use the Hille-Yosida theorem).
Study the control problem

\[(P)\]
\[\inf\{ J(y, u) \mid u \in L^2(0, T; L^2(\ell_1, \ell_2)) , \]
\[(y, u) \text{ satisfies } (TE) \}\].

with

\[ J(y, u) = \frac{1}{2} \int_0^L (y(T) - y_d(T))^2 + \frac{1}{2} \int_0^T \int_{\ell_1}^{\ell_2} u^2, \]

where \(y_d \in C([0, T]; L^2(0, L))\). Prove the existence of a unique solution. Write first order optimality conditions.
Optimal control of a first order hyperbolic system
The state equation

Consider the first order hyperbolic system

\[
\frac{\partial}{\partial t} \begin{bmatrix} z_1(x, t) \\ z_2(x, t) \end{bmatrix} = \frac{\partial}{\partial x} \begin{bmatrix} m_1 z_1 \\ -m_2 z_2 \end{bmatrix} - \begin{bmatrix} a_{11} z_1 + a_{12} z_2 + b_1 u_1 \\ a_{21} z_1 + a_{22} z_2 + b_2 u_2 \end{bmatrix},
\]

in \((0, \ell) \times (0, T)\), with the initial condition

\[z_1(x, 0) = z_{01}(x), \quad z_2(x, 0) = z_{02}(x) \quad \text{in} \quad (0, \ell),\]

and the boundary conditions

\[z_1(\ell, t) = 0, \quad z_2(0, t) = 0 \quad \text{in} \quad (0, T).\]

We refer to this system as the system \((HE)\). This kind of systems intervenes in heat exchangers [9].
We suppose that the constant coefficients $m_1 > 0$, $m_2 > 0$, and that $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ are regular.

**State equation**

We set $Y = L^2(0, \ell) \times L^2(0, \ell)$, and we define the unbounded operator $A$ in $Y$ by

$$D(A) = \{ z \in H^1(0, \ell) \times H^1(0, \ell) \mid z_1(\ell) = 0, z_2(0) = 0 \}$$

and

$$A z = \begin{bmatrix} m_1 \frac{dz_1}{dx} \\ m_2 \frac{dz_2}{dx} \\ -m_2 \frac{dz_2}{dx} \end{bmatrix}.$$
We define the operator $L \in \mathcal{L}(Y)$ by

$$Lz = \begin{bmatrix} -a_{11}z_1 - a_{12}z_2 \\ -a_{21}z_1 - a_{22}z_2 \end{bmatrix}.$$
**Theorem.** The operator \((A, D(A))\) is the infinitesimal generator of a strongly continuous semigroup of contractions on \(Y\).

**Proof.** The theorem relies the Hille-Yosida theorem.

(i) The operator \(A\) is dissipative in \(Y\):

\[
(Az, z) = \int_0^\ell m_1 \frac{dz_1}{dx} z_1 - \int_0^\ell m_2 \frac{dz_2}{dx} z_2
\]

\[
= -\frac{m_1}{2} z_1(0)^2 - \frac{m_2}{2} z_2(\ell)^2 \leq 0.
\]
(ii) For $\lambda > 0$, $f \in L^2(0, \ell)$, $g \in L^2(0, \ell)$, consider the equation

$$z \in D(A), \quad \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

that is

$$\lambda z_1 - m_1 \frac{dz_1}{dx} = f \quad \text{in } (0, \ell), \quad z_1(\ell) = 0,$$

$$\lambda z_2 + m_2 \frac{dz_2}{dx} = g \quad \text{in } (0, \ell), \quad z_2(0) = 0.$$

This equation admits a unique solution $z \in D(A)$. 
Theorem. The operator \((A + L, D(A))\) is the infinitesimal generator of a strongly continuous semigroup on \(Y\).

Theorem. For all \(z_0 = (z_{10}, z_{20}) \in Y\), \(u_1 \in L^2((0, \ell) \times (0, T))\), \(u_2 \in L^2((0, \ell) \times (0, T))\), the system \((HE)\) admits a unique weak solution in \(L^2(0, T; L^2(0, \ell))\), this solution belongs to \(C([0, T]; Y)\) and satisfies

\[
\|z\|_{C([0,T]; Y)} \\
\leq C\left(\|z_0\|_Y + \|u_1\|_{L^2((0,\ell) \times (0,T))} + \|u_2\|_{L^2((0,\ell) \times (0,T))}\right).
\]
The adjoint operator of \((A, D(A))\), with respect to the \(Y\)-topology, is defined by

\[
D(A^*) = \left\{ (\phi, \psi) \in H^1(0, \ell) \times H^1(0, \ell) \mid \phi(0) = 0, \quad \psi(\ell) = 0 \right\},
\]

and

\[
(A^* + L^*) \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} -m_1 \frac{d\phi}{dx} - a_{11}\phi - a_{21}\psi \\ m_2 \frac{d\psi}{dx} - a_{12}\phi - a_{22}\psi \end{bmatrix}.
\]

To study the system \((HE)\), we define the operator \(B \in \mathcal{L}((L^2(0, \ell))^2)\) by

\[
B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b_1 u_1 \\ b_2 u_2 \end{bmatrix}.
\]
The (HE) is of the form

\[ z' = (A + L)z + Bu, \quad z(0) = z_0. \]

**The control problem**

We want to study the control problem

\[
(P) \quad \inf \{ J(z, u) \mid (z, u) \text{ obeys (HE)}, \quad u \in (L^2((0, \ell) \times (0, T)))^2 \},
\]

where

\[
J(z, u) = \frac{1}{2} \int_0^\ell |z(T) - z_d(T)|^2 + \frac{\beta}{2} \int_0^T \int_0^\ell (u_1^2 + u_2^2),
\]

and \( \beta > 0. \) We assume that \( z_d \in C([0, T]; Y). \)
Theorem. Problem $(P)$ admits a unique solution $(\bar{z}, \bar{u})$. Moreover $\bar{u}$ is characterized by

$$\bar{u}_1(x, t) = -\frac{b_1}{\beta} \phi(x, t) \quad \text{and} \quad \bar{u}_2(x, t) = -\frac{b_2}{\beta} \psi(x, t),$$

in $(0, T)$, where $(\phi, \psi)$ is the solution to the adjoint system

$$- \frac{\partial}{\partial t} \begin{bmatrix} \phi(x, t) \\ \psi(x, t) \end{bmatrix} = \frac{\partial}{\partial x} \begin{bmatrix} -m_1 \phi \\ m_2 \psi \end{bmatrix} - \begin{bmatrix} a_{11} \phi + a_{21} \psi \\ a_{12} \phi + a_{22} \psi \end{bmatrix}$$

in $(0, \ell) \times (0, T)$, with the terminal condition

$$\phi(T) = \bar{z}_1(T) - z_{d,1}(T), \quad \psi(T) = \bar{z}_2(T) - z_{d,2}(T)$$

in $(0, \ell)$, and the boundary conditions

$$\phi(0, t) = 0, \quad \psi(\ell, t) = 0 \quad \text{in} \quad (0, T).$$
**Proof.** (i) The existence of a unique solution to $(P)$ is classical and is left as exercise.

(ii) The state equation is of the form

$$z' = (A + L)z + Bu, \quad z(0) = z_0,$$

and the cost functional

$$J(z, u) = \frac{1}{2} \|z(T) - z_d(T)\|_{L^2(0, \ell)}^2 + \frac{\beta}{2} \int_0^T \|u(t)\|_{L^2(0, \ell)}^2 dt.$$

Thus the optimal control $\bar{u}$ is characterized by

$$\bar{u}(t) = -\frac{1}{\beta} B^* p(t),$$

where $p$ is the solution to

$$-p' = (A + L)^* p, \quad p(T) = \bar{z}(T) - z_d(T).$$
Set

\[ p = \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \]

We can verify that \((\phi, \psi)\) is the solution to the adjoint equation corresponding to \(\bar{z}\).

We can prove that

\[ B^*(\phi(t), \psi(t)) = (b_1 \phi(x, t), b_2 \psi(x, t)). \]
(iii) We can directly prove the optimality conditions for problem \((P)\) by using the same method as for the heat and the wave equations. Setting \(F(u) = J(z(z_0, u), u)\), where \(z(z_0, u)\) is the solution to (HE), we have

\[
F'(\bar{u})u = \int_0^\ell (\bar{z}_1(T) - z_{d1}(T))w_{u1}(T)
\]
\[+ \int_0^\ell (\bar{z}_2(T) - z_{d2}(T))w_{u2}(T) + \beta \int_0^T (\bar{u}_1u_1 + \bar{u}_2u_2),
\]

where \(w_u = z(0, u)\), and \(z(0, u)\) is the solution to (HE) for \(z_0 = 0\).

We can establish an integration by parts formula between \(w_u\) and the solution \((\phi, \psi)\) to (AE) to complete the proof.
References


