**Université Paul Sabatier**

# Optimal Control of Partial Differential Equations

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# Contents

# Chapter 1

# Examples of control problems

## 1.1 What is a control problem ?

Roughly speaking a control problem consists of:

- A controlled system, that is an input-output process,
- an observation of the output of the controlled system,
- an objective to be achieved.

In this course we are interested in controlled systems described by partial differential equations. The input can be a function in a boundary condition, an initial condition, a coefficient in a partial differential equation, or any parameter in the equation, and the output is the solution of the partial differential equation. The input is called the control variable, or the **control**, and the output is called the **state** of the system. An observation of the system is a mapping (very often a linear operator) depending on the state.

We can seek for various objectives:

• Minimize a criterion depending on the observation of the state and on the control variable. This is an optimal control problem. The unknown of this minimization problem is the control variable.

• We can look for a control for which the observation belongs to some target. This corresponds to a controllability problem.

• We can look for a control which stabilizes the state or an observation of the state of the system. This is a stabilization problem.

## 1.2 Control of elliptic equations

Elliptic equations may describe an electrical potential, a stationary distribution of temperature, a scattered field, or a velocity potential. We give two examples of optimal control problems, taken from the literature, for systems governed by elliptic equations.

### 1.2.1 Optimal control of current in a cathodic protection system

The problem is treated in [29]. When a metal is placed in a corrosive electrolyte, it tends to ionize and dissolve in the electrolyte. To prevent corrosion process, an other metal (less noble than the previous one) can be placed in contact with the electrolyte. In this device the noble metal plays the role of a cathode, and the other one the role of the anode. A current can be prescribed to the anode to modify the electric field in the electrolyte. This process is known as cathodic protection.

The system can be described by the elliptic equation

$$
\begin{gathered}
-\mathrm{div}(\sigma\nabla\phi) = 0 \qquad \text{in } \Omega, \\
-\sigma\frac{\partial\phi}{\partial n} = i \quad \text{on } \Gamma_a, \qquad -\sigma\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \Gamma_i, \qquad -\sigma\frac{\partial\phi}{\partial n} = f(\phi) \quad \text{on } \Gamma_c,
\end{gathered}
\tag{1.2.1}
$$

where $\phi$ is the electrical potential, $\Omega$ is the domain occupied by the electrolyte, $\Gamma_a$ is a part of the boundary of $\Omega$ occupied by the anode, $\Gamma_c$ is a part of the boundary of $\Omega$ occupied by the cathode, $\Gamma_i$ is the rest of the boundary $\Gamma$, $\Gamma_i = \Gamma \setminus (\Gamma_a \cup \Gamma_c)$. The control function is the current density $i$, the constant $\sigma$ is the conductivity of the electrolyte, the function $f$ is known as the cathodic polarization function, and in general it is a nonlinear function of $\phi$.

The cathode is protected if the electrical potential is closed to a given potential $\bar{\phi}$ on $\Gamma_c$. Thus the cathodic protection can be achived by choosing the current $i$ as the solution to the minimization problem

$$
(P_1) \qquad \inf\{J_1(\phi) \mid (\phi, i) \in H^1(\Omega) \times L^2(\Gamma_a),\ (\phi, i) \text{ satisfies (1.2.1)},\ a \le i \le b\},
$$

where $a$ and $b$ are some bounds on the current $i$, and

$$
J_1(\phi) = \int_{\Gamma_c} (\phi - \bar{\phi})^2.
$$

A compromise between 'the cathodic protection' and 'the consumed energy' can be obtained by looking for a solution to the problem

$$
(P_2) \qquad \inf\{J_2(\phi, i) \mid (\phi, i) \in H^1(\Omega) \times L^2(\Gamma_a),\ (\phi, i) \text{ satisfies (1.2.1)},\ a \le i \le b\},
$$

where

$$
J_2(\phi, i) = \int_{\Gamma_c} (\phi - \bar{\phi})^2 + \beta \int_{\Gamma_a} i^2,
$$

and $\beta$ is a positive constant.

### 1.2.2 Optimal control problem in radiation and scattering

Here the problem consists in determining the surface current of a radiating structure which maximize the radiated far field in some given directions [22]. Let $\Omega \subset \mathbb{R}^N$ be the complementary subset in $\mathbb{R}^N$ of a regular bounded domain ($\Omega$ is called an exterior domain), and let $\Gamma$ its boundary. The radiated field $y$ satisfies the Helmholtz equation

$$
\Delta y + k^2 y = 0, \qquad \text{in } \Omega, \tag{1.2.2}
$$

where $k \in \mathbb{C}$, $\mathrm{Im}k > 0$, and the radiation condition

$$
\frac{\partial y}{\partial r} - iky = O\Big(\frac{1}{|x|^{(N+1)/2}}\Big), \quad \text{when } r = |x| \to \infty. \tag{1.2.3}
$$

The current on the boundary $\Gamma$ is chosen as the control variable, and the boundary condition is:

$$y = u \qquad \text{on } \Gamma. \tag{1.2.4}$$

The solution $y$ to equation (1.2.2)-(1.2.4) satisfies the following asymptotic behaviour

$$y(x) = \frac{e^{ik|x|}}{|x|^{(N-1)/2}} F\Big(\frac{x}{|x|}\Big) + O\Big(\frac{1}{|x|^{(N+1)/2}}\Big), \quad \text{when } r = |x| \to \infty,$$

and $F$ is called the far field of $y$. The optimal control problem studied in [22] consists in finding a control $u$, belonging to $U_{ad}$, a closed convex subset of $L^\infty(\Gamma)$, which maximizes the far field in some directions. The problem can be written in the form

$$(P) \qquad \sup\{J(F_u) \mid u \in U_{ad}\},$$

where $F_u$ is the far field associated with $u$, and

$$J(F) = \int_S \alpha\Big(\frac{x}{|x|}\Big)\Big|F\Big(\frac{x}{|x|}\Big)\Big|^2 dr,$$

$S$ is the unit sphere in $\mathbb{R}^N$, $\alpha$ is the characteristic function of some subset in $S$.

# 1.3 Control of parabolic equations

## 1.3.1 Identification of a source of pollution

Consider a river or a lake with polluted water, occupying a two or three dimensional domain $\Omega$. The control problem consists in finding the source of pollution (which is unknown). The concentration of pollutant $y(x,t)$ can be measured in a subset $\mathcal{O}$ of $\Omega$, during the interval of time $[0,T]$. The concentration $y$ is supposed to satisfy the equation

$$\begin{aligned} &\tfrac{\partial y}{\partial t} - \Delta y + V\cdot\nabla y + \sigma y = s(t)\delta_a \qquad \text{in } \Omega\times]0,T[,\\ &\tfrac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma\times]0,T[, \quad y(x,0) = y_0 \quad \text{in } \Omega, \end{aligned} \tag{1.3.5}$$

where $a \in K$ is the position of the source of pollution, $K$ is a compact subset in $\overline{\Omega}$, $s(t)$ is the flow rate of pollution. The initial concentration $y_0$ is supposed to be known or estimated (it could also be an unknown of the problem). The problem consists in finding $a \in K$ which minimizes

$$\int_0^T \int_{\mathcal{O}} (y - y_{obs})^2,$$

where $y$ is the solution of (1.3.5) and $y_{obs}$ corresponds to the measured concentration. In this problem the rate $s(t)$ is supposed to be known. This problem is taken from [24].

We can imagine other problems where the source of pollution is known but not accessible, and for which the rate $s(t)$ is unknown. In that case the problem consists in finding $s$ satisfying some a priori bounds $s_0 \le s(t) \le s_1$ and minimizing $\int_0^T \int_{\mathcal{O}} (y - y_{obs})^2$.

### 1.3.2 Cooling process in metallurgy

In modern steel casting machine, the design of cooling process leads to rather challenging control problems. The problem is described by a fully nonlinear heat equation of the form

$$\rho(T)c(T)\frac{\partial T}{\partial t} = \mathrm{div}(k(T)\nabla T), \qquad \text{in } \ \Omega \times (0, t_f),$$

where $T$ is the temperature in the domain $\Omega$, $c(T)$ the specific heat capacity, $\rho(T)$ the density, and $k(T)$ the conductivity of the steel at the temperature $T$. The heat extraction is ensured by water sprays corresponding to nonlinear boundary conditions:

$$k(T)\frac{\partial T}{\partial n} = R(T, u), \qquad \text{on } \ \Gamma \times (0, t_f),$$

where $u$ is the control variable, and the radiation law $R$ is a nonlinear function (for example, $R$ can be the Stefan-Boltzmann radiation law). The cost functional can be of the form:

$$J(T, u) = \beta_1 \int_\Omega (T(t_f) - \bar{T})^2 + \beta_2 \int_0^{t_f} |u|^q$$

with $\beta_1 > 0$, $\beta_2 > 0$, $t_f$ is the terminal time of the process, $\bar{T}$ is a desired profile of temperature, the exponent $q$ is chosen in function of the radiation law $R$.

In industrial applications, constraints must be added on the temperature. In that case we shall speak of state constraints. For more details and other examples we refer to [21].

## 1.4 Control of hyperbolic equations

The control of acoustic noise, the stabilization of flexible structures, the identification of acoustic sources can be formulated as control problems for hyperbolic equations. Lot of models have been studied in the literature [16]. The one dimensional models cover the elementary theories of elastic beam motion [28]. Let us present a stabilization problem for the Timoshenko model. The equation of motion of the Timoshenko beam is described by the following set of equations:

$$\rho\frac{\partial^2 u}{\partial t^2} - K\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial \phi}{\partial x}\right) = 0, \qquad \text{in } \ (0, L),$$

$$I_\rho\frac{\partial^2 \phi}{\partial t^2} - EI\frac{\partial^2 \phi}{\partial x^2} + K\left(\phi - \frac{\partial u}{\partial x}\right) = 0, \qquad \text{in } \ (0, L),$$

where $u$ is the deflection of the beam, $\phi$ is the angle of rotation of the beam cross-sections due to bending. The coefficient $\rho$ is the mass density per unit length, $EI$ is the flexural rigidity of the beam, $I_\rho$ is the mass moment of inertia of the beam cross section, and $K$ is the shear modulus. If the beam is clamped at $x = 0$, the corresponding boundary conditions are

$$u(0, t) = 0 \qquad \text{and} \qquad \phi(0, t) = 0 \quad \text{for } t \geq 0.$$

If a boundary control force $f_1$ and a boundary control moment $f_2$ are applied at $x = L$, the boundary conditions are

$$K(\phi(L, t) - u_x(L, t)) = f_1(t) \qquad \text{and} \qquad -EI\phi_x(L, t) = f_2(t) \quad \text{for } t \geq 0.$$

The stabilization problem studied in [25] consists in finding $f_1$ and $f_2$ so that the energy of the beam

$$E(t) = \frac{1}{2}\int_0^L \Big(\rho u_t^2(t) + I_\rho \phi_t^2(t) + K(\phi(t) - u_x(t))^2 + EI\phi_x^2(t)\Big)\,dx.$$

asymptotically and uniformly decays to zero.

## 1.5 Objectives of these lectures

The first purpose of these lectures is to introduce the basic tools to prove the existence of solutions to optimal control problems, to derive first order optimality conditions, and to explain how these optimality conditions may be used in optimization algorithms to compute optimal solutions.

A more advanced objective in optimal control theory consists in calculating feedback laws. For problems governed by linear evolution equations and for quadratic functionals, feedback laws can be determined by solving the so-called *Riccati equations*. This will be done in the specialized course [26]. But we want to introduce right now the results necessary to study Riccati equations. Exploring literature on Riccati equations (for control problems governed by partial differential equations) requires some knowledge on the semigroup theory for evolution equations. This is why we have chosen this approach throughout these lectures to study evolution equations.

The plan of these lectures is as follows. In Chapter 2, we study optimal control problems for linear elliptic equations. On a simple example we explain how the *adjoint state* allows us to calculate the gradient of a functional. For controls in a Dirichlet boundary condition, we also introduce the *transposition method*. This method is next used to study some evolution equations with nonsmooth data.

Some basic results of the semigroup theory are recalled (without proof) in Chapter 4. For a more detailed study we refer to the preliminary lectures by Kesavan [9], and to classical references [8], [18], [2]. In Chapters 5 and 6, we study optimal control problems for the heat equation and the wave equation. We systematically investigate the case of distributed controls, Neumann boundary controls, and Dirichlet boundary controls. The extension to problems governed by abstract evolution equations is continued in Chapters 7 and 8. These two chapters constitute the starting point to study Riccati equations in the second part of the course [26]. Chapter 7 is devoted to bounded control operators (the case of distributed controls), while Chapter 8 is concerned with unbounded control operators (the case of boundary controls or pointwise controls). We show that problems studied in Chapters 5 and 6 correspond to this framework. Other extensions and applications are given.

Many systems are governed by nonlinear equations (see sections 1.2.1 and 1.3.2). Studying these problems requires some additional knowledge. We have only studied two problems governed by nonlinear equations. In Chapter 3, we consider control problems for elliptic equations with nonlinear boundary conditions. In Chapter 9, we study a control problem for a semilinear parabolic equation of Burgers' type, in dimension 2. This model is an interesting introduction for studying flow control problems [27]. Finally Chapter 10 is devoted to numerical algorithms.

For a first introduction to optimal control of partial differential equations, the reader can study the linear-quadratic control problems considered in Chapters 2, 5, and 6.

The reader who is interested by optimal control problems of nonlinear equations, and the corresponding numerical algorithms may complete the previous lectures with Chapters 3, 9, and 10.

Chapters 7 and 8 are particularly recommended as preliminaries to study Riccati equations [26].

# Chapter 2

# Control of elliptic equations

## 2.1 Introduction

In this chapter we study optimal control problems for elliptic equations. We first consider problems with controls acting in a Neumann boundary condition. We derive optimality conditions for problems without control constraints. In section 2.3 we explain how these optimality conditions may be useful to compute the optimal solution. The case of a Dirichlet boundary control is studied in section 2.4. For a control in a Dirichlet boundary condition, we define solutions by the transposition method. Extensions to problems with control constraints are considered in section 2.5. The existence of optimal solutions is proved in section 2.6. In section 2.7, we extend the results of section 2.2 to problems defined by various functionals.

## 2.2 Neumann boundary control

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, with a boundary $\Gamma$ of class $C^2$. We consider the elliptic equation

$$-\Delta z + z = f \quad \text{in } \Omega, \quad \frac{\partial z}{\partial n} = u \quad \text{on } \Gamma. \tag{2.2.1}$$

The function $f$ is a given source term and the function $u$ is a control variable.

A classical control problem consists in finding a control function $u \in L^2(\Gamma)$ which minimizes the cost functional

$$J_1(z,u) = \frac{1}{2}\int_\Omega (z - z_d)^2 + \frac{\beta}{2}\int_\Gamma u^2,$$

where the pair $(z,u)$ satisfies equation (2.2.1). The term $\frac{\beta}{2}\int_\Gamma u^2$ (with $\beta > 0$) is proportional to the consumed energy. Thus, minimizing $J_1$ is a compromise between the energy consumption and finding $u$ so that the distribution $z$ is close to the desired profile $z_d$. In the case of equation (2.2.1) we say that the control function is a *boundary control* because the control acts on $\Gamma$.

Before studying the above control problem, we first recall some results useful for equation (2.2.1). The existence of a unique solution $z \in H^1(\Omega)$ to equation (2.2.1) may be proved with the Lax-Milgram theorem. More precisely we have the following theorem.

**Theorem 2.2.1** ([*12*]) *For every $f \in L^2(\Omega)$ and every $u \in L^2(\Gamma)$, equation (2.2.1) admits a unique weak solution $z(f,u)$ in $H^1(\Omega)$, moreover the operator*

$$(f,u) \mapsto z(f,u)$$

*is linear and continuous from* $L^2(\Omega)\times L^2(\Gamma)$ *into* $H^{3/2}(\Omega)$.

We want to write optimality conditions for the control problem

$$(P_1) \qquad \inf\{J_1(z,u) \mid (z,u)\in H^1(\Omega)\times L^2(\Gamma),\ (z,u) \text{ satisfies } (2.2.1)\}.$$

We suppose that $(P_1)$ admits a unique solution $(\bar z,\bar u)$ (this result is established in section 2.6). We set $F_1(u) = J_1(z(f,u),u)$, where $z(f,u)$ is the solution to equation (2.2.1). From the optimality of $(\bar z,\bar u)$, it follows that

$$\frac{1}{\lambda}\Big(F_1(\bar u+\lambda u)-F_1(\bar u)\Big)\geq 0$$

for all $\lambda>0$ and all $u\in L^2(\Gamma)$. By an easy calculation we obtain

$$F_1(\bar u+\lambda u)-F_1(\bar u)=\frac{1}{2}\int_\Omega (z_\lambda-\bar z)(z_\lambda+\bar z-2z_d)+\frac{\beta}{2}\int_\Gamma(2\lambda u\bar u+\lambda^2u^2),$$

where $z_\lambda = z(f,\bar u+\lambda u)$. The function $w_\lambda = z_\lambda-\bar z$ is the solution to equation

$$-\Delta w+w=0 \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n}=\lambda u \quad \text{on } \Gamma.$$

Due to Theorem 2.2.1 we have

$$\|w_\lambda\|_{H^1(\Omega)}\leq C|\lambda|\|u\|_{L^2(\Gamma)}.$$

Thus the sequence $(z_\lambda)_\lambda$ converges to $\bar z$ in $H^1(\Omega)$ when $\lambda$ tends to zero. Set $\frac{1}{\lambda}w_\lambda = w_u$, the function $w_u$ is the solution to equation

$$-\Delta w+w=0 \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n}=u \quad \text{on } \Gamma. \tag{2.2.2}$$

By passing to the limit when $\lambda$ tends to zero, we finally obtain:

$$0\leq \lim_{\lambda\to 0}\frac{1}{\lambda}\Big(F_1(\bar u+\lambda u)-F_1(\bar u)\Big)=F_1'(\bar u)u=\int_\Omega(\bar z-z_d)w_u+\int_\Gamma \beta u\bar u.$$

Here $F_1'(\bar u)u$ denotes the derivative of $F_1$ at $\bar u$ in the direction $u$. It can be easily checked that $F_1$ is differentiable in $L^2(\Gamma)$. Since $F_1'(\bar u)u\geq 0$ for every $u\in L^2(\Gamma)$, we deduce

$$F_1'(\bar u)u=0 \qquad \text{for all } u\in L^2(\Gamma). \tag{2.2.3}$$

In this form the optimality condition (2.2.3) is not usable. For the computation of optimal controls, we need the expression of $F_1'(\bar u)$. Since $F_1$ is a differentiable mapping from $L^2(\Gamma)$ into $\mathbb{R}$, $F_1'(\bar u)$ may be identified with a function of $L^2(\Gamma)$. Hence, we look for a function $\pi\in L^2(\Gamma)$ such that

$$\int_\Omega(\bar z-z_d)w_u=\int_\Gamma \pi u \qquad \text{for all } u\in L^2(\Gamma).$$

This identity is clearly related to a Green formula. We observe that if $p\in H^1(\Omega)$ is the solution to the equation

$$-\Delta p+p=\bar z-z_d \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n}=0 \quad \text{on } \Gamma, \tag{2.2.4}$$

then we have

$$\int_\Omega(\bar z-z_d)w_u=\int_\Omega(-\Delta p+p)w_u=\int_\Gamma p\frac{\partial w_u}{\partial n}=\int_\Gamma pu.$$

This means that $F_1'(\bar u)=p|_\Gamma+\beta\bar u$, where $p$ is the solution to equation (2.2.4).

**Theorem 2.2.2** *If $(\bar{z}, \bar{u})$ is the solution to $(P_1)$ then $\bar{u} = -\frac{1}{\beta} p|_\Gamma$, where $p$ is the solution to equation (2.2.4).*

*Conversely, if a pair $(\tilde{z}, \tilde{p}) \in H^1(\Omega) \times H^1(\Omega)$ obeys the system*

$$\begin{aligned} -\Delta \tilde{z} + \tilde{z} &= f \quad \text{in } \Omega, & \frac{\partial \tilde{z}}{\partial n} &= -\tfrac{1}{\beta}\tilde{p} \quad \text{on } \Gamma, \\ -\Delta \tilde{p} + \tilde{p} &= \tilde{z} - z_d \quad \text{in } \Omega, & \frac{\partial \tilde{p}}{\partial n} &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{2.2.5}$$

*then the pair $(\tilde{z}, -\frac{1}{\beta}\tilde{p})$ is the optimal solution to problem $(P_1)$.*

**Proof.** The first part of the theorem is already proved. Suppose that $(\tilde{z}, \tilde{p}) \in H^1(\Omega) \times H^1(\Omega)$ obeys the system (2.2.5). Set $\tilde{u} = -\frac{1}{\beta}\tilde{p}$. For every $u \in L^2(\Gamma)$, we have

$$\begin{aligned} F_1(\tilde{u} + u) - F_1(\tilde{u}) &= \frac{1}{2}\int_\Omega (z_u - \tilde{z})(z_u + \tilde{z} - 2z_d) + \frac{\beta}{2}\int_\Gamma (2u\tilde{u} + u^2) \\ &= \frac{1}{2}\int_\Omega (z_u - \tilde{z})^2 + \frac{\beta}{2}\int_\Gamma u^2 + \int_\Omega (z_u - \tilde{z})(\tilde{z} - z_d) + \beta\int_\Gamma u\tilde{u}, \end{aligned}$$

with $z_u = z(f, \tilde{u} + u)$. From the equation satisfied by $\tilde{p}$ and a Green formula it follows that

$$\begin{aligned} \int_\Omega (z_u - \tilde{z})(\tilde{z} - z_d) &= \int_\Omega (z_u - \tilde{z})(-\Delta\tilde{p} + \tilde{p}) \\ &= \int_\Gamma \left(\frac{\partial z_u}{\partial n} - \frac{\partial \tilde{z}}{\partial n}\right)\tilde{p} = -\beta\int_\Gamma u\tilde{u}. \end{aligned}$$

We finally obtain

$$F_1(\tilde{u} + u) - F_1(\tilde{u}) = \frac{1}{2}\int_\Omega (z_u - \tilde{z})^2 + \frac{\beta}{2}\int_\Gamma u^2 \geq 0.$$

Thus $(\tilde{z}, -\frac{1}{\beta}\tilde{p})$ is the optimal solution to problem $(P_1)$. ∎

We give another proof of the second part of Theorem 2.2.2 by using a general result stated below.

**Theorem 2.2.3** *Let $F$ be a differentiable mapping from a Banach space $U$ into $\mathbb{R}$. Suppose that $F$ is convex.*

*(i) If $\bar{u} \in U$ and $F'(\bar{u}) = 0$, then $F(\bar{u}) \leq F(u)$ for all $u \in U$.*

*(ii) If $U_{ad}$ is a closed convex subset in $U$, $F$ is convex, $\bar{u} \in U_{ad}$ and if $F'(\bar{u})(u - \bar{u}) \geq 0$ for all $u \in U_{ad}$, then $F(\bar{u}) \leq F(u)$ for all $u \in U_{ad}$.*

**Proof.** The theorem follows from the convexity inequality

$$F(u) - F(v) \geq F'(v)(u - v) \qquad \text{for all } u \in U, \text{ and all } v \in U.$$

The second statement of the theorem will be used for problems with control constraints. ∎

**Second proof of Theorem 2.2.2.** Due to the previous calculations, for every $\hat{u} \in L^2(\Gamma)$, we have $F'(\hat{u}) = \hat{p}|_\Gamma + \beta\hat{u}$, where $\hat{p}$ is the solution to

$$-\Delta\hat{p} + \hat{p} = \hat{z} - z_d \quad \text{in } \Omega, \quad \frac{\partial\hat{p}}{\partial n} = 0 \quad \text{on } \Gamma,$$

and $\hat{z} = z(f, \hat{u})$.

Thus if $(\tilde{z}, \tilde{p}) \in H^1(\Omega) \times H^1(\Omega)$ satisfies the system (2.2.5), we have $F_1'(-\frac{1}{\beta}\tilde{p}|_\Gamma) = \tilde{p}|_\Gamma + -\beta\frac{1}{\beta}\tilde{p}|_\Gamma = 0$. Due to Theorem 2.2.3, it follows that $-\frac{1}{\beta}\tilde{p}|_\Gamma$ is the optimal control to problem $(P_1)$. ∎

## 2.3 Computation of optimal controls

Theorem 2.2.2 gives a characterization of the optimal solution to problem $(P_1)$. One way to compute this solution consists in solving the optimality system (2.2.5). Another way consists in applying the *conjugate gradient algorithm* to the minimization problem $(P_1)$. Let us recall what is the conjugate gradient method (CGM in brief) for quadratic functionals. We want to calculate the solution to the optimization problem

$$(P_1) \qquad \inf\{F(u) \mid u \in U\},$$

where $U$ is a Hilbert space, $F$ is a quadratic functional

$$F(u) = \frac{1}{2}(u, Qu)_U - (b, u)_U,$$

and $Q \in \mathcal{L}(U)$, $Q = Q^* > 0$, $b \in U$. The idea of the conjugate gradient method consists in solving a sequence of problems of the form

$$(P_{k+1}) \qquad \inf\{F(u) \mid u \in C_k\},$$

where $C_k = u_k + \mathrm{vect}(d_0, \ldots, d_k)$. The next iterate is the solution $u_{k+1}$ to problem $(P_{k+1})$. Since at each step the dimension of the set $C_k$ increases, the method can be complicated. The algorithm simplifies if the different directions $d_0, \ldots, d_k$ are two by two orthogonal with respect to $Q$. It is for example the case if we take $d_0 = -g_0 = -F'(u_0)^*$, and $C_k = u_k + \mathrm{vect}(d_0, \ldots, d_{k-1}, -g_k)$, with $g_k = F'(u_k)^*$. The direction $d_k$ is computed to have $\mathrm{vect}(d_0, \ldots, d_{k-1}, -g_k) = \mathrm{vect}(d_0, \ldots, d_k)$, and $(d_k, d_j)_U = 0$ for all $0 \le j \le k-1$. The corresponding algorithm is given below.

**Conjugate Gradient Algorithm.**

*Initialization.* Choose $u_0$ in $U$. Compute $g_0 = Qu_0 - b$. Set $d_0 = -g_0$ and $n = 0$.

*Step 1.* Compute

$$\rho_n = (g_n, g_n)/(d_n, Qd_n),$$

and

$$u_{n+1} = u_n + \rho_n d_n.$$

Determine

$$g_{n+1} = Qu_{n+1} - b = g_n + \rho_n Q d_n.$$

*Step 2.* If $\|g_{n+1}\|_U/\|g_0\|_U \leq \varepsilon$, stop the algorithm and take $u = u_{n+1}$, else compute

$$\beta_n = (g_{n+1}, g_{n+1})/(g_n, g_n),$$

and

$$d_{n+1} = -g_{n+1} + \beta_n d_n.$$

Replace $n$ by $n+1$ and go to step 1.

We want to apply this algorithm to problem $(P_1)$ of section 2.2. For that we have to identify $J_1(z(f,u),u)$, where $z(f,u)$ is the solution to equation (2.2.1), with a quadratic form on $L^2(\Gamma)$. Since $z(f,u) = w_u + y(f)$ , where $w_u$ is the solution to equation (2.2.2) and where $y(f)$ does not depend on $u$, we can write

$$J_1(z(f,u),u) = \frac{1}{2}\int_\Omega w_u^2 + \frac{\beta}{2}\int_\Gamma u^2 + (b,u)_{L^2(\Gamma)} + c = \frac{1}{2}(Qu,u)_{L^2(\Gamma)} + (b,u)_{L^2(\Gamma)} + c,$$

with $Q = \Lambda^*\Lambda + \beta I$, $b = \Lambda^*(y(f) - z_d)$, $c = \frac{1}{2}\int_\Omega (y(f) - z_d)^2$, and where $\Lambda$ is the bounded operator from $L^2(\Gamma)$ into $L^2(\Omega)$ defined by

$$\Lambda \ : \ u \longmapsto w_u.$$

Thus to apply the CGM to problem $(P_1)$ we have to compute $\Lambda^*\Lambda$ for $d \in L^2(\Gamma)$. By definition of $\Lambda$ we have $\Lambda d = w_d$. With a Green formula, we can easily verify that $\Lambda^*$, the adjoint of $\Lambda$, is defined by

$$\Lambda^* \ : \ g \longmapsto p|_\Gamma,$$

where $p$ is the solution to equation

$$-\Delta p + p = g \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma.$$

Now it is clear that the CGM applied to problem $(P_1)$ is the following algorithm.

**CGM for** $(P_1)$

*Initialization.* Choose $u_0$ in $L^2(\Gamma)$. Compute $z_0 = z(f,u_0)$ and $p_0$, the solution to

$$-\Delta p + p = z_0 - z_d \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma.$$

Set $g_0 = \beta u_0 + p_0|_\Gamma$ , $d_0 = -g_0$ and $n = 0$.

*Step 1.* Compute $\hat{z}_n$ the solution to

$$-\Delta z + z = 0 \quad \text{in } \Omega, \quad \frac{\partial z}{\partial n} = d_n \quad \text{on } \Gamma,$$

and $\hat{p}_n$ the solution to

$$-\Delta p + p = \hat{z}_n \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma.$$

Compute

$$\hat{g}_n = \beta d_n + \hat{p}_n|_\Gamma, \qquad \text{and} \ \ \rho_n = -\int_\Gamma |g_n|^2 / \int_\Gamma \hat{g}_n g_n.$$

Set

$$u_{n+1} = u_n + \rho_n d_n \qquad \text{and} \quad g_{n+1} = g_n + \rho_n \hat{g}_n.$$

*Step 2.* If $\|g_{n+1}\|_{L^2(\Gamma)}/\|g_0\|_{L^2(\Gamma)} \leq \varepsilon$, stop the algorithm and take $u = u_{n+1}$, else compute

$$\beta_n = \|g_{n+1}\|^2_{L^2(\Gamma)}/\|g_n\|^2_{L^2(\Gamma)},$$

and

$$d_{n+1} = -g_{n+1} + \beta_n d_n.$$

Replace $n$ by $n+1$ and go to step 1.

Observe that $g_{n+1} = g_n + \rho_n \hat{g}_n$ is the gradient of $F_1$ at $u_{n+1}$.

## 2.4 Dirichlet boundary control

Now we want to control the Laplace equation by a Dirichlet boundary control, that is

$$-\Delta z = f \quad \text{in } \Omega, \quad z = u \quad \text{on } \Gamma. \tag{2.4.6}$$

We say that a function $z \in H^1(\Omega)$ is a solution to equation (2.4.6) if the equation $-\Delta z = f$ is satisfied in the sense of distributions in $\Omega$ and if the trace of $z$ on $\Gamma$ is equal to $u$. When $f \in H^{-1}(\Omega)$ and $u \in H^{1/2}(\Gamma)$ equation (2.4.6) can be solved as follows. From a trace theorem in $H^1(\Omega)$, we know that there exists a linear continuous operator from $H^{1/2}(\Gamma)$ to $H^1(\Omega)$:

$$u \longmapsto z_u,$$

such that $z_u|_\Gamma = u$. Thus we look for a solution $z$ to equation (2.4.6) of the form $z = z_u + y$, with $y \in H^1_0(\Omega)$. The equation $-\Delta z = f$ is satisfied in the sense of distributions in $\Omega$ and $z = u$ on $\Gamma$ if and only if $y$ is the solution to equation

$$-\Delta y = f + \Delta z_u \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \Gamma.$$

If we identify the distribution $\Delta z_u$ with the mapping

$$\varphi \longmapsto -\int_\Omega \nabla z_u \nabla \varphi,$$

we can check that $-\Delta z_u$ belongs to $H^{-1}(\Omega)$. Hence the existence of $y$ is a direct consequence of the Lax-Milgram theorem. Therefore we have the following theorem.

**Theorem 2.4.1** *For every $f \in H^{-1}(\Omega)$ and every $u \in H^{1/2}(\Gamma)$, equation (2.4.6) admits a unique weak solution $z(f, u)$ in $H^1(\Omega)$, moreover the operator*

$$(f, u) \mapsto z(f, u)$$

*is linear and continuous from $H^{-1}(\Omega) \times H^{1/2}(\Gamma)$ into $H^1(\Omega)$.*

We want to extend the notion of solution to equation (2.4.6) in the case where $u$ belongs to $L^2(\Gamma)$. To do this we introduce the so-called *transposition method*.

**The transposition method**

Suppose that $u$ is regular. Let $z$ be the solution to the equation

$$-\Delta z = 0 \quad \text{in } \Omega, \quad z = u \quad \text{on } \Gamma, \tag{2.4.7}$$

and let $y$ be the solution to

$$-\Delta y = \phi \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \Gamma,$$

where $\phi$ belongs to $L^2(\Omega)$. With the Green formula we have

$$\int_\Omega z\phi = -\int_\Gamma u\frac{\partial y}{\partial n}.$$

Observe that the mapping

$$\Lambda \ : \ \phi \longmapsto -\frac{\partial y}{\partial n}$$

is linear and continuous from $L^2(\Omega)$ to $H^{1/2}(\Gamma)$. Thus $\Lambda$ is a compact operator from $L^2(\Omega)$ to $L^2(\Gamma)$, and its adjoint $\Lambda^*$ is a compact operator from $L^2(\Gamma)$ to $L^2(\Omega)$. Since

$$-\int_\Gamma u\frac{\partial y}{\partial n} = \langle u, \Lambda\phi\rangle_{L^2(\Gamma)} = \langle \Lambda^* u, \phi\rangle_{L^2(\Omega)}$$

for all $\phi \in L^2(\Omega)$, the solution $z$ to equation (2.4.7) obeys

$$z = \Lambda^* u.$$

Up to now, to define $y$, we have supposed that $u$ is regular. However $\Lambda^* u$ is well defined if $u$ belongs to $L^2(\Gamma)$. The transposition method consists in taking $z = \Lambda^* u$ as the solution to equation (2.4.7) in the case where $u$ belongs to $L^2(\Gamma)$. This method is here presented in a particular situation, but it will be used in these lectures in many other situations. The solution to equation (2.4.6), defined by transposition is given below.

**Definition 2.4.1** *A function $z \in L^2(\Omega)$ is a solution to equation (2.4.6) if, and only if,*

$$\int_\Omega z\phi = \langle f, y\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Gamma u\frac{\partial y}{\partial n}$$

*for all $\phi \in L^2(\Omega)$, where $y$ is the solution to*

$$-\Delta y = \phi \quad \textit{in } \Omega, \quad y = 0 \quad \textit{on } \Gamma. \tag{2.4.8}$$

**Theorem 2.4.2** *For every $f \in L^2(\Omega)$ and every $u \in L^2(\Gamma)$, equation (2.4.6) admits a unique weak solution $z(f, u)$ in $L^2(\Omega)$, moreover the operator*

$$(f, u) \mapsto z(f, u)$$

*is linear and continuous from $L^2(\Omega) \times L^2(\Gamma)$ into $L^2(\Omega)$.*

**Proof.** (i) *Uniqueness.* If $(f,u)=0$, we have $\int_\Omega z\phi = 0$ for all $\phi \in L^2(\Omega)$. Thus $z=0$ and the solution to equation (2.4.6) is unique.
(ii) The solution $z$ to equation (2.4.6) in the sense of definition 2.4.1 is equal to $z_1+z_2$, where $z_1 \in L^2(\Omega)$ is the solution to

$$\int_\Omega z\phi = \int_\Omega fy \qquad \text{for all } \phi \in L^2(\Omega),$$

and $z_2 \in L^2(\Omega)$ is the solution to

$$\int_\Omega z\phi = -\int_\Gamma u\frac{\partial y}{\partial n} \qquad \text{for all } \phi \in L^2(\Omega).$$

Therefore $z_2 = \Lambda^* u$, and we can take $z_1$ as the solution to

$$-\Delta z = f \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \Gamma.$$

∎

We study the following control problem

$$(P_2) \qquad \inf\{J_2(z,u) \mid (z,u) \in L^2(\Omega)\times L^2(\Gamma),\ (z,u) \text{ satisfies (2.4.6)}\},$$

with

$$J_2(z,u) = \frac{1}{2}\int_\Omega (z-z_d)^2 + \frac{\beta}{2}\int_\Gamma u^2.$$

To write optimality conditions for $(P_2)$, we consider adjoint equations of the form

$$-\Delta p = g \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma. \tag{2.4.9}$$

**Theorem 2.4.3** *Suppose that $f \in L^2(\Omega)$, $u \in L^2(\Gamma)$, and $g \in L^2(\Omega)$. Then the solution $z$ of equation (2.4.6) and the solution $p$ of (2.4.9) satisfy the following formula*

$$\int_\Omega f\,p = \int_\Omega z\,g + \int_\Gamma u\frac{\partial p}{\partial n}. \tag{2.4.10}$$

**Proof.** The result is nothing else than definition 2.4.1. ∎

**Theorem 2.4.4** *Assume that $f \in L^2(\Omega)$. Let $(\bar z, \bar u)$ be the unique solution to problem $(P_2)$. The optimal control $\bar u$ is defined by $\bar u = \frac{1}{\beta}\frac{\partial p}{\partial n}$, where $p$ is the solution to the equation*

$$-\Delta p = \bar z - z_d \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma. \tag{2.4.11}$$

**Proof.** We leave the reader adapt the proof of Theorem 2.2.2. ∎

## 2.5 Problems with control constraints

In many applications control variables are subject to constraints. For example the control variable must satisfy inequality constraints of the form $a \le u \le b$ , where $a$ and $b$ are two given functions. More generally, we shall consider control constraints of the form $u \in U_{ad}$ , where the set $U_{ad}$ is called the set of admissible controls. We want to write optimality conditions for the corresponding control problems. To explain how to proceed, we consider the particular problem

$$(P_3) \qquad \inf\{J_3(z,u) \mid (z,u) \in H^1(\Omega) \times U_{ad},\ (z,u) \text{ satisfies (2.2.1)}\}.$$

where $J_3 = J_1$, and $U_{ad}$, the set of admissible controls, is a closed convex subset in $L^2(\Gamma)$. We shall prove, in section 2.6, that $(P_3)$ admits a unique solution. We state below a result similar to that of Theorem 2.2.2.

**Theorem 2.5.1** *If $(\bar{z}, \bar{u})$ is the solution to $(P_3)$ then*

$$\int_\Gamma (\beta\bar{u} + p)(u - \bar{u}) \geq 0 \qquad \text{for all } u \in U_{ad}$$

*where $p$ is the solution to equation (2.2.4).*

*Conversely, if a triplet $(\tilde{z}, \tilde{u}, \tilde{p}) \in H^1(\Omega) \times U_{ad} \times H^1(\Omega)$ obeys the system*

$$\begin{aligned} -\Delta\tilde{z} + \tilde{z} = f \quad \text{in } \Omega, \quad \frac{\partial\tilde{z}}{\partial n} = \tilde{u} \quad \text{on } \Gamma, \\ -\Delta\tilde{p} + \tilde{p} = \tilde{z} - z_d \quad \text{in } \Omega, \quad \frac{\partial\tilde{p}}{\partial n} = 0 \quad \text{on } \Gamma, \\ \textstyle\int_\Gamma (\beta\tilde{u} + \tilde{p})(u - \tilde{u}) \geq 0 \qquad \text{for all } u \in U_{ad}, \end{aligned} \tag{2.5.12}$$

*then the pair $(\tilde{z}, \tilde{u})$ is the optimal solution to problem $(P_3)$.*

**Proof.** Setting $F_3(u) = J_3(z(f,u),u)$, where $z(f,u)$ is the solution to equation (2.2.1), we have already shown that $F_3'(\bar{u}) = \beta\bar{u} + p|_\Gamma$. Thus the first part of the theorem follows from the optimality of $\bar{u}$. The second part is a direct consequence of Theorem 2.2.3 (ii). ∎

## 2.6 Existence of solutions

We recall some results of functional analysis needed in the sequel.

**Theorem 2.6.1** *([4, Chapter 3, Theorem 7]) Let $E$ be a Banach space, and let $C \subset E$ be a convex subset. If $C$ is closed in $E$, then $C$ is also closed in $(E, \sigma(E, E'))$ (that is, closed in $E$ endowed with its weak topology).*

**Corollary 2.6.1** *([4, Chapter 3, Corollary 8]) Let $E$ be a Banach space, and let $\varphi : E \mapsto ]-\infty, \infty]$ be a lower semicontinuous convex function. Then $\varphi$ is also lower semicontinuous for the weak topology $\sigma(E, E')$. In particular $\varphi$ is sequentially lower semicontinuous.*

**Theorem 2.6.2** *([4, Chapter 3, Theorem 9]) Let $E$ and $F$ be two Banach spaces, and let $T$ be a continuous linear operator from $E$ into $F$. Then $T$ is also continuous from $(E, \sigma(E, E'))$ into $(F, \sigma(F, F'))$.*

**Corollary 2.6.2** *Let $(u_n)_n$ be a sequence converging to $u$ for the weak topology of $L^2(\Gamma)$. Then the sequence $(z(f, u_n))_n$, where $z(f, u_n)$ is the solution to (2.2.1) corresponding to the control function $u_n$, converges to $z(f, u)$ in $H^1(\Omega)$.*

**Proof.** From Theorems 2.2.1 and 2.6.2, it follows that $(z(f,u_n))_n$ converges to $z(f,u)$ for the weak topology of $H^{3/2}(\Omega)$. Since the imbedding from $H^{3/2}(\Omega)$ into $H^1(\Omega)$ is compact, the proof is complete.

**Theorem 2.6.3** *Problem $(P_3)$ admits a unique solution.*

**Proof.** Set $m = \inf_{u\in U_{ad}} J_3(z(f,u),u)$. Since $0 \le m < \infty$, there exists a minimizing sequence $(u_n)_n \subset U_{ad}$ such that $\lim_{n\to\infty} J_3(z(f,u_n),u_n) = m$. Without loss of generality, we can suppose that $J_3(z(f,u_n),u_n) \le J_3(z(f,0),0)$. Thus the sequence $(u_n)_n$ is bounded in $L^2(\Gamma)$. There exists a subsequence of $(u_n)_n$, still indexed by $n$ to simplify the notation, and a function $u$, such that $(u_n)_n$ converges to $u$ for the weak topology of $L^2(\Gamma)$. Due to Theorem 2.6.1, the limit $u$ belongs to $U_{ad}$. Due to Corollary 2.6.1, the mapping $\|\cdot\|^2_{L^2(\Gamma)}$ is weakly lower semicontinuous in $L^2(\Gamma)$. Thus, we have

$$\int_\Gamma u^2 \le \text{liminf}_{n\to\infty} \int_\Gamma u_n^2.$$

From Corollary 2.6.2, it follows that

$$\int_\Omega (z(f,u) - z_d)^2 \le \text{liminf}_{n\to\infty} \int_\Omega (z(f,u_n) - z_d)^2.$$

Combining these results we obtain

$$J_3(z(f,u),u) \le \text{liminf}_{n\to\infty} J_3(z(f,u_n),u_n) = m.$$

Therefore $(z(f,u),u)$ is a solution to problem $(P_3)$.

*Uniqueness.* Suppose that $(P_3)$ admits two distinct solutions $(z(f,u_1),u_1)$ and $(z(f,u_2),u_2)$. Let us set $u = \frac{1}{2}u_1 + \frac{1}{2}u_2$. Observe that $z(f,u) = \frac{1}{2}z(f,u_1) + \frac{1}{2}z(f,u_2)$. Due to the strict convexity of the functional $(z,u) \mapsto J_3(z,u)$, we verify that $J_3(z(f,u),u) < \frac{1}{2}J_3(z(f,u_1),u_1) + \frac{1}{2}J_3(z(f,u_2),u_2) = m$. We obtain a contradiction with the assumption that '$(z(f,u_1),u_1)$ and $(z(f,u_2),u_2)$ are two solutions of $(P_3)$'. Thus $(P_3)$ admits a unique solution. ∎

## 2.7 Other functionals

### 2.7.1 Observation in $H^1(\Omega)$

We consider the control problem

$(P_4)$ $\quad\quad \inf\{J_4(z,u) \mid (z,u) \in H^1(\Omega)\times L^2(\Gamma),\ (z,u) \text{ satisfies } (2.2.1)\}.$

with

$$J_4(z,u) = \frac{1}{2}\int_\Omega |\nabla z - \nabla z_d|^2 + \frac{\beta}{2}\int_\Gamma u^2,$$

where $z_d$ belongs to $H^1(\Omega)$. As in section 2.6, we can prove that $(P_4)$ admits a unique solution $(\bar z, \bar u)$. We set $F_4(u) = J_4(z(f,u),u)$, where $z(f,u)$ is the solution to (2.2.1). With calculations similar to those in section 2.2, we have

$$F_4'(\bar u)u = \int_\Omega (\nabla \bar z - \nabla z_d)\nabla w_u + \beta\int_\Gamma \bar u u,$$

where $w_u$ is the solution to equation (2.2.2):

$$-\Delta w + w = 0 \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial n} = u \quad \text{on } \Gamma.$$

Let us consider the variational equation

$$\begin{array}{c} \text{Find } p \in H^1(\Omega), \quad \text{such that} \\ \int_\Omega (\nabla p \nabla \phi + p\phi) = \int_\Omega (\nabla \bar{z} - \nabla z_d)\nabla \phi \qquad \text{for all } \phi \in H^1(\Omega). \end{array} \tag{2.7.13}$$

With the Lax-Milgram theorem we can prove that equation (2.7.13) admits a unique solution $p \in H^1(\Omega)$. Taking $\phi = w_u$ in the variational equation (2.7.13), we obtain

$$F_4'(\bar{u})u = \int_\Omega (\nabla \bar{z} - \nabla z_d)\nabla w_u + \beta \int_\Gamma \bar{u}u = \int_\Omega (\nabla p \nabla w_u + p w_u) + \beta \int_\Gamma \bar{u}u.$$

With the Green formula we finally have

$$F_4'(\bar{u})u = \int_\Gamma (p + \beta \bar{u})u.$$

*Interpretation of equation (2.7.13).* Observe that if $p$ is the solution of (2.7.13), then

$$-\Delta p + p = -\mathrm{div}(\nabla \bar{z} - \nabla z_d)$$

in the sense of distributions in $\Omega$. If $\bar{z}$ and $z_d$ belongs to $H^2(\Omega)$, then $p \in H^2(\Omega)$. We can verify that a function $p$ is a solution of (2.7.13) in $H^2(\Omega)$ if and only if

$$-\Delta p + p = -\mathrm{div}(\nabla \bar{z} - \nabla z_d) \quad \text{in } \Omega, \quad \text{and} \quad \frac{\partial p}{\partial n} = (\nabla \bar{z} - \nabla z_d) \cdot n \quad \text{on } \Gamma. \tag{2.7.14}$$

In the case when $\bar{z}$ and $z_d$ do not belong to $H^2(\Omega)$, $(\nabla \bar{z} - \nabla z_d) \cdot n|_\Gamma = \frac{\partial \bar{z}}{\partial n} - \frac{\partial z_d}{\partial n}$ is not necessarily defined. However, we still use the formulation (2.7.14) in place of (2.7.13), even if it is abusive. We can state the following theorem.

**Theorem 2.7.1** *If $(\bar{z}, \bar{u})$ is the solution to $(P_4)$ then $\bar{u} = -\frac{1}{\beta} p|_\Gamma$, where $p$ is the solution to equation (2.7.14).*
*Conversely, if a pair $(\tilde{z}, \tilde{p}) \in H^1(\Omega) \times H^1(\Omega)$ obeys $\tilde{z} = z(f, -\frac{1}{\beta}\tilde{p}|_\Gamma)$ and*

$$-\Delta \tilde{p} + \tilde{p} = -\mathit{div}(\nabla \tilde{z} - \nabla z_d) \quad \text{in } \Omega, \qquad \frac{\partial p}{\partial n} = (\nabla \tilde{z} - \nabla z_d) \cdot n \quad \text{on } \Gamma,$$

*then the pair $(\tilde{z}, -\frac{1}{\beta}\tilde{p})$ is the optimal solution to problem $(P_4)$.*

### 2.7.2 Pointwise observation

We consider the control problem

$$(P_5) \qquad \inf\{J_5(z,u) \mid (z,u) \in H^1(\Omega) \times L^2(\Gamma),\ (z,u) \text{ satisfies (2.2.1)}\}.$$

with

$$J_5(z,u) = \frac{1}{2}\Sigma_{i=1}^k (z(x_i) - z_d(x_i))^2 + \frac{\beta}{2}\int_\Gamma u^2,$$

where $x_1, \dots, x_k$ are given points in $\Omega$.

We know that $z(x_i)$ is not defined if $z \in H^1(\Omega)$, except in dimension one, since in that case we have $H^1(a,b) \subset C([a,b])$ if $-\infty < a < b < \infty$. If $z = z(f,u)$ is the solution to (2.2.1), and if $u \in L^2(\Gamma)$ we know that $z \in H^{3/2}(\Omega)$. Due to the imbedding $H^s(\Omega) \subset C(\overline{\Omega})$ if $s > \frac{N}{2}$, problem $(P_5)$ is well posed if $N = 2$.

First suppose that $N = 2$. Set $F_5(u) = J_5(z(f,u),u)$. We have

$$F_5'(\bar{u})u = \Sigma_{i=1}^k (\bar{z}(x_i) - z_d(x_i))w_u(x_i) + \beta\int_\Gamma \bar{u}u = \Sigma_{i=1}^k \langle w_u, (\bar{z} - z_d)\delta_{x_i}\rangle_{C(\overline{\Omega})\times\mathcal{M}(\overline{\Omega})} + \beta\int_\Gamma \bar{u}u,$$

where $\bar{z} = z(f,\bar{u})$, and $w_u$ is the solution to equation (2.2.2). Thus we have to consider the adjoint equation

$$-\Delta\tilde{p} + \tilde{p} = \Sigma_{i=1}^k (\bar{z} - z_d)\delta_{x_i} \quad \text{in } \Omega, \qquad \frac{\partial\tilde{p}}{\partial n} = 0 \quad \text{on } \Gamma. \tag{2.7.15}$$

Since $\Sigma_{i=1}^k (\bar{z} - z_d)\delta_{x_i} \notin (H^1(\Omega))'$, we cannot study equation (2.7.15) with the Lax-Milgram theorem. To study equation (2.7.15) and optimality conditions for $(P_5)$, we need additional regularity results that we develop in the next chapter (see section 3.5).

## 2.8 Exercises

**Exercise 2.8.1**

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, with a Lipschitz boundary $\Gamma$. Let $\Gamma_1$ be a closed subset in $\Gamma$, and set $\Gamma_2 = \Gamma \setminus \Gamma_1$. We consider the elliptic equation

$$-\Delta z = f \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n} = u \quad \text{on } \Gamma_1, \quad z = 0 \quad \text{on } \Gamma_2. \tag{2.8.16}$$

The function $f \in L^2(\Omega)$ is a given source term and the function $u \in L^2(\Gamma_1)$ is a control variable. Denote by $H^1_{\Gamma_2}(\Omega)$ the space

$$H^1_{\Gamma_2}(\Omega) = \{z \in H^1(\Omega) \mid z|_{\Gamma_2} = 0\}.$$

1 - Prove that equation (2.8.16) admits a unique weak solution in $H^1_{\Gamma_2}(\Omega)$ (give the precise definition of a weak solution).

We want to study the control problem

$(P_6)$ $\qquad \inf\{J_6(z,u) \mid (z,u) \in H^1_{\Gamma_2}(\Omega) \times L^2(\Gamma_1),\ (z,u) \text{ satisfies (2.8.16)}\}.$

with

$$J_6(z,u) = \frac{1}{2}\int_\Omega (z - z_d)^2 + \frac{\beta}{2}\int_{\Gamma_1} u^2,$$

the function $z_d$ belongs to $L^2(\Omega)$, and $\beta > 0$.

2 - Let $(u_n)_n$ be a sequence in $L^2(\Gamma_1)$ converging to $\hat{u}$ for the weak topology of $L^2(\Gamma_1)$. Let $z(u_n)$ be the solution to equation (2.8.16) corresponding to $u_n$. What can we say about the sequence $(z(u_n))_n$ ? Prove that problem $(P_6)$ admits a unique weak solution.

Characterize the optimal control by writing first order optimality conditions.

**Exercise 2.8.2**

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, with a Lipschitz boundary $\Gamma$. Consider the elliptic equation

$$-\Delta z + z = f + \chi_\omega u \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma. \tag{2.8.17}$$

The function $f$ belongs to $L^2(\Omega)$, the control $u \in L^2(\omega)$, and $\omega$ is an open subset in $\Omega$. We want to study the control problem

$$(P_7) \qquad \inf\{J_7(z,u) \mid (z,u) \in H^1(\Omega) \times L^2(\omega),\ (z,u) \text{ satisfies } (2.8.17)\}.$$

with

$$J_7(z,u) = \frac{1}{2}\int_\Gamma (z - z_d)^2 + \frac{\beta}{2}\int_\omega u^2,$$

the function $z_d$ belongs to $H^1(\Omega)$.

1 - Prove that problem $(P_7)$ admits a unique weak solution.

2 - Characterize the optimal control by writing first order optimality conditions.

# Chapter 3

# Control of semilinear elliptic equations

## 3.1 Introduction

In the sequel we suppose that $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$), with a boundary $\Gamma$ of class $C^2$. Let $A$ be an elliptic operator defined by

$$Az = -\Sigma_{i,j=1}^N \partial_i(a_{ij}(x)\partial_j z(x)) + a_0(x)z,$$

$\partial_i$ and $\partial_j$ denote the partial derivatives with respect to $x_i$ and $x_j$. We suppose that $a_{ij} \in C^1(\overline{\Omega})$, $a_0 \in C(\overline{\Omega})$, and

$$\Sigma_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq m|\xi|^2 \qquad \text{and} \quad a_0(x) \geq m > 0,$$

for all $\xi \in \mathbb{R}^N$ and all $x \in \overline{\Omega}$. In this chapter we study optimal control problems for systems governed by elliptic equations of the form

$$Az = f \quad \text{in } \Omega, \quad \frac{\partial z}{\partial n_A} + \psi(z) = u \quad \text{on } \Gamma, \tag{3.1.1}$$

where $\frac{\partial z}{\partial n_A} = \Sigma_{i,j=1}^N (a_{ij}(x)\partial_j z(x))n_i(x)$ denotes the conormal derivative of $z$ with respect to $A$, $n = (n_1, \ldots, n_N)$ is the outward unit normal to $\Gamma$. Typically we shall study the case of a Stefan-Boltzmann boundary condition, that is

$$\psi(z) = |z|^3 z,$$

but what follows can be adapted to other regular nondecreasing functions.

In this chapter we study equation (3.1.1) for control functions $u \in L^s(\Gamma)$ with $s > N - 1$. In that case we obtain solutions to equation (3.1.1) in $C(\overline{\Omega})$. These results are also used in section 3.5 to deal with problems involving pointwise observations. Equation (3.1.1) can as well be studied for control functions $u \in L^2(\Gamma)$. But, due to the nonlinear boundary condition, the analysis is more complicated than for controls in $L^s(\Gamma)$ with $s > N - 1$. Here we do not study the case of $L^2$-controls.

## 3.2 Linear elliptic equations

**Theorem 3.2.1** *For every $f \in L^r(\Omega)$, with $r \geq \frac{2N}{N+2}$ if $N > 2$ and $r > 1$ if $N = 2$, equation*

$$Az = f \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n_A} = 0 \quad \text{on } \Gamma, \tag{3.2.2}$$

*admits a unique solution in $H^1(\Omega)$, this solution belongs to $W^{2,r}(\Omega)$.*

**Proof.** Since $H^1(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega)$ (with a dense imbedding) if $N > 2$, and $H^1(\Omega) \subset L^p(\Omega)$ for any $p < \infty$ (with a dense imbedding) if $N = 2$, we have $L^{\frac{2N}{N+2}}(\Omega) \subset (H^1(\Omega))'$ if $N > 2$, and $L^{p'}(\Omega) \subset (H^1(\Omega))'$ for all $p < \infty$ if $N = 2$. Thus $L^r(\Omega) \subset (H^1(\Omega))'$. The existence of a unique solution in $H^1(\Omega)$ follows from the Lax-Migram theorem. The regularity result in $W^{2,r}(\Omega)$ is proved in [30, Theorem 3.17]. ∎

For any exponent $q$, we denote by $q'$ the conjugate exponent to $q$. When $f \in (W^{1,q'}(\Omega))'$, $q \geq 2$, we replace equation (3.2.2) by the variational equation

$$\text{find } z \in H^1(\Omega) \quad \text{such that} \quad a(z, \phi) = \langle f, \phi \rangle_{(H^1(\Omega))' \times H^1(\Omega)} \quad \text{for all } \phi \in H^1(\Omega), \tag{3.2.3}$$

where $a(z, \phi) = \int_\Omega \Sigma_{i,j=1}^N a_{ij}(x) \partial_j z \partial_i \phi \, dx$.

**Theorem 3.2.2** *For every $f \in (W^{1,q'}(\Omega))'$, with $q \geq 2$, equation (3.2.3) admits a unique solution in $H^1(\Omega)$, this solution belongs to $W^{1,q}(\Omega)$, and*

$$\|z\|_{W^{1,q}(\Omega)} \leq C \|f\|_{(W^{1,q'}(\Omega))'}.$$

**Proof.** As previously we notice that the existence of a unique solution in $H^1(\Omega)$ follows from the Lax-Migram theorem. The regularity result in $W^{1,q}(\Omega)$ is proved in [30, Theorem 3.16]. ∎

With Theorem 3.2.2, we can study elliptic equations with nonhomogeneous boundary conditions.

**Lemma 3.2.1** *If $g \in L^s(\Gamma)$ with $s \geq \frac{2(N-1)}{N}$, the mapping*

$$\phi \longmapsto \int_\Gamma g\phi$$

*belongs to $(W^{1,q'}(\Omega))'$ for all $s \geq \frac{(N-1)q}{N}$.*

**Proof.** If $\phi \in W^{1,q'}(\Omega)$, then $\phi|_\Gamma$ belongs to $W^{1-\frac{1}{q'},q'}(\Gamma) \subset L^{\frac{(N-1)q'}{N-q'}}(\Gamma)$. Thus the mapping $\phi \mapsto \int_\Gamma g\phi$ belongs to $(W^{1,q'}(\Omega))'$ if $s \geq \left(\frac{(N-1)q'}{N-q'}\right)' = \frac{(N-1)q}{N}$. The proof is complete. ∎

**Theorem 3.2.3** *For every $g \in L^s(\Gamma)$, with $s \geq \frac{2(N-1)}{N}$, equation*

$$Az = 0 \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n_A} = g \quad \text{on } \Gamma, \tag{3.2.4}$$

*admits a unique solution in $H^1(\Omega)$, this solution belongs to $W^{1,q}(\Omega)$ with $q = \frac{Ns}{N-1}$, and*

$$\|z\|_{W^{1,q}(\Omega)} \leq C \|g\|_{L^s(\Gamma)}.$$

**Proof.** Obviously $z \in H^1(\Omega)$ is a solution to equation (3.2.4) if and only if

$$a(z, \phi) = \int_\Gamma g\phi \quad \text{for all } \phi \in H^1(\Omega).$$

The existence and uniqueness still follow from the Lax-Milgram theorem. The regularity result is a direct consequence of Lemma 3.2.1 and Theorem 3.2.2. ∎

## 3.3 Semilinear elliptic equations

The Minty-Browder Theorem, stated below, is a powerful tool to study nonlinear elliptic equations.

**Theorem 3.3.1** ([4]) *Let $E$ be a reflexive Banach space, and $\mathcal{A}$ be a nonlinear continuous mapping from $E$ into $E'$. Suppose that*

$$\langle \mathcal{A}(z_1) - \mathcal{A}(z_2), z_1 - z_2 \rangle_{E',E} > 0 \qquad \text{for all } z_1,\ z_2 \in E,\ \text{with } z_1 \neq z_2, \tag{3.3.5}$$

*and*

$$lim_{\|z\|_E \to \infty} \frac{\langle \mathcal{A}(z), z \rangle_{E',E}}{\|z\|_E} = \infty.$$

*Then, for all $\ell \in E'$, there exists a unique $z \in E$ such that $\mathcal{A}(z) = \ell$.*

We want to apply this theorem to the nonlinear equation

$$Az = f \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n_A} + \psi_k(z) = g \quad \text{on } \Gamma, \tag{3.3.6}$$

with $f \in L^r(\Omega)$, $g \in L^s(\Gamma)$, and

$$\psi_k(z) = \begin{cases} \psi(k) + \psi'(k)(z-k) & \text{if } z > k, \\ \psi(z) & \text{if } |z| \leq k, \\ \psi(-k) + \psi'(-k)(z+k) & \text{if } z < -k. \end{cases}$$

We explain below why we first replace $\psi$ by the truncated function $\psi_k$ (see remark after Theorem 3.3.2). To apply Theorem 3.3.1, we set $E = H^1(\Omega)$, and we define $\mathcal{A}$ by

$$\langle \mathcal{A}(z), \phi \rangle_{(H^1(\Omega))', H^1(\Omega)} = a(z, \phi) + \int_\Gamma \psi_k(z)\phi,$$

and $\ell$ by

$$\langle \ell, \phi \rangle = \int_\Omega f\phi + \int_\Gamma g\phi.$$

Condition (3.3.5) is satisfied because $a(z_1 - z_2, z_1 - z_2) \geq m\|z_1 - z_2\|^2_{H^1(\Omega)}$ and $\int_\Gamma \psi_k(z_1 - z_2)(z_1 - z_2) \geq 0$ (indeed, the function $\psi_k$ is increasing). Moreover

$$\frac{\langle \mathcal{A}(z), z \rangle_{(H^1(\Omega))', H^1(\Omega)}}{\|z\|_{H^1(\Omega)}} \geq m\|z\|_{H^1(\Omega)} \to \infty \quad \text{as } \|z\|_{H^1(\Omega)} \to \infty.$$

Let us verify that the mapping

$$z \longmapsto \Big(\phi \mapsto \int_\Gamma \psi_k(z)\phi\Big)$$

is continuous from $H^1(\Omega)$ into $(H^1(\Omega))'$. Let $(z_n)_n$ be a sequence converging to some $z$ in $H^1(\Omega)$. Since $\psi_k$ is a Lipschitz function we have

$$|\int_\Gamma (\psi_k(z_n) - \psi_k(z))\phi| \le C\|z_n - z\|_{L^2(\Gamma)}\|\phi\|_{L^2(\Gamma)} \le C\|z_n - z\|_{H^1(\Omega)}\|\phi\|_{H^1(\Omega)}.$$

This proves the continuity result. Thus all the assumptions of Theorem 3.3.1 are satisfied and we have established the following theorem.

**Theorem 3.3.2** *The nonlinear equation (3.3.6) admits a unique solution in* $H^1(\Omega)$.

**Remark.** We have explicitly used the Lipschitz continuity of $\psi_k$ to prove that $\mathcal{A}$ is a continuous mapping from $H^1(\Omega)$ into $(H^1(\Omega))'$. Observe that the mapping

$$z \longmapsto \Big(\phi \mapsto \int_\Gamma \psi(z)\phi\Big)$$

is not a mapping from $H^1(\Omega)$ into $(H^1(\Omega))'$. This is the reason why we have used the truncated function $\psi_k$. We shall see later how we can go back to the initial equation.

**Theorem 3.3.3** *(Comparison principle) Let* $f \in L^r(\Omega)$, *with* $r \ge \frac{2N}{N+2}$, $g \in L^s(\Gamma)$, *with* $s \ge \frac{2(N-1)}{N}$, *and* $b \in L^\infty(\Gamma)$ *satisfying* $b \ge 0$. *Suppose that* $f \ge 0$ *and* $g \ge 0$. *Then the solution* $z$ *to the equation*

$$Az = f \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n_A} + bz = g \quad \text{on } \Gamma,$$

*is nonnegative.*

**Theorem 3.3.4** *For all* $f \in L^r(\Omega)$, *with* $r > \frac{N}{2}$, *and* $g \in L^s(\Gamma)$, *with* $s > N-1$, *the solution to equation*

$$Az = f \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n_A} = g \quad \text{on } \Gamma, \tag{3.3.7}$$

*belongs to* $C^{0,\nu}(\overline{\Omega})$ *for some* $0 < \nu \le 1$, *and*

$$\|z\|_{C^{0,\nu}(\overline{\Omega})} \le C(\|f\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)}).$$

**Proof.** Let $z_1$ be the solution to

$$Az = f \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n_A} = 0 \quad \text{on } \Gamma,$$

and $z_2$ the solution to

$$Az = 0 \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n_A} = g \quad \text{on } \Gamma.$$

With Theorem 3.2.1 and Theorem 3.2.3, we have $z_1 \in W^{2,r}(\Omega)$ and $z_2 \in W^{1,\frac{Ns}{N-1}}(\Omega)$. Due to Sobolev imbeddings, we have $W^{2,r}(\Omega) \subset W^{1,\frac{Nr}{N-r}}(\Omega)$ if $r < N$, and $W^{2,r}(\Omega) \subset W^{1,q}(\Omega)$ for all $q < \infty$ if $r \ge N$. Observe that $\frac{Nr}{N-r} > N$ because $r > \frac{N}{2}$. Thus we have $W^{2,r}(\Omega) \subset C^{0,\nu}(\overline{\Omega})$ for some $0 < \nu \le 1$. We also have $\frac{Ns}{N-1} > N$, because $s > N-1$, and $W^{1,\frac{Ns}{N-1}}(\Omega) \subset C^{0,\nu}(\overline{\Omega})$ for $\nu = \frac{s-N+1}{s}$. Since $z = z_1 + z_2$, the proof is complete. ∎

**Theorem 3.3.5** *For all $f \in L^r(\Omega)$, with $r > \frac{N}{2}$, and $g \in L^s(\Gamma)$, with $s > N-1$, the solution to equation (3.3.6) belongs to $C(\overline{\Omega})$, and*

$$\|z\|_{C(\overline{\Omega})} \leq C(\|f\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)}),$$

*where the constant $C$ is independent of $k$.*

**Proof.** Due to Theorem 3.3.2 the solution exists and is unique in $H^1(\Omega)$. We can rewrite equation (3.3.6) in the form

$$Az = f \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n_A} + bz = g \quad \text{on } \Gamma,$$

with $b(x) = \int_0^1 \psi_k'(\theta z(x))d\theta$. Observe that $b \in L^\infty(\Gamma)$ because $\psi_k'$ is bounded. Using the positive and the negative parts of $f$ and $g$ ($f = f^+ - f^-$ and $g = g^+ - g^-$), we have $z = z_1 - z_2$, where $z_1$ is the solution to

$$Az = f^+ \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n_A} + bz = g^+ \quad \text{on } \Gamma,$$

and $z_2$ is the solution to

$$Az = f^- \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n_A} + bz = g^- \quad \text{on } \Gamma.$$

Due to the comparison principle, the functions $z_1$ and $z_2$ are nonnegative. Denote by $\hat{z}_1$ the solution to

$$Az = f^+ \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n_A} = g^+ \quad \text{on } \Gamma.$$

Thus we have $0 \leq \hat{z}_1$ (due to the comparison principle), and $w = \hat{z}_1 - z_1$ obeys

$$Aw = 0 \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial n_A} = bz_1 \geq 0 \quad \text{on } \Gamma.$$

Still with the comparison principle it follows that $0 \leq z_1 \leq \hat{z}_1$, and

$$\|z_1\|_{C(\overline{\Omega})} \leq \|\hat{z}_1\|_{C(\overline{\Omega})} \leq C(\|f\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)}),$$

where the constant $C$ is independent of $k$. Similarly we have $0 \leq z_2 \leq \hat{z}_2$, and

$$\|z_2\|_{C(\overline{\Omega})} \leq \|\hat{z}_2\|_{C(\overline{\Omega})} \leq C(\|f\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)}),$$

where the constant $C$ is independent of $k$. Since $\|z\|_{C(\overline{\Omega})} \leq \|z_1\|_{C(\overline{\Omega})} + \|z_2\|_{C(\overline{\Omega})}$, the proof is complete. ∎

**Theorem 3.3.6** *For all $f \in L^r(\Omega)$, with $r > \frac{N}{2}$, $g \in L^s(\Gamma)$, with $s > N-1$, and all $b \geq 0$, $b \in L^\infty(\Omega)$, the solution to equation*

$$Az = f \quad \textit{in } \Omega, \qquad \frac{\partial z}{\partial n_A} + bz = g \quad \textit{on } \Gamma,$$

*belongs to $C(\overline{\Omega})$, and*

$$\|z\|_{C(\overline{\Omega})} \leq C(\|f\|_{L^r(\Omega)} + \|g\|_{L^s(\Gamma)}),$$

*where the constant $C$ is independent of $b$.*

**Proof.** The proof is completely analogous to the previous one. ∎

We give a definition of solution to equation (3.1.1) very similar to the one for equation (3.3.6).

**Definition 3.3.1** *A function $z \in H^1(\Omega)$ is a solution to equation (3.1.1) if $\psi(z) \in L^{\frac{2N}{N+2}}(\Gamma)$ when $N > 2$, $\psi(z) \in L^r(\Gamma)$ with $r > 1$ when $N = 2$, and*

$$a(z, \phi) + \int_\Gamma \psi(z)\phi = \int_\Omega f\phi + \int_\Gamma u\phi \qquad \text{for all} \quad \phi \in H^1(\Omega).$$

**Theorem 3.3.7** *For all $f \in L^r(\Omega)$, with $r > \frac{N}{2}$, and $u \in L^s(\Gamma)$, with $s > N-1$, equation (3.1.1) admits a unique solution in $H^1(\Omega) \cap L^\infty(\Omega)$, this solution belongs to $C(\overline{\Omega})$ and*

$$\|z\|_{C(\overline{\Omega})} + \|z\|_{H^1(\Omega)} \leq C(\|f\|_{L^r(\Omega)} + \|u\|_{L^s(\Gamma)}).$$

*It also belongs to $C^{0,\nu}(\overline{\Omega})$ for some $0 < \nu \leq 1$, and*

$$\|z\|_{C^{0,\nu}(\overline{\Omega})} \leq C\Big(\|f\|_{L^r(\Omega)} + \|u\|_{L^s(\Gamma)} + \psi(\|z\|_{C(\overline{\Omega})})\Big).$$

**Proof.** Let $k > C(\|f\|_{L^r(\Omega)} + \|u\|_{L^s(\Gamma)})$, where $C$ is the constant in the estimate of Theorem 3.3.5. Let $z_k$ be the solution to equation (3.3.6). Due to Theorem 3.3.5 $\|z_k\|_{C(\overline{\Omega})} < k$. Thus $\psi_k(z_k) = \psi(z_k)$. This means that $z_k$ is also a solution to equation (3.1.1). To prove the uniqueness, we argue by contradiction. Suppose that $z_1$ and $z_2$ are two solutions to equation (3.1.1) in $H^1(\Omega) \cap L^\infty(\Omega)$. Then $w = z_1 - z_2$ is the solution to

$$Aw = 0 \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial n_A} + bw = 0 \quad \text{on } \Gamma,$$

with $b(x) = \int_0^1 \psi'(z_2(x) + \theta(z_1(x) - z_2(x)))d\theta$. The functions $z_1$ and $z_2$ being bounded, $b \in L^\infty(\Gamma)$. From the comparison principle, it follows that $w = 0$.

The estimate in $C(\overline{\Omega})$ follows from the estimate of $z_k$. The estimate in $H^1(\Omega)$ follows from the variational formulation. The estimate in $C^{0,\nu}(\overline{\Omega})$ can be deduced from Theorem 3.3.4 by writing the boundary condition in the form $\frac{\partial z}{\partial n_A} = -\psi(z) + u$. The proof is complete. ∎

## 3.4 Control problems

From now on we suppose that the exponent $N - 1 < s < \infty$ is given fixed, and $f \in L^r(\Omega)$ with $r > \frac{N}{2}$ is also given. In this case the nonlinear equation (3.1.1) admits a unique solution for all $u \in L^s(\Gamma)$. We are going to study the control problem

$$(P_1) \qquad \inf\{J_1(z, u) \mid (z, u) \in H^1(\Omega) \times U_{ad},\ (z, u) \text{ satisfies (3.1.1)}\},$$

where $U_{ad}$ is a closed convex subset in $L^s(\Gamma)$, and

$$J_1(z, u) = \frac{1}{2}\int_\Omega (z - z_d)^2 + \beta \int_\Gamma |u|^s.$$

We suppose that either $\beta > 0$ or $U_{ad}$ is bounded in $L^s(\Gamma)$. The function $z_d$ belongs to $L^2(\Omega)$.

### 3.4.1 Existence of solutions

**Theorem 3.4.1** *Let $(u_n)_n$ be a sequence converging to $u$ for the weak topology of $L^s(\Gamma)$. Let $z(f, u_n)$ (resp. $z(f, u)$) be the solution to (3.1.1) corresponding to $u_n$ (resp. $u$). Then $(z(f, u_n))_n$ converges to $z(f, u)$ for the weak topology of $H^1(\Omega)$, and for the strong topology of $C(\overline{\Omega})$.*

**Proof.** Set $z_n = z(f, u_n)$. For any subsequence $(z_{n_k})_k$, due to Theorem 3.3.7 the sequence $(z_{n_k})_k$ is bounded in $H^1(\Omega)$ and in $C^{0,\nu}(\overline{\Omega})$ for some $0 < \nu \le 1$. The imbedding from $C^{0,\nu}(\overline{\Omega})$ into $C(\overline{\Omega})$ is compact. Thus we can extract a subsequence, still indexed by $k$ to simplify the notation, such that $(z_{n_k})_k$ converges to some $z$ for the weak topology of $H^1(\Omega)$, and for the strong topology of $C(\overline{\Omega})$. By passing to the limit in the variational formulation satisfied by $z_{n_k}$, we see that $z = z(f, u)$. The limit being unique, the sequence $(z_n)_n$ has a unique cluster point which proves that all the sequence $(z_n)_n$ converges to $z(f, u)$. ∎

**Theorem 3.4.2** *Problem $(P_1)$ admits at least one solution.*

**Proof.** Let $(u_n)_n$ be a minimizing sequence for $(P_1)$. Since $\beta > 0$ or $U_{ad}$ is bounded in $L^s(\Gamma)$, the sequence $(u_n)_n$ is bounded in $L^s(\Gamma)$. Without loss of generality, we can suppose that $(u_n)_n$ converges to some $u$ for the weak topology of $L^s(\Gamma)$. The function $u$ belongs to $U_{ad}$, because $U_{ad}$ is a closed convex subset in $L^s(\Gamma)$ (see Theorem 2.6.1). Due to Theorem 3.4.1, the sequence $(z(f, u_n))_n$ converges to $z(f, u)$ in $C(\overline{\Omega})$. Thus we have

$$J_1(z(f, u), u) \le \mathrm{liminf}_{n\to\infty} J_1(z(f, u_n), u_n) = \inf(P_1),$$

that is $(z(f, u), u)$ is a solution to problem $(P_1)$.

∎

### 3.4.2 Optimality conditions

**Theorem 3.4.3** *Let $u$ be in $U_{ad}$ and let $(b_k)_k$ be a sequence in $L^\infty(\Gamma)$ converging to $b$ in $L^\infty(\Gamma)$. Let $w_k$ be the solution to*

$$Aw = 0 \quad in\ \Omega, \quad \frac{\partial w}{\partial n_A} + b_k w = u \quad on\ \Gamma. \tag{3.4.8}$$

*Then the sequence $(w_k)_k$ converges to $w$ in $C(\overline{\Omega})$, where $w$ is the solution to*

$$Aw = 0 \quad in\ \Omega, \quad \frac{\partial w}{\partial n_A} + bw = u \quad on\ \Gamma.$$

**Proof.** Due to Theorem 3.3.6, $\|w_k\|_{C(\overline{\Omega})} \le C\|u\|_{L^s(\Gamma)}$ and $C$ is independent of $k$. The function $w_k - w$ is the solution to

$$Az = 0 \quad \text{in } \Omega, \quad \frac{\partial z}{\partial n_A} + bz = (b - b_k)w_k \quad \text{on } \Gamma.$$

Thus we have

$$\|w_k - w\|_{C(\overline{\Omega})} \le C\|(b - b_k)w_k\|_{L^\infty(\Gamma)} \le C\|b - b_k\|_{L^\infty(\Gamma)}\|u\|_{L^s(\Gamma)}.$$

The proof is complete. ∎

**Theorem 3.4.4** *Let $\bar u$ and $u$ be in $U_{ad}$ and let $\lambda > 0$. Set $w_\lambda = \frac{1}{\lambda}(z(f, \bar u + \lambda(u - \bar u)) - z(f, \bar u))$. Then $(w_\lambda)_\lambda$ tends to $w$ for the weak topology of $H^1(\Omega)$, where $w$ is the solution to equation*

$$Aw = 0 \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n_A} + \psi'(z(f, \bar u))w = u - \bar u \quad \text{on } \Gamma. \tag{3.4.9}$$

**Proof.** Set $z_\lambda = z(f, \bar u + \lambda(u - \bar u))$ and $\bar z = z(f, \bar u)$. Writing the equation satisfied by $z_\lambda - z$, we can easily prove that $z_\lambda$ tends to $\bar z$, as $\lambda$ tends to zero. Now we write the equation satisfied by $w_\lambda = \frac{1}{\lambda}(z_\lambda - \bar z)$:

$$Aw = 0 \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n_A} + b_\lambda w = u - \bar u \quad \text{on } \Gamma,$$

with $b_\lambda = \int_0^1 \psi'(\bar z + \theta(z_\lambda - \bar z))d\theta$. Since $z_\lambda$ tends to $\bar z$, $b_\lambda$ tends to $\psi'(\bar z)$ as $\lambda$ tends to zero. Thus we can apply Theorem 3.4.3 to complete the proof. ∎

**Theorem 3.4.5** *If $(\bar z, \bar u)$ is a solution to $(P_1)$ then*

$$\int_\Gamma (\beta s|\bar u|^{s-2}\bar u + p)(u - \bar u) \geq 0 \qquad \text{for all } u \in U_{ad},$$

*where $p$ is the solution to equation*

$$A^* p = \bar z - z_d \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n_{A^*}} + \psi'(\bar z)p = 0 \quad \text{on } \Gamma. \tag{3.4.10}$$

*($A^*$ is the formal adjoint of $A$, that is $A^*p = -\Sigma_{i,j=1}^N \partial_j(a_{ij}\partial_i p) + a_0 p$, and $\frac{\partial p}{\partial n_{A^*}} = \Sigma_{i,j=1}^N (a_{ij}\partial_i p n_j)$.)*

**Proof.** With the notation used in the proof of Theorem 3.4.4, for all $\lambda > 0$, we have

$$0 \leq \frac{J_1(z_\lambda, u_\lambda) - J_1(\bar z, \bar u)}{\lambda} = \int_\Omega \frac{1}{2}(z_\lambda + \bar z - 2z_d)w_\lambda + \frac{\beta}{\lambda}\int_\Gamma (|u_\lambda|^s - |u|^s),$$

with $u_\lambda = \bar u + \lambda(u - \bar u)$. From the convexity of the mapping $u \mapsto |u|^s$ it follows that

$$\int_\Gamma (|u_\lambda|^s - |u|^s) \leq \int_\Gamma s|u_\lambda|^{s-2}u_\lambda\, \lambda(u - \bar u).$$

Thus we have

$$0 \leq \int_\Omega \frac{1}{2}(z_\lambda + \bar z - 2z_d)w_\lambda + \int_\Gamma \beta s|u_\lambda|^{s-2}u_\lambda\,(u - \bar u).$$

Passing to the limit when $\lambda$ tends to zero, it yields

$$0 \leq \int_\Omega (\bar z - z_d)w + \int_\Gamma \beta s|\bar u|^{s-2}\bar u(u - \bar u),$$

where $w$ is the solution to equation (3.4.9). Now using a Green formula between $p$ and $w$ we obtain

$$\int_\Omega (\bar z - z_d)w = \int_\Omega A^* p w = \int_\Gamma \frac{\partial w}{\partial n_A}p - \int_\Gamma \frac{\partial p}{\partial n_{A^*}}w = \int_\Gamma p(u - \bar u).$$

This completes the proof. ∎

## 3.5 Pointwise observation

In the previous chapter, we have already mentioned that pointwise observations are well defined if the state variable is continuous. Due to Theorem 3.3.5 the solution to equation (3.1.1) is continuous if the boundary control $u$ belongs to $L^s(\Gamma)$ with $s > N-1$. We are now in position to study the control problem

$$(P_2) \qquad \inf\{J_2(z,u) \mid (z,u) \in C(\overline{\Omega}) \times U_{ad},\ (z,u) \text{ satisfies } (3.1.1)\}.$$

where $U_{ad}$ is a closed convex subset in $L^s(\Gamma)$, and

$$J_2(z,u) = \frac{1}{2}\int_0^T \Sigma_{i=1}^k (z(x_i) - z_d(x_i))^2 + \beta \int_\Gamma |u|^s,$$

where $x_1, \ldots, x_k$ are given points in $\Omega$, and $z_d \in C(\overline{\Omega})$.

The adjoint equation for $(P_2)$ is of the form

$$A^*p = \Sigma_{i=1}^k (z - z_d)\delta_{x_i} \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n_{A^*}} + bp = 0 \quad \text{on } \Gamma, \tag{3.5.11}$$

with $b \in L^\infty(\Gamma)$, $b \geq 0$. Since $\Sigma_{i=1}^k (z - z_d)\delta_{x_i} \notin (H^1(\Omega))'$, we cannot study equation (3.5.11) with the Lax-Milgram theorem. For simplicity in the notation let us suppose that there is only one observation point $a \in \Omega$. We study equation

$$A^*p = \delta_a \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n_{A^*}} + bp = 0 \quad \text{on } \Gamma, \tag{3.5.12}$$

with the transposition method.

**Definition 3.5.1** *We say that $p \in L^q(\Omega)$, with $q < \frac{N}{N-2}$, is a solution to equation (3.5.12) in the sense of transposition if*

$$\int_\Omega p\phi = y(a),$$

*for all $\phi \in L^{q'}(\Omega)$, where $y$ is the solution to*

$$Ay = \phi \quad \text{in } \Omega, \quad \frac{\partial y}{\partial n_A} + by = 0 \quad \text{on } \Gamma. \tag{3.5.13}$$

Following the transposition method of Chapter 2, we set

$$\Lambda : \phi \longmapsto y,$$

and $p$ is defined by

$$\langle \phi, p\rangle_{L^{q'}(\Omega), L^q(\Omega)} = \langle \Lambda(\phi), \delta_a\rangle_{C(\overline{\Omega}), \mathcal{M}(\overline{\Omega})} = \langle \phi, \Lambda^*(\delta_a)\rangle_{L^{q'}(\Omega), L^q(\Omega)}$$

for all $\phi \in L^{q'}(\Omega)$, that is $p = \Lambda^*(\delta_a)$. By proving that $\Lambda$ is continuous from $L^{q'}(\Omega)$ into $C(\overline{\Omega})$, we show that $\Lambda^*$ is continuous from $\mathcal{M}(\overline{\Omega})$ into $L^{q'}(\Omega)$.

Here, we prefer to construct the solution to equation (3.5.12) by an approximation process, and next use the transposition method to obtain estimates and to prove uniqueness. This approximation process is used to obtain regularity results (see Theorem 3.5.1). We approximate the distribution $\delta_a$ by the sequence of bounded functions $(f_\varepsilon)_\varepsilon$ with $f_\varepsilon = \frac{1}{|B(a,\varepsilon)|}\chi_{B(a,\varepsilon)}$, where $B(a,\varepsilon)$ is the ball centered at $a$ with radius $\varepsilon$, $|B(a,\varepsilon)|$ the measure of the ball, and $\chi_{B(a,\varepsilon)}$ is the characteristic function of $B(a,\varepsilon)$. Let $p_\varepsilon$ be the solution to

$$A^*p = f_\varepsilon \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n_{A^*}} + bp = 0 \quad \text{on } \Gamma. \tag{3.5.14}$$

Using a Green formula, we have

$$\int_\Omega p_\varepsilon \phi = \frac{1}{|B(a,\varepsilon)|}\int_\Omega \chi_{B(a,\varepsilon)} y, \tag{3.5.15}$$

where $y$ is the solution to (3.5.13). If $q' > \frac{N}{2}$, from Theorem 3.3.5 it follows

$$\left|\int_\Omega p_\varepsilon \phi\right| = \frac{1}{|B(a,\varepsilon)|}\left|\int_\Omega \chi_{B(a,\varepsilon)} y\right| \le \|y\|_{C(\overline{\Omega})} \le C\|\phi\|_{L^{q'}(\Omega)},$$

and

$$\|p_\varepsilon\|_{L^p(\Omega)} \le \sup_{\|\phi\|_{L^{q'}(\Omega)}=1}\left|\int_\Omega p_\varepsilon \phi\right| \le C.$$

Thus the sequence $(p_\varepsilon)_\varepsilon$ is bounded in $L^q(\Omega)$. We can extract a subsequence, still indexed by $\varepsilon$ to simplify the notation, such that $(p_\varepsilon)_\varepsilon$ converges to some $p$ for the weak topology of $L^q(\Omega)$. Since $q' > \frac{N}{2}$, $y \in C(\overline{\Omega})$ and

$$\frac{1}{|B(a,\varepsilon)|}\int_\Omega \chi_{B(a,\varepsilon)} y \longrightarrow y(a)$$

as $\varepsilon$ tends to zero. By passing to the limit in (3.5.15), we prove that $p$ is a solution to equation (3.5.12) in the sense of definition 3.5.1.

Let us prove the uniqueness. Let $p_1$ and $p_2$ be two solutions in $L^{q'}(\Omega)$ to equation (3.5.12). We have

$$\int_\Omega (p_1 - p_2)\phi = 0$$

for all $\phi \in L^q(\Omega)$. Thus $p_1 = p_2$.

**Theorem 3.5.1** *Equation (3.5.12) admits a unique solution in* $L^q(\Omega)$ *for all* $q < \frac{N}{N-2}$, *and this solution belongs to* $W^{1,\tau}(\Omega)$ *for all* $\tau < \frac{N}{N-1}$.

**Lemma 3.5.1** *Let* $\phi \in \mathcal{D}(\Omega)$, *and let* $y$ *be the solution to equation*

$$Ay = -\partial_i \phi \quad \text{in } \Omega, \quad \frac{\partial y}{\partial n_A} + by = 0 \quad \text{on } \Gamma, \tag{3.5.16}$$

*where* $i \in \{1,\dots,N\}$. *Then, for all* $\tau < \frac{N}{N-1}$, $y$ *satisfies the estimate*

$$\|y\|_{C(\overline{\Omega})} \le C(\tau)\|\phi\|_{L^{\tau'}(\Omega)}.$$

**Proof.** We have

$$\left|\int_\Omega(-\partial_i\phi\psi)\right| = \left|\int_\Omega \phi\partial_i\psi\right| \leq \|\phi\|_{L^{\tau'}(\Omega)}\|\psi\|_{W^{1,\tau}(\Omega)}.$$

Therefore, the mapping

$$\psi \longrightarrow \int_\Omega(-\partial_i\phi\psi)$$

is an element in $(W^{1,\tau}(\Omega))'$, whose norm in $(W^{1,\tau}(\Omega))'$ is bounded by $\|\phi\|_{L^{\tau'}(\Omega)}$. Now, from Theorem 3.2.2, it follows that

$$\|y\|_{C(\overline{\Omega})} \leq C\|y\|_{W^{1,\tau'}(\Omega)} \leq C\|\phi\|_{L^{\tau'}(\Omega)},$$

because $\tau' > N$. The proof is complete. ∎

**Proof of Theorem 3.5.1.** The first part of the theorem is already proved. Let us prove that $p$ belongs to $W^{1,\tau}(\Omega)$ for all $\tau < \frac{N}{N-1}$. Replacing $\phi$ by $-\partial_i\phi$ in (3.5.15), where $i \in \{1,\ldots,N\}$, we obtain

$$\int_\Omega \partial_i p_\varepsilon\phi = \int_\Omega p_\varepsilon(-\partial_i\phi) = \frac{1}{|B(a,\varepsilon)|}\int_\Omega \chi_{B(a,\varepsilon)}y,$$

where $y$ is the solution to equation (3.5.16). Since $\|y\|_{C(\overline{\Omega})} \leq C\|\phi\|_{L^{\tau'}(\Omega)}$, it yields

$$\|\partial_i p_\varepsilon\|_{L^\tau(\Omega)} \leq \sup_{\|\phi\|_{L^{\tau'}(\Omega)}=1}\left|\int_\Omega \partial_i p_\varepsilon\phi\right| \leq C.$$

This estimate is true for all $i \in \{1,\ldots,N\}$, therefore the sequence $(p_\varepsilon)_\varepsilon$ is bounded in $W^{1,\tau}(\Omega)$. We already know that the sequence $(p_\varepsilon)_\varepsilon$ tends to $p$ for the weak topology of $L^q(\Omega)$ for all $1 < q < \frac{N}{N-2}$. From the previous estimate it follows that the sequence $(p_\varepsilon)_\varepsilon$ tends to $p$ for the weak topology of $W^{1,\tau}(\Omega)$ for all $1 < \tau < \frac{N}{N-1}$. ∎

The proof of existence of solutions to problem $(P_2)$ is similar to the one of Theorem (3.4.2).

**Theorem 3.5.2** *If* $(\bar{z},\bar{u})$ *is a solution to* $(P_2)$ *then*

$$\int_\Gamma(\beta s|\bar{u}|^{s-2}\bar{u} + p)(u - \bar{u}) \geq 0 \qquad \textit{for all } u \in U_{ad},$$

*where* $p$ *is the solution to equation*

$$A^*p = \Sigma_{i=1}^k(\bar{z}(x_i) - z_d(x_i))\delta_{x_i} \quad \textit{in } \Omega, \quad \frac{\partial p}{\partial n_{A^*}} + \psi'(\bar{z})p = 0 \quad \textit{on } \Gamma. \tag{3.5.17}$$

**Proof.** Let $w_u$ be the solution to

$$Aw = 0 \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n_A} + \psi'(\bar{z})w = u \quad \text{on } \Gamma,$$

and $p_\varepsilon$ be the solution to

$$A^*p = \Sigma_{i=1}^k f_{i,\varepsilon} \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n_{A^*}} + \psi'(\bar{z})p = 0 \quad \text{on } \Gamma,$$

where $f_{i,\varepsilon} = \frac{(\bar{z}(x_i)-z_d(x_i))}{|B(x_i,\varepsilon)|}\chi_{B(x_i,\varepsilon)}$. We can easily verify that $w$ and $p_\varepsilon$ satisfy the Green formula

$$\Sigma_{i=1}^k \int_\Omega f_{i,\varepsilon} w_u = \int_\Gamma p_\varepsilon u.$$

The sequence $(p_\varepsilon)_\varepsilon$ is bounded in $W^{1,\tau}(\Omega)$ for all $\tau < N/(N-1)$, and when $\varepsilon$ tends to zero, $(p_\varepsilon)_\varepsilon$ tends to $p$ for the weak topology of $W^{1,\tau}(\Omega)$ for all $1 < \tau < N/(N-1)$. By passing to the limit when $\varepsilon$ tends to zero in the previous formula, we obtain

$$\Sigma_{i=1}^k (\bar{z}(x_i) - z_d(x_i)) w_u(x_i) = \int_\Gamma pu.$$

On the other hand, setting $J_2(z(f,u),u) = F_2(u)$, we have

$$F_2'(\bar{u})u = \Sigma_{i=1}^k (\bar{z}(x_i) - z_d(x_i)) w_u(x_i) + \beta \int_\Gamma s|\bar{u}|^{s-2}\bar{u}u.$$

Using the previous Green formula, we deduce

$$F_2'(\bar{u})u = \int_\Gamma pu + \beta \int_\Gamma |\bar{u}|^{s-2}\bar{u}u.$$

This completes the proof. ∎

## 3.6 Exercises

**Exercise 3.6.1**

We study the optimal control problem of section 1.2.1. We suppose that the electrical potential $\phi$ in a bounded domain $\Omega$ satisfies the elliptic equation

$$\begin{gathered} -\mathrm{div}(\sigma\nabla\phi) = 0 \qquad \text{in } \Omega, \\ -\sigma\frac{\partial\phi}{\partial n} = i \quad \text{on } \Gamma_a, \qquad -\sigma\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \Gamma_i, \qquad -\sigma\frac{\partial\phi}{\partial n} = f(\phi) \quad \text{on } \Gamma_c, \end{gathered} \tag{3.6.18}$$

where $\Gamma_a$ is a part of the boundary of $\Omega$ occupied by the anode, $\Gamma_c$ is a part of the boundary of $\Omega$ occupied by the cathode, $\Gamma_i$ is the rest of the boundary $\Gamma$, $\Gamma_i = \Gamma \setminus (\Gamma_a \cup \Gamma_c)$. The control function is the current density $i$, the constant $\sigma$ is the conductivity of the electrolyte, the function $f$ is supposed to be of class $C^1$, globally Lipschitz in $\mathbb{R}$, and such that $f(r) \geq c_1 > 0$ for all $r \in \mathbb{R}$. We study the control problem

$(P_3)$ $\qquad \inf\{J_3(\phi,i) \mid (\phi,i) \in H^1(\Omega) \times L^2(\Gamma_a),\ (\phi,i) \text{ satisfies (3.6.18)},\ a \leq i \leq b\},$

where

$$J_3(\phi,i) = \int_{\Gamma_c} (\phi - \bar{\phi})^2 + \beta \int_{\Gamma_a} i^2,$$

$a \in L^2(\Gamma_a)$ and $b \in L^2(\Gamma_a)$ are some bounds on the current $i$, and $\beta$ is a positive constant.

Prove that $(P_3)$ has at least one solution. Write the first order optimality condition for the solutions to $(P_3)$.

**Exercise 3.6.2**

We use Theorem 3.5.1 to study problems with pointwise controls. Let $x_1, \dots, x_k$ be given points in $\Omega$, and consider equation

$$-\Delta z = f + \Sigma_{i=1}^k u_i \delta_{x_i} \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \Gamma, \tag{3.6.19}$$

and the control problem

$$(P_4) \qquad \inf\{J_4(z,u) \mid (z,u) \in L^2(\Omega) \times \mathbb{R}^k,\ (z,u) \text{ satisfies (3.6.19)}\},$$

where

$$J_4(z,u) = \int_\Omega (z - z_d)^2 + \Sigma_{i=1}^k u_i^2,$$

$z_d \in L^2(\Omega)$ and $f \in L^2(\Omega)$.

Prove that $(P_4)$ admits a unique solution. Characterize this optimal solution by first order optimality conditions.

# Chapter 4

# Evolution equations

## 4.1 Introduction

The purpose of this chapter is to introduce some existence and regularity results for linear evolution equations. We consider equations of the form

$$z' = Az + f, \qquad z(0) = z_0. \tag{4.1.1}$$

In this setting $A$ is an unbounded operator in a reflexive Banach space $Z$, with domain $D(A)$ dense in $Z$. We suppose that $A$ is the infinitesimal generator of a strongly continuous semigroup on $Z$. This semigroup will be denoted by $(e^{tA})_{t\geq 0}$. In section 4.2, we study the weak solutions to equation (4.1.1) in $L^p(0,T;Z)$. For application to boundary control problems, we have to extend the notion of solutions to the case where $f \in L^p(0,T;(D(A^*))')$. In that case we study the solutions in $L^p(0,T;(D(A^*))')$ (see section 4.3). Before studying equation (4.1.1), let us now recall the Hille-Yosida theorem, which is very useful in applications.

**Theorem 4.1.1** (*[18, Chapter 1, Theorem 3.1], [8, Theorem 4.4.3]*) *An unbounded operator $A$ with domain $D(A)$ in a Banach space $Z$ is the infinitesimal generator of a strongly continuous semigroup of contractions if and only if the two following conditions hold:*
*(i) $A$ is a closed operator and $\overline{D(A)} = Z$,*
*(ii) for all $\lambda > 0$, $(\lambda I - A)$ is a bijective operator from $D(A)$ onto $Z$, $(\lambda I - A)^{-1}$ is a bounded operator on $Z$, and*

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{1}{\lambda}.$$

**Theorem 4.1.2** *Let $(e^{tA})_{t\geq 0}$ be a strongly continuous semigroup in $Z$ with generator $A$. Then there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|e^{tA}\|_{\mathcal{L}(Z)} \leq Me^{\omega t} \qquad \text{for all } t \geq 0.$$

*For all $c \in \mathbb{R}$, $A - cI$ is the infinitesimal generator of a strongly continuous semigroup on $Z$, denoted by $(e^{t(A-cI)})_{t\geq 0}$, which satisfies*

$$\|e^{t(A-cI)}\|_{\mathcal{L}(Z)} \leq Me^{(\omega-c)t} \qquad \text{for all } t \geq 0.$$

The first part of the theorem can be found in [2, Chapter 1, Corollary 2.1], or in [18, Chapter 1, Theorem 2.2]. The second statement follows from that $e^{t(A-cI)} = e^{-ct}e^{tA}$.

## 4.2 Weak solutions in $L^p(0,T;Z)$

We recall the notion of weak solution to equation

$$z' = Az + f, \qquad z(0) = z_0, \tag{4.2.2}$$

where $z_0 \in Z$ and $f \in L^p(0,T;Z)$, with $1 \le p < \infty$.

The adjoint operator of $A$ is an unbounded operator in $Z'$ defined by

$$D(A^*) = \{\zeta \in Z' \mid |\langle \zeta, Az\rangle| \le c\|z\|_Z \text{ for all } z \in D(A)\}$$

and

$$\langle A^*\zeta, z\rangle = \langle \zeta, Az\rangle \quad \text{for all } \zeta \in D(A^*) \text{ and all } z \in D(A).$$

We know that the domain of $A^*$ is dense in $Z'$.

**Definition 4.2.1** *A function $z \in L^p(0,T;Z)$, with $1 \le p < \infty$, is a weak solution to equation (4.2.2) if for every $\zeta \in D(A^*)$, $\langle z(\,\cdot\,), \zeta\rangle$ belongs to $W^{1,p}(0,T)$ and*

$$\frac{d}{dt}\langle z(t), \zeta\rangle = \langle z(t), A^*\zeta\rangle + \langle f(t), \zeta\rangle \quad \text{in } ]0,T[, \qquad \langle z(0), \zeta\rangle = \langle z_0, \zeta\rangle.$$

**Theorem 4.2.1** ([2, Chapter 1, Proposition 3.2]) *For every $z_0 \in Z$ and every $f \in L^p(0,T;Z)$, with $1 \le p < \infty$, equation (4.2.2) admits a unique solution $z(f, z_0) \in L^p(0,T;Z)$, this solution belongs to $C([0,T];Z)$ and is defined by*

$$z(t) = e^{tA}z_0 + \int_0^t e^{(t-s)A} f(s)\, ds.$$

*The mapping $(f, z_0) \mapsto z(f, z_0)$ is linear and continuous from $L^p(0,T;Z) \times Z$ into $C([0,T];Z)$.*

The following regularity result is very useful.

**Theorem 4.2.2** ([2, Chapter 1, Proposition 3.3]) *If $f \in C^1([0,T];Z)$ and $z_0 \in D(A)$, then the solution $z$ to equation (4.2.2) belongs to $C([0,T];D(A)) \cap C^1([0,T];Z)$.*

The adjoint equation for control problems associated with equation (4.2.2) will be of the form

$$-p' = A^*p + g, \qquad p(T) = p_T. \tag{4.2.3}$$

This equation can be studied with the following theorem.

**Theorem 4.2.3** ([18, Chapter 1, Corollary 10.6]) *The family of operator $((e^{tA})^*)_{t\ge 0}$ is a strongly continuous semigroup on $Z'$ with generator $A^*$. Since $e^{tA^*} = (e^{tA})^*$, $(e^{tA^*})_{t\ge 0}$ is called the adjoint semigroup of $(e^{tA})_{t\ge 0}$.*

Due to this theorem and to Theorem 4.2.1, with a change of time variable, it can be proved that if $p_T \in Z'$ and if $g \in L^p(0,T;Z')$, then equation (4.2.3) admits a unique weak solution which is defined by

$$p(t) = e^{(T-t)A^*}p_T + \int_t^T e^{(s-t)A^*} g(s)\, ds.$$

## 4.3 Weak solutions in $L^p(0,T;(D(A^*))')$

When the data of equation (4.2.2) are not regular, it is possible to extend the notion of solution by using duality arguments. It is the main objective of this section. For simplicity we suppose that $Z$ is a Hilbert space (the results can be extended to the case where $Z$ is a reflexive Banach space).

The imbeddings

$$D(A) \hookrightarrow Z \qquad \text{and} \qquad D(A^*) \hookrightarrow Z'$$

are continuous and with dense range. Thus we have

$$D(A) \hookrightarrow Z \hookrightarrow (D(A^*))'.$$

Since the operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup on $Z$, from Theorem 4.2.3 it follows that $(A^*, D(A^*))$ is the infinitesimal generator of a semigroup on $Z'$. Let us denote by $(S^*(t))_{t\geq 0}$ this semigroup.

Recall that the operator $(A_1^*, D(A_1^*))$ defined by

$$D(A_1^*) = D((A^*)^2), \qquad A_1^*z = A^*z \quad \text{for all } z \in D(A_1^*),$$

is the infinitesimal generator of a semigroup on $D(A^*)$ and that this semigroup $(S_1^*(t))_{t\geq 0}$ obeys $S_1^*(t)z = S^*(t)z$ for all $z \in D(A^*)$.

From Theorem 4.2.3 we deduce that $((S_1^*)^*(t))_{t\geq 0}$ is the semigroup on $(D(A^*))'$ generated by $(A_1^*)^*$. We are going to show that $(S_1^*)^*(t)$ is the continuous extension of $S(t)$ to $(D(A^*))'$. More precisely we have the following

**Theorem 4.3.1** *The adjoint of the unbounded operator $(A_1^*, D(A_1^*))$ in $D(A^*)$, is the unbounded operator $((A_1^*)^*, D((A_1^*)^*))$ on $(D(A^*))'$ defined by*

$$D((A_1^*)^*) = Z, \qquad \langle (A_1^*)^*z, y\rangle = \langle z, A_1^*y\rangle \qquad \text{for all } z \in Z \quad \text{and all } y \in D(A_1^*).$$

*Moreover, $(A_1^*)^*z = Az$ for all $z \in D(A)$. The semigroup $((S_1^*)^*(t))_{t\geq 0}$ is the semigroup on $(D(A^*))'$ generated by $(A_1^*)^*$ and*

$$(S_1^*)^*(t)z = S(t)z \qquad \text{for all } z \in Z \quad \text{and all } t \geq 0.$$

**Proof.** Let us show that $D((A_1^*)^*) = Z$. For all $z \in Z$ and all $y \in D(A_1^*)$, we have

$$|\langle z, A_1^*y\rangle_{(D(A^*))',D(A^*)}| = |\langle z, A_1^*y\rangle_{Z,Z'}| \leq \|z\|_Z \|y\|_{D(A^*)}.$$

Consequently

$$Z \subset D((A_1^*)^*). \tag{4.3.4}$$

Let us show the reverse inclusion. Let $z \in Z$ with $z \neq 0$, and let $y_z \in Z'$ be such that

$$\|z\|_Z = \sup_{y\in Z'} \frac{\langle z, y\rangle_{Z,Z'}}{\|y\|_{Z'}} = \frac{\langle z, y_z\rangle_{Z,Z'}}{\|y_z\|_{Z'}}.$$

We have

$$\|z\|_Z = \frac{\langle z, (I - A_1^*)(I - A_1^*)^{-1}y_z\rangle_{Z,Z'}}{\|y_z\|_{Z'}} = \frac{\langle z, (I - A_1^*)\zeta_z\rangle_{Z,Z'}}{\|\zeta_z\|_{D(A^*)}}$$

with $\zeta_z = (I - A_1^*)^{-1} y_z$. We can take

$$\zeta \longmapsto \|(I - A_1^*)^{-1}\zeta\|_{Z'}$$

as a norm on $D(A^*)$. For such a choice $(I - A_1^*)^{-1}$ is an isometry from $Z'$ to $(D(A^*))'$. Thus

$$\sup_{\zeta \in D(A^*)} \frac{\langle z, (I - A_1^*)\zeta\rangle_{Z,Z'}}{\|\zeta\|_{D(A^*)}} = \sup_{y \in Z'} \frac{\langle z, y\rangle_{Z,Z'}}{\|y\|_{Z'}}.$$

Since

$$\|z\|_{D((A_1^*)^*)} = \sup_{\zeta \in D(A^*)} \frac{\langle z, (I - A_1^*)\zeta\rangle_{Z,Z'}}{\|\zeta\|_{D(A^*)}},$$

one has

$$\|z\|_{D((A_1^*)^*)} \le \|z\|_Z. \tag{4.3.5}$$

The equality $D((A_1^*)^*) = Z$ follows from (4.3.4) and (4.3.5).

For all $z \in D(A)$, and all $y \in D(A_1^*)$, we have

$$\langle (A_1^*)^* z, y\rangle = \langle z, A_1^* y\rangle = \langle z, A^* y\rangle = \langle Az, y\rangle.$$

Thus, $(A_1^*)^* z = Az$ for all $z \in D(A)$.

From Theorem 4.2.3 we deduce that $((S_1^*)^*(t))_{t\ge 0}$ is the semigroup on $(D(A^*))'$ generated by $(A_1^*)^*$. To prove that $(S_1^*)^*(t)z = S(t)z$ for all $z \in Z$ and all $t \ge 0$, it is sufficient to observe that

$$\langle (S_1^*)^*(t)z, y\rangle = \langle z, S_1^*(t)y\rangle = \langle z, S^*(t)y\rangle = \langle S(t)z, y\rangle,$$

for all $z \in Z$, all $y \in D(A^*)$, and all $t \ge 0$.

**Remark.** Therefore we can extend the notion of solution for the equation (4.2.2) in the case where $x_0 \in (D(A^*))'$ and $f \in L^p(0, T; (D(A^*))')$, by considering the equation

$$z'(t) = (A_1^*)^* z(t) + f(t) \qquad \text{dans } (0, T), \qquad z(0) = z_0. \tag{4.3.6}$$

It is a usual abuse of notation to replace $A_1^*$ by $A^*$ and to write equation (4.3.6) in the form (cf [2, page 160])

$$z'(t) = (A^*)^* z(t) + f(t) \qquad \text{dans } (0, T), \qquad z(0) = z_0. \tag{4.3.7}$$

Since $(A_1^*)^*$ is an extension of the operator $A$, sometimes equations (4.3.6) or (4.3.7) are written in the form (4.2.2) even if $z_0 \in (D(A^*))'$ and $f \in L^p(0, T; (D(A^*))')$.

**Theorem 4.3.2** *For every $z_0 \in (D(A^*))'$ and every $f \in L^p(0, T; (D(A^*))')$, with $1 \le p < \infty$, equation (4.3.6) admits a unique solution $z(f, z_0) \in L^p(0, T; (D(A^*))')$, this solution belongs to $C([0, T]; (D(A^*))')$ and is defined by*

$$z(t) = e^{t(A_1^*)^*} z_0 + \int_0^t e^{(t-s)(A_1^*)^*} f(s)\, ds.$$

*The mapping $(f, z_0) \mapsto z(f, z_0)$ is linear and continuous from $L^p(0, T; (D(A^*))') \times (D(A^*))'$ into $C([0, T]; (D(A^*))')$.*

**Proof.** The theorem is a direct consequence of Theorems 4.3.1 and 4.2.1. ∎

For simplicity in the notation, we often write $e^{tA}$ in place of $e^{t(A_1^*)^*}$, or $A$ in place of $(A_1^*)^*$.

We often establish identities by using density arguments. The following regularity result will be useful to establish properties for weak solutions to equation (4.3.4).

**Theorem 4.3.3** *If $f$ belongs to $H^1(0,T;(D(A^*))')$ and $z_0$ belongs to $Z$, then the solution $z(f,z_0)$ to equation (4.3.4) belongs to $C^1([0,T];(D(A^*))') \cap C([0,T];Z)$.*

**Proof.** See [2, Chapter 3, Theorem 1.1].

# 4.4 Analytic semigroups

Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup on a Hilbert space $Z$. The resolvent set $\rho(A)$ is the set of all complex numbers $\lambda$ such that the operator $(\lambda I - A) \in \mathcal{L}(D(A), Z)$ has a bounded inverse in $Z$. Since $Z$ is a Hilbert space, and $A$ is a closed operator (because $A$ is the infinitesimal generator of a strongly continuous semigroup), we have the following characterization of $\rho(A)$:

$\lambda \in \rho(A)$ if and only if $R(\lambda, A) = (\lambda I - A)^{-1}$ exists and $\mathrm{Im}(\lambda I - A) = Z$.

The resolvent set of $A$ always contains a real half-line $[a,\infty)$ (see [2, Chapter 1, Proposition 2.2 and Corollary 2.2]).

## 4.4.1 Fractional powers of infinitesimal generators

We follows the lines of [5, Section 7.4]. Let $(e^{tA})_{t\geq 0}$ be a strongly continuous semigroup on $Z$ with infinitesimal generator $A$ satisfying

$$\|e^{tA}\|_{\mathcal{L}(Z)} \leq Me^{-ct} \qquad \text{for all } t \geq 0, \tag{4.4.8}$$

with $c > 0$. We can define fractional powers of $(-A)$ by

$$(-A)^{-\alpha} z = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{tA} z \, dt$$

for some $\alpha > 0$ and all $z \in Z$. The operator $(-A)^{-\alpha}$ obviously belongs to $\mathcal{L}(Z)$. For $0 \leq \alpha \leq 1$, we set

$$(-A)^{\alpha} = (-A)(-A)^{\alpha-1}.$$

The domain of $(-A)^\alpha = (-A)(-A)^{\alpha-1}$ is defined by $D((-A)^\alpha) = \{z \in Z \mid (-A)^{\alpha-1} z \in D(A)\}$.

## 4.4.2 Analytic semigroups

Different equivalent definitions of an analytic semigroup can be given.

**Definition 4.4.1** *Let* $(e^{tA})_{t\geq 0}$ *be a strongly continuous semigroup on* $Z$*, with infinitesimal generator* $A$*. The semigroup* $(e^{tA})_{t\geq 0}$ *is analytic if there exists a sector*

$$\Sigma_{a,\frac{\pi}{2}+\delta} = \{\lambda \in \mathbb{C} \mid |arg(\lambda - a)| < \frac{\pi}{2} + \delta\}$$

*with* $0 < \delta < \frac{\pi}{2}$*, such that* $\Sigma_{a,\frac{\pi}{2}+\delta} \subset \rho(A)$*, and*

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda - a|} \qquad \text{for all } \lambda \in \Sigma_{a,\frac{\pi}{2}+\delta}.$$

It can be proved that the semigroup $(e^{tA})_{t\geq 0}$ satisfies the conditions of definition 4.4.1 if and only if $(e^{tA})_{t\geq 0}$ can be extended to a function $\lambda \mapsto e^{\lambda A}$, where $e^{\lambda A} \in \mathcal{L}(Z)$, analytic in the sector

$$\Sigma_{a,\delta} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda - a)| < \delta\},$$

and strongly continuous in

$$\{\lambda \in \mathbb{C} \mid |\arg(\lambda - a)| \leq \delta\}.$$

Such a result can be find in a slightly different form in [2, Chapter 1, Theorem 2.1]. A theorem very useful for studying regularity of solutions to evolution equations is stated below.

**Theorem 4.4.1** *([18, Chapter 2, Theorem 6.13]) Let* $(e^{tA})_{t\geq 0}$ *be a continuous semigroup with infinitesimal generator* $A$*. Suppose that (4.4.8) is satisfied for some* $c > 0$*. Then* $e^{tA}Z \subset D((-A)^\alpha)$*,* $(-A)^\alpha e^{tA} \in \mathcal{L}(Z)$ *for all* $t > 0$*, and, for all* $0 \leq \alpha \leq 1$*, there exists* $k > 0$ *and* $C(\alpha)$ *such that*

$$\|(-A)^\alpha e^{tA}\|_{\mathcal{L}(Z)} \leq C(\alpha) t^{-\alpha} e^{-kt} \qquad \text{for all } t \geq 0. \tag{4.4.9}$$

A very simple criterion of analitycity is known in the case of real Hilbert spaces.

**Theorem 4.4.2** *([2, Chapter 1, Proposition 2.11]) If* $A$ *is a selfadjoint operator on a real Hilbert space* $Z$*, and if*

$$(Az, z) \leq 0 \qquad \text{for all } z \in D(A),$$

*then* $A$ *generates an analytic semigroup of contractions on* $Z$*.*

# Chapter 5

# Control of the heat equation

## 5.1 Introduction

We begin with distributed controls (section 5.2). Solutions of the heat equation are defined via the semigroup theory, but we explain how we can recover regularity results in $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ (Theorem 5.2.3). Since we study optimal control problems of evolution equations for the first time, we carefully explain how we can calculate the gradient, with respect to the control variable, of a functional depending of the state variable via the adjoint state method. The case of Neumann boundary controls is studied in section 5.3. Estimates in $W(0,T;H^1(\Omega),(H^1(\Omega))')$ are obtained by an approximation process, using the Neumann operator (see the proof of Theorem 5.3.6). Section 5.4 deals with Dirichlet boundary controls. In that case the solutions do not belong to $C([0,T];L^2(\Omega))$, but only to $C([0,T];H^{-1}(\Omega))$. We carefully study control problems for functionals involving observations in $C([0,T];H^{-1}(\Omega))$ (see section 5.4.2).

We only study problems without control constraints. But the extension of existence results and optimality conditions to problems with control constraints is straightforward.

## 5.2 Distributed control

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, with a boundary $\Gamma$ of class $C^2$. Let $T>0$, set $Q=\Omega\times(0,T)$ and $\Sigma=\Gamma\times(0,T)$. We consider the heat equation with a distributed control

$$\frac{\partial z}{\partial t}-\Delta z=f+\chi_\omega u \quad \text{in } Q, \qquad z=0 \quad \text{on } \Sigma, \qquad z(x,0)=z_0 \quad \text{in } \Omega. \tag{5.2.1}$$

The function $f$ is a given source of temperature, $\chi_\omega$ is the characteristic function of $\omega$, $\omega$ is an open subset of $\Omega$, and the function $u$ is a control variable. We consider the control problem

$$(P_1) \qquad \inf\{J_1(z,u)\mid (z,u)\in C([0,T];L^2(\Omega))\times L^2(0,T;L^2(\omega)),\ (z,u) \text{ satisfies } (5.2.1)\},$$

where

$$J_1(z,u)=\frac{1}{2}\int_Q(z-z_d)^2+\frac{1}{2}\int_\Omega(z(T)-z_d(T))^2+\frac{\beta}{2}\int_Q\chi_\omega u^2,$$

and $\beta>0$. In this section, we assume that $f\in L^2(Q)$ and that $z_d\in C([0,T];L^2(\Omega))$.

Before studying the above control problem, we first recall some results useful for the equation

$$\frac{\partial z}{\partial t} - \Delta z = \phi \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = z_0(x) \quad \text{in } \Omega. \tag{5.2.2}$$

**Theorem 5.2.1** *Set $Z = L^2(\Omega)$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $Au = \Delta u$. The operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $L^2(\Omega)$.*

**Proof.** The proof relies on the Hille-Yosida theorem and on regularity properties for solutions to the Laplace equation.

(i) Prove that $A$ is a closed operator. Let $(z_n)_n$ be a sequence in $D(A)$ converging to some $z$ in $L^2(\Omega)$. Suppose that $(\Delta z_n)_n$ converges to some $f$ in $L^2(\Omega)$. We necessarily have $\Delta z = f$ in the sense of distributions in $\Omega$. Due to Theorem 3.2.1, we have $\|z_n - z_m\|_{H^2(\Omega)} \leq C\|\Delta z_n - \Delta z_m\|_{L^2(\Omega)}$ . This means that $(z_n)_n$ is a Cauchy sequence in $H^2(\Omega)$. Hence $z \in H^2(\Omega) \cap H_0^1(\Omega)$. The first condition of Theorem 4.1.1 is satisfied.

(ii) Let $\lambda > 0$ and $f \in L^2(\Omega)$. It is clear that $(\lambda I - A)$ is invertible in $L^2(\Omega)$, and $(\lambda I - A)^{-1} f$ is the solution $z$ to the equation

$$\lambda z - \Delta z = f \qquad \text{in } \Omega, \qquad z = 0 \qquad \text{on } \Gamma.$$

We know that $z \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$\lambda \int_\Omega z^2 + \int_\Omega |\nabla z|^2 = \int_\Omega fz.$$

Thus we have

$$\|z\|_{L^2(\Omega)} \leq \frac{1}{\lambda}\|f\|_{L^2(\Omega)},$$

and the proof is complete. ∎

Equation (5.2.2) may be rewritten in the form of an evolution equation:

$$z' - Az = \phi \quad \text{in } ]0,T[, \qquad z(0) = z_0. \tag{5.2.3}$$

We can easily verify that $D(A^*) = D(A)$ and $A^* = A$, that is $A$ is selfadjoint. Recall that $z \in L^2(0,T; L^2(\Omega))$ is a weak solution to equation (5.2.3) if for all $\zeta \in H^2(\Omega) \cap H_0^1(\Omega)$ the mapping $t \mapsto \langle z(t), \zeta\rangle$ belongs to $H^1(0,T)$, $\langle z(0), \zeta\rangle = \langle z_0, \zeta\rangle$, and

$$\frac{d}{dt}\langle z(t), \zeta\rangle = \langle z(t), A\zeta\rangle + \langle \phi, \zeta\rangle.$$

**Theorem 5.2.2** *(i) For every $\phi \in L^2(Q)$ and every $z_0 \in L^2(\Omega)$, equation (5.2.2) admits a unique weak solution $z(\phi, z_0)$ in $L^2(0,T; L^2(\Omega))$, moreover the operator is linear and continuous from $L^2(Q) \times L^2(\Omega)$ into $W(0,T; H_0^1(\Omega), H^{-1}(\Omega))$.*

*(ii) The operator is also continuous from $L^2(Q) \times H_0^1(\Omega)$ into $L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$.*

**Comments.** Recall that

$$W(0,T; H_0^1(\Omega), H^{-1}(\Omega)) = \Big\{ z \in L^2(0,T; H_0^1(\Omega)) \mid \frac{dz}{dt} \in L^2(0,T; H^{-1}(\Omega)) \Big\}.$$

We say that $\frac{dz}{dt} \in L^2(0,T;H^{-1}(\Omega))$ if

$$\|\frac{d}{dt}\langle z(t),\zeta\rangle\|_{L^2(0,T)} \leq C\|\zeta\|_{H_0^1(\Omega)}, \qquad \text{for all } \ \zeta \in H_0^1(\Omega).$$

**Proof of Theorem 5.2.2.**

(i) Due to Theorem 5.2.1 and Theorem 4.2.1, we can prove that the operator $(\phi, z_0) \mapsto z(\phi, z_0)$ is continuous from $L^2(Q) \times L^2(\Omega)$ into $C([0,T];L^2(\Omega))$. To prove that the solution $z$ belongs to $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$, we can use a density argument. Suppose that $\phi \in C^1([0,T];Z)$ and that $z_0 \in D(A)$. Then $z$ belongs to $C([0,T];D(A)) \cap C^1([0,T];Z)$ (Theorem 4.2.2). In that case we can multiply equation (5.2.2) by $z$, and with integration by parts and a Green formula, we obtain

$$\int_\Omega |z(T)|^2 + 2\int_0^T\int_\Omega |\nabla z|^2 \leq 2\int_0^T\int_\Omega \phi z + \int_\Omega |z_0|^2$$
$$\leq 2\|\phi\|_{L^2(Q)}\|z\|_{L^2(Q)} + \|z_0\|^2_{L^2(\Omega)}.$$

With Poincaré's inequality $\|z\|_{L^2(\Omega)} \leq C_p\|\nabla z\|_{L^2(\Omega)}$, and Young' inequality we deduce

$$\int_0^T\int_\Omega |\nabla z|^2 \leq C_p\|\phi\|^2_{L^2(Q)} + \|z_0\|^2_{L^2(\Omega)}.$$

Therefore the operator $(\phi, z_0) \mapsto z(\phi, z_0)$ is continuous from $L^2(Q)\times L^2(\Omega)$ into $L^2(0,T;H_0^1(\Omega))$. Next, by using the equation and the regularity $z \in L^2(0,T;H_0^1(\Omega))$, we get

$$\frac{d}{dt}\langle z(t),\zeta\rangle = \langle z(t), A\zeta\rangle + \langle\phi,\zeta\rangle = -\int_\Omega \nabla z\nabla\zeta + \int_\Omega \phi\zeta.$$

From which it follows that

$$\|\frac{d}{dt}\langle z(t),\zeta\rangle\|_{L^2(0,T)} \leq \|z\|_{L^2(0,T;H_0^1(\Omega))}\|\zeta\|_{H_0^1(\Omega)} + \|\phi\|_{L^2(\Omega)}\|\zeta\|_{L^2(\Omega)}$$
$$\leq \max(C_p,1)\Big(\|z\|_{L^2(0,T;H_0^1(\Omega))} + \|\phi\|_{L^2(\Omega)}\Big)\|\zeta\|_{H_0^1(\Omega)},$$

for all $\zeta \in H_0^1(\Omega)$. Thus $\frac{dz}{dt}$ belongs to $L^2(0,T;H^{-1}(\Omega))$. The first part of the Theorem is proved.

(ii) The second regularity result is proved in [13], [7]. ∎

Since the solution $z(f,u,z_0)$ to equation (5.2.1) belongs to $C([0,T];L^2(\Omega))$ (when $u \in L^2(0,T;L^2(\omega))$), $J_1(z(f,u,z_0),u)$ is well defined and is finite for any $u \in L^2(0,T;L^2(\omega))$. We first assume that $(P_1)$ admits a unique solution (see Theorem 7.3.1, see also exercise 5.5.1). We set $F_1(u) = J_1(z(f,u,z_0),u)$, and, as in the case of optimal control for elliptic equations, the optimal solution $(z(f,\bar{u},z_0),\bar{u})$ to problem $(P_1)$ is characterized by the equation $F_1'(\bar{u}) = 0$. To compute the gradient of $F_1$ we have to consider adjoint equations of the form

$$-\frac{\partial p}{\partial t} - \Delta p = g \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad p(x,T) = p_T \quad \text{in } \Omega, \tag{5.2.4}$$

with $g \in L^2(Q)$ and $p_T \in L^2(\Omega)$. It is well known that the backward heat equation is not well posed. Due to the condition $p(x,T) = p_T$ equation (5.2.4) is a terminal value problem, which must be integrated backward in time. But equation (5.2.4) is not a backward heat equation because we have $-\frac{\partial p}{\partial t} - \Delta p = g$ and not $\frac{\partial p}{\partial t} - \Delta p = g$ (as in the case of the backward heat equation). Let us explain why the equation is well posed. If $p$ is a solution of (5.2.4) and if we set $w(t) = p(T-t)$, we can check, at least formally, that $w$ is the solution of

$$\frac{\partial w}{\partial t} - \Delta w = g(x, T-t) \quad \text{in } Q, \quad w = 0 \quad \text{on } \Sigma, \quad w(x,0) = p_T \quad \text{in } \Omega. \tag{5.2.5}$$

Since equation (5.2.5) is well posed, equation (5.2.4) is also well posed even if (5.2.4) is a *terminal value problem*. In particular equation (5.2.4) admits a unique weak solution in $L^2(0,T;L^2(\Omega))$, and this solution belongs to $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$. To obtain the expression of the gradient of $F_1$ we need a Green formula which is stated below.

**Theorem 5.2.3** *Suppose that $\phi \in L^2(Q)$, $g \in L^2(Q)$, and $p_T \in L^2(\Omega)$. Then the solution $z$ of equation*

$$\frac{\partial z}{\partial t} - \Delta z = \phi \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = 0 \quad \text{in } \Omega,$$

*and the solution $p$ of (5.2.4) satisfy the following formula*

$$\int_Q \phi\, p = \int_Q z\, g + \int_\Omega z(T) p_T. \tag{5.2.6}$$

**Proof.** If $p_T \in H_0^1(\Omega)$, due to Theorem 5.2.2, $z$ and $p$ belong to $L^2(0,T;D(A)))\cap H^1(0,T;L^2(\Omega))$. In that case, with the Green formula we have

$$\int_\Omega -\Delta z(t) p(t)\, dx = \int_\Omega -\Delta p(t) z(t)\, dx$$

for almost every $t \in [0,T]$, and

$$\int_0^T \int_\Omega \frac{\partial z}{\partial t} p = -\int_0^T \int_\Omega \frac{\partial p}{\partial t} z + \int_\Omega z(T) p_T.$$

Thus formula (5.2.6) is established in the case when $p_T \in H_0^1(\Omega)$ (Theorem 5.2.2 (ii)). If $(p_{Tn})_n$ is a sequence in $H_0^1(\Omega)$ converging to $p_T$ in $L^2(\Omega)$, due to Theorem 5.2.2, $(p_n)_n$, where $p_n$ is the solution to equation (5.2.4) corresponding to $p_{Tn}$, converges to $p$ in $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ when $n$ tends to infinity. Thus, in the case when $p_T \in L^2(\Omega)$, formula (5.2.6) can be deduced by passing to the limit in the formula satisfied by $p_n$.

**The gradient of $F_1$.** Let $(z(f,\bar{u},z_0),\bar{u}) = (\bar{z},\bar{u})$ be the solution to problem $(P_1)$. By a direct calculation we obtain

$$F_1(\bar{u} + \lambda u) - F_1(\bar{u}) = \frac{1}{2}\int_Q (z_\lambda - \bar{z})(z_\lambda + \bar{z} - 2z_d)$$

$$+\frac{1}{2}\int_\Omega (z_\lambda(T) - \bar{z}(T))(z_\lambda(T) + \bar{z}(T) - 2z_d(T)) + \frac{\beta}{2}\int_0^T \int_\omega (2\lambda u\bar{u} + \lambda^2 u^2),$$

where $z_\lambda = z(f, \bar{u} + \lambda u, z_0)$. The function $w_\lambda = z_\lambda - \bar{z}$ is the solution to the equation

$$\frac{\partial w}{\partial t} - \Delta w = \lambda \chi_\omega u \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad w(x,0) = 0 \quad \text{in } \Omega.$$

Due to Theorem 5.2.2 we have

$$\|w_\lambda\|_{W(0,T;H_0^1(\Omega),H^{-1}(\Omega))} \le C|\lambda|\|u\|_{L^2(0,T;L^2(\omega))}.$$

Thus the sequence $(z_\lambda)_\lambda$ converges to $\bar{z}$ in $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ when $\lambda$ tends to zero. Set $w_u = \frac{1}{\lambda} w_\lambda$, the function $w_u$ is the solution to the equation

$$\frac{\partial w}{\partial t} - \Delta w = \chi_\omega u \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad w(x,0) = 0 \quad \text{in } \Omega. \tag{5.2.7}$$

Dividing $F_1(\bar{u} + \lambda u) - F_1(\bar{u})$ by $\lambda$, and passing to the limit when $\lambda$ tends to zero, we obtain:

$$F_1'(\bar{u})u = \int_Q (\bar{z} - z_d) w_u + \int_\Omega (\bar{z}(T) - z_d(T)) w_u(T) + \int_0^T \int_\omega \beta u \bar{u}.$$

To derive the expression of $F_1'(\bar{u})$ we introduce the adjoint equation

$$-\frac{\partial p}{\partial t} - \Delta p = \bar{z} - z_d \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad p(x,T) = \bar{z}(T) - z_d(T) \quad \text{in } \Omega. \tag{5.2.8}$$

With formula (5.2.6) applied to $p$ and $w_u$ we have

$$\int_Q (\bar{z} - z_d) w_u + \int_\Omega (\bar{z}(T) - z_d(T)) w_u(T) = \int_0^T \int_\omega \chi_\omega u p.$$

Hence $F_1'(\bar{u}) = p|_{\omega\times(0,T)} + \beta \bar{u}$ , where $p$ is the solution to equation (5.2.8).

**Theorem 5.2.4** *(i) If $(\bar{z}, \bar{u})$ is the solution to $(P_1)$ then $\bar{u} = -\frac{1}{\beta} p|_{\omega\times(0,T)}$, where $p$ is the solution to equation (5.2.8).*

*(ii) Conversely, if a pair $(\tilde{z}, \tilde{p}) \in W(0,T;H_0^1(\Omega),H^{-1}(\Omega)) \times W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ obeys the system*

$$\begin{aligned} &\frac{\partial z}{\partial t} - \Delta z = f - \frac{1}{\beta}\chi_\omega p \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad z(x,0) = \bar{z}_0 \quad \text{in } \Omega, \\ -&\frac{\partial p}{\partial t} - \Delta p = \bar{z} - z_d \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad p(x,0) = \bar{z}(T) - z_d(T) \quad \text{in } \Omega, \end{aligned} \tag{5.2.9}$$

*then the pair $(\tilde{z}, -\frac{1}{\beta}\tilde{p})$ is the optimal solution to problem $(P_1)$.*

**Proof.** (i) The necessary optimality condition is already proved.

(ii) The sufficient optimality condition can be proved with Theorem 2.2.3. ∎

**Comments.** Before ending this section let us observe that equation (5.2.1) can be written in the form

$$z' = Az + f + Bu, \qquad z(0) = z_0,$$

where $B \in \mathcal{L}(L^2(\Gamma), L^2(\Omega))$ is defined by $Bu = \chi_\omega u$. Control problems governed by such evolutions equations are studied in Chapter 7.

## 5.3 Neumann boundary control

In this section, we study problems in which the control variable acts through a Neumann boundary condition

$$\frac{\partial z}{\partial t} - \Delta z = f \quad \text{in } Q, \quad \frac{\partial z}{\partial n} = u \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega. \tag{5.3.10}$$

**Theorem 5.3.1** *Set $Z = L^2(\Omega)$, $D(A) = \{z \in H^2(\Omega) \mid \frac{\partial z}{\partial n} = 0\}$, $Az = \Delta z$. The operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions in $L^2(\Omega)$.*

**Proof.** The proof still relies on the Hille-Yosida theorem. It is very similar to the proof of Theorem 5.2.1 and is left to the reader. ■

The operator $(A, D(A))$ is selfadjoint in $Z$. Equation

$$\frac{\partial z}{\partial t} - \Delta z = f \quad \text{in } Q, \quad \frac{\partial z}{\partial n} = 0 \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega, \tag{5.3.11}$$

may be written in the form

$$z' = Az + f, \qquad z(0) = z_0. \tag{5.3.12}$$

A function $z \in L^2(0,T;L^2(\Omega))$ is a weak solution to equation (5.3.12) if for all $\zeta \in D(A)$ the mapping $t \mapsto \langle z(t), \zeta\rangle$ belongs to $H^1(0,T)$, $\langle z(0), \zeta\rangle = \langle z_0, \zeta\rangle$, and

$$\frac{d}{dt}\int_\Omega z(t)\zeta = \langle z, A\zeta\rangle + \langle f, \zeta\rangle = \int_\Omega z(t)\Delta\zeta + \int_\Omega f(t)\zeta.$$

**Theorem 5.3.2** *For every $\phi \in L^2(Q)$ and every $z_0 \in L^2(\Omega)$, equation (5.3.11) admits a unique weak solution $z(\phi, z_0)$ in $L^2(0,T;L^2(\Omega))$, moreover the operator*

$$(\phi, z_0) \mapsto z(\phi, z_0)$$

*is linear and continuous from $L^2(Q) \times L^2(\Omega)$ into $W(0,T;H^1(\Omega),(H^1(\Omega))')$.*

Recall that

$$W(0,T;H^1(\Omega),(H^1(\Omega))') = \Big\{z \in L^2(0,T;H^1(\Omega)) \mid \frac{dz}{dt} \in L^2(0,T;(H^1(\Omega))')\Big\}.$$

**Proof.** The existence in $C([0,T];L^2(\Omega))$ follows from Theorem 5.3.1. The regularity in $W(0,T;H^1(\Omega),(H^1(\Omega))')$ can be proved as for Theorem 5.2.2. ■

Similarly we would like to say that a function $z \in L^2(0,T;L^2(\Omega))$ is a weak solution to equation (5.3.10) if for all $\zeta \in D(A)$ the mapping $t \mapsto \langle z(t), \zeta\rangle$ belongs to $H^1(0,T)$, $\langle z(0), \zeta\rangle = \langle z_0, \zeta\rangle$, and

$$\frac{d}{dt}\int_\Omega z(t)\zeta = \int_\Omega z(t)\Delta\zeta + \int_\Omega f\zeta + \int_\Gamma u\zeta.$$

Unfortunately the mapping $\zeta \mapsto \int_\Gamma u\zeta$ is not an element of $L^2(0,T;L^2(\Omega))$, it only belongs to $L^2(0,T;(H^1(\Omega))')$. One way to study equation (5.3.10) consists in using $(A_1^*)^*$ (see Chapter 4), the extension of $A$ to $(D(A^*))' = (D(A))'$ ($A$ is selfadjoint). We can directly improve this result in the following way. We set $\widehat{Z} = (H^1(\Omega))'$. We endow $(H^1(\Omega))'$ with the dual norm of the $H^1$-norm. We can check that the corresponding inner product in $(H^1(\Omega))'$ is defined by

$$\Big(z,\zeta\Big)_{(H^1(\Omega))'} = \int_\Omega z(-\Delta + I)^{-1}\zeta = \int_\Omega (-\Delta + I)^{-1}z\,\zeta,$$

where $(-\Delta + I)^{-1}\zeta$ is the function $w \in H^1(\Omega)$ obeying

$$-\Delta w + w = \zeta \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma.$$

We define the unbounded operator $\widehat{A}$ in $(H^1(\Omega))'$ by $D(\widehat{A}) = H^1(\Omega)$, and

$$\langle \widehat{A}z, \zeta\rangle_{(H^1(\Omega))', H^1(\Omega)} = -\int_\Omega \nabla z \nabla \zeta = \Big(\widehat{A}z, \zeta\Big)_{(H^1(\Omega))'}.$$

**Theorem 5.3.3** *The operator $(\widehat{A}, D(\widehat{A}))$ is the infinitesimal generator of a strongly continuous semigroup of contractions in $(H^1(\Omega))'$.*

**Proof.** The proof still relies on the Hille-Yosida theorem. It is more complicated than the previous ones. It is left to the reader. ∎

We write equation (5.3.10) in the form

$$z' = \widehat{A}z + f + \hat{u}, \qquad z(0) = z_0, \tag{5.3.13}$$

where $\hat{u} \in L^2(0,T;(H^1(\Omega))')$ is defined by $\hat{u} \mapsto \int_\Gamma u\zeta$ for all $\zeta \in H^1(\Omega)$. Due to Theorem 5.3.3 equation (5.3.13), or equivalently equation (5.3.10), admits a unique solution in $L^2(0,T;(H^1(\Omega))')$ and this solution belongs to $C([0,T];(H^1(\Omega))')$. To establish regularity properties of solutions to equation (5.3.10) we need to construct solutions by an approximation process.

**Approximation by regular solutions.**

Recall that the solution to equation

$$\Delta w - w = 0 \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n} = v \quad \text{on } \Gamma, \tag{5.3.14}$$

satisfies the estimate

$$\|w\|_{H^2(\Omega)} \le C\|v\|_{H^{1/2}(\Gamma)}. \tag{5.3.15}$$

Let $u$ be in $L^2(\Sigma)$ and let $(u_n)_n$ be a sequence in $C^1([0,T];H^{1/2}(\Gamma))$, converging to $u$ in $L^2(\Sigma)$. Denote by $w_n(t)$ the solution to equation (5.3.14) corresponding to $v = u_n(t)$. With estimate (5.3.15) we can prove that $w_n$ belongs to $C^1([0,T];H^2(\Omega))$ and that

$$\|w_n\|_{C^1([0,T];H^2(\Omega))} \le C\|u_n\|_{C^1([0,T];H^{1/2}(\Gamma))}.$$

Let $z_n$ be the solution to equation (5.3.10) corresponding to $(f, u_n, z_0)$. Then $y_n = z_n - w_n$ is the solution to

$$\frac{\partial y}{\partial t} - \Delta y = f - \frac{\partial w_n}{\partial t} + \Delta w_n \quad \text{in } Q, \quad \frac{\partial y}{\partial n} = 0 \quad \text{on } \Sigma, \quad y(x,0) = (z_0 - w_n(0))(x) \quad \text{in } \Omega.$$

Since $(z_0 - w_n(0)) \in L^2(\Omega)$ and $f - \frac{\partial w_n}{\partial t} - \Delta w_n$ belongs to $L^2(Q)$, $y_n$ and $z_n$ belong to $W(0,T;H^1(\Omega),(H^1(\Omega))')$. Thus, for every $t \in ]0,T]$, we have

$$\int_\Omega |z_n(t)|^2 + 2\int_0^t \int_\Omega |\nabla z_n|^2 = 2\int_0^t \int_\Omega f z_n + 2\int_0^t \int_\Gamma u_n z_n + \int_\Omega |z_0|^2.$$

We first get

$$\|y\|^2_{C([0,T];L^2(\Omega))} + 2\|\nabla y\|^2_{L^2(0,T;L^2(\Omega))} \leq 2\|f\|_{L^2(Q)}\|y\|_{L^2(Q)} + 2\|u\|_{L^2(\Sigma)}\|y\|_{L^2(\Sigma)} + \|y_0\|^2_{L^2(\Omega)}.$$

Thus with Young's inequality, we obtain

$$\|y\|_{C([0,T];L^2(\Omega))} + \|y\|_{L^2(0,T;H^1(\Omega))} \leq C\Big(\|f\|_{L^2(Q)} + \|u\|_{L^2(\Sigma)} + \|y_0\|_{L^2(\Omega)}\Big).$$

In the same way, we can prove

$$\|z_n - z_m\|_{C([0,T];L^2(\Omega))} + \|z_n - z_m\|_{L^2(0,T;H^1(\Omega))} \leq C\|u_n - u_m\|_{L^2(\Sigma)}.$$

Hence the sequence $(z_n)_n$ converges to some $z$ in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Due to Theorem 5.3.3, we can also prove that the sequence $(z_n)_n$ converges to the solution of equation (5.3.10) in $C([0,T];L^2(\Omega))$. By using the same arguments as for Theorem 5.2.2, we can next prove an estimate in $W(0,T;H^1(\Omega),(H^1(\Omega))')$. Therefore we have established the following theorem.

**Theorem 5.3.4** *For every $f \in L^2(Q)$, every $u \in L^2(\Sigma)$, and every $z_0 \in L^2(\Omega)$, equation (5.3.10) admits a unique weak solution $z(f,u,z_0)$ in $L^2(0,T;L^2(\Omega))$, moreover the operator*

$$(f,u,z_0) \mapsto z(f,u,z_0)$$

*is linear and continuous from $L^2(Q) \times L^2(\Sigma) \times L^2(\Omega)$ into $W(0,T;H^1(\Omega),(H^1(\Omega))')$.*

We now consider the control problem

$$(P_2) \qquad \inf\{J_2(z,u) \mid (z,u) \in C([0,T];L^2(\Omega)) \times L^2(0,T;L^2(\Gamma)),\ (z,u) \text{ satisfies (5.3.10)}\},$$

where

$$J_2(z,u) = \frac{1}{2}\int_Q (z - z_d)^2 + \frac{1}{2}\int_\Omega (z(T) - z_d(T))^2 + \frac{\beta}{2}\int_\Sigma u^2.$$

We assume that $f \in L^2(Q)$, $z_0 \in L^2(\Omega)$, and $z_d \in C([0,T];L^2(\Omega))$. Problem $(P_2)$ admits a unique solution $(\bar{z},\bar{u})$ (see exercise 5.5.2). The adjoint equation for $(P_2)$ is of the form

$$-\frac{\partial p}{\partial t} - \Delta p = g \quad \text{in } Q, \qquad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Sigma, \qquad p(x,T) = p_T \quad \text{in } \Omega. \tag{5.3.16}$$

**Theorem 5.3.5** *Suppose that $u \in L^2(\Sigma)$, $g \in L^2(Q)$, $p_T \in L^2(\Omega)$. Then the solution $z$ of equation*

$$\frac{\partial z}{\partial t} - \Delta z = 0 \quad \text{in } Q, \qquad \frac{\partial z}{\partial n} = u \quad \text{on } \Sigma, \qquad z(0) = 0 \quad \text{in } \Omega,$$

*and the solution $p$ of (5.3.16) satisfy the following formula*

$$\int_\Sigma up = \int_Q z\,g + \int_\Omega z(T)p_T. \tag{5.3.17}$$

**Proof.** We leave the reader adapt the proof of Theorem 5.2.3.

**Theorem 5.3.6** *If* $(\bar{z}, \bar{u})$ *is the solution to* $(P_2)$ *then* $\bar{u} = -\frac{1}{\beta} p|_\Sigma$, *where* $p$ *is the solution to the equation*

$$-\frac{\partial p}{\partial t} - \Delta p = \bar{z} - z_d \quad \text{in } Q, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Sigma, \quad p(x,T) = \bar{z}(T) - z_d(T) \quad \text{in } \Omega. \qquad (5.3.18)$$

*Conversely, if a pair* $(\tilde{z}, \tilde{p}) \in W(0,T; H^1(\Omega), (H^1(\Omega))') \times W(0,T; H^1(\Omega), (H^1(\Omega))')$ *obeys the system*

$$\begin{aligned} &\frac{\partial z}{\partial t} - \Delta z = f \quad \text{in } Q, \quad \frac{\partial z}{\partial n} = -\frac{1}{\beta} p|_\Sigma \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega, \\ -&\frac{\partial p}{\partial t} - \Delta p = z - z_d \quad \text{in } Q, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Sigma, \quad p(T) = z(T) - z_d(T) \quad \text{in } \Omega, \end{aligned} \qquad (5.3.19)$$

*then the pair* $(\tilde{z}, -\frac{1}{\beta}\tilde{p}|_\Sigma)$ *is the optimal solution to problem* $(P_2)$.

**Proof.** We set $F_2(u) = J_2(z(f,u,z_0),u)$. A calculation similar to that of the previous section leads to:

$$F_2'(\bar{u})u = \int_Q (\bar{z} - z_d) w_u + \int_\Omega (\bar{z}(T) - z_d(T)) w_u(T) + \int_\Sigma \beta u \bar{u},$$

where $w_u$ is the solution to the equation

$$\frac{\partial w}{\partial t} - \Delta w = 0 \quad \text{in } Q, \quad \frac{\partial w}{\partial n} = u \quad \text{on } \Sigma, \quad w(x,0) = 0 \quad \text{in } \Omega.$$

With formula (5.3.12) applied to $p$ and $w_u$ we obtain

$$\int_Q (\bar{z} - z_d) w_u + \int_\Omega (\bar{z}(T) - z_d(T)) w_u(T) = \int_\Sigma up.$$

Thus $F_2'(\bar{u}) = p|_\Sigma + \beta \bar{u}$. The end of the proof is similar to that of Theorem 5.2.4. ∎

## 5.4 Dirichlet boundary control

Now we want to control the heat equation by a Dirichlet boundary control, that is

$$\frac{\partial z}{\partial t} - \Delta z = f \quad \text{in } Q, \quad z = u \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega. \qquad (5.4.20)$$

Since we want to study equation (5.4.20) in the case when $u$ belongs to $L^2(\Sigma)$, we have to define the solution to equation (5.4.20) by the transposition method. We follow the method introduced in Chapter 2. We first study the equation

$$\frac{\partial z}{\partial t} - \Delta z = 0 \quad \text{in } Q, \quad z = u \quad \text{on } \Sigma, \quad z(x,0) = 0 \quad \text{in } \Omega. \qquad (5.4.21)$$

Suppose that $u$ is regular enough to define the solution to equation (5.4.21) in a classical sense. Let $y$ be the solution to

$$-\frac{\partial y}{\partial t} - \Delta y = \phi \quad \text{in } Q, \quad y = 0 \quad \text{on } \Sigma, \quad y(x,T) = 0 \quad \text{in } \Omega. \qquad (5.4.22)$$

With a Green formula (which is justified if $z$ and $y$ are regular enough), we can write

$$\int_Q z\phi = -\int_\Sigma u\frac{\partial y}{\partial n} = \langle u, \Lambda\phi\rangle_{L^2(\Sigma)},$$

where $\Lambda\phi = -\frac{\partial y}{\partial n}$. Due to Theorem 5.2.2 we know that the mapping

$$\phi \longmapsto y$$

is linear and continuous from $L^2(Q)$ into $L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))\cap H^1(0,T;L^2(\Omega))$. Thus the operator $\Lambda$ is linear and continuous from $L^2(Q)$ into $L^2(0,T;L^2(\Gamma))$, and $\Lambda^*$ is a linear and continuous operator from $L^2(0,T;L^2(\Gamma))$ into $L^2(Q)$. Since the identity $\int_Q z\phi = \langle u, \Lambda\phi\rangle_{L^2(\Sigma)} = \langle \Lambda^* u, \phi\rangle_{L^2(Q)}$ is satisfied for every $\phi \in L^2(Q)$, we have $z = \Lambda^* u$. For $u \in L^2(\Sigma)$, the solution $z_u$ to equation (5.4.21) is defined by $z_u = \Lambda^* u$. For equation (5.4.20) the definition of solution is stated below.

**Definition 5.4.1** *A function $z \in L^2(Q)$ is a solution to equation (5.4.20) if, and only if,*

$$\int_Q z\phi = \int_Q fy + \int_\Omega z_0 y(0) - \int_\Sigma u\frac{\partial y}{\partial n}$$

*for all $\phi \in L^2(Q)$, where $y$ is the solution to equation (5.4.22).*

Due to the continuity property of $\Lambda^*$, we have the following theorem.

**Theorem 5.4.1** *For every $f \in L^2(Q)$, every $u \in L^2(\Sigma)$, and every $z_0 \in L^2(\Omega)$, equation (5.4.20) admits a unique weak solution $z(f,u,z_0)$ in $L^2(0,T;L^2(\Omega))$, moreover the operator*

$$(f,u,z_0) \mapsto z(f,u,z_0)$$

*is linear and continuous from $L^2(Q)\times L^2(\Sigma)\times L^2(\Omega)$ into $L^2(Q)$.*

### 5.4.1 Observation in $L^2(Q)$

Thanks to Theorem 5.4.1 we can study the following control problem

$$(P_3) \qquad \inf\{J_3(z,u) \mid (z,u) \in L^2(0,T;L^2(\Omega))\times L^2(\Sigma),\ (z,u) \text{ satisfies } (5.4.20)\},$$

with

$$J_3(z,u) = \frac{1}{2}\int_Q (z-z_d)^2 + \frac{\beta}{2}\int_\Sigma u^2.$$

We here suppose that $z_d$ belongs to $L^2(Q)$. Contrary to the case of Neumann boundary controls, we cannot include an observation of $z(T)$ in $L^2(\Omega)$ in the definition of $(P_3)$. To write optimality conditions for $(P_3)$, we consider adjoint equations of the form

$$-\frac{\partial p}{\partial t} - \Delta p = g \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad p(x,T) = 0 \quad \text{in } \Omega. \tag{5.4.23}$$

**Theorem 5.4.2** *If $u \in L^2(\Sigma)$, then the solution $z$ of equation (5.4.21) and the solution $p$ of (5.4.23) satisfy the following formula*

$$\int_Q f\,p = \int_Q z\,g + \int_\Sigma u\frac{\partial p}{\partial n}. \tag{5.4.24}$$

**Proof.** The result directly follows from definition 5.4.1. ∎

**Theorem 5.4.3** *Assume that $f \in L^2(Q)$, $z_0 \in L^2(\Omega)$, and $z_d \in L^2(0,T;L^2(\Omega))$. Let $(\bar z, \bar u)$ be the unique solution to problem $(P_3)$. The optimal control $\bar u$ is defined by $\bar u = \frac{1}{\beta}\frac{\partial p}{\partial n}$, where $p$ is the solution to the equation*

$$-\frac{\partial p}{\partial t} - \Delta p = \bar z - z_d \quad in\ Q, \quad p = 0 \quad on\ \Sigma, \quad p(x,T) = 0 \quad in\ \Omega. \tag{5.4.25}$$

*This necessary optimality condition is also sufficient.*

**Proof.** We set $F_3(u) = J_3(z(f, z_0, u), u)$. Due to Theorem 5.4.2, we have

$$F_3'(\bar u)u = \int_Q (\bar z - z_d)w_u + \beta\int_\Sigma \bar u u = \int_\Sigma \Big(-\frac{\partial p}{\partial n} + \beta \bar u\Big)u.$$

The end of the proof is now classical. ∎

### 5.4.2 Observation in $C([0,T];H^{-1}(\Omega))$

Denote by $\|\cdot\|_{H^{-1}(\Omega)}$ the dual norm of the $H_0^1(\Omega)$-norm, that is the usual norm in $H^{-1}(\Omega)$:

$$\|f\|_{H^{-1}(\Omega)} = \sup_{z\in H_0^1(\Omega)}\frac{\langle f, z\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}}{\|z\|_{H_0^1(\Omega)}}.$$

Let $f$ be in $H^{-1}(\Omega)$ and denote by $(-\Delta)^{-1}f$ the solution to the equation

$$-\Delta z = f \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \Gamma.$$

**Theorem 5.4.4** *The mapping*

$$f \longmapsto |||f|||_{H^{-1}(\Omega)} = \langle f, (-\Delta)^{-1}f\rangle^{1/2}_{H^{-1}(\Omega)\times H_0^1(\Omega)}$$

*is a norm in $H^{-1}(\Omega)$ equivalent to the usual norm.*

**Proof.** We know that $(-\Delta)^{-1}$ is an isomorphism from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$. Thus $f \mapsto \|(-\Delta)^{-1}f\|_{H_0^1(\Omega)}$ is a norm in $H^{-1}(\Omega)$ equivalent to the usual norm. If $f \in H^{-1}(\Omega)$, multiplying the equation $-\Delta((-\Delta)^{-1}f) = f$ by $(-\Delta)^{-1}f$, with a Green formula, we have

$$\int_\Omega |\nabla((-\Delta)^{-1}f)|^2 = \langle f, (-\Delta)^{-1}f\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \le \|(-\Delta)^{-1}f\|_{H_0^1(\Omega)}\|f\|_{H^{-1}(\Omega)}.$$

Since the norm $f \mapsto \|(-\Delta)^{-1}f\|_{H_0^1(\Omega)}$ is equivalent to the norm in $H^{-1}(\Omega)$, we obtain

$$c_1\|f\|^2_{H^{-1}(\Omega)} \le \langle f, (-\Delta)^{-1}f\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \le c_2\|f\|^2_{H^{-1}(\Omega)}.$$

The proof is complete. ∎

**Theorem 5.4.5** *(i) Let $z(f,u,z_0)$ be the solution to equation (5.4.20). The operator*

$$(f,u,z_0)\mapsto z(f,u,z_0),$$

*is linear and continuous from $L^2(Q)\times L^2(\Sigma)\times L^2(\Omega)$ into $C([0,T];H^{-1}(\Omega))$.*
*(ii) If $u\in L^2(\Sigma)$, and if $p_T\in H_0^1(\Omega)$, then the solution $z$ of equation (5.4.21) and the solution $p$ of*

$$-\frac{\partial p}{\partial t}-\Delta p=0 \quad \textit{in } Q,\quad p=0\quad \textit{on } \Sigma,\quad p(x,T)=p_T\quad \textit{in } \Omega,$$

*satisfy the following formula*

$$\langle z(T),p_T\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}=-\int_\Sigma u\frac{\partial p}{\partial n}. \tag{5.4.26}$$

**Proof.** (i) We only need to prove the regularity result for the solution $z$ of equation (5.4.21). For every $\varphi\in H_0^1(\Omega)$ and every $\tau\in]0,T]$, consider the solution $y$ to equation

$$-\frac{\partial y}{\partial t}-\Delta y=0\quad \text{in } Q,\quad y=0\quad \text{on } \Sigma,\quad y(\tau)=\varphi\quad \text{in } \Omega.$$

Due to Theorem 5.2.2, we have

$$\|y\|_{L^2(0,\tau;H^2(\Omega)\cap H_0^1(\Omega))}\le c\|\varphi\|_{H_0^1(\Omega)},$$

and the constant $c$ is independent of $\tau$. Let $(u_n)_n\subset L^2(\Sigma)$ a sequence of regular functions satisfying the compatibility condition $u_n(x,0)=0$, and converging to $u$ in $L^2(\Sigma)$. Denote by $z_n$ the solution to (5.4.21) corresponding to $u_n$. Since $z_n$ is regular, it satisfies the formula

$$\int_\Omega z_n(\tau)\varphi=-\int_{\Gamma\times(0,\tau)}u_n\frac{\partial y}{\partial n}.$$

Thus we have

$$\|z_n(\tau)\|_{H^{-1}(\Omega)}=\sup_{\|\varphi\|_{H_0^1(\Omega)}=1}\Big|\int_{\Gamma\times(0,\tau)}u_n\frac{\partial y}{\partial n}\Big|\le c\|u_n\|_{L^2(\Sigma)},$$

where the constant $c$ is independent of $\tau$. From this estimate it follows that

$$\|z_n-z_m\|_{C([0,T];H^{-1}(\Omega))}=\|z_n-z_m\|_{L^\infty(0,T;H^{-1}(\Omega))}\le c\|u_n-u_m\|_{L^2(\Sigma)}.$$

Therefore the sequence $(z_n)_n$ converges to some $\tilde z$ in $C([0,T];H^{-1}(\Omega))$. Due to Theorem 5.4.1, the sequence $(z_n)_n$ converges to the solution $z$ of equation (5.4.21). We finally have $z=\tilde z\in C([0,T];H^{-1}(\Omega))$.

(ii) Formula (5.4.26) can be established for regular data, and next deduced in the general case from density arguments. ∎

Now we are in position to study the control problem

$$(P_4)\qquad \inf\{J_4(z,u)\mid (z,u)\in L^2(0,T;L^2(\Omega))\times L^2(\Sigma),\ (z,u)\text{ satisfies (5.4.20)}\},$$

with

$$J_4(z,u)=\frac{1}{2}\||z(T)-z_T\||^2_{H^{-1}(\Omega)}+\frac{\beta}{2}\int_\Sigma u^2.$$

The proof of existence and uniqueness of solution to problem $(P_4)$ is standard (see exercise 5.5.3).

**Theorem 5.4.6** *Assume that* $f \in L^2(Q)$, $z_0 \in L^2(\Omega)$, *and* $z_d \in L^2(0,T;L^2(\Omega))$. *Let* $(\bar{z},\bar{u})$ *be the unique solution to problem* $(P_4)$. *The optimal control* $u$ *is defined by* $u = \frac{1}{\beta}\frac{\partial p}{\partial n}$, *where* $p$ *is the solution to the equation*

$$-\frac{\partial p}{\partial t} - \Delta p = 0 \quad in\ Q, \quad p = 0 \quad on\ \Sigma, \quad p(x,T) = (-\Delta)^{-1}(\bar{z}(T) - z_T) \quad in\ \Omega. \tag{5.4.27}$$

**Proof.** We set $F_4(u) = J_4(z(f,z_0,u),u)$. If $w_u$ is the solution to equation 5.4.21, and $p$ the solution to equation 5.4.27, with the formula stated in Theorem 5.4.5(ii), we have

$$F_4(\bar{u})u = \langle w_u(T), (-\Delta)^{-1}(\bar{z}(T) - z_T)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} + \beta\int_\Sigma \bar{u}u.$$

$$= \int_\Sigma \Big( -\frac{\partial p}{\partial n} + \beta\bar{u}\Big)u.$$

The proof is complete. ∎

## 5.5 Exercises

### Exercise 5.5.1

The notation are the ones of section 5.2. Let $(u_n)_n$ be a sequence in $L^2(0,T;L^2(\omega))$, converging to $u$ for the weak topology of $L^2(0,T;L^2(\omega))$. Let $z_n$ be the solution to equation (5.2.1) corresponding to $u_n$, and $z_u$ be the solution to equation (5.2.1) corresponding to $u$. Prove that $(z_n(T))_n$ converges to $z_u(T)$ for the weak topology of $L^2(\Omega)$. Prove that the control problem $(P_1)$ admits a unique solution.

### Exercise 5.5.2

Prove that the control problem $(P_2)$ of section 5.3 admits a unique solution.

### Exercise 5.5.3

The notation are the ones of section 5.4. Let $(u_n)_n$ be a sequence in $L^2(\Sigma)$, converging to $u$ for the weak topology of $L^2(\Sigma)$. Let $z_n$ be the solution to equation (5.4.20) corresponding to $u_n$, and $z_u$ be the solution to equation (5.4.20) corresponding to $u$. Prove that $(z_n(T))_n$ converges to $z_u(T)$ for the weak topology of $H^{-1}(\Omega)$. Prove that the control problem $(P_4)$ admits a unique solution.

### Exercise 5.5.4

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, with a boundary $\Gamma$ of class $C^2$. Let $T > 0$, set $Q = \Omega\times(0,T)$ and $\Sigma = \Gamma \times (0,T)$. We consider a convection-diffusion equation with a distributed control

$$\frac{\partial z}{\partial t} - \Delta z + \vec{V}\cdot\nabla z = f + \chi_\omega u \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega. \tag{5.5.28}$$

The function $f$ belongs to $L^2(Q)$, $\chi_\omega$ is the characteristic function of $\omega$, $\omega$ is an open subset of $\Omega$, and the function $u$ is a control variable. We suppose that $\vec{V} \in (L^\infty(Q))^N$. We want to study the control problem

$$(P_5) \qquad \inf\{J_5(z,u) \mid (z,u) \in C([0,T];L^2(\Omega)) \times L^2(0,T;L^2(\omega)),\ (z,u) \text{ satisfies (5.5.28)}\},$$

where

$$J_5(z,u) = \frac{1}{2}\int_Q (z - z_d)^2 + \frac{1}{2}\int_\Omega (z(T) - z_d(T))^2 + \frac{\beta}{2}\int_Q \chi_\omega u^2,$$

and $\beta > 0$. We assume that $z_d \in C([0,T];L^2(\Omega))$.

We first study equation (5.5.28) by a fixed point method. For that we need a regularity for the heat equation that we state below.

**Regularity result.** *For any $1 < q < \infty$, there exists a constant $C(q)$ such that the solution $z$ to the heat equation*

$$\frac{\partial z}{\partial t} - \Delta z = f \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = 0 \quad \text{in } \Omega,$$

*satisfies*

$$\|z\|_{C([0,T];L^2(\Omega))} + \|z\|_{L^2(0,T;H_0^1(\Omega))} \le C(q)\|f\|_{L^q(0,T;L^2(\Omega))} \quad \text{for all} \quad f \in L^q(0,T;L^2(\Omega)).$$

1 - Now we choose $1 < q < 2$. Let $r$ be defined by $\frac{1}{2} + \frac{1}{r} = \frac{1}{q}$, and $\bar{t} \in ]0,T]$ such that $C(q)\bar{t}^{1/r}\|\vec{V}\|_{(L^\infty(Q))^N} \le \frac{1}{2}$. Let $\phi \in C([0,\bar{t}];L^2(\Omega)) \cap L^2(0,\bar{t};H_0^1(\Omega))$, and denote by $z_\phi$ the solution to equation

$$\frac{\partial z}{\partial t} - \Delta z = f + \chi_\omega u - \vec{V}\cdot\nabla\phi \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega. \tag{5.5.29}$$

Prove that the mapping

$$\phi \longmapsto z_\phi$$

is a contraction in $C([0,\bar{t}];L^2(\Omega)) \cap L^2(0,\bar{t};H_0^1(\Omega))$.

2 - Let $\hat{z}$ be the solution in $C([0,\bar{t}];L^2(\Omega)) \cap L^2(0,\bar{t};H_0^1(\Omega))$ to equation

$$\frac{\partial z}{\partial t} - \Delta z + \vec{V}\cdot\nabla z = f + \chi_\omega u \quad \text{in } \Omega\times(0,\bar{t}), \quad z = 0 \quad \text{on } \Gamma\times(0,\bar{t}), \quad z(x,0) = z_0 \quad \text{in } \Omega.$$

The existence of $\hat{z}$ follows from the previous question. Let $\phi \in C([0,2\bar{t}];L^2(\Omega)) \cap L^2(0,2\bar{t};H_0^1(\Omega))$ such that $\phi = \hat{z}$ on $[0,\bar{t}]$, and denote by $z_\phi$ the solution to equation

$$\frac{\partial z}{\partial t} - \Delta z = f + \chi_\omega u - \vec{V}\cdot\nabla\phi \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega. \tag{5.5.30}$$

Prove that the mapping

$$\phi \longmapsto z_\phi$$

is a contraction in the metric space

$$\{\phi \in C([0,2\bar{t}];L^2(\Omega)) \cap L^2(0,2\bar{t};H_0^1(\Omega)) \mid \phi = \hat{z} \text{ on } [0,\bar{t}]\},$$

for the metric corresponding to the norm of the space $C([0,2\bar{t}];L^2(\Omega))\cap L^2(0,2\bar{t};H_0^1(\Omega))$.

3 - Prove that equation (5.5.28) admits a unique solution in $C([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$, and that this solution obeys

$$\|z\|_{C([0,T];L^2(\Omega))}+\|z\|_{L^2(0,T;H_0^1(\Omega))}\le C(\|f\|_{L^2(Q)}+\|u\|_{L^2(\omega\times(0,T))}+\|z_0\|_{L^2(\Omega)}).$$

4 - Prove that the control problem $(P_5)$ admits a unique solution. Write first order optimality conditions.

**Exercise 5.5.5**

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, with a boundary $\Gamma$ of class $C^2$. Let $T>0$, set $Q=\Omega\times(0,T)$ and $\Sigma=\Gamma\times(0,T)$. We consider the heat equation with a control in a coefficient

$$\left\{\begin{array}{l}\frac{\partial y}{\partial t}-\Delta y+u\,y=f \text{ in } Q,\ T>0,\\ y=0 \text{ on } \Gamma\times]0,T[,\quad y(x,0)=y_0(x) \text{ in } \Omega,\end{array}\right. \tag{5.5.31}$$

avec $f\in L^2(Q)$, $y_0\in L^2(\Omega)$ et

$$u\in U_{ad}=\{u\in L^\infty(Q)\mid 0\le u(x,t)\le M \text{ a.e. in } Q\},\quad M>0.$$

We want to study the control problem

$$(P_6)\qquad \inf\{J_6(y)\mid u\in U_{ad},\ (y,u) \text{ satisfies (5.5.31)}\}$$

avec $J_6(y)=\int_\Omega|y(x,T)-y_d(x)|^2dx$, $y_d$ is a given function in $L^2(\Omega)$.

1 - Prove that equation (5.5.31) admits a unique solution $y_u$ in $C([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ (the fixed point method of the previous exercise can be adapted to deal with equation (5.5.31)). Prove that this solution belongs to $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$.

2 - Let $(u_n)_n\subset U_{ad}$ be a sequence converging to $u$ for the weak star topology of $L^\infty(Q)$. Prove that $(y_{u_n})_n$ converges to $y_u$ for the weak topology of $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$. Prove that $(P_6)$ admits solutions.

3 - Let $u$ and $v$ be two functions in $U_{ad}$. Set $z_\lambda=(y_{u+\lambda v}-y_u)/\lambda$. Prove that $(z_\lambda)_\lambda$ converges, when $\lambda$ tends to zero, to the solution $z_{u,v}$ of the equation

$$\left\{\begin{array}{l}\frac{\partial z}{\partial t}-\Delta z+vy_u+uz=0 \text{ in } \Omega\times]0,T[,\\ z=0 \text{ on } \Gamma\times]0,T[,\quad z(x,0)=0 \text{ in } \Omega.\end{array}\right. \tag{5.5.32}$$

4 - Let $(y_u,u)$ be a solution to problem $(P_6)$. Write optimality conditions for $(y_u,u)$ in function of $z_{u,v-u}$ (for $v\in U_{ad}$). Next, write this optimality condition by introducing the adjoint state associated with $(y_u,u)$.

# Chapter 6

# Control of the wave equation

## 6.1 Introduction

We first begin by problems with a distributed control. We study the wave equation via the semigroup theory with initial data in $H_0^1(\Omega) \times L^2(\Omega)$ (section 6.2.1), and in $L^2(\Omega) \times H^{-1}(\Omega)$ (section 6.2.2). These results are next used to derive optimality conditions in the case of functionals involving observations of the derivative of the state (Theorem 6.3.1). The case of Neumann boundary controls is briefly presented in section 6.4. To obtain fine regularity results in the case of Dirichlet boundary controls, we need a trace regularity result for solutions to the wave equation with homogeneous boundary conditions (Theorem 6.5.1). Equations with nonhomogeneous Dirichlet boundary conditions is studied by the transposition method (Theorem 6.6.1). We derive optimality conditions for functionals involving observations in $C([0,T]; H^{-1}(\Omega))$ (Theorem 6.6.2).

The notation $\Omega$, $\Gamma$, $T$, $Q$, $\Sigma$, as well as the assumptions on $\Omega$ and $\Gamma$, are the ones of the previous chapter.

## 6.2 Existence and regularity results

### 6.2.1 The wave equation in $H_0^1(\Omega) \times L^2(\Omega)$

To study equation

$$\frac{\partial^2 z}{\partial t^2} - \Delta z = f \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = z_0 \text{ and } \frac{\partial z}{\partial t}(x,0) = z_1 \quad \text{in } \Omega, \qquad (6.2.1)$$

with $(z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $f \in L^2(Q)$, we transform the equation in a first order evolution equation. Setting $y = (z, \frac{dz}{dt})$, equation (6.2.1) may be written in the form

$$\frac{dy}{dt} = Ay + F, \quad y(0) = y_0,$$

with

$$Ay = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \Delta y_1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad \text{and} \quad y_0 = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}.$$

Set $Y = H_0^1(\Omega) \times L^2(\Omega)$. The domain of $A$ in $Y$ is $D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

**Theorem 6.2.1** *The operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $Y$.*

**Proof.** The theorem relies on the Hille-Yosida theorem.
(i) The domain $D(A)$ is dense in $Y$. Prove that $A$ is a closed operator. Let $(y_n)_n$ be a sequence converging to $y = (y_1, y_2)$ in $H_0^1(\Omega) \times L^2(\Omega)$, and such that $(Ay_n)_n = (y_{2,n}, \Delta y_{1,n})_n$ converges to $(f, g)$ in $H_0^1(\Omega) \times L^2(\Omega)$. We have $y_2 = f$, and $\Delta y_1 = g$ because $(\Delta y_{1,n})_n$ converges to $\Delta y_1$ in the sense of distributions in $\Omega$. Due to Theorem 3.2.1, we have $\|y_{1,n} - y_{1,m}\|_{H^2(\Omega)} \leq C\|\Delta y_{1,n} - \Delta y_{1,m}\|_{L^2(\Omega)}$. Thus $(y_{1,n})_n$ is a Cauchy sequence in $H^2(\Omega) \cap H_0^1(\Omega)$. Hence $y_1 \in H^2(\Omega) \cap H_0^1(\Omega)$. The first condition of Theorem 4.1.1 is satisfied.
(ii) For $\lambda > 0$, $f \in H_0^1(\Omega)$, $g \in L^2(\Omega)$, consider the equation

$$\lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

that is

$$\begin{array}{rcll} \lambda y_1 - y_2 & = & f & \text{in } \Omega, \\ \lambda y_2 - \Delta y_1 & = & g & \text{in } \Omega. \end{array} \tag{6.2.2}$$

We have

$$\lambda^2 y_1 - \Delta y_1 = \lambda f + g \qquad \text{in } \Omega.$$

This equation admits a unique solution in $H^2(\Omega) \cap H_0^1(\Omega)$. Thus the system (6.2.2) admits a unique solution $y \in D(A)$. From the equation $\lambda y_2 - \Delta y_1 = g$, we deduce

$$\lambda \int_\Omega y_2^2 + \int_\Omega \nabla y_1 \nabla y_2 = \int_\Omega g y_2.$$

Replacing $y_2$ by $\lambda y_1 - f$ in the second term, we obtain

$$\lambda \int_\Omega y_2^2 + \lambda \int_\Omega |\nabla y_1|^2 = \int_\Omega g y_2 + \int_\Omega \nabla y_1 \nabla f$$

$$\leq \Big( \int_\Omega y_2^2 + \int_\Omega |\nabla y_1|^2 \Big)^{1/2} \Big( \int_\Omega g^2 + \int_\Omega |\nabla f|^2 \Big)^{1/2},$$

and

$$\lambda \Big( \int_\Omega y_2^2 + \int_\Omega |\nabla y_1|^2 \Big)^{1/2} \leq \Big( \int_\Omega g^2 + \int_\Omega |\nabla f|^2 \Big)^{1/2}.$$

We can choose $y \mapsto \Big( \int_\Omega y_2^2 + \int_\Omega |\nabla y_1|^2 \Big)^{1/2}$ as a norm on $Y$ and the proof is complete. ∎

**Theorem 6.2.2** *For every $f \in L^2(Q)$, every $z_0 \in H_0^1(\Omega)$, every $z_1 \in L^2(\Omega)$, equation (6.2.1) admits a unique weak solution $z(f, z_0, z_1)$, moreover the operator*

$$(f, z_0, z_1) \mapsto z(f, z_0, z_1)$$

*is linear and continuous from $L^2(Q) \times H_0^1(\Omega) \times L^2(\Omega)$ into $C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$.*

**Proof.** The theorem is a direct consequence of Theorem 4.2.1 and Theorem 6.2.1. ∎

### 6.2.2 The wave equation in $L^2(\Omega) \times H^{-1}(\Omega)$

We study equation (6.2.1) when $(z_0, z_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $f \in L^2(0,T; H^{-1}(\Omega))$. In that case we set $\widehat{Y} = L^2(\Omega) \times H^{-1}(\Omega)$, $D(\widehat{A}) = H_0^1(\Omega) \times L^2(\Omega)$ and

$$\widehat{A}y = \widehat{A}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \tilde{A}y_1 \end{pmatrix},$$

where $(\tilde{A}y_1, \zeta)_{H^{-1}(\Omega)} = -\int_\Omega \nabla y_1 \cdot \nabla(-\Delta)^{-1}\zeta$ and $(-\Delta)^{-1}\zeta$ is the solution $w$ of the equation

$$w \in H_0^1(\Omega), \qquad -\Delta w = \zeta \quad \text{in } \Omega.$$

We have the same kind of result as above.

**Theorem 6.2.3** *The operator $(\widehat{A}, D(\widehat{A}))$ is the infinitesimal generator of a semigroup of contractions on $\widehat{Y}$.*

**Proof.** The theorem still relies on the Hille-Yosida theorem.
(i) The domain $D(\widehat{A})$ is dense in $\widehat{Y}$. As for the proof of Theorem 6.2.1, we prove that $(\widehat{A}, D(\widehat{A}))$ is a closed operator. The first condition of Theorem 4.1.1 is satisfied.
(ii) For $\lambda > 0$, $f \in L^2(\Omega)$, $g \in H^{-1}(\Omega)$, consider the system

$$\begin{array}{rcll} \lambda y_1 - y_2 & = & f & \text{in } \Omega, \\ \lambda y_2 - \Delta y_1 & = & g & \text{in } \Omega. \end{array} \tag{6.2.3}$$

The equation

$$\lambda^2 y_1 - \Delta y_1 = \lambda f + g \qquad \text{in } \Omega,$$

admits a unique solution in $H_0^1(\Omega)$. Thus the system (6.2.3) admits a unique solution $y \in D(A)$. The obtention of the estimate is more delicate than previously. We compose the two members of the first equation by $(-\Delta)^{-1}$, the inverse of the Laplace operator with homogeneous boundary conditions. We have $\lambda(-\Delta)^{-1}y_1 - (-\Delta)^{-1}y_2 = (-\Delta)^{-1}f$ , and we choose $(-\Delta)^{-1}y_2$ as a test function for the second equation:

$$\lambda\langle y_2, (-\Delta)^{-1}y_2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \lambda\langle -\Delta y_1, (-\Delta)^{-1}y_1\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$$

$$= \langle g, (-\Delta)^{-1}y_2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle -\Delta y_1, (-\Delta)^{-1}f\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

Recall that the mapping

$$f \longmapsto |||f|||_{H^{-1}(\Omega)} = \langle f, (-\Delta)^{-1}f\rangle^{1/2}_{H^{-1}(\Omega)\times H_0^1(\Omega)}$$

is a norm in $H^{-1}(\Omega)$ equivalent to the usual norm (Theorem 5.4.4). This norm is associated with the scalar product

$$(f,g) \longmapsto \langle f, (-\Delta)^{-1}g\rangle^{1/2}_{H^{-1}(\Omega)\times H_0^1(\Omega)}.$$

Thus we have

$$\langle g, (-\Delta)^{-1}y_2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \le |||g|||_{H^{-1}(\Omega)}|||y_2|||_{H^{-1}(\Omega)}.$$

We can also verify that

$$\langle -\Delta y_1, (-\Delta)^{-1} y_1 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega y_1^2,$$

and

$$\langle -\Delta y_1, (-\Delta)^{-1} f \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega y_1 f.$$

Collecting together these relations we obtain

$$\lambda |||y_2|||^2_{H^{-1}(\Omega)} + \lambda \|y_1\|^2_{L^2(\Omega)} \le |||g|||_{H^{-1}(\Omega)} |||y_2|||_{H^{-1}(\Omega)} + \|f\|_{L^2(\Omega)} \|y_1\|_{L^2(\Omega)}.$$

We can choose $y \mapsto \left( \|y_1\|^2_{L^2(\Omega)} + |||y_2|||^2_{H^{-1}(\Omega)} \right)^{1/2}$ as a norm on $Y$ and the proof is complete. ∎

**Theorem 6.2.4** *For every $f \in L^2(0, T; H^{-1}(\Omega))$, every $z_0 \in L^2(\Omega)$, every $z_1 \in H^{-1}(\Omega)$, equation (6.2.1) admits a unique weak solution $z(f, z_0, z_1)$, moreover the operator*

$$(f, z_0, z_1) \mapsto z(f, z_0, z_1)$$

*is linear and continuous from $L^2(0, T; H^{-1}(\Omega)) \times L^2(\Omega) \times H^{-1}(\Omega)$ into $C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$.*

**Proof.** The theorem is a direct consequence of Theorem 4.2.3 and Theorem 6.2.1. ∎

In the following, we have to deal with adjoint equations of the form:

$$\frac{\partial^2 p}{\partial t^2} - \Delta p = g \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad p(x, T) = p_T \text{ and } \frac{\partial p}{\partial t}(x, T) = \pi_T \quad \text{in } \Omega, \qquad (6.2.4)$$

with $(p_T, \pi_T) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $g \in L^2(0, T; H^{-1}(\Omega))$.

**Theorem 6.2.5** *Suppose that $f \in L^2(Q)$, $z_0 \in H_0^1(\Omega)$, $z_1 \in L^2(\Omega)$, $g \in L^2(0, T; H^{-1}(\Omega))$, $p_T \in L^2(\Omega)$, every $\pi_T \in H^{-1}(\Omega)$, then the solution $p$ to equation (6.2.2) and the solution $z$ to equation*

$$\frac{\partial^2 z}{\partial t^2} - \Delta z = f \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(0) = 0 \text{ and } \frac{\partial z}{\partial t}(0) = 0 \quad \text{in } \Omega,$$

*satisfy the formula:*

$$\int_Q f p \, dx dt = \int_Q z g \, dx dt + \int_\Omega z_t(T) p_T \, dx - \langle \pi_T, z(T) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}. \qquad (6.2.5)$$

**Proof.** First observe that, due to Theorem 6.2.4, the solution $p$ to equation (6.2.2) belongs to $C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$. Formula (6.2.5) can be first established for regular functions with integrations by parts and a Green formula. It is next derived from this case by using density arguments and a passage to the limit, which is justified due to Theorems 6.2.2 and 6.2.4. ∎

## 6.3 Distributed control

We consider the wave equation with a distributed control

$$\frac{\partial^2 z}{\partial t^2} - \Delta z = f + \chi_\omega u \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = z_0 \text{ and } \frac{\partial z}{\partial t}(x,0) = z_1 \quad \text{in } \Omega, \tag{6.3.6}$$

$\chi_\omega$ is the characteristic function of $\omega$, $\omega$ is an open subset of $\Omega$, $f$ belongs to $L^2(Q)$ and $u \in L^2(0,T;L^2(\omega))$. Due to Theorem 6.2.2, the solution to equation (6.3.6) belongs to $C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$. Thus we can study the following family of problems

$$(P_i) \qquad \inf\{J_i(z,u) \mid (z,u) \in C([0,T];H_0^1(\Omega)) \times L^2(0,T;L^2(\omega)),\ (z,u) \text{ satisfies (6.3.6)}\},$$

with, for $i = 1, \dots, 3$, the functionals $J_i$ are defined by

$$J_1(z,u) = \frac{1}{2}\int_Q (z - z_d)^2 + \frac{1}{2}\int_\Omega (z(T) - z_d(T))^2 + \frac{\beta}{2}\int_Q \chi_\omega u^2,$$

$$J_2(z,u) = \frac{1}{2}\int_\Omega (\nabla z(T) - \nabla z_d(T))^2 + \frac{\beta}{2}\int_Q \chi_\omega u^2,$$

$$J_3(z,u) = \frac{1}{2}\int_\Omega (\frac{\partial z}{\partial t}(T) - \frac{\partial z_d}{\partial t}(T))^2 + \frac{\beta}{2}\int_Q \chi_\omega u^2,$$

where the function $z_d$ belongs to $C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$.

**Theorem 6.3.1** *Assume that $f \in L^2(Q)$, $z_0 \in H_0^1(\Omega)$, $z_1 \in L^2(\Omega)$, and $z_d \in C([0,T];L^2(\Omega)) \cap C^1([0,T];L^2(\Omega))$. For $i = 1, \dots, 3$, problem $(P_i)$ admits a unique solution $(\bar{z}_i, \bar{u}_i)$. Moreover the optimal control $\bar{u}_i$ is defined by $\bar{u}_i = -\frac{1}{\beta}\chi_\omega p_i$, where $p_1$ is the solution to the equation*

$$\frac{\partial^2 p}{\partial t^2} - \Delta p = \bar{z}_1 - z_d \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad p(T) = 0, \ \frac{\partial p}{\partial t}(T) = (\bar{z}_1 - z_d)(T) \quad \text{in } \Omega, \tag{6.3.7}$$

*$p_2$ is the solution to the equation*

$$\frac{\partial^2 p}{\partial t^2} - \Delta p = 0 \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad p(T) = 0 \text{ and } \frac{\partial p}{\partial t}(T) = -\Delta(\bar{z}_2 - z_d)(T) \quad \text{in } \Omega, \tag{6.3.8}$$

*and $p_3$ is the solution to the equation*

$$\frac{\partial^2 p}{\partial t^2} - \Delta p = 0 \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad p(T) = \Big(\frac{\partial \bar{z}_3}{\partial t} - \frac{\partial z_d}{\partial t}\Big)(T) \text{ and } \frac{\partial p}{\partial t}(T) = 0 \quad \text{in } \Omega. \tag{6.3.9}$$

*These necessary optimality conditions are also sufficient.*

**Proof.** Since $\bar{z}_1 - z_d$ belongs to $L^2(Q)$ and $(\bar{z}_1 - z_d)(T)$ belongs to $L^2(\Omega)$ we can apply Theorem 6.2.2 to show that $p_1$ belongs to $C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$. We can identify $-\Delta(\bar{z}_2 - z_d)(T)$ with an element of $H^{-1}(\Omega)$, and $(\bar{z}_3{}' - z_d')(T)$ belongs to $H^{-1}(\Omega)$. Thus $p_2$ and $p_3$ belong to $C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$.

The existence of a unique solution to problem $(P_i)$ can be proved as in the previous chapters. Theorem 7.3.1 can also be applied to prove the existence of a unique solution to problem $(P_1)$. For problems $(P_2)$ and $(P_3)$ the proof must be adapted (see exercise 6.7.1).

Let us establish the optimality conditions for $(P_2)$. As usual we set $F_2(u) = J_2(z(u), u)$, where $z(u)$ is the solution to (6.3.6). We have $F_2'(\bar{u}_2)u = \int_\Omega (\nabla \bar{z}_2(T) - \nabla z_d(T))\nabla w_u(T) + \beta \int_Q \chi_\omega \bar{u}_2 u$, where $w_u$ is the solution to

$$\frac{\partial^2 w}{\partial t^2} - \Delta w = \chi_\omega u \quad \text{in } Q, \quad w = 0 \quad \text{on } \Sigma, \quad w(x,0) = 0 \text{ and } \frac{\partial w}{\partial t}(x,0) = 0 \quad \text{in } \Omega.$$

Since

$$\int_\Omega (\nabla \bar{z}_2(T) - \nabla z_d(T))\nabla w_u(T) = \langle -\Delta(\bar{z}_2(T) - z_d(T)), w_u(T)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)},$$

applying formula (6.2.3) to $p_2$ and $w_u$, we obtain

$$F_2'(\bar{u}_2)u = \int_Q (\chi_\omega(\beta \bar{u}_2 + p_2)u = 0$$

for every $u \in L^2(0, T; L^2(\omega))$. Thus the optimality condition for $(P_2)$ is proved. The proof of the other results is left to the reader. ∎

**Comments.** As for the heat equation with distributed controls, equation (6.3.6) is of the form

$$y' = Ay + F + Bu, \quad y(0) = y_0,$$

with

$$Ay = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \Delta y_1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad y_0 = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \quad \text{and} \quad Bu = \begin{pmatrix} 0 \\ \chi_\omega u \end{pmatrix}.$$

Thus problem $(P_1)$ is a particular case of control problems studied in Chapter 7.

## 6.4 Neumann boundary control

We first study the equation

$$\frac{\partial^2 z}{\partial t^2} - \Delta z = f \quad \text{in } Q, \quad \frac{\partial z}{\partial n} = 0 \quad \text{on } \Sigma, \quad z(x,0) = z_0 \text{ and } \frac{\partial z}{\partial t}(x,0) = z_1 \quad \text{in } \Omega. \tag{6.4.10}$$

We set $D(A) = \{y_1 \in H^2(\Omega) \mid \frac{\partial y_1}{\partial n} = 0\} \times H^1(\Omega)$, $Y = H^1(\Omega) \times L^2(\Omega)$, and

$$Ay = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \Delta y_1 - y_1 \end{pmatrix}, \quad Ly = \begin{pmatrix} 0 \\ y_1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad \text{and} \quad y_0 = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}.$$

Equation (6.4.10) may be written in the form

$$\frac{dy}{dt} = (A + L)y + F, \quad y(0) = y_0.$$

**Theorem 6.4.1** *The operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $Y$.*

**Proof.** We leave the reader adapt the proof of Theorem 6.2.1. ∎

**Theorem 6.4.2** *For every $f \in L^2(Q)$, every $z_0 \in H^1(\Omega)$, every $z_1 \in L^2(\Omega)$, equation (6.4.10) admits a unique weak solution $z(f, z_0, z_1)$, moreover the operator*

$$(f, z_0, z_1) \mapsto z(f, z_0, z_1)$$

*is linear and continuous from $L^2(Q) \times H^1(\Omega) \times L^2(\Omega)$ into $C([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$.*

To study the wave equation with nonhomogeneous boundary conditions, we set $D(\widehat{A}) = H^1(\Omega) \times L^2(\Omega)$, $\widehat{Y} = L^2(\Omega) \times (H^1(\Omega))'$, and

$$\widehat{A}y = \widehat{A}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \tilde{A}y_1 - y_1 \end{pmatrix},$$

where

$$\left(\tilde{A}y_1, \zeta\right)_{(H^1(\Omega))'} = -\int_\Omega \nabla y_1 \cdot \nabla(-\Delta + I)^{-1}\zeta.$$

**Theorem 6.4.3** *The operator $(\widehat{A}, D(\widehat{A}))$ is the infinitesimal generator of a semigroup of contractions on $\widehat{Y}$.*

**Proof.** The proof is similar to the one of Theorem 6.2.3. ∎

Now, we consider the wave equation with a control in a Neumann boundary condition:

$$\frac{\partial^2 z}{\partial t^2} - \Delta z = f \quad \text{in } Q, \quad \frac{\partial z}{\partial n} = u \quad \text{on } \Sigma, \quad z(x,0) = z_0 \text{ and } \frac{\partial z}{\partial t}(x,0) = z_1 \quad \text{in } \Omega. \quad (6.4.11)$$

For any $u \in L^2(\Gamma)$, the mapping $\zeta \mapsto \int_\Gamma u\zeta$ is a continuous linear on $H^1(\Omega)$. Thus it can be identified with an element of $(H^1(\Omega))'$. Thus for $u \in L^2(\Sigma)$, the mapping $\zeta \mapsto \int_\Gamma u(\cdot)\zeta$ is an element of $L^2(0,T; (H^1(\Omega))')$. Let us denote this mapping by $\hat{u}$. We set

$$V = \begin{pmatrix} 0 \\ \hat{u} \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad L\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ y_1 \end{pmatrix}, \quad \text{and} \quad y_0 = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}.$$

Equation (6.4.11) may be written in the form

$$\frac{dy}{dt} = (\widehat{A} + L)y + F + V, \quad y(0) = y_0,$$

with $F$ and $V$ belong to $L^2(0,T; L^2(\Omega)) \times L^2(0,T; (H^1(\Omega))')$, $y_0 \in L^2(\Omega) \times (H^1(\Omega))'$.

**Theorem 6.4.4** *For every $(f, u, z_0, z_1) \in L^2(Q) \times L^2(\Sigma) \times L^2(\Omega) \times (H^1(\Omega))'$, equation (6.4.11) admits a unique weak solution $z(f, u, z_0, z_1)$ in $C([0,T]; L^2(\Omega)) \cap C^1([0,T]; (H^1(\Omega))')$. Moreover the mapping $(f, u, z_0, z_1) \mapsto z(f, u, z_0, z_1)$ is continuous from $L^2(Q) \times L^2(\Sigma) \times L^2(\Omega) \times (H^1(\Omega))'$ into $C([0,T]; L^2(\Omega)) \cap C^1([0,T]; (H^1(\Omega))')$.*

**Proof.** The result is a direct consequence of Theorem 6.4.3. ∎

We consider the control problem

$$(P_4) \qquad \inf\{J_4(z,u) \mid (z,u) \in C([0,T]; L^2(\Omega)) \times L^2(0,T; L^2(\Omega)),\ (z,u) \text{ satisfies } (6.4.11)\},$$

with

$$J_4(z,u) = \frac{1}{2}\int_Q (z - z_d)^2 + \frac{1}{2}\int_\Omega (z(T) - z_d(T))^2 + \frac{\beta}{2}\int_\Sigma u^2.$$

**Theorem 6.4.5** *Assume that* $f \in L^2(0,T;L^2(\Omega))$*,* $z_0 \in L^2(\Omega)$*,* $z_1 \in (H^1(\Omega))'$*, and* $z_d \in C([0,T];L^2(\Omega))$*. Problem* $(P_2)$ *admits a unique solution* $(\bar z, \bar u)$*. Moreover the optimal control* $\bar u$ *is defined by* $\bar u = -\frac{1}{\beta} p|_\Sigma$*, where* $p$ *is the solution to the equation*

$$\begin{aligned} &\frac{\partial^2 p}{\partial t^2} - \Delta p = \bar z - z_d \text{ in } Q, \qquad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Sigma \;=\; \Gamma\times ]0,T[,\\ &p(T) = 0, \quad \frac{\partial p}{\partial t}(T) = \bar z(T) - z_d(T) \qquad\qquad \text{in } \Omega. \end{aligned} \tag{6.4.12}$$

**Proof.** We leave the reader adapt the proof of Theorem 6.3.1. ∎

## 6.5 Trace regularity

To study the wave equation with a control in a Dirichlet boundary condition, we have to establish a sharp regularity result stated below.

**Theorem 6.5.1** *Let* $y$ *be the solution to the equation*

$$\frac{\partial^2 y}{\partial t^2} - \Delta y = f \quad \textit{in } Q, \quad y = 0 \quad \textit{on } \Sigma, \quad y(x,0) = y_0 \textit{ and } \frac{\partial y}{\partial t}(x,0) = y_1 \quad \textit{in } \Omega. \tag{6.5.13}$$

*We have*

$$\|\frac{\partial y}{\partial n}\|_{L^2(\Sigma)} \le C\Big(\|f\|_{L^2(Q)} + \|y_0\|_{H_0^1(\Omega)} + \|y_1\|_{L^2(\Omega)}\Big). \tag{6.5.14}$$

The proof can be found in [14, Theorem 2.2].

## 6.6 Dirichlet boundary control

We consider the wave equation with a control in a Dirichlet boundary condition

$$\frac{\partial^2 z}{\partial t^2} - \Delta z = f \quad \text{in } Q, \quad z = u \quad \text{on } \Sigma, \quad z(x,0) = z_0 \text{ and } \frac{\partial z}{\partial t}(x,0) = z_1 \quad \text{in } \Omega. \tag{6.6.15}$$

As for the heat equation with a Dirichlet boundary control, the solution to equation (6.6.15) is defined by the transposition method.

**Definition 6.6.1** *A function* $z \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$ *is called a weak solution to equation (6.6.15) if, and only if,*

$$\begin{aligned} &\int_Q fy\,dxdt = \int_Q z\varphi\,dxdt + \langle \frac{\partial z}{\partial t}(T), y_T\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \\ &-\langle \frac{\partial z}{\partial t}(0), y(0)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} - \int_\Omega z(T)\nu_T\,dx + \int_\Omega z(0)\frac{\partial y}{\partial t}(0)\,dx + \int_\Sigma \frac{\partial y}{\partial n} u\,dsdt \end{aligned} \tag{6.6.16}$$

*for all* $(\varphi, y_T, \nu_T) \in L^1(0,T;L^2(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega)$*, where* $y$ *is the solution to*

$$\frac{\partial^2 y}{\partial t^2} - \Delta y = \varphi \quad \textit{in } Q, \quad y = 0 \quad \textit{on } \Sigma, \quad y(x,T) = y_T \textit{ and } \frac{\partial y}{\partial t}(x,T) = \nu_T \quad \textit{in } \Omega. \tag{6.6.17}$$

**Theorem 6.6.1** *For every* $(f, u, z_0, z_1) \in L^1(0,T;H^{-1}(\Omega)) \times L^2(\Sigma) \times L^2(\Omega) \times H^{-1}(\Omega)$, *equation (6.6.15) admits a unique weak solution* $z(f, u, z_0, z_1)$ *in* $C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$. *The mapping* $(f, u, y_0, y_1) \mapsto z(f, u, z_0, z_1)$ *is linear and continuous from* $L^1(0,T;H^{-1}(\Omega)) \times L^2(\Sigma) \times L^2(\Omega) \times H^{-1}(\Omega)$ *into* $C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$.

**Proof.** This existence and regularity result can be proved by the transposition method with Theorem 6.5.1.

(i) Due to Theorem 6.2.3, the mapping

$$(f, z_0, z_1) \longmapsto z(f, 0, z_0, z_1)$$

is linear and continuous from $L^1(0,T;H^{-1}(\Omega)) \times L^2(\Omega) \times H^{-1}(\Omega)$ into $C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$. Thus we have only to consider the case where $(f, z_0, z_1) = (0,0,0)$.

(ii) Denote by $\Lambda$ the mapping $\varphi \mapsto \frac{\partial y}{\partial n}$, where $y$ is the solution to equation (6.6.17) corresponding to $(y_T, \nu_T) = (0,0)$. Due to Theorem 6.5.1, $\Lambda$ is a linear operator from $L^2(Q)$ into $L^2(\Sigma)$. If we set $z = \Lambda^* u$, with $u \in L^2(\Sigma)$, we observe that $z \in L^2(Q)$, and $z$ is a solution to equation (6.6.15) with $(f, z_0, z_1) = (0,0,0)$, in the sense of definition 6.6.1. This solution is unique in $L^2(Q)$. Indeed if $z_1$ and $z_2$ are two solutions to equation (6.6.15) with $(f, z_0, z_1) = (0,0,0)$, in the sense of definition 6.6.1, we have

$$\int_\Omega (z_1 - z_2)\varphi = 0 \qquad \text{for all } \ \varphi \in L^2(Q).$$

To prove that $z$ belongs to $C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$, we proceed by approximation. Let $(u_n)_n$ be a sequence of regular functions such that $z_n = z(0, u_n, 0, 0)$ be regular. For $\tau \in ]0,T]$ and $(y_\tau, \nu_\tau) \in H_0^1(\Omega) \times L^2(\Omega)$, we denote by $y(y_\tau, \nu_\tau)$ the solution to the equation

$$\frac{\partial^2 y}{\partial t^2} - \Delta y = 0 \quad \text{in } Q, \quad y = 0 \quad \text{on } \Sigma, \quad y(x,\tau) = y_\tau \text{ and } \frac{\partial y}{\partial t}(x,\tau) = \nu_\tau \quad \text{in } \Omega.$$

With Theorem 6.5.1, we have

$$\|\frac{\partial y(y_\tau, \nu_\tau)}{\partial n}\|_{L^2(\Sigma)} \le C\Big(\|y_\tau\|_{H_0^1(\Omega)} + \|\nu_\tau\|_{L^2(\Omega)}\Big), \tag{6.6.18}$$

where the constant $C$ depends on $T$, but is independent of $\tau$. Since $z_n$ is the solution to equation (6.6.15), according to definition 6.6.1, we have

$$\langle \frac{\partial z_n}{\partial t}(\tau), y_\tau \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \int_\Omega \frac{\partial z_n}{\partial t}(\tau) y_\tau \, dx = -\int_\Sigma \frac{\partial y(y_\tau, 0)}{\partial n} u \, dsdt$$

and

$$\int_\Omega z_n(\tau)\nu_T \, dx = \int_\Sigma \frac{\partial y(0, \nu_\tau)}{\partial n} u_n \, dsdt.$$

From which we deduce:

$$\begin{aligned}\|z_n - z_m\|_{C([0,T];L^2(\Omega))} &= \sup_{\tau\in]0,T]} \sup_{\|\nu_\tau\|_{L^2(\Omega)}=1} \left| \int_\Sigma \frac{\partial y(0,\nu_\tau)}{\partial n}(u_n - u_m)\, dsdt \right| \\ &\le C\|u_n - u_m\|_{L^2(\Sigma)}\end{aligned}$$

and

$$\|\frac{\partial z_n}{\partial t} - \frac{\partial z_n}{\partial t}\|_{C([0,T];H^{-1}(\Omega))} = \sup_{\tau\in]0,T]}\sup_{\|y_\tau\|_{H^1_0(\Omega)}=1}\left|\int_\Sigma \frac{\partial y(y_\tau,0)}{\partial n}(u_n - u_m)\, dsdt\right|$$
$$\leq C\|u_n - u_m\|_{L^2(\Sigma)}.$$

Thus $(z_n)_n$ is a Cauchy sequence in $C([0,T];L^2(\Omega))$, and $(\frac{\partial z_n}{\partial t})_n$ is a Cauchy sequence in $C([0,T];H^{-1}(\Omega))$. It is clear that the limit of the sequence $(z_n)_n$ is $z(0,u,0,0)$, and the limit of the sequence $(\frac{\partial z_n}{\partial t})_n$ is $\frac{\partial z(0,u,0,0)}{\partial t}$. The proof is complete. ∎

We consider the control problem

$(P_5)$ $\quad \inf\{J_5(z,u) \mid (z,u) \in C([0,T];L^2(\Omega)) \times L^2(0,T;L^2(\Gamma)),\ (z,u) \text{ satisfies } (6.6.15)\},$

with

$$J_5(z,u) = \frac{c_1}{2}\int_Q (z-z_d)^2 + \frac{c_2}{2}\|z(T)-z_d(T)\|^2_{L^2(\Omega)} + \frac{c_3}{2}\|\frac{\partial z}{\partial t}(T) - \frac{\partial z_d}{\partial t}(T)\|^2_{H^{-1}(\Omega)} + \frac{\beta}{2}\int_\Sigma u^2,$$

where $c_1$, $c_2$, and $c_3$ are nonnegative constants, and $\beta > 0$.

**Theorem 6.6.2** *Assume that $f \in L^2(Q)$, $z_0 \in L^2(\Omega)$, $z_1 \in H^{-1}(\Omega)$, and $z_d \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$. Problem $(P_5)$ admits a unique solution $(\bar z, \bar u)$. Moreover the optimal control $\bar u$ is defined by $\bar u = \frac{1}{\beta}\frac{\partial p}{\partial n}$, where $p$ is the solution to the equation*

$$\begin{aligned} &\frac{\partial^2 p}{\partial t^2} - \Delta p = c_1(z - z_d) \quad \text{in } Q, \qquad p = 0 \text{ on } \Sigma = \Gamma\times]0,T[, \\ &p(T) = c_3(-\Delta)^{-1}(\frac{\partial z}{\partial t}(T) - \frac{\partial z_d}{\partial t}(T)), \quad \frac{\partial p}{\partial t}(T) = c_2(z(T) - z_d(T)) \text{ in } \Omega. \end{aligned} \tag{6.6.19}$$

**Proof.** Set $F_5(u) = J_5(z(f,z_0,z_1,u),u)$, where $z(f,z_0,z_1,u)$ is the solution to equation (6.6.15). We have

$$F_5(\bar u)u = \int_Q c_1(\bar z - z_d)w_u + \int_\Omega c_2(\bar z(T) - z_d(T))w_u(T)$$
$$+c_3\langle\frac{\partial w_u}{\partial t}(T), (-\Delta)^{-1}(\frac{\partial \bar z}{\partial t}(T) - \frac{\partial z_d}{\partial t}(T))\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} + \beta\int_\Sigma \bar u u,$$

where $w_u = z(0,0,0,u)$. The functions $w_u$ and $p$ satisfy the Green formula

$$\int_Q c_1(\bar z - z_d)w_u + \int_\Omega c_2(\bar z(T) - z_d(T))w_u(T)$$
$$+c_3\langle\frac{\partial w_u}{\partial t}(T), (-\Delta)^{-1}(\frac{\partial \bar z}{\partial t}(T) - \frac{\partial z_d}{\partial t}(T))\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} = -\int_\Sigma \frac{\partial p}{\partial n}u.$$

Observe that $(-\Delta)^{-1}(\frac{\partial z}{\partial t}(T) - \frac{\partial z_d}{\partial t}(T))$ belongs to $H^1_0(\Omega)$, and $(z(T) - z_d(T))$ belongs to $L^2(\Omega)$. Therefore, due to Theorem 6.5.1, $\frac{\partial p}{\partial n}$ belongs to $L^2(\Sigma)$. Since all the terms in the above formula are well defined, this formula can be proved for regular data, and next proved by a passage to the limit. Due to this formula, we have

$$F_5(\bar u)u = -\int_\Sigma \frac{\partial p}{\partial n}u + \beta\int_\Sigma \bar u u,$$

This completes the proof. ∎

## 6.7 Exercises

### Exercise 6.7.1

The notation are the ones of section 6.3. Let $(u_n)_n$ be a sequence in $L^2(\omega)$, converging to $u$ for the weak topology of $L^2(\omega)$. Let $z_n$ be the solution to equation (6.3.6) corresponding to $u_n$, and $z_u$ be the solution to equation (6.3.6) corresponding to $u$. Prove that $(z_n(T))_n$ converges to $z_u(T)$ for the weak topology of $H^1_0(\Omega)$, and that $(\frac{\partial z_n}{\partial t}(T))_n$ converges to $\frac{\partial z_u}{\partial t}(T)$ for the weak topology of $L^2(\Omega)$. Prove that the control problem $(P_2)$ admits a unique solution. Prove that problem $(P_3)$ admits a unique solution.

### Exercise 6.7.2

We study a control problem for the system of the Timoshenko beam (see section 1.4). We consider the following set of equations:

$$\begin{aligned} &\rho\frac{\partial^2 u}{\partial t^2} - K\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial \phi}{\partial x}\right) = 0, \qquad \text{in } (0,L),\\ &I_\rho\frac{\partial^2 \phi}{\partial t^2} - EI\frac{\partial^2 \phi}{\partial x^2} + K\left(\phi - \frac{\partial u}{\partial x}\right) = 0, \qquad \text{in } (0,L), \end{aligned} \tag{6.7.20}$$

with the boundary conditions

$$\begin{aligned} &u(0,t) = 0 \quad \text{and} \quad \phi(0,t) = 0 \quad \text{for } t \geq 0,\\ &K(\phi(L,t) - u_x(L,t)) = f_1(t) \quad \text{and} \quad -EI\phi_x(L,t) = f_2(t) \quad \text{for } t \geq 0. \end{aligned} \tag{6.7.21}$$

and the initial conditions

$$\begin{aligned} &u(x,0) = u_0 \ \text{ pour } \ \frac{\partial u}{\partial t}(x,0) = u_1 \ \text{ pour } x \in (0,L),\\ &\phi(x,0) = \phi_0 \ \text{ and } \ \frac{\partial \phi}{\partial t}(x,0) = \phi_1 \text{ in } (0,L). \end{aligned} \tag{6.7.22}$$

We recall that $u$ is the deflection of the beam, $\phi$ is the angle of rotation of the beam cross-sections due to bending. The coefficient $\rho$ is the mass density per unit length, $EI$ is the flexural rigidity of the beam, $I_\rho$ is the mass moment of inertia of the beam cross section, and $K$ is the shear modulus. We suppose that $u_0 \in H^1_0(0,L)$, $u_1 \in L^2(0,L)$, $\phi_0 \in H^1_0(0,L)$, $\phi_1 \in L^2(0,L)$. The control functions $f_1$ and $f_2$ are taken in $L^2(0,T)$.

To study the system (6.7.20)-(6.7.22), we use a fixed point method as in exercise 5.5.4. The Hille-Yosida theorem could also be used to directly study the system. Denote by $H^1_{\{0\}}(0,L)$ the space of functions $\psi$ in $H^1(0,L)$ such that $\psi(0) = 0$. Let $\tau > 0$, for $\psi \in L^2(0,\tau; H^1_{\{0\}}(0,L))$, we denote by $(u_\psi, \phi_\psi)$ the solution to

$$\begin{aligned} &\rho\frac{\partial^2 u}{\partial t^2} - K\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial \psi}{\partial x}\right) = 0, \qquad \text{in } (0,L),\\ &I_\rho\frac{\partial^2 \phi}{\partial t^2} - EI\frac{\partial^2 \phi}{\partial x^2} + K\left(\phi - \frac{\partial u}{\partial x}\right) = 0, \qquad \text{in } (0,L), \end{aligned} \tag{6.7.23}$$

with the boundary conditions

$$\begin{aligned} &u(0,t) = 0 \quad \text{and} \quad \phi(0,t) = 0 \quad \text{for } t \geq 0,\\ &K(\psi(L,t) - u_x(L,t)) = f_1(t) \quad \text{and} \quad -EI\phi_x(L,t) = f_2(t) \quad \text{for } t \geq 0. \end{aligned} \tag{6.7.24}$$

and the initial conditions

$$\begin{aligned} &u(x,0) = u_0 \ \text{ pour } \ \tfrac{\partial u}{\partial t}(x,0) = u_1 \text{ pour } x \in (0,L),\\ &\phi(x,0) = \phi_0 \ \text{ and } \ \tfrac{\partial \phi}{\partial t}(x,0) = \phi_1 \text{ in } (0,L). \end{aligned} \tag{6.7.25}$$

Prove that, if $\tau > 0$ is small enough, then the mapping

$$\psi \longmapsto \phi_\psi$$

is a contraction in $L^2(0,\tau; H^1_{\{0\}}(0,L))$.
We set

$$E(t) = \frac{1}{2}\int_0^L \Big(\rho u_t^2(t) + I_\rho \phi_t^2(t) + K(\phi(t) - u_x(t))^2 + EI\phi_x^2(t)\Big)\, dx.$$

Let $(u,\phi)$ be the solution to (6.7.20)-(6.7.22) defined on $(0,L)\times(0,\tau)$. Prove that

$$E(0) = E(t) + f_1(t)u(L,t) + f_2(t)\phi(L,t) \quad \text{ for almost all } \ t \in (0,\tau).$$

Prove that the system (6.7.20)-(6.7.22) admits a unique solution $(u,\phi)$ belonging to $(C([0,T]; H^1_{\{0\}}(0,L)) \cap C^1([0,T]; L^2(0,L))) \times (C([0,T]; H^1_{\{0\}}(0,L)) \cap C^1([0,T]; L^2(0,L)))$.
We consider the control problem

$$(P_6) \qquad \inf\{J_6(u,\phi,f_1,f_2) \mid (u,\phi,f_1,f_2) \text{ satisfies } (6.7.20) - (6.7.22)\},$$

with

$$J_6(u,\phi,f_1,f_2) = \frac{1}{2}\int_0^T E(t) + \frac{\beta}{2}\int_0^T (f_1^2 + f_2^2), \qquad \text{with } \beta > 0.$$

Prove that $(P_6)$ admits a unique solution. Write the corresponding first order optimality conditions.

# Chapter 7

# Control of evolution equations with bounded control operators

## 7.1 Introduction

The purpose of this chapter is to extend results obtained for the control of the wave and heat equations to other linear evolution equations. We consider equations of the form

$$z' = Az + Bu + f, \qquad z(0) = z_0. \tag{7.1.1}$$

We have already seen that the controlled equations of chapters 5 and 6 may be written in this form. Other examples will also be considered.

We make the following assumptions.

**Assumption** $(H_1)$
$Z$ and $U$ are two Hilbert spaces.
The unbounded operator $A$, with domain $D(A)$ dense in $Z$, is the infinitesimal generator of a strongly continuous semigroup on $Z$. This semigroup will be denoted by $(e^{tA})_{t\geq 0}$.
The operator $B$ belongs to $\mathcal{L}(U;Z)$.

We here suppose that $B$ is a bounded operator from $U$ into $Z$. The case of unbounded control operators will be studied in the next chapter.

Associated with equation (7.1.1), we shall study the control problem

$$(P) \qquad \inf\{J(z,u) \mid (z,u) \in C([0,T];Z) \times L^2(0,T;U),\ (z,u) \text{ satisfies (7.1.1)}\}.$$

with

$$J(z,u) = \frac{1}{2}\int_0^T |Cz(t) - y_d(t)|_Y^2 + \frac{1}{2}|Dz(T) - y_T|_{Y_T}^2 + \frac{1}{2}\int_0^T |u(t)|_U^2. \tag{7.1.2}$$

This problem is often referred as a 'Linear Quadratic Regulation' problem (LQR problem in short). We make the following assumption on the operators $C$ and $D$.

**Assumption** $(H_2)$
$Y$ and $Y_T$ are Hilbert spaces.
The operator $C$ belongs to $\mathcal{L}(Z;Y)$, and the operator $D$ belongs to $\mathcal{L}(Z;Y_T)$. The function $y_d$ belongs to $L^2(0,T;Y)$ and $y_T \in Y_T$.

**In this chapter we identify $Z'$ with $Z$, and $U'$ with $U$.**

## 7.2 Adjoint equation

The adjoint equation for $(P)$ will be of the form

$$-p' = A^*p + g, \qquad p(T) = p_T. \tag{7.2.3}$$

We state an integration by parts formula between the adjoint state $p$ and the solution $z$ to the equation

$$z' = Az + f, \qquad z(0) = z_0. \tag{7.2.4}$$

**Theorem 7.2.1** *For every $(f, z) \in L^2(0, T; Z) \times Z$, and every $(g, p_T) \in L^2(0, T; Z) \times Z$, the solution $z$ to equation (7.2.4) and the solution $p$ to equation (7.2.3) satisfy the following formula*

$$\int_0^T (f(t), p(t))_Z\, dt = \int_0^T (z(t), g(t))_Z\, dt + (z(T), p_T)_Z - (z_0, p(0))_Z. \tag{7.2.5}$$

**Proof.** Suppose that $(f, z_0)$ and $(g, p_T)$ belong to $C^1([0, T]; Z) \times D(A^*)$. In this case we can write

$$\int_0^T (f(t), p(t))_Z\, dt = \int_0^T (z'(t) - Az(t), p(t))_Z\, dt$$

$$= \int_0^T -(z(t), p'(t))_Z\, dt + (z(T), p_T)_Z - (z_0, p(0))_Z - \int_0^T (Az(t), p(t))_Z\, dt$$

$$= \int_0^T (z(t), g(t))_Z\, dt + (z(T), p_T)_Z - (z_0, p(0))_Z.$$

Thus, formula (7.2.5) can be deduced from this case by using density arguments.

## 7.3 Optimal control

**Theorem 7.3.1** *Assume that $(H_1)$ and $(H_2)$ are satisfied. Problem $(P)$ admits a unique solution $(z, u)$.*

To prove this theorem we need the following lemma.

**Lemma 7.3.1** *Let $(u_n)_n$ be a sequence in $L^2(0, T; U)$ converging to $u$ for the weak topology of $L^2(0, T; U)$. Then $(z(f, u_n, z_0))_n$ (the sequence of solutions to equation (7.1.1) corresponding to $(f, u_n, z_0)$) converges to $z(f, u, z_0)$ for the weak topology of $L^2(0, T; Z)$, and $(z(f, u_n, z_0)(T))_n$ converges to $z(f, u, z_0)(T)$ for the weak topology of $Z$.*

**Proof.** The lemma is a direct consequence of Theorems 4.2.1 and 2.6.2.

**Proof of Theorem 7.3.1.** Let $(u_n)_n$ be a minimizing sequence weakly converging to a function $u$ in $L^2(0, T; U)$. Set $z_n = z(f, u_n, z_0)$ and $z_u = z(f, u, z_0)$. Due to Lemma 7.3.1 and to Theorem 2.6.2, the sequence $(Cz_n)_n$ converges to $Cz_u$ for the weak topology of $L^2(0, T; Y)$, and the sequence $(Dz_n(T))_n$ converges to $Dz_u(T)$ for the weak topology of $Y_T$. Due to Corollary 2.6.1 the mapping $u \mapsto \int_0^T \|u(t)\|_U^2 dt$, is lower semicontinuous for the weak topology of $L^2(0, T; U)$, the mapping $y \mapsto \int_0^T \|y(t)\|_Y^2 dt$, is lower semicontinuous for the weak topology of

$L^2(0,T;Y)$, and the mapping $y \mapsto \|y\|^2_{Y_T}$ is lower semicontinuous for the weak topology of $Y_T$. Combining these arguments we can prove that

$$J(z_u, u) \leq \text{liminf}_{n\to\infty} J(z_n, u_n) = \inf(P).$$

Thus $(z_u, u)$ is a solution to problem $(P)$.

*Uniqueness.* The uniqueness follows from the strict convexity of the mapping $u \mapsto J(z(f,u,z_0),u)$. ∎

**Theorem 7.3.2** *If $(\bar{z}, \bar{u})$ is the solution to $(P)$ then $\bar{u} = -B^*p$, where $p$ is the solution to equation*

$$-p' = A^*p + C^*(C\bar{z} - y_d), \qquad p(T) = D^*(D\bar{z}(T) - y_T). \tag{7.3.6}$$

*Conversely, if a pair $(\tilde{z}, \tilde{p}) \in C([0,T];Z) \times C([0,T];Z)$ obeys the system*

$$\begin{aligned} \tilde{z}' &= A\tilde{z} - BB^*\tilde{p} + f, \qquad &\tilde{z}(0) = z_0, \\ -\tilde{p}' &= A^*\tilde{p} + C^*(C\tilde{z} - y_d), \qquad &\tilde{p}(T) = D^*(D\tilde{z}(T) - y_T), \end{aligned} \tag{7.3.7}$$

*then the pair $(\tilde{z}, -B^*\tilde{p})$ is the optimal solution to problem $(P)$.*

**Proof.** Let $(\bar{z}, \bar{u})$ be the optimal solution to problem $(P)$. Set $F(u) = J(z(f,u),u)$. For every $u \in L^2(0,T;U)$, we have

$$F'(\bar{u})u = \int_0^T (C\bar{z}(t) - y_d, Cw(t))_Y + (D\bar{z}(T) - y_T, Dw(T))_{Y_T} + \int_0^T (\bar{u}(t), u(t))_U$$

$$= \int_0^T \Big(C^*(C\bar{z}(t) - y_d), w(t)\Big)_Z + \Big(D^*(D\bar{z}(T) - y_T), w(T)\Big)_Z + \int_0^T (\bar{u}(t), u(t))_U,$$

where $w$ is the solution to

$$w' = Aw + Bu, \qquad w(0) = 0.$$

Applying formula (7.2.5) to $p$ and $w$, we obtain

$$F'(\bar{u})u = \int_0^T (p(t), Bu(t))_Z + \int_0^T (\bar{u}(t), u(t))_U = \int_0^T (B^*p(t) + \bar{u}(t), u(t))_U.$$

The first part of the Theorem is established. The second part follows from Theorem 2.2.3 (see also the proof of Theorem 2.2.2).

## 7.4 Exercises

### Exercise 7.4.1

Let $L > 0$ and $a$ be a function in $H^1(0,L)$ such that $0 < c_1 \leq a(x)$ for all $x \in H^1(0,L)$. Consider the equation

$$\begin{aligned} &z_t + az_x = f + \chi_{(\ell_1,\ell_2)}u, \qquad \text{in } (0,L)\times(0,T), \\ &z(0,t) = 0, \qquad \text{in } (0,T), \\ &z(x,0) = z_0, \qquad \text{in } (0,L), \end{aligned} \tag{7.4.8}$$

where $f \in L^2((0,L)\times(0,T))$, $\chi_{(\ell_1,\ell_2)}$ is the characteristic function of $(\ell_1,\ell_2) \subset (0,L)$, $u \in L^2((\ell_1,\ell_2)\times(0,T))$, and $z_0 \in L^2(0,L)$.

Prove that equation (7.4.8) admits a unique solution in $C([0,T];L^2(0,L))$ (the Hille-Yosida theorem can be used).

Study the control problem

$$(P) \quad \inf\{J(z,u) \mid (z,u) \in C([0,T];L^2(0,L)) \times L^2(0,T;L^2(\ell_1,\ell_2)),\ (z,u) \text{ satisfies (7.4.8)}\}.$$

with

$$J(z,u) = \frac{1}{2}\int_0^L (z(T) - z_d(T))^2 + \frac{1}{2}\int_0^T \int_{\ell_1}^{\ell_2} u^2,$$

where $z_d \in C([0,T];L^2(0,L))$. Prove the existence of a unique solution. Write first order optimality conditions.

# Chapter 8

# Control of evolution equations with unbounded control operators

## 8.1 Introduction

In this chapter we consider the control problem

$$(P) \qquad \inf\{J(z,u) \mid (z,u) \in C([0,T];Z) \times L^2(0,T;U),\ (z,u) \text{ satisfies (8.1.1)}\}.$$

with

$$J(z,u) = \frac{1}{2}\int_0^T |Cz(t) - y_d|_Y^2 + \frac{1}{2}|Dz(T) - y_T|_{Y_T}^2 + \frac{1}{2}\int_0^T |u(t)|_U^2.$$

and

$$z' = Az + Bu + f, \qquad z(0) = z_0, \tag{8.1.1}$$

in the case when $B$ is an unbounded operator.

**Assumptions.**

As in the previous chapter, $Z$, $Y$, $Y_T$, and $U$ denote Hilbert spaces. The operator $A$, with domain $D(A)$ dense in $Z$, is the infinitesimal generator of a strongly continuous semigroup on $Z$, denoted by $(e^{tA})_{t\geq 0}$. The operator $C$ belongs to $\mathcal{L}(Z;Y)$, and the operator $D$ belongs to $\mathcal{L}(Z;Y_T)$. The function $y_d$ belongs to $L^2(0,T;Y)$, $y_T \in Y_T$ and $f \in L^2(0,T;Z)$.

We denote by $A^*$ the $Z$-adjoint of $A$, and by $(D(A^*))'$ the dual of $D(A^*)$ with respect to the $Z$-topology. We suppose that $B \in \mathcal{L}(U;(D(A^*))')$. Let us give an equivalent formulation of this assumption. Let $\lambda$ be a real in $\rho(A)$ (the resolvent set of $A$). Then $(\lambda I - A) \in \mathcal{L}(D(A);Z)$ has a bounded inverse in $Z$. Moreover, $(\lambda I - (A^*)^*)$, the extension of $(\lambda I - A)$ to $(D(A^*))'$, also denoted by $(\lambda I - A)$ to simplify the notation, has a bounded inverse from $(D(A^*))'$ into $Z$. Thus there exists an operator $B_0 \in \mathcal{L}(U;Z)$ such that $B = (\lambda I - A)B_0$.

We study problem $(P)$ under two kinds of assumptions.

$(\mathcal{HP})$ (*The parabolic case*) The family $(e^{tA})_{t\geq 0}$ is a strongly continuous analytic semigroup on $Z$. We suppose that

$$\|e^{tA}\|_{\mathcal{L}(Z)} \leq Me^{-ct} \qquad \text{for all } t \geq 0, \tag{8.1.2}$$

for some $c > 0$. (Thanks to Theorem 4.1.2 this condition is not restrictive. Indeed by replacing $A$ by $A - \lambda I$ with $\lambda > 0$ big enough, the condition (8.1.2) will be satisfied by $A - \lambda I$.) There exists $B_1 \in \mathcal{L}(U;Z)$ and $0 < \alpha < 1$ such that

$$B = (-A)^{1-\alpha} B_1.$$

$(\mathcal{HH})$ (*The hyperbolic case*) The operator $B^* e^{tA^*}$ admits a continuous extension from $Z$ into $L^2(0,T;U)$, that is there exists a constant $C(T)$ such that

$$\int_0^T \|B^* e^{tA^*} \zeta\|_U^2 \leq C(T) \|\zeta\|_Z^2 \tag{8.1.3}$$

for every $\zeta \in D(A^*)$. In the sequel we denote by $[B^* e^{tA^*}]_e$ the extension of $B^* e^{tA^*}$ to $Z$.

## 8.2 The case of analytic semigroups

We suppose that $(\mathcal{HP})$ is satisfied. We have to distinguish the cases $\alpha > \frac{1}{2}$ and $\alpha \leq \frac{1}{2}$. We are going to see that if $\alpha \leq \frac{1}{2}$ an additional assumption on $D$ is needed in order that the problem $(P)$ be well posed.

### 8.2.1 The case $\alpha > \frac{1}{2}$

**Theorem 8.2.1** *In this section we suppose that $(\mathcal{HP})$ is satisfied with $\alpha > \frac{1}{2}$. For every $z_0 \in Z$, every $f \in L^2(0,T;Z)$ and every $u \in L^2(0,T;U)$, equation (8.1.1) admits a unique weak solution $z(z_0,u,f)$ in $L^2(0,T;Z)$, this solution belongs to $C([0,T];Z)$ and the mapping*

$$(z_0,u,f) \longmapsto z(z_0,u,f)$$

*is continuous from $Z \times L^2(0,T;U) \times L^2(0,T;Z)$ into $C([0,T];Z)$.*

**Proof.** Due to Theorem 4.3.1, we have

$$\begin{aligned} z(t) &= e^{tA} z_0 + \int_0^t e^{(t-\tau)A} (-A)^{1-\alpha} B_1 u(\tau) d\tau \\ &= e^{tA} z_0 + \int_0^t (-A)^{1-\alpha} e^{(t-\tau)A} B_1 u(\tau) d\tau. \end{aligned}$$

Thus $z(t)$ satisfies the estimate

$$|z(t)|_Z \leq C\|z_0\|_Z + \int_0^t |t-\tau|^{\alpha-1} |u(\tau)|_U d\tau.$$

Since the mapping $t \mapsto t^{\alpha-1}$ belongs to $L^2(0,T)$ and the mapping $t \mapsto |u(t)|$ belongs to $L^2(0,T)$, from the above estimate it follows that $t \mapsto |z(t)|$ belongs to $L^\infty(0,T)$. ∎

The adjoint equation for $(P)$ is of the form

$$-p' = A^* p + g, \qquad p(T) = p_T. \tag{8.2.4}$$

**Theorem 8.2.2** *For every $(g, p_T) \in L^2(0,T;Z) \times Z$, the solution $p$ to equation (8.2.4) belongs to $L^2(0,T;D((-A^*)^{1-\alpha}))$.*

**Proof.** We have

$$(-A^*)^{1-\alpha}p(t) = (-A^*)^{1-\alpha}e^{(T-t)A^*}p_T + \int_t^T (-A^*)^{1-\alpha}e^{(s-t)A^*}g(s)ds.$$

Using estimates on analytic semigroups (Theorem 4.4.1), we obtain

$$|(-A^*)^{1-\alpha}p(t)|_Z \leq \frac{C}{(T-t)^{1-\alpha}}|p_T|_Z + \int_t^T \frac{C}{(s-t)^{1-\alpha}}|g(s)|_Z ds.$$

Since the mapping $t \mapsto \frac{C}{t^{1-\alpha}}$ belongs to $L^2(0,T)$ (because $\alpha > \frac{1}{2}$), and the mapping $t \mapsto |g(t)|$ belongs to $L^2(0,T)$, the mapping $t \mapsto \int_t^T \frac{C}{(s-t)^{1-\alpha}}|g(s)|_Z ds$ belongs to $L^\infty(0,T)$. Moreover the mapping $t \mapsto \frac{C}{(T-t)^{1-\alpha}}|p_T|_Z$ belongs to $L^2(0,T)$. ∎

**Theorem 8.2.3** *For every $u \in L^2(0,T;U)$, and every $(g,p_T) \in L^2(0,T;Z) \times Z$, the solution $z$ to equation*

$$z' = Az + Bu, \qquad z(0) = 0,$$

*and the solution $p$ to equation (8.2.4) satisfy the following formula*

$$\int_0^T (B_1u(t), (-A^*)^{1-\alpha}p(t))_Z \, dt = \int_0^T (z(t), g(t))_Z \, dt + (z(T), p_T)_Z. \tag{8.2.5}$$

**Proof.** This formula can be proved if $p$ and $z$ are regular enough to justify integration by parts and tranposition of the operator $A$ (for example if $p$ and $z$ belong to $C^1([0,T]); Z) \cap C([0,T]; D(A)))$. Due to Theorem 8.2.2, formula (8.2.5) can be next obtained by a passage to the limit.

**Theorem 8.2.4** *Assume that $(\mathcal{HP})$ is satisfied with $\alpha > \frac{1}{2}$. Problem $(P)$ admits a unique solution $(z,u)$.*

To prove this theorem we need the following lemma.

**Lemma 8.2.1** *Let $(u_n)_n$ be a sequence in $L^2(0,T;U)$ converging to $u$ for the weak topology of $L^2(0,T;U)$. Then $(z(f,u_n,z_0))_n$ (the sequence of solutions to equation (8.1.1) corresponding to $(f,u_n,z_0)$) converges to $z(f,u,z_0)$ for the weak topology of $L^2(0,T;Z)$, and $(z(f,u_n,z_0)(T))_n$ converges to $z(f,u,z_0)(T)$ for the weak topology of $Z$.*

**Proof.** The lemma is a direct consequence of Theorems 8.2.1 and 2.6.2.

**Proof of Theorem 8.2.4.** The proof is completely analogous to that of Theorem 7.3.1. ∎

**Theorem 8.2.5** *If $(\bar{z},\bar{u})$ is the solution to $(P)$ then $\bar{u} = -B_1^*(-A^*)^{1-\alpha}p$, where $p$ is the solution to equation*

$$-p' = A^*p + C^*(Cz - y_d), \qquad p(T) = D^*(Dz(T) - y_T). \tag{8.2.6}$$

*Conversely, if a pair $(\tilde{z},\tilde{p}) \in C([0,T];Z) \times C([0,T];Z)$ obeys the system*

$$\begin{aligned} &\tilde{z}' = A\tilde{z} - (-A)^{1-\alpha}B_1B_1^*(-A^*)^{1-\alpha}\tilde{p} + f, \qquad \tilde{z}(0) = z_0,\\ &-\tilde{p}' = A^*\tilde{p} + C^*(C\tilde{z} - y_d), \qquad \tilde{p}(T) = D^*(D\tilde{z}(T) - y_T), \end{aligned} \tag{8.2.7}$$

*then the pair $(\tilde{z}, -B_1^*(-A^*)^{1-\alpha}\tilde{p})$ is the optimal solution to problem $(P)$.*

**Proof.** Let $(\bar{z}, \bar{u})$ be the optimal solution to problem $(P)$. Set $F(u) = J(z(f,u),u)$. For every $u \in L^2(0,T;U)$, we have

$$F'(\bar{u})u = \int_0^T (Cz(t) - y_d, Cw(t))_Y + (Dz(T) - y_T, Dw(T))_{Y_T} + \int_0^T (\bar{u}(t), u(t))_U$$

$$= \int_0^T (C^*Cz(t) - y_d, w(t))_Z + (D^*Dz(T) - y_T, w(T))_Z + \int_0^T (\bar{u}(t), u(t))_U,$$

where $w$ is the solution to

$$w' = Aw + Bu, \qquad w(0) = 0.$$

Applying formula (8.2.5) to $p$ and $w$, we obtain

$$F'(\bar{u})u = \int_0^T ((-A^*)^{1-\alpha}p(t), B_1u(t))_Z + \int_0^T (\bar{u}(t), u(t))_U = \int_0^T (B_1^*(-A^*)^{1-\alpha}p(t) + \bar{u}(t), u(t))_U,$$

for all $u \in L^2(0,T;U)$. The first part of the Theorem is established. The second part follows from Theorem 2.2.3 (see also the proof of Theorem 2.2.2).

### 8.2.2 The case $\alpha \le \frac{1}{2}$

**Theorem 8.2.6** *Suppose that $(\mathcal{HP})$ is satisfied with $\alpha \le \frac{1}{2}$. For every $z_0 \in Z$, every $f \in L^2(0,T;Z)$ and every $u \in L^2(0,T;U)$, equation (8.1.1) admits a unique weak solution $z(z_0,u,f)$ belonging to $L^r(0,T;Z)$ for every $r < \frac{2}{1-2\alpha}$, and the mapping*

$$(z_0, u, f) \longmapsto z(z_0, u, f)$$

*is continuous from $Z \times L^2(0,T;U) \times L^2(0,T;Z)$ into $L^r(0,T;Z)$ for every $r < \frac{2}{1-2\alpha}$.*

**Proof.** We know that equation (8.1.1) admits a unique weak solution $z(z_0,u,f)$ in $C([0,T];(D(A^*))')$. Moreover $z$ is defined by

$$z(t) = e^{tA}z_0 + \int_0^t e^{(t-s)(A^*)^*}Bu(s)ds$$

$$= e^{tA}z_0 + \int_0^t e^{(t-s)(A^*)^*}(-A)^{1-\alpha}B_1u(s)ds.$$

Thus we have

$$|z(t)|_Z \le |z_0|_Z + \int_0^t \frac{C}{(t-s)^{1-\alpha}}|u(s)|_U\,ds.$$

Due to Young's inequality for the convolution product, we verify that $t \mapsto |z(t)|_Z$ belongs to $L^r(0,T;Z)$ for every $r < \frac{2}{1-2\alpha}$. ∎

In general the solution to equation (8.1.1) does not belong to $C([0,T];Z)$. Thus we cannot study problem $(P)$ in the case where $D$ belongs to $\mathcal{L}(Z, Y_T)$. We have to make an additional assumption.

$(\mathcal{HD})$ (*Smoothing property of operator $D$*)

There exists $\beta \in ]\frac{1}{2} - \alpha, \frac{1-\alpha}{2}[$ such that

$$(-A^*)^\beta D^*D(-A)^\beta \in \mathcal{L}(Z).$$

**Theorem 8.2.7** *Suppose that $(\mathcal{HP})$ is satisfied with $\alpha \le \frac{1}{2}$. The mapping*

$$(z_0, u, f) \longmapsto z(z_0, u, f)$$

*is continuous from $Z \times L^2(0,T;U) \times L^2(0,T;Z)$ into $C([0,T]; D((-A)^{-\beta}))$.*

**Proof.** From the equality

$$(-A)^{-\beta} z(t) = (-A)^{-\beta} e^{tA} z_0 + \int_0^t e^{(t-s)(A^*)^*} (-A)^{1-\alpha-\beta} B_1 u(s) ds,$$

it follows that

$$|(-A)^{-\beta} z(t)|_Z \le |z_0|_Z + \int_0^t \frac{C}{(t-s)^{1-\alpha-\beta}} |u(s)|_U \, ds.$$

Now the end of the proof is classical. ∎

Since $z$ does not belong to $C([0,T];Z)$, $z(T)$ is not defined in $Z$, it is only defined in $D((-A)^{-\beta}))$ (Theorem 8.2.7). Thus we have to give a precise meaning to $|Dz(T) - y_T|^2_{Y_T}$. We first assume that $y_T$ is of the form $y_T = D(-A)^\beta z_T$ for some $z_T \in D((-A)^\beta)$, and we replace $|Dz(T) - Dz_T|^2_{Y_T}$ in the definition of $J$ by

$$\Big((-A^*)^\beta D^* D(-A)^\beta \Big((-A)^{-\beta} z(T) - z_T\Big), (-A)^{-\beta} z(T) - z_T\Big)_Z.$$

Thus we deal with the control problem

$(\widehat{P})$ $\quad \inf\{\hat{J}(z,u) \mid (z,u) \in C([0,T];Z) \times L^2(0,T;U),\ (z,u) \text{ satisfies } (8.1.1)\}.$

with

$$\hat{J}(z,u) = \frac{1}{2} \int_0^T |Cz(t) - y_d|^2_Y + \frac{1}{2} \int_0^T |u(t)|^2_U$$
$$+ \frac{1}{2} ((-A^*)^\beta D^* D(-A)^\beta ((-A)^{-\beta} z(T) - z_T), (-A)^{-\beta} z(T) - z_T)_Z.$$

**Theorem 8.2.8** *For every $u \in L^2(0,T;U)$, and every $(g, p_T) \in L^2(0,T;Z) \times D((-A^*)^\beta)$, the solution $z$ to equation*

$$z' = Az + Bu, \qquad z(0) = 0,$$

*and the solution $p$ to equation*

$$-p' = A^* p + g, \qquad p(T) = p_T,$$

*satisfy the following formula*

$$\int_0^T (B_1 u(t), (-A^*)^{1-\alpha} p(t))_Z \, dt = \int_0^T (z(t), g(t))_Z \, dt + (z(T), p_T)_Z. \tag{8.2.8}$$

**Proof.** As for Theorem 8.2.3 the proof is straightforward if we prove that the function $p$ belongs to $L^2(0,T;D((-A^*)^{1-\alpha}))$. We have

$$(-A^*)^{1-\alpha} p(t) = (-A^*)^{1-\alpha-\beta} e^{(T-t)A^*} (-A^*)^\beta p_T + \int_t^T (-A^*)^{1-\alpha} e^{(s-t)A^*} g(s) ds.$$

Using estimates on analytic semigroups we obtain

$$|(-A^*)^{1-\alpha}p(t)| \le \frac{C}{(T-t)^{1-\alpha-\beta}}|(-A^*)^{\beta}p_T|_Z + \int_t^T \frac{C}{(s-t)^{1-\alpha}}|g(s)|_Z ds.$$

The mapping $t \mapsto \frac{C}{t^{1-\alpha-\beta}}$ belongs to $L^2(0,T)$. Moreover the mapping $t \mapsto |g(t)|$ belongs to $L^\infty(0,T)$ and the mapping $t \mapsto \frac{C}{t^{1-\alpha}}$ belongs to $L^s(0,T)$ for all $s < \frac{1}{1-\alpha}$. Thus the mapping $t \mapsto \int_t^T \frac{C}{(s-t)^{1-\alpha}}|g(s)|_Z ds$ belongs to $L^\infty(0,T)$. ∎

**Theorem 8.2.9** *Assume that ($\mathcal{HD}$) and ($\mathcal{HP}$) are satisfied with $\alpha \le \frac{1}{2}$. Problem $(\widehat{P})$ admits a unique solution $(z,u)$.*

To prove this theorem we need the following lemma.

**Lemma 8.2.2** *Let $(u_n)_n$ be a sequence in $L^2(0,T;U)$ converging to $u$ for the weak topology of $L^2(0,T;U)$. Then $(z(f,u_n,z_0))_n$ (the sequence of solutions to equation (8.1.1) corresponding to $(f,u_n,z_0)$) converges to $z(f,u,z_0)$ for the weak topology of $L^2(0,T;Z)$, and $(z(f,u_n,z_0)(T))_n$ converges to $z(f,u,z_0)(T)$ for the weak topology of $D((-A)^{-\beta})$.*

**Proof.** The lemma is a direct consequence of Theorems 8.2.7 and 2.6.2. ∎

**Proof of Theorem 8.2.9.** We leave the reader adapt the proof of Theorem 8.2.4. ∎

**Theorem 8.2.10** *If $(\bar z,\bar u)$ is the solution to $(\widehat{P})$ then $\bar u = -B_1^*(-A^*)^{1-\alpha}p$, where $p$ is the solution to equation*

$$-p' = A^*p + C^*(Cz-y_d), \qquad p(T) = D^*D(-A)^{\beta}((-A)^{-\beta}z(T)-z_T). \tag{8.2.9}$$

*Conversely, if a pair $(\tilde z,\tilde p) \in C([0,T];Z)\times C([0,T];Z)\cap L^2(0,T;D((-A^*)^{1-\alpha}))$ obeys the system*

$$\begin{aligned} \tilde z' &= A\tilde z - (-A)^{1-\alpha}B_1B_1^*(-A^*)^{1-\alpha}\tilde p + f, \qquad \tilde z(0)=z_0,\\ -\tilde p' &= A^*\tilde p + C^*(C\tilde z - y_d), \qquad \tilde p(T) = D^*D(-A)^{\beta}((-A)^{-\beta}z(T)-z_T), \end{aligned} \tag{8.2.10}$$

*then the pair $(\tilde z, -B_1^*(-A^*)^{1-\alpha}\tilde p)$ is the optimal solution to problem $(\widehat{P})$.*

**Proof.** Let $(\bar z,\bar u)$ be the optimal solution to problem $(P)$. Set $F(u) = J(z(f,u),u)$. For every $u \in L^2(0,T;U)$, we have

$$F'(\bar u)u = \int_0^T (C^*(Cz(t)-y_d), w(t))_Z$$

$$+((-A^*)^{\beta}D^*D(-A)^{\beta}((-A)^{-\beta}z(T)-z_T),(-A)^{-\beta}w(T))_Z + \int_0^T (\bar u(t),u(t))_U,$$

where $w$ is the solution to

$$w' = Aw + Bu, \qquad w(0)=0.$$

Applying formula (8.2.8) to $p$ and $w$, we obtain

$$F'(\bar u)u = \int_0^T ((-A^*)^{1-\alpha}p(t), B_1u(t))_Z + \int_0^T (\bar u(t),u(t))_U = \int_0^T (B_1^*(-A^*)^{1-\alpha}p(t)+\bar u(t),u(t))_U,$$

for all $u \in L^2(0,T;U)$. The first part of the Theorem is established. The second part follows from Theorem 2.2.3. ∎

## 8.3 The hyperbolic case

In this section we suppose that $(\mathcal{HH})$ is satisfied.

To study equation (8.1.1), we consider the operator defined on $L^2(0,T;U)$ by

$$Lu(t) = \int_0^t e^{(t-\tau)(A^*)^*} Bu(\tau)\, d\tau.$$

**Theorem 8.3.1** *The operator $L$ is continuous from $L^2(0,T;U)$ into $C([0,T];Z)$.*

**Proof.** Set $z(t) = \int_0^t e^{(t-\tau)(A^*)^*} Bu(\tau)\, d\tau$. For all $\zeta \in D(A^*)$, we have

$$\langle z(t), \zeta\rangle_{(D(A^*))', D(A^*)} = \int_0^t (u(\tau), B^* e^{(t-\tau)A^*}\zeta)_U\, d\tau.$$

Thanks to $(\mathcal{HH})$ it follows that

$$|\langle z(t), \zeta\rangle_{(D(A^*))', D(A^*)}|^2 \le \|u\|^2_{L^2(0,T;U)} \int_0^t \|B^* e^{(t-\tau)A^*}\zeta\|^2_U\, d\tau$$

$$\le C\|u\|^2_{L^2(0,T;U)}\|\zeta\|^2_Z \qquad \text{for all } \zeta \in D(A^*).$$

Thus $z$ belongs to $L^\infty(0,T;Z)$. To prove that $z \in C([0,T];Z)$, we proceed by approximation. Let $(u_n)_n$ be a sequence in $C^1([0,T];U)$ converging to $u$ in $L^2(0,T;U)$. Due to the assumption on the operator $B$, there exists $\lambda \in \rho(A)$ such that $B = (\lambda I - A)B_0$, with $B_0 \in \mathcal{L}(U;Z)$. From Theorem 4.2.2 it follows that $\int_0^t e^{(t-\tau)A} B_0 u_n(\tau)\, d\tau \in C([0,T];D(A))$ . Then

$$Lu_n(t) = (\lambda I - A)\int_0^t e^{(t-\tau)A} B_0 u_n(\tau)\, d\tau \in C([0,T];Z).$$

and $(Lu_n)_n$ converges to $Lu$ in $C([0,T];Z)$. ∎

The operator $L^*$, the adjoint of $L$ in the sense that $(Lu, f)_{L^2(0,T;Z)} = (u, L^*f)_{L^2(0,T;U)}$, is defined by

$$L^*f(t) = \int_t^T B^* e^{(\tau-t)A^*} f(\tau)\, d\tau \qquad \text{for } f \in L^1(0,T;D(A^*)).$$

For $f \in L^1(0,T;Z)$, we have to set

$$L^*f(t) = \int_t^T [B^* e^{(\tau-t)A^*}]_e f(\tau)\, d\tau.$$

Thanks to Theorem 8.3.1, it can be shown that the operator $L^*$ is continuous from $L^1(0,T;Z)$ into $L^2(0,T;U)$.

**Theorem 8.3.2** *Suppose that $(\mathcal{HH})$ is satisfied. For every $z_0 \in Z$, every $f \in L^2(0,T;Z)$ and every $u \in L^2(0,T;U)$, equation (8.1.1) admits a unique weak solution $z(z_0,u,f)$ in $L^2(0,T;Z)$, this solution belongs to $C([0,T];Z)$ and the mapping*

$$(z_0, u, f) \longmapsto z(z_0, u, f)$$

*is continuous from $Z \times L^2(0,T;U) \times L^2(0,T;Z)$ into $C([0,T];Z)$.*

**Proof.** The result is a direct consequence of Theorem 8.3.1.

**Theorem 8.3.3** *For every* $u \in L^2(0,T;U)$*, and every* $(g, p_T) \in C([0,T];Z) \times Z$*, the solution* $w$ *to equation*

$$w' = Aw + Bu, \qquad w(0) = 0,$$

*satisfies the following formula*

$$\int_0^T (w(t), g(t))_Z \, dt + (w(T), p_T)_Z$$

$$= \int_0^T \Big([B^* e^{(T-t)A^*}]_e \, p_T, u(t)\Big)_U dt + \int_0^T \Big(\int_t^T [B^* e^{(s-t)A^*}]_e \, g(s)\, ds, u(t)\Big)_U dt, \tag{8.3.11}$$

*where* $[B^* e^{tA^*}]_e$ *denotes the extension of* $B^* e^{tA^*}$ *to* $Z$*. If moreover* $p_T \in D(A^*)$ *and* $g \in L^2(0,T;D(A^*))$*, then the solution* $p$ *to the adjoint equation*

$$-p' = A^* p + g, \qquad p(T) = p_T,$$

*belongs to* $C([0,T];D(A^*))$ *and we have*

$$\int_0^T (w(t), g(t))_Z \, dt + (w(T), p_T)_Z = \int_0^T (u(t), B^* p(t))_U \, dt. \tag{8.3.12}$$

**Proof.** We have

$$\int_0^T (w(t), g(t))_Z \, dt + (w(T), p_T)_Z$$

$$= \Big(p_T, \int_0^T e^{(T-t)(A^*)^*} Bu(t) dt\Big)_Z + \int_0^T \Big(g(s), \int_0^s e^{(s-t)(A^*)^*} Bu(t)\, dt\Big)_Z ds$$

$$= \int_0^T \Big([B^* e^{(T-t)A^*}]_e p_T, u(t)\Big)_U dt + \int_0^T \Big(\int_t^T [B^* e^{(s-t)A^*}]_e g(s) ds, u(t)\Big)_U dt.$$

If $p$ belongs to $C([0,T];D(A^*))$, the mappings $t \mapsto e^{(T-t)A^*} p_T$ and $t \mapsto \int_t^T e^{(s-t)A^*} g(s) ds$ belong to $C([0,T];D(A^*))$, $[B^* e^{(s-t)A^*}]_e \, g(s) = B^* e^{(s-t)A^*} g(s)$ for almost every $s \in (t,T)$, and $[B^* e^{(T-t)A^*}]_e \, p_T = B^* e^{(T-t)A^*} p_T$ for all $t \in [0,T]$. Therefore, (8.3.12) is proved. ∎

**Theorem 8.3.4** *Assume that* ($\mathcal{HH}$) *is satisfied. Problem* $(P)$ *admits a unique solution.*

To prove this theorem we need the following lemma.

**Lemma 8.3.1** *Let* $(u_n)_n$ *be a sequence in* $L^2(0,T;U)$ *converging to* $u$ *for the weak topology of* $L^2(0,T;U)$*. Then* $(z(f, u_n, z_0))_n$ *(the sequence of solutions to equation (8.1.1) corresponding to* $(f, u_n, z_0)$*) converges to* $z(f,u,z_0)$ *for the weak topology of* $L^2(0,T;Z)$*, and* $(z(f,u_n,z_0)(T))_n$ *converges to* $z(f,u,z_0)(T)$ *for the weak topology of* $Z$*.*

**Proof.** The lemma is a direct consequence of Theorems 8.3.2 and 2.6.2.

**Proof of Theorem 8.3.4.** See exercise 8.7.1.

**Theorem 8.3.5** *If $(\bar{z}, \bar{u})$ is the solution to $(P)$ then*

$$\bar{u}(t) = -[B^* e^{(T-t)A^*}]_e\, D^*(Dz(T) - y_T) - \int_t^T [B^* e^{(s-t)A^*}]_e\, C^*(Cz(s) - y_d(s))ds.$$

*If moreover $D^*(Dz(T) - y_T) \in D(A^*)$ and $C^*(Cz - y_d) \in L^2(0, T; D(A^*))$, then $\bar{u} = -B^*p$, where $p$ is the solution to equation*

$$-p' = A^*p + C^*(Cz - y_d), \qquad p(T) = D^*(Dz(T) - z_T). \tag{8.3.13}$$

*Conversely, if a pair $(\tilde{z}, \tilde{u}) \in C([0, T]; Z) \times L^2(0, T; Z)$ obeys the system*

$$\begin{aligned} &\tilde{z}' = A\tilde{z} + B\tilde{u} + f, \qquad \tilde{z}(0) = z_0, \\ &\tilde{u}(t) = -[B^* e^{(T-t)A^*}]_e\, D^*(D\tilde{z}(T) - y_T) - \int_t^T [B^* e^{(s-t)A^*}]_e\, C^*(C\tilde{z}(s) - y_d(s))ds, \end{aligned} \tag{8.3.14}$$

*then the pair $(\tilde{z}, \tilde{u})$ is the optimal solution to problem $(P)$.*
*If a pair $(\tilde{z}, \tilde{p}) \in C([0, T]; Z) \times C([0, T]; D(A^*))$ obeys the system*

$$\begin{aligned} \tilde{z}' &= A\tilde{z} - BB^*\tilde{p} + f, \qquad \tilde{z}(0) = z_0, \\ -\tilde{p}' &= A^*\tilde{p} + C^*(C\tilde{z} - y_d), \qquad \tilde{p}(T) = D^*(D\tilde{z}(T) - y_T), \end{aligned} \tag{8.3.15}$$

*then the pair $(\tilde{z}, -B^*\tilde{p})$ is the optimal solution to problem $(P)$.*

**Proof.** Let $(\bar{z}, \bar{u})$ be the optimal solution to problem $(P)$. Set $F(u) = J(z(f, u), u)$. For every $u \in L^2(0, T; U)$, we have

$$F'(\bar{u})u = \int_0^T (Cz(t) - y_d, Cw(t))_Y + (Dz(T) - y_T, Dw(T))_{Y_T} + \int_0^T (\bar{u}(t), u(t))_U$$

$$= \int_0^T \Big(C^*(Cz(t) - y_d), w(t)\Big)_Z + \Big(D^*(Dz(T) - y_T), w(T)\Big)_Z + \int_0^T (\bar{u}(t), u(t))_U,$$

where $w$ is the solution to

$$w' = Aw + Bu, \qquad w(0) = 0.$$

We obtain the expression of $\bar{u}$ by applying formula (8.3.11). If $D^*(Dz(T) - y_T) \in D(A^*)$ and $C^*(Cz - y_d) \in L^2(0, T; D(A^*))$, the characterization of $\bar{u}$ follows from formula (8.3.12). The first part of the Theorem is established. The second part follows from Theorem 2.2.3 (see also the proof of Theorem 2.2.2).

## 8.4 The heat equation

We are going to see that $(\mathcal{HP})$ is satisfied for the heat equation with a Neumann boundary control in the case when $\alpha > \frac{1}{2}$, and with a Dirichlet boundary control in the case when $\alpha \leq \frac{1}{2}$.

### 8.4.1 Neumann boundary control

We want to write equation (5.3.10) in the form (8.1.1). For this, we set $Z = L^2(\Omega)$ and we define the unbounded operator $A$ in $Z$ by

$$D(A) = \{z \in H^2(\Omega) \mid \frac{\partial z}{\partial n} = 0\}, \qquad Az = \Delta z.$$

We know that $A$ generates an analytic semigroup on $Z$ (see [2]),

$$D((I-A)^\alpha) = H^{2\alpha}(\Omega) \quad \text{if } \alpha \in ]0, \frac{3}{4}[,$$

and

$$D((I-A)^\alpha) = \{z \in H^{2\alpha}(\Omega) \mid \frac{\partial z}{\partial n} = 0\} \quad \text{if } \alpha \in ]\frac{3}{4}, 1[.$$

Consider now the Neumann operator $N$ from $L^2(\Gamma)$ into $L^2(\Omega)$ defined by $N: \ u \mapsto w$ , where $w$ is the solution to

$$\Delta w - w = 0 \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n} = u \quad \text{on } \Gamma.$$

From [12] we deduce that $N \in \mathcal{L}(L^2(\Gamma); H^{\frac{3}{2}}(\Omega))$. This implies that $N \in \mathcal{L}(L^2(\Gamma); D((I-A)^\alpha))$ for all $\alpha \in ]0, \frac{3}{4}[$. We also have $N \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma); H^2(\Omega))$.

Suppose that $u \in C^1([0,T]; H^{\frac{1}{2}}(\Gamma))$ and denote by $z$ the solution to equation (5.3.10). Set $y(x,t) = z(x,t) - (Nu(t))(x)$. Since $u \in C^1([0,T]; H^{\frac{3}{2}}(\Gamma))$, $Nu(\cdot)$ belongs to $C^1([0,T]; H^2(\Omega))$, and we have

$$\frac{\partial y}{\partial t} - \Delta y = f - \frac{\partial Nu}{\partial t} + Nu \quad \text{in } Q, \quad \frac{\partial y}{\partial n} = 0 \quad \text{on } \Sigma, \quad y(x,0) = (z_0 - Nu(0))(x) \quad \text{in } \Omega.$$

Thus $y$ is defined by

$$y(t) = e^{tA}(z_0 - Nu(0)) - \int_0^t e^{(t-s)A} \frac{d}{dt}(Nu(s))ds + \int_0^t e^{(t-s)A}(Nu(s) + f(s))ds.$$

With an integration by parts we check that

$$(y + Nu)(t) = z(t) = e^{tA} z_0 + \int_0^t e^{(t-s)A} f(s)ds + \int_0^t (I-A)e^{(t-s)A} Nu(s)ds.$$

This means that, when $u$ is regular enough, equation (5.3.10) may be written in the form

$$z' = Az + f + (I-A)Nu, \quad z(0) = z_0.$$

It is known that $A$ is selfadjoint in $L^2(\Omega)$, that is $D(A) = D(A^*)$ and $Az = A^*z$ for all $z \in D(A)$. Since $A^* \in \mathcal{L}(D(A^*); L^2(\Omega))$, $(A^*)^* \in \mathcal{L}(L^2(\Omega); D(A^*)')$. Observing that $N \in \mathcal{L}(L^2(\Gamma); H^{\frac{3}{2}}(\Omega))$ and $(I-(A^*)^*) \in \mathcal{L}(L^2(\Omega); D(A^*)')$, we have $(I-(A^*)^*)N \in \mathcal{L}(L^2(\Gamma); D(A^*)')$.

In the case when $u \in L^2(0,T; L^2(\Gamma))$ we consider the equation

$$z' = (A^*)^* z + f + (I - (A^*)^*)Nu, \quad z(0) = z_0. \tag{8.4.16}$$

For every $z_0 \in Z$, $f \in L^2(Q)$, and $u \in L^2(0,T;L^2(\Gamma))$ equation (8.4.16) admits a unique solution in $H^1(0,T;D(A^*)')$ which is

$$z(t) = e^{t(A^*)^*}z_0 + \int_0^t (I-(A^*)^*)e^{(t-s)(A^*)^*}Nu(s)\,ds + \int_0^t e^{(t-s)A}f(s)\,ds. \tag{8.4.17}$$

When $u \in C^1([0,T];H^{\frac{1}{2}}(\Gamma))$, the solutions given by Theorem 5.3.4 and by the formula (8.4.17) coincide. Henceforth, by density arguments it follows that the solution defined by (8.4.17) and the one of Theorem 5.3.4 also coincide when $u \in L^2(0,T;L^2(\Gamma))$. In this case to simplify the writing, we often write

$$z' = Az + f + (I-A)Nu, \quad z(0) = z_0, \tag{8.4.18}$$

in place of (8.4.16). Since $N \in \mathcal{L}(L^2(\Gamma);D((I-A)^\alpha))$ for all $\alpha \in ]0,\frac{3}{4}[$, the operator $(I-A)N = (I-A)^{1-\alpha}(I-A)^\alpha N$ can be decomposed in the form $(I-A)N = (I-A)^{1-\alpha}B_1$, where $B_1 = (I-A)^\alpha N$ belongs to $\mathcal{L}(L^2(\Gamma);L^2(\Omega))$. This decomposition will be very useful to study the Riccati equation corresponding to problem $(P_3)$ (see [26]).

### 8.4.2 Dirichlet boundary control

We set $Z = L^2(\Omega)$ and we define the unbounded operator $A$ in $Z$ by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \qquad Az = \Delta z.$$

We know that $A$ is the infinitesimal generator of an analytic semigroup on $Z$ and that (see [2])

$$D((-A)^\alpha) = H^{2\alpha}(\Omega) \quad \text{if } \alpha \in ]0,\frac{1}{4}[$$

and

$$D((-A)^\alpha) = \{z \in H^{2\alpha}(\Omega) \mid z = 0 \text{ on } \Gamma\} \quad \text{if } \alpha \in ]\frac{1}{4},1[.$$

We define the Dirichlet operator $G$ from $L^2(\Gamma)$ into $L^2(\Omega)$ by $G: \ u \mapsto w$ , where $w$ is the solution to

$$\Delta w = 0 \quad \text{in } \Omega, \quad w = u \quad \text{on } \Gamma.$$

From [12] we deduce that $G \in \mathcal{L}(L^2(\Gamma);H^{\frac{1}{2}}(\Omega))$. This implies that $G \in \mathcal{L}(L^2(\Gamma);D((-A)^\alpha))$ for all $\alpha \in ]0,\frac{1}{4}[$. We also have $G \in \mathcal{L}(H^{\frac{3}{2}}(\Gamma);H^2(\Omega))$.

Suppose that $u \in C^1([0,T];H^{\frac{3}{2}}(\Gamma))$ and denote by $z$ the solution to equation (5.4.20). Set $y(x,t) = z(x,t) - (Gu(t))(x)$. Since $u \in C^1([0,T];H^{\frac{3}{2}}(\Gamma))$, $Gu(\cdot)$ belongs to $C^1([0,T];H^2(\Omega))$. Thus we have

$$\frac{\partial y}{\partial t} - \Delta y = f - \frac{\partial Gu}{\partial t} \quad \text{in } Q, \quad y = 0 \quad \text{on } \Sigma, \quad y(x,0) = (z_0 - Gu(0))(x) \quad \text{in } \Omega.$$

As for Neumann controls we can check that, when $u$ is regular enough, equation (5.4.20) may be written in the form

$$z' = Az + f + (-A)Gu, \quad z(0) = z_0.$$

And in the case when $u \in L^2(0,T;L^2(\Gamma))$ we still continue to use the above formulation even if for a correct writing $A$ should be replaced by its extension $(A^*)^*$.

Since $G \in \mathcal{L}(L^2(\Gamma); D((-A)^\alpha))$ for all $\alpha \in ]0, \frac{1}{4}[$, the operator $(-A)G$ can be decomposed in the form $(-A)G = (-A)^{1-\alpha}B_1$, where $B_1 = (-A)^\alpha G$ belongs to $\mathcal{L}(L^2(\Gamma); L^2(\Omega))$. We shall see that this situation (with $0 < \alpha < \frac{1}{4}$) is more complicated than the previous one where $\alpha$ was allowed to take values greater than $\frac{1}{2}$.

## 8.5 The wave equation

We only treat the case of a Dirichlet boundary control. We first define the unbounded operator $\Lambda$ in $H^{-1}(\Omega)$ by

$$D(\Lambda) = H^1_0(\Omega), \qquad \Lambda z = \Delta z.$$

We set $Z = L^2(\Omega) \times H^{-1}(\Omega)$. We define the unbounded operator $A$ in $Z$ by

$$D(A) = H^1_0(\Omega) \times L^2(\Omega), \qquad A = \begin{pmatrix} 0 & I \\ \Lambda & 0 \end{pmatrix}.$$

Using the Dirichlet operator $G$ introduced in section 8.4.2, equation (6.6.15) may be written in the form

$$\frac{d^2 z}{dt^2} = \Lambda z - \Lambda G u + f, \quad z(0) = z_0, \quad \frac{dz}{dt}(0) = z_1.$$

Now setting $y = (z, \frac{dz}{dt})$, we have

$$\frac{dy}{dt} = Ay + Bu + F, \quad y(0) = y_0, \tag{8.5.19}$$

with

$$Bu = \begin{pmatrix} 0 \\ -\Lambda G u \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad \text{and} \quad y_0 = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}.$$

The adjoint operator of $A$ for the $Z$-topology is defined by

$$D(A^*) = H^1_0(\Omega) \times L^2(\Omega), \qquad A^* = \begin{pmatrix} 0 & -I \\ -\Lambda & 0 \end{pmatrix}.$$

The operator $(A, D(A))$ is a strongly continuous group of contractions on $Z$. Set

$$C(t)z_0 = e^{tA} \begin{pmatrix} z_0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad S(t)z_1 = e^{tA} \begin{pmatrix} 0 \\ z_1 \end{pmatrix}.$$

Since $(e^{tA})_{t\geq 0}$ is a group, we can verify that $C(t) = \frac{1}{2}(e^{tA} + e^{-tA})$. Using equation (8.5.19), we can prove that $S(t)z = \int_0^t C(\tau)z\, d\tau$ and

$$e^{tA} = \begin{pmatrix} C(t) & S(t) \\ \Lambda S(t) & C(t) \end{pmatrix}.$$

We can also check that

$$e^{tA^*} = e^{-tA} = \begin{pmatrix} C(t) & -S(t) \\ -\Lambda S(t) & C(t) \end{pmatrix}.$$

We denote by $B^*$ the adjoint of $B$, where $B$ is an unbounded operator from $L^2(\Gamma)$ into $Z$. Thus $B^*$ is the adjoint of $B$ with respect to the $L^2(\Gamma)$-topology and the $Z$-topology.

**Theorem 8.5.1** *For any $0 < T < \infty$ the operator defined on $D(A^*)$ by $\zeta \mapsto B^* e^{tA^*}\zeta$ admits a continuous extension from $Z$ into $L^2(0,T;L^2(\Gamma))$. In other words there exists a constant $C(T)$, depending on $T$, such that*

$$\int_0^T \|B^* e^{tA^*}\zeta\|^2_{L^2(\Gamma)} \le C(T)\|\zeta\|^2_Z \tag{8.5.20}$$

*for every $\zeta \in D(A^*)$.*

**Proof.** Let us first determine $B^*$. Recall the definition of the scalar product on $Z$:

$$((z_0,z_1),(y_0,y_1))_Z = \int_\Omega z_0 y_0\, dx + \langle(-\Lambda)^{-1}z_1, y_1\rangle_{H^1_0(\Omega),H^{-1}(\Omega)}.$$

For every $(y_0,y_1) \in D(A^*)$, we have

$$(B^*(y_0,y_1),u)_{L^2(\Gamma)} = (y_1, Bu)_{H^{-1}(\Omega)} = \langle(-\Lambda)^{-1}y_1, -\Lambda G u\rangle_{H^1_0(\Omega),H^{-1}(\Omega)} = \int_\Gamma \frac{\partial}{\partial n}\Lambda^{-1}y_1\, u,$$

for all $u \in H^{3/2}(\Gamma)$. Hence

$$B^*(y_0,y_1) = \frac{\partial}{\partial n}\Lambda^{-1}y_1.$$

Due to the expression of $e^{tA^*}$, we have

$$\Lambda^{-1}e^{tA^*}\begin{pmatrix}\zeta_0\\ \zeta_1\end{pmatrix} = \Lambda^{-1}\begin{pmatrix}C(t)\zeta_0 - S(t)\zeta_1\\ -\Lambda S(t)\zeta_0 + C(t)\zeta_1\end{pmatrix} = \begin{pmatrix}C(t)\Lambda^{-1}\zeta_0 - S(t)\Lambda^{-1}\zeta_1\\ -S(t)\zeta_0 + C(t)\Lambda^{-1}\zeta_1\end{pmatrix}.$$

From the previous calculations it follows that condition (8.5.20) is equivalent to

$$\int_\Sigma \left|\frac{\partial}{\partial n}\Big(-S(t)\zeta_0 + C(t)\Lambda^{-1}\zeta_1\Big)\right|^2 \le C(T)(\|\zeta_0\|^2_{L^2(\Omega)} + \|\Lambda^{-1}\zeta_1\|^2_{H^{-1}(\Omega)}). \tag{8.5.21}$$

Let us notice that if we set $\phi_0 = \Lambda^{-1}\zeta_1$, $\phi_1 = -\zeta_0$ and $\phi(t) = -S(t)\zeta_0 + C(t)\Lambda^{-1}\zeta_1$, then $\phi$ is the solution to

$$\frac{\partial^2\phi}{\partial t^2} - \Delta\phi = 0 \quad \text{in } Q, \qquad \phi = 0 \quad \text{on } \Sigma, \qquad \phi(x,0) = \phi_0 \text{ and } \frac{\partial\phi}{\partial t}(x,0) = \phi_1 \quad \text{in } \Omega.$$

Therefore the condition (8.5.21) is equivalent to

$$\int_\Sigma \left|\Big(\frac{\partial\phi}{\partial n}\Big)\right|^2 \le C(T)(\|\phi_0\|^2_{H^1_0(\Omega)} + \|\phi_1\|^2_{L^2(\Omega)}).$$

The proof follows from Theorem 6.5.1.

## 8.6 A first order hyperbolic system

Consider the first order hyperbolic system

$$\frac{\partial}{\partial t}\begin{bmatrix} z_1(x,t)\\ z_2(x,t)\end{bmatrix} = \frac{\partial}{\partial x}\begin{bmatrix} m_1 z_1\\ -m_2 z_2\end{bmatrix} - \begin{bmatrix} b_{11}z_1 + b_{12}z_2\\ b_{21}z_1 + b_{22}z_2\end{bmatrix}, \qquad \text{in } (0,\ell)\times(0,T) \tag{8.6.22}$$

with the initial condition

$$z_1(x,0) = z_{01}(x), \qquad z_2(x,0) = z_{02}(x) \qquad \text{in } (0,\ell), \tag{8.6.23}$$

and the boundary conditions

$$z_1(\ell,t) = u_1(t), \qquad z_2(0,t) = u_2(t) \qquad \text{in } (0,T). \tag{8.6.24}$$

This kind of systems intervenes in heat exchangers [31]. For simplicity we suppose that the coefficients $m_1 > 0$, $m_2 > 0$, $b_{11}$, $b_{12}$, $b_{21}$, $b_{22}$ are constant. We also suppose that

$$b_{11}z_1^2 + b_{21}z_2z_1 + b_{21}z_1z_2 + b_{22}z_2^2 \geq 0 \qquad \text{for all } (z_1,z_2) \in \mathbb{R}^2.$$

Before studying control problems, let us state existence results for the system (8.6.22)-(8.6.24).

### 8.6.1 State equation

We set $Z = L^2(0,\ell) \times L^2(0,\ell)$, and we define the unbounded operator $A$ in $Z$ by

$$D(A) = \{z \in H^1(0,\ell) \times H^1(0,\ell) \mid z_1(\ell) = 0, \quad z_2(0) = 0\}$$

and

$$Az = \begin{bmatrix} m_1\dfrac{dz_1}{dx} - b_{11}z_1 - b_{12}z_2 \\ -m_2\dfrac{dz_2}{dx} - b_{21}z_1 - b_{22}z_2 \end{bmatrix}.$$

We endow $D(A)$ with the norm $\|z\|_{D(A)} = (\|z_1\|^2_{H^1(0,\ell)} + \|z_2\|^2_{H^1(0,\ell)})^{1/2}$.

**Theorem 8.6.1** *For every $(f,g) \in L^2(0,\ell)^2$, the system $Az = (f,g)^T$ admits a unique solution in $D(A)$, and*

$$\|z\|_{D(A)} \leq C(\|f\|_{L^2(0,\ell)} + \|g\|_{L^2(0,\ell)}).$$

**Proof.** Let $A_0$ be the operator defined by $D(A_0) = D(A)$ and $A_0z = (m_1\frac{dz_1}{dx}, -m_2\frac{dz_2}{dx})^T$. It is clear that $A_0$ is an isomorphism from $D(A_0)$ into $L^2(0,\ell)^2$. We rewrite equation $Az = (f,g)^T$ in the form $z - A_0^{-1}Bz = A_0^{-1}(f,g)^T$, where

$$Bz = \begin{bmatrix} b_{11}z_1 + b_{12}z_2 \\ b_{21}z_1 + b_{22}z_2 \end{bmatrix}.$$

If $z \in D(A_0)$, then $Bz \in (H^1(0,\ell))^2$ and $A_0^{-1}Bz \in (H^2(0,\ell))^2 \cap D(A_0)$. Thus the operator $A_0^{-1}B$ is a compact operator in $D(A_0)$. Let us prove that $I - A_0^{-1}B$ is injective. Let $z \in D(A_0)$ be such that $(I - A_0^{-1}B)z = 0$. Then $Az = 0$. Multiplying the first equation in the system $Az = 0$ by $z_1$, the second equation by $z_2$, integrating over $(0,\ell)$, and adding the two equalities, we obtain:

$$m_1z_1(0)^2 + m_2z_2(\ell)^2 + \int_0^\ell b_{11}z_1^2 + b_{21}z_2z_1 + b_{21}z_1z_2 + b_{22}z_2^2 = 0.$$

Thus $z = 0$. Now the theorem follows from the Fredholm Alternative. ∎

**Theorem 8.6.2** *The operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $Z$.*

**Proof.** The theorem relies the Hille-Yosida theorem.
(i) The domain $D(A)$ is dense in $Z$. Prove that $A$ is a closed operator. Let $(z_n)_n = (z_{1,n}, z_{2,n})_n$ be a sequence converging to $z = (z_1, z_2)$ in $Z$, and such that $(Az_n)_n$ converges to $(f, g)$ in $Z$. We have $m_1 \frac{dz_1}{dx} - b_{11}z_1 - b_{12}z_2 = f$, and $-m_2 \frac{dz_2}{dx} - b_{21}z_1 - b_{22}z_2 = g$, because $(\frac{dz_{1,n}}{dx}, \frac{dz_{2,n}}{dx})_n$ converges to $(\frac{dz_1}{dx}, \frac{dz_2}{dx})$ in the sense of distributions in $(0, \ell)$. Due to Theorem 8.6.1, we have

$$\|z_n - z_m\|_{D(A)} \le C\|A(z_n - z_m)\|_{(L^2(0,\ell))^2}.$$

Thus $(z_n)_n$ is a Cauchy sequence in $D(A)$, and $z$, its limit in $Z$, belongs to $D(A)$. The first condition of Theorem 4.1.1 is satisfied.
(ii) For $\lambda > 0$, $f \in L^2(0, \ell)$, $g \in L^2(0, \ell)$, consider the equation

$$z \in D(A), \qquad \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

that is

$$\begin{aligned} \lambda z_1 - m_1 \frac{dz_1}{dx} + b_{11}z_1 + b_{12}z_2 &= f \qquad \text{in } (0,\ell), \quad z_1(\ell) = 0, \\ \lambda z_2 + m_2 \frac{dz_2}{dx} + b_{21}z_1 + b_{22}z_2 &= g \qquad \text{in } (0,\ell), \quad z_2(0) = 0. \end{aligned}$$

As for Theorem 8.6.1, we can prove that this equation admits a unique solution $z \in D(A)$. Multiplying the first equation by $z_1$, the second by $z_2$, and integrating over $(0, \ell)$, we obtain

$$\lambda \int_0^\ell (z_1^2 + z_2^2) + \int_0^\ell (b_{11}z_1^2 + b_{12}z_2z_1 + b_{21}z_1z_2 + b_{22}z_2^2) + m_1z_1(0)^2 + m_2z_2(\ell)^2 = \int_0^\ell (fz_1 + gz_2)$$

$$\le \Big(\int_0^\ell z_1^2 + \int_0^\ell z_2^2\Big)^{1/2}\Big(\int_0^\ell f^2 + \int_0^\ell g^2\Big)^{1/2}.$$

The proof is complete. ∎

**Theorem 8.6.3** *For every $z_0 = (z_{10}, z_{20}) \in Z$, equation*

$$\frac{\partial}{\partial t}\begin{bmatrix} z_1(x,t) \\ z_2(x,t) \end{bmatrix} = \frac{\partial}{\partial x}\begin{bmatrix} m_1 z_1 \\ -m_2 z_2 \end{bmatrix} - \begin{bmatrix} b_{11}z_1 + b_{12}z_2 \\ b_{21}z_1 + b_{22}z_2 \end{bmatrix}, \qquad \text{in } (0,\ell) \times (0,T)$$

*with the initial condition*

$$z_1(x,0) = z_{01}(x), \qquad z_2(x,0) = z_{02}(x) \qquad \text{in } (0,\ell),$$

*and homogeneous boundary conditions*

$$z_1(\ell,t) = 0, \qquad z_2(0,t) = 0 \qquad \text{in } (0,T),$$

*admits a unique weak solution in $L^2(0,T; L^2(0,\ell))$, this solution belongs to $C([0,T]; Z)$ and satisfies*

$$\|z\|_{C([0,T];Z)} \le \|z_0\|_Z.$$

The theorem is a direct consequence of Theorems 8.6.2 and 4.2.1. ∎

The adjoint operator of $(A, D(A))$, with respect to the $Z$-topology, is defined by

$$D(A^*) = \{(\phi, \psi) \in H^1(0, \ell) \times H^1(0, \ell) \mid \phi(0) = 0, \quad \psi(\ell) = 0\},$$

and

$$A^* \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} -m_1 \dfrac{d\phi}{dx} - b_{11}\phi - b_{21}\psi \\ m_2 \dfrac{d\psi}{dx} - b_{12}\phi - b_{22}\psi \end{bmatrix}.$$

To study the system (8.6.22)-(8.6.24), we define the operator $B$ from $\mathbb{R}^2$ into $(D(A^*))'$ by

$$B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} m_1 u_1 \delta_\ell \\ m_2 u_2 \delta_0 \end{bmatrix},$$

where $\delta_\ell$ and $\delta_0$ denote the Dirac distributions at $\ell$ and 0. Since $B \in \mathcal{L}(\mathbb{R}^2; (D(A^*))')$, we can write the system (8.6.22)-(8.6.24) in the form

$$z' = (A^*)^* z + Bu(t), \qquad z(0) = z_0,$$

where $(A^*)^*$ is the extension of $A$ to $(D(A^*))'$, and $u(t) = (u_1(t), u_2(t))^T$. From Theorem 4.3.2, if follows that, for every $u \in (L^2(0, T))^2$, the system (8.6.22)-(8.6.24) admits a unique weak solution $z$ in $L^2(0, T; (D(A^*))')$ which satisfies:

$$\|z\|_{C([0,T];(D(A^*))')} \le C(\|z_0\|_Z + \|u\|_{(L^2(0,T))^2}).$$

Moreover, due to Theorem 4.3.3, if $u \in (H^1(0, T))^2$ the solution $z$ belongs to $C([0, T]; Z)$ and

$$\|z\|_{C([0,T];Z)} \le C(\|z_0\|_Z + \|u\|_{(H^1(0,T))^2}).$$

We would like to prove that $z$ belongs to $C([0, T]; Z)$ when $u \in (L^2(0, T))^2$.

**Theorem 8.6.4** *For every $u = (u_1, u_2) \in \mathbb{R}^2$, and every $(f, g) \in (L^2(0, \ell))^2$, the system*

$$\begin{aligned} m_1 \frac{dz_1}{dx} + b_{11} z_1 + b_{12} z_2 &= f, \qquad z_1(\ell) = u_1, \\ -m_2 \frac{dz_2}{dx} + b_{21} z_1 + b_{22} z_2 &= g, \qquad z_2(0) = u_2, \end{aligned} \tag{8.6.25}$$

*admits a unique solution $z \in (H^1(0, \ell))^2$, and*

$$\|z\|_{(H^1(0,\ell))^2} \le C(|u_1| + |u_2| + \|f\|_{L^2(0,\ell)} + \|g\|_{L^2(0,\ell)}).$$

**Proof.** Let $\Lambda$ be the operator defined by $D(\Lambda) = (H^1(0, \ell))^2$ and $\Lambda z = (m_1 \frac{dz_1}{dx}, -m_2 \frac{dz_2}{dx}, z_1(\ell), z_2(0))^T$. It is clear that $\Lambda$ is an isomorphism from $D(\Lambda)$ into $L^2(0, \ell)^2 \times \mathbb{R}^2$. We rewrite equation (8.6.25) in the form $z - \Lambda^{-1} Dz = \Lambda^{-1}(f, g, u_1, u_2)^T$, where

$$Dz = \begin{bmatrix} b_{11} z_1 + b_{12} z_2 \\ b_{21} z_1 + b_{22} z_2 \\ 0 \\ 0 \end{bmatrix}.$$

If $z \in D(\Lambda)$, then $Dz \in (H^1(0, \ell))^2 \times \mathbb{R}^2$ and $\Lambda^{-1} Dz \in (H^2(0, \ell))^2$. Thus the operator $\Lambda^{-1} D$ is a compact operator in $D(\Lambda)$. The end of the proof is similar to that of Theorem 8.6.1. ∎

**Theorem 8.6.5** *For every $z_0 = (z_{10}, z_{20}) \in Z$, and every $u \in (L^2(0,T))^2$, the solution $z$ to system (8.6.22)-(8.6.24) belongs to $C([0,T];Z) \cap C([0,\ell];L^2(0,T))$ and*

$$\|z\|_{C([0,T];Z)} + \|z\|_{C([0,\ell];(L^2(0,T))^2)} \le C(\|z_0\|_Z + \|u\|_{(L^2(0,T))^2}).$$

**Proof.** Let $(u_n)_n$ be a sequence in $(C^1([0,T]))^2$ converging to $u$ in $(L^2(0,T))^2$, and let $(z_{0,n})_n$ be a sequence $D(A)$ converging to $z_0$ in $Z$. Let $z_n$ be the solution to (8.6.22)-(8.6.24) corresponding to $u_n$ and $z_{0,n}$. Let $w_n(t)$ be the solution to equation $\Lambda w_n(t) = (0,0,u_{1,n}(t),u_{2,n}(t))^T$. With Theorem 8.6.4 we can prove that $w_n$ belongs to $C^1([0,T];(H^1(0,\ell))^2)$. Observe that $z_n = y_n + w_n$, where $y_n = (y_{1,n}, y_{2,n})$ is the solution to equation

$$\frac{\partial}{\partial t}\begin{bmatrix} y_1(x,t) \\ y_2(x,t)\end{bmatrix} = \frac{\partial}{\partial x}\begin{bmatrix} m_1 y_1 \\ -m_2 y_2\end{bmatrix} - \begin{bmatrix} b_{11}y_1 + b_{12}y_2 \\ b_{21}y_1 + b_{22}y_2\end{bmatrix} - \begin{bmatrix} \dfrac{\partial w_{1,n}}{\partial t} \\ \dfrac{\partial w_{2,n}}{\partial t}\end{bmatrix} \quad \text{in } (0,\ell)\times(0,T),$$

with the initial condition

$$y_1(x,0) = z_{01,n}(x) - w_{1,n}(x,0), \qquad y_2(x,0) = z_{02,n}(x) - w_{2,n}(x,0) \qquad \text{in } (0,\ell),$$

and homogeneous boundary conditions

$$y_1(\ell,t) = 0, \qquad y_2(0,t) = 0 \qquad \text{in } (0,T).$$

By Theorem 4.2.2, $y_n$ belongs to $C([0,T];D(A)) \cap C^1([0,T];Z)$. Thus $z_n$ belongs to $C([0,T];(H^1(0,\ell))^2) \cap C^1([0,T];Z)$. Multiplying the first equation of the system by $z_{1,n}$, the second one by $z_{2,n}$, integrating over $(0,\ell)\times(0,t)$, and adding the two equalities, we obtain

$$\int_0^\ell (z_{1,n}(t)^2 + z_{2,n}(t)^2) + 2\int_0^t\int_0^\ell (b_{11}z_{1,n}^2 + b_{12}z_{1,n}z_{2,n} + b_{21}z_{1,n}z_{2,n} + b_{22}z_{2,n}^2)$$
$$+ \int_0^t m_1 z_1^2(0,\tau)d\tau + \int_0^t m_2 z_2^2(\ell,\tau)d\tau = \int_0^\ell (z_{01,n}^2 + z_{02,n}^2) + \int_0^t m_1 u_{1,n}^2 + \int_0^t m_2 u_{2,n}^2.$$

We first deduce

$$\|z_n\|^2_{C([0,T];Z)} \le C(\|u_{1,n}\|^2_{L^2(0,T)} + \|u_{2,n}\|^2_{L^2(0,T)} + \|z_{0,n}\|^2_Z).$$

In the same way, $z_n - z_m$ obeys

$$\|z_n - z_m\|^2_{C([0,T];Z)} \le C(\|u_{1,n} - u_{1,m}\|^2_{L^2(0,T)} + \|u_{2,n} - u_{2,m}\|^2_{L^2(0,T)} + \|z_{0,n} - z_{0,m}\|^2_Z).$$

Thus $(z_n)_n$ is a Cauchy sequence in $C([0,T];Z)$ and the estimate is proved in $C([0,T];Z)$.

To prove the estimate in $C([0,\ell];(L^2(0,T))^2)$, we multiplying the first equation of the system by $z_{1,n}$ and we integrate over $(x,\ell)\times(0,T)$. We multiply the second equation by $z_{2,n}$ and we integrate over $(0,x)\times(0,T)$, and adding the two equalities, we obtain

$$m_1\int_0^T z_{1,n}(x)^2 + m_2\int_0^T z_{2,n}(x)^2 + \int_x^\ell z_{1,n}(T)^2 - \int_x^\ell z_{01,n}^2 + \int_0^x z_{2,n}(T)^2 - \int_0^x z_{02,n}^2$$
$$= m_1\int_0^T u_{1,n}^2 + m_2\int_0^T u_{2,n}^2 - 2\int_0^T\int_x^\ell (b_{11}z_{1,n}^2 + b_{12}z_{1,n}z_{2,n}) - 2\int_0^T\int_0^x (b_{21}z_{1,n}z_{2,n} + b_{22}z_{2,n}^2).$$

Writing the estimate for $z_n - z_m$, we have

$$\|z_n - z_m\|^2_{C([0,\ell];(L^2(0,T))^2)}$$
$$\le C(\|u_{1,n} - u_{1,m}\|^2_{L^2(0,T)} + \|u_{2,n} - u_{2,m}\|^2_{L^2(0,T)} + \|z_{0,n} - z_{0,m}\|^2_Z + \|z_n - z_m\|^2_{C([0,T];Z)}).$$

Thus $(z_n)_n$ converges to $z$ in $C([0,\ell];(L^2(0,T))^2)$. ∎

### 8.6.2 Control problem

We want to study the control problem

$$(P_1) \qquad \inf\{J_1(z,u) \mid (z,u) \in C([0,T];Z) \times L^2(0,T)^2,\ (z,u) \text{ satisfies } (8.6.22) - (8.6.24)\},$$

where

$$J_1(z,u) = \frac{1}{2}\int_0^\ell |z(T) - z_d(T)|^2 + \frac{\beta}{2}\int_0^T (u_1^2 + u_2^2),$$

and $\beta > 0$. We assume that $z_d \in C([0,T];Z)$.

**Theorem 8.6.6** *Problem $(P_1)$ admits a unique solution $(\bar z, \bar u)$. Moreover $\bar u$ is characterized by*

$$\bar u_1(t) = -\frac{m_1}{\beta}\phi(\ell,t) \quad \textit{and} \quad \bar u_2(t) = -\frac{m_2}{\beta}\psi(0,t) \quad \textit{in } (0,T),$$

*where $(\phi,\psi)$ is the solution to the system*

$$-\frac{\partial}{\partial t}\begin{bmatrix} \phi(x,t) \\ \psi(x,t) \end{bmatrix} = \frac{\partial}{\partial x}\begin{bmatrix} -m_1\phi \\ m_2\psi \end{bmatrix} - \begin{bmatrix} b_{11}\phi + b_{21}\psi \\ b_{12}\phi + b_{22}\psi \end{bmatrix}, \qquad \textit{in } (0,\ell)\times(0,T) \qquad (8.6.26)$$

*with the terminal condition*

$$\phi(T) = z_1(T) - z_{d,1}(T), \qquad \psi(T) = z_2(T) - z_{d,2}(T) \qquad \textit{in } (0,\ell), \qquad (8.6.27)$$

*and the boundary conditions*

$$\phi(0,t) = 0, \qquad \psi(\ell,t) = 0 \qquad \textit{in } (0,T). \qquad (8.6.28)$$

**Proof.** (i) The existence of a unique solution to $(P_1)$ is classical and is left to the reader.

(ii) First observe that the solution $(\phi,\psi)$ to system (8.6.26)-(8.6.28) belongs to $C([0,T];Z) \cap C([0,\ell];(L^2(0,T))^2)$ (the proof is similar to that of Theorem 8.6.5). Thus $\phi(\ell,\cdot)$ and $\psi(0,\cdot)$ belong to $L^2(0,T)$.

Let $\zeta \in D(A^*)$, the function $t \mapsto e^{tA^*}\zeta$ is the solution to system

$$-\frac{\partial}{\partial t}\begin{bmatrix} \phi(x,t) \\ \psi(x,t) \end{bmatrix} = \frac{\partial}{\partial x}\begin{bmatrix} -m_1\phi \\ m_2\psi \end{bmatrix} - \begin{bmatrix} b_{11}\phi + b_{21}\psi \\ b_{12}\phi + b_{22}\psi \end{bmatrix}, \qquad \text{in } (0,\ell)\times(0,T) \qquad (8.6.29)$$

with the terminal condition

$$\phi(T) = \zeta_1, \qquad \psi(T) = \zeta_2 \qquad \text{in } (0,\ell), \qquad (8.6.30)$$

and the boundary conditions

$$\phi(0,t) = 0, \qquad \psi(\ell,t) = 0 \qquad \text{in } (0,T). \qquad (8.6.31)$$

We can verify that $B^*e^{tA^*}\zeta = (m_1\phi(\ell,t), m_2\psi(0,t))$, where $(\phi,\psi)$ is the solution to (8.6.29)-(8.6.31). Thus assumption $(\mathcal{HH})$ is satisfied by $(A,B)$ in $Z$, and applying Theorem 8.3.5, we have

$$\bar u = -\frac{1}{\beta}[B^*e^{(T-t)A^*}]_e(\bar z(T) - z_d(T)).$$

Since the solution to system (8.6.26)-(8.6.28) belongs to $C([0,\ell];(L^2(0,T))^2)$, using an approximation process we can prove that

$$[B^* e^{(T-t)A^*}]_e(\bar{z}(T) - z_d(T)) = (m_1\phi(\ell,t), m_2\psi(0,t)),$$

where $(\phi,\psi)$ is the solution to (8.6.26)-(8.6.28).

(iii) We can directly prove the optimality conditions for problem $(P_1)$ by using the method of chapters 5 and 6. Setting $F_1(u) = J_1(z(z_0,u),u)$, where $z(z_0,u)$ is the solution to system (8.6.22)-(8.6.24), we have

$$F_1'(\bar{u})u = \int_0^\ell (\bar{z}_1(T) - z_{d1}(T))w_{u1}(T) + \int_0^\ell (\bar{z}_2(T) - z_{d2}(T))w_{u2}(T) + \beta\int_0^T (\bar{u}_1 u_1 + \bar{u}_2 u_2),$$

where $w_u = z(0,u)$, and $z(0,u)$ is the solution to system (8.6.22)-(8.6.24) for $z_0 = 0$.

We can establish an integration by parts formula between $w_u$ and the solution $(\phi,\psi)$ to system (8.6.26)-(8.6.28) to completes the proof.

## 8.7 Exercises

**Exercise 8.7.1**

Prove the existence of a unique solution to problem $(P)$ of section 8.1 in the case where assumption $(\mathcal{HH})$ is satisfied.

**Exercise 8.7.2**

We consider a one-dimensional linear thermoelastic system

$$\begin{aligned} &z_{tt} - \alpha^2 z_{xx} + \gamma_1\theta_x = 0 \qquad \text{in } (0,L)\times(0,T),\\ &\theta_t + \gamma_2 z_{xt} - k\theta_{xx} = 0 \qquad \text{in } (0,L)\times(0,T), \end{aligned} \tag{8.7.32}$$

with the boundary conditions

$$z(0,t) = z(L,t) = 0 \quad \text{in } (0,T), \qquad \text{and } \theta_x(0,t) = u_1(t),\ \theta_x(L,t) = u_2(t), \tag{8.7.33}$$

and the initial conditions

$$z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x), \quad \text{and } \theta(x,0) = \theta_0(x) \quad \text{in } (0,L), \tag{8.7.34}$$

with $\alpha > 0$, $k > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$. Physically $z$ represents the displacement of a rod and $\theta$ its temperature. By setting $y = (y_1,y_2,y_3) = (z, z_t, \theta)$, system (8.7.32)-(8.7.34) can be written in the form of a first order evolution equation $y' = Ay + Bu$, $y(0) = y_0$. We set

$$A = \begin{pmatrix} 0 & I & 0 \\ \alpha^2\dfrac{d^2}{dx^2} & 0 & -\gamma_1\dfrac{d}{dx} \\ 0 & -\gamma_2\dfrac{d}{dx} & k\dfrac{d^2}{dx^2} \end{pmatrix},$$

and

$$D(A) = \{y \mid y_1 \in H^2 \cap H_0^1(0,L),\ y_2 \in H_0^1(0,L),\ y_3 \in H^2(0,L) \text{ such that } y_{3x}(0) = y_{3x}(L) = 0\}.$$

We endow $Y = H_0^1(0,L) \times L^2(0,L) \times L^2(0,L)$ with the scalar product

$$(y,w) = \int_0^L \Big(\frac{dy_1}{dx}\frac{dw_1}{dx} + y_2 w_2 + \frac{\gamma_1}{\gamma_2} y_3 w_3\Big).$$

1 - Prove that $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup on $Y$.

2 - We suppose that $z_0 \in H_0^1(0,L)$, $z_1 \in L^2(0,L)$, $\theta \in L^2(0,L)$, $u_1 \in L^2(0,T)$, $u_2 \in L^2(0,T)$. Prove that system (8.7.32)-(8.7.34) admits a unique solution $(z, z_t, \theta)$ in $C([0,T]; H_0^1(0,L)) \times C([0,T]; L^2(0,L)) \times C([0,T]; L^2(0,L))$.

3 - Consider the control problem
$(P_2)$
$\inf\{J_2(z,\theta,u) \mid (z, z_t, \theta, u) \in C([0,T]; Y) \times L^2(0,T)^2,\ (z, z_t, \theta, u) \text{ satisfies } (8.7.32) - (8.7.34)\}$,

where

$$J_2(z,\theta,u) = \frac{1}{2}\int_0^T \int_0^L (|z|^2 + |\theta|^2) + \frac{\beta}{2}\int_0^T (u_1^2 + u_2^2),$$

and $\beta > 0$. Prove that $(P_2)$ admits a unique solution. Characterize this solution by establishing first order optimality conditions.

# Chapter 9

# Control of a semilinear parabolic equation

## 9.1 Introduction

In this chapter we study control problems for a semilinear parabolic equation of Burgers' type in dimension 2. For a $L^2$-distributed control, we prove the existence of a unique solution to the state equation in $C([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$. Following the approach of the previous chapters we use the semigroup theory. We first prove the existence of a local solution for initial data in $L^{2p}(\Omega)$ for $p>2$, and next the existence of a global solution by establishing an energy estimate. The existence of a (uinque) solution for $L^2$-initial data is obtained by approximation. The classical method to study this kind of equation is the variational method (also called the Faedo-Galerkin method). This approach is treated in exercise 9.7.2. This chapter can be considered as an introduction to the optimal control of the Navier-Stokes equations [27]. Indeed the proof of optimality conditions is very similar in both cases.

To study the state equation and the control problem, we need additional regularity results on parabolic equations. These results are stated in Appendix (section 9.6).

## 9.2 Distributed control

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$, with a regular boundary $\Gamma$. Let $T>0$, set $Q=\Omega\times(0,T)$ and $\Sigma=\Gamma\times(0,T)$. We consider the equation

$$\frac{\partial z}{\partial t}-\Delta z+\Phi(z)=f+\chi_\omega u \quad \text{in } Q, \quad z=0 \quad \text{on } \Sigma, \quad z(x,0)=z_0 \quad \text{in } \Omega, \tag{9.2.1}$$

with $f\in L^2(Q)$, $u\in L^2(0,T;L^2(\omega))$, $z_0\in L^2(\Omega)$. The function $f$ is a given source term, $\chi_\omega$ is the characteristic function of $\omega$, $\omega$ is an open subset of $\Omega$, and the function $u$ is a control variable. The nonlinear term $\phi$ is defined by $\phi(z)=2z\partial_{x_1}z=\partial_{x_1}(z^2)$. Any other combination of first order partial derivatives may be considered, for example we can as well consider $\phi(z)=\Sigma_{i=1}^2 z\partial_{x_i}z$. We first want to prove the existence of a unique weak solution (in a sense to be precised) to equation (9.2.1). Recall that, if $z_0\in L^2(\Omega)$ and $g\in L^2(0,T;H^{-1}(\Omega))$, equation

$$\frac{\partial z}{\partial t}-\Delta z=g \quad \text{in } Q, \quad z=0 \quad \text{on } \Sigma, \quad z(x,0)=z_0 \quad \text{in } \Omega, \tag{9.2.2}$$

admits a unique weak solution in $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$. Moreover

$$z(t)=e^{tA}z_0+\int_0^t e^{(t-s)A}g(s)\,ds.$$

Observe that if $z$ belongs to $C([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$, then $z$ belonds to $L^4(Q)$ (Theorem 9.6.1), and $\phi(z)$ belongs to $L^2(0,T;H^{-1}(\Omega))$. Thus it is reasonable to consider equation (9.2.1) as a special form of equation (9.2.2) with $g=f+\chi_\omega u-\phi(z)$, and to define weak solutions to equation (9.2.1) in the following manner.

**Definition 9.2.1** *A function $z\in C([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ is a weak solution to equation (9.2.1) if, for every $\zeta\in H_0^1(\Omega)$, the mapping $t\mapsto\langle z(t),\zeta\rangle$ belongs to $H^1(0,T)$, $\langle z(0),\zeta\rangle=\langle z_0,\zeta\rangle$, and*

$$\frac{d}{dt}\langle z(t),\zeta\rangle=\langle\nabla z(t),\nabla\zeta\rangle+\langle f,\zeta\rangle+\langle\chi_\omega u,\zeta\rangle-\langle\phi(z),\zeta\rangle.$$

## 9.3 Existence of solutions for $L^{2p}$-initial data, $p>2$

### 9.3.1 Existence of a local solution

We suppose that $f\in L^2(Q)$, and $z_0\in L^{2p}(\Omega)$ with $p>2$. We want to prove that equation

$$\frac{\partial z}{\partial t}-\Delta z+\frac{1}{2}\partial_{x_1}(z^2)=f\quad\text{in } Q,\quad z=0\quad\text{on }\Sigma,\quad z(x,0)=z_0\quad\text{in }\Omega,\tag{9.3.3}$$

admits a solution in $C([0,\bar t];L^{2p}(\Omega))$ for $\bar t$ small enough.

Let $s>\frac{2p}{p-1}$. Due to Theorem 9.6.3, if $h\in L^s(0,T;L^p(\Omega))$ then $z_h$, the solution to equation (9.2.2) corresponding to $(g,z_0)$ with $z_0=0$, and $g=\partial_{x_1}h$, belongs to $C([0,T];L^{2p}(\Omega))$, and there exists a constant $C(s)$ such that

$$\|z_h\|_{C([0,T];L^{2p}(\Omega))}\le C(s)\|h\|_{L^s(0,T;L^p(\Omega))}.$$

Set $R=\|y\|_{C([0,T];L^{2p}(\Omega))}$, where $y$ is the solution to equation (9.2.2) corresponding to $(g,z_0)$ with $g=f$. Let us fix $s>\frac{2p}{p-1}$ and set $\bar t=(4RC(s))^{-s}$. Let $B(2R)$ be the closed ball in $C([0,\bar t];L^{2p}(\Omega))$, centered at the origin, with radius $2R$. Endowed with the distance associated with the norm $\|\cdot\|_{C([0,\bar t];L^{2p}(\Omega))}$, $B(2R)$ is a complete metric space. For $z\in C([0,\bar t];L^{2p}(\Omega))$, denote by $\Psi(z)$ the solution to equation (9.2.2) corresponding to $(g,z_0)$ with $g=f-\phi(z)$. Let us show that the mapping $z\mapsto\Psi(z)$ is a contraction in $B(2R)$. Let $z\in B(2R)$, then

$$\|\Psi(z)\|_{C([0,\bar t];L^{2p}(\Omega))}\le\|y\|_{C([0,\bar t];L^{2p}(\Omega))}+C(s)\|z^2\|_{L^s(0,\bar t;L^p(\Omega))}$$
$$\le R+C(s)\bar t^{1/s}\|z^2\|_{L^\infty(0,\bar t;L^p(\Omega))}\le R+C(s)\bar t^{1/s}R^2\le 2R.$$

Let $z_1$ and $z_2$ be in $B(2R)$, then

$$\|\Psi(z_1)-\Psi(z_2)\|_{C([0,\bar t];L^{2p}(\Omega))}\le C(s)\|z_1^2-z_2^2\|_{L^s(0,\bar t;L^p(\Omega))}$$

$$\le C(s)\,\bar t^{1/s}2R\|z_1-z_2\|_{L^\infty(0,\bar t;L^{2p}(\Omega))}\le\frac{1}{2}\|z_1-z_2\|_{L^\infty(0,\bar t;L^{2p}(\Omega))}.$$

∎

### 9.3.2 Initial data in $D((-A)^\alpha)$

Set $D(A) = H^2(\Omega)\cap H^1_0(\Omega)$ and $Az = \Delta z$ for $z \in D(A)$. For $\frac{1}{2} < \alpha < 1$, we have $D((-A)^\alpha) = H^{2\alpha}(\Omega) \cap H^1_0(\Omega)$ (see [18]). Suppose that $f \in L^2(Q)$, and $z_0 \in D((-A)^\alpha)$, with $\frac{1}{2} < \alpha < 1$. As in section 9.3.1, we can prove that equation (9.3.3) admits a unique weak solution in $C([0,\hat{t}]; D((-A)^\alpha))$ for $\hat{t}$ small enough. Since $D((-A)^\alpha) \subset L^{2p}(\Omega)$ for all $p \le \infty$, this implies that the solution defined in $C([0,\hat{t}]; D((-A)^\alpha))$ is the same as the solution defined in section 9.3.1.

### 9.3.3 Existence of a global solution

Suppose that $f \in L^2(Q)$, and $z_0 \in D((-A)^\alpha)$, with $\frac{1}{2} < \alpha < 1$. Let $T_{max}$ be such that the solution to equation (9.3.3) exists in $C([0,\tau]; L^{2p}(\Omega))$ for all $p > 2$ and all $\tau < T_{max}$. If $T_{max} = \infty$, we have proved the existence of a global solution. Otherwise, we necessarily have

$$\lim_{\tau\to T_{max}} \|z\|_{C([0,\tau];L^{2p}(\Omega))} = \infty, \tag{9.3.4}$$

for some $p > 2$. Let us show that we have a contradiction. Multiplying the equation by $|z|^{2p-2}z$, and integrating on $(0,\tau)\times\Omega$, we obtain

$$\frac{1}{2p}\int_\Omega |z(\tau)|^{2p} + \int_0^\tau\int_\Omega (2p-1)|\nabla z|^2|z|^{2p-2} = \frac{1}{2p}\int_\Omega |z_0|^{2p} + \int_0^\tau\int_\Omega f|z|^{2p-2}z. \tag{9.3.5}$$

Indeed, with an integration by parts, we get

$$\int_\Omega \partial_{x_1}(z^2)|z|^{2p-2}z = -\frac{2p-1}{2}\int_\Omega |z|^{2p-2}z\partial_{x_1}(z^2). \tag{9.3.6}$$

Thus it yields

$$\int_\Omega |z|^{2p-2}z\partial_{x_1}(z^2) = 0.$$

Moreover

$$\int_\Omega \nabla z\cdot\nabla(|z|^{2p-2}z) = \int_\Omega (2p-1)|\nabla z|^2|z|^{2p-2}. \tag{9.3.7}$$

Formula (9.3.5) is established. It is clearly in contradiction with (9.3.4). Observe that calculations in (9.3.6) and (9.3.7) are justified because $z$ is bounded, and in that case the solution to (9.3.3) belongs to $L^q(0,T;W^{1,q}(\Omega))$ for all $q < \infty$ (apply Theorem 9.6.3). Therefore formulas (9.3.6) and (9.3.7) are meaningful. The regularity in $C([0,T]; L^{2p}(\Omega))$ is not sufficient since in that case $\partial_{x_1}(z^2)|z|^{2p-2}z$ does not belong to $L^1$. It is the reason why we have constructed bounded solutions to justify (9.3.6) and (9.3.7).

■

## 9.4 Existence of a global weak solution for $L^2$-initilal data

**Theorem 9.4.1** *For all $z_0 \in L^2(\Omega)$, all $T > 0$, and all $f \in L^2(0,T;L^2(\Omega))$, equation (9.3.3) admits a unique weak solution in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ in the sense of definition*

*9.2.1. This solution satisfies*

$$\frac{1}{2}\int_\Omega |z(T)|^2 + \int_0^T\int_\Omega |\nabla z|^2 = \frac{1}{2}\int_\Omega |z_0|^2 + \int_0^T\int_\Omega fz.$$

**Proof.** (i) *Existence.* Let $(z_{0n})_n$ be a sequence in $D((-A)^\alpha)$, with $\frac{1}{2} < \alpha < 1$, converging to $z_0$ in $L^2(\Omega)$. Denote by $z_n$ the solution to equation (9.3.3) corresponding to the initial condition $z_{0n}$. Multiplying the first equation in (9.3.3) by $z_n$, and integrating on $(0,\tau)\times\Omega$, we obtain

$$\frac{1}{2}\int_\Omega |z_n(\tau)|^2 + \int_0^\tau\int_\Omega |\nabla z_n|^2 = \frac{1}{2}\int_\Omega |z_{0n}|^2 + \int_0^\tau\int_\Omega fz_n.$$

Thus $(z_n)_n$ is bounded in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$. Next using the equation

$$\frac{d}{dt}\langle z_n(t),\zeta\rangle = \int_\Omega \nabla z_n(t)\nabla\zeta + \langle f,\zeta\rangle - \langle \phi(z_n),\zeta\rangle,$$

satisfied for all $\zeta\in D(A)$, we show that $((\frac{dz}{dt})_n)_n$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. Then, there exits a subsequence, still indexed by $n$ to simplify the notation, such that $(z_n)_n$ converges to some $z$ for the weak topology of $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$. Due to Theorem 9.6.4, we can suppose that $(z_n)_n$ converges to $z$ in $L^2(Q)$. Since the sequence $(z_n)_n$ is bounded in $L^4(Q)$, $(z_n)_n$ also converges to $z$ in $L^r(Q)$ for all $r<4$. Thus, we can pass to the limit in the equation satisfied by $z_n$, and we prove that $z$ is a solution to equation (9.3.3).

(ii) *Uniqueness.* Let $z_1$ and $z_2$ be two solutions to equation (9.3.3). Set $w = z_1 - z_2$. Then $w$ is the solution to

$$\frac{\partial w}{\partial t} - \Delta w = -\frac{1}{2}\partial_{x_1}(wz_1 + wz_2) \quad \text{in } Q, \quad w = 0 \quad \text{on } \Sigma, \quad w(x,0) = 0 \quad \text{in } \Omega. \tag{9.4.8}$$

Multiplying equation (9.4.8) by $w$ and integrating over $\Omega$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |w(t)|^2 + \int_\Omega |\nabla w|^2 = -\frac{1}{2}\int_\Omega \partial_{x_1}(wz_1 + wz_2)w$$

$$= -\frac{1}{4}\int_\Omega \partial_{x_1}(z_1+z_2)w^2 \le \frac{1}{4}\|z_1+z_2\|_{H_0^1(\Omega)}\|w\|^2_{L^4(\Omega)}$$

$$\le \frac{\sqrt{2}}{4}\|z_1+z_2\|_{H_0^1(\Omega)}\|\nabla w\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}.$$

With Young's inequality, we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |w(t)|^2 \le C\|z_1+z_2\|^2_{H_0^1(\Omega)}\int_\Omega |w(t)|^2,$$

and we conclude with Gronwall's lemma. ∎

## 9.5 Optimal control problem

We consider the control problem

$$(P_1) \qquad \inf\{J_1(z,u) \mid (z,u) \in W(0,T;H_0^1(\Omega),H^{-1}(\Omega)) \times U_{ad},\ (z,u) \text{ satisfies } (9.2.1)\},$$

where $U_{ad}$ is a closed convex subset of $L^2(0,T;L^2(\omega))$,

$$J_1(z,u) = \frac{1}{2}\int_Q (z-z_d)^2 + \frac{1}{2}\int_\Omega (z(T)-z_d(T))^2 + \frac{\beta}{2}\int_Q \chi_\omega u^2,$$

and $\beta > 0$. In this section, we assume that $f \in L^2(Q)$ and that $z_d \in C([0,T];L^2(\Omega))$.

We set $Z = W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ . We define the mapping $G$ from $Z \times L^2(0,T;L^2(\omega))$ into $L^2(0,T;H^{-1}(\Omega)) \times L^2(\Omega)$ by

$$G(z,u) = \Big(\frac{\partial z}{\partial t} - \Delta z + \partial_{x_1}(z^2) - f - \chi_\omega u, z(0) - z_0\Big).$$

**Theorem 9.5.1** *The mapping $G$ is of class $C^1$, and for every $(z,u) \in Z \times L^2(0,T;L^2(\omega))$, $G'_z(z,u)$ is an isomorphism from $Z$ into $L^2(0,T;H^{-1}(\Omega)) \times L^2(\Omega)$.*

**Proof.**
(i) *Differentiability of $F$.* The mapping

$$(z,u) \mapsto \Big(\frac{\partial z}{\partial t} - \Delta z - \chi_\omega u, z(0)\Big),$$

is linear and bounded from $Z \times L^2(0,T;L^2(\omega))$ into $L^2(0,T;H^{-1}(\Omega)) \times L^2(\Omega)$. Thus to prove that $G$ is differentiable, we have only to check that

$$\frac{\|\phi(z+h) - \phi(z) - 2\partial_{x_1}(zh)\|_{L^2(0,T;H^{-1}(\Omega))}}{\|h\|_Z} \to 0 \quad \text{as } \|h\|_Z \to 0.$$

Since $\|\phi(z+h) - \phi(z) - 2\partial_{x_1}(zh)\|_{L^2(0,T;H^{-1}(\Omega))} = \|\phi(h)\|_{L^2(0,T;H^{-1}(\Omega))} \le \|h\|_Z^2$, the result is obvious. We can also verify that

$$z \longmapsto (h \mapsto \partial_{x_1}(zh))$$

is differentiable from $Z$ into $\mathcal{L}(Z;L^2(0,T;H^{-1}(\Omega)))$. This means that $G$ is twice differentiable. In fact $G$ is of class $C^\infty$.
(ii) *$G'_z(z,u)$ is an isomorphism from $Z$ into $L^2(0,T;H^{-1}(\Omega)) \times L^2(\Omega)$.* Observe that $G'_z(z,u)w = (\frac{\partial w}{\partial t} - \Delta w + 2\partial_{x_1}(zw), w(0))$. Thus, to prove that $G'_z(z,u)$ is an isomorphism from $Z$ into $L^2(0,T;H^{-1}(\Omega)) \times L^2(\Omega)$, we have only to verify that, for any $(f,z_0) \in L^2(0,T;H^{-1}(\Omega)) \times L^2(\Omega)$, equation

$$\frac{\partial w}{\partial t} - \Delta w + 2\partial_{x_1}(zw) = f \quad \text{in } Q, \quad w = 0 \quad \text{on } \Sigma, \quad w(0) = z_0 \quad \text{in } \Omega,$$

admits a unique solution in $Z$. This clearly follows from Theorem 9.6.5. ∎

**Theorem 9.5.2** *Let $z(u)$ be the solution to equation (9.2.1). The mapping*

$$u \longmapsto z(u),$$

*is of class $C^1$ from $L^2(0,T;L^2(\omega))$ into $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$, and for all $\bar{u}$ and $u$ in $L^2(0,T;L^2(\omega))$, the function $w = \frac{dz}{du}(\bar{u})u$ is the solution to equation*

$$\frac{\partial w}{\partial t} - \Delta w + 2\partial_{x_1}(z(\bar{u})w) = \chi_\omega u \quad \text{in } Q, \quad w = 0 \quad \text{on } \Sigma, \quad w(0) = 0 \quad \text{in } \Omega. \tag{9.5.9}$$

**Proof.** Let $\bar{u} \in L^2(0,T;L^2(\omega))$. We have $G(z(\bar{u}),\bar{u}) = 0$. From Theorem 9.5.1 and from the implicit function theorem, it follows that there exists a neighborhood $V(\bar{u})$ of $\bar{u}$ in $L^2(0,T;L^2(\omega))$, such that the mapping $u \mapsto z(u)$ is of class $C^1$ from $V(\bar{u})$ to $Z$, and

$$G'_z(z(\bar{u}),\bar{u}) \circ \frac{dz}{du}(\bar{u})\, u + G'_u(z(\bar{u}),\bar{u})\, u = 0,$$

for all $u \in L^2(0,T;L^2(\omega))$. If we set $w = \frac{dz}{du}(\bar{u})u$, we have

$$G'_z(z(\bar{u}),\bar{u})w = (\frac{\partial w}{\partial t} - \Delta w + 2\partial_{x_1}(z(\bar{u})w), w(0))$$

and $G'_u(z(\bar{u}),\bar{u})\, u = -\chi_\omega u$. The proof is complete.

**Theorem 9.5.3** *If $(\bar{z},\bar{u})$ is a solution to $(P_1)$ then*

$$\int_Q \chi_\omega(\beta\bar{u} + p)(u - \bar{u}) \geq 0 \qquad \text{for all } u \in U_{ad},$$

*where $p$ is the solution to equation*

$$-\frac{\partial p}{\partial t} - \Delta p - 2\bar{z}\partial_{x_i} p = \bar{z} - z_d \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad p(x,T) = \bar{z}(T) - z_T \quad \text{in } \Omega. \tag{9.5.10}$$

**Proof.** Set $F_1(u) = J_1(z(u),u)$, where $z(u)$ is the solution to equation (9.5.9). From Theorem 9.5.2 it follows that $F_1$ is of class $C^1$ on $L^2(0,T;L^2(\omega))$, and that

$$F_1'(\bar{u})u = \int_Q (\bar{z} - z_d)w + \int_\Omega (\bar{z}(T) - z_d(T))w(T) + \int_\Omega \chi_\omega \beta\bar{u}u,$$

where $w$ is the solution to equation (9.5.9). Since $(\bar{z},\bar{u})$ is a solution to $(P_1)$, $F_1'(\bar{u})(u - \bar{u}) \geq 0$ for all $u \in U_{ad}$. Using a Green formula between $p$ and $w$ we obtain

$$\int_\Omega (\bar{z}(T) - z_d(T))w(T) + \int_Q (\bar{z} - z_d)w = \int_Q \chi_\omega pu,$$

and

$$F_1'(\bar{u})u = \int_Q \chi_\omega pu + \int_\Omega \chi_\omega \beta\bar{u}u.$$

This completes the proof. ∎

## 9.6 Appendix

**Lemma 9.6.1** *For $N = 2$, we have*

$$\|z\|_{L^4(\Omega)} \le 2^{1/4}\|\nabla z\|^{1/2}_{L^2(\Omega)}\|z\|^{1/2}_{L^2(\Omega)},$$

*for all $z \in H^1_0(\Omega)$.*

**Proof.** Let us prove the result for $z \in \mathcal{D}(\Omega)$. We have

$$|z(x)|^2 \le 2\int_{-\infty}^{x_1} |z(\xi_1, x_2)||\partial_1 z(\xi_1, x_2)|d\xi_1,$$

and

$$|z(x)|^2 \le 2\int_{-\infty}^{x_2} |z(x_1, \xi_2)||\partial_1 z(x_1, \xi_2)|d\xi_2.$$

Thus

$$\int_{\mathbb{R}^2} |z(x)|^4 dx \le 4\|z\|^2_{L^2(\Omega)}\|\partial_1 z\|_{L^2(\Omega)}\|\partial_2 z\|_{L^2(\Omega)} \le 2\|z\|^2_{L^2(\Omega)}\|\nabla z\|^2_{L^2(\Omega)}.$$

This completes the proof. ∎

**Theorem 9.6.1** *For $N = 2$, the imbedding from $C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ into $L^4((0,T)\times\Omega)$ is continuous. Moreover we have*

$$\|z\|_{L^4(Q)} \le 2^{1/4}\|z\|^{1/2}_{L^2(0,T;H^1_0(\Omega))}\|z\|^{1/2}_{C([0,T];L^2(\Omega))}.$$

**Proof.** Due to Lemma 9.6.1 we have

$$\int_0^T\int_\Omega |z|^4 \le 2\int_0^T \|z\|^2_{L^2(\Omega)}\|\nabla z\|^2_{L^2(\Omega)} \le 2\|z\|^2_{C([0,T];L^2(\Omega))}\|z\|^2_{L^2(0,T;H^1_0(\Omega))}.$$

The proof is complete. ∎

The other results are stated in dimension $N \ge 2$.

**Theorem 9.6.2** *Set $D(A_p) = W^{2,p}(\Omega)\cap W^{1,p}_0(\Omega)$ and $A_p z = \Delta z$ for $z \in D(A_p)$, with $1 < p < \infty$. The operator $(A_p, D(A_p))$ is the infinitesimal generator of a strongly continuous analytic semigroup on $L^p(\Omega)$.*

See for example [5, Theorem 7.6.1]. This theorem together with properties of fractional powers of $(-A_p)$ can be used to prove the theorem below.

**Theorem 9.6.3** *Let $h \in \mathcal{D}(Q)$, and $z_h$ be the solution to equation*

$$\frac{\partial z}{\partial t} - \Delta z = \partial_{x_i} h \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = 0 \quad \text{in } \Omega, \tag{9.6.11}$$

*where $i \in \{1,\dots,N\}$. Suppose that $1 < s < \infty$ and $1 < p < \infty$. The mapping $h \mapsto z_h$ is continuous from $L^s(0,T;L^p(\Omega))$ into $C([0,T];L^r(\Omega)) \cap L^s(0,T;W^{1,p}(\Omega))$ if*

$$\frac{N}{2p} + \frac{1}{s} < \frac{1}{r} + \frac{1}{2}.$$

**Theorem 9.6.4** *The imbedding from $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ into $L^2(Q)$ is compact.*

**Theorem 9.6.5** *Let $z$ be in $L^4(Q)$. For all $z_0 \in L^2(\Omega)$, all $f \in L^2(Q)$, equation*

$$\frac{\partial w}{\partial t} - \Delta w + 2\partial_{x_1}(zw) = f \quad in\ Q, \quad w = 0 \quad on\ \Sigma, \quad w(0) = z_0 \quad in\ \Omega,$$

*admits a unique solution in $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$. Moreover the mapping $(z_0,f) \mapsto w$ is continuous from $L^2(\Omega)\times L^2(Q)$ into $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$.*

**Proof.** This theorem can be proved by using a fixed point method as in exercise 5.5.4 (see exercise 9.7.1).

## 9.7 Exercises

**Exercise 9.7.1**

Adapt the fixed point method of exercise 5.5.4 to prove Theorem 9.6.5.

**Exercise 9.7.2** *(Variational method)*

We want to give another proof of Theorem 9.4.1. Assumptions and notation are the ones of Theorem 9.4.1. Let $(\psi_n)_n$ be a Hilbertian basis in $H_0^1(\Omega)$, and let $(\phi_n)_n$ be the basis obtained by applying the Gram-Schmidt process to $(\psi_n)_n$ for the scalar product of $L^2(\Omega)$. Thus $(\phi_n)_n$ is a Hilbertian basis in $L^2(\Omega)$ whose elements belong to $H_0^1(\Omega)$. Denote by $H_m = \mathrm{vect}(\psi_0,\ldots,\psi_m)$ the vector space generated by $(\psi_0,\ldots,\psi_m)$. We have

$$\overline{\cap_{m=0}^{\infty} H_m}^{H_0^1} = H_0^1(\Omega),$$

and $\int_\Omega \phi_i\phi_j = \delta_{ij}$. We also assume that the family $(\psi_n)_n$ is orthogonal in $H_0^1(\Omega)$ (which is satisfied if we choose a family of eigenfunctions of the Laplace operator). Denote by $P_m$ the orthogonal projection in $L^2(\Omega)$ on $H_m$. Observe that a function $z$ belongs to $H^1(0,T;H_m)$ if and only if $z$ is of the form $z = \Sigma_{j=0}^m g_j\phi_j$, with $g_j \in H^1(0,T)$.

1 - Prove that the variational equation

$$\begin{aligned}&\text{find } \ z = \Sigma_{j=0}^m g_j\phi_j \in H^1(0,T;H_m) \quad \text{such that}\\ &\frac{d}{dt}\langle z(t),\zeta\rangle = \langle \nabla z(t),\nabla\zeta\rangle + \langle f,\zeta\rangle - \langle \phi(z),\zeta\rangle \text{ and } \ \langle z(0),\zeta\rangle = \langle z_0,\zeta\rangle,\end{aligned} \tag{9.7.12}$$

for all $\zeta \in H_m$, is equivalent to a system of ordinary differential equations in $\mathbb{R}^N$ satisfied by $g = (g_0,\ldots,g_m)^T$. Prove that this system admits a unique solution $g^m = (g_0^m,\ldots,g_m^m)^T$, and that the corresponding function $z_m = \Sigma_{j=0}^m g_j^m\phi_j$ obeys

$$\frac{1}{2}\int_\Omega |z_m(T)|^2 + \int_0^T\int_\Omega |\nabla z_m|^2 = \frac{1}{2}\int_\Omega |z_{0m}|^2 + \int_0^T\int_\Omega f_m z_m,$$

where $z_{0m} = P_m(z_0)$ and $f_m(t) = P_m(f(t))$. Prove that $\|\langle z_m(\cdot),\phi_j\rangle\|_{H^1(0,T)} \le C$, where $C$ is independent of $m$ and $j$.

2 - Using the diagonal process, show that there exists a subsequence $(z_{m_k})_k$, extracted from $(z_m)_m$, and a function $z \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, such that

$$\begin{array}{lll} (z_{m_k})_k & \text{converges to } z & \text{for the weak* topology of } L^\infty(0,T;L^2(\Omega)), \\ (z_{m_k})_k & \text{converges to } z & \text{for the weak topology of } L^2(0,T;H_0^1(\Omega)), \\ (\langle z_{m_k},\phi_j\rangle)_k & \text{converges to } z & \text{for the weak topology of } H^1(0,T) \text{ for all } j \in \mathbb{N}. \end{array} \tag{9.7.13}$$

Show that $z$ is a weak solution to equation (9.3.3) in the sense of Definition 9.2.1. Prove that the solution belongs to $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$, and is unique in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$.

**Exercise 9.7.3**

The notation are the ones of section 9.2. To study the boundary control of Burgers' equation, we recall the definition of anisotropic Sobolev spaces:

$$H^{1,\frac{1}{2}}(Q) = L^2(0,T;H^1(\Omega)) \cap H^{\frac{1}{2}}(0,T;L^2(\Omega)),$$

$$H^{\frac{1}{2},\frac{1}{4}}(\Sigma) = L^2(0,T;H^{\frac{1}{2}}(\Gamma)) \cap H^{\frac{1}{4}}(0,T;L^2(\Omega)).$$

We admit the following result.

*Regularity result* ([13, page 84]) For every $u \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$, the solution to equation

$$\frac{\partial w}{\partial t} - \Delta w = 0 \quad \text{in } Q, \quad w = u \quad \text{on } \Sigma, \quad w(x,0) = 0 \quad \text{in } \Omega, \tag{9.7.14}$$

belongs to $W(0,T;H^1(\Omega),(H^1(\Omega))')$ and

$$\|w\|_{W(0,T;H^1(\Omega),(H^1(\Omega))')} \le C\|u\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}.$$

We want to study a control problem for the equation

$$\frac{\partial z}{\partial t} - \Delta z + \Phi(z) = f \quad \text{in } Q, \quad z = u \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega, \tag{9.7.15}$$

with $\Phi(z) = 2z\partial_{x_1}z$, $f \in L^2(Q)$, $z_0 \in L^2(\Omega)$, and $u \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$.

1 - We look for a solution $z$ to equation (9.7.15) of the form $z = w_u + y$, where $w_u$ is the solution to equation (9.7.14). Write the equation satisfied by $y$, and prove that equation (9.7.15) admits a unique solution in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Prove that this solution belong to $W(0,T;H^1(\Omega),(H^1(\Omega))')$.

2 - Consider the control problem

$$(P_2) \qquad \inf\{J_2(z,u) \mid (z,u) \in W(0,T;H^1(\Omega),(H^1(\Omega))') \times U_{ad},\ (z,u) \text{ satisfies } (9.7.15)\},$$

where $U_{ad}$ is a closed convex subset of $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$,

$$J_2(z,u) = \frac{1}{2}\int_Q (z-z_d)^2 + \frac{1}{2}\int_\Omega (z(T)-z_d(T))^2 + \frac{\beta}{2}\int_\Sigma u^2,$$

with $\beta > 0$ and $z_d \in C([0,T]; L^2(\Omega))$. Prove the existence of a solution to problem $(P_2)$. Write optimality conditions.

2 - Consider the following variant of problem $(P_2)$

$$(P_3) \qquad \inf\{J_3(z,u) \mid (z,u) \in W(0,T; H^1(\Omega), (H^1(\Omega))') \times U_{ad},\ (z,u) \text{ satisfies } (9.7.15)\},$$

where $U_{ad}$ is a closed convex subset of $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$,

$$J_3(z,u) = \frac{1}{2}\int_Q |\nabla z - \nabla z_d|^2 + \frac{1}{2}\int_\Omega (z(T) - z_d(T))^2 + \frac{\beta}{2}\int_\Sigma u^2,$$

with $\beta > 0$ and $z_d \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$. Prove the existence of a solution to problem $(P_3)$. Write optimality conditions.

# Chapter 10

# Algorithms for solving optimal control problems

## 10.1 Introduction

In section 10.2.1, we first recall the Conjugate Gradient Method (CGM in brief) for quadratic functionals. We next explain how this algorithm can be used for control problems studied in chapter 7. For functionals which are not necessarily quadratic we introduce the Polak-Ribiere algorithm, the Fletcher-Reeves algorithm, and Quasi-Newton methods. These algorithms can be used for control problems governed by semilinear equations such as the ones studied in chapter 3. For linear-quadratic problems with bound constraints on the control variable we introduce in section 10.4 a projection method due to Bertsekas. For other problems with control constraints we describe the Gradient Method with projection in section 10.5.1. We end this chapter with the Sequential Quadratic Programming Method (SQP method), which is a particular implementation of the Newton method applied to the optimality system of control problems.

## 10.2 Linear-quadratic problems without constraints

### 10.2.1 The conjugate gradient method for quadratic functionals

In chapter 2 we have applied the Conjugate Gradient Method to control problems governed by elliptic equations. In this section, we want to apply the CGM to control problems governed by evolution equations. Let us recall the algorithm for quadratic functionals. Consider the optimization problem

$$(P_1) \qquad \inf\{F(u) \mid u \in U\},$$

where $U$ is a Hilbert space and $F$ is a quadratic functional

$$F(u) = \frac{1}{2}(u, Qu)_U - (b, u)_U.$$

In this setting $Q \in \mathcal{L}(U)$, $Q = Q^* > 0$, $b \in U$, and $(\cdot, \cdot)_U$ denotes the scalar product in $U$. For simplicity we write $(\cdot, \cdot)$ in place of $(\cdot, \cdot)_U$.

Let us recall the GC algorithm:

**Algorithm 1.**

*Initialization.* Choose $u_0$ in $U$. Compute $g_0 = Qu_0 - b$. Set $d_0 = -g_0$ and $n = 0$.

*Step 1.* Compute

$$\rho_n = (g_n, g_n)/(d_n, Qd_n),$$

and

$$u_{n+1} = u_n + \rho_n d_n.$$

Determine

$$g_{n+1} = Qu_{n+1} - b = g_n + \rho_n Q d_n.$$

*Step 2.* If $\|g_{n+1}\|_U/\|g_0\|_U \leq \varepsilon$, stop the algorithm and take $u = u_{n+1}$, else compute

$$\beta_n = (g_{n+1}, g_{n+1})/(g_n, g_n),$$

and

$$d_{n+1} = -g_{n+1} + \beta_n d_n.$$

Replace $n$ by $n+1$ and go to step 1.

### 10.2.2 The conjugate gradient method for control problems

We want to apply the CGM to problems studied in chapter 7. The state equation is of the form

$$z' = Az + Bu + f, \qquad z(0) = z_0, \tag{10.2.1}$$

and the control problem is defined by

$$(P_2) \qquad \inf\{J(z,u) \mid (z,u) \in C([0,T];Z) \times L^2(0,T;U),\ (z,u) \text{ satisfies } (10.2.1)\}.$$

with

$$J(z,u) = \frac{1}{2}\int_0^T |Cz(t) - y_d(t)|_Y^2 + \frac{1}{2}|Dz(T) - y_T|_{Y_T}^2 + \frac{1}{2}\int_0^T |u(t)|_U^2. \tag{10.2.2}$$

Assumptions are the ones of chapter 7. We have to identify problem $(P_2)$ with a problem of the form $(P_1)$. Let $z_u$ be the solution to equation (10.2.1), and set $F(u) = J(z_u, u)$. Observe that $(z_u, z_u(T)) = (\Lambda_1 u, \Lambda_2 u) + \zeta(f, z_0)$, where $\Lambda_1$ is a bounded linear operator from $L^2(0,T;U)$ to $L^2(0,T;Z)$, and $\Lambda_2$ is a bounded linear operator from $L^2(0,T;U)$ to $Z$. We must determine the quadratic form $Q$ such that

$$\frac{1}{2}\int_0^T |Cz_u(t) - y_d(t)|_Y^2 + \frac{1}{2}|Dz_u(T) - y_T|_{Y_T}^2 + \frac{1}{2}\int_0^T |u(t)|_U^2 = \frac{1}{2}(u, Qu)_U - (b,u)_U + c.$$

Since $(z_u, z_u(T)) = (\Lambda_1 u, \Lambda_2 u) + \zeta(f, z_0)$, we have

$$Q = \Lambda_1^* \widehat{C}^* \widehat{C} \Lambda_1 + \Lambda_2^* D^* D \Lambda_2 + I,$$

where $\widehat{C} \in \mathcal{L}(L^2(0,T;Z); L^2(0,T;Y))$ is defined by $(\widehat{C}z)(t) = Cz(t)$ for all $z \in L^2(0,T;Z)$, and $\widehat{C}^* \in \mathcal{L}(L^2(0,T;Y); L^2(0,T;Z))$ is the adjoint of $\widehat{C}$. In the CGM we have to compute

$Qd$ for some $d \in L^2(0,T;U)$. Observe that $(\Lambda_1 d, \Lambda_2 d)$ is equal to $(w_d, w_d(T))$, where $w_d$ is the solution to

$$w' = Aw + Bd, \qquad w(0) = 0. \tag{10.2.3}$$

Moreover, using formula (7.2.5), we can prove that $\Lambda_1^* g = B^* p_1$, where $p_1$ is the solution to equation

$$-p' = A^* p + g, \qquad p(T) = 0, \tag{10.2.4}$$

and $\Lambda_2^* p_T = B^* p_2$, where $p_2$ is the solution to equation

$$-p' = A^* p, \qquad p(T) = p_T. \tag{10.2.5}$$

Thus $\Lambda_1^* \widehat{C}^* \widehat{C} \Lambda_1 d + \Lambda_2^* D^* D \Lambda_2 d$ is equal to $B^* p$, where $p$ is the solution to

$$-p' = A^* p + C^* C w_d, \qquad p(T) = D^* D w_d(T), \tag{10.2.6}$$

where $w_d$ is the solution to equation (10.2.3).

If we apply Algorithm 1 to problem $(P_2)$ we obtain:

**Algorithm 2.**

*Initialization.* Choose $u_0$ in $L^2(0,T;U)$. Denote by $z^0$ the solution to the state equation

$$z' = Az + Bu_0 + f, \qquad z(0) = z_0.$$

Denote by $p^0$ the solution to the adjoint equation

$$-p' = A^* p + C^*(Cz^0 - y_d), \qquad p(T) = D^*(Dz^0(T) - y_T).$$

Compute $g_0 = B^* p^0 + u_0$ , set $d_0 = -g_0$ and $n = 0$.

*Step 1.* To compute $Qd_n$, we calculate $w_n$ the solution to equation

$$w' = Aw + Bd_n, \qquad w(0) = 0.$$

We compute $p_n$ the solution to equation

$$-p' = A^* p + C^* C w_n, \qquad p(T) = D^* D w_n(T).$$

We have $Qd_n = B^* p_n + d_n$. Set $\bar{g}_n = B^* p_n + d_n$. Compute

$$\rho_n = -(g_n, g_n)/(\bar{g}_n, g_n),$$

and

$$u_{n+1} = u_n + \rho_n d_n.$$

Determine

$$g_{n+1} = g_n + \rho_n \bar{g}_n.$$

*Step 2.* If $\|g_{n+1}\|_{L^2(0,T;U)}/\|g_0\|_{L^2(0,T;U)} \le \varepsilon$, stop the algorithm and take $u = u_{n+1}$, else compute

$$\beta_n = (g_{n+1}, g_{n+1})/(g_n, g_n),$$

and

$$d_{n+1} = -g_{n+1} + \beta_n d_n.$$

Replace $n$ by $n+1$ and go to step 1.

## 10.3 Control problems governed by semilinear equations

Let us first recall the extension of the CGM to non-quadratic functionals. Consider the problem

$$(P_3) \qquad \inf\{F(u) \mid u \in U\},$$

where $F$ is a differentiable mapping.

**Algorithm 3.** (Polak-Ribiere)

*Initialization.* Choose $u_0$ in $U$. Compute $g_0 = F'(u_0)$. Set $d_0 = -g_0$ and $n = 0$.

*Step 1.* Determine $\rho_n = \text{argmin}_{\rho \geq 0} F(u_n + \rho d_n)$ and $u_{n+1} = u_n + \rho_n d_n$.

Compute $g_{n+1} = F'(u_{n+1})$.

*Step 2.* If $\|g_{n+1}\|_U / \|g_0\|_U \leq \varepsilon$, stop the algorithm and take $u = u_{n+1}$, else compute

$$\beta_n = \frac{(g_{n+1}, g_{n+1} - g_n)}{(g_n, g_n)}.$$

Set $d_{n+1} = -g_{n+1} + \beta_n d_n$. Replace $n$ by $n+1$. If $(d_n, g_n) < 0$ go to step 1, else set $d_n = -g_n$ and go to step 1.

**Remark.** In step 1, we have to calculate the solution to the one-dimensional minimization problem

$$\inf\{F(u_n + \rho d_n) \mid \rho \geq 0\}.$$

It is called the 'step length computation'. Different algorithms can be used to replace an exact step length computation by an approximate one, or by some heuristic rules known as *step-length criteria* (see [34]).

The algorithm below is a variant of the Polak-Ribiere algorithm.

**Algorithm 4.** (Fletcher-Reeves)

This algorithm corresponds to the previous one in which we replace the computation of $\beta_n$ by

$$\beta_n = \frac{(g_{n+1}, g_{n+1})}{(g_n, g_n)}.$$

**Quasi-Newton methods.** The Quasi-Newton methods can be applied to minimize non-quadratic functionals. The most popular one, the Broyden-Fletcher-Goldfarb-Shanno algorithm is described below.

**Algorithm 5.** (BFGS)

*Initialization.* Choose $u_0$ in $U$. Set $H_0 = I$ and $n = 0$ ($I$ denotes the identity in $U$).

*Step 1.* Compute $d_n = -H_n^{-1} F'(u_n)^*$.

*Step 2.* Compute $\lambda_n \in ]0, 1]$ such that

$$F(u_n + \lambda_n d_n) = \min\{F(u_n + \lambda d_n) \mid \lambda \in ]0, 1]\}.$$

*Step 3.* Set $u_{n+1} = u_n + \lambda_n d_n$. If $|u_{n+1} - u_n|_U \leq \varepsilon$, stop the algorithm, else compute

$$s_n = u_{n+1} - u_n, \quad \gamma_n = F'(u_{n+1})^* - F'(u_n)^*,$$

and

$$H_{n+1} = H_n + \frac{\gamma_n \gamma_n^*}{\langle s_n, \gamma_n \rangle} - \frac{H_n s_n (H_n s_n)^*}{\langle s_n, H_n s_n \rangle}.$$

Replace $n$ by $n+1$ and go to step 1.

**Comments.** The terminology *Quasi-Newton method* comes from that the update rule of $H_n$ in step 3 is based on a secant approximation of the Hessian operator $F''(u_n)$ . In the initialization procedure $H_0 = I$ can be replaced by $H_0 = F''(x_0)$ if the computation of the Hessian operator is not too expensive or too complicated. A direct update of the matrix $H_n^{-1}$ can be performed. It corresponds to a variant of the above method, where in step 3 the update of $H_n$ is replaced by

$$H_{n+1}^{-1} = H_n^{-1} + \frac{(s_n - H_n^{-1}\gamma_n)s_n^* + s_n(s_n - H_n^{-1}\gamma_n)^*}{\langle s_n, \gamma_n \rangle} - \frac{\langle s_n - H_n^{-1}\gamma_n, \gamma_n \rangle}{\langle s_n, \gamma_n \rangle^2} s_n s_n^*.$$

For more details on quasi-Newton methods we refer to [35].

These algorithms can be applied to control problems governed by semilinear evolution equations of the form

$$z' = Az + \phi(z) + Bu, \qquad z(0) = z_0, \tag{10.3.7}$$

or by semilinear elliptic equations of the form

$$Az = f, \qquad \frac{\partial z}{\partial n_A} + \phi(z) = u, \tag{10.3.8}$$

where $A$ is a uniformly elliptic operator. (Control of semilinear elliptic equations has been studied in chapter 3.) Let us explain how algorithms 3-5 can be applied to the control problem

$(P_3)$ $\qquad \inf\{J(z,u) \mid (z,u) \in C([0,T];Z) \times L^2(0,T;U),\ (z,u) \text{ satisfies } (10.3.7)\}.$

with

$$J(z,u) = \frac{1}{2}\int_0^T |Cz(t) - y_d(t)|_Y^2 + \frac{1}{2}|Dz(T) - y_T|_{Y_T}^2 + \frac{1}{2}\int_0^T |u(t)|_U^2.$$

Assumptions on $C$, $D$, $Y$, $Y_T$ are the ones of chapter 7. The nonlinear function $\phi$ is for example the one of chapter 3. For algorithms 3-5, we have to compute the gradient of $F(u) = J(z_u, u)$, where $z_u$ is the solution to equation (10.3.7).

For a given $u \in U$, $F'(u)$ is computed as follows. We first solve equation (10.3.7). Next we solve the adjoint equation

$$-p' = A^*p + \phi'(z_u)^*p + C^*(Cz_u - y_d), \qquad p(T) = D^*(Dz_u(T) - y_T).$$

We have

$$F'(u) = B^*p + u.$$

## 10.4 Linear-quadratic problems with control constraints

We consider the problem

$(P_4)$ $\qquad \inf\{F(u) \mid u \in U_{ad}\},$

where $U_{ad}$ is a closed convex subset of $U$ and $F$ is a quadratic functional on $U$. For computational considerations, we have to approximate the control set $U_{ad}$ by a finite dimensional set. Let $U_{ad}^m$ be such a finite dimensional approximation of $U_{ad}$, and suppose that $U_{ad}^m$ is a closed convex subset in $\mathbb{R}^m$. Let us denote by $(P^m)$ the corresponding finite dimensional optimization problem

$$(P^m) \qquad \inf\{F(u) \mid u \in U_{ad}^m\}.$$

We only treat the case where $U_{ad}^m$ is defined by bound constraints, that is

$$U_{ad}^m = \{v \in \mathbb{R}^m \mid u_a^j \le v^j \le u_b^j \text{ for all } j = 1, \ldots, m\}.$$

A projection algorithm due to Bertsekas [32] is an efficient method for solving problem with bound constraints. The algorithm is the following.

**Algorithm 6.** Choose two fixed positive numbers $\varepsilon$ and $\sigma$. We denote by $u_n = (u_n^1, \cdots, u_n^m)^T$ the vector representing the current iterate, and let $I = \{1, \cdots, m\}$ be the index set associated with $u_n$.

1 - Choose $u_0 = (u_0^1, \cdots, u_0^m)^T$, and set $n = 0$.

2 - Compute $F'(u_n) = (\partial_1 F(u_n), \ldots, \partial_n F(u_n))$.

3 - Define the sets of strongly active inequalities

$$I_a^\sigma = \{j \in I \mid u_n^j = u_a^j \text{ and } \partial_j F(u_n) > \sigma\},$$
$$I_b^\sigma = \{i \in I \mid u_n^j = u_b^j \text{ and } \partial_j F(u_n) < -\sigma\}.$$

4 - Set $\hat{u}_n^j = u_n^j$ for all $j \in I_a^\sigma \cup I_b^\sigma$.

5 - Solve the unconstrained problem

$$(P_{aux}) \qquad \inf\{F(u) \mid u \in \mathbb{R}^m \text{ and } u^j = \hat{u}^j \text{ for all } j \in I_a^\sigma \cup I_b^\sigma\}.$$

Denote by $v_n$ the vector solution to $(P_{aux})$.

6 - Set $u_{n+1} = P_{[u_a,u_b]} v_n$, where $P_{[u_a,u_b]}$ denotes the projection onto $[u_a^1, u_b^1] \times \ldots [u_a^m, u_b^m]$.

7 - If $\|u_{n+1} - u_n\| \ge \varepsilon$, then replace $n$ by $n+1$ and go to 2. Otherwise stop the iteration.

The auxiliary problem $(P_{aux})$ may be solved by the CGM.

## 10.5 General problems with control constraints

### 10.5.1 Gradient method with projection

We consider the problem

$$(P_5) \qquad \inf\{F(u) \mid u \in U_{ad}\},$$

where $U_{ad}$ is a closed convex subset of $U$ and $F$ is a differentiable functional on $U$, which is not necessarily quadratic.

Denote by $P_{U_{ad}}$ the projection on the convex set $U_{ad}$. For all $\tilde{u} \in U$, $P_{U_{ad}}\tilde{u}$ is characterized by

$$(\tilde{u} - P_{U_{ad}}\tilde{u}, u - P_{U_{ad}}\tilde{u}) \leq 0 \quad \text{for all } u \in U_{ad}.$$

If $\bar{u}$ is a solution to $(P_5)$, then

$$(F'(\bar{u}), u - \bar{u}) \geq 0 \quad \text{for all } u \in U_{ad}.$$

This optimality condition is equivalent to

$$(\bar{u} - \rho F'(\bar{u}) - \bar{u}, u - \bar{u}) \leq 0 \quad \text{for all } u \in U_{ad},$$

where $\rho$ is any positive number. We can verify that this variational inequality is equivalent to

$$\bar{u} = P_{U_{ad}}(\bar{u} - \rho F'(\bar{u})), \tag{10.5.9}$$

where $P_{U_{ad}}$ is the projection on the convex set $U_{ad}$. Thus $\bar{u}$ is a fixed point of the mapping $\Phi_\rho$ defined by

$$\Phi_\rho(u) = P_{U_{ad}}(u - \rho F'(u)).$$

The gradient method with projection consists in calculating a fixed point of $\Phi_\rho$. The corresponding algorithm is the following:

**Algorithm 7.**

*Initialization.* Choose $u_0$ in $U_{ad}$, and $\rho > 0$. Set $n = 0$.

*Step 1.* Set $\tilde{u}_n = P_{U_{ad}}(u_n - \rho F'(u_n))$, $v_n = \tilde{u}_n - u_n$.

*Step 2.* Compute $\lambda_n \in ]0, 1]$ such that

$$F(u_n + \lambda_n v_n) = \min\{F(u_n + \lambda v_n) \mid \lambda \in ]0, 1]\}.$$

*Step 3.* Set $u_{n+1} = u_n + \lambda_n v_n$. If $|u_{n+1} - u_n|_U \leq \varepsilon$, stop the algorithm, else replace $n$ by $n + 1$ and go to step 1.

Convergence results have been proved for Algorithm 7 in the case when $F$ is convex (see [21]).

### 10.5.2 The sequential quadratic programming method

The sequential quadratic programming method (SQP method in brief) is a particular implementation of the Newton method applied to the optimality system. Let us explain the Newton method for the control problem

$$(P_6) \qquad \inf\{J(z, u) \mid (z, u) \in C([0, T]; Z) \times U_{ad},\ (z, u) \text{ satisfies (10.3.7)}\}.$$

where

$$J(z, u) = \frac{1}{2}\int_0^T |Cz(t) - y_d(t)|_Y^2 + \frac{1}{2}|Dz(T) - y_T|_{Y_T}^2 + \frac{1}{2}\int_0^T |u(t)|_U^2, \tag{10.5.10}$$

$U_{ad}$ is a closed convex subset of $L^2(0,T;U)$, and the state equation is of the form

$$z' = Az + \phi(z) + Bu, \qquad z(0) = z_0. \tag{10.5.11}$$

Assumptions on $C$, $D$, $Y$, $Y_T$ are the ones of chapter 7. We suupse that $(A, D(A))$ is the generator of a strongly continuous semigroup on the Hilbert space $Y$, and (for simplicity) that $\phi$ is Lipschitz on $Y$.

The optimality system for $(P_6)$ satisfied by a solution $(\bar{z}, \bar{u})$ consists of the equations

$$\begin{aligned}
&-\bar{p}' = A^*\bar{p} + \phi'(\bar{z})^*\bar{p} + C^*(C\bar{z} - y_d), \qquad \bar{p}(T) = D^*(D\bar{z}(T) - y_T),\\
&\bar{z}' = A\bar{z} + \phi(\bar{z}) + B\bar{u}, \qquad \bar{z}(0) = z_0,\\
&\textstyle\int_0^T (B^*\bar{p} + \bar{u})(u - \bar{u}) \geq 0 \qquad \text{for all } u \in U_{ad}.
\end{aligned} \tag{10.5.12}$$

The Newton method applied to the system (10.5.12) corresponds to the following algorithm:

**Algorithm 8.**

*Initialization.* Set $n = 0$. Choose $u_0$ in $U_{ad}$, compute $\hat{z}_0$ the solution to the state equation for $u = u_0$, and $\hat{p}_0$ the solution to the adjoint equation

$$-p' = A^*p + \phi'(\hat{z}_0)^*p + C^*(C\hat{z}_0 - y_d), \qquad p(T) = D^*(D\hat{z}_0(T) - y_T).$$

*Step 1.* Compute $(u_{n+1}, \hat{z}_{n+1}, \hat{p}_{n+1}) \in U_{ad} \times C([0,T];Z) \times C([0,T];Z)$ the solution to the system

$$\begin{aligned}
&-p' = A^*p + \phi'(\hat{z}_n)^*p + (\phi''(\hat{z}_n)(z - \hat{z}_n))^*\hat{p}_n + C^*(Cz - y_d),\\
&p(T) = D^*(Dz(T) - y_T),\\
&z' = Az + \phi(\hat{z}_n) + \phi'(\hat{z}_n)(z - \hat{z}_n) + Bu, \qquad z(0) = z_0,\\
&\textstyle\int_0^T (B^*p + u, v - u)_U \geq 0 \qquad \text{for all } v \in U_{ad}.
\end{aligned} \tag{10.5.13}$$

*Step 2.* If $|u_{n+1} - u_n|_U \leq \varepsilon$, stop the algorithm, else replace $n$ by $n+1$ and go to step 1.

Observe that the mapping $\phi$ must necessarily be of class $C^2$. The convergence of the Newton method is studied in [33]. Roughly speaking, if $\phi'$ satisfy some Lipschitz property, and if the optimality system (10.5.12) is strongly regular in the sense of Robinson (see [33]), then there exists a neighborhood $\bar{V}$ of $(\bar{z}, \bar{u}, \bar{p})$ such that for any starting point in $\bar{V}$ the Newton algorithm is quadratically convergent.

The SQP method corresponds to the previous algorithm in which $(u_{n+1}, \hat{z}_{n+1})$ is computed by solving the 'Linear-Quadratic' problem

$$(QP_{n+1}) \qquad \begin{aligned}
&\text{Minimize} \qquad J'(\hat{z}_n, u_n)(z - \hat{z}_n, u - u_n) + \frac{1}{2}\langle \hat{p}_n, \phi''(\hat{z}_n)(z - \hat{z}_n)^2\rangle,\\
&\text{subject to} \qquad z' = Az + \phi(\hat{z}_n) + \phi'(\hat{z}_n)(z - \hat{z}_n) + Bu, \qquad z(0) = z_0,\\
&\phantom{\text{subject to}} \qquad u \in U_{ad},
\end{aligned}$$

and $\hat{p}_{n+1}$ is the solution to the adjoint equation for $(\mathrm{QP}_{n+1})$ associated with $(u_{n+1}, \hat{z}_{n+1})$. For problems with bound constraints this 'Linear-Quadratic' problem may be solved by Algorithm 6.

If the optimal solution $(\bar{z}, \bar{u}, \bar{p})$ satisfies a sufficient second order optimality condition, and if the optimality system (10.5.12) is strongly regular in the sense of Robinson, then the SQP method and the Newton method are equivalent ([33]).

## 10.6 Algorithms for discrete problems

For numerical computations, we have to write discrete approximations to control problems. Suppose that equation

$$z' = Az + Bu + f, \qquad z(0) = z_0, \tag{10.6.14}$$

is approximated by an implicit Euler scheme

$$\begin{aligned} & z^0 = z_0, \\ & \text{for } n = 1, \dots, M, \quad z^n \text{ is the solution to} \\ & \tfrac{1}{\Delta t}(z^n - z^{n-1}) = Az^n + Bu^n + f^n, \end{aligned} \tag{10.6.15}$$

where $f^n = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} f(t)\,dt$, $u^n = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} u(t)\,dt$, $t_n = n\Delta t$, and $T = M\Delta t$. To approximate the functional

$$J(z,u) = \frac{1}{2}\int_0^T |Cz(t) - y_d(t)|_Y^2 + \frac{1}{2}|Dz(T) - y_T|_{Y_T}^2 + \frac{1}{2}\int_0^T |u(t)|_U^2,$$

we set

$$J_M(z,u) = \frac{1}{2}\Delta t\sum_{n=1}^{M} |Cz^n - y_d^n|_Y^2 + \frac{1}{2}|Dz^M - y_T|_{Y_T}^2 + \frac{1}{2}\Delta t\sum_{n=1}^{M} |u^n|_U^2,$$

with $z = (z^0, \dots, z^M)$, $u = (u^1, \dots, u^M)$, $y_d^n = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} y_d(t)\,dt$. We can define a discrete control problem associated with $(P_2)$ as follows:

$$(P_M) \qquad \inf\{J_M(z,u) \mid (z,u) \in Z^{M+1} \times U^M,\ (z,u) \text{ satisfies (10.6.15)}\}.$$

To apply the CGM to problem $(P_M)$, we have to compute the gradient of the mapping $u \mapsto J_M(z_u, u)$, where $z_u$ is the solution to (10.6.15) corresponding to $u$. Set $F_M(u) = J_M(z_u, u)$. We have

$$F_M'(\bar{u})u = \Delta t\sum_{n=1}^{M}(C\bar{z}^n - y_d^n, Cw_u^n)_Y + (D\bar{z}^M - y_T, Dw_u^M)_{Y_T} + \Delta t\sum_{n=1}^{M}(\bar{u}^n, u^n)_U,$$

where $\bar{z} = z_{\bar{u}}$ and $w = (w^0, \dots, w^M) \in Z^{M+1}$ is defined by

$$\begin{aligned} & w^0 = 0, \\ & \text{for } n = 1, \dots, M, \quad w^n \text{ is the solution to} \\ & \tfrac{1}{\Delta t}(w^n - w^{n-1}) = Aw^n + Bu^n. \end{aligned} \tag{10.6.16}$$

To find the expression of $F_M'(\bar{u})$, we have to introduce an adjoint equation. Let $p = (p^0, \dots, p^M)$ be in $Z^{M+1}$, or in $D(A^*)^{M+1}$ if we want to justify the calculations. Taking a weak formulation of the different equations in (10.6.16), we can write

$$\frac{1}{\Delta t}((w^n - w^{n-1}), p^{n-1})_Z - (w^n, A^*p^{n-1})_Z = (Bu^n, p^{n-1})_Z = (u^n, B^*p^{n-1})_U.$$

Now, by adding the different equalities, we find the adjoint equation by identifying

$$\Delta t \sum_{n=1}^{M} (C\bar{z}^n - y_d^n, Cw_u^n)_Y + (D\bar{z}^M - y_T, Dw_u^M)_{Y_T}$$

with

$$\Delta t \sum_{n=1}^{M} (u^n, B^* p^{n-1})_U.$$

More precisely, if $p = (p^0, \dots, p^M)$ is defined by

$$\begin{array}{l} p^M = D^*(D\bar{z}^M - y_T), \\ \text{for } n = 1, \dots, M, \quad p^n \text{ is the solution to} \\ \frac{1}{\Delta t}(-p^n + p^{n-1}) = A^* p^{n-1} + C^*(C\bar{z}^n - y_d^n), \end{array} \tag{10.6.17}$$

then

$$F'_M(\bar{u})u = \Delta t \sum_{n=1}^{M} (u^n, B^* p^{n-1})_U + \Delta t \sum_{n=1}^{M} (\bar{u}^n, u^n)_U.$$

Observe that the above identification is not justified since $D^*(D\bar{z}^M - y_T)$ does not necessarily belong to $D(A^*)$. In practice, a 'space-discretization' is also performed. This means that equation (10.6.15) is replaced by a system of ordinary differential equations, the operator $A$ is replaced by an operator belonging to $\mathcal{L}(\mathbb{R}^\ell)$, where $\ell$ is the dimension of the discrete space, and the above calculations are justified for the corresponding discrete problem.

## 10.7 Exercises

**Exercise 10.7.1**

Apply the conjugate gradient method to problem $(P_4)$ of chapter 5. In particular identify the bounded operator $\Lambda$ from $L^2(\Sigma)$ into $L^2(\Omega)$, and its adjoint $\Lambda^*$, such that

$$|||w_u(T)|||^2_{H^{-1}(\Omega)} = \int_\Omega |\Lambda u|^2$$

where $w_u$ is the solution to equation

$$\frac{\partial w}{\partial t} - \Delta w = 0 \quad \text{in } Q, \quad w = u \quad \text{on } \Sigma, \quad w(0) = 0 \quad \text{in } \Omega.$$

**Exercise 10.7.2**

Apply the conjugate gradient method to problem $(P_5)$ of chapter 6.

**Exercise 10.7.3**

Apply the SQP method to problem $(P_1)$ of chapter 3. In particular, prove that the Linear-Quadratic problem $(QP_{n+1})$ of the SQP method is well posed.

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