Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

BOUNDARY FEEDBACK STABILIZATION OF THE TWO DIMENSIONAL NAVIER-STOKES EQUATIONS WITH FINITE DIMENSIONAL CONTROLLERS

Abstract. We study the boundary stabilization of the two-dimensional Navier-Stokes equations about an unstable stationary solution by controls of finite dimension in feedback form. The main novelty is that the linear feedback control law is determined by solving an optimal control problem of finite dimension. More precisely, we show that, to stabilize locally the Navier-Stokes equations, it is sufficient to look for a boundary feedback control of finite dimension, able to stabilize the projection of the linearized equation onto the unstable subspace of the linearized Navier-Stokes operator. The feedback operator is obtained by solving an algebraic Riccati equation in a space of finite dimension, that is to say a matrix Riccati equation.

Jean-Pierre Raymond

Université de Toulouse & CNRS
Institut de Mathématiques, UMR 5219
31062 Toulouse Cedex 9, France

Laetitia Thevenet

Université de Toulouse & CNRS
Institut de Mathématiques, UMR 5219
31062 Toulouse Cedex 9, France

1. Introduction.

Control of fluid flows by feedback is a challenging problem both from the theoretical and numerical points of view, see [15, 35] and the references therein. In this paper, we are interested in determining boundary feedback control laws of finite dimension able to stabilize the two dimensional Navier-Stokes equations in a neighborhood of an unstable stationary solution.

The system that we are going to consider may be written in the form

$$\mathbf{z}' = A\mathbf{z} + F(\mathbf{z} + L\mathbf{u}) + B_m\mathbf{u}, \quad \mathbf{z}(0) = \mathbf{z}_0, \tag{1.1}$$

where $(A, D(A))$ is the infinitesimal generator of an analytic semigroup in a real Hilbert space Z, the control operator B_m belongs to $\mathcal{L}(U, (D(A^*))')$, U is another Hilbert space, $L \in \mathcal{L}(U, Z)$. We further assume that the resolvent of A is compact in Z, and the nonlinear mapping F obeys $F(0) = 0$ and $F'(0) = 0$. We are interested in finding a control $\mathbf{u}$, in feedback form and of finite dimension, that is of the form

$$\mathbf{u}(t) = \sum_{i=1}^{K} \big(\mathbf{z}(t), \xi_i\big)_Z \, \zeta_i, \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary: 93B52, 93C20, 93D15; Secondary: 35Q30, 76D55, 76D05, 76D07.

Key words and phrases. Dirichlet control, feedback control, stabilization, Navier-Stokes equations, Oseen equations, Riccati equation, finite dimensional controllers.

Both authors are supported by the ANR-project CORMORED BLAN-08-0115-02.

so that the closed loop system

$$\mathbf{z}' = A\mathbf{z} + F\left(\mathbf{z} + \Sigma_{i=1}^{K}\big(\mathbf{z}(t), \xi_i\big)_Z\, L\zeta_i\right) + \Sigma_{i=1}^{K}\big(\mathbf{z}(t), \xi_i\big)_Z\, B_m\zeta_i, \quad \mathbf{z}(0) = \mathbf{z}_0, \quad (1.3)$$

is locally exponentially stable for all $\mathbf{z}_0$ in some ball $B(0, R)$ in Z. We are going to see that, when equation (1.1) corresponds to the Navier-Stokes equations, the functions $(\xi_i)_{1\le i\le K}$ may be chosen of the form $\Pi Q B_m\zeta_i$, where $\zeta_i = B_m^*\varepsilon_i$ and ε_i are eigenfunctions or generalized eigenfunctions of A^*, Q is a projection operator and Π can be found by solving a Riccati equation of finite dimension. For that, we apply the following programme.

(i) We decompose the original state space Z into a finite dimensional unstable subspace Z_u and an infinite dimensional stable subspace Z_s. Both subspaces are invariant under e^{tA}.

(ii) We project the linearized system

$$\mathbf{z}' = A\mathbf{z} + B_m\mathbf{u}, \quad \mathbf{z}(0) = \mathbf{z}_0, \quad (1.4)$$

onto the stable and the unstable subspaces. The projection is defined firstly in the complexified space $Z + iZ$ and next in the real space Z. We prove that the projected system onto the unstable subspace is stabilizable by controls $\mathbf{u}$ of the form (1.2).

(iii) We determine a feedback control law, of the form (1.2), able to stabilize the finite dimensional projected unstable system by solving a Riccati equation of finite dimension.

(iv) We prove that the feedback law determined in step (iii) stabilizes locally the nonlinear system (1.1).

The idea of using finite dimensional controllers for the stabilization of linear parabolic systems goes back to R. Triggiani [33] (see also [26]). The idea of using linear feedback law for proving local stabilization of semilinear partial differential equations is not new and goes back to I. Lasiecka [25]. In that case the proof of local stabilization relies on a fixed point argument, see [24]. In the context of an internal control of the Navier-Stokes equations such an approach has been developed in [8]. The drawback of this method is that it does not give an obvious Lyapunov functional for the closed loop nonlinear system. This explains why other approaches have been investigated. They consist in solving the optimal control problem

$$(\mathcal{P}) \qquad \inf\Big\{J(\mathbf{z}, \mathbf{u}) \mid (\mathbf{z}, \mathbf{u}) \text{ is solution to (1.4)}\Big\},$$

with a cost functional J is of the form

$$J(\mathbf{z}, \mathbf{u}) = \frac{1}{2}\int_0^\infty |C\mathbf{z}(t)|_Y^2\, dt + \frac{1}{2}\int_0^\infty |\mathbf{u}(t)|_U^2\, dt,$$

where C is a linear operator (not necessarily bounded) from Z into another Hilbert space Y. We consider the cases when $(\mathcal{P})$ admits a unique solution. The value function of problem $(\mathcal{P})$ is of the form

$$\mathbf{z}_0 \longmapsto \frac{1}{2}(\Pi\mathbf{z}_0, \mathbf{z}_0)_Z,$$

and the optimal pair $(\mathbf{z}_{\mathbf{z}_0}, \mathbf{u}_{\mathbf{z}_0})$ obeys the feedback law $\mathbf{u}_{\mathbf{z}_0}(t) = -B_m^*\Pi\mathbf{z}_{\mathbf{z}_0}(t)$. The operator Π may be bounded or unbounded in Z depending on the choice of C. This choice is crucial both for the characterization of Π and for proving that the linear feedback law stabilizes locally the nonlinear system (1.1). In [6, 12] for the internal stabilization of the Navier-Stokes equations and in [9] for the boundary stabilization of the Navier-Stokes equations, the observation operator C is chosen so that the feedback generator $A_\Pi = A - B_m B_m^* \Pi$ is dissipative with respect to the product $(\Pi\cdot, \cdot)_Z$ and the value function of problem $(\mathcal{P})$ is a Lyapunov function of the closed loop nonlinear system. This choice for C is refered as the high gain functional approach [10, 11]. In the case of an internal control [6], the operator Π is unbounded in Z and it is the unique solution to an algebraic Riccati equation which is well posed in $D(A)$ (a similar result is obtained in [12] for controls of finite dimension). In the case of a boundary control [9], Π is still an unbounded operator in Z, but it is not characterized by a Riccati equation satisfied in Z or in $D(A)$. (The equation is satisfied only in $D(A_\Pi^2)$ which is not known and therefore this equation is not useful.) Moreover the 'high gain functional approach' for a boundary control has been developed only in the case of tangential controls. It is not known if such an approach is still valid for controls which are not tangential to the boundary. The drawback with tangential controls is that the stabilizability is not necessarily guarenteed. It is established only under a smallness condition on the data (see [34, Theorem 2.2, and assumption (1.7) of Theorem 1.2]).

The approach consisting in taking C equal either to the identity in Z or to a smoothing operator (the low gain functional approach) has been developed for the boundary control of the Navier-Stokes equations in [29], for the two dimensional case, and in [30], for the three dimensional case. In those cases the Riccati equation is well posed in Z. In [29], we have proved that when $C = I$ the operator Π is bounded from Z into $D(A)$ (which is slightly better than the results already known in the literature [27, Theorem 2.2.1 (a_3)]). In [30, 31] we have shown, for the first time to the authors knowledge, that the smoothing properties of Π can be improved by taking an operator C more smoothing than the identity. The existence of Lyapunov functionals for the corresponding closed loop nonlinear systems has been solved very recently by M. Badra [2, 3, 4].

However in all these approaches [12, 9, 29, 30], the pair (A, C) is completely detectable, and the feedback control laws determined in [6, 12], in the case of an internal control, and in [29], in the case of a boundary control, are obtained by solving an algebraic Riccati equation stated in a space of infinite dimension.

In the present paper we introduce a qualitative jump by showing that it is possible to stabilize a nonlinear system with a linear feedback law determined with an operator C for which the pair (A, C) is no longer completely detectable. In addition, the Riccati equation that we have to solve is of finite dimension (the equation is stated in $\mathbb{R}^{K\times K}$ where K is the dimension of the unstable space of the linearized Navier-Stokes operator). To the authors knowledge, this type of result is completely new in the context of local stabilization of nonlinear systems. (Of course for linear systems the idea goes back to [32].) Furthermore, following [23], we could even take $C = \mathbf{0}$. In that case the feedback law is determined by looking for the maximal solution to a degenerate Riccati equation.

For numerical calculations an approximation scheme has to be used for solving the Navier-Stokes system. To determine the Riccati equation that we have to solve in the present paper, one needs to determine a few eigenvalues and the corresponding eigenfunctions and generalized eigenfunctions of the operators A and A^*. Even if these numerical calculations can be delicate, they are much easier than solving an algebraic Riccati equation of high dimension.

Let us describe more precisely our problem. Let Ω be a bounded and connected domain in $\mathbb{R}^2$ with boundary Γ of class C^4, $\nu > 0$, and consider a couple $(\mathbf{w}, \chi)$ – a velocity field and a pressure – solution to the stationary Navier-Stokes equations in Ω

$$-\nu\Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla\chi = \mathbf{f} \quad \text{and} \quad \operatorname{div}\mathbf{w} = 0 \text{ in } \Omega, \quad \mathbf{w} = \mathbf{u}_s^\infty \quad \text{on } \Gamma. \tag{1.5}$$

We assume that $\mathbf{w}$ belongs to $\mathbf{V}^3(\Omega) = \{\mathbf{v} \in H^3(\Omega;\mathbb{R}^2) \mid \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\}$, and that it is an unstable solution of the Navier-Stokes equations. We consider the control system

$$\begin{aligned} &\frac{\partial \mathbf{z}}{\partial t} - \nu\Delta\mathbf{z} + (\mathbf{z}\cdot\nabla)\mathbf{z} + \nabla q = 0, \quad \operatorname{div}\mathbf{z} = 0 \quad \text{in } \Omega\times(0,\infty),\\ &\mathbf{z} = \mathbf{u}_s^\infty + M\hat{\mathbf{u}} \ \text{ on } \Sigma_\infty = \Gamma\times(0,\infty), \quad \mathbf{z}(0) = \mathbf{w} + \mathbf{y}_0 \text{ in } \Omega, \end{aligned} \tag{1.6}$$

and the corresponding system satisfied by $\hat{\mathbf{y}} = \mathbf{z} - \mathbf{w}$

$$\begin{aligned} &\frac{\partial \hat{\mathbf{y}}}{\partial t} - \nu\Delta\hat{\mathbf{y}} + (\hat{\mathbf{y}}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\hat{\mathbf{y}} + (\hat{\mathbf{y}}\cdot\nabla)\hat{\mathbf{y}} + \nabla\hat{p} = 0, \quad \text{in } \Omega\times(0,\infty),\\ &\operatorname{div}\hat{\mathbf{y}} = 0 \quad \text{in } \Omega\times(0,\infty),\\ &\hat{\mathbf{y}} = M\hat{\mathbf{u}} \ \text{ on } \Sigma_\infty, \quad \hat{\mathbf{y}}(0) = \mathbf{y}_0 \text{ in } \Omega. \end{aligned} \tag{1.7}$$

The operator $M \in \mathcal{L}(L^2(\Gamma;\mathbb{R}^2))$ is used to localize the control in a part of the boundary (see section 2.1). In order to stabilize $\hat{\mathbf{y}}$ with a prescribed exponential decay rate $e^{-\alpha t}$, $\alpha > 0$, we set

$$\mathbf{y} = e^{\alpha t}\hat{\mathbf{y}}, \quad p = e^{\alpha t}\hat{p}, \quad \mathbf{u} = e^{\alpha t}\hat{\mathbf{u}}.$$

Then, $(\mathbf{y}, p)$ is solution to the system

$$\begin{aligned} &\frac{\partial \mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} - \alpha\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{y} + e^{-\alpha t}(\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = 0,\\ &\operatorname{div}\mathbf{y} = 0 \text{ in } \Omega\times(0,\infty),\\ &\mathbf{y} = M\mathbf{u} \ \text{ on } \Sigma_\infty, \quad \mathbf{y}(0) = \mathbf{y}_0 \text{ in } \Omega. \end{aligned} \tag{1.8}$$

In [29], the first author has determined a linear feedback law able to stabilize the nonlinear system (1.8). As mentioned above, in [29] the Riccati equation is stated in a space of infinite dimension. In the present paper, we want to find a control $\mathbf{u}$ of the form

$$\mathbf{u}(t,x) = \sum_{i=1}^{n_c} v_i(t)\zeta_i(x) \in L^2(0,\infty;L^2(\Gamma;\mathbb{R}^2)), \tag{1.9}$$

able to stabilize equation (1.8) and for which $v_i \in L^2(0,\infty)$, $i = 1,\dots,n_c$, are written in feedback form. (The functions $\{\zeta_i\}_{i=1}^{n_c}$ are not a priori known and have to be determined). Let us explain how we proceed. Following [28, 29], we first write the linearized equation associated with (1.8) in the form

$$\begin{aligned} &P\mathbf{y}' = AP\mathbf{y} + BM\mathbf{u} = AP\mathbf{y} + (\lambda_0 I - A)PD_AM\mathbf{u} \quad \text{in } (0,\infty),\\ &P\mathbf{y}(0) = \mathbf{y}_0,\\ &(I-P)\mathbf{y} = (I-P)D_AM\mathbf{u} \quad \text{in } (0,\infty), \end{aligned} \tag{1.10}$$

where P is the so-called Helmholtz or Leray projection operator, A is the linearized Navier-Stokes operator, D_A is a Dirichlet operator and B is a control operator (see section 2). Let us notice that the associated nonlinear system

$$P\mathbf{y}' = AP\mathbf{y} + BM\mathbf{u} + F(P\mathbf{y} + (I-P)D_AM\mathbf{u}) \quad \text{in } (0,\infty), \quad P\mathbf{y}(0) = \mathbf{y}_0,$$

is of the form (1.1) if we set $\mathbf{z} = P\mathbf{y}$, $B_m = BM$, $L = (I-P)D_AM$ and $F(\mathbf{y}) = e^{-\alpha t}P((\mathbf{y}\cdot\nabla)\mathbf{y})$.

Next, we decompose $\mathbf{V}^0_n(\Omega) = \Big\{\mathbf{y} \in L^2(\Omega;\mathbb{R}^2) \mid \ \text{div}\ \mathbf{y} = 0 \ \text{ in } \Omega,\ \mathbf{y}{\cdot}\mathbf{n} = 0 \text{ on } \Gamma\Big\}$ in the form

$$\mathbf{V}^0_n(\Omega) = \mathbf{Y}_{\alpha^-} \oplus \mathbf{Y}_\alpha,$$

where $\mathbf{Y}_\alpha \subset D(A)$ is the finite dimensional unstable susbspace of A and $\mathbf{Y}_{\alpha^-}$ is the stable subspace. Similarly, we have

$$\mathbf{V}^0_n(\Omega) = \mathbf{Y}^*_{\alpha^-} \oplus \mathbf{Y}^*_\alpha,$$

where $\mathbf{Y}^*_\alpha \subset D(A^*)$ is the finite dimensional unstable susbspace of A^* and $\mathbf{Y}^*_{\alpha^-}$ is the stable subspace. In section 3, we prove that there exist a basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_K\}$ of $\mathbf{Y}_\alpha$ and a basis $\{\varepsilon_1, \cdots, \varepsilon_K\}$ of $\mathbf{Y}^*_\alpha$ such that

$$\Big(\mathbf{e}_i, \varepsilon_j\Big) = \int_\Omega \mathbf{e}_i(x)\varepsilon_j(x)dx = \delta^j_i,$$

where δ^j_i is the Kronecker symbol. This type of result is already established for parabolic equations in [19] and in [18, 20] for linearized Navier-Stokes equations. Let Q denote the projection onto $\mathbf{Y}_\alpha$ along $\mathbf{Y}_{\alpha^-}$. With such a choice for the basis of $\mathbf{Y}_\alpha$ and $\mathbf{Y}^*_\alpha$, we obtain a very simple expression of Q and of Q^*, the adjoint of Q. We consider the system

$$QP\mathbf{y}' = AQP\mathbf{y} + QBM\mathbf{u}, \quad QP\mathbf{y}(0) = Q\mathbf{y}_0. \tag{1.11}$$

In section 4, we prove that equation (1.11) is stabilizable by a control $\mathbf{u}$ of the form (1.9), where $\{\zeta_1, \cdots, \zeta_{n_c}\}$ is a basis of $\mathcal{U} = \text{vect}\Big\{MB^*\varepsilon_j \mid j = 1, \cdots, K\Big\}$. In section 5, we introduce a linear quadratic control problem of which the Riccati equation is

$$\begin{aligned} &\Pi = \Pi^* \in \mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}^*_\alpha), \quad \Pi \geq 0, \\ &\Pi A_\alpha + A^*_\alpha \Pi - \Pi QB\widetilde{M}\widetilde{M}^* B^* Q^* \Pi + Q^* Q = 0, \end{aligned} \tag{1.12}$$

where $\widetilde{M}$ is defined in (4.6). Let us notice that it is a finite dimensional algebraic Riccati equation. Finally, in section 6 we show that the feedback law

$$v_i(t) = -\Big(\Pi Q\, BM\zeta_i, P\mathbf{y}(t)\Big)_{\mathbf{V}^0_n(\Omega)} = -\int_\Omega \Pi Q\, BM\zeta_i\, P\mathbf{y}(t)\, dx,$$

with $\mathbf{u}$ given by (1.9) stabilizes the Navier-Stokes equation locally about $\mathbf{w}$.

Even if this result seems interesting, we would like to explain what is its practical interest for numerical computations. Such an approach consisting in decoupling the linearized Navier-Stokes equations into a stable and an unstable part has been used by S. Ahuja and C. W. Rowley in [1] to design reduced order models. Here, our goal is to use this decomposition to define a finite dimensional Riccati equation. Even if the domain of stability of the feedback law determined here is small, the result is still interesting. Indeed, there is an efficient algorithm to solve large scale Riccati equations, the so-called Newton–Kleinman method (see [14], and the references therein). The drawback of the Newton–Kleinman method is that it requires an initial guess for which the corresponding closed loop system is stable. The feedback that we determine may provide such an initial guess. By this way, we can hope to enlarge the domain of stability of the feedback law. Let us finally mention that, following [2, 3, 4], it should be possible to define a Lyapunov function of the closed loop system obtained by coupling the Navier-Stokes equations with the finite dimensional feedback control that we have determined.

Some months after the submission of this paper, we were informed of [5]. One objective in [5] is to use the tools introduced in the present paper to characterize the minimal dimension of static and dynamic controllers able to stabilize system (1.8). Here our purpose is mainly to characterize feedback controllers of finite dimension by a matrix Riccati equation (see section 5.3).

2. Functional framework.

2.1. Notation and assumptions.

Let us introduce the following spaces $H^s(\Omega;\mathbb{R}^N) = \mathbf{H}^s(\Omega)$, $L^2(\Omega;\mathbb{R}^N) = \mathbf{L}^2(\Omega)$, the same notation conventions will be used for trace spaces and for the spaces $H^s_0(\Omega;\mathbb{R}^N)$. We also introduce different spaces of free divergence functions and some corresponding trace spaces

$$\mathbf{V}^s(\Omega) = \Big\{ \mathbf{y} \in \mathbf{H}^s(\Omega) \mid \ \mathrm{div}\ \mathbf{y} = 0 \ \text{ in } \Omega,\ \langle \mathbf{y}\cdot\mathbf{n}, 1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0 \Big\}, \quad s \geq 0,$$

$$\mathbf{V}^s_n(\Omega) = \Big\{ \mathbf{y} \in \mathbf{H}^s(\Omega) \mid \ \mathrm{div}\ \mathbf{y} = 0 \ \text{ in } \Omega,\ \mathbf{y}\cdot\mathbf{n} = 0 \text{ on } \Gamma \Big\} \quad \text{for } s \geq 0,$$

$$\mathbf{V}^s_0(\Omega) = \Big\{ \mathbf{y} \in \mathbf{H}^s(\Omega) \mid \ \mathrm{div}\ \mathbf{y} = 0 \ \text{ in } \Omega,\ \mathbf{y} = 0 \text{ on } \Gamma \Big\} \quad \text{for } s > 1/2,$$

$$\mathbf{V}^s(\Gamma) = \Big\{ \mathbf{y} \in \mathbf{H}^s(\Gamma) \mid \langle \mathbf{y}\cdot\mathbf{n}, 1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0 \Big\} \quad \text{for } s \geq -1/2\,.$$

In the above setting $\mathbf{n}$ denotes the unit normal to Γ outward Ω. We shall use the following notation $\Sigma_\infty = \Gamma \times (0,\infty)$. We also set

$$\mathbf{V}^{s,\sigma}(\Omega\times(0,\infty)) = H^\sigma(0,\infty;\mathbf{V}^0(\Omega)) \cap L^2(0,\infty;\mathbf{V}^s(\Omega)) \quad \text{for } s,\ \sigma \geq 0,$$

and

$$\mathbf{V}^{s,\sigma}(\Sigma_\infty) = H^\sigma(0,\infty;\mathbf{V}^0(\Gamma)) \cap L^2(0,\infty;\mathbf{V}^s(\Gamma)) \quad \text{for } s,\ \sigma \geq 0.$$

For an open subset Γ_c of Γ, we introduce a weight function $m \in C^2(\Gamma)$ with values in $[0,1]$, with support in Γ_c, equal to 1 in Γ_0, where Γ_0 is an open subset in Γ_c. Associated with this function m, we introduce the operator $M \in \mathcal{L}(\mathbf{V}^0(\Gamma))$ defined by

$$M\mathbf{u}(x) = m(x)\mathbf{u}(x) - \frac{m}{\int_\Gamma m}\left(\int_\Gamma m\mathbf{u}\cdot\mathbf{n}\right)\mathbf{n}(x).$$

By this way, we can replace the condition $\mathrm{supp}(\mathbf{u}) \subset \Gamma_c$ by considering a boundary condition of the form

$$\mathbf{z} - \mathbf{w} = M\hat{\mathbf{u}} \quad \text{on} \quad \Sigma_\infty.$$

For all $\psi \in H^{1/2+\varepsilon'}(\Omega)$, with $\varepsilon' > 0$, we denote by $c(\psi)$ the constant defined by

$$c(\psi) = \frac{1}{|\Gamma|}\int_\Gamma \psi, \tag{2.1}$$

where $|\Gamma|$ is the $(N-1)$-dimensional Lebesgue measure of Γ. Let us recall that P, the so-called Leray or Helmholtz projector, is the orthogonal projection in $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}^0_n(\Omega)$.

2.2. Properties of some operators.

In this subsection we briefly recall the definitions and properties of some operators already used in [29]. The proof of these results can be found in [29]. We denote by $(A, D(A))$ and $(A^*, D(A^*))$ the unbounded operators in $\mathbf{V}^0_n(\Omega)$ defined by

$$D(A) = \mathbf{H}^2(\Omega)\cap\mathbf{V}^1_0(\Omega), \qquad A\mathbf{y} = \nu P\Delta\mathbf{y} + \alpha\mathbf{y} - P((\mathbf{w}\cdot\nabla)\mathbf{y}) - P((\mathbf{y}\cdot\nabla)\mathbf{w}),$$

$$D(A^*) = \mathbf{H}^2(\Omega)\cap\mathbf{V}^1_0(\Omega), \quad A^*\mathbf{y} = \nu P\Delta\mathbf{y} + \alpha\mathbf{y} + P((\mathbf{w}\cdot\nabla)\mathbf{y}) - P((\nabla\mathbf{w})^T\mathbf{y}).$$

Since $\mathbf{w} \in \mathbf{V}^3(\Omega)$ and $\mathrm{div}\ \mathbf{w} = 0$, we can verify that there exists $\lambda_0 > 0$ in the resolvent set of A satisfying

$$\big((\lambda_0 I - A)\mathbf{y}, \mathbf{y}\big)_{\mathbf{V}^0_n(\Omega)} \geq \frac{\nu}{2}|\mathbf{y}|^2_{\mathbf{V}^1_0(\Omega)} \qquad \text{for all } \mathbf{y} \in D(A),$$

and (2.2)

$$\big((\lambda_0 I - A^*)\mathbf{y}, \mathbf{y}\big)_{\mathbf{V}^0_n(\Omega)} \geq \frac{\nu}{2}|\mathbf{y}|^2_{\mathbf{V}^1_0(\Omega)} \qquad \text{for all } \mathbf{y} \in D(A^*).$$

Theorem 2.1. *The unbounded operator* $(A - \lambda_0 I)$ *(respectively* $(A^* - \lambda_0 I)$*) with domain* $D(A - \lambda_0 I) = D(A)$ *(respectively* $D(A^* - \lambda_0 I)$*) is the infinitesimal generator of a bounded analytic semigroup on* $\mathbf{V}^0_n(\Omega)$*. Moreover, we have*

$$D((\lambda_0 I - A)^\theta) = D((\lambda_0 I - A^*)^\theta) = [\mathbf{V}^0_n(\Omega), D(A)]_\theta$$

for all $0 \leq \theta \leq 1$*.*

Observe that the semigroups $(e^{t(A-\lambda_0 I)})_{t\geq 0}$ and $(e^{t(A^*-\lambda_0 I)})_{t\geq 0}$ are exponentially stable on $\mathbf{V}^0_n(\Omega)$ and that

$$\|e^{t(A-\lambda_0 I)}\|_{\mathcal{L}(\mathbf{V}^0_n(\Omega))} \leq Ce^{-\omega t} \quad \text{and} \quad \|e^{t(A^*-\lambda_0 I)}\|_{\mathcal{L}(\mathbf{V}^0_n(\Omega))} \leq Ce^{-\omega t},$$

for all $\omega < \nu/2$ (see [13, Chapter 1, Theorem 2.12]).

Let us introduce D_A and D_p, two Dirichlet operators associated with A, defined as follows (see [29, p. 796]). For $\mathbf{u} \in \mathbf{V}^0(\Gamma)$, we set $D_A\mathbf{u} = \mathbf{y}$ and $D_p\mathbf{u} = q$ where $(\mathbf{y}, q)$ is the unique solution in $\mathbf{V}^{1/2}(\Omega) \times (H^{1/2}(\Omega)/\mathbb{R})'$ to the equation

$$\lambda_0\mathbf{y} - \nu\Delta\mathbf{y} - \alpha\mathbf{y} + (\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} + \nabla q = 0 \quad \text{in } \Omega,$$
$$\operatorname{div} \mathbf{y} = 0 \quad \text{in } \Omega, \quad \mathbf{y} = \mathbf{u} \quad \text{on } \Gamma.$$

Lemma 2.2. (i) *The operator* D_A *is a bounded operator from* $\mathbf{V}^0(\Gamma)$ *into* $\mathbf{V}^0(\Omega)$*, moreover it satisfies*

$$|D_A\mathbf{u}|_{\mathbf{V}^{s+1/2}(\Omega)} \leq C(s)|\mathbf{u}|_{\mathbf{V}^s(\Gamma)} \qquad \text{for all } 0 \leq s \leq 2\,.$$

(ii) *The operator* $D^*_A \in \mathcal{L}(\mathbf{V}^0(\Omega), \mathbf{V}^0(\Gamma))$*, the adjoint operator of* $D_A \in \mathcal{L}(\mathbf{V}^0(\Gamma), \mathbf{V}^0(\Omega))$*, is defined by*

$$D^*_A\mathbf{g} = -\nu\frac{\partial\mathbf{z}}{\partial\mathbf{n}} + \pi\mathbf{n} - c(\pi)\mathbf{n}, \tag{2.3}$$

where $(\mathbf{z}, \pi)$ *is the solution of*

$$\lambda_0\mathbf{z} - \nu\Delta\mathbf{z} - \alpha\mathbf{z} - (\mathbf{w}\cdot\nabla)\mathbf{z} + (\nabla\mathbf{w})^T\mathbf{z} + \nabla\pi = \mathbf{g} \quad \operatorname{div}\mathbf{z} = 0 \quad \text{in } \Omega,$$
$$\mathbf{z} = 0 \quad \text{on } \Gamma, \tag{2.4}$$

and $c(\pi)$ *is defined by* (2.1)*.*

Lemma 2.3. *The operator* $M \in \mathcal{L}(\mathbf{V}^0(\Gamma))$ *is symmetric.*

We introduce the operator $B = (\lambda_0 I - A)PD_A \in \mathcal{L}(\mathbf{V}^0(\Gamma), (D(A^*))')$.

Proposition 2.4. *The operator adjoint* $B^* \in \mathcal{L}(D(A^*), \mathbf{V}^0(\Gamma))$ *satisfies* $B^*\mathbf{\Phi} = D^*_A(\lambda_0 I - A^*)\mathbf{\Phi}$ *and*

$$B^*\mathbf{\Phi} = -\nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}} + \psi\mathbf{n} - c(\psi)\mathbf{n},$$

for all $\mathbf{\Phi} \in D(A^*)$*, with*

$$\nabla\psi = (I - P)\Big[\nu\Delta\mathbf{\Phi} + (\mathbf{w}\cdot\nabla)\mathbf{\Phi} - (\nabla\mathbf{w})^T\mathbf{\Phi}\Big],$$

and $c(\psi)$ *defined by* (2.1)*. Moreover, the following estimate holds*

$$|B^*\mathbf{\Phi}|_{\mathbf{V}^{s-3/2}(\Gamma)} \leq C|\mathbf{\Phi}|_{\mathbf{V}^s(\Omega)\cap\mathbf{V}^1_0(\Omega)},$$

for all $\mathbf{\Phi} \in \mathbf{V}^s(\Omega) \cap \mathbf{V}^1_0(\Omega)$ *with* $s \geq 2$*.*

3. **Projected systems.** In order to introduce the generalized eigenfunctions of the operator A, we consider the complexified space

$$V_n^0(\Omega) = \mathbf{V}_n^0(\Omega) \oplus i\mathbf{V}_n^0(\Omega).$$

The first equation in (1.10) may be extended to spaces of functions with complex values as follows

$$Py' = APy + BMu, \quad Py(0) = y_0, \tag{3.1}$$

where y_0, y and u are now functions with complex values.

3.1. **The resolvent of the operator A.** We first study the resolvent of the operator A.

Lemma 3.1. *The resolvent of A is compact and the spectrum of A is discrete.*

Proof. See [18, Lemma 3.1]. □

Now, we give a decomposition of the resolvent of A by using Laurent series. Let λ_j belong to the spectrum of A. For λ in the neighbourhood of λ_j, the resolvent of A can be expressed in a Laurent series

$$\begin{aligned} &R(\lambda, A) = \sum_{k=-\infty}^{+\infty} (\lambda - \lambda_j)^k R_k(\lambda_j) \\ &\text{with} \quad R_k(\lambda_j) = \frac{1}{2\pi i} \int_{|\lambda-\lambda_j|=\varepsilon} \frac{R(\lambda, A)}{(\lambda - \lambda_j)^{k+1}} d\lambda, \quad \text{and } \varepsilon > 0 \text{ small enough.} \end{aligned} \tag{3.2}$$

Lemma 3.2. *The expansion (3.2) of the resolvent in a Laurent series in a neighbourhood of λ_j contains finitely many terms with negative power of $\lambda - \lambda_j$, that is*

$$R(\lambda, A) = \sum_{k=-m(\lambda_j)}^{+\infty} (\lambda - \lambda_j)^k R_k(\lambda_j). \tag{3.3}$$

Proof. The proof is done in [19, Lemma 3.3]. □

Since the spectrum of A is a pointwise spectrum, we may always choose $\alpha > 0$ such that

$$\cdots \le \Re\lambda_{N_\alpha+1} < 0 < \Re\lambda_{N_\alpha} \le \cdots \le \Re\lambda_1$$

for some $N_\alpha \in \mathbb{N}^*$. We consider the continuous contour γ_0 in the complex half-plane $\{\lambda \in \mathbb{C} \mid \Re\lambda \le 0\}$ made up of a segment of the line $\{\Re\lambda = 0\}$ and the two branches of γ on rays $\{\ \mathrm{Arg}\lambda = \pm\theta\}$ with $\theta > \frac{\pi}{2}$. Thanks to this new contour, we obtain another expression of the semigroup given in the following lemma.

Lemma 3.3. *The semigroup e^{tA} may be written in the form*

$$e^{tA} = \frac{1}{2\pi i} \int_{\gamma_0} (\lambda I - A)^{-1} e^{\lambda t} d\lambda + \sum_{j=1}^{N_\alpha} e^{\lambda_j t} \sum_{n=1}^{m(\lambda_j)} \frac{t^{n-1}}{(n-1)!} R_{-n}(\lambda_j).$$

Proof. See [19, p. 603]. □

3.2. **Canonical Systems.** For $1 \leq j \leq N_\alpha$, we set

$$E(\lambda_j) = \text{Ker } (A - \lambda_j I) \quad \text{ and } \quad \ell(j) = \text{ dim } E(\lambda_j).$$

$E(\lambda_j)$ is the eigenspace associated to the eigenvalue λ_j and $\ell(j)$ is the geometric multiplicity of λ_j. We also introduce the generalized eigenspace

$$G(\lambda_j) = \text{Ker } ((A - \lambda_j I)^{m(\lambda_j)}) \quad \text{ and } \quad N(\lambda_j) = \text{ dim } G(\lambda_j),$$

where $m(\lambda_j)$ is the multiplicity of the pole λ_j of the resolvent (see Lemma 3.2) and $N(\lambda_j)$ is the algebraic multiplicity of λ_j.

If λ_j is an eigenvalue of A, then $\overline{\lambda_j}$ is an eigenvalue of A^*. Since A has real coefficients, $\overline{\lambda_j}$ is an eigenvalue of A and λ_j is an eigenvalue of A^*. We also introduce the generalized eigenspaces of A^*

$$G^*(\lambda_j) = \text{Ker } ((A^* - \lambda_j I)^{m(\lambda_j)}).$$

Let us define the multiplicity of an eigenvector.

Definition 3.4. (See [18, 19]). We say that $(e_1^k, e_2^k, \cdots, e_r^k)$ forms a chain of generalized eigenvectors, when the following relations hold

$$(\lambda_j I - A)e_1^k = 0, \quad (\lambda_j I - A)e_2^k + e_1^k = 0, \quad \cdots, \quad (\lambda_j I - A)e_r^k + e_{r-1}^k = 0.$$

If the maximal order of the chain of generalized eigenvectors corresponding to e_1^k is m then the number m is called the multiplicity of the eigenvector e_1^k.

We consider special bases of generalized eigenvectors.

Definition 3.5. (See [18, 19]). A basis of $G(\lambda_j)$ of the form

$$\left\{ e_i^k \mid k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k^j \right\}$$

is called a canonical system if $\left\{ e_1^k \mid k = 1, \cdots, \ell(j) \right\}$ is a basis of $E(\lambda_j)$ defined in the following way

- e_1^1 is an eigenvector with maximum possible multiplicity m_1^j,
- e_1^k is an eigenvector with maximum possible multiplicity m_k^j such that e_1^k is not linearly expressible in terms of $e_1^1, \cdots, e_1^{k-1}$,

and, for $k = 1, \cdots, \ell(j)$ and $i = 2, \cdots, m_k^j$, $(A - \lambda_j I)e_i^k = e_{i-1}^k$.

Obviously, we have $m(\lambda_j) = \max(m_1^j, \cdots, m_{\ell(j)}^j)$ and $N(\lambda_j) = \sum_{k=1}^{\ell(j)} m_k^j$. We remark that if λ_j is an eigenvalue of the operator A with multiplicity $m(\lambda_j)$, then $\overline{\lambda_j}$ is an eigenvalue of the operator A^* with the same multiplicity. That is why, we can define another canonical system associated to $\overline{\lambda_j}$ for A^*

$$\left\{ \varepsilon_i^k \mid k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k^j \right\},$$

where $\left\{ \varepsilon_1^k \mid k = 1, \cdots, \ell(j) \right\}$ is a basis of Ker $(A^* - \overline{\lambda_j} I)$ such that

- ε_1^1 is an eigenvector with maximum possible multiplicity m_1^j,
- ε_1^k is an eigenvector with maximum possible multiplicity m_k^j such that ε_1^k is not linearly expressible in terms of $\varepsilon_1^1, \cdots, \varepsilon_1^{k-1}$,

and, for $k = 1, \cdots, \ell(j)$ and $i = 2, \cdots, m_k^j$, $(A^* - \overline{\lambda_j} I)\varepsilon_i^k = \varepsilon_{i-1}^k$.

In what follows, we denote by $\Big(\cdot, \cdot\Big)$ the complex inner product in $V_n^0(\Omega)$, that is

$$\Big(f, g\Big) = \int_\Omega f\, \overline{g}\, dx.$$

Definition 3.6. For a couple $(e_i, \varepsilon_j) \in (V_n^0(\Omega))^2$, we denote by $e_i \varepsilon_j$ the operator defined by

$$(e_i \varepsilon_j) f = \Big(f, \varepsilon_j\Big)\, e_i$$

for all $f \in V_n^0(\Omega)$.

Theorem 3.7. *For any canonical system* $\Big\{\varepsilon_i^k \mid k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k^j\Big\}$ *of* A^* *corresponding to the eigenvalue* $\overline{\lambda_j}$*, there is a uniquely determined canonical system* $\Big\{e_i^k \mid k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k^j\Big\}$ *of* A *for* λ_j *such that the principal part of the resolvent can be expressed in the following way in a neighbourhood of* λ_j

$$\sum_{p=-m(\lambda_j)}^{-1} (\lambda - \lambda_j)^p R_p(\lambda_j)$$
$$= \sum_{k=1}^{\ell(j)} \left(\frac{e_1^k \varepsilon_1^k}{(\lambda - \lambda_j)^{m_k^j}} + \frac{e_1^k \varepsilon_2^k + e_2^k \varepsilon_1^k}{(\lambda - \lambda_j)^{m_k^j - 1}} + \cdots + \frac{e_1^k \varepsilon_{m_k^j}^k + e_2^k \varepsilon_{m_k^j - 1}^k + \cdots + e_{m_k^j}^k \varepsilon_1^k}{(\lambda - \lambda_j)} \right).$$

Proof. See [19, Theorem 3.1] or [18, 22]. □

3.3. **The complex projected system.** We consider the space $Z_\alpha = \oplus_{j=1}^{N_\alpha} G(\lambda_j)$. We denote by $N = \sum_{j=1}^{N_\alpha} N(\lambda_j)$ its dimension. With [21, p. 178 – 182], we first notice that the space $V_n^0(\Omega)$ can be decomposed as follows

$$V_n^0(\Omega) = Z_{\alpha^-} \oplus Z_\alpha,$$

where Z_{α^-} is the stable space of A, that is to say $Z_{\alpha^-} \cap D(A)$ is invariant under A. Similarly, we have the decomposition

$$V_n^0(\Omega) = Z_{\alpha^-}^* \oplus Z_\alpha^*,$$

where $Z_\alpha^* = \oplus_{j=1}^{N_\alpha} G^*(\lambda_j)$ and where $Z_{\alpha^-}^* \cap D(A^*)$ is invariant under A^*.

The space Z_α will be equipped with the norm

$$|y|_{Z_\alpha} = |y|_{V_n^0(\Omega)}.$$

Let γ_α be a simple closed curve enclosing $(\lambda_1, \cdots, \lambda_{N_\alpha})$ but no other point of the spectrum of A, and oriented counterclockwise. The operator

$$P_\alpha = \frac{1}{2\pi i} \int_{\gamma_\alpha} (\lambda I - A)^{-1} d\lambda$$

is the projection onto Z_α parallel to Z_{α^-} (see [21, p. 178–182]).

Lemma 3.8. *For all* $j \in \{1, \cdots, N_\alpha\}$*, we consider* $\big\{\varepsilon_i^k(\overline{\lambda_j}) \mid k = 1, \cdots, \ell(j),$ $i = 1, \cdots, m_k^j\big\}$*, a canonical system of* A^* *corresponding to the eigenvalue* $\overline{\lambda_j}$*, and* *the corresponding canonical system of* A, $\left\{e_i^k(\lambda_j) \mid k=1,\cdots,\ell(j),\ i=1,\cdots,m_k^j\right\}$, *associated with* λ_j *and determined in Theorem 3.7. Then, we have*

$$\left(e_{i_1}^{k_1}(\lambda_{j_1}), \varepsilon_{i_2}^{k_2}(\overline{\lambda_{j_2}})\right) = \delta_{j_1}^{j_2}\,\delta_{k_1}^{k_2}\,\delta_{m_{k_1}^{j_1}+1-i_1}^{i_2},$$

for all $j_1, j_2 \in \{1,\cdots,N_\alpha\}$, $k_1 \in \{1,\cdots,\ell(j_1)\}$, $k_2 \in \{1,\cdots,\ell(j_2)\}$, $i_1 \in \left\{1,\cdots,m_{k_1}^{j_1}\right\}$, $i_2 \in \left\{1,\cdots,m_{k_2}^{j_2}\right\}$ (*where the Kronecker symbols* $\delta_{j_1}^{j_2}$, $\delta_{k_1}^{k_2}$ *and* $\delta_{m_{k_1}^{j_1}+1-i_1}^{i_2}$ *are equal to* 1 *if the index is equal to the exponent and* 0 *otherwise*).

Proof. Let j be in $\{1,\cdots,N_\alpha\}$. With the definition of P_α and the definition of $R_{-1}(\lambda_j)$ in (3.2), we have

$$P_\alpha = \sum_{j=1}^{N_\alpha} \frac{1}{2\pi i}\int_{|\lambda-\lambda_j|=\varepsilon} (\lambda I - A)^{-1} d\lambda = \sum_{j=1}^{N_\alpha} R_{-1}(\lambda_j).$$

Let us set $z = e_{i_1}^{k_1}(\lambda_{j_1})$, $j_1 \in \{1,\cdots,N_\alpha\}$, $k_1 \in \{1,\cdots,\ell(j_1)\}$ and $i_1 \in \left\{1,\cdots,m_{k_1}^{j_1}\right\}$. We have $P_\alpha(z) = z$ because z belongs to Z_α. Due to Theorem 3.7, we obtain

$$\begin{aligned} &R_{-1}(\lambda_j)(z)\\ &= \textstyle\sum_{k=1}^{\ell(j)} \left(z, \varepsilon_{m_k^j}^k(\overline{\lambda_j})\right) e_1^k(\lambda_j) + \left(z, \varepsilon_{m_{k-1}^j}^k(\overline{\lambda_j})\right) e_2^k(\lambda_j) + \cdots + \left(z, \varepsilon_1^k(\overline{\lambda_j})\right) e_{m_k^j}^k(\lambda_j). \end{aligned}$$

Since $\left\{e_i^k(\lambda_j) \mid j=1,\cdots,N_\alpha,\ k=1,\cdots,\ell(j),\ i=1,\cdots,m_k^j\right\}$ is a basis of Z_α, we clearly obtain the result and the proof is complete. □

Remark 3.9. With Lemma 3.8, we obtain

$$P_\alpha z = \sum_{j=1}^{N_\alpha}\sum_{k=1}^{\ell(j)}\sum_{i=1}^{m_k^j} \left(z, \varepsilon_{m_k^j+1-i}^k(\overline{\lambda_j})\right) e_i^k(\lambda_j).$$

Since $\dim(Z_\alpha) < \infty$, we can extend, in a continuous way, the operator P_α to $(D(A^*))'$ as follows

$$P_\alpha z = \sum_{j=1}^{N_\alpha}\sum_{k=1}^{\ell(j)}\sum_{i=1}^{m_k^j} \left\langle z, \varepsilon_{m_k^j+1-i}^k(\overline{\lambda_j})\right\rangle_{(D(A^*))',D(A^*)} e_i^k(\lambda_j) \quad \text{for all} \quad z \in (D(A^*))'.$$

Let us observe that the operator P_α belongs to $\mathcal{L}((D(A^*))', Z_\alpha)$.

Remark 3.10. Due to Lemma 3.3, we notice that

$$P_\alpha e^{At} = \sum_{j=1}^{N_\alpha} e^{\lambda_j t} \sum_{n=1}^{m(\lambda_j)} \frac{t^{n-1}}{(n-1)!} R_{-n}(\lambda_j).$$

The system (3.1) projected onto Z_α along to Z_{α^-} is

$$P_\alpha y' = AP_\alpha y + P_\alpha BMu, \quad P_\alpha y(0) = P_\alpha y_0. \tag{3.4}$$

From the definition of P_α, it follows that

$$P_\alpha BMu = \sum_{j=1}^{N_\alpha}\sum_{k=1}^{\ell(j)}\sum_{i=1}^{m_k^j} \left(u, MB^*\varepsilon_{m_k^j+1-i}^k(\overline{\lambda_j})\right) e_i^k(\lambda_j),$$

where

$$\Big(u, MB^*\varepsilon^k_{m^j_k+1-i}(\overline{\lambda_j})\Big) = \Big(u, MB^*\varepsilon^k_{m^j_k+1-i}(\overline{\lambda_j})\Big)_{V^0(\Gamma)} = \int_\Gamma u\,\overline{MB^*\varepsilon^k_{m^j_k+1-i}(\overline{\lambda_j})}.$$

We can rewrite it as a system of ordinary differential equations in $\mathbb{R}^N$ with $N = \sum_{j=1}^{N_\alpha} N(\lambda_j)$. For that, we introduce the coordinates of $P_\alpha y$ in the basis $(e^k_i(\lambda_j))_{1\le j\le N_\alpha,\,1\le k\le \ell(j),\ 1\le i\le m^j_k}$

$$P_\alpha y = \sum_{j=1}^{N_\alpha}\sum_{k=1}^{\ell(j)}\sum_{i=1}^{m^j_k} y^k_i(\lambda_j)e^k_i(\lambda_j).$$

For $1 \le j \le N_\alpha$, we denote by $Y(\lambda_j) \in \mathbb{R}^{N(\lambda_j)}$, the vectors

$$Y(\lambda_j) = \Big((y^1_i(\lambda_j))_{1\le i\le m^j_1}, \cdots, (y^{\ell(j)}_i(\lambda_j))_{1\le i\le m^j_{\ell(j)}}\Big),$$

and the vector $Y \in \mathbb{R}^N$ defined by

$$Y = (Y(\lambda_1), \cdots, Y(\lambda_{N_\alpha}))^T.$$

Similarly, we denote by $U(\lambda_j) \in \mathbb{R}^{N(\lambda_j)}$, the vectors

$$U(\lambda_j) = \left(\Big(u, MB^*\varepsilon^1_{m^j_1+1-i}(\overline{\lambda_j})\Big)_{1\le i\le m^j_1}, \cdots, \Big(u, MB^*\varepsilon^{\ell(j)}_{m^j_{\ell(j)}+1-i}(\overline{\lambda_j})\Big)_{1\le i\le m^j_{\ell(j)}}\right),$$

and the vector $\mathcal{V}(u) \in \mathbb{R}^N$ defined by $\mathcal{V}(u) = (U(\lambda_1), \cdots, U(\lambda_{N_\alpha}))^T$. Due to Lemma 3.8, we can observe that $P_\alpha y$ is solution to system (3.4) if and only if $Y(t)$, the vector of coordinates of $P_\alpha y$, is solution of the differential system

$$Y' = JY + \mathcal{V}(u), \qquad Y(0) = \Big(y_0, \varepsilon^k_{m^j_k+1-i}(\overline{\lambda_j})\Big)_{1\le j\le N_\alpha,\,1\le k\le \ell(j),\,1\le i\le m^j_k}, \tag{3.5}$$

where $J = \mathrm{diag}(J_1, \cdots, J_{N_\alpha})$ and, for $1 \le j \le N_\alpha$, J_j is constituted of Jordan blocks associated with λ_j. If in (3.5) we take u of the form

$$u(x,t) = \sum_{m=1}^{K} v_m(t)\,\zeta_m(x),$$

with $\zeta_m \in \mathrm{vec}\{MB^*\varepsilon^k_i(\overline{\lambda_j}) \mid 1 \le k \le \ell(j),\ 1 \le j \le N_\alpha,\ 1 \le i \le m^j_k\}$, then system (3.5) is of the form

$$Y' = JY + \mathcal{B}V, \tag{3.6}$$

with $V = (v_1, \cdots, v_K)^T$, $\mathcal{B} \in \mathbb{C}^{N\times K}$, $N = \sum_{j=1}^{N_\alpha} N(\lambda_j)$,

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}(\overline{\lambda_1}) \\ \vdots \\ \mathcal{B}(\overline{\lambda_{N_\alpha}}) \end{bmatrix}, \quad \mathcal{B}(\overline{\lambda_j}) = \begin{bmatrix} \mathcal{B}^1(\overline{\lambda_j}) \\ \vdots \\ \mathcal{B}^{\ell(j)}(\overline{\lambda_j}) \end{bmatrix}, \quad \mathcal{B}^k(\overline{\lambda_j}) = \begin{bmatrix} \mathcal{B}^k_1(\overline{\lambda_j}) \\ \vdots \\ \mathcal{B}^k_{m^j_k}(\overline{\lambda_j}) \end{bmatrix},$$

with $1 \le j \le N_\alpha$, $1 \le k \le \ell(j)$, and for $1 \le i \le m^j_k$,

$$\mathcal{B}^k_i(\overline{\lambda_j}) = \Big[(\zeta_1, M^*B^*\varepsilon^k_{m^j_k+1-i}(\overline{\lambda_j}))_{V^0(\Gamma)}, \cdots, (\zeta_K, M^*B^*\varepsilon^k_{m^j_k+1-i}(\overline{\lambda_j}))_{V^0(\Gamma)}\Big] \in \mathbb{C}^K.$$

We do not study here the controllability of system (3.6). We shall prove the controllability of the corresponding real system in section 4 for a particular family $(\zeta_1, \cdots, \zeta_K)$.

3.4. **The real projected system.** We recall that $\Re V_n^0(\Omega) = \mathbf{V}_n^0(\Omega)$ and we introduce the subspaces $\mathbf{Y}_\alpha = \Re Z_\alpha$ and $\mathbf{Y}_{\alpha^-} = \Re Z_{\alpha^-}$. We have

$$\mathbf{V}_n^0(\Omega) = \mathbf{Y}_{\alpha^-} \oplus \mathbf{Y}_\alpha,$$

$\mathbf{Y}_\alpha \subset \cap D(A)$ is invariant under A and $\mathbf{Y}_{\alpha^-} \cap D(A)$ is invariant under A too. We also introduce $\mathbf{Y}_\alpha^* = \Re Z_\alpha^*$ and $\mathbf{Y}_\alpha^* = \Re Z_{\alpha^-}^*$, and we have

$$\mathbf{V}_n^0(\Omega) = \mathbf{Y}_{\alpha^-}^* \oplus \mathbf{Y}_\alpha^*.$$

It is obvious that $\mathbf{Y}_\alpha^* \subset D(A^*)$ and $\mathbf{Y}_{\alpha^-}^* \cap D(A^*)$ are both invariant under A^*.

We follow the proof of [19, Lemma 6.2] to construct two bi-orthogonal families that will be very useful to define the projection on $\mathbf{Y}_\alpha$ parallel to $\mathbf{Y}_{\alpha^-}$ and the projection on $\mathbf{Y}_\alpha^*$ parallel to $\mathbf{Y}_{\alpha^-}^*$.

Since the operator A has real coefficients, λ_j and $\overline{\lambda_j}$ either both are or both are not eigenvalues of A. Moreover $e_i^k(\lambda_j)$ is a corresponding eigenfunction associated with λ_j (or a generalized eigenfunction) if and only if $\overline{e_i^k(\lambda_j)}$ is an eigenfunction (or a generalized eigenfunction) associated with $\overline{\lambda_j}$. A similar assertion applies to eigenvalues and generalized functions of the operator A^*. Due to that, we can choose a canonical system $(\overline{e_i^k(\lambda_j)})_{i,k}$ associated to $\overline{\lambda_j}$ such that

$$\overline{e_i^k(\lambda_j)} = e_i^k(\overline{\lambda_j}).$$

As a consequence, if λ_j is real, the chosen canonical system associated to λ_j is real too. Similarly, we choose the eigenfunctions and generalized eigenfunctions of the operator A^* such that

$$\overline{\varepsilon_i^k(\lambda_j)} = \varepsilon_i^k(\overline{\lambda_j}).$$

Let us consider the sets

$$\mathcal{F}_1 = \{j \in \{1, \cdots, N_\alpha\} \mid \Im\lambda_j > 0\} \quad \text{and} \quad \mathcal{F}_2 = \{j \in \{1, \cdots, N_\alpha\} \mid \Im\lambda_j = 0\}.$$

Then, we set

$$\mathcal{B}_{1,1} = \left\{\sqrt{2}\,\Re e_i^k(\lambda_j), \quad \sqrt{2}\,\Im(e_i^k(\lambda_j)) \mid j \in \mathcal{F}_1,\ k = 1, \cdots, \ell(j),\ i = 1 \cdots m_k^j\right\}$$

and

$$\mathcal{B}_{1,2} = \left\{e_i^k(\lambda_j) \mid j \in \mathcal{F}_2,\ k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k^j\right\}.$$

We also set

$$\mathcal{B}_{2,1} = \left\{\sqrt{2}\,\Re\varepsilon^k_{m_k^j+1-i}(\lambda_j),\ \sqrt{2}\,\Im\varepsilon^k_{m_k^j+1-i}(\lambda_j) \mid j \in \mathcal{F}_1,\ k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k^j\right\}$$

and

$$\mathcal{B}_{2,2} = \left\{\varepsilon^k_{m_k^j+1-i}(\lambda_j) \mid j \in \mathcal{F}_2,\ k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k^j\right\}.$$

Let us notice that

$$\mathcal{B}_{2,1} = \left\{\sqrt{2}\,\Re\varepsilon^k_{m_k^j+1-i}(\overline{\lambda_j}),\ \sqrt{2}\,\Im\varepsilon^k_{m_k^j+1-i}(\overline{\lambda_j}) \mid j \in \mathcal{F}_1,\ k = 1, \cdots, \ell(j),\ i = 1, \cdots, m_k^j\right\}.$$

The families $\mathcal{B}_1 = \mathcal{B}_{1,1} \cup \mathcal{B}_{1,2}$ and $\mathcal{B}_2 = \mathcal{B}_{2,1} \cup \mathcal{B}_{2,2}$ are linearly independent. From Lemma 3.8 (see also [19, Lemma 6.2]), it follows that

$$\left(\sqrt{2}\,\Re e_{i_1}^{k_1}(\lambda_{j_1}), \sqrt{2}\,\Re\varepsilon_{i_2}^{k_2}(\overline{\lambda_{j_2}})\right) = \left(\sqrt{2}\,\Im e_{i_1}^{k_1}(\lambda_{j_1}), \sqrt{2}\,\Im\varepsilon_{i_2}^{k_2}(\overline{\lambda_{j_2}})\right) = \delta_{j_1}^{j_2}\,\delta_{k_1}^{k_2}\,\delta^{i_2}_{m_{k_1}^{j_1}+1-i_1},$$

$$\left(\Re e^{k_1}_{i_1}(\lambda_{j_1}), \Im \varepsilon^{k_2}_{i_2}(\overline{\lambda_{j_2}})\right) = 0 \quad \text{and} \quad \left(\Im e^{k_1}_{i_1}(\lambda_{j_1}), \Re \varepsilon^{k_2}_{i_2}(\overline{\lambda_{j_2}})\right) = 0,$$

for all $j_1, j_2 \in \mathcal{F}_1$, $k_1 \in \{1, \cdots, \ell(j_1)\}$, $k_2 \in \{1, \cdots, \ell(j_2)\}$, $i_1 \in \left\{1, \cdots, m^{j_1}_{k_1}\right\}$, $i_2 \in \left\{1, \cdots, m^{j_2}_{k_2}\right\}$,

$$\left(e^{k_1}_{i_1}(\lambda_{j_1}), \varepsilon^{k_2}_{i_2}(\overline{\lambda_{j_2}})\right) = \delta^{j_2}_{j_1}\, \delta^{k_2}_{k_1}\, \delta^{i_2}_{m^{j_1}_{k_1}+1-i_1},$$

for all $j_1, j_2 \in \mathcal{F}_2$, $k_1 \in \{1, \cdots, \ell(j_1)\}$, $k_2 \in \{1, \cdots, \ell(j_2)\}$, $i_1 \in \left\{1, \cdots, m^{j_1}_{k_1}\right\}$, $i_2 \in \left\{1, \cdots, m^{j_2}_{k_2}\right\}$. Moreover the vectors belonging to $\mathcal{B}_{1,1}$ are orthogonal to vectors belonging to $\mathcal{B}_{2,2}$ and the vectors belonging to $\mathcal{B}_{1,2}$ are orthogonal to vectors belonging to $\mathcal{B}_{2,1}$. Thus we can rewrite the families $\mathcal{B}_1$ and $\mathcal{B}_2$ in the form

$$\mathcal{B}_1 = \left\{\mathbf{e}_1, \cdots, \mathbf{e}_K\right\} \quad \text{and} \quad \mathcal{B}_2 = \left\{\varepsilon_1, \cdots, \varepsilon_K\right\} \tag{3.7}$$

so that

$$\left(\mathbf{e}_i, \varepsilon_j\right) = \delta^j_i, \tag{3.8}$$

where δ^j_i is a Kronecker symbol and $\left(\cdot, \cdot\right)$ is the inner product in the real space $\mathbf{V}^0_n(\Omega)$.

Let us consider the operator Q defined by

$$Q\mathbf{f} = \Re(P_\alpha \mathbf{f}), \qquad \text{for all } \mathbf{f} \in \mathbf{V}^0_n(\Omega).$$

From the definition of P_α and from (3.8), it follows that

$$Q\mathbf{f} = \sum_{j=1}^{K} \left(\mathbf{f}, \varepsilon_j\right) \mathbf{e}_j \qquad \text{for all } \mathbf{f} \in \mathbf{V}^0_n(\Omega).$$

Since $\mathbf{Y}_\alpha = \Re Z_\alpha$ and $\mathbf{Y}_{\alpha^-} = \Re Z_{\alpha^-}$, it is clear that Q is the projection onto $\mathbf{Y}_\alpha$ parallel to $\mathbf{Y}_{\alpha^-}$. Moreover, we have

$$\mathbf{Y}_\alpha = \mathrm{vect}\left\{\mathbf{e}_j \mid j = 1, \cdots, K\right\} \quad \text{and} \quad \mathbf{Y}^*_\alpha = \mathrm{vect}\left\{\varepsilon_j \mid j = 1, \cdots, K\right\}.$$

Proposition 3.11. *The adjoint operator $Q^* \in \mathcal{L}(\mathbf{V}^0_n(\Omega))$ is defined by*

$$Q^*\mathbf{f} = \sum_{j=1}^{K} \left(\mathbf{f}, \mathbf{e}_j\right) \varepsilon_j \quad \textit{for all} \quad \mathbf{f} \in \mathbf{V}^0_n(\Omega)$$

and it is the projection onto $\mathbf{Y}^*_\alpha$ parallel to $\mathbf{Y}^*_{\alpha^-}$.

Proof. It is a direct consequence of the definition of the operator Q. □

Proposition 3.12. *We can characterize the space $\mathbf{Y}_{\alpha^-}$ as follows*

$$\mathbf{Y}_{\alpha^-} = \left\{\mathbf{f} \in \mathbf{V}^0_n(\Omega) \mid \left(\mathbf{f}, \mathbf{g}\right) = 0 \quad \text{for all} \quad \mathbf{g} \in \mathbf{Y}^*_\alpha\right\}.$$

Proof. Let $\mathbf{f}$ belong to $\mathbf{Y}_{\alpha^-}$. We have $Q\mathbf{f} = 0$. From the definition of Q, since $\{\mathbf{e}_1, \cdots, \mathbf{e}_K\}$ is a basis of $\mathbf{Y}_\alpha$, we obtain $\left(\mathbf{f}, \varepsilon_j\right) = 0$, for all $j = 1, \cdots, K$, and therefore

$$\mathbf{Y}_{\alpha^-} \subset \left\{\mathbf{f} \in \mathbf{V}^0_n(\Omega) \mid \left(\mathbf{f}, \mathbf{g}\right) = 0 \quad \text{for all} \quad \mathbf{g} \in \mathbf{Y}^*_\alpha\right\}.$$

Let $\mathbf{f} \in \mathbf{V}_n^0(\Omega)$ be such that $\Big(\mathbf{f}, \mathbf{g}\Big) = 0$ for all $\mathbf{g} \in \mathbf{Y}_\alpha^*$. Obviously, we have $\Big(\mathbf{f}, \varepsilon_j\Big) = 0$, for all $j = 1, \cdots, K$. From the definition of the operator Q, we obtain $Q\mathbf{f} = 0$. Thus, we have $\mathbf{f} = (I - Q)\mathbf{f}$ and the proof is complete. □

Remark 3.13. Similarly, we can characterize the space $\mathbf{Y}_{\alpha^-}^*$ as follows

$$\mathbf{Y}_{\alpha^-}^* = \Big\{\mathbf{\Phi} \in \mathbf{V}_n^0(\Omega) \mid \Big(\mathbf{\Phi}, \mathbf{h}\Big) = 0 \quad \text{for all} \quad \mathbf{h} \in \mathbf{Y}_\alpha\Big\}.$$

Corollary 3.14. *We can identify the dual of $\mathbf{Y}_\alpha$ with $\mathbf{Y}_\alpha^*$, and the dual of $\mathbf{Y}_{\alpha^-}$ with $\mathbf{Y}_{\alpha^-}^*$.*

Proof. The space $\mathbf{V}_n^0(\Omega)$ is identified with its dual. The space $\mathbf{Y}_\alpha$ is a closed subspace of $\mathbf{V}_n^0(\Omega)$. Thus

$$(\mathbf{Y}_\alpha)' = \mathbf{V}_n^0(\Omega)/(\mathbf{Y}_\alpha)^\perp = \mathbf{V}_n^0(\Omega)/\mathbf{Y}_{\alpha^-}^*,$$

where $(\mathbf{Y}_\alpha)^\perp = \Big\{\mathbf{\Phi} \in \mathbf{V}_n^0(\Omega) \mid \Big(\mathbf{\Phi}, \mathbf{h}\Big) = 0 \quad \text{for all} \quad \mathbf{h} \in \mathbf{Y}_\alpha\Big\} = \mathbf{Y}_{\alpha^-}^*$. Since $\mathbf{V}_n^0(\Omega) = \mathbf{Y}_\alpha^* \oplus \mathbf{Y}_{\alpha^-}^*$, the subspace $\mathbf{V}_n^0(\Omega)/\mathbf{Y}_{\alpha^-}^*$ is isomorph to $\mathbf{Y}_\alpha^*$ and $(\mathbf{Y}_\alpha)'$ may be identified with $\mathbf{Y}_\alpha^*$. The proof is similar to show that the dual of $\mathbf{Y}_{\alpha^-}$ can be identified with $\mathbf{Y}_{\alpha^-}^*$. □

From their definitions, Q and Q^* are linear and continuous from $\mathbf{V}_n^0(\Omega)$ to respectively $\mathbf{Y}_\alpha$ and $\mathbf{Y}_\alpha^*$. Since $\mathbf{Y}_\alpha$ and $\mathbf{Y}_\alpha^*$ are spaces of finite dimension, we can extend the operator Q to $(D(A^*))'$ and Q^* to $(D(A))'$ in the following way.

Remark 3.15. The operators $Q \in \mathcal{L}((D(A^*))', \mathbf{Y}_\alpha)$ and $Q^* \in \mathcal{L}((D(A))', \mathbf{Y}_\alpha^*)$ may be defined by

$$Q\mathbf{f} = \sum_{j=1}^K \Big\langle \mathbf{f}, \varepsilon_j \Big\rangle_{(D(A^*))', D(A^*)} \mathbf{e}_j \quad \text{and} \quad Q^*\mathbf{f} = \sum_{j=1}^K \Big\langle \mathbf{f}, \mathbf{e}_j \Big\rangle_{(D(A))', D(A)} \varepsilon_j,$$

for all $\mathbf{f}$ in $(D(A^*))' = (D(A))'$.

We consider the subspace $\mathcal{U}$ of $L^2(\Gamma)$ defined by

$$\mathcal{U} = \text{vect}\Big\{MB^*\varepsilon_j \mid j = 1, \cdots, K\Big\}. \tag{3.9}$$

We denote by $\{\zeta_1, \cdots \zeta_{n_c}\}$ a basis of $\mathcal{U}$.

Remark 3.16. From Remark 3.15 and the definition of the space $\mathcal{U}$ we can deduce that MB^*Q^* belongs to $\mathcal{L}((D(A))', \mathcal{U})$.

For notational simplicity, we still denote by A_α the restriction of A to $\mathbf{Y}_\alpha$. Let A_{α^-} be the unbounded operator in $\mathbf{Y}_{\alpha^-}$ defined by

$$D(A_{\alpha^-}) = D(A) \cap \mathbf{Y}_{\alpha^-}, \quad A_{\alpha^-}\mathbf{y} = A\mathbf{y} \quad \text{for all} \quad \mathbf{y} \in D(A_{\alpha^-}).$$

It is easy to check that the adjoint of $(A_{\alpha^-}, D(A_{\alpha^-}))$ is the unbounded operator $(A_{\alpha^-}^*, D(A_{\alpha^-}^*))$ in $\mathbf{Y}_{\alpha^-}^*$ defined by

$$D(A_{\alpha^-}^*) = D(A^*) \cap \mathbf{Y}_{\alpha^-}^*, \quad A_{\alpha^-}^*\mathbf{y} = A^*\mathbf{y} \quad \text{for all} \quad \mathbf{y} \in D(A_{\alpha^-}^*).$$

The space $D(A)$ (respectively $D(A^*)$) is equipped with the norm $\mathbf{y} \mapsto |(\lambda_0 I - A)\mathbf{y}|_{\mathbf{V}_n^0(\Omega)}$ (respectively $\mathbf{y} \mapsto |(\lambda_0 I - A^*)\mathbf{y}|_{\mathbf{V}_n^0(\Omega)}$). Let us recall that $D(A) = D(A^*)$, and actually the two norms are equivalent. The space $D(A_{\alpha^-})$ is a closed subspace of $D(A)$ and it is dense in $\mathbf{Y}_{\alpha^-}$. Thus, we shall equip $D(A_{\alpha^-})$ with the norm of

$D(A)$. Similarly, $D(A^*_{\alpha^-})$ is closed in $D(A^*)$ and dense in $\mathbf{Y}^*_{\alpha^-}$, and $D(A^*_{\alpha^-})$ will be equipped with the norm of $D(A^*)$.

The operator $A^*_{\alpha^-}$ can be considered either as an unbounded operator in $\mathbf{Y}^*_{\alpha^-}$ (or in $\mathbf{V}^0_n(\Omega)$) with domain $D(A^*_{\alpha^-})$ or as an isomorphism from $D(A^*_{\alpha^-})$ to $\mathbf{Y}^*_{\alpha^-}$. Similarly, the operator A_{α^-} can be considered either as an unbounded operator in $\mathbf{Y}_{\alpha^-}$ (or in $\mathbf{V}^0_n(\Omega)$) with domain $D(A_{\alpha^-})$ or as an isomorphism from $D(A_{\alpha^-})$ to $\mathbf{Y}_{\alpha^-}$. Since $A^*_{\alpha^-} \in \mathrm{Isom}(D(A^*_{\alpha^-}), \mathbf{Y}^*_{\alpha^-})$, we have $(A^*_{\alpha^-})^* \in \mathrm{Isom}(\mathbf{Y}_{\alpha^-}, (D(A^*_{\alpha^-}))')$ (here, $\mathrm{Isom}(E, F)$ denotes the space of isomorphisms from E onto F). The operator $(A^*_{\alpha^-})^*$ can also be viewed as an unbounded operator in $(D(A^*_{\alpha^-}))'$ with domain $\mathbf{Y}_{\alpha^-}$. As in [13, Chapter 3, p. 160], it can be shown that this unbounded operator is the extension of A_{α^-} to $(D(A^*_{\alpha^-}))'$. For simplicity, it will be still denoted by A_{α^-}. Let us observe that we have the following decomposition

$$D(A) = \mathbf{Y}_\alpha \oplus D(A_{\alpha^-}) \quad \text{and} \quad D(A^*) = \mathbf{Y}^*_\alpha \oplus D(A^*_{\alpha^-}). \tag{3.10}$$

In Proposition 3.19, we are going to see that $(I - Q)$ belongs to $\mathcal{L}((D(A^*))', (D(A^*_{\alpha^-}))')$. For that, we need a precise characterization of $(D(A^*_{\alpha^-}))'$, which is given in Proposition 3.18.

Lemma 3.17. *The space $(D(A^*))'$ can be decomposed as follows*

$$(D(A^*))' = \mathbf{Y}_\alpha \oplus \overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}},$$

where $\overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}}$ denotes the closure of $\mathbf{Y}_{\alpha^-}$ in the norm $(D(A^*))'$.

Proof. Step 1. We first prove the identity $\overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}} = E_\alpha$, where

$$E_\alpha = \left\{ \mathbf{f} \in (D(A^*))' \mid \Big\langle \mathbf{f}, \mathbf{g} \Big\rangle_{(D(A^*))', D(A^*)} = 0 \quad \text{for all } \mathbf{g} \in \mathbf{Y}^*_\alpha \right\}.$$

Let $\mathbf{f}$ belong to $\overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}}$. There exists $(\mathbf{f}_n)_{n\in\mathbb{N}}$, such that $\mathbf{f}_n$ belongs to $\mathbf{Y}_{\alpha^-}$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \mathbf{f}_n = \mathbf{f}$ in $(D(A^*))'$. From Proposition 3.12, for all $n \in \mathbb{N}$, we have

$$\Big\langle \mathbf{f}_n, \mathbf{g} \Big\rangle_{(D(A^*))', D(A^*)} = \Big(\mathbf{f}_n, \mathbf{g} \Big) = 0 \quad \text{for all} \quad \mathbf{g} \in \mathbf{Y}^*_\alpha.$$

We show that $\overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}} \subset E_\alpha$ by passing to the limit in the previous identity when n tends to infinity. Let $\mathbf{f}$ belong to E_α. There exists $(\mathbf{f}_n)_{n\in\mathbb{N}}$, such that $\mathbf{f}_n$ belongs to $\mathbf{V}^0_n(\Omega)$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \mathbf{f}_n = \mathbf{f}$ in $(D(A^*))'$. For all $n \in \mathbb{N}$, we have $(I - Q)\mathbf{f}_n \in \mathbf{Y}_{\alpha^-}$. Moreover, from the definition of E_α and (3.8), we can check that $Q\mathbf{f} = 0$. Thus, we have $(I - Q)\mathbf{f}_n - \mathbf{f} = (I - Q)(\mathbf{f}_n - \mathbf{f})$. It follows that $(I - Q)\mathbf{f}_n$ tends to $\mathbf{f}$ in $(D(A^*))'$ when n tends to infinity and the equality $\overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}} = E_\alpha$ is proved.

Step 2. We show that $(D(A^*))' = \mathbf{Y}_\alpha \oplus E_\alpha$. From Remark 3.15, for all $\mathbf{f} \in (D(A^*))'$, we have $Q\mathbf{f} \in \mathbf{Y}_\alpha$. Since $(I - Q)\mathbf{f} \in E_\alpha$ for all $\mathbf{f} \in (D(A^*))'$, the proof is complete. □

Since the space $D(A_{\alpha^-})$ is continuously and densely imbedded in $\mathbf{Y}_{\alpha^-}$ and $D(A^*_{\alpha^-})$ is continuously and densely imbedded in $\mathbf{Y}^*_{\alpha^-}$, by duality we have

$$\mathbf{Y}^*_{\alpha^-} \hookrightarrow (D(A_{\alpha^-}))' \quad \text{and} \quad \mathbf{Y}_{\alpha^-} \hookrightarrow (D(A^*_{\alpha^-}))',$$

with dense and continuous imbeddings if $(D(A_{\alpha^-}))'$ is equipped with the dual norm of $D(A_{\alpha^-})$ and if $(D(A^*_{\alpha^-}))'$ is equipped with the dual norm of $D(A^*_{\alpha^-})$. Since the two norms $|\cdot|_{D(A)}$ and $|\cdot|_{D(A^*)}$ are equivalent, we have the following proposition.

Proposition 3.18. *We have the identity* $(D(A^*_{\alpha^-}))' = \overline{\mathbf{Y}_{\alpha^-}}^{|\cdot|_{(D(A^*))'}}$.

Proposition 3.19. *The operator* $(I-Q)$ *is linear and continuous from* $(D(A^*))'$ *to* $(D(A^*_{\alpha^-}))'$.

Proof. Due to Proposition 3.18 and Lemma 3.17, we have $(I-Q)\mathbf{f} \in (D(A^*_{\alpha^-}))'$ for all $\mathbf{f}$ in $(D(A^*))'$. We notice that for all $\mathbf{\Phi} \in D(A^*_{\alpha^-})$, we have $\mathbf{\Phi} = (I-Q^*)\mathbf{\Phi}$. Then, we have

$$\Big\langle (I-Q)\mathbf{f}, \mathbf{\Phi} \Big\rangle_{(D(A^*_{\alpha^-}))', D(A^*_{\alpha^-})} = \Big\langle \mathbf{f}, \mathbf{\Phi} \Big\rangle_{(D(A^*))', D(A^*)}$$

for all $\mathbf{\Phi} \in D(A^*_{\alpha^-})$, and the proof is complete. ☐

Let us set

$$\mathbf{y}_\alpha = Q\mathbf{y} \quad \text{and} \quad \mathbf{y}_{\alpha^-} = (I-Q)\mathbf{y}.$$

The linearized equation

$$\mathbf{y}' = A\mathbf{y} + BM\mathbf{u} \quad \text{in } (0,\infty), \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{3.11}$$

may be split as follows

$$\begin{aligned} &\mathbf{y}_\alpha' = A_\alpha \mathbf{y}_\alpha + QBM\mathbf{u} \quad \text{in } (0,\infty), \quad \mathbf{y}_\alpha(0) = Q\mathbf{y}_0, \\ &\mathbf{y}_{\alpha^-}' = A_{\alpha^-}\mathbf{y}_{\alpha^-} + (I-Q)BM\mathbf{u} \quad \text{in } (0,\infty), \quad \mathbf{y}_{\alpha^-}(0) = (I-Q)\mathbf{y}_0. \end{aligned} \tag{3.12}$$

4. **Stabilizability of the real projected system by finite dimensional controls.** In this section, we study the controllability of the projected system

$$\begin{aligned} &{\mathbf{y}_\alpha}' = A_\alpha \mathbf{y}_\alpha + QBM\mathbf{u} = A_\alpha \mathbf{y}_\alpha + \sum_{i=1}^{n_c} v_i(t) QBM\zeta_i \quad \text{in } (0,\infty), \\ &\mathbf{y}_\alpha(0) = Q\mathbf{y}_0, \end{aligned} \tag{4.1}$$

where $\{\zeta_1, \cdots, \zeta_{n_c}\}$ is the basis of $\mathcal{U}$ defined in (3.9). We can rewrite this system as a differential system in $\mathbb{R}^K$ by introducing the coordinates of $\mathbf{y}_\alpha$ in the basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_K\}$

$$\mathbf{y}_\alpha = \sum_{j=1}^K y_j\, \mathbf{e}_j.$$

From Remark 3.15, it follows that

$$QBM\zeta_i = \sum_{j=1}^K \Big\langle BM\zeta_i, \varepsilon_j \Big\rangle_{(D(A^*))', D(A^*)} \mathbf{e}_j = \sum_{j=1}^K \Big(\zeta_i, MB^*\varepsilon_j \Big)_{\mathbf{V}^0(\Gamma)} \mathbf{e}_j.$$

Thus, by setting $Y = \mathrm{col}[y_1, \cdots, y_K]$ and $V = \mathrm{col}[v_1, \cdots, v_{n_c}]$, equation (4.1) can be written as follows

$$Y' = \widehat{A}Y + \widehat{B}V \quad \text{in } (0,\infty), \tag{4.2}$$

where $\widehat{A}$ is of the form

$$\widehat{A} = \begin{pmatrix} \Lambda_1 & & & \\ & \Lambda_2 & 0 & \\ & 0 & \ddots & \\ & & & \Lambda_{N_\alpha} \end{pmatrix}$$

and

$$\widehat{B} = \begin{pmatrix} \Big(\zeta_1, MB^*\varepsilon_1\Big)_{\mathbf{V}^0(\Gamma)} & \cdots & \Big(\zeta_{n_c}, MB^*\varepsilon_1\Big)_{\mathbf{V}^0(\Gamma)} \\ \vdots & & \vdots \\ \Big(\zeta_1, MB^*\varepsilon_K\Big)_{\mathbf{V}^0(\Gamma)} & \cdots & \Big(\zeta_{n_c}, MB^*\varepsilon_K\Big)_{\mathbf{V}^0(\Gamma)} \end{pmatrix}.$$

In Proposition 4.2 (see below), we prove that system (4.1) (and therefore system (4.2) too) is stabilizable. If we assume that the family

$$\{MB^*\varepsilon_j \mid j = 1, \cdots, K\} \tag{4.3}$$

is linearly independent, then $\text{rank}(\widehat{B}) = K$ and the Kalman controllability condition is satisfied, but this condition is not necessary for the controllability of system (4.2) (see Remark 4.1 below).

Remark 4.1. As noticed above, the linearly independence of the family (4.3) is not a necessary condition to the controllability. To illustrate this, let us consider the case where the family (4.3) is linearly dependent. To simplify, let us choose $K = 2$, $n_c = 1$ and

$$\widehat{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

The matrix $\widehat{B} \in \mathbb{R}^{2\times 1}$ is given by

$$\widehat{B} = \begin{pmatrix} \Big(\zeta_1, MB^*\varepsilon_1\Big)_{\mathbf{V}^0(\Gamma)} \\ \Big(\zeta_1, MB^*\varepsilon_2\Big)_{\mathbf{V}^0(\Gamma)} \end{pmatrix}.$$

We can choose $\zeta_1 = MB^*\varepsilon_2$. We have $(\zeta_1, MB^*\varepsilon_2)_{\mathbf{V}^0(\Gamma)} \neq 0$ and the Kalman controllability criterion is satisfied.

Let us now give a direct proof of the stabilizability of system (4.1) based on the fact that the system (3.11) is stabilizable by a control of infinite dimension.

Proposition 4.2. *System* (4.1) *is exactly controllable.*

Proof. Since system (4.1) is of finite dimension, it is sufficient to prove that it is completely stabilizable. Thus we have to prove that, for all $\rho > 0$, there exists a constant $C > 0$ and a control $V = (v_1, \cdots, v_{n_c})^T \in L^2(0, \infty; \mathbb{R}^{n_c})$ such that the solution $\mathbf{y}_\alpha$ to (4.1) obeys

$$|\mathbf{y}_\alpha(t)|_{\mathbf{V}^0_n(\Omega)} \leq Ce^{-\rho t}|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}.$$

Step 1. It has been proved in [29], that system (3.11) is completely stabilizable. Thus for a given $\rho > 0$, there exists a control $\tilde{\mathbf{u}}$ such that the solution $\mathbf{y}_{\tilde{\mathbf{u}}}$ of (3.11) obeys $|\mathbf{y}_{\tilde{\mathbf{u}}}(t)|_{\mathbf{V}^0_n(\Omega)} \leq Ce^{-\rho t}|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}$. Since Q is a continuous operator from $\mathbf{V}^0_n(\Omega)$ to $\mathbf{Y}_\alpha$, we have

$$|Q\mathbf{y}_{\tilde{\mathbf{u}}}(t)|_{\mathbf{V}^0_n(\Omega)} \leq Ce^{-\rho t}|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}. \tag{4.4}$$

Since Q and A_α commute, we have

$$Q\mathbf{y}_{\tilde{\mathbf{u}}}(t) = e^{A_\alpha t}Q\mathbf{y}_0 + \int_0^t e^{A_\alpha(t-\tau)}QBM\tilde{\mathbf{u}}(\tau)d\tau.$$

Moreover, the solution to the real projected system (4.1) obeys

$$\mathbf{y}_\alpha(t) = e^{A_\alpha t} Q\mathbf{y}_0 + \int_0^t e^{A_\alpha(t-\tau)} QBM\mathbf{u}(\tau)d\tau.$$

We look for $\mathbf{u} \in L^2(0,\infty;\mathcal{U})$ such that $\mathbf{y}_\alpha = Q\mathbf{y}_{\tilde{\mathbf{u}}}$. Both solutions are equal when

$$QBM\mathbf{u}(\tau) = QBM\tilde{\mathbf{u}}(\tau), \qquad \text{for all} \qquad \tau \in (0,\infty). \tag{4.5}$$

Step 2. From the definition of the operator Q, condition (4.5) is satisfied if and only if

$$\Big(\mathbf{u}(\tau) - \tilde{\mathbf{u}}(\tau), MB^*\varepsilon_j\Big)_{\mathbf{V}^0(\Gamma)} = \Big\langle BM\mathbf{u}(\tau) - BM\tilde{\mathbf{u}}(\tau), \varepsilon_j\Big\rangle_{(D(A^*))', D(A^*)} = 0,$$

for all $j = 1, \cdots, K$ and all $\tau \in (0,\infty)$. We choose $\mathbf{u}(\tau)$ equal to the orthogonal projection in $\mathbf{V}^0(\Gamma)$ of $\tilde{\mathbf{u}}(\tau)$ onto $\mathcal{U}$. Thus, condition (4.5) is satisfied and both solutions $Q\mathbf{y}_{\tilde{\mathbf{u}}}$ and $\mathbf{y}_\alpha$ to (4.1) are equal. Finally, with (4.4), we obtain the desired estimate. □

We denote by $\widetilde{M} \in \mathcal{L}(\mathbb{R}^{n_c}, \mathbf{V}^0(\Gamma))$ the operator defined by

$$(\widetilde{M}\mathbf{v})(x) = \sum_{i=1}^{n_c} v_i M\zeta_i(x) \quad \text{for all} \quad \mathbf{v} = (v_1, \cdots, v_{nc}) \in \mathbb{R}^{n_c}. \tag{4.6}$$

Proposition 4.3. *The adjoint operator* $\widetilde{M}^*$ *belongs to* $\mathcal{L}(\mathbf{V}^0(\Gamma), \mathbb{R}^{n_c})$ *and for all* $\mathbf{u} \in \mathbf{V}^0(\Gamma)$, *we have*

$$(\widetilde{M}^*\mathbf{u})_i = \Big(\zeta_i, M\mathbf{u}\Big)_{\mathbf{V}^0(\Gamma)}, \quad \textit{for all } i = 1, \cdots, n_c.$$

Proof. This is a direct consequence of the definition of $\widetilde{M}$. □

5. Feedback control of the real projected system.

The aim of this section is to study the finite dimensional control problem $(\mathcal{P}^\infty_{\mathbf{y}_0})$ stated in subsection 5.2. More precisely we want to characterize its optimal solution via a feedback law defined thanks to a finite dimensional Riccati equation (see (5.9) and the corresponding matrix equation (5.15)). Equation (5.9) is stated over a finite dimensional space since $(\mathcal{P}^\infty_{\mathbf{y}_0})$ is a finite dimensional control problem. To achieve this goal we could obviously use results from the existing literature [36, 27]. In (5.9) we look for a solution Π belonging to $\mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}_\alpha^*)$ because we have not identified the dual of $\mathbf{Y}_\alpha$ with itself. The approach in [36, 27] consists in looking for an operator Π defined in a space which is identified with its dual. Here we follow the lines of [29] where, by studying a family of finite time horizon control problems $(\mathcal{P}^k_{\mathbf{y}_0})$, we clearly understand why Π belongs to $\mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}_\alpha^*)$, and why we recover results which are very similar to that in [36, 27].

In what follows we only state the results since the proof can easily be adapted from [29].

5.1. A finite time horizon control problem.

For all $\mathbf{y}_0$ in $\mathbf{Y}_\alpha$, we consider the following optimal control problem

$$(\mathcal{P}^k_{\mathbf{y}_0}) \qquad \inf\Big\{ I_k(\mathbf{y},\mathbf{v}) \;\mid\; (\mathbf{y},\mathbf{v}) \text{ satisfies (5.1)}, \mathbf{v} \in L^2(0,k;\mathbb{R}^{n_c})\Big\},$$

where

$$I_k(\mathbf{y},\mathbf{v}) = \frac{1}{2}\int_0^k \int_\Omega |Q\mathbf{y}|^2\, dxdt + \frac{1}{2}\int_0^k |\mathbf{v}(t)|^2\, dt,$$

and

$$\mathbf{y}' = A_\alpha \mathbf{y} + QB\widetilde{M}\mathbf{v} \quad \text{in } (0,k), \qquad \mathbf{y}(0) = \mathbf{y}_0. \tag{5.1}$$

The space $\mathbf{Y}_\alpha$ is equipped with the norm $|\cdot|_{\mathbf{V}^0_n(\Omega)}$. Thus, we have $|\mathbf{y}|^2_{\mathbf{Y}_\alpha} = \int_\Omega |\mathbf{y}|^2 dx = \int_\Omega |Q\mathbf{y}|^2 dx$ for all $\mathbf{y} \in \mathbf{Y}_\alpha$.

Remark 5.1. To simplify the notations, we shall denote by $\mathbf{y}$ the solution to equation (5.1), but we have to keep in mind that it represents $\mathbf{y}_\alpha = Q\mathbf{y}$. Then, of course we have $Q\mathbf{y} = \mathbf{y}$ and the cost functional can also be written

$$I_k(\mathbf{y},\mathbf{v}) = \frac{1}{2}\int_0^k \int_\Omega |\mathbf{y}|^2\, dxdt + \frac{1}{2}\int_0^k |\mathbf{v}(t)|^2\, dt.$$

Proposition 5.2. *Problem $(\mathcal{P}^k_{\mathbf{y}_0})$ admits a unique solution $(\mathbf{y},\mathbf{v})$ where*

$$\mathbf{v}(t) = -\widetilde{M}^* B^* Q^* \mathbf{\Phi}(t), \tag{5.2}$$

and $\mathbf{\Phi}$ is solution to the equation

$$-\mathbf{\Phi}' = A^*_\alpha \mathbf{\Phi} + Q^* Q\mathbf{y} \quad \text{in } (0,k), \quad \mathbf{\Phi}(k) = 0. \tag{5.3}$$

Conversely the system

$$\begin{aligned} &\mathbf{y}' = A_\alpha \mathbf{y} - QB\widetilde{M}\widetilde{M}^* B^* Q^* \mathbf{\Phi} \quad \text{in } (0,k), \quad \mathbf{y}(0) = \mathbf{y}_0, \\ &-\mathbf{\Phi}' = A^*_\alpha \mathbf{\Phi} + Q^* Q\mathbf{y} \quad \text{in } (0,k), \quad \mathbf{\Phi}(k) = 0, \end{aligned} \tag{5.4}$$

admits a unique solution $(\mathbf{y},\mathbf{\Phi}) \in C^1([0,k];\mathbf{Y}_\alpha) \times C^1([0,k];\mathbf{Y}^*_\alpha)$ and $(\mathbf{y}, -\widetilde{M}^* B^* Q^* \mathbf{\Phi})$ is the solution to problem $(\mathcal{P}^k_{\mathbf{y}_0})$.

Proof. The proof follows the lines of [29, Theorem 3.1]. □

As in [29, Corollary 3.8], with this proposition we obtain the following corollary.

Corollary 5.3. *The value of the infimum of $(\mathcal{P}^k_{\mathbf{y}_0})$ is given by*

$$\inf(\mathcal{P}^k_{\mathbf{y}_0}) = \frac{1}{2}\Big\langle \mathbf{y}_0, \mathbf{\Phi}(0)\Big\rangle_{\mathbf{Y}_\alpha,\mathbf{Y}^*_\alpha},$$

where $(\mathbf{y},\mathbf{\Phi})$ is solution to system (5.4).

We define the operator $\Pi(k) \in \mathcal{L}(\mathbf{Y}_\alpha,\mathbf{Y}^*_\alpha)$ by

$$\Pi(k)\mathbf{y}_0 = \mathbf{\Phi}(0),$$

where $(\mathbf{y},\mathbf{\Phi})$ is solution to system (5.4).

Theorem 5.4. *The solution $(\mathbf{y},\mathbf{v})$ to problem $(\mathcal{P}^k_{\mathbf{y}_0})$ belongs to $C^1([0,k];\mathbf{Y}_\alpha) \times C([0,k];\mathbb{R}^{n_c})$ and it obeys the feedback formula*

$$\mathbf{v}(t) = -\widetilde{M}^* B^* Q^* \Pi(t)\mathbf{y}(t).$$

Moreover, the optimal cost is given by

$$J(\mathbf{y},\mathbf{v}) = \frac{1}{2}\Big\langle \mathbf{y}_0, \Pi(k)\mathbf{y}_0\Big\rangle_{\mathbf{Y}_\alpha,\mathbf{Y}^*_\alpha}.$$

5.2. **An infinite time horizon control problem.** For all $\mathbf{y}_0$ in $\mathbf{Y}_\alpha$, we now consider the infinite time horizon control problem

$$(\mathcal{P}^\infty_{\mathbf{y}_0}) \qquad \inf\Big\{I(\mathbf{y},\mathbf{v}) \ \mid \ (\mathbf{y},\mathbf{v}) \text{ satisfies } (5.5), \mathbf{v}\in L^2(0,\infty;\mathbb{R}^{n_c})\Big\},$$

where

$$I(\mathbf{y},\mathbf{v}) = \frac{1}{2}\int_0^\infty\int_\Omega |Q\mathbf{y}|^2\,dxdt + \frac{1}{2}\int_0^\infty |\mathbf{v}(t)|^2_{\mathbb{R}^{n_c}}\,dt,$$

and

$$\mathbf{y}' = A_\alpha\mathbf{y} + QB\widetilde{M}\mathbf{v} \quad \text{in } (0,\infty), \qquad \mathbf{y}(0)=\mathbf{y}_0. \tag{5.5}$$

Theorem 5.5. *For all $\mathbf{y}_0$ in $\mathbf{Y}_\alpha$, problem $(\mathcal{P}^\infty_{\mathbf{y}_0})$ admits a unique solution $(\mathbf{y}_{\mathbf{y}_0},\mathbf{v}_{\mathbf{y}_0})$. Moreover, denoting by $(\mathbf{y}^k_{\mathbf{y}_0},\mathbf{v}^k_{\mathbf{y}_0})$ the solution to the finite time horizon control problem $(\mathcal{P}^k_{\mathbf{y}_0})$, and by $(\tilde{\mathbf{y}}^k_{\mathbf{y}_0},\tilde{\mathbf{v}}^k_{\mathbf{y}_0})$ their extensions by zero to the interval (k,∞) we have*

$$(\tilde{\mathbf{y}}^k_{\mathbf{y}_0},\tilde{\mathbf{v}}^k_{\mathbf{y}_0}) \to (\mathbf{y}_{\mathbf{y}_0},\mathbf{v}_{\mathbf{y}_0}), \qquad \textit{in } \ L^2(0,\infty;\mathbf{Y}_\alpha)\times L^2(0,\infty;\mathbb{R}^{n_c}).$$

Proof. We have proved in section 4 that there exists a control $\mathbf{v}$ such that the projected system (5.5) is stabilizable by finite dimensional controllers. This implies that $(\mathbf{y}_\mathbf{v},\mathbf{v})$ obeys

$$I(\mathbf{y}_\mathbf{v},\mathbf{v})<\infty,$$

where $\mathbf{y}_\mathbf{v}$ is the solution of equation (5.5). The existence of a unique solution $(\mathbf{y}_{\mathbf{y}_0},\mathbf{v}_{\mathbf{y}_0})$ to $(\mathcal{P}^\infty_{\mathbf{y}_0})$ follows from classical arguments. The convergence of $(\tilde{\mathbf{y}}^k_{\mathbf{y}_0},\tilde{\mathbf{v}}^k_{\mathbf{y}_0})$ towards $(\mathbf{y}_{\mathbf{y}_0},\mathbf{v}_{\mathbf{y}_0})$ follows the proof of [29, Theorem 4.1]. □

Theorem 5.6. *There exists $\Pi\in\mathcal{L}(\mathbf{Y}_\alpha,\mathbf{Y}^*_\alpha)$ satisfying $\Pi=\Pi^*\geq 0$ and*

$$\inf(\mathcal{P}^\infty_{\mathbf{y}_0}) = \frac{1}{2}\Big\langle \mathbf{y}_0,\Pi\mathbf{y}_0\Big\rangle_{\mathbf{Y}_\alpha,\mathbf{Y}^*_\alpha}.$$

Proof. See [29, Theorem 4.1]. The operator Π is obtained as the limit of $\Pi(k)$ when k tends to infinity. □

Theorem 5.7. *For every $\mathbf{y}_0$ in $\mathbf{Y}_\alpha$, the system*

$$\begin{aligned}
&\mathbf{y}' = A_\alpha\mathbf{y} - QB\widetilde{M}\widetilde{M}^*B^*Q^*\mathbf{\Phi} \quad \textit{in } (0,\infty), \quad \mathbf{y}(0)=\mathbf{y}_0\\
&-\mathbf{\Phi}' = A^*_\alpha\mathbf{\Phi} + Q^*Q\mathbf{y} \quad \textit{in } (0,\infty), \quad \mathbf{\Phi}(\infty)=0\\
&\mathbf{\Phi}(t) = \Pi\mathbf{y}(t) \quad \textit{for all } t\in(0,\infty),
\end{aligned} \tag{5.6}$$

admits a unique solution $(\mathbf{y},\mathbf{\Phi})$ in $H^1(0,\infty;\mathbf{Y}_\alpha)\times H^1(0,\infty;\mathbf{Y}^*_\alpha)$. This solution satisfies

$$\|\mathbf{y}\|_{H^1(0,\infty;\mathbf{Y}_\alpha)} + \|\mathbf{\Phi}\|_{H^1(0,\infty;\mathbf{Y}^*_\alpha)} \leq C|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}.$$

The pair $(\mathbf{y},-\widetilde{M}^*B^*Q^*\mathbf{\Phi})$ is the solution of problem $(\mathcal{P}^\infty_{\mathbf{y}_0})$.

Proof. The existence of a solution follows the lines of [29].

Step 1. We prove the uniqueness of this solution. We denote by $(\mathbf{y},\mathbf{v})$ the solution to problem $(\mathcal{P}^\infty_{\mathbf{y}_0})$. Adapting the proof of [29, Lemma 4.2], we can check that

$$\mathbf{v} = -\widetilde{M}^*B^*Q^*\mathbf{\Phi},$$

where $(\mathbf{y},\mathbf{\Phi})$ is solution to (5.7). Thus, with Theorem 5.6, we have

$$\int_0^\infty |\mathbf{y}(t)|^2_{\mathbf{Y}_\alpha}dt + \int_0^\infty |\widetilde{M}^*B^*Q^*\mathbf{\Phi}(t)|^2_{\mathbb{R}^{n_c}} = \frac{1}{2}\Big\langle \mathbf{y}_0,\Pi\mathbf{y}_0\Big\rangle_{\mathbf{Y}_\alpha,\mathbf{Y}^*_\alpha}. \tag{5.7}$$

It follows that if $\mathbf{y}_0 = 0$, then $\mathbf{y} = 0$ and since $\mathbf{\Phi}(t) = \Pi\mathbf{y}(t)$, the uniqueness is proved.

Step 2. We prove the final estimate. Let us denote by $(\mathbf{y}, \mathbf{\Phi})$ the solution of system (5.6). With (5.7), we have

$$\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{Y}_\alpha)} \le C|\mathbf{y}_0|_{\mathbf{Y}_\alpha}.$$

Moreover, since $\mathbf{\Phi} = \Pi\mathbf{y}$ and $\Pi \in \mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}_\alpha^*)$, it follows that

$$\|\mathbf{\Phi}\|_{L^2(0,\infty;\mathbf{Y}_\alpha^*)} \le C|\mathbf{y}_0|_{\mathbf{Y}_\alpha}.$$

From the equation satisfied by $\mathbf{\Phi}$, we deduce that $\mathbf{\Phi}$ belongs to $H^1(0,\infty;\mathbf{Y}_\alpha^*)$, and

$$\|\mathbf{\Phi}\|_{H^1(0,\infty;\mathbf{Y}_\alpha^*)} \le C(\|\mathbf{\Phi}\|_{L^2(0,\infty;\mathbf{Y}_\alpha^*)} + \|\mathbf{y}\|_{L^2(0,\infty;\mathbf{Y}_\alpha)}).$$

Since $\mathbf{\Phi} \in H^1(0,\infty;\mathbf{Y}_\alpha^*)$, we can verify that $\mathbf{v} = -\widetilde{M}^*B^*Q^*\mathbf{\Phi}$ belongs to $H^1(0,\infty;\mathbb{R}^{n_c})$ and

$$\|\mathbf{v}\|_{H^1(0,\infty;\mathbb{R}^{n_c})} \le C\|\mathbf{\Phi}\|_{H^1(0,\infty;\mathbf{Y}_\alpha^*)}. \tag{5.8}$$

Then, obviously we have $B\widetilde{M}\mathbf{v} \in H^1(0,\infty;(D(A^*))')$ and

$$\|B\widetilde{M}\mathbf{v}\|_{H^1(0,\infty;(D(A^*))')} \le C\|\mathbf{\Phi}\|_{H^1(0,\infty;\mathbf{Y}_\alpha^*)}.$$

Finally, since Q and is bounded and linear from $(D(A^*))'$ to $\mathbf{Y}_\alpha$ we obtain

$$\|QB\widetilde{M}\mathbf{v}\|_{H^1(0,\infty;\mathbf{Y}_\alpha)} \le C\|\mathbf{\Phi}\|_{H^1(0,\infty;\mathbf{Y}_\alpha^*)}.$$

Using the equation satisfied by $\mathbf{y}$, we deduce that $\mathbf{y}$ belongs to $H^1(0,\infty;\mathbf{Y}_\alpha)$, and

$$\|\mathbf{y}\|_{H^1(0,\infty;\mathbf{Y}_\alpha)} \le C(\|\mathbf{\Phi}\|_{L^2(0,\infty;\mathbf{Y}_\alpha^*)} + \|\mathbf{y}\|_{L^2(0,\infty;\mathbf{Y}_\alpha)}).$$

With all these estimates, we obtain

$$\|\mathbf{y}\|_{H^1(0,\infty;\mathbf{Y}_\alpha)} + \|\mathbf{\Phi}\|_{H^1(0,\infty;\mathbf{Y}_\alpha^*)} \le C(\|\mathbf{\Phi}\|_{L^2(0,\infty;\mathbf{Y}_\alpha^*)} + \|\mathbf{y}\|_{L^2(0,\infty;\mathbf{Y}_\alpha)}) \le C|\mathbf{y}_0|_{\mathbf{Y}_\alpha},$$

and the proof is complete. □

Remark 5.8. From estimate (5.8) and Theorem 5.7, we deduce that

$$\|\mathbf{v}\|_{H^1(0,\infty;\mathbb{R}^{n_c})} \le C|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)}$$

for every $\mathbf{y}_0$ in $\mathbf{Y}_\alpha$, where $\mathbf{v} = -\widetilde{M}^*B^*Q^*\mathbf{\Phi}$ and $\mathbf{\Phi}$ solution to the system (5.6). By iterating the argument used in the previous proof, we can prove that $(\mathbf{y}, \mathbf{\Phi})$ actually belongs to $H^r(0,\infty;\mathbf{Y}_\alpha) \times H^r(0,\infty;\mathbf{Y}_\alpha^*)$ for all $r > 0$.

Let us consider the algebraic Riccati equation

$$\begin{aligned} &\Pi \in \mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}_\alpha^*), \quad \Pi = \Pi^*, \quad \Pi \ge 0, \\ &\Pi A_\alpha + A_\alpha^*\Pi - \Pi QB\widetilde{M}\widetilde{M}^*B^*Q^*\Pi + Q^*Q = 0. \end{aligned} \tag{5.9}$$

Let us make some comments. We shall say that $\Pi = \Pi^* \ge 0$ when

$$\Big\langle \Pi\mathbf{y}, \mathbf{z}\Big\rangle_{\mathbf{Y}_\alpha^*,\mathbf{Y}_\alpha} = \Big\langle \mathbf{y}, \Pi\mathbf{z}\Big\rangle_{\mathbf{Y}_\alpha,\mathbf{Y}_\alpha^*} \quad \text{and} \quad \Big\langle \Pi\mathbf{y}, \mathbf{y}\Big\rangle_{\mathbf{Y}_\alpha^*,\mathbf{Y}_\alpha} \ge 0,$$

for all $\mathbf{y}$, $\mathbf{z} \in \mathbf{Y}_\alpha$. We shall say that an operator $\Pi \in \mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}_\alpha^*)$ obeys the second equation in (5.9) when

$$\Big\langle \Pi A_\alpha\mathbf{y}, \mathbf{z}\Big\rangle_{\mathbf{Y}_\alpha^*,\mathbf{Y}_\alpha} + \Big\langle A_\alpha^*\Pi\mathbf{y}, \mathbf{z}\Big\rangle_{\mathbf{Y}_\alpha^*,\mathbf{Y}_\alpha} - \big(\widetilde{M}^*B^*\Pi\mathbf{y}, \widetilde{M}^*B^*\Pi\mathbf{z}\big)_{\mathbb{R}^{n_c}} + \Big\langle Q^*Q\mathbf{y}, \mathbf{z}\Big\rangle_{\mathbf{Y}_\alpha^*,\mathbf{Y}_\alpha} = 0,$$

for all $\mathbf{y}$, $\mathbf{z} \in \mathbf{Y}_\alpha$. (Let us notice that $B^*\Pi\mathbf{y} = B^*Q^*\Pi\mathbf{y}$ and $B^*\Pi\mathbf{z} = B^*Q^*\Pi\mathbf{z}$.)

To prove that the operator $\Pi \in \mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}_\alpha^*)$, determined in Theorem 5.6, is the unique solution to equation (5.9), it is sufficient to adapt the classical proofs to the case where $\mathbf{Y}_\alpha$ is not identified with its dual (see e.g. [36, 27]).

From Theorem 5.7, it follows that, for all $\mathbf{y}_0 \in \mathbf{Y}_\alpha$, the evolution equation

$$\mathbf{y}' = A_\alpha \mathbf{y} - QB\widetilde{M}\widetilde{M}^* B^* Q^* \Pi \mathbf{y} \quad \text{in } (0,\infty), \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{5.10}$$

admits at least one weak solution belonging to $H^1(0,\infty;\mathbf{Y}_\alpha)$. Moreover, we can check that this solution is unique. Due to Theorem 5.7, this solution is equal to $\mathbf{y}_{\mathbf{y}_0}$, where $(\mathbf{y}_{\mathbf{y}_0}, \mathbf{v}_{\mathbf{y}_0})$ is the solution of problem $(\mathcal{P}^\infty_{\mathbf{y}_0})$. Then, with Theorem 5.6 we have

$$\int_0^\infty |\mathbf{y}_{\mathbf{y}_0}(t)|^2_{\mathbf{Y}_\alpha} \le C|\mathbf{y}_0|^2 < \infty. \tag{5.11}$$

Let us define the operator $A_\Pi \in \mathcal{L}(\mathbf{Y}_\alpha)$ by

$$A_\Pi \mathbf{y} = A_\alpha \mathbf{y} - QB\widetilde{M}\widetilde{M}^* B^* Q^* \Pi \mathbf{y} \quad \text{for all} \quad \mathbf{y} \in \mathbf{Y}_\alpha.$$

Remark 5.9. The semigroup $(e^{tA_\Pi})_{t\ge 0}$ satisfies

$$|e^{tA_\Pi}\mathbf{f}|_{\mathbf{V}^0_n(\Omega)} \le Ce^{-\beta t}|\mathbf{f}|_{\mathbf{V}^0_n(\Omega)}, \quad \text{for all} \quad \mathbf{f} \in \mathbf{Y}_\alpha, \tag{5.12}$$

for some $\beta > 0$.

Proof. The operator A_α belongs to $\mathcal{L}(\mathbf{Y}_\alpha)$. From Remark 3.16, $MB^*Q^*\Pi$ is bounded and linear from $\mathbf{Y}_\alpha$ to $\mathcal{U}$. It follows that $B\widetilde{M}\widetilde{M}^*B^*Q^*\Pi$ belongs to $\mathcal{L}(\mathbf{Y}_\alpha, (D(A^*))')$. Using Remark 3.15, A_Π belongs to $\mathcal{L}(\mathbf{Y}_\alpha)$. Estimate (5.12) is a consequence of (5.11). □

Let us come back to the equation satisfied by $\mathbf{y}$. From Theorem 5.7, we can give the expression of the feedback control

$$\mathbf{v} = -\widetilde{M}^* B^* Q^* \Pi \mathbf{y}. \tag{5.13}$$

Thus, the linearized equation becomes

$$\mathbf{y}' = A\mathbf{y} - B\widetilde{M}\widetilde{M}^* B^* Q^* \Pi Q \mathbf{y},$$

that is to say, using the definition of $\widetilde{M}$

$$\mathbf{y}' = A\mathbf{y} + BM\mathbf{u},$$

with

$$\mathbf{u} = \sum_{i=1}^{n_c} v_i \zeta_i \quad \text{ and } \mathbf{v} = (v_1, \cdots, v_{n_c}) = -\widetilde{M}^* B^* Q^* \Pi Q \mathbf{y}. \tag{5.14}$$

Remark 5.10. From the definition of $\widetilde{M}^*$, we deduce the expression of the feedback law

$$v_i(t) = -\Big(\zeta_i, MB^*Q^*\Pi QP\mathbf{y}(t)\Big)_{\mathbf{V}^0(\Gamma)} = -\int_\Gamma \zeta_i(x) MB^*Q^*\Pi QP\mathbf{y}(t,x)dx$$
$$= -\Big(\Pi QBM\zeta_i, P\mathbf{y}(t)\Big)_{\mathbf{V}^0_n(\Omega)} = -\int_\Omega \Pi QBM\zeta_i(x)\, P\mathbf{y}(t,x)dx,$$

for all $i = 1 \cdots n_c$.

5.3. **The matrix Riccati equation corresponding to (5.9).** In equation (5.9), the operator Π belongs to $\mathcal{L}(\mathbf{Y}_\alpha, \mathbf{Y}_\alpha^*)$. Let us denote by $\widehat{\Pi}$ the matrix of Π when $\{\mathbf{e}_1, \cdots, \mathbf{e}_K\}$ is chosen as the basis of $\mathbf{Y}_\alpha$ and $\{\varepsilon_1, \cdots, \varepsilon_K\}$ is chosen as the basis of $\mathbf{Y}_\alpha^*$. We want to determine the equation satisfied by $\widehat{\Pi}$. We set

$$\Pi\mathbf{e}_j = \textstyle\sum_{i=1}^K \pi_{ij}\varepsilon_i, \quad A_\alpha\mathbf{e}_j = \sum_{i=1}^K a_{ij}\mathbf{e}_i, \quad \widehat{A} = (a_{ij})_{1\le i,j\le K},$$
$$E = (e_{ij})_{1\le i,j\le K} = ((\mathbf{e}_j, \mathbf{e}_i))_{1\le i,j\le K}.$$

We have

$$(\Pi\mathbf{e}_j, \mathbf{e}_i) = \pi_{ij} \quad \text{and} \quad A_\alpha^*\varepsilon_j = \sum_{k=1}^K a_{jk}\varepsilon_k.$$

We here assume that the basis $\{\zeta_1, \cdots \zeta_{n_c}\}$ of $\mathcal{U}$ is orthonormal in $\mathbf{V}^0(\Gamma)$. We set

$$\widetilde{M}^* B^*\varepsilon_j = \sum_{k=1}^{n_c} b_{jk}\zeta_k, \quad \widehat{B} = (b_{ij})_{1\le i\le K,\, 1\le j\le n_c} \quad \text{and} \quad \widehat{B}^T = (b_{ji})_{1\le j\le n_c,\, 1\le i\le K}.$$

Let us notice that the matrices $\widehat{A}$ and $\widehat{B}$ are the ones introduced in section 4. From equation (5.9), it follows that

$$(\Pi A_\alpha\mathbf{e}_j, \mathbf{e}_i)_{\mathbf{V}_n^0(\Omega)} + (A_\alpha^*\Pi\mathbf{e}_j, \mathbf{e}_i)_{\mathbf{V}_n^0(\Omega)} - \Big(\widetilde{M}^* B^*\Pi\mathbf{e}_j, \widetilde{M}^* B^*\Pi\mathbf{e}_i\Big)_{\mathbf{V}^0(\Gamma)} + (Q\mathbf{e}_j, Q\mathbf{e}_i)_{\mathbf{V}_n^0(\Omega)} = 0,$$

for all $1 \le i, j \le K$, that is

$$\widehat{\Pi}\,\widehat{A} + \widehat{A}^T\widehat{\Pi} - \widehat{\Pi}\widehat{B}\widehat{B}^T\widehat{\Pi} + E = 0. \tag{5.15}$$

This is the matrix Riccati equation that we have to solve to determine a feedback law stabilizing system (1.8).

6. **Stabilizabilition of the Navier-Stokes equations by finite dimensional controllers in feedback form.** In this section, using the expression of the feedback control given by (5.14), we consider the system

$$\begin{aligned} &P\mathbf{y}' = AP\mathbf{y} - \textstyle\sum_{i=1}^{n_c} \big(\Pi QBM\zeta_i, P\mathbf{y}\big)_{\mathbf{V}_n^0(\Omega)} BM\zeta_i + PF(\mathbf{y}),\\ &P\mathbf{y}(0) = \mathbf{y}_0,\\ &(I-P)\mathbf{y} = -\textstyle\sum_{i=1}^{n_c} \big(\Pi QBM\zeta_i, P\mathbf{y}\big)(I-P)D_A M\zeta_i \quad \text{in } (0,\infty),\end{aligned} \tag{6.1}$$

where

$$F(\mathbf{y}) = -e^{-\alpha t}(\mathbf{y}\cdot\nabla)\mathbf{y}.$$

Writing $\mathbf{f} = F(\mathbf{y})$ and $\mathbf{y}$ instead of $P\mathbf{y}$, we first have to study the nonhomogeneous equation

$$\mathbf{y}' = A\mathbf{y} - \sum_{i=1}^{n_c} \big(\Pi QBM\zeta_i, P\mathbf{y}\big) BM\zeta_i + \mathbf{f}, \qquad \mathbf{y}(0) = \mathbf{y}_0.$$

We recall that this equation may be written in the form

$$\mathbf{y}' = A\mathbf{y} - B\widetilde{M}\widetilde{M}^* B^* Q^*\Pi Q\mathbf{y} + \mathbf{f}, \qquad \mathbf{y}(0) = \mathbf{y}_0. \tag{6.2}$$

To study such an equation, we will need the following lemma.

Lemma 6.1. *Let X be a Hilbert space, and suppose that $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup of negative type. Then, the mapping*

$$\begin{array}{ccc} L^2(0,\infty;X)\cap H^1(0,\infty;(D(\mathcal{A}^*))') & \mapsto & L^2(0,\infty;(D(\mathcal{A}^*))')\times[(D(\mathcal{A}^*))',X]_{1/2} \\ \mathbf{y} & \mapsto & (\mathbf{y}'-\mathcal{A}\mathbf{y},\mathbf{y}(0)) \end{array}$$

is an isomorphism.

Proof. The proof is a direct consequence of [13, Chapter 3, p. 165] and [13, Chapter 1, p. 108 and p. 80]. □

In the following, we introduce the notation

$$\mathcal{V}^\theta(\Omega)=D\left((\lambda_0 I-A^*)^{\theta/2}\right)\quad\text{and}\quad\mathcal{V}^{-\theta}(\Omega)=(\mathcal{V}^\theta(\Omega))'\qquad\text{for}\quad 0\le\theta\le 2.$$

6.1. **Studying of the linearized problem with a nonhomogeneous source term.** In this subsection, we study equation (6.2). We assume that

$$\mathbf{f}\in L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega)),\quad \mathbf{y}_0\in\mathbf{V}^\varepsilon_n(\Omega)\quad\text{with } 0\le\varepsilon<1/2. \tag{6.3}$$

Lemma 6.2. *Let us suppose that (6.3) is satisfied. Then, equation (6.2) admits a unique solution $\mathbf{y}$ in $L^2(0,\infty;\mathbf{V}^0_n(\Omega))$ which obeys*

$$\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))}\le C\big(|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}+\|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}\big).$$

Proof. Let us split equation (6.2) as follows

$$\begin{aligned} &\mathbf{y}_\alpha'=A_\alpha\mathbf{y}_\alpha-QB\widetilde{M}\widetilde{M}^*B^*Q^*\Pi\mathbf{y}_\alpha+Q\mathbf{f}\quad\text{ in }(0,\infty),\\ &\mathbf{y}_\alpha(0)=Q\mathbf{y}_0\\ &\mathbf{y}_{\alpha^-}'=A_{\alpha^-}\mathbf{y}_{\alpha^-}-(I-Q)B\widetilde{M}\widetilde{M}^*B^*Q^*\Pi\mathbf{y}_\alpha+(I-Q)\mathbf{f}\quad\text{ in }(0,\infty),\\ &\mathbf{y}_{\alpha^-}(0)=(I-Q)\mathbf{y}_0. \end{aligned}\tag{6.4}$$

We consider the first equation of this system. We notice that it can be written in the form

$$\mathbf{y}_\alpha{}'=A_\Pi\mathbf{y}_\alpha+Q\mathbf{f},\qquad\mathbf{y}_\alpha(0)=Q\mathbf{y}_0,\tag{6.5}$$

where the operator A_Π is defined in section 5. Due to Remark 5.9, the solution to equation (6.5) obeys

$$|\mathbf{y}_\alpha(t)|_{\mathbf{Y}_\alpha}\le C(e^{-\beta t}|Q\mathbf{y}_0|_{\mathbf{Y}_\alpha}+\int_0^t e^{-\beta(t-\tau)}|Q\mathbf{f}(\tau)|_{\mathbf{Y}_\alpha}d\tau),$$

for some $\beta>0$. It follows that

$$\|\mathbf{y}_\alpha\|_{L^2(0,\infty;\mathbf{Y}_\alpha)}\le C\big(|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}+\|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}\big).\tag{6.6}$$

Let us consider the second equation of system (6.4). We can notice that $B\widetilde{M}\widetilde{M}^*B^*Q^*\Pi\mathbf{y}_\alpha$ belongs to $L^2(0,\infty;(D(A^*))')$. Finally, with Proposition 3.19 we have

$$\tilde{\mathbf{f}}=-(I-Q)B\widetilde{M}\widetilde{M}^*B^*Q^*\Pi\mathbf{y}_\alpha+(I-Q)\mathbf{f}\in L^2(0,\infty;(D(A^*_{\alpha^-}))').$$

Since $A-\lambda_0 I$ generates an analytic semigroup on $\mathbf{V}^0_n(\Omega)$, the operator A_{α^-}, with domain $D(A_{\alpha^-})$ in $\mathbf{Y}_{\alpha^-}$, generates an analytic semigroup on $\mathbf{Y}_{\alpha^-}$. From [32, Proposition 2.2], A_{α^-} satisfies the spectrum determined growth assumption on $\mathbf{Y}_{\alpha^-}$. Then, A_{α^-} is of negative type, since

$$\sup\,\mathrm{Re}\,\sigma(A_{\alpha^-})\le\Re\lambda_{N_{\alpha+1}}<0.$$

We can notice that $(I-Q)\mathbf{y}_0$ belongs to $\mathbf{Y}_{\alpha^-} \subset [(D(A^*_{\alpha^-}))', \mathbf{Y}_{\alpha^-}]_{1/2}$. Using Lemma 6.1 with $X = \mathbf{Y}_{\alpha^-}$ and $\mathcal{A} = A_{\alpha^-}$, it follows that the solution $\mathbf{y}_{\alpha^-}$ belongs to $L^2(0,\infty;\mathbf{Y}_{\alpha^-})$ and we have

$$\begin{aligned}\|\mathbf{y}_{\alpha^-}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} &\leq C\big(|(I-Q)\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)} + \|\tilde{\mathbf{f}}\|_{L^2(0,\infty;(D(A^*_{\alpha^-}))')}\big)\\ &\leq C\big(|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))} + \|\mathbf{y}_\alpha\|_{L^2(0,\infty;\mathbf{Y}_\alpha)}\big).\end{aligned}$$

Using estimate (6.6) on $\mathbf{y}_\alpha$, we have

$$\|\mathbf{y}_{\alpha^-}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} \leq C\big(|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}\big),$$

and the proof is complete. □

Corollary 6.3. *Let us consider the solution $\mathbf{y}_\alpha$ of (6.5). The control*

$$\mathbf{u} = \sum_{i=1}^{n_c} v_i\zeta_i, \qquad \textit{with } \mathbf{v} = (v_1,\cdots,v_{n_c}) = -\widetilde{M}^* B^* Q^* \Pi \mathbf{y}_\alpha$$

belongs to $\mathbf{V}^{2,1}(\Sigma_\infty)$ and

$$\|\mathbf{u}\|_{\mathbf{V}^{2,1}(\Sigma_\infty)} \leq C(|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}).$$

Proof. We have already proved in Lemma 6.2 that the solution to equation

$$\mathbf{y}_\alpha{}' = A_\Pi \mathbf{y}_\alpha + Q\mathbf{f}, \qquad \mathbf{y}_\alpha(0) = Q\mathbf{y}_0,$$

belongs to $L^2(0,\infty;\mathbf{Y}_\alpha)$. Since $A_\Pi \in \mathcal{L}(\mathbf{Y}_\alpha)$, we clearly obtain that $A_\Pi\mathbf{y}_\alpha \in L^2(0,\infty;\mathbf{Y}_\alpha)$. Moreover, $Q\mathbf{f}$ belongs to $L^2(0,\infty;\mathbf{Y}_\alpha)$ since $\mathbf{f} \in L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))$ and $Q \in \mathcal{L}((D(A^*))', \mathbf{Y}_\alpha)$. Then, we can conclude that $\mathbf{y}_\alpha \in H^1(0,\infty;\mathbf{Y}_\alpha)$, and we have

$$\|\mathbf{y}_\alpha\|_{H^1(0,\infty;\mathbf{Y}_\alpha)} \leq C(\|\mathbf{y}_\alpha\|_{L^2(0,\infty;\mathbf{Y}_\alpha)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}).$$

Moreover, $\mathbf{v}$ belongs to $H^1(0,\infty;\mathbb{R}^{n_c})$ since $\widetilde{M}^*B^*Q^*\Pi$ is a continous and linear operator from $H^1(0,\infty;\mathbf{Y}_\alpha)$ to $H^1(0,\infty;\mathbb{R}^{n_c})$. Thus, we have

$$\|\mathbf{v}\|_{H^1(0,\infty;\mathbb{R}^{n_c})} \leq C(\|\mathbf{y}_\alpha\|_{L^2(0,\infty;\mathbf{Y}_\alpha)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}).$$

Since Ω is an open subset of class C^4, the space $\mathcal{U}$ is included in $H^{5/2}(\Gamma)$. Then, we have proved that $\mathbf{u} \in \mathbf{V}^{2,1}(\Sigma_\infty)$ and that

$$\|\mathbf{u}\|_{\mathbf{V}^{2,1}(\Sigma_\infty)} \leq C(\|\mathbf{y}_\alpha\|_{L^2(0,\infty;\mathbf{Y}_\alpha)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}).$$

With estimate (6.6), the proof is complete. □

Theorem 6.4. *Let us assume that (6.3) is satisfied. Equation (6.2) admits a unique solution $\mathbf{y}$ in the space $\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))$, it obeys*

$$\|\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))} \leq C_1(\varepsilon)\big(|\mathbf{y}_0|_{\mathbf{V}^\varepsilon_n(\Omega)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}\big).$$

Proof. With Lemma 6.2, we know that

$$\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} \leq C\big(|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)} + \|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}\big).$$

Let us set $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$, where $\mathbf{y}_1$ is solution to

$$\mathbf{y}_1' = (A-\lambda_0)\mathbf{y}_1 + BM\mathbf{u} \quad \text{in } (0,\infty), \quad \mathbf{y}(0) = 0, \tag{6.7}$$

and $\mathbf{y}_2$ is solution to

$$\mathbf{y}_2' = (A-\lambda_0)\mathbf{y}_2 + \lambda_0\mathbf{y} + \mathbf{f} \quad \text{in } (0,\infty), \quad \mathbf{y}(0) = \mathbf{y}_0. \tag{6.8}$$

Due to [29, Lemma 8.3], since $\mathbf{u}$ belongs to $\mathbf{V}^{2,1}(\Sigma_\infty)$, $\mathbf{y}_1$ belongs to $\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))$ for $0\le\varepsilon<1/2$, and we have

$$\|\mathbf{y}_1\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon}(\Omega\times(0,\infty))}\le C\|\mathbf{u}\|_{\mathbf{V}^{2,1}(\Sigma_\infty)}.$$

Let us consider the equation for $\mathbf{y}_2$. We can check that, for $0\le\varepsilon<1/2$, we have

$$\mathbf{V}^\varepsilon_n(\Omega)=[[D(A^*),\mathbf{V}^0_n(\Omega)]'_{1/2},[\mathbf{V}^0_n(\Omega),D(A)]_{1/2}]_{(1+\varepsilon)/2}.$$

Furthermore,

$$\lambda_0\mathbf{y}+\mathbf{f}\in L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))=L^2(0,\infty;[D(A^*),\mathbf{V}^0_n(\Omega)]'_{(1+\varepsilon)/2}).$$

By using an interpolation result, with Lemma 6.1 and [13, chapter 3], it follows that

$$\mathbf{y}_2\in L^2(0,\infty;[\mathbf{V}^0_n(\Omega),D(A)]_{(1+\varepsilon)/2})\cap H^1(0,\infty;[D(A^*),\mathbf{V}^0_n(\Omega)]'_{(1+\varepsilon)/2}).$$

As $[\mathbf{V}^0_n(\Omega),D(A)]_{(1+\varepsilon)/2}\subset\mathbf{V}^{1+\varepsilon}(\Omega)$, we clearly obtain that $\mathbf{y}_2\in L^2(0,\infty;\mathbf{V}^{1+\varepsilon}(\Omega))$. By interpolation, $\mathbf{y}_2$ belongs to $H^{1/2+\varepsilon/2}(0,\infty;\mathbf{V}^0_n(\Omega))$ and it obeys

$$\begin{aligned}&\|\mathbf{y}_2\|_{V^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))}\\&\le C(|\mathbf{y}_0|_{\mathbf{V}^\varepsilon_n(\Omega)}+\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))}+\|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}).\end{aligned}\tag{6.9}$$

The solution $\mathbf{y}=\mathbf{y}_1+\mathbf{y}_2$ belongs to $\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))$ for $0\le\varepsilon<1/2$, and we have

$$\|\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))}\le C(|\mathbf{y}_0|_{\mathbf{V}^\varepsilon_n(\Omega)}+\|\mathbf{u}\|_{\mathbf{V}^{2,1}(\Sigma_\infty)}+\|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}).$$

Due to Corollary 6.3, the proof is complete. □

6.2. **Stabilization of the two dimensional Navier-Stokes equations.**

Theorem 6.5. *For all $0\le\varepsilon<1/2$, there exist $\mu_0>0$ and a nondecreasing function η from $\mathbb{R}^+$ into itself, such that if $\mu\in(0,\mu_0)$ and $|\mathbf{y}_0|_{\mathbf{V}^\varepsilon_n(\Omega)}\le\eta(\mu)$, then equation (6.1) admits a unique solution in the set*

$$D_\mu=\Big\{\mathbf{y}\in\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))\mid\|\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))}\le\mu\Big\}.$$

Moreover $(I-P)\mathbf{y}$ belongs to $H^{1/2+\varepsilon/2}(0,\infty;\mathbf{V}^{1/2}(\Omega))\cap L^2(0,\infty;\mathbf{V}^{1+\varepsilon}(\Omega))$.

From Theorem 6.5, the solution of (1.7) obeys

$$\|e^{\alpha(\cdot)}\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))}\le\mu.$$

It remains to show Theorem 6.5. For that, we will need some lemmas.

Lemma 6.6. *If $\mathbf{z}$ belongs to $\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))$ with $0\le\varepsilon<1/2$, then*

$$\|PF(\mathbf{z})\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}\le C_2\|\mathbf{z}\|^2_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(Q)}.$$

Proof. This proof can be adapted from [29, Lemma 6.4]. □

Lemma 6.7. *The mapping PF is locally Lipschitz continuous from $\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))$ into the space $L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))$. More precisely, we have*

$$\begin{aligned}&\|PF(\mathbf{z}_1)-PF(\mathbf{z}_2)\|_{L^2(0,\infty;\mathcal{V}^{-1+\varepsilon}(\Omega))}\\&\le C_2(\|\mathbf{z}_1\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))}+\|\mathbf{z}_2\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))})\\&\qquad\times\|\mathbf{z}_1-\mathbf{z}_2\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))}.\end{aligned}$$

for all $\mathbf{z}_1$ and $\mathbf{z}_2\in\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))$.

Proof. See [29, Lemma 6.5]. □

Lemma 6.8. *If* $\mathbf{y}$ *belongs to* $\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))$ *for some* $0\le\varepsilon<1/2$*, then*

$$\|(I-P)D_AM\mathbf{u}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))}\le C_3\|\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))}$$

with

$$\mathbf{u}=\sum_{i=1}^{n_c}v_i\zeta_i,\qquad \textit{with } \mathbf{v}=(v_1,\cdots,v_{n_c})=-\widetilde{M}^*B^*Q^*\Pi QP\mathbf{y}.$$

Proof. Clearly, $P\mathbf{y}$ belongs to $H^{1/2+\varepsilon/2}(0,\infty;\mathbf{V}_n^0(\Omega))$. From Remark 3.16, it follows that $MB^*Q^*\Pi QP\mathbf{y}$ belongs to $H^{1/2+\varepsilon/2}(0,\infty;\mathcal{U})$ and we have

$$\|(I-P)D_AM\mathbf{u}\|_{H^{1/2+\varepsilon/2}(0,\infty;\mathbf{V}_n^0(\Omega))}\le C_3\|\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))}.$$

Let us show the estimate in $L^2(0,\infty;\mathbf{H}^{1+\varepsilon}(\Omega))$. Obviously, $\mathbf{v}$ belongs to $L^2(0,\infty;\mathbb{R}^{n_c})$. Since Ω is an open subset of class C^4, we have already proved that $\mathcal{U}\subset H^{5/2}(\Gamma)$. It clearly follows that $\mathbf{u}$ belongs to $L^2(0,\infty;H^{5/2}(\Gamma))$ and that $(I-P)D_AM\mathbf{u}\in L^2(0,\infty;H^{1+\varepsilon}(\Omega))$. Thus, we have

$$\|(I-P)D_AM\mathbf{u}\|_{L^2(0,\infty;\mathbf{H}^{1+\varepsilon}(\Omega))}\le C_3\|\mathbf{y}\|_{\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Omega\times(0,\infty))}.$$

□

Proof. **Proof of Theorem 6.5.** The proof follows the lines of [29, Theorem 6.1]. □

REFERENCES

[1] S. Ahuja, C. W. Rowley, *Low-Dimensional Models for Feedback Stabilization of Unstable Steady States*, AIAA, 2008-553, 46th Aerospace Sciences Meeting and Exhibit, 2008.

[2] M. Badra, *Stabilisation par feedback et approximation des équations de Navier-Stokes*, PhD Thesis, Université Paul Sabatier, Toulouse, 2006.

[3] M. Badra, *Local stabilization of the Navier-Stokes equations with a feedback controller localized in an open subset of the domain*, Numer. Funct. Anal. Optim., Vol. **28** (2007), 559–589.

[4] M. Badra, *Lyapunov function and local feedback boundary stabilization of the Navier-Stokes equations*, SIAM J. Control Optim, 43 (2009), 1797–1830.

[5] M. Badra, T. Takahashi, *Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamic controllers. Application to the Navier-Stokes system*, hal-00431041, November 2009.

[6] V. Barbu, *Feedback stabilization of the Navier-Stokes equations*, ESAIM COCV, Vol. 9 (2003), 197–206.

[7] V. Barbu, *The internal stabilization by noise of the linearized Navier-Stokes equations*, to appear in ESAIM COCV, 2009.

[8] V. Barbu, S. S. Sritharan, H^∞ *- control theory of fluid dynamics*, Proc. R. Soc. Lond. A 454 (1998), 3009–3033.

[9] V. Barbu, I. Lasiecka, R. Triggiani, *Tangential boundary stabilization of Navier-Stokes equations*, Memoirs of the A.M.S., 2006, number 852.

[10] V. Barbu, I. Lasiecka, R. Triggiani, *Abstract settings for tangential boundary stabilization of Navier-Stokes equations by high and low-gain feedback controllers*, Nonlinear Anal. 64 (2006), 2704–2746.

[11] V. Barbu, I. Lasiecka, R. Triggiani, *Local exponential stabilization strategies for the Navier-Stokes equations, $d=2,3$, via feedback stabilization of its linearization*, Int. Series of Num. Math., Vol. 155, 2007, Birkhäuser Verlag, 13–46.

[12] V. Barbu, R. Triggiani, *Internal stabilization of Navier-Stokes equations with finite-dimensional controllers*, Indiana University Mathematics Journal, Vol. 53 (2004), 1443–1494.

[13] A. Bensoussan, G. Da Prato, M. C. Delfour, S. K. Mitter, *Representation and Control of Infinite Dimensional Systems*, Vol. 1, Birkhäuser, 1992.

[14] J. A. Burns, E. W. Sachs, L. Zietsman, *Mesh independence of Kleinman-Newton iterations for Riccati equations in Hilbert space*, SIAM J. Control Optim., Vol. 47 (2008), 2663–2692.

[15] C. Cao, I. G. Kevrekidis, E. S. Titi, *Numerical criterion for the stabilization of steady states of the Navier-Stokes equations*, Indiana Univ. Math. J., 50 (2001), 37–96.

[16] C. Fabre, G. Lebeau, *Prolongement unique des solutions de l'équation de Stokes*, Comm. Partial Differential Equations 21 (1996), 573–596.

[17] C. Fabre, G. Lebeau, *Régularité et unicité pour le problème de Stokes*, Comm. Partial Differential Equations, 27 (2002), 437–475.

[18] A. V. Fursikov, *Stabilizability of two-dimensional Navier-Stokes equations with help of a boundary feedback control*, J. Math. Fluid Mech., 3 (2001), 259–301.

[19] A. V. Fursikov, *Stabilizability of a quasi-linear parabolic equation by means of a boundary control with feedback*, Sbornik Mathematics, 192:4 (2001), 593–639.

[20] A. V. Fursikov, *Stabilization for the 3D Navier-Stokes system by feedback boundary control*, Discrete and Cont. Dyn. Systems, 10 (2004), 289–314.

[21] T. Kato, Perturbation theory for linear operators, Reprint of the 1980 Edition, Springer-Verlag, 1995.

[22] M. V. Keldysh, *On completeness of the eigenfunctions of certain classes of non-self-adjoint linear operators*, Uspekhi Mat. Nauk (1971), 15–41.

[23] S. Kesavan, J.-P. Raymond, *On a degenerate algebraic Riccati equation, to appear in Control and Cybernetics*, 2009.

[24] I. Lasiecka, *Stabilization of hyperbolic and parabolic systems with nonlinear perturbed boundary conditions*, J. Diff. Eq., 75 (1988), 53–87.

[25] I. Lasiecka, *Exponential stabilization of hyperbolic systems with nonlinaer, unbounded perturbations–Riccati operator approach*, Applicable An., 42 (1991), 243–261.

[26] I. Lasiecka, R. Triggiani, *Stabilization and structural assignment of Dirichlet boundary feedback parabolic equations*, SIAM J. Control and Optim., 21 (1983), 766–803.

[27] I. Lasiecka, R. Triggiani, *Control Theory for Partial Differential Equations*, Vol. 1, Cambridge University Press, 2000.

[28] J.-P. Raymond, *Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions*, Annales de l'IHP, An. non lin., Vol. 6 (2007), 921–951.

[29] J.-P. Raymond, *Boundary feedback stabilization of the two dimensional Navier-Stokes equations*, SIAM J. Control and Optim., Vol. 45 (2006), 790–828.

[30] J.-P. Raymond, *Feedback boundary stabilization of the three dimensional incompressible Navier-Stokes equations*, J. Math. Pures Appl., 87 (2007), 627–669.

[31] J.-P. Raymond, *A family of stabilization problems for the Oseen equations*, Control of coupled partial differential equations, Internat. Ser. Numer. Math., Vol. 155, (2007), 269–291.

[32] R. Triggiani, *On the stabilizability problem in Banach space*, J. Math. Anal. Appl. 52 (1975), 383–403.

[33] R. Triggiani, *Boundary feedback stabilizability of parabolic equations*, Appl. Math. Optim. 6 (1980), no. 3, 201–220.

[34] R. Triggiani, *Unique continuation of boundary over-determined Stokes and Oseen eigenproblems*, Discrete and Cont. Dyn. Systems, Series S, 2 (2009), 647–677.

[35] R. Vasquez, M. Krstic, *Control of turbulent and magnetohydrodynamic channel flows*, Birkhäuser, 2008.

[36] J. Zabczyk, *Mathematical control theory: An Introduction*, Birkhäuser, 1992.

E-mail address: raymond@math.univ-toulouse.fr

E-mail address: Laetitia.Thevenet@math.univ-toulouse.fr