# METRICS and COHOMOLOGY

**Theorem** (the $\partial\bar{\partial}$-lemma)

Let $X$ be a compact complex manifold.

If $X$ carries a Kähler metric, then

$X$ is a $\partial\bar{\partial}$-manifold.

## Sketch of proof

1. Harmonic theory for the Laplacians

$$\Delta = dd^* + d^* d : C^\infty_k(X, \mathbb{C}) \longrightarrow C^\infty_k(X, \mathbb{C})$$

$$\Delta' = \partial\partial^* + \partial^*\partial : C^\infty_{p,q}(X, \mathbb{C}) \longrightarrow C^\infty_{p,q}(X, \mathbb{C})$$

$$\Delta'' = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : C^\infty_{p,q}(X, \mathbb{C}) \longrightarrow C^\infty_{p,q}(X, \mathbb{C})$$

All these Laplacians are elliptic, self-adjoint and $\geq 0$:

$$\langle\langle \Delta u, u \rangle\rangle \geq 0 \quad \forall u$$

the $L^2$-inner product

induced by a given

Hermitian metric $\omega$

$$\langle\langle \Delta' u, u \rangle\rangle, \langle\langle \Delta'' u, u \rangle\rangle \geq 0$$

Therefore, from

$X$ compact + $\Delta, \Delta', \Delta''$ elliptic,

we get 3-space decomposition:

$$C^\infty_k(X, \mathbb{C}) = \underbrace{\ker \Delta \oplus \operatorname{im} d}_{= \ker d} \oplus \operatorname{im} d^*$$

$$C^\infty_{p,q}(X, \mathbb{C}) = \ker \Delta' \oplus \operatorname{im} \partial \oplus \operatorname{im} \partial^*$$

$$\underbrace{\hspace{4cm}}_{= \ker \partial}$$

$$C^\infty_{p,q}(X,\mathbb{C}) = \underbrace{\ker \Delta'' \oplus \operatorname{im} \bar\partial}_{= \ker \bar\partial} \oplus \operatorname{im} \bar\partial^*$$

- Suppose $\exists\ \omega$ Kähler metric on $X$.

2 Crucial fact

$$\boxed{\Delta' = \Delta'' = \frac{1}{2}\Delta} \qquad (*)$$

This follows from the Kähler commutation relations:

(i) $\boxed{\partial^* = i[\Lambda, \bar\partial]}$

where $\Lambda = \Lambda_\omega = (\omega \wedge \cdot)^*$ the adjoint w.r.t. the pointwise inner product $\langle \cdot , \cdot \rangle_\omega$.

We get the other three commutation relations by taking conjugate and adjoints:

(ii) $\boxed{\bar{\partial}^* = -i[\Lambda, \partial]}$

(iii) $\boxed{\partial = -i[\bar{\partial}^*, \omega \wedge \cdot\,]}$

(iv) $\boxed{\bar{\partial} = i[\partial^*, \omega \wedge \cdot\,]}$

where the anti-commutator $[\cdot , \cdot]$ is

$$[A, B] = AB - (-1)^{ab} BA$$

$$\begin{cases} a = \deg A & = AB - BA \quad \text{if } a \text{ or } b \text{ is even} \\ b = \deg B & = AB + BA \quad \text{if } a, b \text{ are odd} \end{cases}$$

These are easily proved by integration by parts if

$$\omega = \sum_{j=1}^{n} i\, dz_j \wedge d\bar{z}_j$$

In general, $\omega$ Kähler $\Rightarrow \exists\, z_1, \dots, z_n$

local coordinates
(geodesic coordinates)

s.t.

$$\omega = \sum_{j,k=1}^{n} \left(1 + \delta_{jk} + O(|z|^2)\right) i\, dz_j \wedge d\bar{z}_k$$

Proof of (*) as a consequence of Kähler commutation

We have:

$$\Delta' = [\partial, \partial^*] = i\left[\partial, [\Lambda, \bar{\partial}]\right]$$

(i) commutation relation

The Jacobi identity yields:

$$-[\partial,[\Lambda,\bar\partial]] + [\Lambda,[\bar\partial,\partial]] + [\bar\partial,[\partial,\Lambda]] = 0$$

$\bar\partial\partial+\partial\bar\partial=0$

$$\Delta' = i[\partial,[\Lambda,\bar\partial]] = i\cancel{[\Lambda,[\bar\partial,\partial]]} + [\bar\partial,\underbrace{i[\partial,\Lambda]}_{=-i[\Lambda,\partial]=\bar\partial^*}]$$

(i) commutation relation.

$$= [\bar\partial,\bar\partial^*] = \Delta''$$

q.e.d.

## (3) End of proof of the $\partial\bar\partial$-lemma

Let $\alpha \in C^\infty_{p,q}(X, \mathbb{C})$ s.t. $d\alpha = 0$

$(\iff \partial\alpha = 0$ and $\bar\partial\alpha = 0)$

Suppose that $\boxed{\alpha \in \operatorname{Im} \partial}$.

Want to prove that $\alpha \overset{?}{\in} \operatorname{Im} \bar\partial$

We know that:

$$\begin{array}{ccccc} \ker \partial & = & \ker \Delta' & \oplus & \operatorname{Im} \partial \\ \cup\!\!\!\shortmid & & \| & & \\ \alpha & & \| \leftarrow \text{because } \omega \text{ is Kähler} & & \\ \cap & & & & \\ \ker \bar\partial & = & \ker \Delta'' & \oplus & \operatorname{Im} \bar\partial \end{array}$$

Since $\alpha \in \ker \partial$, we have the equivalence:

$$\alpha \in \operatorname{im} \partial \iff \alpha \perp \ker \Delta'$$

$$\iff \alpha \perp \ker \Delta''$$

$\omega$ is Kähler

$$\iff \alpha \in \operatorname{im} \bar{\partial}$$

The remaining implications are proved similarly.

q.e.d.

## II Other types of special Hermitian metrics

Let $X$ be a compact complex manifold, $n = \dim_{\mathbb{C}} X$.

Let $\omega$ be a Hermitian metric on $X$.

$\omega^{n-1} > 0$ $(n-1, n-1)$-form on $X$

(1,1) $d\omega = 0 \implies \exists \alpha^{0,2} \in C^\infty_{0,2}(X, \mathbb{C})$ s.t.

$(\omega$ Kähler$)$ $d\Big(\underbrace{\overline{\alpha^{0,2}}}_{(2,0)} + \omega + \underbrace{\alpha^{0,2}}_{(0,2)}\Big) = 0$ $\implies \partial\bar\partial\omega = 0$

$\Downarrow$

($\omega$ is Hermitian-symplectic) (H-S)

($\omega$ is SKT) (pluriclosed)

$\omega$ is strongly Gauduchon
(s G) (P. 2009)

$(n-1,n-1)$ $d\,\omega^{n-1} = 0 \Rightarrow \exists\, \Gamma^{n-2,n} \in C^{\infty}_{n-2,n}$ s.t. $\Rightarrow \partial\bar\partial\,\omega^{n-1} = 0$

$\omega$ is balanced $\quad d\big(\underbrace{\overline{\Gamma^{n-2,n}}}_{(n,n-2)} + \underbrace{\omega^{n-1}}_{(n-1,n-1)} + \underbrace{\Gamma^{n-2,n}}_{(n-2,n)}\big) = 0$ $\quad \omega$ is Gauduchon ('77)

Gauduchon '77 $\Leftrightarrow$ $\boxed{\partial\omega^{n-1} \in \operatorname{Im}\bar\partial}$

- Leibniz rule $\quad d(\omega^{n-1}) = (n-1)\,\omega^{n-2} \wedge \underbrace{d\omega}_{\substack{\shortparallel \\ 0}}$ if $\omega$ is Kähler

$\Leftrightarrow \sqrt{(n-1)!}\; d(\star\omega) = 0$

$\Leftrightarrow \underbrace{(-\star d \star)}_{\substack{\shortparallel \\ d^{\star}}}(\omega) = 0$

the Hodge star operator induced by $\omega$

$d^{\star}$ the adjoint of $d$.

Conclusion $\quad \omega$ is balanced $\Leftrightarrow d^{\star}\omega = 0$
$(d\,\omega^{n-1} = 0) \qquad (\omega$ is co-closed$)$

**Properties** (i) Gauduchon metrics <u>always</u> exist!

Moreover

**Gauduchon** (1977)

$\forall\ \omega$ Hermitian metric on $X$,

$\exists\ \tilde{\omega}$ <u>Gauduchon metric</u>.

$\exists\ f : X \longrightarrow (0, +\infty)$ $C^\infty$ function

s.t. $\boxed{\tilde{\omega} = f\,\omega}$ i.e. $\tilde{\omega}$ is conformally equivalent to $\omega$.

Moreover, $f$ is <u>unique</u> up to positive multiplicative constants.

<u>Idea of proof</u> : Using an elliptic operator

+

harmonic theory

+

the maximum principle.

One can prove the existence of a <u>Gauduchon metric</u> (without the conformal class information)

Using only the <u>Hahn-Banach separation</u> theorem.

q.e.d.

(ii) All the other special metrics defined above <u>need not exist</u>.

We get:

· <u>Kähler</u> manifolds,

· <u>balanced</u> manifolds,

· <u>sG</u> manifolds,

⋮

There are not many compact Kähler manifolds in $\dim_{\mathbb{C}} \geq 3$.

<u>Fundamental fact</u>

(Kodaira's classification + Miyaoka + Siu)

(Buchdahl (1999) and Lamari (1999))

**Theorem** If $\dim_{\mathbb{C}} X = 2$, then

$$\boxed{\exists\, \omega \text{ \underline{Kähler} metric on } X \iff b_1 \text{ is \underline{even}}}$$

Kähler property for surfaces is topological

Proof of the trivial "$\Rightarrow$" implication

$\exists\, \omega$ Kähler metric $\Rightarrow X$ is a $\partial\bar\partial$ - manifold

$$\Rightarrow H^1_{DR}(X,\mathbb{C}) \simeq H^{1,0}_{\bar\partial}(X,\mathbb{C}) \oplus H^{0,1}_{\bar\partial}(X,\mathbb{C})$$

(Hodge decomposition)

$$\Rightarrow b_1 = \underbrace{h^{1,0}_{\bar\partial}}_{=} + \underbrace{h^{0,1}_{\bar\partial}}_{} = 2\, h^{0,1}_{\bar\partial} \qquad \text{q.e.d.}$$

by Hodge symmetry

Observation if $\dim_{\mathbb{C}} X \geq 3$, then

$b_1$ even $\not\Rightarrow$ $\exists$ $\omega$ Kähler metric

$\Longleftarrow$

· if $\dim_{\mathbb{C}} X \geq 3$, there are far more non-Kähler manifolds than Kähler ones

Another example of topological/metric interplay

Prop (P. 2009) if $X$ is a $\partial\bar{\partial}$-manifold, then every Gauduchon metric on $X$ is strongly Gauduchon.

In particular, SG-metrics exist.

**Proof** Let $\omega$ be Gauduchon. Then

$$\partial\bar\partial\,\omega^{n-1} = 0 \iff \underbrace{\partial\,\omega^{n-1}}_{(n,n-1)\text{-form}} \in \operatorname{Ker}\bar\partial$$

$$\Rightarrow \underbrace{\partial\,\omega^{n-1}}_{(n,n-1)} \in \operatorname{Ker} d \cap \operatorname{Im}\partial$$

$$\underset{X\ \partial\bar\partial\text{-manifold}}{\Longrightarrow} \boxed{\partial\,\omega^{n-1} \in \operatorname{Im}\bar\partial} \overset{\text{def}}{\iff} \omega \text{ is SG.}$$

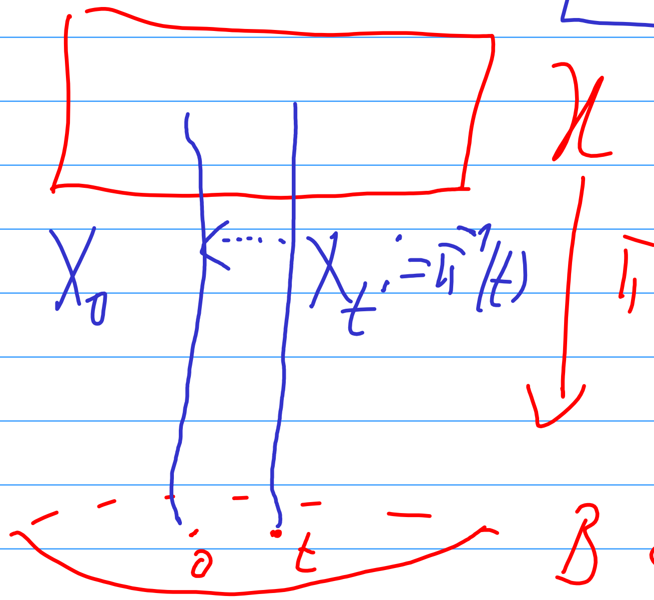
q.e.d

## III Deformation of complex structures

**Def** A holomorphic family of compact complex manifolds is a proper holomorphic submersion

$$\Pi : \underbrace{\mathcal{X}}_{\text{a complex manifold}} \longrightarrow \underbrace{B}_{\text{an open ball in some } \mathbb{C}^N}$$

$$\boxed{X_0 = \lim_{B \ni t \to 0} X_t}$$




$B \subset \mathbb{C}^N$

$X_t$ = the <u>fibre</u> over $t$

Each $X_t$ is a <u>compact</u> <u>complex manifold</u>

Let $n = \dim_{\mathbb{C}} X_t$, $\forall t \in B$

<u>Fundamental fact</u> (Ehresmann 1947)

$\forall t_1, t_2 \in B$, $\exists\ X_{t_1} \overset{C^\infty}{\cong} X_{t_2}$

diffeomorphism

in particular, $\forall k = 0, 1, \ldots, 2n$,

$$\boxed{H^k_{DR}(X_t, \mathbb{C}) = H^k_{DR}(X, \mathbb{C})}, \forall t \in B$$

the $C^\infty$ manifold underlying all the fibres $X_t$

However, the complex structure $J_t$ of $X_t$ depends, in general, on $t \in B$.

Then,

$$B \ni t \longmapsto H^{p,q}_{\bar\partial}(X_t, \mathbb{C})$$

$$\longmapsto H^{p,q}_{BC}(X_t, \mathbb{C})$$

$$\longmapsto H^{p,q}_{A}(X_t, \mathbb{C})$$

non-constant maps (in general)

# Fundamental question

(1) Openness

$X_0$ Kähler $\partial\bar\partial$ balanced $\Rightarrow$ ? $\forall t \sim 0$, $X_t$ is again Kähler / $\partial\bar\partial$ / balanced / sG / ...

2) Closedness

$\forall t \in B \setminus \{0\}$, $X_t$ is Kähler / $\partial\bar\partial$ / balanced / sG / ... $\Rightarrow$ ? $X_0$ is again Kähler / balanced / sG / ...

Theorem (openness)

(i) (Kodaira-Spencer 1960)

$X_0$ Kähler $\Rightarrow$ $X_t$ is Kähler $\forall t \sim 0$

(ii) C.C. Wu (2007), Angella-Tomassini (2013)

$X_0$ $\partial\bar\partial$-manifold $\Rightarrow$ $X_t$ $\partial\bar\partial$-manifold

$\forall t \sim 0$

(iii) P. (2009)

$X_0$ sG $\Rightarrow$ $X_t$ sG $\forall t \sim 0$

Prop d(ii) Let $\omega_0$ be an HS on $X_0$.

$\Updownarrow$

$\exists\, \Omega \in C^\infty_{2n-2}(X, \mathbb{R})$ s.t.

$$\begin{cases} \cdot\ d\Omega = 0 \\ \cdot\ \bar{\Omega} = \Omega \\ \cdot\ \Omega_0^{n-1,n-1} = \omega_0^{n-1} > 0 \implies \Omega_t^{n-1,n-1} > 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \forall t \sim 0 \end{cases}$$

continuity of $\Omega_t^{n-1,n-1}$

the $(n-1,n-1)$-component of $\Omega$ w.r.t. $J_0$

Consider the pure-type decomposition

$$\Omega = \Omega_t^{n,n-2} + \Omega_t^{n-1,n-1} + \Omega_t^{n-2,n}$$

$\Omega_t^{n-1,n-1} > 0$

w.r.t. $J_t$, $\forall t \in B$

$\exists!\ \omega_t^{n-1}$, $\underbrace{\omega_t}_{(1,1)} > 0$

$\omega_t$ is an $\partial G$ metric on $X_t$, $\forall t \sim 0$

by construction

q.e.d

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$\omega$ lck + balanced $\Rightarrow$ $\omega$ Kähler

$$\underbrace{\partial\omega}_{(2,1)} = (\partial\omega)_{prim} + \omega \wedge \underbrace{\alpha}_{(1,0)}$$

the $\omega$/Lefschetz decomposition

$\cdot$ $\omega$ balanced $\Rightarrow$ $\partial\omega$ primitive

$\Leftrightarrow$ $\partial\omega = (\partial\omega)_{prim}$

$\cdot$ $\omega$ lck $\Rightarrow$ $(\partial\omega)_{prim} = 0$

$\Rightarrow$ $(\partial\omega)_{prim} = 0$, $\omega \wedge \alpha = 0$

$\Rightarrow$ $\partial\omega = 0$