Cohomology

X a compact complex manifold,

$\dim_{\mathbb{C}} X = n$

$d = \underbrace{\partial}_{(1,0)} + \underbrace{\bar\partial}_{(0,1)}$ the complex structure of X

$$d^2 = 0 \Rightarrow \begin{cases} \partial^2 = 0 & (2,0) \\ \partial\bar\partial + \bar\partial\partial = 0 & (1,1) \\ \bar\partial^2 = 0 & (0,2) \end{cases}$$

d: depends on the C^∞ structure

1) The De Rham cohomology of X

We have a complex:

$$\cdots \xrightarrow{d} C^\infty_{k-1}(X,\mathbb{C}) \xrightarrow{d} \underbrace{C^\infty_k(X,\mathbb{C})}_{\text{smooth } k\text{-forms on } X} \xrightarrow{d} C^\infty_{k+1}(X,\mathbb{C}) \xrightarrow{d} \cdots$$

$$\boxed{\operatorname{im} d \subset \ker d}$$

We need not have equality.

<u>Def</u> $\forall k \in \{0, 1, \ldots, 2n = \dim_{\mathbb{R}} X\}$,

we set

$$H^k_{DR}(X, \mathbb{C}) := \frac{\ker(d : C^\infty_k \to C^\infty_{k+1})}{\operatorname{im}(d : C^\infty_{k-1} \to C^\infty_k)}$$

the <u>De Rham cohomology group</u> of degree k of X

a $\mathbb{C}$-vector space

Fundamental fact: if X is <u>compact</u>,

$$\boxed{\dim_{\mathbb{C}} H^k_{DR}(X, \mathbb{C}) \underset{\text{finite}}{<} \infty}$$

<u>Idea of proof</u> We consider the d-Laplacian

$$\Delta = dd^* + d^* d : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$$

<u>elliptic</u>, self-adjoint, ≥ 0 differential operator

where d^* = the adjoint of d w.r.t. a <u>Hermitian metric ω on X</u>

<u>Fundamental PDE fact</u>

- $\dim \ker \Delta < +\infty$ (finite)
- $\mathrm{Im}\,\Delta$ is <u>closed</u>

<u>Hodge isomorphism</u>

$$\underbrace{H^k_{DR}(X, \mathbb{C})}_{\text{an algebraic object}} \simeq \underbrace{\ker(\Delta : C^\infty_k(X,\mathbb{C}) \to C^\infty_k(X,\mathbb{C}))}_{\text{the } \Delta\text{-harmonic space}}$$

Terminology $\forall k$, we set

$$b_k := \dim_{\mathbb{C}} H^k_{DR}(X, \mathbb{C})$$

the k^{th} Betti number

of X

These are topological invariants of X

Example When $k=1$,

$H^1_{DR}(X, \mathbb{C})$ = the Abelianization of the fundamental ~~group~~ $\pi_1(X)$ of X

In particular, if X is simply connected,

$$\boxed{b_1(X) = 0}$$

2) $\boxed{\bar{\partial}^2 = 0}$, hence we get the

<u>Dolbeault cohomology</u> of X

(depends on the <u>complex structure</u> of X)

The Dolbeault <u>complex</u>:

$$C^\infty_{p,q-1}(X,\mathbb{C}) \xrightarrow{\bar{\partial}} C^\infty_{p,q}(X,\mathbb{C}) \xrightarrow{\bar{\partial}} C^\infty_{p,q+1}(X,\mathbb{C})$$

$$\operatorname{im}\left(\bar{\partial} : C^\infty_{p,q-1} \to C^\infty_{p,q}\right) \subset \ker\left(\bar{\partial} : C^\infty_{p,q} \to C^\infty_{p,q+1}\right)$$

We define: $\forall p, q \in \{0, 1, \dots, n\}$,

$$H^{p,q}_{\bar{\partial}}(X,\mathbb{C}) := \frac{\ker\left(\bar{\partial} : C^\infty_{p,q} \to C^\infty_{p,q+1}\right)}{\operatorname{im}\left(\bar{\partial} : C^\infty_{p,q-1} \to C^\infty_{p,q}\right)}$$

the Dolbeault cohomology group of bidegree (p,q)

Obs $\forall h$,

infinite-dim vector spaces

$$\boxed{C^\infty_h(X,\mathbb{C}) = \bigoplus_{p+q=h} C^\infty_{p,q}(X,\mathbb{C})}$$

$\exists!\ \alpha = \alpha^{h,0} + \alpha^{h-1,1} + \dots + \alpha^{0,h}$

at the level of forms

Fundamental question Do we have the same thing in cohomology?

$$(*)\quad \boxed{\underbrace{H^h_{DR}(X,\mathbb{C})}_{\text{the } C^\infty \text{ structure of } X} \simeq \bigoplus_{p+q=h} \underbrace{H^{p,q}_{\bar\partial}(X,\mathbb{C})}_{\text{the complex structure of } X} \qquad \forall h \in \{0,1,\dots,2n\}}$$

canonically isomorphic

Answer: $\begin{cases} \cdot \text{ in general, no!} \\ \cdot \text{ yes, if } X \text{ has an } \underline{\text{important property}} \end{cases}$

Def Let X be a compact complex manifold

$n = \dim_{\mathbb{C}} X$.

We say that X is a $\partial\bar\partial$-manifold

$\overset{\text{def}}{\iff} \forall\, p, q = 0, 1, \dots, n$

$\forall\, \alpha \in C^{\infty}_{p,q}(X, \mathbb{C})$ s.t. $\boxed{d\alpha = 0}$

the following equivalences hold:

$$\boxed{\begin{gathered} \alpha \in \operatorname{Im} d \iff \alpha \in \operatorname{Im} \partial \iff \alpha \in \operatorname{Im} \bar\partial \\ \iff \alpha \in \operatorname{Im}(\partial\bar\partial) \end{gathered}}$$

(a symmetry property for the complex structure)

Theorem Let X be a compact $\partial\bar{\partial}$-manifold, $\dim_{\mathbb{C}} X = n$.

Then:

1) $\forall p,q,\ \forall [\alpha] \in H^{p,q}_{\bar{\partial}}(X,\mathbb{C})$

$\exists \tilde{\alpha} \in [\alpha]$ s.t. $\boxed{d\tilde{\alpha} = 0}$

2) $\forall k$, the following linear map:

$$\bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(X,\mathbb{C}) \longrightarrow H^k_{DR}(X,\mathbb{C})$$

$$\left([\alpha^{p,q}]_{\bar{\partial}}\right)_{p,q} \longmapsto \left\{\sum_{p+q=k} \alpha^{p,q}\right\}_{DR}$$

$d\alpha^{p,q} = 0$

is <u>well-defined</u> (independent of the choice of <u>d-closed</u> representatives $\alpha^{p,q}$ of the Dolbeault classes)

and is an <u>isomorphism</u> of vector spaces.

This property of X is called the <u>Hodge decomposition</u> property of X.

3) $\forall p, q = 0, 1, \dots, n,$ the map:

$$(\alpha) \quad \boxed{\begin{aligned} H^{p,q}_{\bar\partial}(X, \mathbb{C}) &\longrightarrow \overline{H^{q,p}_{\bar\partial}(X, \mathbb{C})} \\ [\alpha^{p,q}]_{\bar\partial} &\longmapsto [\overline{\alpha^{p,q}}]_{\bar\partial} \end{aligned}}$$

$\bar\partial \alpha^{p,q} = 0 \iff \partial \overline{\alpha^{p,q}} = 0$

in general, $\overline{\alpha^{p,q}}$ is not $\bar\partial$-closed.

We choose $\alpha^{p,q}$ s.t. $\boxed{d\alpha^{p,q} = 0}$

$\Updownarrow$

$\left\{ \begin{array}{l} \boxed{\partial \alpha^{p,q} = 0}, \\ \bar{\partial} \alpha^{p,q} = 0 \end{array} \right.$

$\Rightarrow \boxed{\overline{\bar{\partial} \alpha^{p,q}} = 0}$

The ~~map~~ (*) defined using $\alpha^{p,q} \in \ker d$ is well-defined and an isomorphism.

This property of X is called the

Hodge symmetry

property.

Proof (i) let $\alpha \in C^\infty_{p,q}(X, \mathbb{C})$ s.t.

$$\boxed{\bar\partial \alpha = 0}$$

representing the Dolbeault class $[\alpha]_{\bar\partial}$.

We want: $\exists \underbrace{\tilde\alpha}_{(p,q)} = \alpha + \bar\partial \underbrace{\beta}_{(p,q-1)} \in [\alpha]_{\bar\partial}$

s.t. $d\underbrace{\tilde\alpha}_{(p,q)} = 0 \iff \begin{cases} \partial \tilde\alpha = 0 \\ \boxed{\bar\partial \tilde\alpha = 0} \end{cases}$ automatic because

$\bar\partial \alpha = 0$

$\bar\partial(\bar\partial \beta) = \bar\partial^2 \beta = 0$

We want

$\partial \tilde\alpha = 0 \iff \partial \alpha + \partial \bar\partial \beta = 0$

$\iff \boxed{\partial \bar\partial \beta = -\partial \alpha}$

Such a β exists iff

$$\partial\alpha \in \mathrm{Im}(\partial\bar\partial)$$

Now
- $\underbrace{\partial\alpha}_{(p,q)}$ is a $(p+1,q)$-form
- $d(\partial\alpha) = \underbrace{\partial^2\alpha}_{=0} + \underbrace{\bar\partial\partial\alpha}_{= -\partial(\underbrace{\bar\partial\alpha}_{=0})} = 0.$
- $\partial\alpha \in \mathrm{Im}\,\partial$

Since X is a $\partial\bar\partial$-manifold,

$$\boxed{\partial\alpha \in \mathrm{Im}(\partial\bar\partial)}$$

q.e.d.

Bott-Chern and Aeppli cohomologies

$$H^{p,q}_{BC}(X,\mathbb{C}) := \frac{\ker\partial \cap \ker\bar\partial}{\operatorname{Im}(\partial\bar\partial)}$$ the *finest* cohomology

$$H^{p,q}_{A}(X,\mathbb{C}) := \frac{\ker(\partial\bar\partial)}{\operatorname{Im}\partial + \operatorname{Im}\bar\partial}$$ the *coarsest* *cohomology*

<u>Immediate observation</u> [No assumption on X.] The following <u>canonical</u> linear maps:

$$H^{p,q}_{BC}(X,\mathbb{C}) \xrightarrow{T^{p,q}} H^{p,q}_{\bar\partial}(X,\mathbb{C}) \longrightarrow H^{p,q}_{A}(X,\mathbb{C})$$

$$[\alpha]_{BC} \longmapsto [\alpha]_{\bar\partial} \longmapsto [\alpha]_{A}$$

any representative

are <u>well defined</u>

Same goes for

$$H^{p,q}_{BC}(X, \mathbb{C}) \longrightarrow H^{p+q}_{DR}(X, \mathbb{C})$$

$$[\alpha]_{BC} \longmapsto \{\alpha\}_{DR}$$

<u>Proof</u> · if $\left\{ \begin{array}{l} \partial \alpha = 0 \\ \boxed{\bar{\partial} \alpha = 0} \end{array} \right.$ $\Rightarrow \alpha$ represents a Dolbeault class $[\alpha]_{\bar{\partial}}$

α representing a $[\alpha]_{BC}$

· Check that $T^{p,q}$ is independent of the choice of representative

Let $\alpha, \tilde{\alpha} \in \mathcal{C}^{\infty}_{p,q}(X, \mathbb{C})$ s.t.

$$\begin{cases} \partial\alpha = \bar\partial\alpha = 0 \\ \partial\tilde\alpha = \bar\partial\tilde\alpha = 0 \end{cases}$$

and $\exists\, \beta$ s.t.

$$\boxed{\tilde\alpha = \alpha + \partial\bar\partial\beta}$$

(α and $\tilde\alpha$ represent the same Bott-Chern class)

Need to check that

$$[\tilde\alpha]_{\bar\partial} \overset{?}{=} [\alpha]_{\bar\partial}$$

$$\left(\Leftarrow \boxed{\tilde\alpha - \alpha \overset{?}{\in} \operatorname{Im} \bar\partial} \right)$$

We have: $\tilde\alpha - \alpha = \partial\bar\partial\beta = \bar\partial(-\partial\beta) \in \operatorname{Im}\bar\partial$

q.e.d.

Theorem: If X is a $\partial\bar{\partial}$-manifold, all the above canonical maps are isomorphisms.

II The metric point of view

Question: $\exists$ a class of metrics on X implying that X is a $\partial\bar{\partial}$-manifold?

Definition: X a complex manifold, $n = \dim_{\mathbb{C}} X$.

A Hermitian metric for X is the following data:

- $\forall x \in X$, $\exists$ an inner product

$$\langle \cdot, \cdot \rangle_x$$

on the holomorphic tangent space

$$T_x^{1,0} X$$

- the map

$$X \ni x \longmapsto \langle \cdot, \cdot \rangle_x \text{ is } C^\infty$$

In local coordinates, the metric g looks like

$$\boxed{g = \sum_{j,k=1}^{n} g_{j\bar{k}} \, dz_j \otimes d\bar{z}_k}$$

Meaning: $\forall \xi = \sum_{j=1}^{n} \xi_j \, dz_j \in T_x^{1,0} X$

$\forall \eta = \sum_{k=1}^{n} \eta_k \, d\bar{z}_k \in T_x^{1,0} X$,

We have:

$$g(x)(\xi,\eta) = \boxed{\langle \xi, \eta \rangle_x = \sum_{j,k=1}^{n} g_{j\bar{k}} \, \xi_j \, \bar{\eta}_k}$$

Note that we necessarily have:

$$\boxed{\overline{g_{j\bar{k}}} = g_{k\bar{j}}} \quad \forall j, k$$

(i.e. the coefficient matrix

$(g_{j\bar{k}})_{j,k}$ is Hermitian)

$\Uparrow$

$\langle \cdot, \cdot \rangle > 0$ (inner product) $\iff (g_{j\bar{k}})_{j,k} > 0$

Second point of view on Hermitian metric

Theorem The following map is bijective:

$$\underbrace{\mathrm{Herm}(X)}_{\text{Hermitian metrics on } X} \ni g = \sum_{j,k=1}^{n} g_{j\bar{k}}\, dz_j \otimes d\bar{z}_k$$

$$\Big\downarrow$$

$$\underbrace{\omega_g}_{C^\infty\ (1,1)\text{-form on } X} := \sum_{j,k=1}^{n} g_{j\bar{k}}\, \underbrace{i\, dz_j \wedge d\bar{z}_k}$$

> 0 $\left(\overset{\text{def}}{\iff} (g_{j\bar{k}})_{j,k} > 0 \text{ : independent of the choice of complex coordinates} \right)$

Proof Easy:

$$W_g(\xi,\eta) = \sum_{j,k=1}^{n} g_{jk} \underbrace{\begin{vmatrix} \xi_j & \bar\xi_k \\ \eta_j & \bar\eta_k \end{vmatrix}}_{\displaystyle = \xi_j \bar\eta_k - \underbrace{\bar\xi_k \eta_j}_{}}$$

The bijection is (invariantly written):

$$\boxed{g = -2\,\mathrm{Im}\, W_g}$$

q.e.d.

Immediate observation

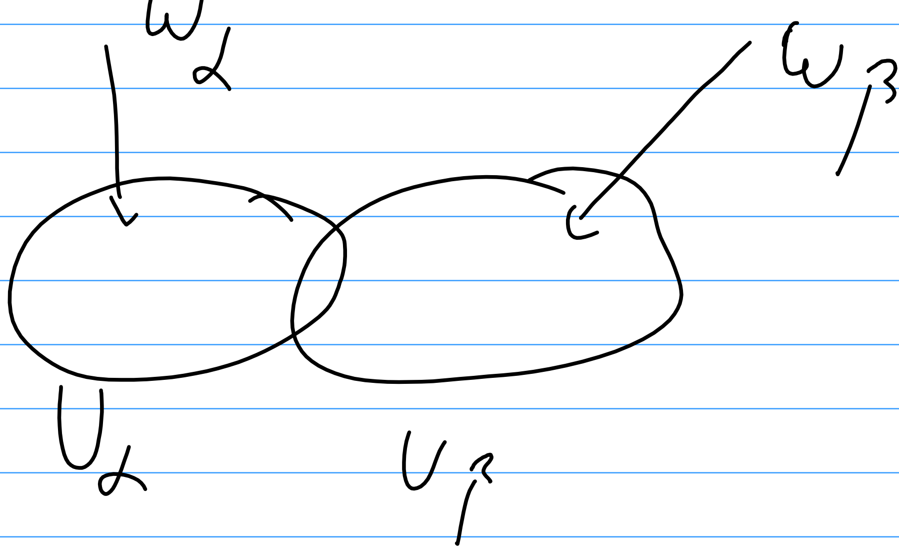
Hermitian metrics always exist on X

Proof Locally: take any >0 matrix

$$\left(\omega_{j\bar{k}}\right)_{j,k=1,\dots,n} > 0$$

Put $\omega := \sum_{j,k=1}^{n} \omega_{j\bar{k}}\, i\, dz_j \wedge d\bar{z}_k.$

We can globalize this using a partition of unity:



Put $\omega := \sum_\alpha \theta_\alpha \omega_\alpha$, where $(\theta_\alpha)_\alpha$ is a partition of unity.

<u>Aim</u> : to study the geometry of X by studying the <u>special</u> types of Hermitian metrics it supports

<u>Def</u> A Hermitian metric ω on X is <u>Kähler</u> $\overset{\text{def}}{\Longleftrightarrow}$

$$\boxed{d\omega = 0}$$

(the symplectic condition)

Properties $\forall k = 0, 1, \ldots, n$, we get:

$$\{\omega^k\}_{DR} \in H^{2k}_{DR}(X, \mathbb{C})$$

$$d\omega^k = k\,\omega^{k-1} \wedge \underbrace{d\omega}_{=0} = 0$$

Crucially if X is compact,

$$\{\underbrace{\omega^k}\}_{DR} \neq 0 \quad \forall k = 0, 1, \ldots, n.$$

(k,k) form, hence (k,k) form

In particular,

$$\boxed{b_{2k} \neq 0}$$

Therefore, if $\exists k = 0, 1, \ldots, n$ s.t. $b_{2k} = 0$, then $\nexists$ complex structure on X supporting a Kähler metric.

<u>Proof</u> If $\omega^k \in \operatorname{im} d$, then

$$\omega^n \in \operatorname{im} d$$
$$\|$$
$$\underbrace{\omega^k}_{\in \operatorname{im} d} \wedge \underbrace{\omega^{n-k}}_{\in \ker d}$$

Then:

$$0 = \int_X \omega^n > 0$$

<u>Contradiction</u>

because $\omega^n \in \operatorname{im} d$ + Stokes

because $\omega > 0$
$\Updownarrow$
$\det(\omega_{i\bar{j}}) > 0$

$$\frac{\omega^n}{n!} = \det\left(\omega_{j\bar{k}}\right)_{j,k} \, i\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge$$

$$i\, dz_n \wedge d\bar{z}_n$$