Vector bundles

Let X be a topological space

$$\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}$$

Definition A $\mathbb{K}$-topological vector bundle on X is a

· topological space E equipped with a continuous map

$$p : E \longrightarrow X$$

s.t. $\forall x \in X$, the fibre of x

$$E_x := p^{-1}(x)$$

is a $\mathbb{K}$-vector space

and such that it satisfies the local triviality axiom:

• $\exists X = \bigcup_{\alpha \in I} U_\alpha$ an open covering of X

such that

$$\exists \theta_\alpha : E|_{U_\alpha} \xrightarrow{\simeq} U_\alpha \times \mathbb{K}^n \quad \text{homeomorphism}$$

for some $n \in \mathbb{N}^\alpha$

and such that

$$\forall x \in X, \quad \underbrace{E_x}_{\mathbb{K}\text{-vector space}} \underset{\substack{\simeq \\ \text{isomorphism of vector spaces}}}{\xrightarrow{\theta_\alpha}} \{x\} \times \mathbb{K}^n \simeq \underbrace{\mathbb{K}^n}_{\mathbb{K}\text{-vector space of dim } n}$$

Terminology

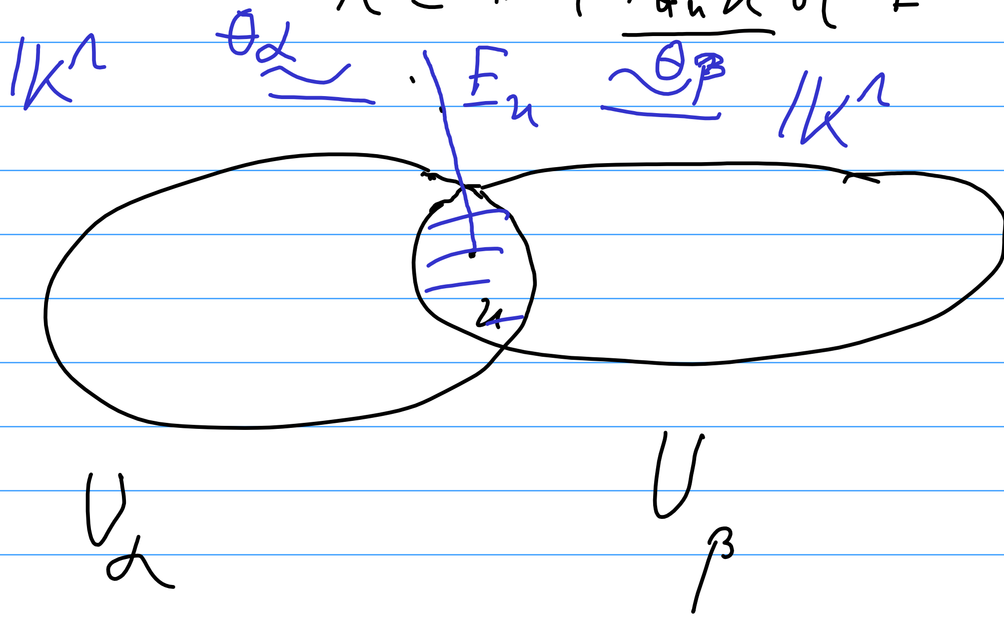
• E = the total space of the vector bundle

• X = the base of the vector bundle

$$\cdot\ \theta_\alpha : E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{K}^n$$

is a <u>local trivialisation</u> of E.

$\cdot\ n =$ the <u>rank</u> of E



<u>Transition automorphisms (matrices)</u>

$$\begin{tikzcd}
 & (U_\alpha \cap U_\beta) \times \mathbb{K}^n \\
E|_{U_\alpha \cap U_\beta} \arrow[ur, "\theta_\alpha", "\sim"'] \arrow[dr, "\theta_\beta"', "\wr"] & \\
 & (U_\alpha \cap U_\beta) \times \mathbb{K}^n \arrow[uu, "\theta_{\alpha\beta} := \theta_\alpha \circ \theta_\beta^{-1}"']
\end{tikzcd}$$

At the level of the fibres : $\forall x \in U_\alpha \cap U_\beta$

$$\begin{tikzcd} \mathbb{K}^n \arrow[r, "\simeq", "\theta_\beta^{-1}"'] \arrow[rr, bend right, "\theta_{\alpha\beta}"'] & E_x \arrow[r, "\simeq", "\theta_\alpha"'] & \mathbb{K}^n \end{tikzcd}$$

$$\begin{array}{c} \Psi \\ \xi \xmapsto{\simeq} g_{\alpha\beta}(x)\,\xi \end{array}$$

We get a linear map

$$g_{\alpha\beta}(x) : \mathbb{K}^n \longrightarrow \mathbb{K}^n$$

an <u>isomorphism</u>

Thus, $\forall x \in U_\alpha \cap U_\beta$, $g_{\alpha\beta}(x) \in GL_n(\mathbb{K})$

invertible $n \times n$ matrices with entries in $\mathbb{K}$

the transition automorphism from $(U_\alpha, \theta_\alpha)$ to (U_β, θ_β).

<u>Consequence</u>

$$\underbrace{\theta_{\alpha\beta}}_{\theta_\alpha \circ \underbrace{\theta_\beta^{-1}}} \circ \underbrace{\theta_{\beta\gamma}}_{\underbrace{\theta_\beta} \circ \theta_\gamma^{-1}} = \underbrace{\theta_{\alpha\gamma}}_{\theta_\alpha \circ \theta_\gamma^{-1}} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \quad \forall \alpha, \beta, \gamma$$

$$\Downarrow$$

$$(*) \quad \boxed{g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)} \quad \forall x \in U_\alpha \cap U_\beta \cap U_\gamma \quad \forall \alpha, \beta, \gamma.$$

composition of automorphisms of $\mathbb{K}^n$
||
product of matrices in $GL_n(\mathbb{K})$

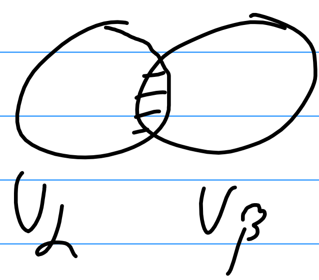
(*) is called the cocycle condition

Conversely: construction of the vector bundle from its transition matrices

Data:

- $X = \bigcup_{\alpha \in I} U_\alpha$ open covering of X
- $\forall \alpha, \beta \in I$, let

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{K}) \quad \text{continuous}$$



such that (*) is satisfied.

Then, we set:

$$E = \coprod_{\alpha \in I} \left(U_\alpha \times \mathbb{K}^n \right) \Big/ \sim ,$$

where the equivalence relation $\sim$

is defined by:

$$(\alpha, x, \xi) \sim (\beta, y, \eta) \overset{\text{def}}{\iff} \begin{cases} x = y \in U_\alpha \cap U_\beta \\ \xi = g_{\alpha\beta}(x)\,\eta \end{cases}$$

$\alpha, \beta \in I$

$x, y \in U_\alpha \cap U_\beta$

$\xi, \eta \in \mathbb{K}^n$

· We set: $p : E \longrightarrow X$

$$(\alpha, x, \xi) \longmapsto x$$

$$\boxed{\xi \in E_x}$$

· The <u>$\mathbb{K}$-vector space structure</u> of E_x is:

$$(\alpha, x, \underbrace{\xi_1}_{\in \mathbb{K}^n}) + (\alpha, x, \underbrace{\xi_2}_{\in \mathbb{K}^n}) := (\alpha, x, \xi_1 + \xi_2)$$

Examples of topological vector bundles

1) The constant vector bundle of rank n over X

$$E = X \times \mathbb{K}^n$$

$X = X$, $\theta = id$ (the unique global trivialization)

$$E_x \simeq \mathbb{K}^n \quad \forall x \in X$$

$$(x, \xi) + (x, \eta) = (x, \xi + \eta)$$

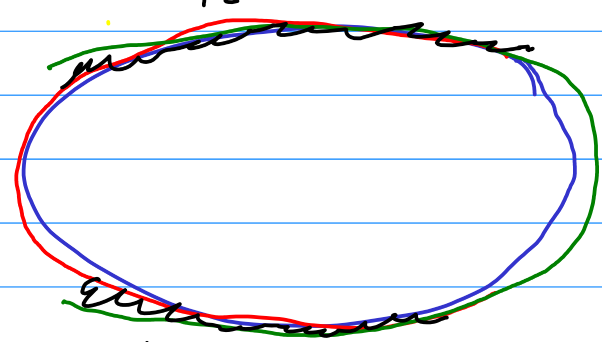
$$\lambda \cdot (x, \xi) = (x, \lambda \xi) \quad \forall \lambda \in \mathbb{K}.$$

2) The Möbius band

$X = S^1$ the unit circle in $\mathbb{C}$

$E \longrightarrow S^1$ an $\mathbb{R}$-vector bundle of

rank $r = 1$



$U_1, U_2 \subset S^1$

open subsets

s.t. $U_1 \cap U_2 =$

$U_{12}^+ \cup U_{12}^-$

Take: $\forall x \in U_1 \cap U_2$,

$g_{12}(x) \in \mathbb{R}^\times \; (= GL_1(\mathbb{R}))$

$$g_{12}(x) = \begin{cases} 1, & \text{if } x \in U_{12}^+ \\ -1, & \text{if } x \in U_{12}^- \end{cases}$$

$S^1 = U_1 \cup U_2$

Extra structures

Let X be a C^∞ / complex manifold.

Def We say that E is a C^∞ / holomorphic vector bundle if one of the two equivalent properties holds:

(i) E is a C^∞ / complex manifold

$p : E \longrightarrow X$ C^∞ / holomorphic

$\theta_\alpha : E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{K}^n$ C^∞ diffeo / biholomorphism

$\forall \alpha$

($\mathbb{K} = \mathbb{C}$ in the holomorphic case)

$\Downarrow$

(ii) the transition automorphisms

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{K})$$

are C^∞ (resp. <u>holomorphic</u>).

<u>Fundamental example of a vector bundle</u>

The <u>tangent bundle</u> of a manifold X

$$TX \xrightarrow{p} X$$

$X = \bigcup_\alpha U_\alpha$ open covering by coordinate charts

$$\tau_\alpha : U_\alpha \xrightarrow{\simeq} V_\alpha \subset \mathbb{R}^n$$

C^∞ diffeomorphism

$$\zeta_{\alpha\beta} := \zeta_\alpha \circ \zeta_\beta^{-1} : \overbrace{\zeta_\beta(U_\alpha \cap U_\beta)}^{\substack{\zeta_\beta(x) \\ \cap}} \to \zeta_\alpha(U_\alpha \cap U_\beta)$$

$$\cap \qquad\qquad\qquad \cap$$

$$\mathbb{R}^n \qquad\qquad\qquad \mathbb{R}^n$$

C^∞ diffeomorphism.

The <u>tangent bundle</u> of X is defined by the <u>transition matrices</u>:

$$g_{\alpha\beta}(x) := d\zeta_{\alpha\beta}(\zeta_\beta(x)) : \mathbb{R}^n \to \mathbb{R}^n$$

automorphism

$\forall x \in U_\alpha \cap U_\beta$

In other words:

$$TX = \coprod_{\alpha \in I} (U_\alpha \times \mathbb{R}^n) \Big/ \sim$$

Thus: $\boxed{\operatorname{rank}(TX) = \dim X = n}$

Operations on vector bundles: mirror the operations on vector spaces

1) The <u>direct sum</u>

$E, F \longrightarrow X$, $r_E = \operatorname{rk}(E)$, $r_F = \operatorname{rk}(F)$

$E \oplus F \longrightarrow X$

$\operatorname{rank}(E \oplus F) = \operatorname{rank}(E) + \operatorname{rank}(F)$

$$g_{\alpha\beta} \oplus h_{\alpha\beta} = \begin{pmatrix} \overbrace{\boxed{g_{\alpha\beta}}}^{r_E \times r_E} & 0 \\ 0 & \underbrace{\boxed{h_{\alpha\beta}}}_{r_F \times r_F} \end{pmatrix}$$

a block-diagonal matrix

2) <u>The dual vector bundle</u>

Let $E \longrightarrow X$ be a vector bundle,

$\operatorname{rk} E = n$

We want to define the <u>dual vector bundle</u>

$$E^* \longrightarrow X$$

s.t. $E^*_x = (E_x)^* \quad \forall x \in X$

$$\begin{tikzcd}
 & (U_\alpha \cap U_\beta) \times \mathbb{K}^n \\
E|_{U_\alpha \cap U_\beta} \arrow[ur, "\theta_\alpha", "\cong"'] \arrow[dr, "\theta_\beta"', "\cong"] & \\
 & (U_\alpha \cap U_\beta) \times \mathbb{K}^n
\end{tikzcd}$$

We want:

$$E^\alpha|_{U_\alpha \cap U_\beta} \xrightarrow[\underset{\left({}^t\theta_\alpha\right)^{-1}}{\parallel}]{\theta_\alpha^\alpha} (U_\alpha \cap U_\beta) \times \mathbb{K}^n$$

<u>Basic thing</u>

If $u : A \longrightarrow B$ linear map of vector spaces,

then the <u>dual</u> linear map

$$u^* = \underbrace{{}^t u}_{\text{the transpose of the matrix of } u} : B^\alpha \longrightarrow A^\alpha$$

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[rr, bend right, "\ell \circ u"'] & B \arrow[r, "\ell"] & \mathbb{K} \end{tikzcd}$$

$$B^\alpha \ni \ell \longmapsto \ell \circ u \in A^\alpha$$

We get the transition matrices for E^*:

$$\left(t_{g_{\alpha\beta}}(x)\right)^{-1}$$

$$\mathrm{rk}(E^{\alpha}) = \mathrm{rk}(E) = r.$$

$\bar{4}$ Continuation of yesterday's discussion

Let X be a C^{∞} manifold,

$$\dim_{\mathbb{R}} X = 2n \qquad (\text{even})$$

Let $x \in X$ be a point and let

$V := T_x X$ the tangent space at x to X

(= the fibre over x of the tangent bundle TX)

Suppose $\exists\ J_x : T_x X \longrightarrow T_x X$

s.t. $J_x^2 = -\mathrm{id}_{T_x X}$.

Such a J_x is called a <u>complex structure</u> on TX.

We consider the <u>complexified</u> situation:

$$\mathbb{C}T_x X := T_x X \otimes_{\mathbb{R}} \mathbb{C}$$

$$\tilde{J}_x : \mathbb{C}T_x X \longrightarrow \mathbb{C}T_x X$$

$$\tilde{J}_x(u + iv) = J_x u + i J_x v$$

$$\forall u, v \in T_x X$$

Let $x_1, y_1, \dots, x_n, y_n$ be local real

coordinates on X

about x

Put: $z_j := x_j + i\, y_j \quad \forall j = 1, \dots, n$

We define:

$$\left\{ \begin{aligned} \frac{\partial}{\partial z_j} &:= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j}\right) \\ \frac{\partial}{\partial \bar{z}_j} &:= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\,\frac{\partial}{\partial y_j}\right) \end{aligned} \right\} \quad j = 1, \dots, n$$

We get:

$$T^{1,0}_x X = \mathbb{C}\left\langle \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right\rangle$$

$(V^{1,0})$

$$T_x^{0,1}X = \mathbb{C}\left\langle \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right\rangle$$

$(V^{0,1})$

We have:

$$\boxed{\mathbb{C}T_x X = T_x^{1,0}X \oplus T_x^{0,1}X = T_x X \oplus \overline{T_x X}}$$

<u>Def</u> An <u>almost complex structure</u> on X is a collection of complex structures

$$J_x : T_x X \to T_x X$$

varying in a <u>C^∞ way</u> with the point $x \in X$.

if an almost complex structure on X arises, as above, from local real coordinates on X, it is called a

<u>complex structure</u>.

<u>Fact</u> Not every <u>even-dimensional</u> real manifold admits a complex structure.

We get the <u>cotangent bundle</u>

$$T^*X$$

of any C^∞ manifold X

as the <u>dual</u> of its <u>tangent bundle</u> TX.

Bidegrees of differential forms on a complex manifold

$$\mathbb{C}T^*X = \mathrm{Hom}(\mathbb{C}TX \longrightarrow \mathbb{C})$$

complexified cotangent bundle

$$= T^*X \oplus \overline{T^*X}$$

$$= \Lambda^{1,0} T^*X \oplus \Lambda^{0,1} T^*X$$

When

$$T^*X = \{ \ell : TX \longrightarrow \mathbb{C} \;/\; \mathbb{C}\text{-linear} \}$$

$$\simeq (T^{1,0}X)^* = \Lambda^{1,0} T^*X$$

$$\overline{T}^*X = \{ \ell : TX \to \mathbb{C} \,/\, \mathbb{C}\text{-antilinear} \}$$

$$\cong (T^{0,1}X)^* = \Lambda^{0,1} T^* X$$

Terminology

(a) The elements of $\Lambda^{1,0} T^* X$ are called

<u>(1,0)-forms on X</u>.

<u>Locally</u> $u = \sum_{j=1}^{n} \underbrace{u_j}_{\uparrow} \underbrace{dz_j}_{\substack{\shortparallel \\ dx_j + i\, dy_j}}$ (1,0)-forms

functions

$$\Lambda^{1,0} T^* X = \mathbb{C}\langle dz_1, \ldots, dz_n \rangle$$

$$T^{1,0} X = \mathbb{C}\left\langle \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \right\rangle$$

(b) The elements of $\Lambda^{0,1}T^{*}X$

$$= \mathbb{C}\langle \underbrace{d\bar z_1}_{\substack{\shortparallel \\ dx_1 - i\,dy_1}}, \dots, \underbrace{d\bar z_n}_{\substack{\shortparallel \\ dx_n - i\,dy_n}} \rangle$$

are called <u>(0,1)-forms</u> on X.

<u>Locally</u> $u = \sum_{j=1}^{n} \underbrace{u_j}_{C^\infty \text{ functions}} d\bar z_j$

Moreover, $\forall k \in \{0, 1, \dots, 2n\}$,

$$\Lambda^k(\mathbb{C}\otimes TX)^{*} = \bigoplus_{p+q} \Lambda^{p,q}T^{*}X$$

$$\Lambda^{p,q}T^{*}X = \mathbb{C}\langle \underbrace{dz_{i_1}\wedge\dots\wedge dz_{i_p}}_{dz_I}\wedge d\bar z_{j_1}\wedge\dots\wedge d\bar z_{j_q}\rangle$$

the vector bundle of k-forms on X

where $I = (1 \leq i_1 < \dots < i_p \leq n)$

is a p-multi-index

$$K = (1 \leq k_1 < \dots < k_q \leq n)$$

$$\Lambda^{p,q} T^\alpha X = \mathbb{C} \langle dz_I \wedge d\bar{z}_K \rangle$$

Examples

1) $\forall f : X \longrightarrow \mathbb{R}$ is a C^1 function

(0-form)

then

the partial derivative of f in the x_j-direction

$$\underbrace{df}_{\text{1-form}} \underset{\substack{\uparrow \\ \text{in real coordinates}}}{=} \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j + \sum_{j=1}^n \frac{\partial f}{\partial y_j} dy_j$$

$$\overline{\partial}\,\text{?}\ \ \underbrace{\sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j}_{=\,\partial f} + \underbrace{\sum_{j=1}^{n} \frac{\partial f}{\partial \bar z_j} d\bar z_j}_{=\,\bar\partial f}$$

in complex coordinates

where
$$\begin{cases} \dfrac{\partial f}{\partial z_j} := \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x_j} - i\,\dfrac{\partial f}{\partial y_j}\right) \\[2ex] \dfrac{\partial f}{\partial \bar z_j} := \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x_j} + i\,\dfrac{\partial f}{\partial y_j}\right) \end{cases}$$

2) Let u be a $\underline{(p,q)\text{-form}}$ on X

(it belongs to $\Lambda^{p,q} T^\alpha X$)

Locally: $u = \sum_{\substack{|I|=p \\ |J|=q}} \underbrace{u_{IJ}}_{\text{function}} dz_I \wedge d\bar z_J$

We have:

$$\underbrace{\partial u}_{(p+1,q)\text{-form}} := \sum_{\substack{|I|=p \\ |J|=q}} \underbrace{\underbrace{\partial u_{IJ}}_{(1,0)\text{-form}} \wedge \underbrace{dz_I}_{(p,0)}}_{(p+1,0)} \wedge \underbrace{d\bar{z}_J}_{(0,q)}$$

$$\underbrace{\bar{\partial} u}_{(p,q+1)\text{-form.}} := \sum_{\substack{|I|=p \\ |J|=q}} \underbrace{\bar{\partial} u_{IJ}}_{(0,1)\text{-form}} \wedge \underbrace{dz_I}_{(p,0)} \wedge \underbrace{d\bar{z}_J}_{(0,q)}$$

<u>Obs</u> (trivial)

$$\left\{ \begin{array}{l} \partial^2 = 0 \\ \bar{\partial}^2 = 0 \end{array} \right\} \Leftarrow \boxed{d^2 = 0}$$

$$\boxed{d = \partial + \bar{\partial}}$$