CHAPTER 1 : Basics on holomorphic functions

$\mathbb{C}^n \ni (z_1, \ldots, z_n)$ the standard coordinates

$z_j = x_j + i y_j$, $j = 1, \ldots, n$

$(x_1, \ldots, x_n, y_1, \ldots, y_n)$ real coordinates on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$

<u>Def</u> Let $\Omega \subset \mathbb{C}^n$ open

$$f : \Omega \longrightarrow \mathbb{C}.$$

f is <u>holomorphic</u> $\overset{def}{\Longleftrightarrow}$ f is <u>continuous</u> and <u>separately holomorphic</u>:

$\forall j$, $z_j \longmapsto f(z_1, \ldots, z_{j-1}, \underline{z_j}, z_{j+1}, \ldots, z_n)$

is <u>holomorphic</u>.

Let $\Delta(c, R) := D(c_1, R_1) \times \ldots \times D(c_n, R_n)$

where $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$, $R = (R_1, \ldots, R_n)$

the polydisc of center c and multiradius R.

The <u>distinguished boundary</u>:

$$\Gamma(c,R) = \Gamma(c_1,R_1)\times \dots \times \Gamma(c_n,R_n)$$

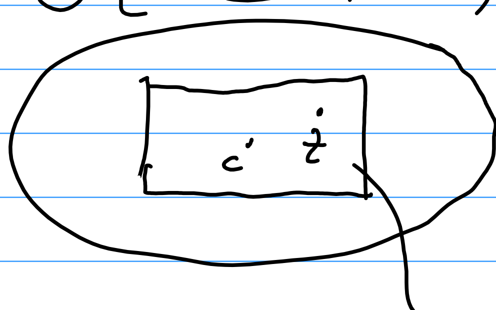
the product of the <u>circles</u>

$$\Gamma(c,R) \subsetneq \partial D(c,R) = \bigcup_j \{ z \in \overline{D(c,R)} \,/\, z_j \in \Gamma(c_j,R_j)\}$$

the topological boundary (of $\partial D(c,R)$)

Theorem (Cauchy formula)

$\Omega \subset \mathbb{C}^n$, open



$D(c,R)$ a polydisc

s.t. $\overline{D(c,R)} \subset \Omega$.

Let $f : \Omega \longrightarrow \mathbb{C}$, a holomorphic function.

Then: $\forall z \in D(c, R)$,

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma(c,R)} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \ldots d\zeta_n$$

Proof: induction on $n \geq 1$ using the Cauchy formula in one variable.

q.e.d.

Consequences of the Cauchy formula

1) Theorem (power series expansion)

Let $f : \Omega \longrightarrow \mathbb{C}$ holomorphic.

$\Omega \subseteq \mathbb{C}^n$

Let $\overline{\Delta(c,R)} \subset \Omega$ be a closed polydisc.

Then: $\forall z \in \Delta(c,R)$,

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z-c)^\alpha =$$

$$= \sum_{\alpha_1, \dots, \alpha_n = 0}^{+\infty} a_{\alpha_1 \dots \alpha_n} \underbrace{(z_1 - c_1)^{\alpha_1} \cdots (z_n - c_n)^{\alpha_n}}_{= (z-c)^\alpha},$$

Where

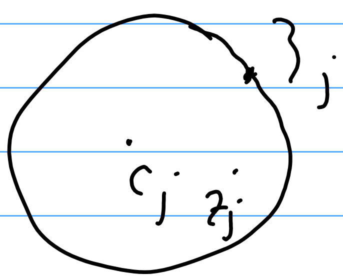
$$a_\alpha = \frac{f^{(\alpha)}(c)}{\alpha!} = \frac{1}{\alpha_1! \cdots \alpha_n!} \frac{\partial^{\alpha_1 + \dots + \alpha_n} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(c)$$

$$= \frac{1}{\alpha_1! \cdots \alpha_n!} \, \frac{1}{(2\pi i)^n} \int \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - c_1)^{\alpha_1+1} \cdots (\zeta_n - c_n)^{\alpha_n+1}} \, d\zeta_1 \dots d\zeta_n$$

Cauchy formula $\qquad (\zeta_1, \dots, \zeta_n) \in \Gamma(c, R)$

Sketch of proof

$$\frac{1}{\zeta_j - z_j} = \frac{1}{(\zeta_j - c_j) - (z_j - c_j)} = \frac{1}{\zeta_j - c_j} \, \frac{1}{1 - \frac{z_j - c_j}{\zeta_j - c_j}}$$



$$|z_j - c_j| < R_j = |\zeta_j - c_j|$$

$$\frac{|z_j - c_j|}{|\zeta_j - c_j|} < 1 \quad !!$$

We get:

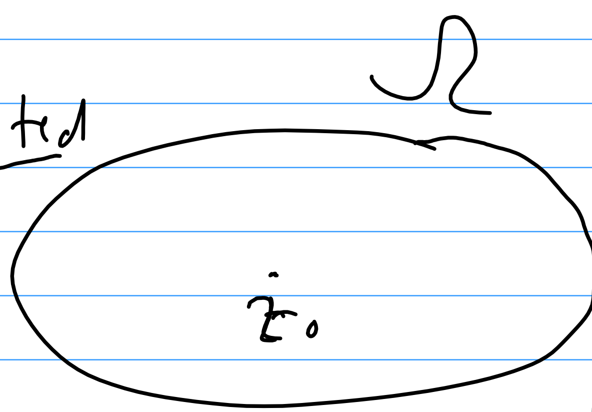
$$\frac{1}{\zeta_j - z_j} = \frac{1}{\zeta_j - c_j} \sum_{\alpha_j = 0}^{+\infty} \left(\frac{z_j - c_j}{\zeta_j - c_j} \right)^{\alpha_j}$$

The statement follows from this and from the Cauchy formula.

q.e.d.

Theorem (analytic continuation)

Let $\Omega \subset \mathbb{C}^n$ open, and $f : \Omega \to \mathbb{C}$ holomorphic.
(Ω connected)



Suppose $\exists\, z_0 \in \Omega$ s.t. $f^{(\alpha)}(z_0) = 0$

$\forall\, \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$

Then: $f(z) = 0 \quad \forall z \in \Omega$.

Proof Let

$$\Omega_0 := \{ z \in \Omega \mid f(z) = 0 \text{ and } f^{(\alpha)}(z) = 0 \quad \forall \alpha \in \mathbb{N}^n \} \subset \Omega.$$

Want to prove: $\Omega_0 = \Omega$.

We see that:

- $\Omega_0 \neq \emptyset$ since $z_0 \in \Omega_0$. (hypothesis)

- Ω_0 is open in Ω thanks to the series expansion property of f.

(Take a polydisc $\overline{\mathbb{D}(z_0, R)} \subset \Omega$ and we have $f^{(\alpha)}(z_0) = 0 \quad \forall \alpha \in \mathbb{N}^n$.)

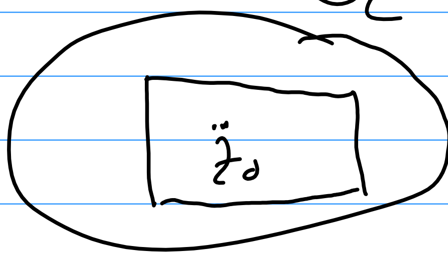
· Ω_α is closed in Ω.

(obvious, holds for any C^0 f)

q.e.d.

Theorem (the Cauchy inequalities)

$\Omega \subset \mathbb{C}^n$, open



Let $f: \Omega \to \mathbb{C}$ holomorphic function.

Then: $\forall z_0 \in \Omega$

$\forall \overline{\Delta(z_0, R)} \subset \Omega$, closed polydisc,

$\forall \alpha \in \mathbb{N}^n$,

$$\left| f^{(\alpha)}(z_0) \right| \leq \frac{\alpha!}{R^\alpha} \sup_{\Gamma(z_0, R)} |f|$$

Recall $R^\alpha := R_1^{\alpha_1} \cdots R_n^{\alpha_n}$.

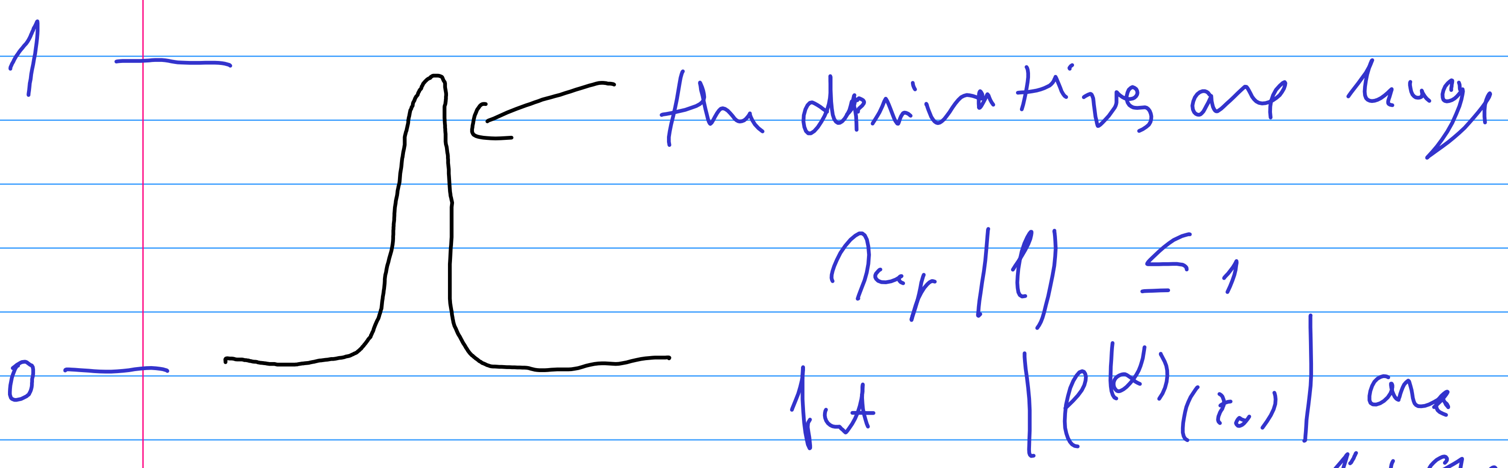
in particular, moving the center z_0, we get:

$$\sup_{D(z_0,r)} |f^{(\alpha)}| \leq \frac{\text{Const}}{C_{r,R}} \sup_{D(z_0,R)} |f|$$

the bigger polydisc

whenever $r_1 < R_1, \ldots, r_n < R_n$

in particular, if f is holomorphic, one cannot have



$\sup |f| \leq 1$

but $|f^{(\alpha)}(z_0)|$ are huge

Sketch of proof An immediate consequence

$$f^{(\alpha)}(z) = \frac{\alpha!}{(2\pi i)^n} \int \frac{f(\zeta_1, \dots, \zeta_n)\, d\zeta_1 \dots d\zeta_n}{(\zeta_1 - z_1)^{\alpha_1+1} \dots (\zeta_n - z_n)^{\alpha_n+1}}$$

↑ derivative in the $P(z_0, R)$

boundary of polydisc (?)

Hence:

$$\left| f^{(\alpha)}(z) \right| \leq \frac{\alpha!}{(2\pi)^n} \sup_{P(z_0,R)} |f| \frac{1}{R_1^{\alpha_1+1} \dots R_n^{\alpha_n+1}} R_1 \dots R_n (2\pi)^n$$

q.e.d.

The topology of the space

$$\mathcal{O}(\Omega) := \{ f : \Omega \to \mathbb{C} \,/\, f \text{ holomorphic} \}$$

$$\subset C^\infty(\Omega, \mathbb{C}) \subset C^0(\Omega, \mathbb{C})$$

C^∞ function — continuous function

· The topology of $C^\infty(\Omega ; \mathbb{C})$:

$f_j \xrightarrow[j \to +\infty]{} f$ in C^∞ $\overset{def}{\iff}$ $\forall K \subset \Omega$ compact

$f_j \xrightarrow[j \to +\infty]{} f$ uniformly on K

$\forall \alpha$, $D^\alpha f_j := f_j^{(\alpha)} \xrightarrow[j \to +\infty]{} D^\alpha f$ uniformly on K

the topology of uniform convergence on compacts of all the derivatives

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· Topology of $C^0(\Omega ; \mathbb{C})$: the topology of uniform convergence on compacts of the functions

$$f_j \xrightarrow[j\to+\infty]{C^0} f \overset{\text{def}}{\iff} \forall K \subset \Omega \text{ compact}$$

$$f_j \xrightarrow[j\to+\infty]{} f \text{ uniformly on } K$$

For C^∞ functions, the C^∞ convergence is strictly stronger than the C^0 convergence.

Corollary of the Cauchy inequalities

On $\mathcal{O}(\Omega)$, the C^∞ topology coincides with the C^0 topology.

We endow $\mathcal{O}(\Omega)$ with this ($C^\infty = C^0$) topology.

Theorem (Montel)

$\mathcal{O}(\Omega)$ has the Heine-Borel property:

the compacts of $\mathcal{O}(\Omega)$ coincide with

the subsets of $\mathcal{O}(\Omega)$ that are closed

and bounded

Reformulation of Montel

if $(f_j)_{j\in\mathbb{N}} \subset \mathcal{O}(\Omega)$ that is

uniformly bounded on the compacts

$K \subset \Omega$,

then $\exists$ $(f_{j_\nu})_{\nu\in\mathbb{N}}$ a convergent subsequence

<u>Proof</u> Cauchy inequalities imply that

$(f_j)_{j\in\mathcal{F}'}$ is <u>equicontinuous</u> on <u>compacts</u> $K\subset\Omega$

Then, apply <u>Ascoli</u> to conclude.

q.e.d.

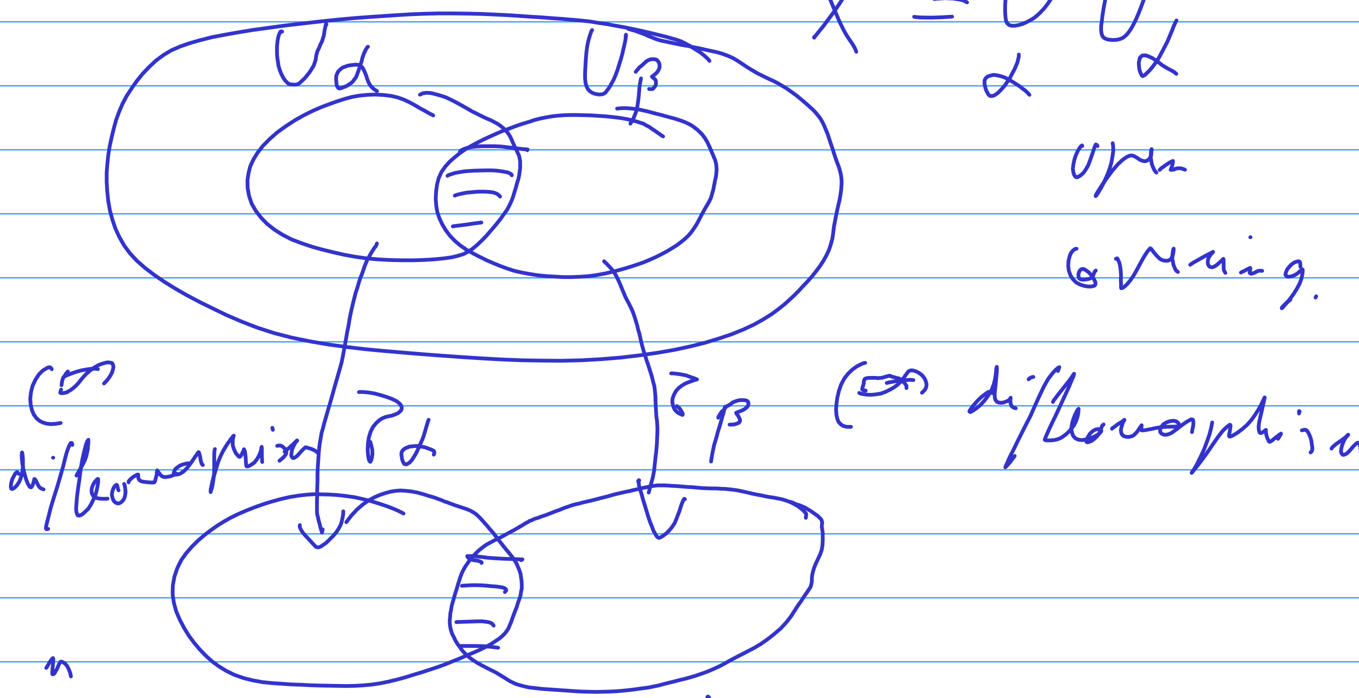
II <u>Complex manifolds and complex structures</u>

1

<u>Def</u> A <u>complex manifold</u> is a differentiable manifold whose changes of charts are <u>holomorphic</u>.

Explicitly let X be a C^∞ manifold. Let $(U_\alpha, \varphi_\alpha)$ be a C^∞ atlas on X.

$X = \bigcup_\alpha U_\alpha$ open covering.



$\mathbb{R}^{2n} \simeq \mathbb{C}^n \supset V_\alpha$ open $\qquad V_\beta \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$ open

$$\mathbb{C}^n \supset V_\alpha \cap V_\beta \xrightarrow{\varphi_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{\varphi_\beta} V_\alpha \cap V_\beta$$

$\varphi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1}$ $\underline{C^\infty \text{ diffeo}}$ $\cap \, \mathbb{C}^n$

$\tau_{\alpha\beta}$ = the change of chart from

(U_α, τ_α) to (U_β, τ_β)

We say that X is a complex manifold if $\tau_{\alpha\beta}$ is holomorphic $\forall \alpha, \beta$

(2) Second point of view: a complex manifold is a C^∞ manifold endowed with a complex structure.

Notion of complex structure

(a) Abstract definition of a complex structure

Def Let V be an $\mathbb{R}$-vector space s.t.

$\dim_{\mathbb{R}} V = 2n$, for some $n \in \mathbb{N}$.

A complex structure on V is any $\mathbb{R}$-linear map

$J : V \longrightarrow V$ s.t.

$$\boxed{J^2 = -\mathrm{Id}_V}$$

Intuitively J is the multiplication by i for complex numbers

Obs · J is an $\mathbb{R}$-automorphism of V

Proof J is *injective* because:

$$Ju = 0 \Rightarrow \underbrace{J^2}_{-1} u = 0 \Rightarrow u = 0.$$

$$\underbrace{}_{-u}$$

q.e.d.

Let (V, J) be an $\mathbb{R}$-vector space equipped with a complex structure,

$\dim_{\mathbb{R}} V = 2n$

Lemma $\exists$ an $\mathbb{R}$-basis

$$\{\underbrace{e_1, \ldots, e_n}, \underbrace{f_1, \ldots, f_n}\}$$

s.t. $\begin{cases} J e_k = f_k \\ J f_k = -e_k \end{cases}$, $\forall k = 1, n$

Proof. Let $e_1 \in V \setminus \{0\}$ arbitrary.

Put $\boxed{f_1 := J e_1}$

Then: $\underbrace{J f_1} = J^2(e_1) = \underbrace{-e_1}$

. We show that e_1 and f_1 are not $\mathbb{R}$-proportional.

Suppose the opposite: $\exists \lambda \in \mathbb{R}^*$ s.t.

$$\underbrace{f_1}_{= J e_1} = \lambda e_1 \Rightarrow \underbrace{J e_1}_{= -e_1} = \lambda \underbrace{J e_1}_{= f_1} = \lambda \underbrace{f_1}_{= \lambda^2 e_1}$$

$\Rightarrow \lambda^2 = -1$ impossible since $\lambda \in \mathbb{R}$.

Thus $\{e_1, f_1\}$ is a free family.

· Take any $e_2 \in V \setminus \mathbb{R}\langle e_1, f_1 \rangle$

Put $f_2 := J e_2$ and continue as above.

We are done after n iterations.

q.e.d.

Now, extend J to its complexification $\tilde{J}$

$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ $\quad$ $\mathbb{C}$ - vector space

$\dim_{\mathbb{C}} V_{\mathbb{C}} = 2n$

Define $\tilde{J} : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$

$$\tilde{J}(u+iv) \overset{def.}{=} Ju + iJv$$

$u, v \in V$

$\tilde{J}$ is $\mathbb{C}$-linear, by definition

Consider the vectors:

$$\begin{cases} u_j := \frac{1}{2}(e_j - i f_j) \in V_{\mathbb{C}} \\ v_j := \frac{1}{2}(e_j + i f_j) \in V_{\mathbb{C}} \end{cases} \quad \forall j = 1, \dots, n.$$

Obs $\tilde{J}^2 = -id_{V_{\mathbb{C}}}$ (obvious)

$\Downarrow$

$\tilde{J}$ has two eigenvalues : i and $-i$

each of <u>multiplicity n</u>

The corresponding <u>eigenspaces</u>:

$$V^{1,0} := \ker(\tilde{J} - i\,\mathrm{id}_{V_{\mathbb{C}}}) \qquad i$$

$$V^{0,1} := \ker(\tilde{J} + i\,\mathrm{id}_{V_{\mathbb{C}}}) \qquad -i$$

These are $\mathbb{C}$-vector spaces,

$$\dim_{\mathbb{C}} V^{1,0} = \dim_{\mathbb{C}} V^{0,1} = n$$

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

<u>Obs</u>
- $\tilde{J} u_j = i u_j \qquad \forall j = 1, \dots, n$
- $\tilde{J} v_j = -i v_j \qquad \forall j = 1, \dots, n$

We have:

$$\begin{cases} V^{1,0} = \mathbb{C}\langle u_1, \dots, u_n \rangle \\ V^{0,1} = \mathbb{C}\langle v_1, \dots, v_n \rangle \end{cases}$$

<u>Lemma</u> The map

$$V \xrightarrow{T^{1,0}} V^{1,0}$$

$\mathbb{C}$-vector space

$$\xi \longmapsto \frac{1}{2}(\xi - iJ\xi)$$

is an $\mathbb{R}$-isomorphism of $\mathbb{R}$-vector spaces.

We have:

$$\begin{cases} T^{1,0}(e_j) = u_j \\ T^{1,0}(f_j) = i u_j \end{cases}$$

$T^{1,0}$ equips V with a complex structure.

__Lemma__ The map

$$V \xrightarrow{T^{0,1}} V^{0,1}$$

$\mathbb{C}$-vector space

$$\xi \longmapsto \frac{1}{2}(\xi + iJ\xi)$$

is an __$\mathbb{R}$-isomorphism__ of $\mathbb{R}$-vector spaces

We have: $\begin{cases} T^{0,1}(e_j) = v_j \\ T^{0,1}(\ell_j) = -i v_j \end{cases}$

Again, $T^{0,1}$ equips V with a structure as a __$\mathbb{C}$-vector space__

We denote the resulting G-vector space by

$$\overline{V}$$