

KALKA : Degeneration openness of Kobayashi hyperbolicity (see Brody from p. 5 below)

+ BRODY : Reparametrisation Lemma + Degeneration Openness + Entire Curves
 HIGHER SCHOOL OF ECONOMICS
 NATIONAL RESEARCH UNIVERSITY
 LABORATORY OF ALGEBRAIC GEOMETRY AND ITS APPLICATIONS

- 1 - Definition $D_n := \{z \in \mathbb{C} / |z| < n\} \subset \mathbb{C}$
 $D := D_1$, $D^k := \{z \in \mathbb{C}^k / |z_i| < 1 \forall i=1, \dots, k\}$
 the unit polydisc

I PRELIMINARIES

Def (Kobayashi-Pogoda pseudo-metric)

* a complete warped, $m = \dim X$

= Pogoda's infinitesimal version of the Kobayashi metric.

$\forall (x, \xi) \in T_x^{lo} X$ (i.e. $x \in X$, $\xi \in T_x^{lo} X$), we put

$$F_x(x, \xi) := \inf_R \left\{ \frac{1}{R} \mid \exists f: D_R \xrightarrow{\text{holomorphic}} X, f(0) = x, \left(\frac{df}{dz} \right)_0 \left(\frac{d}{dz} \Big|_{z=0} \right) = \xi \right\}$$

$$\stackrel{''}{=} f'(0) = \frac{df}{dz}(0)$$

$$= \inf \left\{ \frac{1}{c} \mid \exists f: D_1 \xrightarrow{\text{holomorphic}} X, f(0) = x, \left(\frac{df}{dz} \right)_0 \left(\frac{d}{dz} \Big|_{z=0} \right) = c\xi \right\}$$

$$\stackrel{''}{=} f'(0) = \frac{df}{dz}(0)$$

Obs $\forall D_R \ni z = R\zeta$, with $\zeta \in D_1$, then: $\frac{d}{dz} = \frac{1}{R} \frac{d}{d\zeta}$

if $\xi = (\xi_1, \dots, \xi_m)$, then:

$$\frac{df}{dz}(0) = \left(\frac{df_1}{dz}(0), \dots, \frac{df_m}{dz}(0) \right)$$

$$= \frac{df_1}{dz}(0) \frac{2}{\partial z_1} + \dots + \frac{df_m}{dz}(0) \frac{2}{\partial z_m}$$

$$\left(\frac{df}{dz} \right)_0 \left(\frac{d}{dz} \Big|_{z=0} \right) = \frac{1}{R} \left(\frac{df}{d\zeta} \right)_0 \left(\frac{d}{d\zeta} \Big|_{\zeta=0} \right)$$

$$\stackrel{''}{=} \xi \Rightarrow \left(\frac{df}{d\zeta} \right)_0 \left(\frac{d}{d\zeta} \Big|_{\zeta=0} \right) = R\xi$$

where $(z_1, \dots, z_m)$ are local holomorphic coordinates on X

Properties of $F_x(n, \cdot)$ (Gjden 1971) - 2 -

(I)

(i) $F_x(n, q) \geq 0 \quad \forall q \in T_x^{lo} X$

(ii) $F_x(\lambda q) = |\lambda| F_x(n, q) \quad \forall \lambda \in \mathbb{C}$

$\Rightarrow F_x$ is a Fisher
pseudo-metric

Notation Fix a Hermitian metric ω on X .

let $S^{lo} X \rightarrow X$ be the holomorphic sphere bundle w.r.t. ω

Thus, F_x can be thought of as defined on $S^{lo} X$.

(iii) $\forall n \in \mathbb{N}, F_x(n, \cdot) : T_x^{lo} X \rightarrow [0, +\infty)$ is upper semi-continuous

(II) Suppose that X is tact $\left(\Rightarrow \right)$ $O(D, X) := \{ p : D \xrightarrow{hol} X \}$ is a

holoid family $\left(\begin{array}{l} \text{def} \\ = \text{pre-compact family} \\ = \text{relatively compact family} \end{array} \right)$

$\forall (p_j)_{j \in \mathbb{N}} \subset O(D, X) \quad \exists (p_{j_k})_{k \in \mathbb{N}}$

sequence

subsequence

s.t. $(p_{j_k})_{k \in \mathbb{N}}$ is either relatively compact

or compactly divergent.

s.t. $(p_j)_{j \in \mathbb{N}}$ converges uniformly on compacts

to some $f : D \xrightarrow{hol} X$

$\forall K \subset D$
compact

$\forall K' \subset X$
compact

$\exists \nu_0 \in \mathbb{N}$ s.t. $\{ p_{j_k}(K) \cap K' = \emptyset \}$
 $\forall \nu \geq \nu_0$

does not occur if
 X is a metric space



Then: (i) F_x is continuous on $T^{1,0}X$

$$(ii) \quad \forall n \in \mathbb{N}, \quad \boxed{F_x(n, \xi) = 0 \Leftrightarrow \xi = 0}$$

$$\forall \xi \in T^{1,0}_n X$$

(i) $\Rightarrow F_x$ is a metric on X if X is totally

Obs: (i) says that

X totally $\Rightarrow X$

hyperbolic

The converse is also

true (asserted, without proof, by Kato, (a) on p. 246)

Prop (Pyden 1979) F_x is the infinitesimal version of the

Nagasaki pseudo-metric ^(Nagasaki pseudo-distance) in the following sense: $\forall p, q \in X$,

$$d_x(p, q) = \inf_{\gamma \in P(p, q)} \int_0^1 F_x \left(\gamma(t), \underbrace{\left(\frac{d\gamma}{dt} \right)_t}_{\gamma'(t)} \right) dt,$$

where $P(p, q) := \{ \gamma: [0, 1] \rightarrow X \text{ piecewise } C^\infty \text{ curves s.t. } \gamma(0) = p, \gamma(1) = q \}$

Def Let X be a complex manifold

X is Nagasaki hyperbolic $\stackrel{\text{def}}{\Leftrightarrow} d_x$ is a distance

$\stackrel{\text{def}}{\Leftrightarrow} d_x(p, q) > 0 \quad \forall p, q \in X \text{ s.t. } p \neq q$

Obs X is holomorphic hyperbolic $\Leftrightarrow \forall (n_0, \rho_0) \in T^{1,0} X$ with $\rho_0 \neq 0$

$$\left\{ \begin{array}{l} \exists U \subset T^{1,0} X \setminus (X \times \{0\}) \\ \quad \cup \\ (n_0, \rho_0) \text{ open} \\ \exists C > 0 \text{ constant} \end{array} \right.$$

$$\text{n.t. } F_X(n, \rho) \geq C \quad \forall (n, \rho) \in U$$

(i.e. $F_X(n, \rho)$ is locally bounded below
by a positive constant as (n, ρ) varies
over $T^{1,0} X$ with $\rho \neq 0$)

Arzela-Ascoli Theorem (cf. Kobayashi's hyperbolicity book, Chapter 1, section 3)

X a locally compact, separable ^{topological} space

(T, d_T) locally compact metric space

let $\mathcal{F} \subset \mathcal{C}(X, T)$ $(= \{f: X \rightarrow T \mid f \text{ continuous}\})$ equipped with the
topology of uniform convergence on
compact = the compact-open
topology (because T is a metric space)

We have the equivalence:

$\mathcal{F}$ is relatively compact
in $\mathcal{C}(X, T)$

$\Leftrightarrow \left\{ \begin{array}{l} (a) \mathcal{F} \text{ is } \underline{\text{equicontinuous}} \text{ at every point } x \in X \\ (b) \forall n \in \mathbb{N}, \text{ the set} \end{array} \right.$

$\{f(x) \mid f \in \mathcal{F}\}$ is relatively
compact in T

(def $\Leftrightarrow \forall (f_n)_{n \in \mathbb{N}}$ sequence

$\exists (f_{n_k})_{k \in \mathbb{N}}$ subsequence

$\exists f \in \mathcal{C}(X, T)$

s.t. $f_{n_k} \xrightarrow[n \rightarrow \infty]{\text{unif. on compacts}} f$

obs if T is compact, (b) is
automatically satisfied.

II BRIDY (1978) : Degeneration operators of the Kobayashi hyperbolicity
+ extra things

① Basic notions

Let $D_n := \{z \in \mathbb{C} \mid |z| < n\} \subset \mathbb{C}$

$$\omega_n := \frac{n^4}{(n^2 - |z|^2)^2} i dz \wedge d\bar{z} = i \left(\frac{n^2}{n^2 - |z|^2} dz \right) \wedge \left(\frac{n^2}{n^2 - |z|^2} d\bar{z} \right)$$

$\Rightarrow$ (1,1)-form on D_n

Up to a constant, there is the Poincaré metric on D_n :

$$ds_n^2 = \gamma_n := \frac{4n^2}{(n^2 - |z|^2)^2} i dz \wedge d\bar{z}$$

$$\omega_n = \frac{n^2}{4} \gamma_n$$

The curvature of γ_n is -1 .

Hence, the curvature of ω_n is

$$-\frac{4}{n^2}$$

Obs / ω_n is complete on D_n .

$\omega_n(0) = i dz \wedge d\bar{z}$ is the Euclidean metric at $z=0$

ω_n is invariant under the automorphisms of D_n .

$$\omega_n\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \left|\frac{\partial}{\partial z}\right|_{\omega_n}^2 = \frac{n^4}{(n^2 - |z|^2)^2} \cdot \text{So, } \left|\frac{\partial}{\partial z}\right|_{\omega_n} = \frac{n^2}{n^2 - |z|^2} \text{ hence } \left|\frac{n^2 - |z|^2}{n^2} \frac{\partial}{\partial z}\right|_{\omega_n} = 1$$

Let: $D := D_1 \subset \mathbb{R}^2$

and $\omega := \omega_1$

the open unit disc

$\omega_1(p, q) \stackrel{\text{def}}{=} \text{the distance between points } p, q \in D_1 \text{ w.r.t. the metric } \omega_1.$

Def Let X be a connected complete manifold.

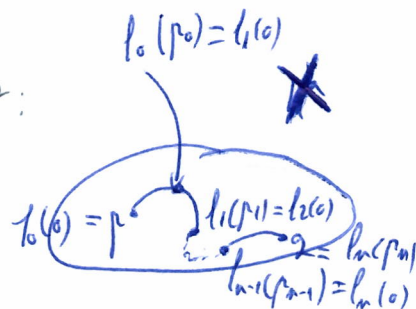
(a) Let $p, q \in X$, arbitrary points.

A chain $\{l_i\}$ connecting the points p, q is a finite collection of homeomorphic maps:

$$l_i: \Delta \rightarrow X$$

together with a collection of points $p_i \in \Delta$ such that:

$$\left\{ \begin{array}{l} l_0(o) = p, \quad l_0(p_0) = l_1(o) \\ \vdots \\ l_{n-1}(p_{n-1}) = l_n(o), \quad l_n(p_n) = q \end{array} \right.$$



(b) $\forall p, q \in X$, we define:

$$d_X(p, q) = \inf \sum_{i=0}^n \omega(o, p_i)$$

over all chains connecting p and q .

the Holroyd's pseudo-distance

(Remark: no metric is needed on X to define d_X)

(c) X is hyperbolic $\stackrel{\text{def}}{\Rightarrow} d_X$ is an actual metric. (distance)
 (i.e. $\forall p, q \in X$ s.t. $p \neq q$,
 $d_X(p, q) > 0$)

Proposition Let X, Y be connected complex manifolds

(a) $\forall F: X \rightarrow Y$ holomorphic map $\left\{ \begin{array}{l} \Rightarrow \{F \circ \gamma_i\}_i \text{ is a chain on } Y \\ \forall \{\gamma_i\} \text{ chain on } X \end{array} \right.$

(b) From (a), we get:

$$d_Y(F(p), F(q)) \leq d_X(p, q) \quad \forall p, q \in X$$

(i.e. any holomorphic map is distance decreasing)
 Kobayashi pseudo

Proof There are more chains in Y than in X moreover,
 $\text{len}(\text{biggest}) \leq \text{len}(\text{smallest})$

(c) From (b), we get:

$\forall F: X \rightarrow Y$ bi-holomorphic map, F is a (d_X, d_Y) -isometry.

(d) When $X = D$, $d_D = \omega$ (i.e. the Kobayashi metric of the disc coincides with the Poincaré metric ω).

Obs (d) follows from the Schwarz-Pick lemma:

$\forall f: D \rightarrow D$ holomorphic, we have:

$$\boxed{\omega(f(p), f(q)) \leq \omega(p, q)} \quad \forall p, q \in D.$$

$$(\Rightarrow \omega \leq d_D)$$

The reverse inequality ($d_D \leq \omega$) is obvious.

(e) ~~The~~ The Kobayashi pseudo-metric of the complex line $\mathbb{C}$ is identically 0.

$$\boxed{d_{\mathbb{C}} \equiv 0}$$

Proof $\forall p, q \in \mathbb{C}$ ~~the map~~

$\forall \varepsilon > 0$

$$\Delta \ni z \xrightarrow{\text{hol}} p + \frac{q-p}{\varepsilon} z \in \mathbb{C}$$

is such that

$$\left\{ \frac{0}{\varepsilon} \right\} \xrightarrow{\text{hol}} \frac{p}{\varepsilon} \quad \text{hence} \quad 0 \leq d_{\mathbb{C}}(p, q) \leq \omega(0, \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{g.l.d.}$$

Observation 1 (immediate consequence of (e))

Let X be a complex manifold.

if X contains a complex line (i.e. $\exists f: \mathbb{C} \rightarrow X$ non-constant holomorphic map)

then X is not hyperbolic.

Proof • f non-constant $\Leftrightarrow \exists z_1, z_2 \in \mathbb{C}$ s.t. $z_1 \neq z_2$ and $f(z_1) \neq f(z_2)$.

$$\text{Then: } \underbrace{d_X(f(z_1), f(z_2))}_{\text{points in } X} \leq d_{\mathbb{C}}(z_1, z_2) = 0$$

$\uparrow$
 $d_{\mathbb{C}} \equiv 0$

Hence, $d_X(f(z_1), f(z_2)) = 0$, so d_X is not a distance.

g.l.d.



-9-

Heads-up (see below) $\left\{ \begin{array}{l} \cdot \text{ if } X \text{ is compact, the converse also holds.} \\ \cdot \text{ if } X \text{ is not compact, the converse fails.} \end{array} \right.$

② Imp/interior formulation of hyperbolicity

let (X, γ) be a connected complex manifold with a Hermitian metric.
let $d_\gamma(p, q) =$ the distance w.r.t. γ between points $p, q \in X$.

Notation: $\forall f: D \xrightarrow{C^\infty} X$, put $f'(z_0) := \underbrace{\left(df_{z_0} \right) \left(\frac{\partial}{\partial z} \Big|_{z=z_0} \right)}_{\text{not } \parallel} \in T_{f(z_0)} X$
 $\forall z_0 \in D$,
 $\text{Hol}(D, X) := \left\{ f: D \xrightarrow{\text{holomorphic}} X \right\}$

Lemma 2 let (X, γ) be a compact connected complex Hermitian manifold.
Ob " $\Leftarrow$ " does not need X to be compact
" $\Rightarrow$ " needs X to be compact

Then: X is hyperbolic $\Leftrightarrow \sup_{f \in \text{Hol}(D, X)} |f'(0)|_\gamma < +\infty$

because D is homogeneous + ω is invariant

$$\sup_{f \in \text{Hol}(D, X)} |df_{z_0}|_{\omega, \gamma} = \sup_{f \in \text{Hol}(D, X)} |df_{z_0}|_{\omega, \gamma} = \sup_{f \in \text{Hol}(D, X)} \left| \left(df_{z_0} \right) \left(\frac{\partial}{\partial z} \Big|_{z=z_0} \right) \right|_\gamma$$

$\forall z \in D$ $\left| \frac{\partial}{\partial z} \right|_\omega = 1$

Proof " $\Leftarrow$ ". Suppose that $\sup_{f \in \text{Hol}(D, X)} \left| (d_0 f) \left(\frac{\partial}{\partial z} \Big|_{z=0} \right) \right|_Y = c < +\infty$.

$$\sup_{f \in \text{Hol}(D, X)} \left| (d_0 f) \left(\frac{\partial}{\partial z} \Big|_{z=0} \right) \right|_Y$$

$\forall a \in D$ but $\forall v \in T_a D$

D is homogeneous



$$= \left| (d_0 f) \left(\frac{\partial}{\partial z} \Big|_{z=0} \right) \right|_Y$$

$$\left| d_0 f \Big|_{w, Y} = \sup_{v \in T_w D} \left| (d_0 f)(v) \right|_Y$$

$$= \sup_{v \in T_0 D} \left| (d_0 f)(v) \right|_Y = c$$

$$|v|_W = 1$$

operator norm
Any such v is $v = \lambda \frac{\partial}{\partial z} \Big|_{z=0}$ with $\lambda \in \mathbb{C}$, $|\lambda| = 1$

$$\forall p \neq q \in X, \quad d_X(p, q) \geq \frac{1}{c} d_Y(p, q)$$

$$d_X(p, q) \geq \frac{1}{c} d_Y(p, q)$$

$$\sup_{f \in \text{Hol}(D, X)} \left| d_0 f \Big|_{w, Y} \right|$$

$$\text{if } \begin{cases} p = f(o) \\ q = f(p_1) \end{cases} \text{ then } d_Y(p, q) = d_Y(f(o), f(p_1)) \leq c d_D(o, p_1)$$

(with $p_1 \in D$)

$$\omega(o, p_1)$$

induced

$$\text{if } \begin{cases} p = f(o), & f_0(p_0) = f_1(o) \\ \vdots \\ f_{n-1}(p_{n-1}) = f_n(o), & f_n(p_n) = q \end{cases}$$

with $f_0, \dots, f_n : D \xrightarrow{\text{hol}} X$

and $p_0, \dots, p_n \in D$,

$$f_{n-1}(p_{n-1}) = f_n(o), \quad f_n(p_n) = q$$

$$\leq c \omega(o, p_0) \leq c \omega(o, p_1)$$

$$\leq c \omega(o, p_{n-1})$$

$$\text{we have: } d_Y(p, q) \leq d_Y(f_0(o), f_0(p_0)) + d_Y(f_1(o), f_1(p_1)) + \dots + d_Y(f_{n-1}(o), f_{n-1}(p_{n-1}))$$

$$+ d_Y(f_n(o), f_n(p_n))$$

$$\leq c \omega(o, p_n)$$

$$\text{So, } d_Y(p, q) \leq c \sum_{i=0}^n \omega(o, p_i)$$

$\forall \{f_i\}$ chain connecting p and q .

$$\text{Then, } d_Y(p, q) \leq c d_X(p, q)$$



-11-

• Since d_f is a genuine distance on X , $d_f(p, q) > 0 \quad \forall p \neq q \in X$

• We get: $d_X(p, q) \geq \frac{1}{2} d_f(p, q) > 0 \quad \forall p \neq q \in X$.

Thus, d_X is a distance on X , so X is hyperbolic.

" $\Rightarrow$ " Suppose X is hyperbolic.

By contradiction: suppose $\sup_{\{z \in \text{Hd}(D, X)\}} \left| (d \circ l) \left(\frac{\partial}{\partial z} \Big|_{z=0} \right) \right|_Y = +\infty$

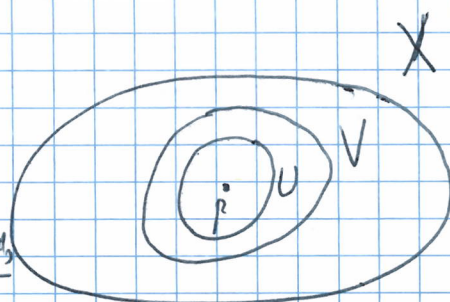
" $\| d \circ l \|_{w, Y}$ since $\left| \frac{\partial}{\partial z} \Big|_{z=0} \right|_w = 1$

$\Downarrow$
 $\exists (l_n)_{n \in \mathbb{N}^*} \subset \text{Hd}(D, X)$ s.t. $\left| (d \circ l_n) \left(\frac{\partial}{\partial z} \Big|_{z=0} \right) \right|_Y \xrightarrow{n \rightarrow +\infty} +\infty$

Since $\{l_n(0) \mid n \in \mathbb{N}^*\} \subset \underbrace{X}_{\text{compact}}$, we may assume that $\exists p \in X$ s.t.
(after passing to a subsequence)

$$\boxed{l_n(0) \xrightarrow{n \rightarrow +\infty} p \in X}$$

• Let U and $V \supset \bar{U}$ be two open coordinate neighborhoods of p .



The Cauchy inequalities satisfied by the holomorphic maps $f_n: D \rightarrow X$ imply:

$$\forall m \in \mathbb{N}^* \quad \exists n_m \in \mathbb{N} \text{ s.t. } (1) \quad \boxed{f_n(D_{\frac{1}{m}}) \cap \partial U \neq \emptyset} \quad \forall n \geq n_m$$

(indeed, $|d \circ f_n|_{\omega, \gamma} \leq \frac{1}{\frac{1}{m}} \sup_{z \in D_{\frac{1}{m}}} |f_n''| = m \sup_{z \in D_{\frac{1}{m}}} |f_n''(z)|_{\gamma}$)

$\downarrow_{n \rightarrow +\infty}$

$+\infty$

Therefore, $\forall m \in \mathbb{N}^*$ (fixed), $\sup_{z \in D_{\frac{1}{m}}} |f_n''(z)|_{\gamma} \xrightarrow{n \rightarrow +\infty} +\infty$

Since, $f_n(o) \xrightarrow{n \rightarrow +\infty} p \in X$ (no loss)

$\Rightarrow f_n(D_{\frac{1}{m}})$ gets arbitrarily large as $n \rightarrow +\infty$, hence it meets ∂U .

∂U is compact

(1) implies that

$$\forall m \in \mathbb{N}^* \quad \exists x_m \in f_{n_m}(D_{\frac{1}{m}}) \cap \partial U \quad \Rightarrow$$

after passing to a subsequence, we may assume that

$$\boxed{\exists x_m \xrightarrow{m \rightarrow +\infty} x \in \partial U}$$

We get:

$$(0 \leq) \quad \boxed{d_{\gamma}(f_{n_m}(o), x_m) \leq \omega(o, \frac{1}{m}) \xrightarrow{m \rightarrow +\infty} 0}$$

d_{γ} is continuous since X is hyperbolic

$\downarrow_{m \rightarrow +\infty}$

$(0 \leq) \quad d_{\gamma}(p, x)$



-13-

This implies: $d_x(p, \pi) = 0$

This contradicts the hyperbolicity assumption on x since

$$\partial U \ni \pi \neq p \in U$$

↑
open

2.e.d.

(3) Brody's Fundamental Lemma

~~Goal~~ Extract relatively compact (w.r.t. the topology of uniform convergence on compact)

families of maps from the less manageable family $Hd(D, X)$

idea for achieving this: get a bound on $|f'(z)|_Y$ over compact subsets

of D in terms of $|f'(0)|_Y$ ^{norm} $(\|df|_{w,y}\|)$ (or on $|d_z f|_{w,y}$)

Since such a bound can hold for arbitrary functions in $Hd(D, X)$, we will associate with every $f \in Hd(D, X)$ a correction $\tilde{f} \in Hd(D, X)$ that satisfies such a bound.

Preliminary (trivial) observation

the operator norm of $d_z f$

$$\forall f: D_n \xrightarrow{C^p} (X, Y), \quad \|d_z f\|_{w,y} = |f'(z)|_Y \left(\frac{\lambda^2 - |z|^2}{\lambda^2} \right)$$

$$\forall z \in D_n$$

Recall $f'(z) := (d_z f) \left(\frac{\partial}{\partial z} \Big|_z \right) \in T_{f(z)} X$

$\text{Proof } \left| \frac{d\phi}{dz} \right|_{\omega_{\lambda, \gamma}} = \sup_{v \neq 0} \frac{|(d_z \phi)(v)|_{\gamma}}{|v|_{\omega_{\lambda}}}$

$\mathbb{D} = \mathbb{T}^1 \setminus \{1\} \Rightarrow v \neq 0$

$\frac{|(d_z \phi)(v)|_{\gamma}}{|v|_{\omega_{\lambda}}} = \frac{|(d_z \phi)(\frac{\partial}{\partial z} z)|_{\gamma}}{|\frac{\partial}{\partial z} z|_{\omega_{\lambda}}}$

any v is of the slope $v = \lambda \frac{\partial}{\partial z} z$, with $\lambda \in \mathbb{C}$

Constant when v varies

$\uparrow \quad \left| \phi'(z) \right|_{\gamma} \left(\frac{\lambda^2 - |z|^2}{\lambda^2} \right)$

s.t.d.

$\left| \frac{\partial}{\partial z} z \right|_{\omega_{\lambda}} = \frac{\lambda^2}{\lambda^2 - |z|^2} \quad \forall z \in \mathbb{D}_{\lambda}$

Brody's Fundamental Lemma

Let (X, γ) be a complex Hermitian manifold. Let $\lambda > 0$ be a constant.

Then: $\forall \phi: \mathbb{D}_{\lambda} \xrightarrow{\text{hol}} X$

$\forall c \in \mathbb{R} \text{ s.t. } \left| \phi'(0) \right|_{\gamma} \geq c \geq 0$

$\exists \tilde{\phi}: \mathbb{D}_{\lambda} \xrightarrow{\text{hol}} X \text{ s.t.}$

$\sup_{z \in \mathbb{D}_{\lambda}} \left| \frac{d\tilde{\phi}}{dz} \right|_{\omega_{\lambda, \gamma}} = \left| \frac{d\tilde{\phi}}{dz} \right|_{\omega_{\lambda, \gamma}} = c$

$\left| \tilde{\phi}'(z) \right|_{\gamma} \left(\frac{\lambda^2 - |z|^2}{\lambda^2} \right) = \left| \tilde{\phi}'(0) \right|_{\gamma}$

Proof We will first arrange to have

$$\sup_{z \in D_n} \left| \tilde{f}'(z) \right|_r \left(\frac{n^2 - |z|^2}{n^2} \right) = c$$

and then we will force this supremum to occur at 0.

• $\forall t \in [0, 1]$, consider the map

$$\left\{ \begin{array}{l} f_t : D_n \rightarrow X \\ f_t(z) := f(tz) \end{array} \right\} \quad \text{holomorphic} \quad (\text{a shrinking of } f)$$

$$\text{let } \lambda(t) := \sup_{z \in D_n} \left| f_t'(z) \right|_r \left(\frac{n^2 - |z|^2}{n^2} \right) \quad \forall t \in [0, 1]$$

$$\text{We have: } \left\{ \begin{array}{l} \lambda(0) = 0 \\ \uparrow \\ t_0 = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} \lambda(1) = \sup_{z \in D_n} \left| f'(z) \right|_r \left(\frac{n^2 - |z|^2}{n^2} \right) \geq \left| f'(0) \right|_r \geq c > 0 \\ \uparrow \\ f_1 = f \end{array} \right. \quad \text{by hypothesis}$$

$$\sup_{z \in D_n} \left| f_t'(z) \right|_r < +\infty \quad \forall 0 \leq t < 1$$

(because $f': D_n \xrightarrow{\text{hol}} X$ may be unbounded on D_n , but is bounded on any smaller disc)

This is why f_t is introduced.

But we may have $\sup_{z \in D_n} |f'_1(z)| = +\infty$, i.e. $f_1 = f'$

Since we also have

$$\frac{\lambda^2 - |z|^2}{\lambda^2} \xrightarrow{z \rightarrow \partial D_\lambda} 0,$$

we get: $\begin{cases} \cdot \lambda(t) < +\infty \quad \forall t \in [0, 1) \\ \cdot [0, 1) \ni t \mapsto \lambda(t) \text{ is continuous} \end{cases}$ [Actually, $\forall 0 \leq t_1 < t_2 < 1$, $\lambda(t_1) \leq \lambda(t_2)$]

We also have: $[0, 1] \ni t \mapsto \lambda(t)$ is continuous (even at $t=1$)

We may have: $\lambda(1) = +\infty$.

By continuity, we get: $\exists t_0 \in [0, 1]$ s.t. $\lambda(t_0) = c$.

Then, the map $f_t : D_\lambda \rightarrow \mathbb{C}$ satisfies the condition $\sup_{z \in D_\lambda} \underbrace{\left| f'_{t_0}(z) \right|_g \left(\frac{\lambda^2 - |z|^2}{\lambda^2} \right)}_{\lambda(t_0)} = c$.

• if $t_0 = 1$, we may take $\tilde{f} = f_{t_0} = f_1 = f$.

• if $t_0 < 1$, the supremum $\lambda(t_0)$ is attained at some point $z_0 \in \overline{D_{t_0, \lambda}} \subset D_\lambda$.

Let $L : D_\lambda \rightarrow D_\lambda$ a holomorphic automorphism s.t.

$$L(0) = z_0.$$

Set

$$\tilde{f} := f_{t_0} \circ L$$

$$\text{Since } \left| f'_{t_0}(z) \right|_g \left(\frac{\lambda^2 - |z|^2}{\lambda^2} \right) = \left| d_z f_{t_0} \right|_{\omega_\lambda, g} \quad \text{and since } L^* \omega_\lambda = \omega_\lambda$$



- 17 -

We conclude that the quantity

$$\left| f'_{t_0}(z) \right|_g \left(\frac{\lambda^2 - |z|^2}{\lambda^2} \right)$$

~~$$\left| \tilde{f}'(z) \right|_g \left(\frac{\lambda^2 - |z|^2}{\lambda^2} \right)$$~~

~~$$|z| \in \mathbb{D}_\lambda$$~~

is invariant under L

Thus, ~~the~~ $\left| \tilde{f}'(z) \right|_g \left(\frac{\lambda^2 - |z|^2}{\lambda^2} \right)$

attains its supremum at $z=0$. q.e.d.

Body of Brady's Fundamental Lemma

Body's Fundamental Lemma prompts one to consider the following classes of maps:

$$\forall \lambda > 0$$

$$\forall c > 0,$$

let $\mathcal{F}_{\lambda, c} := \left\{ f: \mathbb{D}_\lambda \xrightarrow{\text{holomorphic}} X \mid \sup_{z \in \mathbb{D}_\lambda} \left| \frac{df}{dz} \right|_{\omega_\lambda, g} = \left| \frac{df}{dz} \right|_{\omega_\lambda, g} = c \right\}$

$$\left| \frac{df}{dz} \right|_{\omega_\lambda, g} = \left| \frac{df}{dz} \right|_{\omega_\lambda, g} = c$$

$\forall (X, g)$ complex Hermitian manifold

$$\text{Hd}(\mathbb{D}_\lambda, X) \supset$$

$$\left| f'(z) \right|_g \left(\frac{\lambda^2 - |z|^2}{\lambda^2} \right) \quad \parallel \quad \left| f'(0) \right|_{\omega_\lambda, g}$$

if X is given different complex structures $(J_s)_{s \in S}$ forming a C^∞ family and an associated C^∞ family of Hermitian metrics $(g_s)_{s \in S}$,

We set:

$$\mathcal{F}_{\lambda, c, D} := \left\{ f: D_\lambda \xrightarrow{\text{hol}} X_D \mid \sup_{z \in D_\lambda} \left(\frac{|f'(z)|}{\omega_{\lambda, D}} = |d_o f|_{\omega_{\lambda, D}} = c \right) \right\} \quad \text{Chol}(D_\lambda, X_D)$$

$|f'(z)|_{D_\lambda} \left(\frac{\lambda^2 - |z|^2}{\lambda^2} \right) = c$

where $X_D := (X, J_D)$

Gallery of Brady's Fundamental lemma

let $(X_D, J_D) = (X, J_D, \gamma_D)_{D \in S}$ be a C^∞ Gallery of compact Complex Hermitian manifolds.

key assumption

Then: $\forall c \in [0, +\infty)$

$\forall \lambda > 0$

~~with $\lambda \rightarrow 0$ if $D=0$ is constant, we are given a sequence $(f_m)_{m \in \mathbb{N}}$ of $f_m \in \mathcal{F}_{\lambda, c, D_m}$ with $D_m \rightarrow 0$ as $m \rightarrow +\infty$~~

~~$\forall (f_m)_{m \in \mathbb{N}} \subset \mathcal{F}_{\lambda, c, D_m}$ with $S \ni D_m \rightarrow 0 \in S$ as $m \rightarrow +\infty$~~

$(S_0, f_m: D_\lambda \xrightarrow{\text{hol}} (X_{D_m}, \gamma_{D_m}) \quad \forall m \in \mathbb{N}^*)$

$\exists (f_{n_m})_{m \in \mathbb{N}^*}$ subsequence of $(f_m)_{m \in \mathbb{N}^*}$

$\exists f: D_\lambda \xrightarrow{\text{hol}} (X_0, \gamma_0)$

st. $f_{n_m} \xrightarrow[m \rightarrow +\infty]{\text{unif. on compacts}} f$

In other words The sets $\mathcal{F}_{\lambda, c, D}$ are relatively compact

Proof $\forall K \subset D_\lambda$
compact

$\exists a_K > 0$ constant st. $\frac{\lambda^2 - |z|^2}{\lambda^2} \geq a_K > 0 \quad \forall z \in K.$

Hence: $\forall K \subset D_\lambda$
compact

$\sup_{z \in K} |f'(z)|_{D_\lambda} \leq \frac{c}{a_K} \quad \forall f \in \mathcal{F}_{\lambda, c, D}$

-19-

Since the family of metrics $(\rho_n)_{n \in \mathbb{N}}$ is at least continuous and $X (=$ the underlying C^∞ manifold of the X_n 's) is compact, we get:

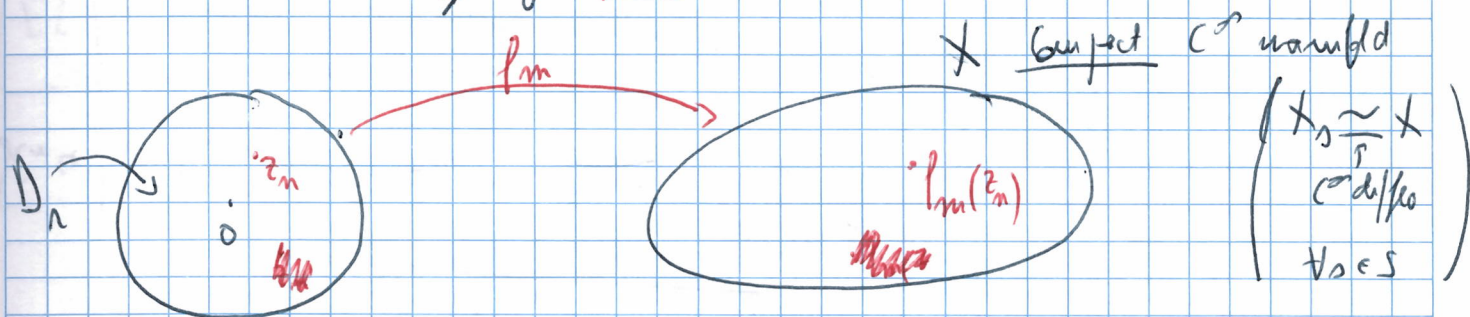
$$\sup_{z \in K} |f'(z)| \leq \frac{C'}{a_K} \quad \forall f \in \bar{F}_{1, C'} \quad (\text{for some constant } C' > 0)$$

$K \subset D_n$, compact $\forall n \gg 0$.

$\Downarrow$
the family $\bigcup_{n \in \mathbb{N}} \bar{F}_{1, C'}^{(n)}$ is equicontinuous on K .
a small neighbourhood of 0 in S

(The argument shows that $\sup_{z \in K} |df|_{z, \rho_0} \leq \frac{C''}{a_K} \quad \forall f \in \bar{F}_{1, C'}^{(n)} \quad \forall n \gg 0$)

$\Downarrow$
the ~~sequence~~ $(f_m)_{m \geq m_0}$ is equicontinuous on K .



Let $(z_n)_{n \in \mathbb{N}^*} \subset D_n$ be a dense sequence of points.

We have: $\forall n \in \mathbb{N}^*$, $(f_m(z_n))_{m \geq m_0} \subset X$ is a sequence of points in X .

X compact $\Rightarrow \exists (f_{m_k}(z_n))_{k \in \mathbb{N}}$ a convergent subsequence of points in X .

By a diagonal procen, we may assume that

$$\forall n \in \mathbb{N}^d, \left(f_m(z_n) \right)_{m \geq m_0} \text{ converges (in } X) \text{ as } m \rightarrow +\infty.$$

$$\text{So, } \forall n \in \mathbb{N}^d \exists f(z_n) \in X \text{ s.t.}$$

$$\boxed{f_m(z_n) \xrightarrow{m \rightarrow +\infty} f(z_n)}$$

$$\text{Since } \left\{ \begin{array}{l} (z_n)_{n \in \mathbb{N}^d} \subset D_1 \text{ is dense} \\ \text{and} \end{array} \right.$$

$$\left(f_m \right)_{m \geq m_0} \text{ is equicontinuous on } K \subset D_1 \text{ compact}$$

$$\text{the function } f: \left\{ z_n \mid n \in \mathbb{N}^d \right\} \longrightarrow X$$

$$\bigcap D_1$$

extends continuously to the whole of D_1 to yield

$$f: D_1 \xrightarrow{C^0} X$$

Indeed: let $z \in D_1$ $\xrightarrow[\text{density}]{d(z_n)_{n \in \mathbb{N}^d}}$

we may assume that $z_n \xrightarrow{m \rightarrow +\infty} z$

$$\implies f_m \text{ is continuous } \forall m \geq m_0$$

$$\forall m \geq m_0,$$

$$\boxed{f_m(z_n) \xrightarrow{m \rightarrow +\infty} f_m(z)}$$



-21-

$(l_m)_{m \geq m_0}$
 $\xrightarrow{\text{is}}$
equicontinuous
at z

$\forall \varepsilon > 0 \exists m_\varepsilon \in \mathbb{N}^d$ (independent of $m \geq m_0$) s.t.

$$d_{\gamma_0}(l_m(z), l_m(z_{n_i})) < \varepsilon \quad \forall m \geq m_\varepsilon$$

$$\forall m \geq m_0$$

We get: $\forall m_1, m_2$,

$$d_{\gamma_0}(l(z_{n_1}), l(z_{n_2})) \leq d_{\gamma_0}(l(z_{n_1}), l_m(z_{n_1})) + d_{\gamma_0}(l_m(z_{n_1}), l_m(z_{n_2})) + d_{\gamma_0}(l_m(z_{n_2}), l(z_{n_2}))$$

(1) $\forall m \geq m_\varepsilon$ $< \varepsilon$

(2) $\forall m \geq m_\varepsilon$ $< \varepsilon$

ELEMENTARY ARGUMENT

after choosing and fixing an $m \geq m_\varepsilon + \max\{m_\varepsilon(1), m_\varepsilon(2)\}$

We get: $\forall \varepsilon > 0 \exists m_\varepsilon \in \mathbb{N}^d$ s.t.

$$d_{\gamma_0}(l(z_{n_1}), l(z_{n_2})) < 4\varepsilon$$

$$\forall n_1, n_2 \geq m_\varepsilon$$

Cauchy and (X, γ)

Then, the sequence $(l(z_{n_i}))_{i \in \mathbb{N}} \subset (X, \gamma)$ is Cauchy $\Rightarrow$ the sequence

converges, hence complete.

$(f(z_n))_{n \in \mathbb{N}} \subset X$ is convergent. Denote its limit by $f(z) \in X$.

Since $z \in K$ ~~was~~ way arbitrary, we have ~~proved~~ extended f continuously to the whole of K .

Since $K \subset X$ is an arbitrary compact, we have extended f continuously to the whole of X .

This part can be avoided by involving the Arzelà-Ascoli Theorem (The above argument actually improves Arzelà-Ascoli.)
(see p. 4 above)

We get $f_m \xrightarrow{m \rightarrow \infty} f$ uniformly on compact sets

$\Downarrow$

f is holomorphic since the f_m 's are holomorphic.

q.e.d.

④ Deformation openness of the hyperbolicity property

Theorem 3 let $(X_s, Y_s)_{s \in S} = (X, Y, Y_s)_{s \in S}$ be a C^∞ family of compact complex Hermitian manifolds
(Bredy 1978)

let $D(s) := \sup_{f \in H^1(D, Y_s)} |f'(0)|_{Y_s}$, $\forall s \in S$.

$|df|_{\omega, Y_s}$

Then, the map $D: S \rightarrow (0, +\infty]$ is continuous.

Ob (trivial) We always have $D(s) > 0 \forall s \in S$.



- Remark {
- The lower semicontinuity of D was proved by Kalka (1976)
 - The upper semicontinuity of D was proved by Brady (1978)
- (see below)

Corollary of the upper semicontinuity of $D: S \rightarrow (0, +\infty]$

Under the hypotheses of Theorem 3, we have the implication:

$$X_0 \text{ is } \underline{\text{hyperbolic}} \Rightarrow X_s \text{ is } \underline{\text{hyperbolic}} \quad \forall s \sim 0.$$

Proof We know from Lemma 2 that: $\forall s \in S$,

$$X_s \text{ is } \underline{\text{hyperbolic}} \Leftrightarrow D(s) < +\infty$$

Suppose X_0 is hyperbolic. Then $D(0) < +\infty$.

$\Downarrow$

$$\exists a \in \mathbb{R} \text{ s.t. } D(0) < a < +\infty$$

$\Downarrow D$ is upper semicontinuous

$$D(s) < a < +\infty \quad \forall s \sim 0$$

$\Downarrow$ Lemma 2

$$X_s \text{ is } \underline{\text{hyperbolic}} \quad \forall s \sim 0$$

qed.

Proof of the upper semi continuity part of Theorem 3

Let $s_0 \in S$, arbitrary. We will prove that D is upper semi continuous at s_0 .

. if $D(s_0) = +\infty$, there is nothing to prove.

. Suppose that $D(s_0) < +\infty$. We need to prove the following thing: $\forall s_i \xrightarrow{S} s_0$,

$$\textcircled{1} \quad \left\{ \begin{array}{l} \forall +\infty > a > D(s_0) \quad \exists i_a \in \mathbb{N} \text{ s.t.} \\ a > D(s_i) \quad \forall i \geq i_a \end{array} \right.$$

To prove this, it suffices to prove the following thing:

$$\textcircled{2} \quad \left\{ \begin{array}{l} \forall s_i \xrightarrow{S} s_0 \text{ s.t. } D(s_i) \geq c \geq 0 \quad \forall i, \text{ we have: } D(s_0) \geq c. \\ \bigcap S \\ \forall c \in \mathbb{R} \end{array} \right.$$

$$\textcircled{2} \Rightarrow \textcircled{1}$$

Suppose $\textcircled{2}$ holds but $\textcircled{1}$ fails. Then:

$$\left. \begin{array}{l} \exists a \in \mathbb{R} \text{ s.t. } D(s_0) < a < +\infty \quad (\Rightarrow a > 0) \\ \exists i_n \xrightarrow{\mathbb{N}^2} +\infty \text{ s.t. } D(s_{i_n}) \geq a \quad \forall n \in \mathbb{N}^2 \end{array} \right\} \text{the failure of (1)}$$

$$\Downarrow \textcircled{2}$$

$$D(s_0) \geq a \quad \text{contradiction.}$$

Proof of $\textcircled{2}$ $\underbrace{D(s_i) \geq c \geq 0}_{\parallel} \forall i$ means:

$$\sup_{f \in \text{Hd}(D, X_{s_i})} |f'(0)|_{X_{s_i}}$$



-25-

Let $\varepsilon > 0$. From $\sup_{f \in H^1(D, x_{0,i})} |f'(0)|_{x_{0,i}} > c - \varepsilon > 0$, we get:

$\forall i, \exists f_i : D \xrightarrow{\text{hol}} x_{0,i}$ s.t. $|f_i'(0)|_{x_{0,i}} > c - \varepsilon > 0$.

$\Downarrow$ Brody's Fundamental Lemma

$\forall i, \exists \tilde{f}_i : D \xrightarrow{\text{hol}} x_{0,i}$ s.t.

$$\sup_{z \in D} |\tilde{f}_i'(z)|_{x_{0,i}} (1 - |z|^2) = |\tilde{f}_i'(0)|_{x_{0,i}} = c - \varepsilon$$

$\Downarrow$ Corollary of Brody's Fundamental Lemma

$\exists (\tilde{f}_{i_k})_k$ subsequence of $(\tilde{f}_i)_i$ s.t.

$$\tilde{f}_{i_k} \xrightarrow[k \rightarrow \infty]{} \tilde{f} \quad \text{Uniformly on compact}$$

Because map $\tilde{f} : D \xrightarrow{\text{hol}} x_{0,0}$ (because $\rho_{i_k} \xrightarrow[k \rightarrow \infty]{} \rho_0$)

$$\begin{aligned} &\Downarrow \\ &\tilde{f}_{i_k}' \xrightarrow[k \rightarrow \infty]{} \tilde{f}' \quad \text{Uniformly on compact} \\ &\Downarrow | \tilde{f}_{i_k}'(0) |_{x_{0,i_k}} = c - \varepsilon \quad \forall k \end{aligned}$$

$$|\tilde{f}'(0)|_{\gamma_{\lambda_0}} = c - \varepsilon$$

$$\Downarrow$$

$$\boxed{D(\lambda_0) \geq c - \varepsilon}$$

- since $\varepsilon > 0$ was arbitrary, we get $\boxed{D(\lambda_0) \geq c}$

This proves (2)

end

q.e.d.

⑤ Hyperbolicity and complex lines

Theorem 4 let X be a compact complex manifold.

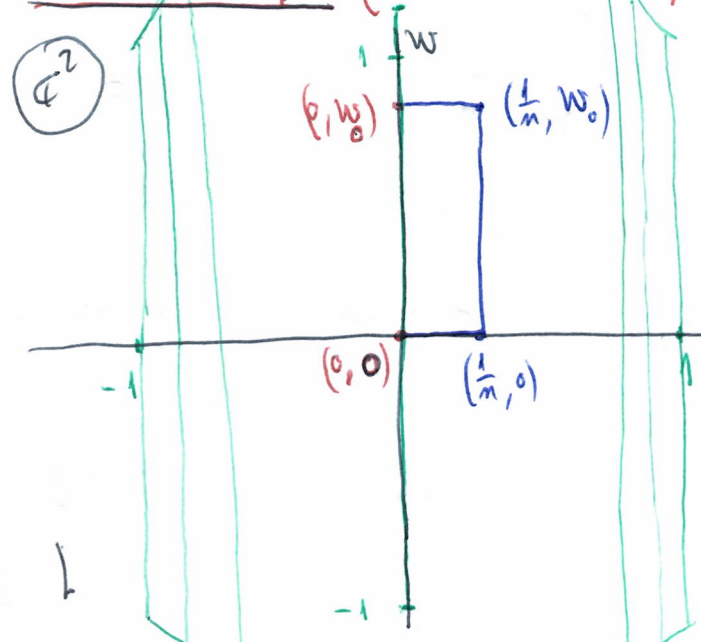
Then: X is hyperbolic $\Leftrightarrow \forall f: \mathbb{C} \xrightarrow{\text{holomorphic}} X, f = \text{const.}$

Obs (1) " $\Rightarrow$ " is always true (even if X is not compact)

See Obs 1 (it follows from $d_{\mathbb{C}} \equiv 0$ and $0 \leq d_X(f(z_1), f(z_2)) \leq d_{\mathbb{C}}(z_1, z_2) = 0$
 $\forall z_1, z_2 \in \mathbb{C}$)

(2) " $\Leftarrow$ " is false when X is non-compact

Counter-example (Eisenman-Taylor, or Kobayashi's book, p. 130)



Let $X := \left\{ (z, w) \in \mathbb{C}^2 \mid |z| < 1, |zw| < 1, \bigcap_{\mathbb{C}^2} |w| < 1 \text{ if } z=0 \right\}$



-27-

Let $\mathbb{C}^2 \xrightarrow{\pi_1} \mathbb{C} \ni z$ be the projection
 $\pi_2 \downarrow$

$w \in \mathbb{C}$

X does not contain any non-constant complex line

Included, suppose $\exists f: \mathbb{C} \xrightarrow{\text{hol.}} X \subset \mathbb{C}^2$

Then: $\pi_1 \circ f$ is a bounded entire function (because $|\pi_1(f(z))| < 1$ ~~for all~~ $\forall z \in \mathbb{C}$)

$$\boxed{|\pi_1 \circ f| = \text{const}} \quad (\Leftrightarrow \exists z_0 \in \mathbb{C} \text{ s.t. } f(\mathbb{C}) \subset \{z_0\} \times \mathbb{C})$$

$$\downarrow$$

$$\left\{ \begin{array}{l} |(\pi_2 \circ f)(z)| < \frac{1}{|z_0|} \quad \forall z \in \mathbb{C}, \text{ if } z_0 \neq 0 \\ < 1 \quad \forall z \in \mathbb{C}, \text{ if } z_0 = 0 \end{array} \right.$$

$\pi_2 \circ f: \mathbb{C} \xrightarrow{\text{hol.}} \mathbb{C}$ is a bounded entire function

$$\boxed{|\pi_2 \circ f| = \text{const}} \quad (\Leftrightarrow \exists w_0 \in \mathbb{C} \text{ s.t. } f(\mathbb{C}) \subset \mathbb{C} \times \{w_0\})$$

~~Therefore f is constant~~

We get: $f(\mathbb{C}) \subset (\{z_0\} \times \mathbb{C}) \cap (\mathbb{C} \times \{w_0\}) = \{(z_0, w_0)\}$

So, $f: \mathbb{C} \rightarrow X$ is constant.

X is not hyperbolic.

To see this, we notice that

$$(0,0) \neq (0,w_0) \in X \quad (\forall w_0 \in \mathbb{D}), \text{ but } d_X((0,0), (0,w_0)) = 0.$$

the Kobayashi pseudo-distance in X
between these points

Indeed, consider the connecting chain given by:

$$\left. \begin{array}{l} f_0: \mathbb{D} \longrightarrow X \\ z \longmapsto (z, 0) \end{array} \right\} \text{ with } p_0 = \frac{1}{n} \quad \text{so } \begin{cases} f_0(0) = (0, 0) \\ f_0(\frac{1}{n}) = (\frac{1}{n}, 0) \end{cases}$$

$$\left. \begin{array}{l} f_1: \mathbb{D} \longrightarrow X \\ z \longmapsto (\frac{1}{n}, nz) \end{array} \right\} \text{ with } p_1 = \frac{w_0}{n} \quad \text{so } \begin{cases} f_1(0) = (\frac{1}{n}, 0) \\ f_1(\frac{w_0}{n}) = (\frac{1}{n}, w_0) \end{cases}$$

$$\left. \begin{array}{l} f_2: \mathbb{D} \longrightarrow X \\ z \longmapsto (\frac{z}{2} + \frac{1}{n}, w_0) \end{array} \right\} \text{ with } p_2 = -\frac{2}{n} \quad \text{so } \begin{cases} f_2(0) = (\frac{1}{n}, w_0) \\ f_2(-\frac{2}{n}) = (0, w_0) \end{cases}$$

We get:

$$0 \leq d_X((0,0), (0,w_0)) \leq \omega\left(0, \left(\frac{1}{n}\right)\right) + \omega\left(0, \left(\frac{w_0}{n}\right)\right) + \omega\left(0, \left(-\frac{2}{n}\right)\right) \xrightarrow{n \rightarrow +\infty} 0$$

$$\begin{array}{c} \downarrow p_0 \\ \downarrow n \rightarrow +\infty \\ 0 \end{array}$$

$$\begin{array}{c} \downarrow p_1 \\ \downarrow n \rightarrow +\infty \\ 0 \end{array}$$

$$\begin{array}{c} \downarrow p_2 \\ \downarrow n \rightarrow +\infty \\ 0 \end{array}$$

because ω is
a distance

Hence: $d_X((0,0), (0,w_0)) = 0.$

q.e.d.



-29-

Proof of "⇐" of Theorem 4

Suppose that $\forall f: \mathbb{C} \xrightarrow{\text{hol}} X$, $\ell = \text{const}$.
We will prove that X is hyperbolic.

By contradiction: suppose X is not hyperbolic.

Let γ be a Hermitian metric on X . (arbitrary).

Lemma 2 + X non-hyperbolic $\Rightarrow \exists (f_i)_{i \in \mathbb{N}} \subset \text{Hol}(\mathbb{D}, X)$ n.t.

$$\left| f_i'(0) \right|_{\gamma} \nearrow_{i \rightarrow \infty} +\infty$$

$\Downarrow$ after rescaling

$$\forall i \in \mathbb{N}^* \exists f_i \in \text{Hol}(\mathbb{D}_{\lambda_i}, X) \text{ n.t. } \left\{ \begin{array}{l} |f_i'(0)|_{\gamma} = 1 \\ \lambda_i \nearrow_{i \rightarrow \infty} +\infty \end{array} \right.$$

$\Downarrow$ Brody's Fundamental Lemma

$$\forall i \in \mathbb{N}^*, \exists \tilde{f}_i \in \text{Hol}(\mathbb{D}_{\lambda_i}, X) \text{ n.t.}$$

$$\sup_{z \in \mathbb{D}_{\lambda_i}} |d_z \tilde{f}_i|_{\omega_{\lambda_i, \gamma}} = |d_0 \tilde{f}_i|_{\omega_{\lambda_i, \gamma}} = 1$$

$$\left(\begin{array}{l} (\Rightarrow) \tilde{f}_i \in \mathcal{F}_{\lambda_i, 1} \quad \forall i \in \mathbb{N}^* \\ \Downarrow \text{trivial} \\ \forall j > i, \tilde{f}_j|_{\mathbb{D}_{\lambda_i}} \in \mathcal{F}_{\lambda_i, 1} \\ (\lambda_j > \lambda_i) \end{array} \right)$$

Colloquy to

Brady's Fundamental
Lemma

• For $i=1$, $\mathcal{F}_{\lambda_1,1}$ is compact (this is where we need X to be compact)

Hence, the sequence

$$(\tilde{f}_i|_{D_{\lambda_1}})_{i \in \mathbb{N}^*} \subset \mathcal{F}_{\lambda_1,1}$$

has a subsequence

$$(\tilde{f}_{i_j}|_{D_{\lambda_1}})_{j \in \mathbb{N}^*}$$

$$\text{s.t. } \tilde{f}_{i_j} \xrightarrow{j \rightarrow \infty} h_1$$

$\cap$
 $\text{Hol}(D_{\lambda_1}, X)$

Uniformly on the compact of D_{λ_1}

• For $i=2$, $\mathcal{F}_{\lambda_2,1}$ is compact. Hence, the sequence $(\tilde{f}_{i_j}|_{D_{\lambda_2}})_{j \in \mathbb{N}^*}$
(Recall: $\lambda_i \xrightarrow{i \rightarrow \infty} \lambda_\infty$)

has a subsequence $(\tilde{f}_{i_{j_k}}|_{D_{\lambda_2}})_{k \in \mathbb{N}^*}$

s.t.

$$\tilde{f}_{i_{j_k}} \xrightarrow{k \rightarrow \infty} h_2 \in \text{Hol}(D_{\lambda_2}, X)$$

Uniformly on the compact of D_{λ_2}

We prove: $h_2|_{D_{\lambda_1}} = h_1$

• Continuing inductively, we get a sequence $(h_k)_{k \in \mathbb{N}^*}$ of maps

$$h_k \in \text{Hol}(D_{\lambda_k}, X) \text{ s.t.}$$

$$h_k|_{D_{\lambda_{k-1}}} = h_{k-1}$$

$\forall k \geq 2$



-31-

Hence, we get a map

$$h \in \text{Hd}(\mathbb{C}, X) \quad \text{s.t.}$$

$$\left(\text{because } \bigcup_{k=1}^{\infty} D_{\lambda_k} = \mathbb{C} \text{ as } \lambda_k \xrightarrow{h \rightarrow \infty} \infty \right)$$

$$\boxed{h|_{D_{\lambda_k}} = h_k} \quad \forall k \in \mathbb{N}^+$$

$$\left| d_0 \tilde{f}_i \right|_{w, \gamma} = 1 \quad \forall i \in \mathbb{N}^+$$

$$\left| \tilde{f}_i'(a) \right|_{\gamma}$$

$$h \text{ is } \underline{\text{non-constant}} \Leftrightarrow \boxed{|d_0 h|_{w, \gamma} = 1} \Leftrightarrow \left| d_0 h_k \right|_{w, \gamma} = 1 \quad \forall k \in \mathbb{N}^+$$

Thus, we have got a non-constant holomorphic map $h: \mathbb{C} \rightarrow X$.

This contradicts the hypothesis.

q.e.d.

III KALKRA

Goal: to prove the lower semicontinuity part (= the remaining part, after Body proved the other part) of Theorem 3. We will prove it in the following lemma

Theorem 5 (Kalkra 1976)

Let $(x_s)_{s \in S} = (x, \overset{\text{Hilb}}{p_s})_{s \in S}$ be a family of compact complex Hermitian manifolds, where $S \subset \mathbb{R}^k$ is some locally open set.

Suppose that x_0 is hyperbolic.

Then: $\forall \varepsilon > 0 \exists \underset{\substack{\varepsilon \\ \downarrow \\ 0}}{S'_\varepsilon} \subset S$ s.t.

$$\forall s \in S'_\varepsilon$$

$$\forall (x, \rho) \in \underbrace{S'^\circ_{\varepsilon} x_0}_{\text{total}} \subset T'^\circ x_0$$

$$F_{x_s}(x, \rho_s(\rho)) \leq F_{x_0}(x, \rho) + \varepsilon,$$

where $\rho_s: \mathbb{C}T_x \longrightarrow T'^\circ x_s$ is the ρ_s -projection onto $T'^\circ x_s$.

$x_0 \simeq \underbrace{x}_{\text{diff}} \quad s \in S$

the holomorphic sphere bundle w.r.t. some fixed Hermitian metric g_0 at $x_0 \rightarrow (x_0)_{s \in S}$ a family of Hermitian metrics at $(x_s)_{s \in S}$

This is the upper semicontinuity of F_{x_s} w.r.t. $s \in S$. ($\Rightarrow$ the lower

semicontinuity of $\frac{1}{F_{x_s}}$ = "the quantity

considered by Body" up to a small modification)



Theorem 5 will follow from two preliminary results (Theorems 6 and 7 below).

① Kalho's ^{best} preliminary results

Theorem 6 (Kalho 1976)

Let $(X_s)_{s \in S} = (X, \mathcal{I}_s, \varphi_s)_{s \in S}$ be a C^∞ family of ^{Hermitian} complex manifolds.

Then: $\forall f_0: \Delta \rightarrow X_0$ holomorphic map

$$\forall s \in S$$

$$\exists \mathcal{S}' = \mathcal{S}'(f_0) \subset \mathcal{S} \quad \text{d.f. } \forall s \in \mathcal{S}'$$

(depending on f_0 and n)

holomorphic map

s.t. (1) $f_s = f_0|_{\Delta_s}$ when $s = 0$

(2) f_s depends C^∞ on $s \in \mathcal{S}'$

(3) $f_s(0) = f_0(0)$, $\forall s \in \mathcal{S}'$

(4) $\underbrace{\begin{pmatrix} df_s \\ 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z} \big|_{z=0} \end{pmatrix}}_{f'_s(0)} = P_s \left(\underbrace{\begin{pmatrix} df_0 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z} \big|_{z=0} \end{pmatrix}}_{f'_0(0)} \right) \in T_0^{1,0} X_0$, $\forall s \in \mathcal{S}'$

if not

the main property of the deformed map f_s (i.e. the tangent vector at $z=0$ does not change) (Kalho)

Preparation to the proof of Theorem 6

Proposition 7 (General facts)

Let X be a compact hyperbolic complex manifold. Then:

(a) $\underbrace{Hd(D, X)}_{\text{the open unit disc in } \mathbb{C}} := \left\{ f: \underbrace{D}_{\text{the open unit disc in } \mathbb{C}} \xrightarrow{\text{hol}} X \right\}$ is compact (for the topology of convergence on compacts)

(b) $\forall (x, \xi) \in T^{1,0} X$,

$\exists f_{(x, \xi)}: \underbrace{D}_{\text{the open disc in } \mathbb{C} \text{ of radius } F_X(x, \xi)^{-1} \text{ centered at } 0}} \xrightarrow{\text{hol}} X$ with $\begin{cases} f(0) = x \\ \underbrace{f'(0)}_{\left(\frac{d}{dz} f \right) \Big|_{z=0}} = \xi \end{cases}$

i.e. the inf in $F_X(x, \xi)$ is a min

Proof (a). X compact hyperbolic $\Rightarrow \sup_{f \in Hd(D, X)} \|f'(0)\|_Y < +\infty$ (for some fixed Hermitian metric Y on X)

lemma 2



$Hd(D, X)$ is equicontinuous at $z=0$.

Fixing an arbitrary point $z \in D$ and having z play the role of 0 , we also have:

X compact hyperbolic $\Rightarrow \sup_{f \in Hd(D, X)} \left\| \frac{d}{dz} f \right\|_{w, Y} < +\infty$



-35-

Prove (y this) (as noticed after the statement of Lemma 2)

$$\sup_{f \in \text{Hol}(\mathbb{D}, X)} \|d_0\|_{w, \gamma} = \sup_{f \in \text{Hol}(\mathbb{D}, X)} \|d_f\|_{w, \gamma}$$

$$\forall z \in \mathbb{D}$$

because $\mathbb{D}$ is homogeneous and w is invariant.



$\text{Hol}(\mathbb{D}, X)$ is equicontinuous at every $z \in \mathbb{D}$

(i.e. X is compact hyperbolic)

|| Arzela-Ascoli (+ compact)

$\text{Hol}(\mathbb{D}, X)$ is relatively compact in $\ell(\mathbb{D}, X)$

(i.e. $\text{Hol}(\mathbb{D}, X)$ is compact for the topology of $\ell(\mathbb{D}, X)$)

(b) follows easily from (a). (compactness of $\text{Hol}(\mathbb{D}, X)$)

Proposition 8 Let $(\Omega_s)_{s \in S}$ be a C^∞ family of complex manifolds s.t.

- Ω_0 is a strongly pseudo-convex domain with C^∞ boundary in a complex manifold X_0 .
- $\Omega_s = (\Omega_0, \mathcal{I}_s)$ $\forall s \in S$ s.t. $\mathcal{I}_s$ depends C^∞ on s and Ω_s has C^∞ boundary.

Then: $\forall \mathcal{I}_0 \in \mathcal{O}(\Omega_0) \cap C^\infty(\bar{\Omega}_0) := \left\{ \mathcal{I} : \Omega_0 \xrightarrow{\text{hol}} \mathbb{C}^n \mid \mathcal{I} \text{ extends } C^\infty \text{ to } \bar{\Omega}_0 \right\}$
 $\exists \mathcal{S}' = \mathcal{S}'(\mathcal{I}_0) \subset \mathcal{S}$ s.t. $\mathcal{S}'$ is open.

$\forall s \in \mathcal{S}'$, $\exists \mathcal{I}_s \in \mathcal{O}(\Omega_s) \cap C^\infty(\bar{\Omega}_s)$ s.t.

- $\mathcal{I}_s = \mathcal{I}_0$ when $s=0$
- $\|\mathcal{I}_s - \mathcal{I}_0\|_{W^k} \leq c(\mathcal{I}_0) \|\varphi(s)\|_{W^k} \quad \forall k \in \mathbb{N}$

the Solobov norm in any given C^∞ family of Hermitian metrics $(g_s)_{s \in S}$ on $(\Omega_s)_{s \in S}$.
 where $\varphi(s) \in C_{0,1}^\infty(\Omega_0, T^{1,0}\Omega_0)$ is the form defining the deformation $(\mathcal{I}_s)_{s \in S}$ of the complex structure $\mathcal{I}_0$.
 or a proper metric on $S \times \Omega_0$.

In other words if the complex structure of a strongly pseudo-convex domain with C^∞ boundary deforms C^∞ , any holomorphic function that is C^∞ up to the boundary deforms C^∞ in a C^∞ way to a holomorphic function that are C^∞ up to the boundary.



-37

Proof We apply Volkmann's solution of the $\bar{\partial}$ -Neumann problem in a strongly pseudoconvex domain to solve the equation:

$$\bar{\partial}_D u_D = \bar{\partial}_D \gamma_0, \quad D \in S.$$

We get a solution u_D s.t. $\|u_D\|_{W^h} \leq C_1 \|\bar{\partial}_D \gamma_0\|_{W^h}$ (Hörmander's given by Gehring)

Put: $\gamma_D := \gamma_0 - u_D, \quad D \in S.$

We have: $\bar{\partial}_D \gamma_D = \bar{\partial}_D \gamma_0 - \bar{\partial}_D u_D = 0$

and $\|\gamma_D - \gamma_0\|_{W^h} = \|u_D\|_{W^h} \leq C_1 \|\bar{\partial}_D \gamma_0\|_{W^h} \leq C_2 \|\gamma_0\|_{W^h} \|\varphi(s)\|_{W^h}$

$C(h, \gamma_0)$

|| not

$$\bar{\partial}_D = \sum_i \left(\frac{\partial}{\partial \bar{z}_i} + \sum_l \varphi_l^i(s) \frac{\partial}{\partial \bar{z}_l} \right) d\bar{z}_i$$

$\frac{\partial \gamma_0}{\partial \bar{z}_l}$ depends only on $\bar{z}$, which is ~~not~~ independent of $D \in S$ (no, we have no variable variables $(D, \bar{z}) \in S \times R_0$)

g.c.d.

Remark 9 (1) In Proposition 8, the same result holds when

$$z_0 \in H^0(\Omega_0, T^{1,0}\Omega_0)$$

$$\cap C^\infty(\bar{\Omega}_0, T^{1,0}\bar{\Omega}_0)$$

(i.e. z_0 is holomorphic vector field that is C^∞ up to the boundary.

In this case, we first project z_0 to $T^{1,0}\Omega_0$ and then proceed as in the above proof by solving the equation

$$\bar{\partial}_\Omega u_\Omega = \bar{\partial}_\Omega (P_\Omega z_0)$$

(2) If $\Omega \subset \mathbb{C}^n$ is a domain in $\mathbb{C}^n$, since $T^{1,0}\mathbb{C}^n$ is globally trivial, we can prescribe the value at a point for the deformed vector field z_Ω :

$$z_\Omega(0) = (P_\Omega z_0)(0)$$

by replacing u_Ω with $u_\Omega - u_\Omega(0)$.

↑
the constant solution. (The constant solution is well defined since $T^{1,0}\mathbb{C}^n$ is globally trivial.)

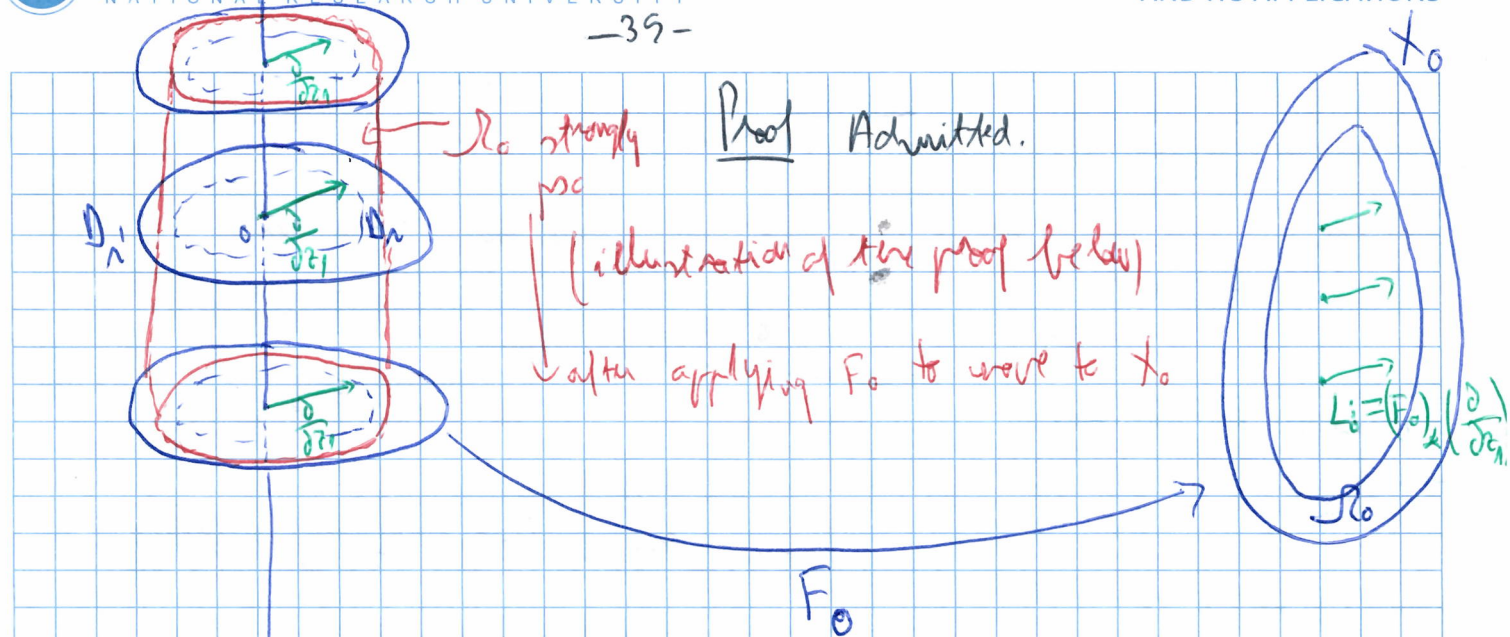
~~Now~~ Reyden's Theorem 10 (1974)

Let $f: D \hookrightarrow X$ a holomorphic embedding, $n = \dim_{\mathbb{C}} X$.

a complex manifold

Then: $\forall \epsilon > 0 \quad \exists F: D_n \times \Delta^{n-1} \hookrightarrow X$ a holomorphic embedding

s.t. $F|_{D_n \times \{0\}} = f|_{D_n}$.



Proof: admitted.

Proof of Theorem 6

- Fix of an embedding Suppose $f_0: D \hookrightarrow X_0$ is a holomorphic embedding.
Let $0 < \lambda < 1$. Fix an arbitrary λ' , s.t. $0 < \lambda < \lambda' < 1$.
- Apply Reydan's Theorem 10: $\exists F_0: D_{\lambda'} \times D^{n-1} \hookrightarrow X_0$ holomorphic embedding
s.t. $F_0|_{D_{\lambda'} \times \{0\}} = f_0|_{D_{\lambda'}}$
 $\Downarrow$ F_0 is an embedding
 $\exists \Omega_0$ a strongly pseudoconvex domain with C^∞ boundary s.t.
 $F_0(D_{\lambda'} \times D^{n-1}) \subset \Omega_0 \subset F_0(D_{\lambda'} \times D^{n-1}) \subset X_0$
- Consider the holomorphic vector field on Ω_0 :

$$L_0 := (F_0)_* \left(\frac{\partial}{\partial z_1} \right) \in H^0(\mathcal{R}_0, T^{1,0}\mathcal{R}_0)$$

an extension of $(f_0)_* \left(\frac{\partial}{\partial z} \right)$ from $f_0(D_1)$ to $\mathcal{R}_0$.

• Apply Proposition 8 + Remarks:

We need Poincaré to get a vector field on the open subset $\mathcal{R}_0$.

(no variation of the complex structure yet)

$\exists L_\delta \in H^0(\mathcal{R}_\delta, T^{1,0}\mathcal{R}_\delta)$ $(\eta_0, \bar{\partial}_\delta L_\delta = 0, \text{ i.e. } L_\delta \text{ is } \eta_\delta\text{-holomorphic})$

s.t.

- $L_\delta(f_0(0)) = P_\delta \left((d_0 f_0) \left(\frac{\partial}{\partial z} \Big|_0 \right) \right) \leftarrow \left(\text{prescribing the value at the point } f_0(0) \in X_0 \text{ for the deformed vector field } L_\delta \right)$
- $L_\delta = L_0 \text{ when } \delta = 0$
- $\|L_\delta - L_0\|_{W^k} \leq C_k \|\varphi(\delta)\|_{W^k}$

deforming the vector field L_0

Consider the system of ~~ODE's~~ ODE's on D :

$$\begin{aligned} (d_2 f_\delta) \left(\frac{\partial}{\partial z} \right) &= L_\delta(f_\delta(z)) \\ f_\delta(0) &= f_0(0) \end{aligned}$$

We construct the deformations f_δ of f_0 as "integral curves" of the deformed vector fields L_δ .

Since L_δ is close to L_0 and since the system $(\mathcal{A}_\delta)$ admits a ^{holomorphic} solution (f_0) for $\delta=0$, the standard theory of ODE's ensures the existence of a holomorphic solution of $(\mathcal{A}_\delta)$ for every $\delta \sim 0$.



-41-

General case Let $f_0: D \rightarrow X_0$ be an arbitrary holomorphic map.

let $0 < \lambda < 1$.

consider the graph map

$$\begin{aligned} g_{f_0}: D &\hookrightarrow D \times X_0 \\ z &\mapsto (z, f_0(z)) \end{aligned}$$

This is an embedding.

We apply the previous case (of an embedding) to g_{f_0} and the degeneration

$$(D \times X_0)_{s \in S} \text{ of } D \times X_0$$

(This degeneration is 0 in the first factor and the given degeneration in the second factor.)

We get maps $g_{f_0}: D_\lambda \xrightarrow{l_\lambda} D \times X_\lambda$, $\forall \lambda > 0$.

$$\begin{array}{ccc} & & \downarrow \mu_{X_\lambda} \\ M_{X_\lambda} \circ l_\lambda & =: & f_\lambda \end{array}$$

q.e.d.

② Kelkals second preliminary result

Theorem 11 (Kalka 1976: Unifgen version of Theorem 6 when X_0 is assumed compact hyperbolic)

Under the assumptions of Theorem 6, assume furthermore that X_0 is compact hyperbolic.

Then: $\exists \mathcal{S}' \subset \mathcal{S}$ (depending only on n and the degeneration)
 $\mathcal{S}'$ is open
 $\forall 0 < \epsilon < 1$,

such that $\forall f_s: D \rightarrow X_s$ holomorphic map

~~$f_s \in \mathcal{S}'$~~ $\forall s \in \mathcal{S}'$

$\exists f_0: D_n \rightarrow X_0$ holomorphic map s.t.

- $$\left\{ \begin{array}{l} (1) f_s = f_0|_{D_n} \text{ when } s=0 \\ (2) f_s \text{ depends on } s \in \mathcal{S}' \\ (3) f_s(0) = f_0(0), \quad \forall s \in \mathcal{S}' \\ (4) \underbrace{\left(d_0 f_s \right) \left(\frac{\partial}{\partial z} \Big|_{z=0} \right)}_{\text{not } f'_0(0)} = P_s \left(\underbrace{\left(d_0 f_0 \right) \left(\frac{\partial}{\partial z} \Big|_{z=0} \right)}_{\text{not } f'_0(0)} \right) \in T_0^{1,0} X_s, \quad \forall s \in \mathcal{S}' \end{array} \right.$$

Main gain in Theorem 11 compared to Theorem 6

$\mathcal{S}'$ is independent of f_0



Preliminaries to the proof of Theorem 11

Lemma 12 (Boyd 1974) cf. Boyd's Theorem 10)

Let $f: D \hookrightarrow X$ be a holomorphic embedding,
a complex manifold

Then: $\forall 0 < \lambda < 1 \quad \exists \gamma_{\lambda, n} \subset X$
open Stein submanifold

$$f(\bar{D}_\lambda) \subset \gamma_{\lambda, n}$$

s.t.

valid if
 X is
Stein

Proof: admitted

Idea: Use Boyd's Lemma 12 to improve on Boyd's Theorem 10 to make the extension map F depend continuously on $f \in \text{Hol}(D, X)$.

$$\text{Hol}(D_n \times D^{n'}, X)$$

Lemma 13 (Nolte 1976)

Let $f: D \hookrightarrow S$ be a holomorphic embedding, $n := \dim_{\mathbb{C}} S$,
a Stein manifold

Then: $\exists F: D \times D^{n-n'} \hookrightarrow S$ a holomorphic embedding

s.t. $F|_{D \times \{0\}} = f$ and F can be chosen to depend continuously on f .

Proof. We know from the general theory of Stein manifolds that

$$\exists H \in \mathbb{C}P^k$$

$$\exists \varphi: S \hookrightarrow \mathbb{C}^H$$

a holomorphic embedding s.t.

$\varphi(S)$ is a closed submanifold of $\mathbb{C}^H$.

Consider

$$\begin{array}{ccccc} D & \xrightarrow{f} & S & \xrightarrow{\varphi} & \mathbb{C}^H \\ \text{hol} & & \text{hol} & & \\ \text{embedding} & & \text{embedding} & & \end{array}$$

$\Downarrow$
 $\varphi \circ f$ hol embedding.

$$z \mapsto (\varphi \circ f)(z) \stackrel{\text{not}}{=} (f_1(z), \dots, f_H(z))$$

We will explicitly construct an extension for $\varphi \circ f$ (into $\mathbb{C}^N$).

$$\text{Let } D \times D^{n-1} = \{ (z, w_1, \dots, w_{n-1}) \in \mathbb{C}^n \mid |z| < 1, |w_j| < 1 \ \forall j=1, \dots, n-1 \} \subset \mathbb{C}^n$$

Put $\tilde{F}: D \times D^{n-1} \longrightarrow \mathbb{C}^N$

$$\tilde{F}(z, w_1, \dots, w_{n-1}) := \underbrace{(\varphi \circ f)(z)}_{(f_1(z), \dots, f_H(z))} + \sum_{i=1}^{n-1} w_i \cdot \underbrace{\left(0, \dots, 0, -f_{H+1}'(z), \dots, -f_N'(z) \right)}_{\substack{(i-1) \text{ slots} \\ \uparrow \\ i \text{th slot}}}$$

We want this to be $(\varphi \circ f)(z)$ when $(w_1, \dots, w_{n-1}) = (0, \dots, 0) \in \mathbb{C}^{n-1}$

$$= \underbrace{(\varphi \circ f)(z)}_{(f_1(z), \dots, f_H(z))} + w_1 \underbrace{(-f_{H+1}'(z), -f_{H+2}'(z), \dots, -f_N'(z))}_{\in \mathbb{C}^N} + w_2 \underbrace{(0, -f_{H+1}'(z), \dots, -f_N'(z))}_{\in \mathbb{C}^N} + \dots + w_{n-1} \underbrace{(0, \dots, 0, -f_{H+1}'(z), \dots, -f_N'(z))}_{\substack{(n-2) \text{ slots} \\ \uparrow \\ (n-1) \text{th slot}}}$$



-45-

We have: $\tilde{F}$ depends continuously on f (because $f_j \xrightarrow{j \rightarrow \infty} f \Rightarrow Df_j \xrightarrow{j \rightarrow \infty} Df$)
 $\cap$
 $\text{Hd}(0,5) \quad \text{Hd}(0,5) \quad \neq$
 $\tilde{F}$ is an embedding of $D \times \{ (w_1, \dots, w_{n-1}) \mid |w_1| < \varepsilon_1, \dots, |w_{n-1}| < \varepsilon_{n-1} \}$
 $\cap$
 $D_{\varepsilon_1} \times \dots \times D_{\varepsilon_{n-1}}$
 into $\mathbb{C}^N$

Indeed, we have:

$$\text{Jac } \tilde{F} = \begin{pmatrix} f_1'(z) - w_1 f_N''(z) \\ f_2'(z) - w_1 f_{N-1}''(z) - w_2 f_N''(z) \\ \vdots \\ f_N'(z) - w_1 f_1''(z) - w_2 f_2''(z) - \dots - w_{n-1} f_{n-1}''(z) - f_1'(z) \end{pmatrix}$$

$\uparrow \frac{\partial}{\partial z}$

$$\begin{pmatrix} -f_N'(z) & 0 & \dots & 0 \\ -f_{N-1}'(z) & \vdots & & \\ \vdots & 0 & & \\ -f_1'(z) & -f_N'(z) & \dots & -f_1'(z) \end{pmatrix}$$

$\uparrow \frac{\partial}{\partial w_1} \quad \dots \quad \frac{\partial}{\partial w_{n-1}}$

$N \times 1$ matrix $N \times (n-1)$ matrix

We get:

$$(\text{Jac } \tilde{F})(z, 0, \dots, 0) = \begin{pmatrix} f_1'(z) & -f_N'(z) & 0 & \dots & 0 \\ f_2'(z) & -f_{N-1}'(z) & -f_N'(z) & & \\ \vdots & \vdots & \vdots & & \\ f_N'(z) & -f_1'(z) & -f_2'(z) & \dots & -f_{n-1}'(z) \end{pmatrix}$$

$\left. \begin{matrix} (n-2) \text{ rows} \\ (n-1)^{\text{th}} \text{ row} \\ N^{\text{th}} \text{ row} \end{matrix} \right\}$

For example, when $N=3$, we have:

$$\det(\text{Jac } \tilde{F})(z, 0, \dots, 0) = \begin{vmatrix} l_1'(z) & -l_3'(z) & 0 \\ l_2'(z) & -l_2'(z) & -l_3'(z) \\ l_3'(z) & -l_1'(z) & -l_2'(z) \end{vmatrix} = l_1'(z)(l_2'(z))^2 + (l_3'(z))^3 - (l_2'(z))^2 l_3'(z) - (l_1'(z))^2 l_3'(z)$$

$$= (l_2')^2 (l_1' - l_3') + l_3' (l_3' - l_1') (l_3' + l_1')$$

$$= (l_1' - l_3') [(l_2')^2 - (l_3')^2 - l_1' l_3']$$

For $\epsilon_1, \dots, \epsilon_{m-1} \ll 1$, $\tilde{F}(\mathbb{D} \times \underbrace{\mathbb{D}_{\epsilon_1} \times \dots \times \mathbb{D}_{\epsilon_{m-1}}}_{\epsilon_{m-1}})$ can be made to lie in an arbitrarily prescribed neighborhood of $\mathcal{C}(S)$ in $\mathbb{C}^t$ (divides)

Now, by the holomorphic retraction theorem (Gunning-Petri book):

$\exists \beta: U \rightarrow \mathcal{C}(S)$ holomorphic retraction (So, β has maximal rank.)

$$\begin{array}{ccccc} \bigcap_{\mathbb{C}^N} & & & & \\ & \tilde{F} & \rightarrow & U & \xrightarrow{\beta} & \mathcal{C}(S) \\ \underbrace{\mathbb{D} \times \underbrace{\mathbb{D}_{\epsilon_1} \times \dots \times \mathbb{D}_{\epsilon_{m-1}}}_{\epsilon_{m-1}}}_{\bigcap_{\mathbb{C}^m}} & & & \bigcap_{\mathbb{C}^N} & & \bigcap_{\mathbb{C}^N} \end{array}$$

Let $\boxed{F := \beta \circ \tilde{F}}$

We make a change of scale in the w-variable.

i.e. d.

We can solve

$$w^i = w^i(z, w^j), \quad j \geq m$$

by the implicit function theorem



-47-

Lemma 14 (Helmer 1976: the first uniformity result, a kind of uniform improvement of Fogden's lemma 12 in the case of compact hyperbolic manifolds)

Let X be a compact hyperbolic complex manifold.
 $0 < \lambda < 1$

Then: $\exists \lambda < +\infty, \exists S_{1,\lambda}, \dots, S_{h,\lambda} \subset D \times X$
 $\bigcap_{i=1}^h S_{i,\lambda}$ open Stein submanifolds of $D \times X$

s.t. $\forall f: D \rightarrow X$ holomorphic map, $\exists i \in \{1, \dots, h\}$ s.t.

$$\text{Graph}(f|_{D_\lambda}) \subset S_{i,\lambda}$$

Proof: $G(\text{Hol}(D, X)) \subset \text{Hol}(D, D \times X)$

the space of graphs of maps $f \in \text{Hol}(D, X)$ $\xrightarrow{\sim}$ $\text{Hol}(D, X)$ homeomorphic compact because X is compact hyperbolic (see Proposition 7)

$\forall f \in \text{Hol}(D, X)$, let $\begin{cases} g_f: D \rightarrow D \times X \\ z \mapsto (z, f(z)) \end{cases}$ a holomorphic embedding

We have: $g_f(D) = \text{Graph}(f)$

By Ryden's Lemma 12 applied to each $g_f: D \hookrightarrow D \times X$, hol embedding, we get.

$$\forall f: D \xrightarrow{\text{hol}} X \quad \exists S_{f,n} \subset D \times X \quad \text{s.t.} \\ \text{open Stein submanifold}$$

$$\boxed{\begin{array}{c} g_f(\overline{D}_n) \subset S_{f,n} \\ \text{"} \\ \text{Graph}(f|_{\overline{D}_n}) \end{array}}$$

Now, $\forall f: D \xrightarrow{\text{hol}} X$, let

$$U_f := \left\{ g \in \text{Hd}(D, X) \mid \text{Graph}(g|_{\overline{D}_n}) \subset S_{f,n} \right\} \subset \text{Hd}(D, X) \\ \text{open}$$

On the other hand,

$$\text{Hd}(D, X) = \bigcup_{f \in \text{Hd}(D, X)} U_f, \quad \{U_f\}_{f \in \text{Hd}(D, X)} \text{ is an open cover of } \text{Hd}(D, X) \\ \text{compact}$$

$$\Downarrow \\ \exists h \in \mathbb{H}^d, \quad \exists f_1, \dots, f_h \in \text{Hd}(D, X) \text{ s.t.}$$

$$\boxed{\text{Hd}(D, X) = U_{f_1} \cup \dots \cup U_{f_h}}$$

(finite subcover)

$$\text{We put: } \begin{cases} S_{1,n} := S_{f_1,n} \\ \vdots \\ S_{h,n} := S_{f_h,n} \end{cases}$$

I.e.d.



$S = [0, 1] \subset \mathbb{R}$ (the def. 1.1.1)

First $0 < \alpha < 1$ - 49-

Use any answer that ~~does not work~~

Proof of Theorem 11

Apply Lemma 14 to $X = X_0$

Compact hyperbolic

||

$\exists h \in \mathbb{N}^k, \exists S_{i, n'}, \dots, S_{h, n'} \subset D \times X_0$

open Stein submanifolds of $D \times X_0$

s.t. $\forall f: D \rightarrow X_0$ holomorphic map, $\exists i \in \{1, \dots, h\}$ s.t.

$$\text{Graph}(f|_{D_{n'}}) \subset S_{i, n'}$$

let $\Delta_{i, n'}(f_0) := \sup \{0 < \delta \leq 1 \mid f_0 \text{ can be deformed to } f_\delta: D_n \rightarrow X_0\}$

if we take the extension F_0 of g_{f_0} into $S_{i, n'}$

Lemma 13 + method of proof of Thm 6

$\Delta_{i, n'}$ is a continuous function on those maps whose graphs lie in $S_{i, n'}$.

the class $\mathcal{F}_{i, n'}$ of maps

Define the function

$$\Delta_n: \text{Hol}(D, X_0) \rightarrow (0, 1]$$

$$\Delta_n(f_0) := \min_i \Delta_{i, n'}(f_0) \quad \text{if } f_0 \in \mathcal{F}_{i, n'}$$

Lemma
Semi-Continuity

$$\forall (f_j)_{j \in \mathbb{N}} \subset \bigcap_{i=1}^h \mathcal{F}_{i, n'} \text{ s.t. } f_j \xrightarrow{j \rightarrow \infty} f \in \bigcap_{i=1}^h \mathcal{F}_{i, n'}, \text{ then we have:}$$

$$\rho_n(l) \leq \lim_{j \rightarrow \infty} \rho_n(l_j)$$

$\Downarrow$

$$\boxed{\rho_n(l_i) \geq c > 0}$$

indeed, suppose that $\rho_n(l_{i_j}) \xrightarrow{j \rightarrow \infty} 0$ (choose sequence $(l_{i_j})_{j \in \mathbb{N}} \subset \text{Hd}(0, x_0)$)

$\Downarrow \text{Hd}(0, x_0) \text{ compact}$

we may assume that

$$\exists l_{i_j} \xrightarrow{j \rightarrow \infty} l \in \text{Hd}(0, x_0)$$

$\Downarrow$

$$\rho_n(l) = 0 \quad \text{Contradiction to}$$

Theorem 6.

q.e.d.

③ Proof of Theorem 5 (Kallu's main result)

$$X_0 \text{ compact hyperbolic} \implies F_{X_0} \text{ is continuous on } S^{1,0} X_0$$

$\uparrow$
Ragden 1971

(the tubular neighborhood of X_0)

$$\Downarrow S^{1,0} X_0 \text{ compact}$$

$\exists C_1, C_2 > 0$ constants s.t.

$$\boxed{0 < C_1 \leq F_{X_0}(u, \theta) \leq C_2 < +\infty \quad \forall (u, \theta) \in S^{1,0} X_0}$$

$\uparrow$
 X_0 compact hyperbolic



-51-

Let $(u, \eta) \in S^{k_0} x_0$.

$\Downarrow$ (by) Proposition 7

$$\exists f_0 : D_{F_{x_0}(u, \eta)^{-1}} \xrightarrow{\text{hol}} x_0 \quad \text{s.t.} \quad \begin{cases} f_0(0) = u \\ f'_0(0) = \eta \end{cases}$$

We use Theorem 11 to deform f_0 :

$$\exists \delta > 0 \text{ (sufficiently small) s.t. } \forall \tilde{f}_0 : D \xrightarrow{\text{hol}} x_0, \\ \text{if } |s| < \delta, \exists \tilde{f}_s : D_{1-\varepsilon h} \xrightarrow{\text{hol}} x_0 \text{ s.t. } (\tilde{f}_s)_s \text{ satisfies} \\ (1)-(4) \text{ in Thm 11}$$

with $h \in \mathbb{R}^+$ to be determined

s.t. $0 < 1 - \varepsilon h < 1$

Put $R := \frac{1}{F_{x_0}(u, \eta)^*}$

By a ~~change~~ change of scale, every $f_0 : D_R \xrightarrow{\text{hol}} x_0$ s.t. $\begin{cases} f_0(0) = u \\ f'_0(u) = \eta \end{cases}$

can be deformed to $f_s : D_{R(1-\varepsilon h)} \xrightarrow{\text{hol}} x_0$, with $|s| < \delta$.

To prove the result, we need:

$$\frac{1}{R(1-\varepsilon h)} < \underbrace{\left(\frac{1}{R} + \varepsilon \right)}_{F_{x_0}(u, \eta) + \varepsilon} \quad \Rightarrow \quad h < \frac{R}{1 + \varepsilon R}$$

Take $h := \frac{1}{C_2} \cdot \frac{C_1}{C_1 + \varepsilon}$

i.e.d.