Hodge Theory of Compact Complex Manifolds

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GEOMETRIC PROPERTIES OF COMPACT COMPLEX MANIFOLDS UNDER DEFORMATIONS OF COMPLEX STRUCTURES

(I) Smooth families of operators

We now introduce the analogues of a family of manifolds for vector bundles, sections thereof and differential operators.

Definition. *Let X be a compact oriented differentiable manifold X and let $\Delta\subset\mathbb{R}^N$ be a small open subset, for some integer $N\geq 1$.*

(i) We say that $(B_t)_{t\in\Delta}$ *is a* C^∞ **family of C^∞ complex vector bundles** $B_t \longrightarrow X$ *over* X *(or that* B_t **varies C^∞** *with* $t \in \Delta$*) if there exists a* C^∞ *complex vector bundle* $\pi : \mathcal{B} \longrightarrow X \times \Delta$ *such that*

$$B_t = \pi^{-1}(X \times \{t\}) = \mathcal{B}_{|X\times\{t\}}, \qquad t \in \Delta.$$

(ii) Let $(B_t)_{t\in\Delta}$ *be a* C^∞ *family of* C^∞ *complex vector bundles* $B_t \longrightarrow X$ *as in (i).*

(a) For every $t \in \Delta$*, let* $\psi_t \in L(B_t) = C^\infty(X,\, B_t)$ *be a smooth global section of* B_t*.*

We say that $(\psi_t)_{t\in\Delta}$ *is a* C^∞ **family of sections** *(or that* ψ_t **varies C^∞** *with* $t \in \Delta$*) if there exists a* C^∞ *section* $\widetilde{\psi} \in C^\infty(X \times \Delta,\, \mathcal{B})$ *of* $\mathcal{B}$ *such that*

$$\psi_t = \widetilde{\psi}_{|X\times\{t\}}, \qquad t \in \Delta.$$

(b) For every $t \in \Delta$, let $E_t : L(B_t) \longrightarrow L(B_t)$ be a linear operator on B_t.

We say that $(E_t)_{t\in\Delta}$ is a C^∞ **family of linear operators** *(or that E_t* **varies** C^∞ *with $t \in \Delta$) if for every C^∞ family $(\psi_t)_{t\in\Delta}$ of sections $\psi_t \in L(B_t)$, the family $(E_t\psi_t)_{t\in\Delta}$ is a C^∞ family of sections.*

(c) For every $t \in \Delta$, let h_t be a Hermitian metric on B_t in the sense that $h_t = \langle\ ,\ \rangle_t = (\langle\ ,\ \rangle_{t,\,x})_{x\in X}$ is a family of positive definite inner products on the fibres $(B_t)_x$ of B_t that vary in a C^∞ way with the point $x \in X$.

We say that $(h_t)_{t\in\Delta}$ is a C^∞ **family of (fibre) metrics** *(or that h_t* **varies** C^∞ *with $t \in \Delta$) if there exists a Hermitian metric h on the vector bundle $\mathcal{B}$ such that*

$$h_t = h_{|X\times\{t\}}, \qquad t \in \Delta.$$

The Kodaira-Spencer fundamental theorems (1960) on families of elliptic operators

Let:

• $(B_t, h_t)_{t\in\Delta}$ *be a C^∞ family of Hermitian C^∞ complex vector bundles on a compact Riemannian manifold (X, g);*

• $(E_t, h_t)_{t\in\Delta}$ *be a C^∞ family of self-adjoint* **elliptic** *linear differential operators $E_t : L(B_t) \longrightarrow L(B_t)$ of even order m;*

• $(\lambda_h(t))_{h\in\mathbb{N}^\star}$ *be, for every fixed $t \in \Delta$, the eigenvalues of E_t and let $(e_h(t))_{h\in\mathbb{N}^\star}$ be the corresponding eigensections $e_h(t) \in L(B_t)$ such that:*

$$\cdot \; E_t\, e_h(t) = \lambda_h(t)\, e_h(t), \qquad h \in \mathbb{N}^\star,\; t \in \Delta;$$

$\cdot$ $(e_h(t))_{h\in\mathbb{N}^\star}$ *is an orthonormal basis of $L(B_t)$,* $\quad t \in \Delta;$

$$\cdot\ \lambda_1(t) \le \cdots \le \lambda_h(t) \le \dots \quad \textit{and} \quad \lim_{h\to+\infty} \lambda_h(t) = +\infty.$$

Then, the following statements hold.

Theorem A *For every $h \in \mathbb{N}^\star$, the function $\Delta \ni t \mapsto \lambda_h(t)$ is* **continuous**.

Theorem B *The function*

$$\Delta \ni t \mapsto dim \ker E_t$$

is **upper-semicontinuous**.

Theorem C *If $dim \ker E_t$ is* **independent of** $t \in \Delta$, *then $(F_t)_{t\in\Delta}$ is a C^∞* **family of linear operators**, *where $F_t : L(B_t) \longrightarrow \ker E_t$ is the orthogonal projection w.r.t. the L^2 inner product $\langle\langle\ ,\ \rangle\rangle_t$, for every $t \in \Delta$.*

Theorem D *If dim* $\ker E_t$ *is* **independent of** $t \in \Delta$, *then* $(G_t)_{t\in\Delta}$ *is a* C^∞ **family of linear operators**, *where* $G_t := E_t^{-1}$ *is the Green operator of* E_t *for every* $t \in \Delta$.

Before moving on to the proofs, let us point out the

Observation. *Having fixed an arbitrary* $h \in \mathbb{N}^\star$, *the function* $\Delta \ni t \mapsto \lambda_h(t)$ **need not be differentiable**.

Example. Notice that the eigenvalues of the operator $E_t := \begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix}$ are $\lambda_1(t) = 1 + \sqrt{t}$ and $\lambda_1(t) = 1 - \sqrt{t}$, which are not differentiable functions of t. □

Preliminary steps in the proofs of Theorems A, B, C, D

Step 1. It can be shown that the vector bundles B_t are C^∞ *isomorphic* to B_0 for all $t \in \Delta$ sufficiently close to 0.

Therefore, we may assume without loss of generality that all the B_t's coincide with a fixed C^∞ vector bundle $B \longrightarrow X$, after possibly shrinking Δ about 0.

In particular, henceforth, we place ourselves in the situation

$$E_t : L(B) \longrightarrow L(B), \qquad t \in \Delta.$$

However, the Hermitian fibre metric h_t on B depends on $t \in \Delta$ and so do $\langle\ ,\ \rangle_t$, $\langle\langle\ ,\ \rangle\rangle_t$ and $||\ ||_t$, but they are mutually equivalent by uniform multiplicative constants.

Step 2. The following technical result is key.

Theorem. *Let* $(E_t)_{t\in\Delta}$ *be a* C^∞ *family of* **elliptic**, *not necessarily self-adjoint, differential operators of* **even order** m. *Suppose that* $E_t : L(B) \longrightarrow L(B)$ *is* **bijective** *for all* $t \in \Delta$.

If there exists a constant $c > 0$ **independent** *of* $t \in \Delta$ *such that*

$$||E_t\psi||_0 \geq c\,||\psi||_0 \quad \textit{for all } \psi \in L(B),$$

the inverse operator E_t^{-1} *varies in a* C^∞ *way with* $t \in \Delta$.

The key point: the uniformity of the constant c.

A constant depending on t with this property always exists thanks to E_t being elliptic and to X being compact, as follows from the *a priori estimate*.

Key ingredient in the proof of the above theorem: the following **a priori estimate** w.r.t. **Sobolev norms** in families of *elliptic* operators.

Theorem. *Let* $(E_t)_{t\in\Delta}$ *be a* C^∞ *family of* **elliptic** *linear differential operators* $E_t : L(B) \longrightarrow L(B)$ *of even order* m*.*

Then, for every $k \in \mathbb{N}$*, there exists a constant* $c_k > 0$ **independent of** $t \in \Delta$ *such that the following* **uniform a priori estimate** *holds:*

$$||\psi||^2_{k+m} \leq c_k \, (||E_t\psi||^2_k + ||\psi||^2_0) \qquad (1)$$

for every $\psi \in L(B)$ *and every* $t \in \Delta$*.*

Step 3. Henceforth, we shall assume that each operator E_t is **self-adjoint** (and, of course, also **elliptic**).

The main technique for the proofs of Theorems A, B, C, D consists in considering, for every $\zeta \in \mathbb{C}$, the elliptic differential operator

$$E_t(\zeta) := E_t - \zeta : L(B) \longrightarrow L(B), \qquad t \in \Delta,$$

to which the following simple but critical observation and several of the above preliminary results will be applied.

Observation. *If* $\zeta \notin \mathit{Spec}\,(E_t) := \{\lambda_1(t), \lambda_2(t), \dots\}$, *then*

$$E_t(\zeta) : L(B) \longrightarrow L(B)$$

is **bijective**.

The key technical result: the hypothesis of an earlier theorem is **uniformly** satisfied by the operators $E_t(\zeta) : L(B) \longrightarrow L(B)$ when $t \in \Delta$ and $\zeta \notin \mathrm{Spec}\,(E_t)$ vary very little.

Lemma. *Let* $t_0 \in \Delta$ *and let* $\zeta_0 \in \mathbb{C} \setminus \mathit{Spec}\,(E_{t_0})$. *Then, there exist constants* $\delta, c > 0$ *such that, for all* $t \in \Delta$ *with* $|t - t_0| < \delta$ *and all* $\zeta \in \mathbb{C}$ *with* $|\zeta - \zeta_0| < \delta$, *the following inequality holds:*

$$||E_t(\zeta)\,\psi||_0 \geq c\,||\psi||_0$$

for all $\psi \in L(B)$.

Step 4. The next goal is to express the **spectral projection operators** by a **Cauchy integral formula**.

Let

$$W := \Big\{(t,\,\zeta) \in \Delta \times \mathbb{C} \mid \zeta \notin \operatorname{Spec} E_t\Big\} \subset \Delta \times \mathbb{C}.$$

An earlier result implies that W is **open** in $\Delta \times \mathbb{C}$ (because we have seen that $\ker E_t(\zeta) = \{0\}$, which amounts to $\zeta \notin \operatorname{Spec} E_t$).

Meanwhile, $E_t(\zeta) : L(B) \longrightarrow L(B)$ is *bijective* for all $(t, \zeta) \in W$.

Let

$$G_t(\zeta) := E_t(\zeta)^{-1} : L(B) \longrightarrow L(B), \qquad (t, \zeta) \in W,$$

be its inverse. From earlier results we get the following crucial piece of information which is the culmination of the above technical work.

Conclusion. $G_t(\zeta)$ **varies in a C^∞ way** *with* $(t, \zeta) \in W$.

Now, fix an arbitrary $t_0 \in \Delta$ and pick a **Jordan curve** C (i.e. a closed simple curve C in the complex plane) such that

$$C \cap \operatorname{Spec} E_{t_0} = \emptyset. \tag{2}$$

Such a curve exists because $\operatorname{Spec} E_{t_0} \subset \mathbb{R}$ is discrete. As is well known, C divides the plane $\mathbb{C}$ into two disjoint regions: the *interior* of C, denoted by $\operatorname{int}(C)$, and the *exterior* of C, denoted by $\operatorname{ext}(C)$.

Property (2) means that $\{t_0\} \times C \subset W$. Since W is open in $\Delta \times C$, there exists $\delta > 0$ such that $[t_0 - \delta, t_0 + \delta] \times C \subset W$. For any $t \in (t_0 - \delta, t_0 + \delta)$, we put:

$$\mathbb{F}_t(C) := \bigoplus_{\lambda(t) \in \operatorname{int}(C)} \mathcal{H}_{\lambda(t)}(E_t) \subset L(B),$$

where $\mathcal{H}_{\lambda(t)}(E_t)$ is the $\lambda(t)$-eigenspace of E_t.

Note that, by *ellipticity* of E_t and *compactness* of X, the $\mathbb{C}$-vector space $\mathbb{F}_t(C)$ is *finite dimensional*.

Furthermore, for any $t \in (t_0 - \delta,\, t_0 + \delta)$, we let

$$F_t(C) : L(B) \longrightarrow \mathbb{F}_t(C)$$

be the $L^2_{h_t}$*-orthogonal projection* onto $\mathbb{F}_t(C)$.

The following simple **Cauchy integral formula** for **orthogonal spectral projectors** will play a key role in the sequel.

Lemma. *The orthogonal projector* $F_t(C) : L(B) \longrightarrow \mathbb{F}_t(C)$ *satisfies the following formula:*

$$F_t(C)\,\psi = -\frac{1}{2\pi i} \int\limits_{\zeta \in C} G_t(\zeta)\,\psi\,d\zeta,$$

for all $\psi \in L(B)$ *and all* $t \in (t_0 - \delta,\, t_0 + \delta)$.

Proof. Let $\psi = \sum_{h=1}^{+\infty} a_h e_h(t) \in L(B)$. Since

$$E_t(\zeta)\,\psi = \sum_{h=1}^{+\infty} (\lambda_h(t) - \zeta)\,a_h e_h(t)$$

and $G_t(\zeta) = E_t(\zeta)^{-1}$, we get:

$$-G_t(\zeta)\,\psi = \sum_{h=1}^{+\infty} \frac{a_h}{\zeta - \lambda_h(t)}\,e_h(t), \qquad \zeta \in C.$$

Therefore,

$$
\begin{aligned}
-\frac{1}{2\pi i}\int\limits_{\zeta\in C} G_t(\zeta)\,\psi\,d\zeta &= \sum_{h=1}^{+\infty}\left(\frac{1}{2\pi i}\int\limits_{\zeta\in C}\frac{1}{\zeta-\lambda_h(t)}\,d\zeta\right)a_h e_h(t)\\
&= \sum_{\lambda_h(t)\in \mathrm{int}\,(C)} a_h e_h(t) = F_t(C)\,\psi,
\end{aligned}
$$

where the following elementary fact on the winding number of a Jordan curve around a point in the complex plane has been used:

$$
\begin{aligned}
&\frac{1}{2\pi i}\int\limits_{\zeta\in C}\frac{1}{\zeta-\lambda_h(t)}\,d\zeta = 1 \text{ if } \;\lambda_h(t)\in \mathrm{int}\,(C) \quad \text{and}\\
&\frac{1}{2\pi i}\int\limits_{\zeta\in C}\frac{1}{\zeta-\lambda_h(t)}\,d\zeta = 0 \quad \text{if} \quad \lambda_h(t)\in \mathrm{ext}\,(C).
\end{aligned}
$$

□

Corollary. *For any* $t_0 \in \Delta$ *and any Jordan curve* $C \subset \mathbb{C}$ *s.t.*

$$C \cap Spec\, E_{t_0} = \emptyset,$$

the orthogonal projector $F_t(C)$ *varies in a* C^∞ *way with* $t \in (t_0 - \delta,\, t_0 + \delta)$ *if* $\delta > 0$ *is* **small enough**.

Proof. Let $(\psi_t)_{t \in (t_0 - \delta,\, t_0 + \delta)}$ be a C^∞ family of sections $\psi_t \in L(B)$. Then, by the Cauchy formula, we have

$$F_t(C)\,\psi_t = -\frac{1}{2\pi i} \int\limits_{\zeta \in C} G_t(\zeta)\,\psi_t\, d\zeta, \qquad t \in (t_0 - \delta,\, t_0 + \delta). \tag{3}$$

Since $G_t(\zeta)$ varies in a C^∞ way with $(t,\, \zeta) \in W$ (see above) and ψ_t varies in a C^∞ way with $t \in (t_0 - \delta,\, t_0 + \delta)$, we conclude that $F_t(C)\,\psi_t$ varies in a C^∞ way with $t \in (t_0 - \delta,\, t_0 + \delta)$. □

The following consequence is the cornerstone of much of what follows.

Corollary. *For any* $t_0 \in \Delta$ *and any Jordan curve* $C \subset \mathbb{C}$ *s.t.*

$$C \cap \mathit{Spec}\, E_{t_0} = \emptyset,$$

the number $\mathit{dim}_{\mathbb{C}} \mathbb{F}_t(C)$ *of eigenvalues, counted with multiplicities, of* E_t *lying in* $\mathit{int}(C)$ *is* **independent** *of* $t \in (t_0 - \delta,\, t_0 + \delta)$ *if* $\delta > 0$ *is* **small enough**.

Proof. Let $d := \dim_{\mathbb{C}} \mathbb{F}_{t_0}(C)$ and let $\{e_1, \ldots, e_d\}$ be a basis of $\mathbb{F}_{t_0}(C)$. There are two inequalities to prove.

- The inequality $\dim_{\mathbb{C}} \mathbb{F}_t(C) \geq \dim_{\mathbb{C}} \mathbb{F}_{t_0}(C)$ for all $t \in (t_0 - \delta,\, t_0 + \delta)$ (if $\delta > 0$ is small enough) is immediate to prove.

Indeed, since $F_t(C) : L(B) \longrightarrow \mathbb{F}_t(C)$ varies in a C^∞ way with $t \in (t_0 - \delta,\, t_0 + \delta)$ (see above), $F_t(C)\, e_j$ varies in a C^∞ way with $t \in (t_0 - \delta,\, t_0 + \delta)$, for every $j \in \{1, \ldots, d\}$. Meanwhile, the property of linear independence is stable under small continuous deformations.

Therefore, since the $e_j = F_{t_0}(C)\, e_j$, with $j \in \{1, \ldots, d\}$, are linearly independent and since the $F_t(C)\, e_j$ vary continuously (even in a C^∞ way) with t, the $F_t(C)\, e_j$, with $j \in \{1, \ldots, d\}$, remain linearly independent elements of $\mathbb{F}_t(C)$ for all t sufficiently close to t_0. Thus, $\dim_{\mathbb{C}} \mathbb{F}_t(C) \geq d$ for all t close enough to t_0.

• The reverse inequality $\dim_{\mathbb{C}} \mathbb{F}_t(C) \leq \dim_{\mathbb{C}} \mathbb{F}_{t_0}(C)$ for all $t \in (t_0 - \delta,\, t_0 + \delta)$ (if $\delta > 0$ is small enough) can be proved by contradiction.

The proof uses the *Sobolev inequality* and elliptic estimates. $\square$

Proof of Theorem B. Recall that we set $\mathbb{F}_t := \ker E_t$ for all $t \in \Delta$. Fix $t_0 \in \Delta$. We have to prove that

$$\exists\ \delta > 0 \quad \text{such that} \quad \dim \mathbb{F}_t \leq \dim \mathbb{F}_{t_0} \quad \forall t \in (t_0 - \delta,\, t_0 + \delta).$$

For any $\varepsilon > 0$, let $C_\varepsilon := C(0, \varepsilon) \subset \mathbb{C}$ be the circle of radius ε centred at the origin in the complex plane. Since $\mathrm{Spec}\, E_{t_0}$ is discrete, $\mathbb{F}_{t_0} = \mathbb{F}_{t_0}(C_\varepsilon)$ (i.e. 0 is the only eigenvalue of E_{t_0} lying in $\mathrm{int}\,(C_\varepsilon)$) if ε is small enough.

The above Corollary applied to C_ε yields:

$$\dim \mathbb{F}_t(C_\varepsilon) = \dim \mathbb{F}_{t_0}(C_\varepsilon), \qquad t \in (t_0 - \delta,\, t_0 + \delta),$$

if $\delta > 0$ is small enough. Since $\dim \mathbb{F}_{t_0}(C_\varepsilon) = \dim \ker E_{t_0}$ and since $\mathbb{F}_t = \ker E_t \subset \mathbb{F}_t(C_\varepsilon)$ for all t, we infer that

$$\dim \mathbb{F}_t \leq \dim \mathbb{F}_t(C_\varepsilon) = \dim \mathbb{F}_{t_0}(C_\varepsilon) = \dim \mathbb{F}_{t_0} \quad \forall t \in (t_0 - \delta,\, t_0 + \delta).$$

$\square$

(II) Deformation openness results

Two points of view are possible.

Definition. (i) *A given property* (P) *of a compact complex manifold is said to be* **open** *under holomorphic deformations if for every holomorphic family of compact complex manifolds* $(X_t)_{t\in B}$ *and for every* $t_0 \in B$*, the following implication holds:*

X_{t_0} *has property* (P) $\implies$ X_t *has property* (P) *for all* $t \in B$ *sufficiently close to* t_0.

(ii) A given property (P) of a compact complex manifold is said to be **closed** *under holomorphic deformations if for every holomorphic family of compact complex manifolds $(X_t)_{t\in B}$ and for every $t_0 \in B$, the following implication holds:*

$$X_t \text{ has property } (P) \text{ for all } t \in B \setminus \{t_0\} \implies X_{t_0} \text{ has property } (P).$$

Theorem. *Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds $X_t := \pi^{-1}(t)$, with $\dim_{\mathbb{C}} X_t = n$ for all $t \in B$. Fix an arbitrary bidegree (p, q).*

(i) The functions: $B \ni t \longmapsto h^{p,\,q}(t) := \dim_{\mathbb{C}} H^{p,\,q}_{\bar\partial}(X_t, \mathbb{C})$,

$$B \ni t \longmapsto h^{p,\,q}_{BC}(t) := \dim_{\mathbb{C}} H^{p,\,q}_{BC}(X_t, \mathbb{C}),$$

$$B \ni t \longmapsto h^{p,\,q}_{A}(t) := \dim_{\mathbb{C}} H^{p,\,q}_{A}(X_t, \mathbb{C}),$$

are **upper-semicontinuous**.

(ii) If the Hodge number $h^{p,\,q}(t)$ is **independent of** $t \in B$, *then the map*

$$B \ni t \longmapsto H^{p,\,q}_{\bar\partial}(X_t, \mathbb{C})$$

defines a C^∞ **vector bundle** *on B.*

The analogous statement holds for $h^{p,\,q}_{BC}(t)$ and $h^{p,\,q}_{A}(t)$.

(1) The first main consequence of the upper-semicontinuity of the Hodge numbers under deformations is the **deformation openness** of the **Frölicher degeneration property at** E_1.

Theorem. *Let* $\pi : \mathcal{X} \longrightarrow B$ *be a holomorphic family of compact complex manifolds* $X_t := \pi^{-1}(t)$*, with* $t \in B$*. Fix an arbitrary reference point* $0 \in B$*.*

If the **Frölicher spectral sequence of** X_0 **degenerates at** E_1*, then, for all* $t \in B$ *sufficiently close to* 0*, we have:*

(a) the **Frölicher spectral sequence of** X_t **degenerates at** E_1*;*

(b) $h^{p,\,q}(t) = h^{p,\,q}(0)$ *for every bidegree* $(p,\,q)$*.*

Proof. We know that the hypothesis $E_1(X_0) = E_\infty(X_0)$ is equivalent to the numerical identities:

$$b_k = \sum_{p+q=k} h^{p,\,q}(0), \qquad\qquad k \in \{0, 1, \ldots, 2n\}, \tag{4}$$

where $b_k := \dim_{\mathbb{C}} H^k_{DR}(X,\, \mathbb{C})$ is the k-th Betti number of the fibres.

For every $t \in B$ sufficiently close to 0, we get:

$$b_k \stackrel{(i)}{\leq} \sum_{p+q=k} h^{p,\,q}(t) \stackrel{(ii)}{\leq} \sum_{p+q=k} h^{p,\,q}(0) \stackrel{(iii)}{=} b_k, \tag{5}$$

where (i) is the dimension inequality that is valid on any manifold, (ii) is the upper-semicontinuity property of the above theorem, while (iii) features above.

Thus, inequalities (i) and (ii) must be equalities for every $t \in B$ sufficiently close to 0.

Now, (i) being an equality for every degree k is equivalent to

$$E_1(X_t) = E_\infty(X_t),$$

while (ii) being an equality for every degree k is equivalent to

$$h^{p,\,q}(t) = h^{p,\,q}(0)$$

for every bidegree $(p,\,q)$. $\square$

(2) The second main consequence of the upper-semicontinuity of the Hodge numbers under deformations is the **deformation openness** of the **$\partial\bar\partial$-property** of compact complex manifolds.

Theorem. (Wu 2006, Angella-Tomassini 2013)

Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds $X_t := \pi^{-1}(t)$, with $t \in B$. Fix an arbitrary reference point $0 \in B$.

If the fibre X_0 is a **$\partial\bar\partial$-manifold**, *then, for all $t \in B$ sufficiently close to 0, we have:*

(a) the fibre X_t is a **$\partial\bar\partial$-manifold**;

(b) $h^{p,\,q}_{BC}(t) = h^{p,\,q}_{BC}(0)$ and $h^{p,\,q}_{A}(t) = h^{p,\,q}_{A}(0)$ for every bidegree $(p,\,q)$.

Proof. By Angella-Tomassini (2013), the $\partial\bar{\partial}$-assumption on X_0 is equivalent to the identities:

$$\sum_{p+q=k} (h^{p,\,q}_{BC}(0) + h^{p,\,q}_{A}(0)) = 2b_k, \qquad k \in \{0, 1, \dots, 2n\}.$$

Meanwhile, the upper-semicontinuity properties yield:

$$h^{p,\,q}_{BC}(0) \geq h^{p,\,q}_{BC}(t) \quad \text{and} \quad h^{p,\,q}_{A}(0) \geq h^{p,\,q}_{A}(t)$$

for all bidegrees $(p,\,q)$ and all $t \in B$ sufficiently close to 0.

Finally, by Angella-Tomassini (2013), we always have the inequalities:

$$\sum_{p+q=k} (h^{p,\,q}_{BC}(t) + h^{p,\,q}_{A}(t)) \geq 2b_k, \qquad t \in B, \quad k \in \{0, 1, \dots, 2n\}.$$

Putting together all these pieces of information, we get:

$$2b_k \overset{(i)}{\leq} \sum_{p+q=k} (h^{p,\,q}_{BC}(t) + h^{p,\,q}_{A}(t)) \overset{(ii)}{\leq} \sum_{p+q=k} (h^{p,\,q}_{BC}(0) + h^{p,\,q}_{A}(0)) = 2b_k,$$

for all $k \in \{0, 1, \dots, 2n\}$ and all $t \in B$ sufficiently close to 0. Hence, both of the above inequalities must be equalities.

In particular, inequalities (i) being equalities for all $k \in \{0, 1, \dots, 2n\}$ and all $t \in B$ sufficiently close to 0 amounts to X_t being a $\partial\bar\partial$-manifold for all $t \in B$ sufficiently close to 0, thanks again to Angella-Tomassini (2013). This proves (a).

Meanwhile, inequalities (ii) being equalities for all bidegrees $(p,\, q)$ and all $t \in B$ sufficiently close to 0 proves (b). $\square$

(3) Deformation openness of the Kähler property (Kodaira-Spencer 1960)

Let us start with a very simple but crucial observation.

Lemma. *Let ω be a Hermitian metric on a compact complex manifold X. The equivalence holds:*

$$\omega \text{ is Kähler} \iff \Delta_{BC}\omega = 0,$$

where $\Delta_{BC} : C^\infty_{1,\,1}(X,\,\mathbb{C}) \longrightarrow C^\infty_{1,\,1}(X,\,\mathbb{C})$ is the Bott-Chern Laplacian induced by ω.

Proof. We know that

$$\ker \Delta_{BC} = \ker \partial \cap \ker \bar{\partial} \cap \ker(\partial\bar{\partial})^{\star}.$$

So, one implication of the above equivalence is obvious:

$$\text{if } \Delta_{BC}\omega = 0, \text{ then } \partial\omega = 0, \text{ which means that } \omega \text{ is Kähler.}$$

Suppose now that ω is Kähler, namely $d\omega = 0$. This implies $\partial\omega = 0$ and $\bar{\partial}\omega = 0$. To prove that $(\partial\bar{\partial})^{\star}\omega = 0$, we will use the standard formulae:

$$\star\star = (-1)^k \,\mathrm{Id} \quad \text{on k-forms;} \qquad \partial^{\star} = -\star\bar{\partial}\star, \qquad \bar{\partial}^{\star} = -\star\partial\star$$

and the standard formula:

$$\star\omega = \frac{\omega^{n-1}}{(n-1)!}, \tag{6}$$

where $\star = \star_\omega$ is the Hodge star operator induced by the Hermitain metric ω.

We get the equivalences:

$$(\partial\bar{\partial})^\star \omega = 0 \iff \star\partial\bar{\partial}(\star\omega) = 0 \iff \partial\bar{\partial}\frac{\omega^{n-1}}{(n-1)!} = 0,$$

where the second one uses the fact that $\star$ is an isomorphism.

Now, the last identity holds since

$$\bar{\partial}\omega^{n-1} = (n-1)\,\omega^{n-2} \wedge \bar{\partial}\omega = 0.$$

Indeed, $\bar{\partial}\omega = 0$ by the Kähler assumption on ω. $\square$

Theorem. (Kodaira-Spencer 1960)

Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds $X_t := \pi^{-1}(t)$, with $t \in B$. Fix an arbitrary reference point $0 \in B$.

(a) If the fibre X_0 is a **Kähler manifold***, then the fibre X_t is a* **Kähler manifold** *for all $t \in B$ sufficiently close to 0.*

(b) Moreover, given any **Kähler metric** *ω_0 on X_0, there exists a small neighbourhood U of 0 in B and a C^∞ family $(\omega_t)_{t\in U}$ of* **Kähler metrics** *on the respective fibres X_t whose member for $t = 0$ is ω_0.*

Proof. Since (b) implies (a), we will prove (b).

Let ω_0 be a Kähler metric on X_0. In particular, ω_0 is a smooth J_0-type $(1, 1)$-form on X_0, hence a smooth 2-form on X (the C^∞ manifold underlying the fibres X_t for $t \in B$ close to 0.)

For every $t \in B$, let ω_t be the J_t-type $(1, 1)$-component of the 2-form ω_0. Clearly, the member for $t = 0$ of the family of forms $(\omega_t)_{t\in B}$ is ω_0. Moreover, the ω_t's vary in a C^∞ way with t because they are the J_t-type $(1, 1)$-components of a fixed 2-form and the J_t's depend in a (at least) C^∞ way on t.

Now, ω_0 is *positive definite* because it is a metric on X_0. By continuity w.r.t. t, ω_t remains *positive definite* for all $t \in U$ if the neighbourhood U of 0 in B is small enough. Hence, ω_t is a Hermitian metric on X_t for every $t \in U$, so $(\omega_t)_{t \in U}$ is a C^∞ family of Hermitian metrics on the respective fibres X_t, whose member for $t = 0$ is the original Kähler metric ω_0.

We have to change the metrics ω_t with $t \in U \setminus \{0\}$ to make them Kähler. The above Lemma tells us that this amounts to making the ω_t's *Bott-Chern harmonic* w.r.t. themselves (i.e. for the Bott-Chern Laplacians induced by the ω_t's).

Let us therefore consider the *$L^2_{\omega_t}$-orthogonal projectors:*

$$F_t : C^\infty_{1,\,1}(X_t,\,\mathbb{C}) \longrightarrow \mathcal{H}^{1,\,1}_{\Delta_{BC}}(X_t,\,\mathbb{C}), \qquad\qquad t \in U,$$

onto the kernels of the Bott-Chern Laplacians $\Delta_{BC,\,t}$ induced by the ω_t's in J_t-bidegree $(1,\,1)$.

The crucial piece of information that we need at this point is the non-jumping of a certain cohomology space dimension. Since X_0 is a $\partial\bar\partial$-manifold (because it is even Kähler, by hypothesis), the dimension $h^{1,\,1}_{BC}(t)$ of $\mathcal{H}^{1,\,1}_{\Delta_{BC}}(X_t,\,\mathbb{C})$ (= the dimension of $H^{1,\,1}_{BC}(X_t,\,\mathbb{C})$, thanks to the Hodge isomorphism) is *independent of $t \in U$* if the neighbourhood U of 0 in B is *small enough*.

Therefore, by Theorem C, *F_t varies in a C^∞ way with $t \in U$.*

Now, put

$$\widetilde{\omega}_t := \frac{1}{2}(F_t\omega_t + \overline{F_t\omega_t}), \qquad t \in U.$$

The J_t-type (1, 1)-forms $\widetilde{\omega}_t$ have the following properties:

(i) $\widetilde{\omega}_t$ is a *real* form (i.e. it equals its conjugate) for every $t \in U$;

(ii) $\widetilde{\omega}_t$ varies in a C^∞ way with $t \in U$, because F_t and ω_t do;

(iii) $\widetilde{\omega}_0 = \omega_0$ because $F_0\omega_0 = \omega_0$ (recall that ω_0 is *Kähler* on X_0 and the above Lemma applies) and ω_0 is real;

(iv) $\widetilde{\omega}_t$ is *positive definite* on X_t for all $t \in U$ (shrink U about 0 if necessary), because $\widetilde{\omega}_0$ is and $\widetilde{\omega}_t$ varies (at least) continuously with $t \in U$;

(v) $\widetilde{\omega}_t \in \ker \partial_t$ for all $t \in U$, because $F_t\omega_t \in \mathcal{H}^{1,\,1}_{\Delta_{BC}}(X_t,\,\mathbb{C}) = \ker \partial_t \cap \ker \bar\partial_t \cap \ker(\partial_t\bar\partial_t)^\star \subset \ker \partial_t \cap \ker \bar\partial_t$.

(Note that the Bott-Chern harmonic space $\mathcal{H}^{1,\,1}_{\Delta_{BC}}(X_t,\,\mathbb{C})$ in (v) is defined by the Hermitian metric ω_t, rather than $\widetilde{\omega}_t$.)

Properties (i)-(v) amount to saying that

$$(\widetilde{\omega}_t)_{t\in U}$$

is a C^∞ family of *Kähler metrics* on the respective fibres X_t, whose member for $t = 0$ is the originally given Kähler metric ω_0 on X_0. $\square$