Hodge Theory of Compact Complex Manifolds

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Dan Popovici

Université Paul Sabatier, Toulouse, France

DEFORMATIONS OF COMPLEX STRUCTURES

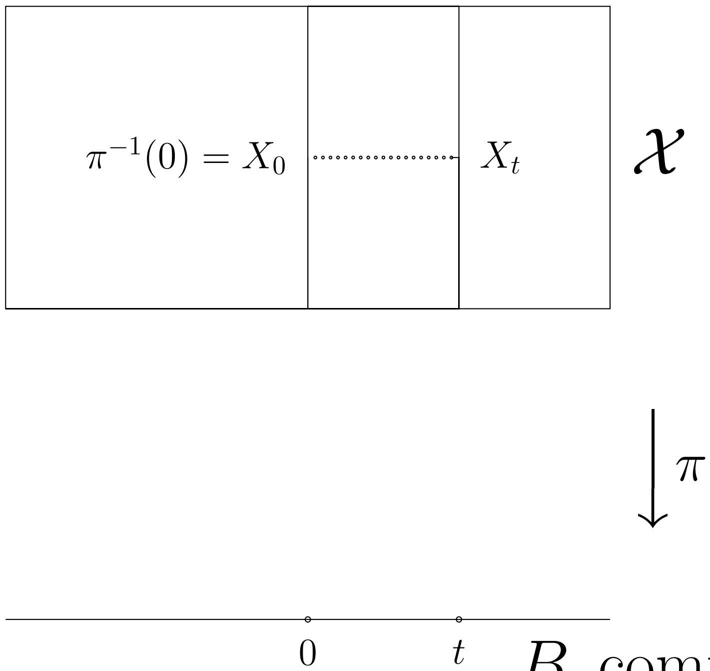
Definition. *A* **holomorphic family** *of compact complex manifolds is a* **proper holomorphic submersion** $\pi : \mathcal{X} \longrightarrow B$ *between complex manifolds* $\mathcal{X}$ *and* B*.*

$\mathcal{X}$ = the *total space*

B = the *base* of the family

$\forall\, t \in B$, $X_t := \pi^{-1}(t) \subset \mathcal{X}$ is a compact complex manifold
(= the *fibre* above t)

Thus, $(X_t)_{t\in B}$ is a family $(X_t)_{t\in B}$ of equidimensional compact complex manifolds parametrised by the points of the base B.



$\mathcal{X}$ complex manifold

B complex manifold

(with a base point $0 \in B$)

We usually let $m = \dim_{\mathbb{C}} B$ and $n = \dim_{\mathbb{C}} X_t$ for $t \in B$.

Common situation: when the base B is an open ball about the origin in some $\mathbb{C}^m$ or, more generally, when a base point $0 \in B$ has been fixed.

We can then take the fibre above $0 \in B$ as a reference fibre and view the fibres X_t for $t \in B$ sufficiently close to 0 as *small deformations* of X_0.

If t is allowed to lie anywhere in B, the family $\pi : \mathcal{X} \longrightarrow B$ can be seen as a family of *holomorphic deformations* of X_0.

(I) Ehresmann's theorem (1947) *(i) Every holomorphic family of compact complex manifolds is* **locally C^∞ trivial** *in the following sense.*

There exists a C^∞ manifold X such that every point $t_0 \in B$ has an open neighbourhood $U \subset B$ for which there exists a C^∞ **diffeomorphism**

$$T : \mathcal{X}_U \longrightarrow X \times U \quad \text{such that} \quad pr_2 \circ T = \pi,$$

where $\mathcal{X}_U = \pi^{-1}(U) \subset \mathcal{X}$ and $pr_2 : X \times U \longrightarrow U$ is the projection on the second factor.

(ii) If the base B is **contractible**, *the family is even* **globally C^∞ trivial** *in the sense that there exists a C^∞ manifold X and a C^∞* **diffeomorphism**

$$T : \mathcal{X} \longrightarrow X \times B \quad \textit{such that} \quad pr_2 \circ T = \pi,$$

where $pr_2 : X \times B \longrightarrow B$ is the projection on the second factor.

(iii) Suppose that the base B of the family is an open ball about the origin in some $\mathbb{C}^m$.

The local trivialisation $T = (T_0, \pi) : \mathcal{X} \longrightarrow X_0 \times B$ of (i), obtained after possibly replacing B by a neighbourhood U of $0 \in B$, can be chosen such that the fibres of the map $T_0 : \mathcal{X} \longrightarrow X_0$ are **complex submanifolds** *of $\mathcal{X}$.*

Consequence

-the Dolbeault, E_r, Bott-Chern and Aeppli cohomology spaces of the fibres X_t, as well as their dimensions, depend on $t \in B$:

$$H^{p,\,q}_{\bar\partial}(X_t,\,\mathbb{C}), \quad E_r^{p,\,q}(X_t), \quad H^{p,\,q}_{BC}(X_t,\,\mathbb{C}), \quad H^{p,\,q}_{A}(X_t,\,\mathbb{C}).$$

-the De Rham cohomology of X_t is *locally constant*, so we can identify:

$$H^k_{DR}(X_t,\,\mathbb{C}) = H^k_{DR}(X,\,\mathbb{C}), \qquad k \in \{0, \dots, 2n\},$$

for all t in a small enough neighbourhood of any given point $t_0 \in B$.

(II) The Kodaira-Spencer map

Let $\pi : \mathcal{X} \longrightarrow B$ be a *holomorphic family* of compact complex manifolds. Fix an arbitrary base point $0 \in B$. The differential map

$$d\pi : T^{1,\,0}\mathcal{X} \longrightarrow \pi^\star(T^{1,\,0}B)$$

is a morphism of *holomorphic vector bundles* over $\mathcal{X}$. Since $X_0 = \pi^{-1}(0) \subset \mathcal{X}$, we have

$$T^{1,\,0}X_0 = \ker\left((d\pi)_{|X_0}\right),$$

so we get an *exact sequence* of *holomorphic vector bundles* over X_0:

$$0 \longrightarrow T^{1,\,0}X_0 \longrightarrow T^{1,\,0}\mathcal{X}_{|X_0} \stackrel{d\pi}{\longrightarrow} \pi^\star(T^{1,\,0}B)_{|X_0} \longrightarrow 0.$$

Meanwhile, $\pi^\star(T^{1,\,0}B)_{|X_0} = X_0 \times T_0^{1,\,0}B$ is the *trivial* holomorphic vector bundle over X_0 of fibre $T_0^{1,\,0}B$.

This exact sequence defines an extension of the holomorphic vector bundle $T^{1,0}X_0$ by the trivial holomorphic vector bundle of fibre $T_0^{1,0}B$. This extension is equivalent to the *connecting morphism*

$$\rho : T_0^{1,0}B = H^0(X_0, \pi^\star(T^{1,0}B)_{|X_0}) \longrightarrow H^1(X_0, \mathcal{O}_{X_0}(T^{1,0}X_0)) \simeq H^{0,1}(X_0, T^{1,0}X_0)$$

that is part of the long exact sequence associated with the above short exact sequence.

Definition 0.1. *The linear map*

$$\rho : T_0^{1,0}B \longrightarrow H^{0,1}(X_0, T^{1,0}X_0)$$

is called the **Kodaira-Spencer map at** 0 *of the family* $\pi : \mathcal{X} \longrightarrow B$.

The main interest in the Kodaira-Spencer map stems from the following loosely stated principle that will be made precise in the next two subsections.

Fact. *The Kodaira-Spencer map at* 0 *can be seen as the* **differential** *at* $t = 0$ *of the map*

$$B \ni t \mapsto J_t, \tag{1}$$

where J_t *is the complex structure of the fibre* X_t*.*

In other words, the Kodaira-Spencer map is the classifying map for the 1-*st order deformations* of (the complex structure of) X_0.

Analytic approach to the Kodaira-Spencer map

There exists a vector-valued form

$$\psi(t) \in C^{\infty}_{0,\,1}(X_0,\, T^{1,\,0}X_0)$$

such that

$$\bar{\partial}_t \simeq \bar{\partial}_0 - \psi(t), \qquad t \in B,\, t \sim 0$$

in the following sense.

Theorem. *After possibly shrinking B about 0, for every $t \in B$ and for every locally defined $\mathbb{C}$-valued C^{∞} function f on $X := X_0$, the following equivalence holds:*

$$f \ \ is\ \ J_t - holomorphic \iff (\bar{\partial}_0 - \psi(t))\, f \equiv 0. \qquad (2)$$

Theorem. *(a) The* $T^{1,\,0}X_0$*-valued* J_0*-*$(0,\,1)$*-form* $\frac{\partial\psi(t)}{\partial t}_{|t=0}$ *is* **$\bar\partial_0$-closed**, *hence it defines a cohomology class*

$$\left\{\frac{\partial\psi(t)}{\partial t}_{|t=0}\right\}_{\bar\partial_0} \in H^{0,\,1}(X_0,\,T^{1,\,0}X_0).$$

(b) The following identity holds:

$$\rho\left(\frac{\partial}{\partial t}_{|t=0}\right) = -\left\{\frac{\partial\psi(t)}{\partial t}_{|t=0}\right\}_{\bar\partial_0},$$

where $\rho : T_0^{1,\,0}B \longrightarrow H^{0,\,1}(X_0,\,T^{1,\,0}X_0)$ *is the Kodaira-Spencer map at* 0 *of the family* $\pi : \mathcal{X} \longrightarrow B$.

Therefore, the following piece of notation is justified.

Notation. *Let* $\pi : \mathcal{X} \longrightarrow B$ *be a holomorphic family of compact complex manifolds. We fix a reference point* $0 \in B$ *and let* $\rho : T_0^{1,0}B \longrightarrow H^{0,1}(X_0, T^{1,0}X_0)$ *be the Kodaira-Spencer map at* 0.

For every holomorphic vector field $\frac{\partial}{\partial t} \in \Gamma(U, T^{1,0}B)$ *on some small open neighbourhood* $U \subset B$ *of* 0, *we put*

$$\frac{\partial X_t}{\partial t}_{|t=0} := \rho\left(\frac{\partial}{\partial t}_{|t=0}\right).$$

The Kodaira-Nirenberg-Spencer existence theorem

We now take the opposite point of view to the previous one.

Question. *Let X be a compact complex manifold and let $\theta \in H^{0,\,1}(X,\, T^{1,\,0}X)$.*

When does there exist a holomorphic family of compact complex manifolds $\pi : \mathcal{X} \longrightarrow B$, with B a small open disc about 0 in $\mathbb{C}$, such that

$$\pi^{-1}(0) = X \quad \text{and} \quad \frac{\partial X_t}{\partial t}_{|t=0} = -\theta?$$

Put differently: when can X be deformed in the *direction* of the given $-\theta$? A posteriori, $-\theta$ will be the tangent vector at 0 to B.

Obstructions to deforming a given complex structure

We need to construct vector-valued forms $\psi(t) \in C^{\infty}_{0,\,1}(X,\, T^{1,\,0}X)$ depending *holomorphically* on $t \in B$ and satisfying the *integrability condition*:

$$\bar{\partial}_0 \psi(t) = \frac{1}{2}\,[\psi(t),\, \psi(t)], \qquad t \in B,$$

for all $t \in B$ close to 0, such that $\psi(0) = 0$.

The *integrability condition* is equivalent to

$$\bar{\partial}_t^2 = 0,$$

where (recall)

$$\bar{\partial}_t \simeq \bar{\partial}_0 - \psi(t), \qquad t \in B,\, t \sim 0.$$

We need to construct $\psi(t)$ as a *convergent* power series

$$\psi(t) = \psi_1(t) + \sum_{\nu=2}^{+\infty} \psi_\nu(t), \tag{3}$$

where, for every $\nu \in \mathbb{N}^\star$, the vector-valued form

$$\psi_\nu(t) = \sum_{\nu_1+\cdots+\nu_m=\nu} \psi_{\nu_1\ldots\nu_m}\, t_1^{\nu_1} \ldots t_m^{\nu_m} \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$$

is a *homogeneous polynomial* of degree ν in the variables $t = (t_1, \ldots, t_m) \in B \subset \mathbb{C}^m$.

In particular, we are looking to construct vector-valued forms

$$\psi_{\nu_1\ldots\nu_m} \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$$

for $(\nu_1, \ldots, \nu_m) \in \mathbb{N}^m$.

The *integrability condition* is equivalent to the following system of equations:

$$(\text{Eq.}\,1) \quad \bar{\partial}\psi_1(t) = 0$$

$$(\text{Eq.}\,\nu) \quad \bar{\partial}\psi_\nu(t) = \frac{1}{2}\sum_{\mu=1}^{\nu-1}[\psi_\mu(t),\, \psi_{\nu-\mu}(t)], \qquad \text{with} \qquad \nu \geq 2, \quad (4)$$

that must be satisfied for all $t \in B$ sufficiently close to 0. Note that, for every $\nu \geq 1$, the terms featuring in $(\text{Eq.}\,\nu)$ are homogeneous polynomials of degree ν in $t = (t_1, \ldots, t_m) \in B$. This is an inductively defined system of equations in that, for every $\nu \geq 2$, the right-hand side term of $(\text{Eq.}\,\nu)$ is determined by the solutions ψ_λ of the previous equations $(\text{Eq.}\,\lambda)$ with $\lambda \leq \nu - 1$.

Suppose, furthermore, that a vector-valued form $\theta \in H^{0,1}(X, T^{1,0}X)$ has been given beforehand and that we are looking to deform $X = X_0$ in the *direction of* $-\theta$.

To make a choice, suppose that $(\partial/\partial t)_{|t=0} = (\partial/\partial t_k)_{|t=0}$ for some $k \in \{1, \dots, m\}$.

Then, we have the following extra condition on $\psi(t)$:

$$\left\{\frac{\partial\psi(t)}{\partial t_k}_{|t=0}\right\}_{\bar\partial} = \theta, \qquad \text{or equivalently} \qquad \{\psi_{0\dots1\dots0}\}_{\bar\partial} = \theta,$$

with 1 in the k-th slot in $\psi_{0\dots1\dots0}$.

Construction of $\psi_1(t)$. Note that $\bar\partial\psi_{0\dots 1\dots 0} = 0$ (with 1 in the k-th slot) for all $k \in \{1, \dots, m\}$, because $\bar\partial\psi_1(t) = 0$ for all t close to 0, by (Eq. 1). Since, ideally, we would like to reach every $\theta \in H^{0,1}(X, T^{1,0}X)$ (i.e. to deform X in all possible directions), we let

$$\{\beta_1, \dots, \beta_m\}$$

be a collection of m $\bar\partial$-closed vector-valued forms $\beta_\lambda \in C^\infty_{0,1}(X, T^{1,0}X)$ such that the set of their cohomology classes

$$\left\{\{\beta_1\}_{\bar\partial}, \dots, \{\beta_m\}_{\bar\partial}\right\}$$

is a *basis* of $H^{0,1}(X, T^{1,0}X)$, and we let

$$\psi_1(t) = \beta_1\, t_1 + \cdots + \beta_m\, t_m \in C^\infty_{0,1}(X, T^{1,0}X) \cap \ker\bar\partial$$

for a priori arbitrary complex variables $t_1, \dots, t_m \in \mathbb{C}$ such that $(t_1, \dots, t_m)$ is as close as will be necessary to $0 \in \mathbb{C}^m$.

In other words, we choose $\psi_{0\ldots 1\ldots 0} = \beta_\lambda$ (with 1 in the λ-th slot) for every $\lambda \in \{1, \ldots, m\}$.

In this way, $\psi_1(t)$ satisfies (Eq. 1) for all $t = (t_1, \ldots, t_m) \in \mathbb{C}^m$ and $\psi(t)$ can be made to satisfy the condition

$$\left\{ \frac{\partial \psi(t)}{\partial t_k}_{|t=0} \right\}_{\bar\partial} = \theta$$

for any pregiven choice of $\theta \in H^{0,\,1}(X,\, T^{1,\,0}X)$ after the $\psi_\nu(t)$'s with $\nu \geq 2$ have been constructed.

Construction of $(\psi_\nu(t))_{\nu\geq 2}$.

Lemma. *For every $\nu \geq 2$, the vector-valued form on the right-hand side of equation $(Eq.\,\nu)$ is* **$\bar\partial$-closed**.

Conclusion. *All the* **obstructions** *to solving the equations $(Eq.\,\nu)_{\nu\in\mathbb{N}^\star}$ lie in $H^{0,\,2}(X,\,T^{1,\,0}X)$.*

In other words, the right-hand side terms of equations (Eq. ν) define cohomology classes

$$\left\{\frac{1}{2}\sum_{\mu=1}^{\nu-1}[\psi_\mu(t),\,\psi_{\nu-\mu}(t)]\right\}_{\bar\partial}\in H^{0,\,2}(X,\,T^{1,\,0}X),\qquad \nu\geq 2.$$

These classes vanish in $H^{0,\,2}(X,\,T^{1,\,0}X) \iff$ the r.h.s. of equations $(\text{Eq.}\,\nu)_{\nu\geq 2}$ are $\bar\partial$-exact $\iff$ the equations (Eq. ν) are solvable.

The *qualitative obstructions* found above are the only obstructions to deforming the complex structure of X.

In other words, if all the equations $(\text{Eq. } \nu)_{\nu \geq 2}$ are solvable, their solutions $(\psi_\nu)_{\nu \geq 2}$ can always be chosen such that the power series defining $\psi(t)$ converges absolutely.

This is the content of the following important **existence theorem of Kodaira-Nirenberg-Spencer (1958)**.

Theorem. *Let X be a compact complex manifold such that*

$$H^{0,\,2}(X,\,T^{1,\,0}X) = 0.$$

Then, there exists a holomorphic family $\pi : \mathcal{X} \longrightarrow B \subset \mathbb{C}^m$ of compact complex manifolds, where $m := dim_{\mathbb{C}} H^{0,\,1}(X,\,T^{1,\,0}X)$ and B is a small open ball about the origin in $\mathbb{C}^m$, such that:

(i) $\pi^{-1}(0) = X$;

(ii) the Kodaira-Spencer map at 0

$$\rho : T_0^{1,\,0}B \longrightarrow H^{0,\,1}(X,\,T^{1,\,0}X), \qquad \frac{\partial}{\partial t}_{|t=0} \mapsto \frac{\partial X_t}{\partial t}_{|t=0},$$

is an **isomorphism**.

In other words, if the space $H^{0,\,2}(X,\,T^{1,\,0}X)$ that contains all the qualitative obstructions to locally deforming X *vanishes*, then X can, indeed, be deformed in all the available directions (parametrised by $H^{0,\,1}(X,\,T^{1,\,0}X)$).

Even more striking is the following

Bogomolov-Tian-Todorov Theorem. *Let X be a* **$\partial\bar\partial$-manifold** *whose canonical bundle K_X is* **trivial**.
Then, the Kuranishi family of X is **unobstructed**.

Calabi-Yau manifolds

Definition. *A compact complex manifold X is said to be a* **Calabi-Yau manifold** *if its canonical bundle K_X is* **trivial**.

Let $n = \dim_{\mathbb{C}} X$. Recall that the *canonical bundle* of X is the holomorphic line bundle of $(n,\, 0)$-forms on X:

$$K_X := \Lambda^{n,\,0} T^\star X = \det(\Lambda^{1,\,0} T^\star X) = -\det(T^{1,\,0} X).$$

Thus, if $(z_1, \dots, z_n)$ is a system of local holomorphic coordinates on X, the holomorphic n-form $dz_1 \wedge \cdots \wedge dz_n$ defines a local holomorphic frame of K_X.

As with any holomorphic line bundle, the triviality is equivalent to the existence of a *non-vanishing* global holomorphic section:

K_X is trivial

$$\iff \exists\, u \in H^0(X,\, K_X) \simeq H^{n,\,0}_{\bar\partial}(X,\, \mathbb{C}) \quad \text{such that} \quad u(x) \neq 0 \ \forall\, x \in X$$
$$\iff \exists\, u \in C^\infty_{n,\,0}(X,\, \mathbb{C}) \quad \text{such that} \quad \bar\partial u = 0 \quad \text{and} \quad u(x) \neq 0 \ \forall\, x \in X.$$

When K_X is trivial, the Hodge number $h^{n,\,0}_{\bar\partial} = 1$, so the non-vanishing holomorphic n-form u on X is *unique up to a multiplicative constant*. Such a form will be called a *Calabi-Yau form*. Note that $H^{n,\,0}_{\bar\partial}(X,\, \mathbb{C}) = C^\infty_{n,\,0}(X,\, \mathbb{C}) \cap \ker\bar\partial$ since, for bidegree reasons, the only $\bar\partial$-exact $(n,\, 0)$-form is zero. So, every $u \in C^\infty_{n,\,0}(X,\, \mathbb{C}) \cap \ker\bar\partial$ identifies with $[u]_{\bar\partial} \in H^{n,\,0}_{\bar\partial}(X,\, \mathbb{C}) \simeq H^0(X,\, K_X)$.

Lemma and Definition. *Suppose that* K_X *is* **trivial** *and let* u *be a* **Calabi-Yau form** *on* X. *Then, for every* $q = 0, \dots, n$, u *defines an isomorphism (that will be called the* **Calabi-Yau isomorphism***):*

$$T_u \ : \ C^\infty_{0,\,q}(X,\, T^{1,\,0}X) \xrightarrow{\cdot\lrcorner u} C^\infty_{n-1,\,q}(X,\, \mathbb{C})$$

mapping any $\theta \in C^\infty_{0,\,q}(X,\, T^{1,\,0}X)$ *to* $T_u(\theta) := \theta\lrcorner u$, *where the operation denoted by* $\cdot\lrcorner$ *combines the contraction of* u *by the* $(1,\,0)$-*vector field component of* θ *with the exterior multiplication by the* $(0,\,q)$-*form component.*

Lemma and Definition. *Suppose that* K_X *is* **trivial** *and let* u *be a* **Calabi-Yau form** *on* X. *Then, when* $q = 1$, *the isomorphism* T_u *satisfies:*

$$T_u(\ker \bar\partial) = \ker \bar\partial \quad \text{and} \quad T_u(Im\,\bar\partial) = Im\,\bar\partial. \tag{5}$$

Hence T_u *induces an isomorphism in cohomology*

$$T_{[u]} \; : \; H^{0,\,1}(X,\, T^{1,\,0}X) \stackrel{\cdot\lrcorner[u]}{\longrightarrow} H^{n-1,\,1}(X,\, \mathbb{C}) \tag{6}$$

defined by $T_{[u]}([\theta]) = [\theta\lrcorner u]$ *for all* $[\theta] \in H^{0,\,1}(X,\, T^{1,\,0}X)$.

The isomorphism $T_{[u]}$ *will be called the* **Calabi-Yau isomorphism in cohomology**.

Definition. *Suppose that* K_X *is* **trivial** *and let* u *be a* **Calabi-Yau form** *on* X. *For all* $q_1, q_2 \in \{0, \dots, n\}$, *define the following bracket:*

$$[\cdot,\,\cdot] : C^\infty_{n-1,\,q_1}(X,\,\mathbb{C}) \times C^\infty_{n-1,\,q_2}(X,\,\mathbb{C}) \longrightarrow C^\infty_{n-1,\,q_1+q_2}(X,\,\mathbb{C}),$$

$$[\zeta_1,\,\zeta_2] := T_u\bigg[T_u^{-1}\zeta_1,\,T_u^{-1}\zeta_2\bigg],$$

where the operation $[\ ,\ ]$ *on the right-hand side combines the Lie bracket of the* $T^{1,\,0}X$*-parts of* $T_u^{-1}\zeta_1 \in C^\infty_{0,\,q_1}(X,\,T^{1,\,0}X)$ *and* $T_u^{-1}\zeta_2 \in C^\infty_{0,\,q_2}(X,\,T^{1,\,0}X)$ *with the wedge product of their* $(0,\,q_1)$*- and respectively* $(0,\,q_2)$*-form parts.*

Sketch of proof of the Bogomolov-Tian-Todorov Theorem.

The main ingredient is the

Tian-Todorov Lemma. *Let X be a compact complex manifold ($n = dim_{\mathbb{C}} X$) such that K_X is* **trivial**. *Then, for any forms $\zeta_1, \zeta_2 \in C^\infty_{n-1,\,1}(X,\,\mathbb{C})$ such that $\partial\zeta_1 = \partial\zeta_2 = 0$, we have*

$$[\zeta_1,\,\zeta_2] \in Im\,\partial.$$

More precisely, the identity

$$[\theta_1 \lrcorner u,\,\theta_2 \lrcorner u] = -\partial(\theta_1 \lrcorner(\theta_2 \lrcorner u)) \tag{7}$$

holds for $\theta_1, \theta_2 \in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$ whenever $\partial(\theta_1 \lrcorner u) = \partial(\theta_2 \lrcorner u) = 0$.

How this is applied

Let $[\eta] \in H^{0,\,1}(X,\, T^{1,\,0}X)$ be an arbitrary nonzero class. Pick any d-closed representative w_1 of the class $[\eta]\lrcorner[u] \in H^{n-1,\,1}(X,\, \mathbb{C})$. Such a d-closed representative exists thanks to the $\partial\bar\partial$ assumption on X.

Since T_u is an isomorphism, there is a unique $\Phi_1 \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$ such that $\Phi_1\lrcorner u = w_1$. Now $\bar\partial w_1 = 0$, so the former equality in (5) implies that $\bar\partial\Phi_1 = 0$. Moreover, since $[\Phi_1\lrcorner u] = [w_1]$, (6) implies that $[\Phi_1] = [\eta] \in H^{0,\,1}(X,\, T^{1,\,0}X)$ and this is the original class we started off with. However, Φ_1 need not be the Δ''-harmonic representative of the class $[\eta]$ in the non-Kaehler case (in contrast to the Kähler case of [Tia87] and [Tod89]).

Meanwhile, by the choice of w_1, we have

$$\partial(\Phi_1 \lrcorner u) = 0,$$

so the Tian-Todorov Lemma applied to $\zeta_1 = \zeta_2 = \Phi_1 \lrcorner u$ yields $[\Phi_1 \lrcorner u,\, \Phi_1 \lrcorner u] \in \operatorname{Im} \partial$. On the other hand, $[\Phi_1 \lrcorner u,\, \Phi_1 \lrcorner u] \in \ker \bar\partial$ (easy to see). By the $\partial\bar\partial$-property of X applied to the $(n-1,\, 2)$-form $1/2\, [\Phi_1 \lrcorner u,\, \Phi_1 \lrcorner u]$, there exists $\psi_2 \in C^\infty_{n-2,\,1}(X,\, \mathbb{C})$ such that

$$\bar\partial\partial\psi_2 = \frac{1}{2}\, [\Phi_1 \lrcorner u,\, \Phi_1 \lrcorner u].$$

We can choose ψ_2 of minimal L^2-norm with this property (i.e. $\psi_2 \in \operatorname{Im}(\partial\bar\partial)^\star$, see the orthogonal three-space decomposition for the Aeppli cohomology).

Put $w_2 := \partial\psi_2 \in C^\infty_{n-1,\,1}(X,\,\mathbb{C})$. Since T_u is an isomorphism, there is a unique $\Phi_2 \in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$ such that $\Phi_2 \lrcorner u = w_2$. Implicitly, $\partial(\Phi_2 \lrcorner u) = 0$.

Moreover, we get

$$(\bar\partial\Phi_2)\lrcorner u = \bar\partial(\Phi_2\lrcorner u) = \frac{1}{2}\,[\Phi_1\lrcorner u,\,\Phi_1\lrcorner u] = \frac{1}{2}\,[\Phi_1,\,\Phi_1]\lrcorner u.$$

Hence

$$\text{(Eq. 1)} \qquad\qquad \bar\partial\Phi_2 = \frac{1}{2}\,[\Phi_1,\,\Phi_1].$$

We can now continue inductively.

Suppose we have constructed $\Phi_1, \ldots, \Phi_{N-1} \in C^{\infty}_{0,\,1}(X,\, T^{1,\,0}X)$ such that

$$\partial(\Phi_k \lrcorner u) = 0 \quad \text{and} \quad \bar\partial(\Phi_k \lrcorner u) = \frac{1}{2}\sum_{l=1}^{k-1}[\Phi_l \lrcorner u,\, \Phi_{k-l} \lrcorner u], \qquad 1 \leq k \leq N-1.$$

Since T_u is an isomorphism, the latter identity above is equivalent to

$$\text{(Eq. } (k-1)) \qquad \bar\partial\Phi_k = \frac{1}{2}\sum_{l=1}^{k-1}[\Phi_l,\, \Phi_{k-l}], \qquad 1 \leq k \leq N-1.$$

Then, again we have

$$\frac{1}{2}\sum_{l=1}^{N-1}[\Phi_l \lrcorner u,\, \Phi_{N-l} \lrcorner u] \in \ker\bar\partial.$$

On the other hand, since $\Phi_1 \lrcorner u, \dots, \Phi_{N-1} \lrcorner u \in \ker \partial$, the Tian-Todorov Lemma gives

$$[\Phi_l \lrcorner u, \, \Phi_{N-l} \lrcorner u] \in \operatorname{Im} \partial \quad \text{for all} \quad l = 1, \dots, N-1.$$

Thanks to the last two relations, the $\partial\bar\partial$-property of X implies the existence of a form $\psi_N \in C^\infty_{n-2,\,1}(X, \, \mathbb{C})$ such that

$$\bar\partial\partial\psi_N = \frac{1}{2} \sum_{l=1}^{N-1} [\Phi_l \lrcorner u, \, \Phi_{N-l} \lrcorner u].$$

We can choose ψ_N of minimal L^2-norm with this property (i.e. $\psi_N \in \operatorname{Im}(\partial\bar\partial)^\star$). Letting $w_N := \partial\psi_N \in C^\infty_{n-1,\,1}$, there exists a unique $\Phi_N \in C^\infty_{0,\,1}(X, \, T^{1,\,0}X)$ such that $\Phi_N \lrcorner u = w_N$. Implicitly

$$\partial(\Phi_N \lrcorner u) = 0.$$

We also have $\bar\partial(\Phi_N \lrcorner u) = \frac{1}{2}\sum_{l=1}^{N-1}[\Phi_l \lrcorner u,\, \Phi_{N-l} \lrcorner u]$ by construction. Since T_u is an isomorphism, this amounts to

$$\text{(Eq. } (N-1)) \qquad \bar\partial\Phi_N = \frac{1}{2}\sum_{l=1}^{N-1}[\Phi_l,\, \Phi_{N-l}].$$

We have thus shown inductively that the equation (Eq. k) is solvable for every $k \in \mathbb{N}^\star$.

This implies the convergence of the power series

$$\Phi(t) := \Phi_1\, t + \Phi_2\, t^2 + \cdots + \Phi_N\, t^N + \ldots$$

in all the Hölder norms $|\ \ |_{k+\alpha}$, with $k \geq 2$ and $\alpha \in (0,\, 1)$, for all $t \in \mathbb{C}$ such that $|t| < \varepsilon_k$, because the ψ_ν's have been chosen of minimal L^2 norms with their respective properties. $\square$