Hodge Theory of Compact Complex Manifolds

CIMPA School "Complex Analysis, Geometry and Dynamics"

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Lecture No. 2

Dan Popovici

Université Paul Sabatier, Toulouse, France

Context and motivation

X a compact complex manifold, $\dim_{\mathbb{C}} X = n$

Definition. *X is a* **$\partial\bar\partial$-manifold** *if*

$\forall p, q,\ \forall u \in C^\infty_{p,\,q}(X,\,\mathbb{C})$ *s.t.* $du = 0$, *we have equivalences:*

$$u \in Im\, d \iff u \in Im\, \partial \iff u \in Im\, \bar\partial \iff u \in Im\, (\partial\bar\partial).$$

The idea goes back to Deligne-Griffiths-Morgan-Sullivan 1975.

Standard fact. ([DGMS75]) X *is a* **$\partial\bar\partial$-manifold** $\iff$ $\forall k \in \{0, 1, \dots, 2n\}$, *the identity induces an* **isomorphism**

$$H^k_{DR}(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,\,q}_{\bar\partial}(X, \mathbb{C}) \qquad (\textit{Hodge decomposition})$$

in the following sense:

- $\forall (p,\, q)$ *s.t.* $p+q=k$, *every class* $[\alpha^{p,\,q}]_{\bar\partial} \in H^{p,\,q}_{\bar\partial}(X, \mathbb{C})$ *can be represented by a* d**-closed** $(p,\, q)$*-form* $\alpha^{p,\,q}$*;*

- *the linear map*

$$\bigoplus_{p+q=k} H^{p,\,q}_{\bar\partial}(X, \mathbb{C}) \ni \sum_{p+q=k} [\alpha^{p,\,q}]_{\bar\partial} \longmapsto \left\{\sum_{p+q=k} \alpha^{p,\,q}\right\}_{DR} \in H^k_{DR}(X, \mathbb{C})$$

is independent of the choices of d**-closed** *representatives* $\alpha^{p,\,q}$ *of the classes* $[\alpha^{p,\,q}]_{\bar\partial}$ *(i.e.* **well-defined***) and* **bijective**.

(i.e. X is *cohomologically Kähler*)

The Frölicher spectral sequence

Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$.

Page 0: the **Dolbeault complex**, i.e.

$$\dots \stackrel{d_0}{\longrightarrow} E_0^{p,\,q-1} \stackrel{d_0}{\longrightarrow} E_0^{p,\,q} \stackrel{d_0}{\longrightarrow} E_0^{p,\,q+1} \stackrel{d_0}{\longrightarrow} \dots,$$

with $E_0^{p,\,q} := C^\infty_{p,\,q}(X,\,\mathbb{C})$ (smooth $(p,\,q)$-forms on X) and $d_0 := \bar\partial$. Put

$$E_1^{p,\,q} := \ker d_0^{p,\,q}/\mathrm{Im}\, d_0^{p,\,q-1} = H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}).$$

Page 1: the cohomology spaces of page 0, i.e.

$$\dots \stackrel{d_1}{\longrightarrow} E_1^{p-1,\,q} \stackrel{d_1}{\longrightarrow} E_1^{p,\,q} \stackrel{d_1}{\longrightarrow} E_1^{p+1,\,q} \stackrel{d_1}{\longrightarrow} \dots,$$

with differential defined as $d_1([\alpha]_{\bar\partial}) := [\partial\alpha]_{\bar\partial}$.

Page r:

$$\cdots \xrightarrow{d_r} E_r^{p-r,\,q+r-1} \xrightarrow{d_r} E_r^{p,\,q} \xrightarrow{d_r} E_r^{p+r,\,q-r+1} \xrightarrow{d_r} \cdots.$$

So, d_r is of bidegree $(r,\,-r+1)$ for every $r \in \mathbb{N}^\star$. Put

$$E_{r+1}^{p,\,q} := \ker d_r^{p,\,q}/\mathrm{Im}\, d_r^{p-r,\,q+r-1}.$$

Fact (Frölicher 1955): This spectral sequence converges to the **De Rham cohomology** of X, i.e. there are **(non-canonical) isomorphisms**:

$$H^k_{DR}(X,\,\mathbb{C}) \simeq \bigoplus_{p+q=k} E_\infty^{p,\,q}, \qquad k = 0, \dots, 2n,$$

where $E_\infty^{p,\,q} = \cdots = E_{r+2}^{p,\,q} = E_{r+1}^{p,\,q} = E_r^{p,\,q}$ for all p, q and where $r \geq 1$ is the smallest positive integer such that the spectral sequence degenerates at E_r. (We write $E_r(X) = E_\infty(X)$.)

Thus, the degeneration at E_r is a purely **numerical property**:

$$E_r(X) = E_\infty(X) \iff b_k = \sum_{p+q=k} \dim_{\mathbb{C}} E_r^{p,\,q} \quad \forall k = 0, \dots, 2n.$$

In particular,

$$\sum_{p+q=k} h^{p,\,q} \geq \cdots \geq \sum_{p+q=k} \dim_{\mathbb{C}} E_l^{p,\,q} \geq \sum_{p+q=k} \dim_{\mathbb{C}} E_{l+1}^{p,\,q} \geq \cdots \geq b_k.$$

Hence, the following implications hold:

$$E_1(X) = E_\infty(X) \implies E_2(X) = E_\infty(X) \implies \cdots \implies E_r(X) = E_\infty(X)$$

•

Relations to other properties

- If X is a $\partial\bar{\partial}$**-manifold**, then $E_1(X) = E_\infty(X)$.

- The converse is false.

e.g. If $\dim_{\mathbb{C}} X = 2$ (i.e. a complex surface), then

· $E_1(X) = E_\infty(X)$

· X is a $\partial\bar{\partial}$**-manifold** $\iff$ X is **Kähler**.

- The property $E_1(X) = E_\infty(X)$ does not imply either the Hodge symmetry or the canonical Hodge decomposition. It only implies the much weaker numerical Hodge decomposition.

Standard facts.

- The following implications hold:

X is compact *Kähler* $\implies$ X is *class* $\mathcal{C}$ $\implies$ X is a $\partial\bar{\partial}$*-manifold*

$\implies$ $E_1(X) = E_\infty(X)$ (in the Frölicher spectral sequence – FSS)

If $n \geq 3$, all the implications are strict.

• If X is a $\partial\bar\partial$-manifold, X has the *Hodge symmetry* property:

for all p, q,

(i) every class $[\alpha^{p,\,q}]_{\bar\partial} \in H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ can be represented by a ***d*-closed** $(p,\,q)$-form $\alpha^{p,\,q}$;

(ii) the linear map

$$H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) \ni [\alpha^{p,\,q}]_{\bar\partial} \longmapsto \overline{[\overline{\alpha^{p,\,q}}]_{\bar\partial}} \in \overline{H^{q,\,p}_{\bar\partial}(X,\,\mathbb{C})}$$

is independent of the choices of ***d*-closed** representatives $\alpha^{p,\,q}$ of the classes $[\alpha^{p,\,q}]_{\bar\partial}$ (i.e. **well-defined**) and **bijective**.

(A) First type of operations on compact manifolds

Modifications : $\sigma : \widetilde{X} \to X$ **holomorphic, bimeromorphic**

Examples : (i) X is called **Moishezon** if $\exists\, \sigma : \widetilde{X} \to X$ modification with $\widetilde{X}$ **projective** ;

Recall: $\widetilde{X}$ **projective** $\overset{\text{def}}{\Longleftrightarrow} \exists\, N \in \mathbb{N}^\star$ s.t. $\widetilde{X} \hookrightarrow \mathbb{CP}^N$
(embedding as a closed submanifold)

(ii) X is called **class** $\mathcal{C}$ if $\exists\, \sigma : \widetilde{X} \to X$ modification with $\widetilde{X}$ compact **Kähler**.

Implications (all are strict)

X **Kähler**

$\Nearrow$ $\Searrow$

X **projective** X **class C**

$\Searrow$ $\Nearrow$

X **Moishezon**

Demailly-Paun (2001) : X is *class* $\mathcal{C}$ $\iff$ $\exists T$ **Kähler current** on X (i.e. $dT = 0$ and $T > 0$).

Moishezon (1967) : if X is **Moishezon** and **non-projective**, then X is **not Kähler**.

Examples.

(1) The twistor space X of any *K3 surface* has $E_1(X) = E_\infty(X)$ but is *not a* $\partial\bar{\partial}$*-manifold*.

(*no Hodge symmetry* – P. 2011)

(2) Let $X = G/H$, also denoted $I^{(3)}$, be the **Iwasawa manifold**, where

$$G := \left\{ M = \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \; ; \; z_1, z_2, z_3 \in \mathbb{C} \right\} \subset GL_3(\mathbb{C})$$

and $H \subset G$ is its discrete subgroup $\Gamma \subset G$ of matrices with entries $z_1, z_2, z_3 \in \mathbb{Z}[i]$.

$I^{(3)}$ is a compact complex manifold, $\dim_{\mathbb{C}} I^{(3)} = 3$.

There exist C^{∞} (1, 0)-forms α, β, γ on X, induced resp. by dz_1, dz_2, $dz_3 - z_1 dz_2$ (look at $M \mapsto M^{-1}dM$) satisfying:

$$\bar\partial\alpha = \bar\partial\beta = \bar\partial\gamma = 0$$

but

$$\partial\alpha = \partial\beta = 0 \quad \text{and} \quad \partial\gamma = -\alpha \wedge \beta \neq 0.$$

Therefore, $E_1(X) \neq E_\infty(X)$. In particular, X is *not a $\partial\bar\partial$-manifold*.

• However, $E_2(X) = E_\infty(X)$. This leads to a Hodge theory for X if the $E_1^{p,\,q}(X)$'s are replaced by the $E_2^{p,\,q}(X)$'s.

(exploited in P. 2018: "Non-Kähler Mirror Symmetry of the Iwasawa Manifold")

Implications (all are strict)

X **balanced**

$\Nearrow$

$\Searrow$

X *class* $\mathcal{C}$ $\qquad\qquad$ X **sG**

$\Searrow$

$\Nearrow$

$\mathbf{X}\ \partial\bar\partial$

$\Searrow$

$\mathbf{E_1(X) = E_\infty(X)}$
(Frölicher spectral sequence)

Bott-Chern and Aeppli cohomologies

• The **Bott-Chern cohomology**

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) = \frac{\ker\partial\cap\ker\bar\partial}{\operatorname{Im}\partial\bar\partial}.$$

The **Aeppli cohomology**

$$H^{p,\,q}_{A}(X,\,\mathbb{C}) = \frac{\ker\partial\bar\partial}{\operatorname{Im}\partial + \operatorname{Im}\bar\partial}.$$

Bott-Chern and Aeppli Laplacians

- **Bott-Chern case** (Kodaira-Spencer 1960)

The 4-th order **Bott-Chern Laplacian**

$$\Delta_{BC} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C})$$

is defined as

$$\Delta_{BC} := \partial^\star\partial + \bar\partial^\star\bar\partial + (\partial\bar\partial)(\partial\bar\partial)^\star + (\partial\bar\partial)^\star(\partial\bar\partial) + (\partial^\star\bar\partial)^\star(\partial^\star\bar\partial) + (\partial^\star\bar\partial)(\partial^\star\bar\partial)^\star.$$

- **Aeppli case** (Schweitzer 2017)

The 4-th order **Aeppli Laplacian**

$$\Delta_A : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C})$$

is defined as

$$\Delta_A := (\partial\bar\partial)^\star(\partial\bar\partial) + \partial\partial^\star + \bar\partial\bar\partial^\star + (\partial\bar\partial)(\partial\bar\partial)^\star + (\partial\bar\partial^\star)(\partial\bar\partial^\star)^\star + (\partial\bar\partial^\star)^\star(\partial\bar\partial^\star).$$

• **Ellipticity**

The key property of these differential operators is

Theorem. Δ_{BC} and Δ_A are **elliptic**.

Idea of proof.

• We may assume, without loss of generality, that we are in an open subset of $\mathbb{C}^n$ and that the metric ω is the *standard* one:

$$\omega = \sum_{j=1}^{n} i dz_j \wedge d\bar{z}_j.$$

Indeed, the ellipticity of Δ_{BC} depends solely on its principal part, which remains unchanged if a different metric is chosen. Metric changes affect only the lower order terms.

• We use the following expressions of $\partial^\star$ and $\bar\partial^\star$ in local coordinates w.r.t. the *standard metric*:

$$\partial = \sum_{j=1}^n dz_j \wedge \frac{\partial}{\partial z_j}, \qquad \text{hence} \qquad \partial^\star = -\sum_{l=1}^n \frac{\partial}{\partial \bar z_l}\Big(\frac{\partial}{\partial z_l} \lrcorner \cdot \Big)$$

$$\bar\partial = \sum_{k=1}^n d\bar z_k \wedge \frac{\partial}{\partial \bar z_k}, \qquad \text{hence} \qquad \bar\partial^\star = -\sum_{r=1}^n \frac{\partial}{\partial z_r}\Big(\frac{\partial}{\partial \bar z_r} \lrcorner \cdot \Big),$$

where $\xi \lrcorner \cdot$ is the *contraction* (a zero-th order operator) of differential forms by the vector field ξ.

These formulae follow from the following easy-to-check formulae:

$$\Big(\frac{\partial}{\partial z_j}\Big)^\star = -\frac{\partial}{\partial \bar z_j} \quad \text{and} \quad \Big(\frac{\partial}{\partial z_j} \lrcorner \cdot \Big)^\star = dz_j \wedge \cdot$$

and their conjugates.

• Computations lead to the following formula for the principal symbol of the Bott-Chern Laplacian:

$$\sigma_{\Delta_{BC}}(x;\,(\xi,\,\eta))\,u(x) = \left[\frac{1}{4}\sum_{j=1}^{n}(\xi_j^2+\eta_j^2)\right]^2 u(x),$$

for all forms u and all points $(x;\,(\xi,\,\eta)) \in {}^{\mathbb{R}}TX$ in the real tangent bundle of X, where we put

$$(\xi,\,\eta) = \sum_{j=1}^{n}\xi_j(x)\,\frac{\partial}{\partial x_j} + \sum_{j=1}^{n}\eta_j(x)\,\frac{\partial}{\partial y_j} \in {}^{\mathbb{R}}T_xX.$$

In particular, $\sigma_{\Delta_{BC}}(x;\,(\xi,\,\eta))$ is injective for all $x \in X$ and all $(\xi,\,\eta) \neq 0$. Consequently, Δ_{BC} is elliptic. $\Box$

• **Consequences of ellipticity**

(a) Hodge isomorphisms

Bott-Chern case

Corollary *Let* (X,ω) *be a* **compact** *Hermitian manifold with* $dim_{\mathbb{C}}X = n$*. Fix arbitrary* $p, q \in \{0, \dots, n\}$*.*

(1) The following L^2_ω**-orthogonal three-space decomposition** *holds:*

$$C^{\infty}_{p,\,q}(X,\mathbb{C}) = \ker \Delta_{BC} \oplus Im\,\partial\bar\partial \oplus (Im\,\partial^{\star} + Im\,\bar\partial^{\star}).$$

(2) Moreover

$$\ker \partial \cap \ker \bar{\partial} = \ker \Delta_{BC} \oplus Im\, \partial\bar{\partial},$$

yielding the **Hodge isomorphism**

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\,\mathbb{C}),$$

where

$$\mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\,\mathbb{C}) = \ker(\Delta_{BC} : C^{\infty}_{p,\,q}(X,\,\mathbb{C}) \to C^{\infty}_{p,\,q}(X,\,\mathbb{C}))$$

is the **Bott-Chern harmonic space***.*

In particular, $dim\, H^{p,\,q}_{BC}(X,\,\mathbb{C}) < +\infty.$

(3) We also have:

$$\begin{aligned} Im\,\Delta_{BC} &= Im\,\partial\bar{\partial} \oplus (Im\,\partial^\star + Im\,\bar{\partial}^\star) \\ \ker(\partial\bar{\partial})^\star &= \ker\Delta_{BC} \oplus (Im\,\partial^\star + Im\,\bar{\partial}^\star). \end{aligned}$$

Hence

$$\ker\Delta_{BC} = \ker\partial \cap \ker\bar{\partial} \cap \ker(\partial\bar{\partial})^\star.$$

Aeppli case

Corollary *Let* (X, ω) *be a* **compact** *Hermitian manifold with* $dim_{\mathbb{C}} X = n$. *Fix arbitrary* $p, q \in \{0, \dots, n\}$.

(1) The following L^2_ω**-orthogonal three-space decomposition** *holds:*

$$C^\infty_{p,\,q}(X, \mathbb{C}) = \ker \Delta_A \oplus (Im\, \partial + Im\, \bar\partial) \oplus Im\, (\partial\bar\partial)^\star.$$

(2) Moreover

$$\ker(\partial\bar\partial) = \ker \Delta_A \oplus (Im\,\partial + Im\,\bar\partial),$$

yielding the **Hodge isomorphism**

$$H^{p,\,q}_A(X,\,\mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta_A}(X,\,\mathbb{C}),$$

where

$$\mathcal{H}^{p,\,q}_{\Delta_A}(X,\,\mathbb{C}) = \ker(\Delta_A : C^{\infty}_{p,\,q}(X,\,\mathbb{C}) \to C^{\infty}_{p,\,q}(X,\,\mathbb{C}))$$

is the **Aeppli harmonic space**.

In particular, $dim\,H^{p,\,q}_A(X,\,\mathbb{C}) < +\infty$.

(3) We also have

$$\begin{aligned} Im\,\Delta_A &= (Im\,\partial + Im\,\bar\partial) \oplus Im\,(\partial\bar\partial)^\star \\ \ker\partial^\star \cap \ker\bar\partial^\star &= \ker\Delta_A \oplus Im\,(\partial\bar\partial)^\star. \end{aligned}$$

Hence

$$\ker\Delta_A = \ker(\partial\bar\partial) \cap \ker\partial^\star \cap \ker\bar\partial^\star.$$

(b) Bott-Chern/Aeppli duality

There is a canonical non-degenerate **duality** between the Bott-Chern and the Aeppli cohomologies of complementary bidegrees.

Theorem *Let* X *be a* **compact** *complex manifold with* $\dim_{\mathbb{C}} X = n$*. Then, for all* $p, q \in \{0, \dots, n\}$*, the bilinear pairing*

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \times H^{n-p,\,n-q}_{A}(X,\,\mathbb{C}) \to \mathbb{C}, \qquad ([\alpha]_{BC},\,[\beta]_A) \mapsto \int\limits_X \alpha\wedge\beta,$$

is **well-defined** *and* **non-degenerate**.

Thus, $H^{p,\,q}_{BC}(X,\,\mathbb{C})$ *is the dual of* $H^{n-p,\,n-q}_{A}(X,\,\mathbb{C})$.

Sketch of proof.

- **Observation** *Under the Hodge star isomorphism*

$$\star = \star_\omega : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{n-q,\,n-p}(X,\,\mathbb{C}), \qquad u \wedge \star\bar{v} = \langle u,\, v\rangle_\omega \, dV_\omega, \quad (1)$$

the Bott-Chern and Aeppli three-space decompositions are related by the following **isomorphisms***:*

$$\begin{aligned} \star : \mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\,\mathbb{C}) &\longrightarrow \mathcal{H}^{n-q,\,n-p}_{\Delta_A}(X,\,\mathbb{C}) \\ \star : Im\,(\partial\bar\partial) &\longrightarrow Im\,(\partial\bar\partial)^\star \\ \star : (Im\,\partial^\star + Im\,\bar\partial^\star) &\longrightarrow (Im\,\partial + Im\,\bar\partial). \end{aligned}$$

Sketch of proof of Observation.

The inclusions "$\subset$" follow easily from the following formulae (easy verification):

$$\star\star = (-1)^k \,\mathrm{Id} \qquad \text{on } k\text{-forms};$$

$$\partial^\star = - \star \bar\partial \star \qquad \bar\partial^\star = - \star \partial \star, \qquad d^\star = - \star d \star .$$

Then, we get the following equivalences for every form u:

$$\begin{aligned} u \in \mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\,\mathbb{C}) &\iff \partial u = 0, \bar\partial u = 0, (\partial\bar\partial)^\star u = 0 \\ &\iff \bar\partial^\star(\star u) = 0, \partial^\star(\star u) = 0, \partial\bar\partial(\star u) = 0 \\ &\iff \star u \in \mathcal{H}^{n-q,\,n-p}_{\Delta_A}(X,\,\mathbb{C}). \end{aligned}$$

End of proof of the Bott-Chern/Aeppli duality.

- *Well-definedness*

If α is changed to another representative $\alpha + \partial\bar\partial u$ of the same Bott-Chern cohomology class, then

$$\int_X (\alpha + \partial\bar\partial u) \wedge \beta = \int_X \alpha \wedge \beta \pm \int_X u \wedge \partial\bar\partial\beta = \int_X \alpha \wedge \beta$$

since $\partial\bar\partial\beta = 0$ (because β represents an Aeppli class).

On the other hand, if β is changed to another representative $\beta + \partial\xi + \bar\partial\zeta$ of the same Aeppli cohomology class, then

$$\int_X \alpha\wedge(\beta+\partial\xi+\bar\partial\zeta) = \int_X \alpha\wedge\beta \pm \int_X \partial\alpha\wedge\xi \pm \int_X \bar\partial\alpha\wedge\zeta = \int_X \alpha\wedge\beta$$

since $\partial\alpha = 0$ and $\bar\partial\alpha = 0$ (because α represents a Bott-Chern class).

Conclusion: the bilinear map in the statement is independent of the choices of representatives of the cohomology classes involved.

• *Non-degeneracy*

Fix an arbitrary Hermitian metric ω on X.

Let $[\alpha]_{BC} \in H^{p,\,q}_{BC}(X,\,\mathbb{C})$ be a non-zero class. Thanks to the Hodge isomorphism for the Bott-Chern cohomology, this class contains a unique (and necessarily non-zero) Bott-Chern harmonic representative. Let us call it $\alpha \in \mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\,\mathbb{C}) \setminus \{0\}$.

We must have $\star\alpha \in \mathcal{H}^{n-q,\,n-p}_{\Delta_A}(X,\,\mathbb{C})$. Then, we also have

$$\star\bar\alpha \in \mathcal{H}^{n-p,\,n-q}_{\Delta_A}(X,\,\mathbb{C})$$

(immediate verification).

Thus, $\star\bar\alpha$ defines a class in $H^{n-p,\,n-q}_{A}(X,\,\mathbb{C})$ and, under the pairing in the statement, we get

$$(\alpha,\,\star\bar\alpha) \mapsto \int\limits_X \alpha\wedge\star\bar\alpha = \int\limits_X |\alpha|^2_\omega\,dV_\omega = ||\alpha||^2_\omega \neq 0.$$

Similarly, let $[\beta]_A \in H_A^{n-p,\, n-q}(X,\, \mathbb{C})$ be a non-zero class and let β be its Aeppli harmonic representative. Then, $\beta \neq 0$ and $\star\bar\beta$ is Bott-Chern harmonic of bidegree $(p,\, q)$. Since

$$(\star\bar\beta,\, \beta) \mapsto \int\limits_X \star\bar\beta\wedge\beta = \int\limits_X |\beta|^2_\omega\, dV_\omega = ||\beta||^2_\omega \neq 0,$$

we are done. $\square$