Hodge Theory of Compact Complex Manifolds

CIMPA School “Complex Analysis, Geometry and Dynamics”

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Lecture No. 1

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Context and motivation

X a **compact complex manifold**, $\dim_{\mathbb{C}} X = n$

This means that X is a compact differentiable (C^∞) manifold equipped with a **holomorphic atlas** with values in $\mathbb{C}^n$, namely with

- an open cover $(U_\alpha)_\alpha$

and

- C^∞ maps $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{C}^n$ such that the transition maps

$$\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$$

are *holomorphic*.

Equivalently, a complex manifold is a C^∞-differentiable manifold X equipped with a **complex structure**.

This is an *almost complex structure*, namely an endomorphism

$$J : TX^{\mathbb{R}} \longrightarrow TX^{\mathbb{R}}$$

of the real tangent bundle such that

$$J^2 = -\mathrm{Id},$$

which is further required to be *integrable* (in the sense that what is called its *Nijenhuis tensor* N_J vanishes).

Alternatively, the complex structure can be seen as a splitting

$$d = \partial + \bar\partial$$

of the Poincaré differential operator

$$d : C^\infty_k(X,\, \mathbb{C}) \longrightarrow C^\infty_{k+1}(X,\, \mathbb{C})$$

of order one acting on the $\mathbb{C}$-valued C^∞ differential forms of any degree $k \in \{0, \dots, 2n\}$ on X into two differential operators of order one:

$$\partial : C^\infty_{p,\,q}(X,\, \mathbb{C}) \longrightarrow C^\infty_{p+1,\,q}(X,\, \mathbb{C})$$
$$\bar\partial : C^\infty_{p,\,q}(X,\, \mathbb{C}) \longrightarrow C^\infty_{p,\,q+1}(X,\, \mathbb{C})$$

acting on the $\mathbb{C}$-valued C^∞ differential forms of any bidegree $(p,\, q)$, with $p, q \in \{0, \dots, n\}$, on X.

For any complex structure $d = \partial + \bar{\partial}$, one has

$$\bar{\partial}^2 = 0,$$

a property that is equivalent to the *integrability condition*.

This further implies that

$$\partial^2 = 0 \quad \text{and} \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

At the local level

If $(z_1, \ldots, z_n)$ are local holomorphic coordinates on an open subset $U \subset X$, we have

$$z_k = x_k + i\, y_k$$

for every k and

$$(x_1, y_1, \ldots, x_n, y_n)$$

is a system of local C^∞ real coordinates on U.

The 1-forms

$$dz_k := dx_k + idy_k$$

are said to be of *bidegree* (or *type*) (1, 0), while the 1-forms

$$d\bar{z}_k := dx_k - idy_k$$

are said to be of *bidegree* (or *type*) (0, 1).

For any $p, q \in \{0, \ldots, n\}$, with $p + q = k \in \{0, \ldots, 2n\}$, the differential forms of bidegree (or type) $(p,\, q)$ are those k-forms that are generated (locally on U) by exterior products of p dz_j's and q $d\bar{z}_k$'s:

$$u = \sum_{|J|=p,\, |K|=q} u_{I\bar{J}}\, dz_J \wedge d\bar{z}_K,$$

where the coefficients $u_{I\bar{J}}$ are C^∞ $\mathbb{C}$-valued functions on U, while $J := (1 \leq j_1 < \cdots < j_p \leq n)$ and $K := (1 \leq k_1 < \cdots < k_q \leq n)$ are multi-indices of lengths p, resp. q. One puts:

$$dz_J := dz_{j_1} \wedge \cdots \wedge dz_{j_p}$$

and

$$d\bar{z}_K := d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_q}.$$

A C^∞ $(p,\ q)$-form on X is a globally and intrinsically defined object.

Its local shape (1) transforms, under a change of local holomorphic coordinates from $(z_1, \dots, z_n)$ on some open subset $U \subset X$ to $(w_1, \dots, w_n)$ on some open subset $V \subset X$, according to the usual rules of calculus, starting from the identities:

$$dz_j = \sum_{k=1}^{n} \frac{\partial z_j}{\partial w_k}\, dw_k \quad \text{and} \quad d\bar{z}_j = \sum_{k=1}^{n} \frac{\partial \bar{z}_j}{\partial \overline{w}_k}\, d\overline{w}_k$$

on $U \cap V$ for every $j \in \{1, \dots, n\}$.

The vector fields:

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j}\right)$$

on U are said to be of type (1, 0), while the vector fields

$$\frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\,\frac{\partial}{\partial y_j}\right)$$

on U are said to be of type (0, 1).

The differential of a C^1 function $f : U \longrightarrow \mathbb{C}$ is the 1-form on U given by

$$\begin{aligned} df &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\, dx_j + \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}\, dy_j \\ &= \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\, dz_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}\, d\bar{z}_j = \partial f + \bar{\partial} f, \end{aligned}$$

where we put $\partial f := \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\, dz_j$ (a $(1,\ 0)$-form) and $\bar{\partial} f := \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}\, d\bar{z}_j$ (a $(0,\ 1)$-form).

Moreover, a C^∞ function $f : U \longrightarrow \mathbb{C}$ is *holomorphic* if and only if

$$\frac{\partial f}{\partial \bar{z}_j} = 0$$

for all $j \in \{1, \ldots, n\}$, namely if and only if $\bar{\partial} f = 0$.

These are the **Cauchy-Riemann equations**.

For an arbitrary C^1 form u of bidegree (or type) $(p,\, q)$ on X, ∂u and $\bar\partial u$ are a $(p+1,\, q)$-form, resp. a $(p,\, q+1)$-form, on X.

In local coordinates, they are obtained by applying ∂, resp. $\bar\partial$, to the coefficients of u written locally in the form (1), so we get:

$$\partial u = \sum_{|J|=p,\, |K|=q} \partial u_{I\bar J} \wedge dz_J \wedge d\bar z_K \quad \text{and} \quad \bar\partial u = \sum_{|J|=p,\, |K|=q} \bar\partial u_{I\bar J} \wedge dz_J$$

where we have (see above):

$$\partial u_{I\bar J} = \sum_{j=1}^{n} \frac{\partial u_{I\bar J}}{\partial z_j}\, dz_j \quad \text{and} \quad \bar\partial u_{I\bar J} = \sum_{j=1}^{n} \frac{\partial u_{I\bar J}}{\partial \bar z_j}\, d\bar z_j.$$

We stress again that $(p,\, q)$-forms and k-forms on X (in particular, the 1-form $df = \partial f + \bar\partial f$ for any C^1 function $f : X \to \mathbb{C}$) are globally and intrinsically defined objects on X.

Indeed, if $TX^{\mathbb{R}}$ denotes the real tangent bundle and $(TX^{\mathbb{R}})^\star$ the real cotangent bundle of X, the complexified exterior algebra $\Lambda^\bullet(\mathbb{C} \otimes TX)^\star := \mathbb{C} \otimes_{\mathbb{R}} \Lambda^\bullet(TX^{\mathbb{R}})^\star$ splits canonically at every point of X as

$$\Lambda^k(\mathbb{C} \otimes TX)^\star = \sum_{p+q=k} \Lambda^{p,\, q} T^\star X, \qquad 0 \leq k \leq 2n,$$

where the space of $(p,\, q)$-forms is defined pointwise as

$$\Lambda^{p,\, q} T^\star X := \Lambda^p T^\star X \otimes \Lambda^q \overline{T^\star X},$$

where $T^\star X$ is the holomorphic cotangent bundle of X (generated locally by the $(1,\, 0)$-forms $dz_1, \dots, dz_n$) and $\overline{T^\star X}$ is the anti-holomorphic cotangent bundle of X (generated locally by the $(0,\, 1)$-forms $d\bar z_1, \dots, d\bar z_n$).

In particular, every k-form α splits uniquely into pure-type forms $\alpha^{p,\,q}$ of respective bidegrees $(p,\,q)$:

$$\alpha = \sum_{p+q=k} \alpha^{p,\,q}.$$

The forms $\alpha^{p,\,q}$ are called the pure-type components of α.

Thus, for every k, we get a splitting:

$$C^\infty_k(X,\,\mathbb{C}) = \bigoplus_{p+q=k} C^\infty_{p,\,q}(X,\,\mathbb{C}).$$

(1) Metric point of view

Hermitian metric on a given complex manifold X:

C^∞, positive definite, (1, 1)-form ω on X

In local coordinates $(z_1, \ldots, z_n)$ on some open subset $U \subset X$, it has the shape

$$\omega = \sum_{j,\,k=1}^{n} \omega_{j\bar{k}}\, i dz_j \wedge d\bar{z}_k,$$

where the $\omega_{j\bar{k}}$'s are $\mathbb{C}$-valued C^∞ functions on U such that the matrix $(\omega_{j\bar{k}})_{1\le j,\,k\le n}$ is *positive definite* at every point of U.

This is equivalent to saying that ω defines a pointwise (positive definite) inner product

$$\langle \cdot\,,\,\cdot \rangle_\omega : T^{1,0}X \times T^{1,0}X \longrightarrow \mathbb{C}$$

on the holomorphic tangent bundle of X:

$$\langle \cdot\,,\,\cdot \rangle_{\omega,\,x} : T^{1,0}_x X \times T^{1,0}_x X \longrightarrow \mathbb{C}, \qquad x \in X,$$

and that the inner product $\langle \cdot\,,\,\cdot \rangle_{\omega,\,x}$ on $T^{1,0}_x X$ depends in a C^∞ way on $x \in X$.

Explicitly, $T^{1,0}X$ is generated by $\partial/\partial z_1, \dots, \partial/\partial z_n$ on U and

$$\left\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \right\rangle_\omega = \omega_{j\bar{k}}, \qquad 1 \leq j,\, k \leq n,$$

at every point $x \in X$.

The inner product $\langle \cdot\,,\,\cdot \rangle_\omega$ induces by duality a pointwise inner product, denoted by the same symbol, on the holomorphic cotangent bundle $\Lambda^{1,\,0}T^\star X$ (generated by $dz_1, \ldots, dz_n$ on U), given by

$$\langle dz_j\,,\, dz_k \rangle_\omega = \omega^{j\bar{k}}, \qquad 1 \leq j,\, k \leq n,$$

at every point $x \in X$, where the matrix $(\omega^{j\bar{k}})_{1 \leq j,\, k \leq n}$ is the transpose of the inverse of $(\omega_{j\bar{k}})_{1 \leq j,\, k \leq n}$ at every point.

By conjugation, we get an induced inner product on $\Lambda^{0,\,1}T^\star X$ and thus also on

$$\mathbb{C}T^\star X = \Lambda^{1,\,0}T^\star X \oplus \Lambda^{0,\,1}T^\star X$$

by putting

$$\langle dz_j\,,\, d\bar{z}_k \rangle_\omega = 0$$

for all j, k.

More generally, ω induces a pointwise inner product on $\Lambda^{p,\,q}T^\star X$ for every bidegree $(p,\,q)$. From this, we get an L^2 *inner product* on the space of global C^∞ $(p,\,q)$-forms on X, defined by

$$\langle\langle u,\,v\rangle\rangle_\omega := \int\limits_X \langle u(x)\,,\,v(x)\rangle_\omega\,dV_\omega,$$

where $dV_\omega := \omega^n/n!$ is the volume form on X induced by ω.

The L^2 inner product defined by a given Hermitian metric ω induces **formal adjoints**

$$d^\star = d^\star_\omega : C^\infty_k(X,\,\mathbb{C}) \longrightarrow C^\infty_{k-1}(X,\,\mathbb{C}),$$
$$\partial^\star = \partial^\star_\omega : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p-1,\,q}(X,\,\mathbb{C}),$$
$$\bar\partial^\star = \bar\partial^\star_\omega : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q-1}(X,\,\mathbb{C})$$

of the differential operators d, ∂ and $\bar\partial$.

An important notion is the following

Definition. *A* **Kähler metric** *on a complex manifold X is a Hermitian metric ω on X such that $d\omega = 0$.*

A complex manifold X is said to be a **Kähler manifold** *if a Kähler metric exists on X.*

Examples.

(1) *Complex projective spaces* $\mathbb{CP}^n = \mathbb{P}^n$;

(2) *Projective manifolds*: those compact complex manifolds X that can be embedded as closed submanifolds into some complex projective space:

there exists an integer $N \geq 1$ such that $X \hookrightarrow \mathbb{CP}^N$.

These are the main objects of study in *complex algebraic geometry*.

Kodaira's Embedding Theorem

A compact complex manifold X is **projective** *if and only if X carries an* **integral** *Kähler class $\{\omega\} \in H^2(X, \mathbb{Z})$.*

More generally, X is said to be **Moishezon** if it is *bimeromorphically equivalent* to a projective manifold, namely if there exists a projective manifold $\widetilde{X}$ and a holomorphic bimeromorphic map (called a *modification*)

$$\mu : \widetilde{X} \longrightarrow X.$$

Intuitively, Moishezon manifolds are those compact complex manifolds that admit "many" (in a precise sense) divisors (= formal linear combinations with integer coefficients of complex hypersurfaces of X).

A transcendental analogue of Moishezon manifolds is provided by the **(Fujiki) class C manifolds**.

These are the compact complex manifolds that are *bimeromorphically equivalent* to compact Kähler manifolds.

Specifically, a compact complex manifold X belongs to this class if and only if there exists a compact Kähler manifold $\widetilde{X}$ and a holomorphic bimeromorphic map (called a *modification*)

$$\mu : \widetilde{X} \longrightarrow X.$$

We have implications:

$$\begin{array}{ccccc} & & X \textbf{ Kähler} & & \\ & \Nearrow & & \Searrow & \\ X \textbf{ projective} & & & & X \textbf{ class C}. \\ & \Searrow & & \Nearrow & \\ & & X \textbf{ Moishezon} & & \end{array}$$

All these implications are *strict* when $\dim_{\mathbb{C}} X \geq 3$.

Classification of compact complex manifolds from the point of view of special metrics they support

$$
\begin{array}{lll}
d\omega = 0 & \Longrightarrow \exists\, \rho^{0,\,2} \in C^\infty_{0,\,2}(X,\,\mathbb{C}) \text{ s.t.} & \Longrightarrow \partial\bar\partial\omega = 0 \\
 & d(\overline{\rho^{0,\,2}} + \omega + \rho^{0,\,2}) = 0 & \\
(\omega \text{ is } \textbf{Kähler}) & (\omega \text{ is } \textbf{Hermitian-symplectic (H-S)}) & (\omega \text{ is } \textbf{SKT}) \\
\Downarrow & & \\
d\omega^{n-1} = 0 & \Longrightarrow \exists\, \Omega^{n-2,\,n} \in C^\infty_{n-2,\,n}(X,\,\mathbb{C}) \text{ s.t.} & \Longrightarrow \partial\bar\partial\omega^{n-1} = 0 \\
 & d(\overline{\Omega^{n-2,\,n}} + \omega^{n-1} + \Omega^{n-2,\,n}) = 0 & \\
(\omega \text{ is } \textbf{balanced}) & (\omega \text{ is } \textbf{strongly Gauduchon (sG)}) & (\omega \text{ is } \textbf{Gauduchon}).
\end{array}
$$

(2) Cohomological point of view

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$. Thanks to the integrability property $d^2 = 0$, $\partial^2 = 0$, $\bar\partial^2 = 0$, each of the differential operators $d, \partial, \bar\partial$ induces a *complex*:

-the **De Rham complex** of X:

$$\cdots \stackrel{d}{\longrightarrow} C^\infty_{k-1}(X,\, \mathbb{C}) \stackrel{d}{\longrightarrow} C^\infty_k(X,\, \mathbb{C}) \stackrel{d}{\longrightarrow} C^\infty_{k+1}(X,\, \mathbb{C}) \stackrel{d}{\longrightarrow} \cdots,$$

giving rise to the **De Rham cohomology** spaces of X:

$$H^k_{DR}(X, \mathbb{C}) := \frac{\ker\,(d : C^\infty_k(X,\, \mathbb{C}) \longrightarrow C^\infty_{k+1}(X,\, \mathbb{C}))}{\mathrm{Im}\,(d : C^\infty_{k-1}(X,\, \mathbb{C}) \longrightarrow C^\infty_k(X,\, \mathbb{C}))}, \qquad k \in \{0, \dots, 2n\},$$

depending only on the *differential structure* of X;

-for every fixed $q \in \{0, \dots, n\}$, the **conjugate Dolbeault complex** of X:

$$\cdots \stackrel{\partial}{\longrightarrow} C^\infty_{p-1,\,q}(X,\,\mathbb{C}) \stackrel{\partial}{\longrightarrow} C^\infty_{p,\,q}(X,\,\mathbb{C}) \stackrel{\partial}{\longrightarrow} C^\infty_{p+1,\,q}(X,\,\mathbb{C}) \stackrel{\partial}{\longrightarrow} \cdots,$$

giving rise to the **conjugate Dolbeault cohomology** spaces of X:

$$H^{p,\,q}_{\partial}(X, \mathbb{C}) := \frac{\ker\,(\partial : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p+1,\,q}(X,\,\mathbb{C}))}{\operatorname{Im}\,(\partial : C^\infty_{p-1,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}))}, \quad p, q \in \{0, \dots, n\},$$

depending on the *complex structure* of X;

-for every fixed $p \in \{0, \dots, n\}$, the **Dolbeault complex** of X:

$$\cdots \stackrel{\bar\partial}{\longrightarrow} C^\infty_{p,\,q-1}(X,\,\mathbb{C}) \stackrel{\bar\partial}{\longrightarrow} C^\infty_{p,\,q}(X,\,\mathbb{C}) \stackrel{\bar\partial}{\longrightarrow} C^\infty_{p,\,q+1}(X,\,\mathbb{C}) \stackrel{\bar\partial}{\longrightarrow} \cdots ,$$

giving rise to the **Dolbeault cohomology** spaces of X:

$$H^{p,\,q}_{\bar\partial}(X, \mathbb{C}) := \frac{\ker(\bar\partial : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q+1}(X,\,\mathbb{C}))}{\mathrm{Im}\,(\bar\partial : C^\infty_{p,\,q-1}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}))}, \qquad p, q,$$

depending on the *complex structure* of X.

The *compactness* of X implies the *finite dimensionality* (as $\mathbb{C}$-vector spaces) of all of the above cohomology spaces whose dimensions are important geometric invariants of a compact complex manifold.

Of particular interest are:

-the **Betti numbers**:

$$b_k = b_k(X) := \dim_{\mathbb{C}} H^k_{DR}(X, \mathbb{C}), \qquad k \in \{0, \ldots, 2n\},$$

depending only on the differential structure of X (so, they are *topological invariants*)

-the **Hodge numbers**:

$$h^{p,\,q} = h^{p,\,q}_{\bar\partial}(X) := \dim_{\mathbb{C}} H^{p,\,q}_{\bar\partial}(X, \mathbb{C}), \qquad p, q \in \{0, \ldots, n\},$$

depending on the complex structure of X.

Two other cohomologies that play a key role in non-Kähler complex geometry are

-the **Bott-Chern cohomology**, whose spaces are defined as

$$H^{\bullet,\bullet}_{BC}(X,\mathbb{C}) := \frac{\ker\,\partial\cap\ker\,\bar\partial}{\mathrm{Im}\,(\partial\bar\partial)},$$

and

-the **Aeppli cohomology**, whose spaces are defined as

$$H^{\bullet,\bullet}_{A}(X,\mathbb{C}) := \frac{\ker\,(\partial\bar\partial)}{\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial}.$$

We denote by $\{\alpha\}_{DR}$, $[\alpha]_{\partial}$, $[\alpha]_{\bar\partial}$, $[\alpha]_{BC}$, $[\alpha]_A$ the De Rham, conjugate Dolbeault, Dolbeault, Bott-Chern, respectively Aeppli *cohomology class* of a given form α that represents such a class.

There are well-defined, **canonical** linear maps induced by the identity among these cohomologies:

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \to H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) \to H^{p,\,q}_{A}(X,\,\mathbb{C}), \qquad [\alpha]_{BC} \mapsto [\alpha]_{\bar\partial} \mapsto [\alpha]_A,$$

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \to H^{p,\,q}_{\partial}(X,\,\mathbb{C}) \to H^{p,\,q}_{A}(X,\,\mathbb{C}), \qquad [\alpha]_{BC} \mapsto [\alpha]_{\partial} \mapsto [\alpha]_A,$$

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \to H^{p+q}_{DR}(X,\,\mathbb{C}) \to H^{p,\,q}_{A}(X,\,\mathbb{C}), \qquad [\alpha]_{BC} \mapsto \{\alpha\}_{DR} \mapsto [\alpha^{p,\,q}]$$

where, for the last map, $\alpha^{p,\,q}$ denotes the $(p,\,q)$-type component of the $(p+q)$-form $\alpha = \sum_{r+s=p+q} \alpha^{r,\,s}$. By *canonical* we mean that these maps depend only on the complex structure of X, so, in particular, they are independent of the choice of a Hermitian metric.

However, these maps need not be either injective or surjective on an arbitrary X.

One of the remarkable properties of a class of compact complex manifolds (the so-called **$\partial\bar\partial$-manifolds**) that strictly contains the Kähler class is that all the maps on the first two rows above are *isomorphisms*, while the two maps on the third row are *injective*, respectively *surjective*.

In particular, on a compact Kähler manifold X, the Dolbeault, conjugate Dolbeault, Bott-Chern and Aeppli cohomologies are *canonically isomorphic*. For this reason, the subscript can be dropped in that case, so $H^{p,\,q}(X,\,\mathbb{C})$ stands for any (usually Dolbeault in practice) of these cohomology groups of bidegree $(p,\,q)$ on a compact Kähler X.

(3) Interplay between the metric and the cohomological points of view

Another basic idea of Hodge theory: to interpret the various cohomology spaces as *harmonic spaces*, namely as the kernels of certain elliptic differential operators called *Laplacians*.

Suppose X is a *compact* complex manifold on which a Hermitian metric ω has been fixed. Using the L^2 inner product induced by ω on the spaces of C^∞ forms on X, one defines *first-order differential operators* $d^\star$, $\partial^\star$, $\bar\partial^\star$ as the adjoints of $d, \partial, \bar\partial$ which, in turn, induce **Laplace-Beltrami operators**:

$$\Delta = \Delta_\omega := dd^\star + d^\star d : C^\infty_k(X,\,\mathbb{C}) \rightarrow C^\infty_k(X,\,\mathbb{C}),$$
$$\Delta' = \Delta'_\omega := \partial\partial^\star + \partial^\star\partial : C^\infty_{p,\,q}(X,\,\mathbb{C}) \rightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}),$$
$$\Delta'' = \Delta''_\omega := \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial : C^\infty_{p,\,q}(X,\,\mathbb{C}) \rightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}),$$
$$\Delta_{BC} := \partial^\star\partial+\bar\partial^\star\bar\partial+(\partial\bar\partial)(\partial\bar\partial)^\star+(\partial\bar\partial)^\star(\partial\bar\partial)+(\partial^\star\bar\partial)^\star(\partial^\star\bar\partial)+(\partial^\star\bar\partial)(\partial^\star\bar\partial)^\star,$$
$$\Delta_A := (\partial\bar\partial)^\star(\partial\bar\partial)+\partial\partial^\star+\bar\partial\bar\partial^\star+(\partial\bar\partial)(\partial\bar\partial)^\star+(\partial\bar\partial^\star)(\partial\bar\partial^\star)^\star+(\partial\bar\partial^\star)^\star(\partial\bar\partial^\star),$$

in every (bi-)degree.

Note that Δ, Δ' and Δ'' are of order 2, while Δ_{BC} and Δ_A (called the Bott-Chern, respectively the Aeppli, Laplacian) are of order 4. Each of them is adapted to one type of cohomology on X.

They all turn out to be *elliptic* and this, together with the compactness of X, leads to **Hodge isomorphisms**:

$$H^k_{DR}(X,\, \mathbb{C}) \simeq \mathcal{H}^k_{\Delta}(X,\, \mathbb{C}),$$

$$H^{p,\,q}_{\partial}(X,\, \mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta'}(X,\, \mathbb{C}), \qquad H^{p,\,q}_{\bar\partial}(X,\, \mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta''}(X,\, \mathbb{C}),$$

$$H^{p,\,q}_{BC}(X,\, \mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\, \mathbb{C}), \qquad H^{p,\,q}_{A}(X,\, \mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta_{A}}(X,\, \mathbb{C}),$$

where $\mathcal{H}^k_P(X,\, \mathbb{C})$ and $\mathcal{H}^{p,\,q}_P(X,\, \mathbb{C})$ stand for the kernels of P in degree k and bidegree $(p,\, q)$, where $P \in \{\Delta, \Delta', \Delta'', \Delta_{BC}, \Delta_A\}$.

These statements follow from the following

Theorem (fundamental facts of Hodge theory) *Let* (X, ω) *be a* **compact** *Hermitian manifold with* $\dim_{\mathbb{C}} X = n$. *Then:*

(1) *the differential operators* Δ, Δ' *and* Δ'' *are* **elliptic** *(i.e. their principal symbols are injective at every point);*

(2) *the kernels of* Δ, Δ' *and* Δ'' *are* **finite dimensional**, *while their images are* **closed** *and finite codimensional in* $C^\infty_k(X, \mathbb{C})$ *(for* Δ*), resp. in* $C^\infty_{p,q}(X, \mathbb{C})$ *(for* Δ' *and* Δ''*).*

Moreover, for all $k \in \{0, \ldots, 2n\}$ and all $p, q \in \{0, \ldots, n\}$, the following **orthogonal** *(for the L^2_ω-norm)* **two-space decompositions** *hold:*

$$C^\infty_k(X, \mathbb{C}) = \ker \Delta \oplus Im\, \Delta, \qquad C^\infty_{p,\,q}(X, \mathbb{C}) = \ker \Delta' \oplus Im\, \Delta',$$

$$C^\infty_{p,\,q}(X, \mathbb{C}) = \ker \Delta'' \oplus Im\, \Delta'',$$

where all the kernels and images involved are taken in the respective (bi)degrees.

(3) *furthermore, the following* L^2_ω**-orthogonal two-space decompositions** *hold:*

$$Im\,\Delta = Im\,d \oplus Im\,d^\star, \qquad Im\,\Delta' = Im\,\partial \oplus Im\,\partial^\star, \qquad Im\,\Delta'' = Im\,\bar\partial \oplus Im\,\bar\partial^\star.$$

Hence, we get the following L^2_ω**-orthogonal three-space decompositions***:*

$$C^\infty_k(X,\,\mathbb{C}) = \ker\Delta \oplus Im\,d \oplus Im\,d^\star,$$

$$C^\infty_{p,\,q}(X,\,\mathbb{C}) = \ker\Delta' \oplus Im\,\partial \oplus Im\,\partial^\star,$$

$$C^\infty_{p,\,q}(X,\,\mathbb{C}) = \ker\Delta'' \oplus Im\,\bar\partial \oplus Im\,\bar\partial^\star,$$

in which we further have:

$$\ker d = \ker\Delta \oplus Im\, d \qquad \textit{and} \qquad \ker d^\star = \ker\Delta \oplus Im\, d^\star,$$

$$\ker \partial = \ker\Delta' \oplus Im\, \partial \qquad \textit{and} \qquad \ker \partial^\star = \ker\Delta' \oplus Im\, \partial^\star,$$

$$\ker \bar\partial = \ker\Delta'' \oplus Im\, \bar\partial \qquad \textit{and} \qquad \ker \bar\partial^\star = \ker\Delta'' \oplus Im\, \bar\partial^\star$$

in all the degrees $k \in \{0, \dots, 2n\}$ *and all the bidegrees* $(p,\, q)$ *with* $p, q \in \{0, \dots, n\}$.

Let us only point out that conclusion (2) above follows from **Gårding's estimate** (or the **a priori estimate**, depending on the terminology being used) satisfied by any *elliptic* operator on a *compact* manifold (without boundary).

Conclusion (3) further follows from the *integrability* of the operators $d, \partial, \bar{\partial}$, namely $d^2 = 0$, $\partial^2 = 0$ and $\bar{\partial}^2 = 0$.

As an immediate consequence, one gets the **Hodge isomorphisms** that display the De Rham, Dolbeault and conjugate Dolbeault cohomology groups as isomorphic to the spaces of **Δ-harmonic, Δ''-harmonic** and resp. **Δ'-harmonic spaces** of forms of the same (bi)degrees.

Corollary. *Let* $(X,\,\omega)$ *be a* **compact** *Hermitian manifold with* $dim_{\mathbb{C}} X = n$. *Then, for all* $k \in \{0, \dots, 2n\}$ *and all* $p, q \in \{0, \dots, n\}$, *the following* **Hodge isomorphisms** *hold:*

$$H^k_{DR}(X,\,\mathbb{C}) \simeq \mathcal{H}^k_{\Delta}(X,\,\mathbb{C}),$$

$$H^{p,\,q}_{\partial}(X,\,\mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta'}(X,\,\mathbb{C}),$$

$$H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta''}(X,\,\mathbb{C}),$$

where $\mathcal{H}^k_{\Delta}(X,\,\mathbb{C}) := \ker(\Delta : C^{\infty}_k(X,\,\mathbb{C}) \to C^{\infty}_k(X,\,\mathbb{C}))$, $\mathcal{H}^{p,\,q}_{\Delta'}(X,\,\mathbb{C}) := \ker(\Delta' : C^{\infty}_{p,\,q}(X,\,\mathbb{C}) \to C^{\infty}_{p,\,q}(X,\,\mathbb{C}))$ *and* $\mathcal{H}^{p,\,q}_{\Delta''}(X,\,\mathbb{C}) := \ker(\Delta'' : C^{\infty}_{p,\,q}(X,\,\mathbb{C}) \to C^{\infty}_{p,\,q}(X,\,\mathbb{C}))$.