Hermitian-symplectic and Kähler Metrics on Compact Complex Manifolds

based on

A Generalised Volume Invariant for Aeppli Cohomology Classes of Hermitian-Symplectic Metrics

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(I) Introduction

Classification of compact complex manifolds

X compact complex manifold, $n = \dim_{\mathbb{C}} X$

Complex structure : $d = \partial + \bar{\partial}$

Idea

the **transcendental** methods, introduced for the study of possibly non-algebraic manifolds, are also relevant in the study of **projective manifolds**.

Context

(a) Complex algebraic geometry

• X is said to be **projective** $\overset{\text{def}}{\Longleftrightarrow} \exists\, N \in \mathbb{N}^\star$ s.t. $X \hookrightarrow \mathbb{CP}^N$

(embedding as a closed submanifold)

• More generally, X is said to be **Moishezon** if there exists

$$\sigma : \widetilde{X} \to X$$

holomorphic and bimeromorphic map with $\widetilde{X}$ **projective**.

(b) **Complex analytic and differential geometry**

Let $\omega > 0$ C^∞ **positive definite** $(1, 1)$**-form** on X.

(**Hermitian metric**, always exists)

Locally : $\omega = \sum_{j,k} \omega_{j\bar{k}}\, i dz_j \wedge d\bar{z}_k,$

where the $\omega_{j\bar{k}}$'s are C^∞ functions;

• ω is said to be **Kähler** if $d\omega = 0$ (need not exist)

• X is a **Kähler manifold** if $\exists$ a Kähler metric ω.

• A compact Kähler manifold need not have any complex submanifolds other than the **points** (e.g. **neither curves**, **nor hypersurfaces**.)

In this **transcendental** context, the objects of study are often **analytic** generalisations of algebraic objects. Examples :

(i) **closed positive currents** (e.g. of bidegree (1, 1)) :

$$T = \sum_{j,\,k} T_{j\bar{k}}\, i dz_j \wedge d\bar{z}_k,$$

where the $T_{j\bar{k}}$'s are complex measures ;

(local shape)

Closed positive currents generalise subvarieties.

Example : if $Y \subset X$ is a subvariety, $\dim_{\mathbb{C}} Y = p$, one defines

$$C_p^\infty(X, \mathbb{C}) \ni \gamma \xrightarrow{[Y]} \int_Y \gamma_{|Y} \in \mathbb{C}$$

the **current of integration** on Y.

(ii) **special Hermitian metrics** (not necessarily Kähler) and their **cohomology classes**.

Transcendental analogue of Moishezon manifolds

X is said to be **of class $\mathcal{C}$** if there exists a **holomorphic and bimeromorphic** map

$$\sigma : \widetilde{X} \to X$$

with $\widetilde{X}$ compact **Kähler**.

Demailly-Paun (2001) : X is of *class* $\mathcal{C}$ $\iff$ $\exists\, T$ **Kähler current** on X (i.e. $dT = 0$ and $T > 0$).

Implications (all strict)

X **Kähler**

$\Nearrow$ $\qquad\qquad$ $\Searrow$

X **projective** $\qquad\qquad\qquad\qquad$ X **class C**

$\Searrow$ $\qquad\qquad$ $\Nearrow$

X **Moishezon**

(1) **Metric point of view**

If $n = \dim_{\mathbb{C}} X \geq 3$, few manifolds X are Kähler.

Further examples of special metrics:

(i) ω is said to be **Gauduchon** if $\partial\bar{\partial}\omega^{n-1} = 0$ (always exists) ;

(ii) ω is said to be **strongly Gauduchon** if $\partial\omega^{n-1}$ is $\bar{\partial}$-exact
(P. 2009) (need not exist) ;

(iii) ω is said to be **balanced** if $d\omega^{n-1} = 0$
(Gauduchon 1977) (need not exist).

(2) **Cohomological point of view**

- **De Rham** cohomology group:

$$H^k_{DR}(X, \mathbb{C}) := \frac{\ker d}{\operatorname{Im} d} \qquad \text{(depends only on the differential structure)}$$

- **Dolbeault** cohomology group:

$$H^{p,\,q}_{\bar\partial}(X, \mathbb{C}) := \frac{\ker \bar\partial}{\operatorname{Im} \bar\partial} \qquad \text{(depends on the complex structure)}$$

- **Bott-Chern** cohomology group:

$$H^{p,\,q}_{BC}(X,\mathbb{C}) := \frac{\ker\partial\cap\ker\bar\partial}{\mathrm{Im}\,(\partial\bar\partial)} \qquad \text{(depends on the complex structure)}$$

- **Aeppli** cohomology group:

$$H^{p,\,q}_{A}(X,\mathbb{C}) := \frac{\ker(\partial\bar\partial)}{\mathrm{Im}\,\partial+\mathrm{Im}\,\bar\partial} \qquad \text{(depends on the complex structure)}$$

Tools: examples of PDE's

(1) The **Cauchy-Riemann equation**

$$\bar\partial u = v,$$

where v is a given C^∞ $(p,\, q)$-form on a compact Hermitian manifold $(X,\, \omega)$. We look for C^∞ $(p,\, q-1)$-form solutions u.

The solution of **minimal L^2_ω-norm** is

$$u = \Delta''^{-1}\bar\partial^\star v,$$

where Δ''^{-1} is the **Green operator** of the $\bar\partial$-Laplacian

$$\Delta'' = \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial : C^\infty_{p,\,q-1}(X,\, \mathbb{C}) \longrightarrow C^\infty_{p,\,q-1}(X,\, \mathbb{C}).$$

(2) The **Monge-Ampère equation**

$$(\omega + i\partial\bar{\partial}\varphi)^n = dV,$$

where (X, ω) : compact Hermitian manifold, $n = \dim_{\mathbb{C}} X$;

$dV > 0$: C^∞ (n, n)-form (**volume form**) such that

$$\int_X \omega^n = \int_X dV.$$

We are looking for C^∞ **solutions**

$$\varphi : X \longrightarrow \mathbb{R}$$

such that $\omega + i\partial\bar{\partial}\varphi > 0$.

(a) If ω is **Kähler**, the Monge-Ampère equation was solved by Yau (1978).

Result: we get a Kähler metric $\omega_\varphi := \omega + i\partial\bar{\partial}\varphi$ with prescribed volume form and lying in a given cohomology class $\{\omega\}_{DR} \in H^2_{DR}(X, \mathbb{R})$.

(b) If ω is arbitrary **Hermitian**, the Monge-Ampère equation was solved by Cherrier (1987), Guan-Li (2006) and Tosatti-Weinkove (2010).

My work with Dinew: we use this Hermitian Monge-Ampère equation to try proving the existence of *critical points* for a *functional* that we have introduced.

(3) A **new Monge-Ampère-type equation** (P. 2015)

$$\left[\left(\omega^{n-1}+i\partial\bar\partial\varphi\wedge\omega^{n-2}+\frac{i}{2}\,(\partial\varphi\wedge\bar\partial\omega^{n-2}-\bar\partial\varphi\wedge\partial\omega^{n-2})\right)^{\frac{1}{n-1}}\right]^n = e^f\,\omega^n,$$

where ω is a *Gauduchon metric* given beforehand on X.

We are looking for C^∞ **solutions**

$$\varphi : X \longrightarrow \mathbb{R}$$

such that $\omega^{n-1}+i\partial\bar\partial\varphi\wedge\omega^{n-2}+\frac{i}{2}\,(\partial\varphi\wedge\bar\partial\omega^{n-2}-\bar\partial\varphi\wedge\partial\omega^{n-2})>0.$

Result (P. 2015, Tosatti-Weinkove 2015, Tosatti-Szekelyhidi-Weinkove 2017) : we get a Gauduchon metric with prescribed volume form and lying in a given Aeppli cohomology class.

(3) **Metric + cohomological point of view**

X **balanced**

$\Nearrow$

$\Searrow$

X *class* $\mathcal{C}$ $\qquad\qquad$ X **sG**

$\Nearrow$

$\Searrow$

$\mathbf{X}$ $\partial\bar\partial$

$\Searrow$

$\mathbf{E_1(X) = E_\infty(X)}$
(Frölicher spectral sequence)

Hodge Theory for the Frölicher spectral sequence

(1) The *first page* (Dolbeault cohomology) : *classical* case

$$H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) \simeq \ker(\Delta'' : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}))$$

(*Hodge isomorphism*)

where

$$\Delta'' = \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial$$

is the $\bar\partial$-Laplacian (*elliptic differential* operator of order 2).

(2) The *second page* (P. 2016) : $\exists$ *Hodge isomorphism*

$$E_2^{p,\,q}(X) \simeq \ker(\widetilde{\Delta} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}))$$

where

$$\widetilde{\Delta} = \partial p''\partial^\star + \partial^\star p''\partial + \Delta''$$

is a *pseudodifferential* operator (P. 2016) and

$$C^\infty_{p,\,q}(X,\,\mathbb{C}) = \ker\Delta'' \oplus \operatorname{Im}\bar\partial \oplus \operatorname{Im}\bar\partial^\star \xrightarrow{p''} \ker\Delta''$$

is the *orthogonal projection*.

(II) Work with Dinew

Hermitian-symplectic geometry

X a compact complex manifold, $\dim_{\mathbb{C}} X = n$

Goal: study the geometry of X in terms of the metrics it supports.

Let $\omega > 0$ a C^∞ **positive definite** $(1, 1)$**-form** on X.

(**a Hermitian metric**, always exists)

A few classes of special Hermitian metrics

(Except for the Gauduchon metrics, they need not exist on a given X.)

$d\omega = 0$	$\Longrightarrow \exists\, \rho^{0,2} \in C^\infty_{0,2}(X, \mathbb{C})$ s.t. $d(\overline{\rho^{0,2}} + \omega + \rho^{0,2}) = 0$	$\Longrightarrow \partial\bar\partial\omega = 0$
(ω is Kähler)	(ω is Hermitian-symplectic) (Streets-Tian '10)	(ω is SKT)
$\Downarrow$		
$d\omega^{n-1} = 0$	$\Longrightarrow \exists\, \Omega^{n-2,n} \in C^\infty_{n-2,n}(X, \mathbb{C})$ s.t. $d(\overline{\Omega^{n-2,n}} + \omega^{n-1} + \Omega^{n-2,n}) = 0$	$\Longrightarrow \partial\bar\partial\omega^{n-1} = 0$
(ω is balanced) (Gauduchon '77)	(ω est strongly Gauduchon (sG)) (P. '09)	(ω is Gauduchon).

Hermitian-symplectic (H-S) metrics and manifolds

• Intrinsic characterisation

Theorem 0.1 *(Sullivan 1976) Let X be a compact complex manifold with $dim_{\mathbb{C}}X = n$.*

X is Hermitian-symplectic $\iff$ X supports no non-zero current T of bidegree $(n-1, n-1)$ such that $T \geq 0$ and T is d-exact.

• Li-Zhang '09, Streets-Tian '10, Dinew-P. '20: when $\dim_{\mathbb{C}}X = 2$,

X is Hermitian-symplectic $\iff$ X is Kähler.

• **Higher-dimensional H-S manifolds**

They are poorly understood.

Question 0.2 *(Streets-Tian '10) Do there exist non-Kähler Hermitian-symplectic complex manifolds X with $dim_{\mathbb{C}} X \geq 3$?*

· The general case of this question is still open.

· It has been answered negatively for a handful of special classes of manifolds, including:

-all nilmanifolds endowed with an invariant complex structure (Enrietti-Fino-Vezzoni '12);

-all twistor spaces (Verbitsky '14).

· If the answer is affirmative, this would be a Kählerianity criterion for manifolds.

Cf. the Demaily-Paun ('04) Kählerianity criterion for cohomology classes on a given compact Kähler manifold.

· This problem lies at the interface between symplectic and complex Hermitian geometries.

· The Streets-Tian question is complementary to Donaldson's earlier

Question 0.3 *(Donaldson '06) If J is an almost-complex structure on a compact* 4*-manifold which is tamed by a symplectic form, is there a symplectic form compatible with J?*

This is *Donaldson's tamed-to-compatible conjecture*.

Preliminaries: relations between special metrics

(1) H-S and sG manifolds

Proposition 0.4 *(Yau-Zhao-Zheng '19; Dinew-P. '20) Every compact complex manifold X that admits a* **Hermitian-symplectic** *metric also admits a* **strongly Gauduchon (sG)** *metric.*

(2) **Balanced SKT metrics**

Proposition 0.5 *(Ivanov- Papadopoulos '13, P. '15) If a Hermitian metric ω on a compact complex manifold X is both* **SKT** *and* **balanced**, *then ω is* **Kähler**.

Proof. The *SKT* assumption on ω translates to:

$$\partial\bar\partial\omega = 0 \iff \partial\omega \in \ker\bar\partial \iff \star(\partial\omega) \in \ker\partial^\star,$$

where we use the standard formula $\partial^\star = -\star\bar\partial\star$ involving the Hodge-star operator induced by ω.

The *balanced* assumption on ω translates to:

$$d\omega^{n-1} = 0 \iff \partial\omega^{n-1} = 0 \iff \omega^{n-2}\wedge\partial\omega = 0 \iff \partial\omega \text{ is primitive}.$$

Moreover, since $\partial\omega$ is *primitive* when ω is balanced, we get:

$$\star(\partial\omega) = i\,\frac{\omega^{n-3}}{(n-3)!}\wedge\partial\omega = \frac{i}{(n-2)!}\,\partial\omega^{n-2} \in \operatorname{Im}\partial.$$

Hence, if ω is both *SKT* and *balanced*, we get:

$$\star(\partial\omega) \in \ker \partial^\star \cap \operatorname{Im} \partial = \{0\},$$

where the last identity follows from $\ker \partial^\star \perp \operatorname{Im} \partial$. Hence

$$\partial\omega = 0,$$

meaning that ω is *Kähler*. $\square$

Our approach and results

Question (strengthening of Streets-Tian)

Does there exist a Kähler metric in the Aeppli cohomology class of every Hermitian-symplectic metric?

Recall

ω is H-S $\implies$ ω is SKT $\quad$ (i.e. $\partial\bar\partial\omega = 0$).

Hence, ω defines an *Aeppli cohomology class*:

$$\{\omega\}_A \in H^{1,\,1}_A(X,\,\mathbb{C}) := \frac{\ker\,(\partial\bar\partial)}{\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial}.$$

The *Hermitian-symplectic* condition can be expressed as follows.

Lemma 0.6 *Let ω be a Hermitian metric on a compact complex manifold X.*

(I) The following statements are **equivalent***.*

(a) ω is **Hermitian-symplectic***.*

(b) There exists a form $\rho^{2,\,0}\in C^\infty_{2,\,0}(X,\,\mathbb{C})$ satisfying the equations:

$$(i)\ \partial\rho^{2,\,0}=0 \quad \textit{and} \quad (ii)\ \bar\partial\rho^{2,\,0}+\partial\omega=0.$$

(I) A new energy functional

Lemma and Definition 0.7 *For every Hermitian-symplectic metric ω on X, there exists a unique smooth $(2,\,0)$-form $\rho_\omega^{2,\,0}$ on X such that:*

$$(i)\ \partial\rho_\omega^{2,\,0} = 0 \quad \textit{and} \quad (ii)\ \bar\partial\rho_\omega^{2,\,0} = -\partial\omega$$

and

$$(iii)\ \rho_\omega^{2,\,0} \in Im\,\partial_\omega^\star + Im\,\bar\partial_\omega^\star.$$

Property (iii) ensures that $\rho_\omega^{2,\,0}$ has **minimal L^2_ω-norm** *among all the $(2,\,0)$-forms satisfying properties (i) and (ii).*

We call $\rho_\omega^{2,\,0}$ the **$(2,\,0)$-torsion form** *and its conjugate $\rho_\omega^{0,\,2}$ the* **$(0,\,2)$-torsion form** *of the Hermitian-symplectic metric ω.*

One has the explicit **Neumann-type formula***:*

$$\rho_\omega^{2,\,0} = -\Delta_{BC}^{-1}[\bar\partial^\star\partial\omega + \bar\partial^\star\partial\partial^\star\partial\omega], \tag{1}$$

where Δ_{BC}^{-1} *is the Green operator of the Bott-Chern Laplacian* Δ_{BC} *induced by* ω*, while* $\partial^\star = \partial_\omega^\star$ *and* $\bar\partial^\star = \bar\partial_\omega^\star$ *are the formal adjoints of* ∂*, resp.* $\bar\partial$*, w.r.t. the* L^2 *inner product defined by* ω*.*

For every Hermitian-symplectic Aeppli class $\{\omega\}_A$, *we denote by*

$$\mathcal{S}_{\{\omega\}} := \left\{\omega + \partial\bar{u} + \bar\partial u \;\mid\; u \in C^\infty_{1,\,0}(X,\,\mathbb{C}) \;\; \textit{such that}\; \omega + \partial\bar{u} + \bar\partial u > 0\right\}$$
$$\subset \; \{\omega\}_A \cap C^\infty_{1,\,1}(X,\,\mathbb{R})$$

the set of all (necessarily H-S) metrics in $\{\omega\}_A$.

It is an open convex subset of the real affine space

$$\{\omega\}_A \cap C^\infty_{1,\,1}(X,\,\mathbb{R}) = \{\omega + \partial\bar{u} + \bar\partial u \;\mid\; u \in C^\infty_{1,\,0}(X,\,\mathbb{C})\}.$$

Definition 0.8 *Let X be a compact complex Hermitian-symplectic manifold with $\dim_{\mathbb{C}} X = n$. For the Aeppli cohomology class $\{\omega_0\}_A$ of any Hermitian-symplectic metric ω_0, we define the following* **energy functional***:*

$$F : \mathcal{S}_{\{\omega_0\}} \to [0, +\infty), \qquad F(\omega) = \int\limits_X |\rho_\omega^{2,\,0}|^2_\omega \, dV_\omega = ||\rho_\omega^{2,\,0}||^2_\omega,$$

where $\rho_\omega^{2,\,0}$ is the $(2,\,0)$-torsion form of the H-S metric $\omega \in \mathcal{S}_{\{\omega_0\}}$, while $|\ \ |_\omega$ is the pointwise norm and $||\ \ ||_\omega$ is the L^2 norm induced by ω.

The introduction of this functional is justified by

Lemma 0.9 *Fix a Hermitian-symplectic Aeppli class* $\{\omega_0\}_A$. *For any H-S metric* $\omega \in \mathcal{S}_{\{\omega_0\}}$, *the following equivalence holds:*

$$\omega \text{ is Kähler} \iff F(\omega) = 0.$$

Proof. *If* ω *is Kähler,* $\partial\omega = 0$ *and the minimal* L^2*-norm solution of the equation* $\bar\partial\rho = 0$ *vanishes. Thus* $\rho_\omega^{2,\,0} = 0$, *hence* $F(\omega) = 0$.

Conversely, if $F(\omega) = 0$, *then* $\rho_\omega^{2,\,0}$ *vanishes identically on* X, *hence* $\partial\omega = -\bar\partial\rho_\omega^{2,\,0} = 0$, *so* ω *is Kähler.* □

Computation of the critical points of F

Theorem 0.10 *The differential at ω of F is given by the formula:*

$$(d_\omega F)(\gamma) = -2\,Re\,\langle\langle u,\,\bar\partial^\star\omega\rangle\rangle_\omega + 2\,Re\int\limits_X u\wedge\rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\bar\partial\left(\frac{\omega^{n-3}}{(n-3)!}\right),$$

for every $(1,\,1)$-form $\gamma = \partial\bar u + \bar\partial u$.

In particular, when $n=3$, this formula reduces to:

$$(d_\omega F)(\gamma) = -2\,Re\,\langle\langle u,\,\bar\partial^\star\omega\rangle\rangle_\omega.$$

Corollary 0.11 *Suppose $n = 3$. Then a Hermitian-symplectic metric ω on a compact complex manifold X of dimension 3 is a* **critical point** *of the energy functional F* **if and only if** *ω is* **Kähler***.*

Proof. "$\Longleftarrow$" *If ω is Kähler, $\partial\omega = 0$. Hence, $\rho_\omega^{2,\,0} = 0$, so $F(\omega) = 0$ and ω is a minimum for F.*

"$\Longrightarrow$" *A metric* ω *is a critical point of* F *if and only if*

$$(d_\omega F)(\gamma) = 0$$

for every $\gamma = \partial\bar{u} + \bar{\partial}u$. *By the above discussion, this amounts to* $Re\,\langle\langle u,\, \bar{\partial}^\star\omega\rangle\rangle_\omega = 0$ *for every* $(1,\,0)$*-form* u.

Thus, if ω *is a critical point of* F, *by taking* $u = \bar{\partial}^\star\omega$ *we get*

$$\bar{\partial}^\star\omega = 0.$$

This is equivalent to ω *being balanced. However,* ω *is already SKT since it is Hermitian-symplectic, so* ω *must be Kähler.* $\square$

Corollary 0.12 *Let X be a compact complex manifold of dimension $n=3$ admitting Hermitian-symplectic metrics.*

Then, for any Aeppli-cohomologous Hermitian-symplectic metrics ω and ω_η:

$$\omega_\eta = \omega + \partial\bar\eta + \bar\partial\eta > 0, \qquad \text{with } \eta \in C^\infty_{1,\,0}(X,\,\mathbb{C}),$$

the respective (2, 0)-torsion forms $\rho_\omega^{2,\,0}$ and $\rho_\eta^{2,\,0} := \rho_{\omega_\eta}^{2,\,0}$ satisfy the identity:

$$||\rho_\eta^{2,\,0}||^2_{\omega_\eta} + \int\limits_X \frac{\omega_\eta^3}{3!} = ||\rho_\omega^{2,\,0}||^2_\omega + \int\limits_X \frac{\omega^3}{3!}$$

and are related by

$$\rho_\eta^{2,\,0} = \rho_\omega^{2,\,0} + \partial\eta.$$

Proof of the first statement. *In an arbitrary dimension* n*, we compute the differential of the map*

$$\mathcal{S}_{\{\omega_0\}} \ni \omega \mapsto \int\limits_X \frac{\omega^n}{n!} := Vol_\omega(X)$$

when the metric ω *varies in its Aeppli cohomology class* $\{\omega_0\}_A$*. For any real, Aeppli null-cohomologous* $(1,\,1)$*-form* $\gamma = \partial\bar{u} + \bar\partial u$

(with $u \in C^{\infty}_{1,0}(X, \mathbb{C})$), we have

$$\begin{aligned}
\frac{d}{dt}_{|t=0} \int\limits_X \frac{(\omega + t\gamma)^n}{n!} &= \frac{1}{(n-1)!} \int\limits_X \omega^{n-1} \wedge \gamma = 2\, Re \int\limits_X \bar\partial u \wedge \frac{\omega^{n-1}}{(n-1)!} \\
&= 2\, Re \int\limits_X u \wedge \bar\partial \star \omega = 2\, Re \int\limits_X u \wedge \star \bigg(- \star \bar\partial \star \omega \bigg) \\
&= 2\, Re \int\limits_X u \wedge \star \partial^\star \omega = 2\, Re \int\limits_X u \wedge \star \overline{\bar\partial^\star \omega} \\
&= 2\, Re \langle\langle u,\, \bar\partial^\star \omega \rangle\rangle.
\end{aligned}$$

The last quantity is $-(d_\omega F)(\gamma)$ *when* $n = 3$. □

(II) Generalised volume of Hermitian-symplectic Aeppli classes

The main takeaway from the last corollary is that the sum

$$F(\omega) + Vol_\omega(X)$$

(where $Vol_\omega(X) := \int_X \omega^3/3!$*) remains* **constant** *when* ω *ranges over the (necessarily Hermitian-symplectic) metrics in the Aeppli cohomology class of a fixed Hermitian-symplectic metric* ω_0*.*

Definition 0.13 *Let* X *be a* 3*-dimensional compact Hermitian-symplectic manifold. For any H-S metric* ω *on* X*, the constant*

$$A = A_{\{\omega\}_A} := F(\omega) + Vol_\omega(X) > 0$$

depending only on $\{\omega\}_A$ *is called the* **generalised volume** *of the H-S Aeppli class* $\{\omega\}_A$*.*

Digression

Definition 0.14 *If ω is an H-S metric on a compact complex manifold X with $dim_{\mathbb{C}}X = 3$ and $\rho_\omega^{2,\,0}$ is the $(2,\,0)$-torsion form of ω, we define the following volume form on X:*

$$d\widetilde{V}_\omega := (1 + |\rho_\omega^{2,\,0}|_\omega^2)\, dV_\omega.$$

Its volume depends only on the H-S Aeppli class:

$$\int\limits_X d\widetilde{V}_{\omega_1} = \int\limits_X d\widetilde{V}_{\omega_2} = A, \qquad \text{for all metrics } \omega_1, \omega_2 \in \{\omega\}_A,$$

where $A = A_{\{\omega\}_A} > 0$ is the generalised volume of the H-S Aeppli class $\{\omega\}_A$.

Therefore, it seems natural to consider the Monge-Ampère equation:

$$\frac{(\omega + i\partial\bar{\partial}\varphi)^3}{3!} = b\, d\widetilde{V}_\omega,$$

subject to the condition $\omega + i\partial\bar{\partial}\varphi > 0$, *where* $b > 0$ *is a given constant.*

By Tosatti-Weinkove '10, $\exists! b > 0$ *such that this equation is solvable. Moreover, for that* b, *the solution* $\omega + i\partial\bar{\partial}\varphi > 0$ *is unique.*

Note that

$$b = \frac{Vol_{\omega + i\partial\bar{\partial}\varphi}(X)}{A_{\{\omega\}_A}} \in (0,\ 1]$$

since $A_{\{\omega\}_A} = F(\omega + i\partial\bar{\partial}\varphi) + Vol_{\omega+i\partial\bar{\partial}\varphi}(X) \geq Vol_{\omega+i\partial\bar{\partial}\varphi}(X)$. *We hope that this can shed some light on the mysterious constant* b.

(III) Obstruction to the existence of a Kähler metric in a given Hermitian-symplectic Aeppli class

Theorem and Definition 0.15 *(a) The (0, 2)-torsion form* $\rho_\omega^{0,\,2}$ *of any H-S metric* ω *is* E_2**-closed***: it defines an* E_2*-cohomology class*

$$\{\rho_\omega^{0,\,2}\}_{E_2} \in E_2^{0,\,2}(X)$$

on the second page of the Frölicher spectral sequence of X*.*

(b) When $\dim_{\mathbb{C}} X = 3$, $\{\rho_\omega^{0,\,2}\}_{E_2}$ *depends only on the Aeppli class* $\{\omega\}_A$.

(a consequence of $F(\omega) + Vol_\omega(X) = Const$ *when the H-S metric* ω *varies in a given Aeppli class)*

$$\{\rho_\omega^{0,\,2}\}_{E_2} \stackrel{def}{=} \text{ the } E_2\textbf{-torsion class} \text{ of the H-S Aeppli class } \{\omega\}_A.$$

(c) When $dim_{\mathbb{C}} X = 3$*, the* **vanishing** *of the* E_2*-torsion class:*

$$\{\rho_\omega^{0,\,2}\}_{E_2} = 0 \in E_2^{0,\,2}(X)$$

is a necessary condition for the Aeppli class $\{\omega\}_A$ *to contain a Kähler metric.*

Natural question. *Do there exist compact* 3*-dimensional Hermitian-symplectic manifolds on which all or some* E_2*-torsion classes are non-vanishing?*