Higher-Degree Holomorphic Contact Structures

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Standard notion

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n = 2p+1$.

A *holomorphic contact structure* on X: $\eta \in C^\infty_{1,\,0}(X,\, \mathbb{C})$ such that

$$\bar\partial\eta = 0 \quad \text{and} \quad \eta \wedge (\partial\eta)^p \neq 0 \quad \text{at every point of } X.$$

More generally, one may have $\eta \in C^\infty_{1,\,0}(X,\, L)$, where L is a holomorphic line bundle over X.

Intuitively, the $2p+1$ dimensions of X are split into two classes: the direction of η and the $2p$ directions of $(\partial\eta)^p$.

We replace this splitting by the splitting $2p+1 = p + (p+1)$.

Definition (Kasuya-P-Ugarte. 2025)

Let X be a compact complex manifold with $dim_{\mathbb{C}} X = n = 2p+1$.

(1) *A* **holomorphic p-contact structure** *on X is a form*

$$\Gamma \in C^{\infty}_{p,\,0}(X,\,\mathbb{C}) \text{ such that}$$

(a) $\bar\partial\Gamma = 0$ *and* (b) $\Gamma \wedge \partial\Gamma \neq 0$ *at every point of X.*

(2) *We say that X is a* **holomorphic p-contact manifold** *if there exists a holomorphic p-contact structure Γ on X.*

The opposite notion

Definition (Kasuya-P-Ugarte. 2025)

Let X be a compact complex manifold with $dim_{\mathbb{C}} X = n = 2p+1$.

(1) *A* **holomorphic p-no-contact structure** *on X is a form*

$$\Gamma \in C^\infty_{p,\,0}(X,\,\mathbb{C}) \text{ such that}$$

$$(a)\quad \bar\partial\Gamma = 0 \qquad \text{and} \qquad (b)\quad \partial\Gamma = \Gamma\wedge\zeta \quad \text{for some } \zeta\in C^\infty_{1,\,0}(X,\,\mathbb{C}).$$

(2) *We say that X is a* **holomorphic p-no-contact manifold** *if there exists a holomorphic p-no-contact structure Γ on X.*

Observation

If p is odd, the condition

$$\partial\Gamma = \Gamma \wedge \zeta$$

implies

$$\Gamma \wedge \partial\Gamma = (\Gamma \wedge \Gamma) \wedge \zeta = 0.$$

In this sense, a holomorphic p-no-contact structure is the opposite of a holomorphic p-contact one.

Background

Demailly (2002): given

- X is *Kähler* ;
- $\Gamma \in C^{\infty}_{p,\,0}(X,\, L^{-1})$ with $L \to X$ a *pseudo-effective* holomorphic line bundle such that $\bar{\partial}\Gamma = 0$;

we have:

$$\partial\Gamma = -\partial\varphi \wedge \Gamma$$

(locally) almost everywhere, where φ is a local weight of a Hermitian fibre metric h on L such that

$$i\Theta_h(L) \geq 0$$

(the positivity assumption on L on the curvature of L)

Initial observations

- **Cohomology**

Suppose (X, Γ) *is holomorphic p-contact. Then*

$$E_1(X) \neq E_\infty(X)$$

in the Frölicher spectral sequence.

In particular, X *is not a* $\partial\bar\partial$*-manifold. Hence,* X *is never Kähler.*

Proof. Γ represents a Dolbeault cohomology class

$$[\Gamma]_{\bar\partial} \in H^{p,\,0}_{\bar\partial}(X,\,\mathbb{C}) = E_1^{p,\,0}(X).$$

Its image under the differential

$$d_1 : E_1^{p,\,0}(X) \longrightarrow E_1^{p+1,\,0}(X)$$

is

$$d_1([\Gamma]_{\bar\partial}) = [\partial\Gamma]_{\bar\partial} \in E_1^{p+1,\,0}(X).$$

If we had $E_1(X) = E_\infty(X)$, we would have

$$d_1 = 0$$

(identically zero). In particular, we would get

$$[\partial\Gamma]_{\bar\partial} = 0 \in E_1^{p+1,\,0}(X) = H_{\bar\partial}^{p+1,\,0}(X,\,\mathbb{C}),$$

so there would exist

$$\zeta \in C^\infty_{p+1,\,-1}(X,\,\mathbb{C}) \text{ such that } \partial\Gamma = \bar\partial\zeta.$$

Then, $\zeta = 0$ (for bidegree reasons), hence also

$$\partial\Gamma = \bar\partial\zeta = 0.$$

This is impossible under our assumption $\Gamma \wedge \partial\Gamma \neq 0$ at every point of X. $\square$

Thus, the best possible degeneration of the Frölicher spectral sequence of any holomorphic p-contact manifold X can occur at the second page.

Many holomorphic p-contact manifolds X are *page*-1-$\partial\bar\partial$-*manifolds* (P.-Stelzig-Ugarte 2020).

RECALL

This means that:

$E_2(X) = E_\infty(X)$ and the De Rham cohomology of X is *pure*:

$$H^k_{dR}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,\,q}_{dR}(X, \mathbb{C}), \qquad k = 0, \dots, 2n.$$

This is equivalent to X having a *canonical Hodge decomposition* in which the E_1-spaces $E_1^{p,\,q}(X)$ (= the Dolbeault cohomology groups $H^{p,\,q}(X,\,\mathbb{C})$ of X) have been replaced by the E_2-spaces $E_2^{p,\,q}(X)$ in the sense that the identity map induces *canonical isomorphisms*:

$$H^k_{dR}(X,\,\mathbb{C}) \simeq \bigoplus_{p+q=k} E_2^{p,\,q}(X), \qquad k = 0, 1, \dots, 2n.$$

Explicitly, X is a *page-*1*-$\partial\bar\partial$-manifold* iff:

• each cohomology class in every space $E_2^{p,\,q}(X)$ can be represented by a smooth d-closed $(p,\,q)$-form

and

• for every $k\in\{0,\dots,2n\}$, the linear map

$$\bigoplus_{p+q=k}E_2^{p,\,q}(X)\ni\bigg(\{\alpha^{p,\,q}\}_{E_2}\bigg)_{p+q=k}\longmapsto\bigg\{\sum_{p+q=k}\alpha^{p,\,q}\bigg\}_{dR}\in H^k_{dR}(X,\,\mathbb{C})$$

is *well defined* (in the sense that it is independent of the choices of d-closed pure-type representatives $\alpha^{p,\,q}$ of their respective E_2-classes) and *bijective*.

- **Dimension**

If $(X,\,\Gamma)$ *is a compact* **holomorphic p-contact manifold** *with* $dim_{\mathbb{C}}X = n = 2p+1$*, then* p *is* **odd**.

In other words, $\dim_{\mathbb{C}}X \equiv 3 \mod 4$.

Proof. If p were even, we would get: $\frac{1}{2}\,\partial(\Gamma^2) = \Gamma\wedge\partial\Gamma$. This is a non-vanishing holomorphic $(n,\,0)$-form on X since Γ is a holomorphic p-contact structure. Hence

$$dV_\Gamma := i^{n^2}\,(\Gamma\wedge\partial\Gamma)\wedge(\overline{\Gamma}\wedge\bar{\partial}\overline{\Gamma}) = \frac{i^{n^2}}{4}\,\partial\bigg(\Gamma^2\wedge\bar{\partial}\overline{\Gamma}^2\bigg)$$

would be a strictly positive ∂-exact $(n,\,n)$-form on X.

We would then get (using Stokes to infer the equality below):

$$0 < \int_X dV_\Gamma = 0,$$

a contradiction. □

The sheaves $\mathcal{F}_\Gamma$ and $\mathcal{G}_\Gamma$

Definition

Let (X, Γ) *be a compact* **holomorphic** p**-contact manifold** *with* $dim_{\mathbb{C}} X = n = 2p + 1$.

(i) We let $\mathcal{F}_\Gamma$ *be the sheaf of germs of holomorphic* $(1, 0)$*-vector fields* ξ *such that* $\xi \lrcorner \Gamma = 0$.

(ii) We let $\mathcal{G}_\Gamma$ *be the sheaf of germs of holomorphic* $(1, 0)$*-vector fields* ξ *such that* $\xi \lrcorner \partial\Gamma = 0$.

• The $\mathcal{O}_X$-module $\mathcal{F}_\Gamma$ is the kernel of the induced morphism of locally free sheaves associated with the holomorphic vector bundle morphism:

$$T^{1,\,0}X \longrightarrow \Lambda^{p-1,\,0}T^\star X, \qquad \xi \longmapsto \xi\lrcorner\Gamma.$$

• The $\mathcal{O}_X$-module $\mathcal{G}_\Gamma$ is the kernel of the induced morphism of locally free sheaves associated with the holomorphic vector bundle morphism:

$$T^{1,\,0}X \longrightarrow \Lambda^{p,\,0}T^\star X, \qquad \xi \longmapsto \xi\lrcorner\partial\Gamma,$$

In particular, $\mathcal{F}_\Gamma$ and $\mathcal{G}_\Gamma$ are coherent, torsion-free analytic sheaves. They need not be locally free.

• The sheaf $\mathcal{F}_\Gamma$ is *not integrable*. This is the point of a holomorphic p-contact structure Γ.

• However, $\mathcal{G}_\Gamma$ displays the opposite behaviour.

Observation

The subsheaf $\mathcal{G}_\Gamma$ *of* $\mathcal{O}(T^{1,0}X)$ *is* **integrable** *in the sense that*

$$[\mathcal{G}_\Gamma, \mathcal{G}_\Gamma] \subset \mathcal{G}_\Gamma,$$

where $[\,\cdot\,,\,\cdot\,]$ *is the Lie bracket of* $T^{1,0}X$*.*

Proof. Since $d\partial\Gamma = 0 = d(d\Gamma)$ (because $\bar{\partial}\Gamma = 0$), the Cartan formula reads:

$$0 = d(\partial\Gamma)(\xi_0, \ldots, \xi_{p+1}) = \sum_{j=0}^{p+1}(-1)^j\, \xi_j \cdot (\partial\Gamma)(\xi_0, \ldots, \widehat{\xi_j}, \ldots, \xi_{p+1})$$

$$+ \sum_{0\leq j<k\leq p+1} (-1)^{j+k}\, (\partial\Gamma)([\xi_j,\, \xi_k],\, \xi_0, \ldots, \widehat{\xi_j}, \ldots, \widehat{\xi_k}, \ldots, \xi_{p+1})$$

for all $(1,\, 0)$-vector fields $\xi_0, \ldots, \xi_{p+1}$.

If two among $\xi_0, \ldots, \xi_{p+1}$, say ξ_l and ξ_s for some $l < s$, are (local) sections of $\mathcal{G}_\Gamma$, then:

- all the terms in the first sum on the r.h.s. above vanish;
- all the terms in the second sum on the r.h.s. above vanish, except

possibly the term

$$(-1)^{l+s}\,(\partial\Gamma)([\xi_l,\,\xi_s],\,\xi_0,\ldots,\widehat{\xi_l},\ldots,\widehat{\xi_s},\ldots,\xi_{p+1}).$$

Thus, this last term must vanish as well for all (1, 0)-vector fields $\xi_0,\ldots,\widehat{\xi_l},\ldots,\widehat{\xi_s},\ldots,\xi_{p+1}$.

This means that

$$[\xi_l,\,\xi_s]\lrcorner\partial\Gamma = 0,$$

which amounts to $[\xi_l,\,\xi_s]$ being a (local) section of $\mathcal{G}_\Gamma$, for all pairs $\xi_l,\,\xi_s$ of (local) sections of $\mathcal{G}_\Gamma$.

This proves the integrability of $\mathcal{G}_\Gamma$. □

Proposition

Let $(X,\,\Gamma)$ *be a compact* **holomorphic p-contact manifold** *with* $dim_{\mathbb{C}}X = n = 2p+1$.

(i) The sum $\mathcal{F}_\Gamma \oplus \mathcal{G}_\Gamma \subset \mathcal{O}(T^{1,\,0}X)$ *is direct.*

(ii) If there exists a **holomorphic contact** *structure* $\eta \in C^\infty_{1,\,0}(X,\,\mathbb{C})$, *the sheaf of germs* $\mathcal{F}_\eta$ *of holomorphic* $(1,\,0)$*-vector fields* ξ *such that* $\xi \lrcorner \eta = 0$ *and the sheaf of germs* $\mathcal{G}_\eta$ *of holomorphic* $(1,\,0)$*-vector fields* ξ *such that* $\xi \lrcorner \partial\eta = 0$ *are* **locally free** *of respective ranks* $n-1$ *and* 1, *while* $T^{1,\,0}X$ *has a direct-sum splitting*

$$T^{1,\,0}X = \mathcal{F}_\eta \oplus \mathcal{G}_\eta$$

with $\mathcal{F}_\eta$ *and* $\mathcal{G}_\eta$ *viewed as holomorphic vector subbundles of* $T^{1,\,0}X$.

Furthermore, the holomorphic line bundle $\mathcal{G}_\eta$ *is* **trivial** *and* η *is a non-vanishing global holomorphic section of its dual* $\mathcal{G}^\star_\eta$.

(iii) Suppose that $\mathcal{F}_\Gamma$ *and* $\mathcal{G}_\Gamma$ *are* **locally free** *and*

$$T^{1,\,0}X = \mathcal{F}_\Gamma \oplus \mathcal{G}_\Gamma.$$

For $s = 0, \dots, n$ *and* $\varphi \in C^\infty_{0,\,s}(X,\,T^{1,\,0}X)$*, let*

$$\varphi = \varphi_\Gamma + \varphi_{\partial\Gamma}$$

be the induced splitting of φ *into pieces* $\varphi_\Gamma, \varphi_{\partial\Gamma} \in C^\infty_{0,\,s}(X,\,T^{1,\,0}X)$ *such that* $\varphi_\Gamma \lrcorner \Gamma = 0$ *and* $\varphi_{\partial\Gamma} \lrcorner \partial\Gamma = 0$*.*

Then, for any $\theta \in C^\infty_{0,\,q}(X,\,T^{1,\,0}X)$ *such that*

$$\theta \in Im\,\bar\partial \quad \text{and} \quad \theta \lrcorner \Gamma = 0$$

and for any $\varphi \in C^\infty_{0,\,q-1}(X,\,T^{1,\,0}X)$ *such that*

$$\bar\partial\varphi = \theta,$$

one has $\bar\partial\varphi_\Gamma = \theta$ *and* $\bar\partial\varphi_{\partial\Gamma} = 0$*.*

The analogous statement is true when $\mathcal{F}_\Gamma$ is replaced by $\mathcal{G}_\Gamma$.

UPSHOT

One can define directional analogues of the $T^{1,\,0}X$-valued $\bar\partial$-cohomology:

$$H^{0,\,q}_{\Gamma}(X,\,T^{1,\,0}X) \quad \text{and} \quad H^{0,\,q}_{\partial\Gamma}(X,\,T^{1,\,0}X).$$

There is a **natural isomorphism**

$$H^{0,\,q}_{\bar\partial}(X,\,T^{1,\,0}X) \simeq H^{0,\,q}_{\Gamma}(X,\,T^{1,\,0}X) \oplus H^{0,\,q}_{\partial\Gamma}(X,\,T^{1,\,0}X).$$

Examples of holomorphic p-contact manifolds

(I) Initial observations

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n = 2p+1$ and p odd.

(1) *If $n=3$ (and $p=1$), a form $\Gamma \in C^\infty_{1,\,0}(X,\,\mathbb{C})$ is a holomorphic contact structure on X if and only if it is a holomorphic 1-contact structure.*

(2) *Let s be the positive integer such that $p = 2s-1$. For any holomorphic contact structure $\eta \in C^\infty_{1,\,0}(X,\,\mathbb{C})$ on X, the form*

$$\Gamma := \eta \wedge (\partial\eta)^{s-1} \in C^\infty_{p,\,0}(X,\,\mathbb{C})$$

is a holomorphic p-contact structure on X.

(II) Examples in dimension 3

- The **Iwasawa manifold** $X = G/\Gamma$:

$G = (\mathbb{C}^3, \star)$ is a nilpotent complex Lie group ($=$ Heisenberg group), where the group operation is defined by

$$(\zeta_1, \zeta_2, \zeta_3) \star (z_1, z_2, z_3) = (\zeta_1 + z_1, \zeta_2 + z_2, \zeta_3 + z_3 + \zeta_1 z_2),$$

$\Gamma \subset G$ is the lattice of $(z_1, z_2, z_3) \in G$ with $z_1, z_2, z_3 \in \mathbb{Z}[i]$.

The cohomology of X is generated by three holomorphic $(1, 0)$-forms α, β, γ on X that satisfy the structure equations:

$$\partial\alpha = \partial\beta = 0 \qquad \text{and} \qquad \partial\gamma = -\alpha\wedge\beta \neq 0 \text{ everywhere on } X.$$

Thus, $\Gamma := \gamma$ defines a **holomorphic contact structure** ($=$ a **holomorphic 1-contact structure)** since

$$\gamma\wedge\partial\gamma = -\alpha\wedge\beta\wedge\gamma \neq 0 \quad \text{at every point of } X.$$

• The **Nakamura manifolds** $X = G/\Gamma$:

$G = (\mathbb{C}^3, \star)$ is a solvable complex Lie group, where the group operation is defined by

$$(\zeta_1,\, \zeta_2,\, \zeta_3) \star (z_1,\, z_2,\, z_3) = (\zeta_1 + z_1,\, \zeta_2 + e^{-\zeta_1} z_2,\, \zeta_3 + e^{\zeta_1} z_3),$$

$\Gamma \subset G$ is a lattice.

The cohomology of X is generated by three holomorphic $(1,\, 0)$-forms $\varphi_1, \varphi_2, \varphi_3$ on X that satisfy the structure equations:

$$\partial\varphi_1 = 0, \qquad \partial\varphi_2 = \varphi_1 \wedge \varphi_2 \qquad \text{and} \qquad \partial\varphi_3 = -\varphi_1 \wedge \varphi_3.$$

Hence:

• φ_2 and φ_3 define **holomorphic 1-no-contact structures**;

• $\Gamma_1 := \varphi_2 + \varphi_3$ and $\Gamma_2 := \varphi_2 - \varphi_3$ define **holomorphic contact structures** (= **holomorphic 1-contact structures**).

• The manifolds $X = SL(2,\,\mathbb{C})/\Gamma$:

$SL(2,\,\mathbb{C})$ is a semi-simple complex Lie group;

$\Gamma \subset SL(2,\,\mathbb{C})$ is a lattice.

The cohomology of X is generated by three holomorphic $(1,\,0)$-forms $\alpha,\,\beta,\,\gamma$ on X that satisfy the structure equations:

$$\partial\alpha = \beta\wedge\gamma, \qquad \partial\beta = \gamma\wedge\alpha, \qquad \partial\gamma = \alpha\wedge\beta.$$

Hence, each of the forms α, β, γ defines a **holomorphic contact structure** (= a **holomorphic 1-contact structure)** on X.

Proposition

Let $X = G/\Gamma$ *be the quotient of a nilpotent Lie group of real dimension* 6 *endowed with a left invariant complex structure.*
If X *has a holomorphic contact structure, then* X *is:*

- *either the Iwasawa manifold*
- *or the non-complex parallelisable nilmanifold* $X = G/\Gamma$*, where:*

$G = (\mathbb{C}^3, \star)$ *is given by the non-holomorphic group operation:*

$$(\zeta_1, \zeta_2, \zeta_3)\star(z_1, z_2, z_3) = \big(\zeta_1+z_1,\, \zeta_2+z_2+\overline{\zeta}_1\, z_1,\, \zeta_3+z_3-\zeta_1\, z_2-\overline{\zeta}_1\, z_1(\zeta_1+\frac{z_1}{2})$$

the lattice $\Gamma \subset G$ *consists of* $(z_1, z_2, z_3) \in G$ *with* $z_1 \in 2\mathbb{Z}[i]$ *and* $z_2, z_3 \in \mathbb{Z}[i]$*;*

Hence:

• *$X = G/\Gamma$ is a compact complex 3-fold equipped with a basis of $(1,\,0)$-forms φ_1, φ_2, φ_3 on X that satisfy the structure equations:*

$$d\varphi_1 = 0, \qquad d\varphi_2 = \varphi_1 \wedge \overline{\varphi}_1, \qquad d\varphi_3 = \varphi_1 \wedge \varphi_2.$$

• *φ_3 defines a* **holomorphic contact structure** *(= a* **holomorphic 1-contact structure)** *on X since $\bar\partial\varphi_3 = 0$ and*

$$\varphi_3 \wedge \partial\varphi_3 = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \neq 0.$$

(III) Examples in arbitrary dimensions $n \equiv 3 \ mod \ 4$

Higher-dimensional Iwasawa-type manifolds $X = G/\Gamma$

Proposition

Let $n = 2p + 1 = 4l + 3$ and let G be the nilpotent n-dimensional complex Lie group equipped with the complex structure defined by either of the following two classes of structure equations involving a basis of holomorphic $(1, 0)$-forms $\varphi_1, \ldots, \varphi_n$:

(Class I)
$$d\varphi_1 = d\varphi_2 = 0,$$
$$d\varphi_3 = \varphi_2 \wedge \varphi_1, \ldots, d\varphi_n = \varphi_{n-1} \wedge \varphi_1;$$

(Class II)
$$d\varphi_1 = \cdots = d\varphi_{n-1} = 0,$$
$$d\varphi_n = \varphi_1 \wedge \varphi_2 + \varphi_3 \wedge \varphi_4 + \cdots + \varphi_{n-2} \wedge \varphi_{n-1}.$$

Then:

(1) *if* G *is in* **class I**, *given any* $(4l+1)\times(4l+1)$ *invertible upper triangular matrix* $A=(a_{ij})_{3\leq i,j\leq 4l+3}$, *we define:*

- *the* $(1,\,0)$*-forms* $\gamma_u=\sum_{i=u}^{4l+3}a_{ui}\,\varphi_i$, *for every* $u\geq 3$;
- *the* $(p=2l+1,\,0)$*-form* Γ_l *given by*

$$\begin{aligned}\Gamma_l = \gamma_3\, \wedge\, &(\gamma_4\wedge\gamma_5+\gamma_5\wedge\gamma_6+\gamma_6\wedge\gamma_7)\wedge\ldots\\ &\wedge\,(\gamma_{4l}\wedge\gamma_{4l+1}+\gamma_{4l+1}\wedge\gamma_{4l+2}+\gamma_{4l+2}\wedge\gamma_{4l+3}).\end{aligned}$$

For any co-compact lattice $\Lambda\subset G$, *the* $(p=2l+1,\,0)$*-form* Γ_l *defines a* **holomorphic p-contact structure** *on the compact nilmanifold* $X=G/\Lambda$.

Meanwhile, X *does not admit any holomorphic contact structure.*

(2) *if G is in* **class II**, *for any co-compact lattice $\Lambda \subset G$, the $(p, 0)$-form*

$$\Gamma = \varphi_n \wedge (\partial \varphi_n)^l = \varphi_n \wedge \left(\sum_{j=0}^{2l} \varphi_{2j+1} \wedge \varphi_{2j+2} \right)^l$$

defines a **holomorphic p-contact structure** *on the compact nilmanifold $X = G/\Lambda$.*

This is induced by the **standard contact structure** φ_n *on X.*

(IV) Examples with holomorphic s-symplectic manifolds

Standard notion

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n = 2s$.

A *holomorphic symplectic structure*: $\omega \in C^\infty_{2,\,0}(X,\, \mathbb{C})$ such that

$(i)\ d\omega = 0; \qquad (ii)\ \bar\partial\omega = 0; \qquad (iii)\ \omega^s \neq 0$ at every point of X.

Observation

(i) implies (ii), but we stress that, besides being holomorphic, ω is required to be d-closed.

Definition (Kasuya-P-Ugarte. 2025)

Let X be a compact complex manifold with $dim_{\mathbb{C}} X = 2s$.

(1) *A* **holomorphic s-symplectic structure** *on X is a smooth $(s,\,0)$-form $\Omega \in C^\infty_{s,\,0}(X,\,\mathbb{C})$ such that*

$$(i)\ \bar\partial\Omega = 0; \quad \text{and} \quad (ii)\ \Omega\wedge\Omega \neq 0 \ \text{ at every point of } X.$$

(2) *We say that X is a* **holomorphic s-symplectic manifold** *if there exists a holomorphic s-symplectic structure on X.*

Note that we do not require holomorphic s-symplectic structures Ω to be d-closed.

First structure theorem (Kasuya-P-Ugarte. 2025)

Let:

- G *be a nilpotent complex Lie group with* $\dim_{\mathbb{C}} G = n = 2s = 4l$*;*
- Λ *be a co-compact lattice in* G*;*
- $\{\varphi_1, \dots, \varphi_{4l}\}$ *be a* $\mathbb{C}$*-basis of the dual vector space* $\mathfrak{g}^\star$ *of the Lie algebra* $\mathfrak{g}$ *of* G*;*
- $Y = G/\Lambda$ *is the induced quotient compact complex* n*-dimensional manifold.*

We still denote by $\{\varphi_1, \dots, \varphi_{4l}\}$ *the induced* $\mathbb{C}$*-basis of* $H^{1,\,0}_{\bar\partial}(Y, \mathbb{C})$.

(1) *The* $(s,\,0)$*-form*

$$\Omega := \varphi_1 \wedge \cdots \wedge \varphi_{2l} + \varphi_{2l+1} \wedge \cdots \wedge \varphi_{4l}$$

is a **holomorphic s-symplectic structure** *on* Y.

(2) *Set* $p := 2l+1$ *and consider the* $(2p+1)$*-dimensional compact complex nilmanifold* X *defined by a basis*

$$\{\pi^\star\varphi_1, \dots, \pi^\star\varphi_{4l}, \varphi_{4l+1}, \varphi_{4l+2}, \varphi_{4l+3}\}$$

of holomorphic $(1,\,0)$*-forms whose first* $4l$ *members are the pull-backs under the natural projection* $\pi : X \longrightarrow Y$ *of the forms considered under* (1) *and the three extra members satisfy the structure equations on* X*:*

$$\partial\varphi_{4l+1} = \partial\varphi_{4l+2} = 0 \quad \text{and} \quad \partial\varphi_{4l+3} = \varphi_{4l+1}\wedge\varphi_{4l+2} + \pi^\star\sigma,$$

where σ *is any rational d-closed* $(2,\,0)$*-form on* Y*.*

Then, the $(p,\,0)$*-form*

$$\Gamma := \pi^\star\Omega\wedge\varphi_{4l+3}$$

is a **holomorphic** p**-contact structure** *on* X*.*

Observations

- *When* $\sigma \neq 0$, $X = Y \times I^{(3)}$ *(where* $I^{(3)} =$ *Iwasawa manifold);*

- *When* $\sigma \neq 0$, *we get further holomorphic p-contact manifolds* X *from holomorphic s-symplectic manifolds* Y *through the above construction besides the products* $Y \times I^{(3)}$.

- *In every complex dimension* $4l$ *with* $l \geq 2$ *there are holomorphic s-symplectic manifolds that are not holomorphic symplectic.*

- *Let* X *be a holomorphic p-contact manifold and* Y *a holomorphic s-symplectic manifold. Then,* $Z = X \times Y$ *is a holomorphic* $(p+s)$*-contact manifold.*

Second structure theorem (Kasuya-P-Ugarte. 2025)

Let $\pi : X \longrightarrow Y$ be a surjective holomorphic submersion between compact complex manifolds with $dim_{\mathbb{C}} X = n = 2p+1 = 4l+3$ and $dim_{\mathbb{C}} Y = 4l = 2s$. Suppose that:

(a) X has a **holomorphic p-contact structure** $\Gamma \in C^{\infty}_{p,\,0}(X,\, \mathbb{C})$;

(b) $T^{1,\,0}X = \mathcal{E} \oplus \mathcal{H}$, where $\mathcal{E}$ and $\mathcal{H}$ are holomorphic subbundles of $T^{1,\,0}X$ such that $\mathcal{H}$ is **Frobenius integrable** *and the leaves of the foliation it induces on X are the fibres of $\pi : X \longrightarrow Y$;*

(c) there exist holomorphic vector fields $\eta_1, \eta_2, \eta_3 \in H^0(X,\, \mathcal{H})$ that globally trivialise $\mathcal{H}$ such that

$$\eta_1, \eta_2 \in H^0(X,\, \mathcal{F}_\Gamma), \qquad \psi_3 \wedge \Gamma = 0$$

(where $\psi_1,\, \psi_2,\, \psi_3 \in H^{1,\,0}_{\bar\partial}(X,\, \mathbb{C})$ is the global frame of $\mathcal{H}^\star$ dual to the global frame $\{\eta_1,\, \eta_2,\, \eta_3\}$ of $\mathcal{H}$) and the following relations are

satisfied:

$$[\eta_1,\,\eta_2] = -[\eta_2,\,\eta_1] = \eta_3 \qquad \text{and} \qquad [\eta_j,\,\eta_k] = 0$$

for all $(j,\,k) \notin \{(1,\,2),\,(2,\,1)\}$.

Then, there exists a **holomorphic s-symplectic structure** $\Omega \in C^\infty_{p-1,\,0}(Y,\,\mathbb{C}) = C^\infty_{2l,\,0}(Y,\,\mathbb{C}) = C^\infty_{s,\,0}(Y,\,\mathbb{C})$ *on* Y *such that*

$$\Gamma = \pi^\star\Omega \wedge \psi_3 \qquad \text{on } X.$$

Essential horizontal deformations

Definition (Kasuya-P-Ugarte. 2025)

The $\mathbb{C}$-vector subspace

$$E_2^{0,\,1}(X,\,\mathcal{F}_\Gamma) := \left\{[\theta]_{\bar\partial} \in H_\Gamma^{0,\,1}(X,\,T^{1,\,0}X) \mid \bar\partial\theta = 0,\;\; \theta\lrcorner\Gamma = 0,\;\; \theta\lrcorner\partial\Gamma \in \mathcal{Z}_2^{p,\,1}(X)\right\}$$

is called the space of (infinitesimal) **essential horizontal deformations** *of X.*

We say that the essential horizontal deformations of X are **unobstructed** *if every class $[\theta_1]_{\bar\partial} \in E_2^{0,\,1}(X,\,\mathcal{F}_\Gamma)$ has a representative $\psi_1 \in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$ such that all the equations*

$$\bar\partial\psi_\nu = \frac{1}{2}\sum_{\mu=1}^{\nu-1}[\psi_\mu,\,\psi_{\nu-\mu}] \qquad (Eq.\ (\nu)), \qquad \nu\geq 2,$$

admit solutions $\psi_\nu \in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$ with the property

$$\psi_\nu\lrcorner u_\Gamma \in \ker\partial$$

for every $\nu \geq 2$.

Notation $u_\Gamma := \Gamma\wedge\partial\Gamma \in C^\infty_{n,\,0}(X,\,\mathbb{C})$ is a non-vanishing form

the Calabi-Yau form of the holomorphic p-contact manifold $(X,\,\Gamma)$

Theorem (Kasuya-P-Ugarte. 2025)

Let (X, Γ) *be a compact* **holomorphic** p**-contact manifold** *with* $dim_{\mathbb{C}} X = n = 2p+1$ *such that* $\mathcal{F}_\Gamma$ *and* $\mathcal{G}_\Gamma$ *are* **locally free** *and* $T^{1,\,0}X = \mathcal{F}_\Gamma \oplus \mathcal{G}_\Gamma$.

Suppose that (X, Γ) *is a* **partially** $\mathcal{F}_\Gamma$**-directional page-**1**-**$\partial\bar\partial$**-manifold** *and a* **partially vertically** $\partial\bar\partial$**-manifold**. *Further suppose that the sheaf* $\mathcal{F}_\Gamma$ *is* **cohomologically integrable in bidegree** $(0,\,1)$ *and* **constantly maximally non-integrable in bidegree** $(0,\,1)$.

Then, the **essential horizontal deformations** *of* X *are* **unobstructed**.

p-contact hyperbolicity

Theorem (Kasuya-P-Ugarte. 2025)

Let $(X,\,\Gamma)$ be a compact **holomorphic p-contact manifold** *with $dim_{\mathbb{C}}X = n = 2p+1$ such that $\mathcal{F}_\Gamma$ and $\mathcal{G}_\Gamma$ are* **locally free** *of respective ranks $p+1$ and p and $T^{1,\,0}X = \mathcal{F}_\Gamma\oplus\mathcal{G}_\Gamma$.*

Then, there exists no holomorphic map $f:\mathbb{C}^{p+1}\longrightarrow X$ such that:

(i) f is non-degenerate at some point $x_0\in\mathbb{C}^{p+1}$;

(ii) f satisfies, for some, hence every, Hermitian metric ω, the slow growth condition

$$\liminf_{r\to+\infty}\frac{A_{\omega,\,f}(S_r)}{Vol_{\omega,\,f}(B_r)} = 0,$$

where the (ω, f)-area $A_{\omega,\, f}(S_r)$ *of the Euclidean sphere* $S_r \subset \mathbb{C}^{p+1}$ *is defined by*

$$A_{\omega,\, f}(S_r) := \int\limits_{S_r} d\sigma_{\omega,\, f,\, r} > 0, \qquad r > 0,$$

while the (ω, f)-volume of the Euclidean ball $B_r \subset \mathbb{C}^{p+1}$ *is defined by*

$$Vol_{\omega,\, f}(B_r) := \int\limits_{B_r} f^{\star}\omega_{p+1}.$$

(iii) $f^{\star}\Gamma = 0$ *(i.e. f is* **horizontal***).*

Main ingredient in the proof

A result from Kasuya-P. (2023) on partial hyperbolicity:

Any holomorphic map $f : \mathbb{C}^{p+1} \longrightarrow X$ *with the above properties (i) and (ii) induces an Ahlfors current* T *on* X*.*

This means: T is a strongly semi-positive current on X

- of bidimension $(p+1,\, p+1)$ (= of bidegree $(p,\, p)$);
- of mass 1 with respect to any pregiven Hermitian metric ω;
- having the extra key property $dT = 0$.

It is obtained as the weak limit of a subsequence $(T_{r_\nu})_\nu$ of the family of currents

$$T_r := \frac{1}{\mathrm{Vol}_{\omega,\,f}(B_r)}\, f_\star[B_r], \qquad r > 0,$$

where $[B_r]$ is the current of integration on the ball $B_r \subset \mathbb{C}^{p+1}$.

Application to our setting

Stokes and $dT = 0$ and $\partial\Gamma \wedge \bar\partial\overline{\Gamma} \in \mathrm{Im}\,(\partial\bar\partial)$ yield:

$$\int_X T \wedge i^{(p+1)^2}\, \partial\Gamma \wedge \bar\partial\overline{\Gamma} = 0.$$

By the positivity properties of T and $i^{(p+1)^2}\, \partial\Gamma \wedge \bar\partial\overline{\Gamma}$, we infer:

$$T \wedge i^{(p+1)^2}\, \partial\Gamma \wedge \bar\partial\overline{\Gamma} = 0 \qquad \text{on } X.$$