A Generalised Volume Invariant for Aeppli Cohomology Classes of Hermitian-Symplectic Metrics

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Context and motivation

X a compact complex manifold, $\dim_{\mathbb{C}} X = n$

Goal: study the geometry of X in terms of the metric structures it supports.

Let $\omega > 0$ be a C^∞ **positive definite** $(1, 1)$**-form** on X.

(**Hermitian metric**, always exists)

Some classes of special Hermitian metrics

(Except for Gauduchon metrics, they need not exist on a given X.)

$d\omega = 0$	$\Longrightarrow$	$\exists\, \rho^{0,2} \in C^\infty_{0,2}(X, \mathbb{C})$ s.t. $d(\overline{\rho^{0,2}} + \omega + \rho^{0,2}) = 0$	$\Longrightarrow$	$\partial\bar\partial\omega = 0$
(ω is Kähler)		(ω is Hermitian-symplectic) (Streets-Tian '10)		(ω is SKT)
$\Downarrow$				
$d\omega^{n-1} = 0$	$\Longrightarrow$	$\exists\, \Omega^{n-2,n} \in C^\infty_{n-2,n}(X, \mathbb{C})$ s.t. $d(\overline{\Omega^{n-2,n}} + \omega^{n-1} + \Omega^{n-2,n}) = 0$	$\Longrightarrow$	$\partial\bar\partial\omega^{n-1} = 0$
(ω is balanced) (Gauduchon '77)		(ω is strongly Gauduchon (sG)) (P. '09)		(ω is Gauduchon).

Hermitian-symplectic (H-S) metrics and manifolds

- Intrinsic characterisation

Theorem 0.1 *(Sullivan 1976) Let X be a compact complex manifold with $dim_{\mathbb{C}} X = n$.*

X is Hermitian-symplectic $\iff$ X carries no non-zero current T of bidegree $(n-1, n-1)$ such that $T \geq 0$ and T is d-exact.

- Li-Zhang '09, Streets-Tian '10, Dinew-P. '20: when $\dim_{\mathbb{C}} X = 2$,

X is Hermitian-symplectic $\iff$ X is Kähler.

• Higher-dimensional Hermitian-symplectic manifolds

They are poorly understood.

Question 0.2 *(Streets-Tian '10) Do there exist non-Kähler Hermitian-symplectic complex manifolds X with $dim_{\mathbb{C}} X \geq 3$?*

· The general case of this question is still open.

· It has been answered negatively for a handful of special classes of manifolds, including:

-all nilmanifolds endowed with an invariant complex structure (Enrietti-Fino-Vezzoni '12);

-all twistor spaces (Verbitsky '14).

· If the answer is affirmative, this would be a Kählerianity criterion for manifolds.

Cf. the Demaily-Paun ('04) Kählerianity criterion for cohomology classes on a given compact Kähler manifold.

· This problem lies at the interface between symplectic and complex Hermitian geometries.

· The Streets-Tian question is complementary to Donaldson's earlier

Question 0.3 *(Donaldson '06) If J is an almost-complex structure on a compact 4-manifold which is tamed by a symplectic form, is there a symplectic form compatible with J?*

This is *Donaldson's tamed-to-compatible conjecture*.

Preliminaries: relations between special metrics

(1) H-S and sG manifolds

Proposition 0.4 *(Yau-Zhao-Zheng '19; Dinew-P. '20) Every compact complex manifold X that admits a* **Hermitian-symplectic** *metric also admits a* **strongly Gauduchon (sG)** *metric.*

Proof. Suppose that an H-S structure exists on X. Thus, there is a real C^∞ d-closed 2-form

$$\widetilde{\omega} = \rho^{2,\,0} + \omega + \rho^{0,\,2}$$

such that its (1, 1)-component $\omega > 0$. Then, $d\widetilde{\omega}^{n-1} = 0$ and

$$\begin{aligned} \widetilde{\omega}^{n-1} &= [\omega + (\rho^{2,\,0} + \rho^{0,\,2})]^{n-1} \\ &= \sum_{k=0}^{n-1} \sum_{l=0}^{k} \binom{n-1}{k} \binom{k}{l} (\rho^{2,\,0})^l \wedge (\rho^{0,\,2})^{k-l} \wedge \omega^{n-k-1}. \end{aligned}$$

The $(n-1,\, n-1)$-component of $\widetilde{\omega}^{n-1}$ is the sum of the terms for which $l = k - l$, i.e.

$$\Omega^{n-1,\,n-1} = \omega^{n-1} + \sum_{l=1}^{[\frac{n-1}{2}]} \binom{n-1}{2l} \binom{2l}{l} (\rho^{2,\,0})^l \wedge (\rho^{0,\,2})^l \wedge \omega^{n-2l-1}.$$

From $d\widetilde{\omega}^{n-1} = 0$, we deduce that $\partial\bar{\partial}\Omega^{n-1,\,n-1} = 0$. Thus, it suffices to check that $\Omega^{n-1,\,n-1} > 0$. This follows from $\omega^{n-1} > 0$ and from the real $(n-1,\, n-1)$-form $(\rho^{2,\,0})^l \wedge (\rho^{0,\,2})^l \wedge \omega^{n-2l-1}$ being weakly (semi)-positive at every point of X. $\square$

(2) **Balanced SKT metrics**

Proposition 0.5 *(Ivanov- Papadopoulos '13, P. '15) If a Hermitian metric ω on a compact complex manifold X is both* **SKT** *and* **balanced**, *then ω is* **Kähler**.

Proof. The *SKT* assumption on ω translates to:

$$\partial\bar\partial\omega = 0 \iff \partial\omega \in \ker\bar\partial \iff \star(\partial\omega) \in \ker\partial^\star,$$

where we use the standard formula $\partial^\star = -\star\bar\partial\star$ involving the Hodge-star operator induced by ω.

The *balanced* assumption on ω translates to:

$$d\omega^{n-1} = 0 \iff \partial\omega^{n-1} = 0 \iff \omega^{n-2}\wedge\partial\omega = 0 \iff \partial\omega \text{ is primitive.}$$

Moreover, since $\partial\omega$ is *primitive* when ω is balanced, we get:

$$\star(\partial\omega) = i\,\frac{\omega^{n-3}}{(n-3)!}\wedge\partial\omega = \frac{i}{(n-2)!}\,\partial\omega^{n-2} \in \operatorname{Im}\partial.$$

Hence, if ω is both *SKT* and *balanced*, we get:

$$\star(\partial\omega) \in \ker \partial^\star \cap \operatorname{Im} \partial = \{0\},$$

where the last identity follows from $\ker \partial^\star \perp \operatorname{Im} \partial$. Hence

$$\partial\omega = 0,$$

meaning that ω is *Kähler*. □

Our approach and results

Question **(strengthening of Streets-Tian)**

Does there exist a Kähler metric in the Aeppli cohomology class of every Hermitian-symplectic metric?

Recall

ω is H-S $\implies$ ω is SKT (i.e. $\partial\bar\partial\omega = 0$).

Hence, ω defines an *Aeppli cohomology class*:

$$\{\omega\}_A \in H^{1,\,1}_A(X,\,\mathbb{C}) := \frac{\ker(\partial\bar\partial)}{\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial}.$$

The *Hermitian-symplectic* condition can be expressed as follows.

Lemma 0.6 *Let ω be a Hermitian metric on a compact complex manifold X.*

(I) The following statements are **equivalent***.*

(a) ω is **Hermitian-symplectic***.*

(b) There exists a form $\rho^{2,\,0} \in C^\infty_{2,\,0}(X,\,\mathbb{C})$ satisfying the equations:

$$(i)\ \partial\rho^{2,\,0} = 0 \quad \text{and} \quad (ii)\ \bar\partial\rho^{2,\,0} + \partial\omega = 0.$$

(c) There exists a form $\rho^{0,\,2} \in C^\infty_{0,\,2}(X,\,\mathbb{C})$ satisfying the equations:

$$(iii)\ \bar\partial\rho^{0,\,2} = 0 \quad \text{and} \quad (iv)\ \partial\rho^{0,\,2} + \bar\partial\omega = 0.$$

(II) If $dim_{\mathbb{C}} X = 3$, the equivalences under (I) simplify to:

ω *is* **Hermitian-symplectic** $\iff \partial\omega \in Im\,\bar\partial \iff \bar\partial\omega \in Im\,\partial$.

Proof of (II). We show that $\partial\rho^{2,\,0} = 0$ whenever $\bar\partial\rho^{2,\,0} + \partial\omega = 0$.

On the one hand:

$$\int\limits_X i\partial\rho^{2,\,0}\wedge\bar\partial\rho^{0,\,2} = -\int\limits_X \bar\partial(i\partial\rho^{2,\,0}\wedge\rho^{0,\,2}) = 0.$$

On the other hand:

$$i\partial\rho^{2,\,0}\wedge\bar\partial\rho^{0,\,2} = \partial\rho^{2,\,0}\wedge\star\overline{\partial\rho^{2,\,0}} = |\partial\rho^{2,\,0}|^2_\omega\,dV_\omega \geq 0.$$

$\square$

(I) A new energy functional

Lemma and Definition 0.7 *For every Hermitian-symplectic metric ω on X, there exists a unique smooth $(2,\,0)$-form $\rho_\omega^{2,\,0}$ on X such that:*

$$(i)\ \partial\rho_\omega^{2,\,0} = 0 \quad \textit{and} \quad (ii)\ \bar\partial\rho_\omega^{2,\,0} = -\partial\omega$$

and

$$(iii)\ \rho_\omega^{2,\,0} \in Im\,\partial_\omega^\star + Im\,\bar\partial_\omega^\star.$$

Property (iii) ensures that $\rho_\omega^{2,\,0}$ has **minimal L^2_ω-norm** *among all the $(2,\,0)$-forms satisfying properties (i) and (ii).*

We call $\rho_\omega^{2,\,0}$ the **$(2,\,0)$-torsion form** *and its conjugate $\rho_\omega^{0,\,2}$ the* **$(0,\,2)$-torsion form** *of the Hermitian-symplectic metric ω.*

One has the explicit **Neumann-type formula***:*

$$\rho_\omega^{2,0} = -\Delta_{BC}^{-1}[\bar\partial^\star\partial\omega + \bar\partial^\star\partial\partial^\star\partial\omega], \tag{1}$$

where Δ_{BC}^{-1} *is the Green operator of the Bott-Chern Laplacian* Δ_{BC} *induced by* ω*, while* $\partial^\star = \partial^\star_\omega$ *and* $\bar\partial^\star = \bar\partial^\star_\omega$ *are the formal adjoints of* ∂*, resp.* $\bar\partial$*, w.r.t. the* L^2 *inner product defined by* ω*.*

Observation 0.8 *When* $dim_{\mathbb{C}}X = 3$*, formula (1) simplifies to*

$$\rho_\omega^{2,0} = -\Delta''^{-1}\bar\partial^\star(\partial\omega), \tag{2}$$

where $\Delta''^{-1} = \Delta_\omega''^{-1}$ *is the Green operator of the* $\bar\partial$*-Laplacian* $\Delta'' = \Delta''_\omega := \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial$ *induced by* ω *via* $\bar\partial^\star = \bar\partial^\star_\omega$*.*

For every *Hermitian-symplectic Aeppli class* $\{\omega\}_A$, we denote by

$$\mathcal{S}_{\{\omega\}} := \left\{\omega + \partial\bar{u} + \bar\partial u \ \mid \ u \in C^\infty_{1,\,0}(X,\,\mathbb{C}) \text{ such that } \omega + \partial\bar{u} + \bar\partial u > 0\right\}$$
$$\subset \{\omega\}_A \cap C^\infty_{1,\,1}(X,\,\mathbb{R})$$

the set of all (necessarily H-S) metrics in $\{\omega\}_A$.

It is an *open convex subset* of the real affine space

$$\{\omega\}_A \cap C^\infty_{1,\,1}(X,\,\mathbb{R}) = \{\omega + \partial\bar{u} + \bar\partial u \ \mid \ u \in C^\infty_{1,\,0}(X,\,\mathbb{C})\}.$$

Definition 0.9 *Let X be a compact complex Hermitian-symplectic manifold with $dim_{\mathbb{C}} X = n$. For the Aeppli cohomology class $\{\omega_0\}_A$ of any Hermitian-symplectic metric ω_0, we define the following* **energy functional***:*

$$F : \mathcal{S}_{\{\omega_0\}} \to [0, +\infty), \qquad F(\omega) = \int\limits_X |\rho_\omega^{2,\,0}|^2_\omega \, dV_\omega = ||\rho_\omega^{2,\,0}||^2_\omega,$$

where $\rho_\omega^{2,\,0}$ is the $(2,\,0)$-torsion form of the H-S metric $\omega \in \mathcal{S}_{\{\omega_0\}}$, while $|\;\;|_\omega$ is the pointwise norm and $||\;\;||_\omega$ is the L^2 norm induced by ω.

The introduction of this functional is justified by

Lemma 0.10 *Fix a Hermitian-symplectic Aeppli class* $\{\omega_0\}_A$. *For any H-S metric* $\omega \in \mathcal{S}_{\{\omega_0\}}$, *the following equivalence holds:*

$$\omega \ \textit{is Kähler} \iff F(\omega) = 0.$$

Proof. If ω is Kähler, $\partial\omega = 0$ and the minimal L^2-norm solution of the equation $\bar\partial\rho = 0$ vanishes. Thus $\rho_\omega^{2,\,0} = 0$, hence $F(\omega) = 0$.

Conversely, if $F(\omega) = 0$, then $\rho_\omega^{2,\,0}$ vanishes identically on X, hence $\partial\omega = -\bar\partial\rho_\omega^{2,\,0} = 0$, so ω is Kähler. □

Computation of the critical points of F

Theorem 0.11 *The differential at ω of F is given by the formula:*

$$(d_\omega F)(\gamma) = -2\,Re\,\langle\langle u,\,\bar\partial^\star\omega\rangle\rangle_\omega + 2\,Re\int\limits_X u\wedge\rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\bar\partial\bigg(\frac{\omega^{n-3}}{(n-3)!}\bigg),$$

for every $(1,\,1)$-form $\gamma = \partial\bar u + \bar\partial u$.

In particular, when $n=3$, this formula reduces to:

$$(d_\omega F)(\gamma) = -2\,Re\,\langle\langle u,\,\bar\partial^\star\omega\rangle\rangle_\omega.$$

Corollary 0.12 *Suppose* $n = 3$. *Then a Hermitian-symplectic metric* ω *on a compact complex manifold* X *of dimension* 3 *is a* **critical point** *of the energy functional* F **if and only if** ω *is* **Kähler**.

Proof. "$\Longleftarrow$" If ω is Kähler, $\partial\omega = 0$. Hence, $\rho_\omega^{2,\,0} = 0$, so $F(\omega) = 0$ and ω is a minimum for F.

"$\Longrightarrow$" A metric ω is a critical point of F if and only if $(d_\omega F)(\gamma) = 0$ for every $\gamma = \partial\bar{u} + \bar\partial u$. By the above discussion, this amounts to $\text{Re}\,\langle\langle u,\, \bar\partial^\star\omega\rangle\rangle_\omega = 0$ for every $(1,\, 0)$-form u.

Thus, if ω is a critical point of F, by taking $u = \bar\partial^\star\omega$ we get $\bar\partial^\star\omega = 0$. This is equivalent to ω being balanced. However, ω is already SKT since it is Hermitian-symplectic, so ω must be Kähler. $\square$

Corollary 0.13 *Let X be a compact complex manifold of dimension $n=3$ admitting Hermitian-symplectic metrics.*

Then, for any Aeppli-cohomologous Hermitian-symplectic metrics ω and ω_η:

$$\omega_\eta = \omega + \partial\bar\eta + \bar\partial\eta > 0, \qquad \textit{with } \eta \in C^\infty_{1,\,0}(X,\,\mathbb{C}),$$

the respective $(2,\,0)$-torsion forms $\rho_\omega^{2,\,0}$ and $\rho_\eta^{2,\,0} := \rho_{\omega_\eta}^{2,\,0}$ satisfy the identity:

$$||\rho_\eta^{2,\,0}||^2_{\omega_\eta} + \int\limits_X \frac{\omega_\eta^3}{3!} = ||\rho_\omega^{2,\,0}||^2_\omega + \int\limits_X \frac{\omega^3}{3!}$$

and are related by

$$\rho_\eta^{2,\,0} = \rho_\omega^{2,\,0} + \partial\eta.$$

Proof of the first statement. In an arbitrary dimension n, we compute the differential of the map

$$\mathcal{S}_{\{\omega_0\}} \ni \omega \mapsto \int\limits_X \frac{\omega^n}{n!} := \mathrm{Vol}_\omega(X)$$

when the metric ω varies in its Aeppli cohomology class $\{\omega_0\}_A$. For any real, Aeppli null-cohomologous $(1,\,1)$-form $\gamma = \partial\bar{u} + \bar\partial u$ (with

$u \in C^{\infty}_{1,\,0}(X,\, \mathbb{C}))$, we have

$$
\begin{aligned}
\frac{d}{dt}_{|t=0} \int\limits_X \frac{(\omega + t\gamma)^n}{n!} &= \frac{1}{(n-1)!} \int\limits_X \omega^{n-1} \wedge \gamma = 2\,\mathrm{Re} \int\limits_X \bar\partial u \wedge \frac{\omega^{n-1}}{(n-1)!} \\
&= 2\,\mathrm{Re} \int\limits_X u \wedge \bar\partial \star \omega = 2\,\mathrm{Re} \int\limits_X u \wedge \star\Big(-\star\bar\partial \star \omega\Big) \\
&= 2\,\mathrm{Re} \int\limits_X u \wedge \star\partial^\star\omega = 2\,\mathrm{Re} \int\limits_X u \wedge \star\overline{\bar\partial^\star\omega} \\
&= 2\,\mathrm{Re}\,\langle\langle u,\, \bar\partial^\star\omega\rangle\rangle.
\end{aligned}
$$

The last quantity is $-(d_\omega F)(\gamma)$ when $n = 3$. $\square$

(II) Generalised volume of Hermitian-symplectic Aeppli classes

The main takeaway from the last corollary is that the sum

$$F(\omega) + \mathrm{Vol}_\omega(X)$$

(where $\mathrm{Vol}_\omega(X) := \int_X \omega^3/3!$) remains **constant** when ω ranges over the (necessarily Hermitian-symplectic) metrics in the Aeppli cohomology class of a fixed Hermitian-symplectic metric ω_0.

Definition 0.14 *Let X be a 3-dimensional compact Hermitian-symplectic manifold. For any H-S metric ω on X, the constant*

$$A = A_{\{\omega\}_A} := F(\omega) + Vol_\omega(X) > 0$$

depending only on $\{\omega\}_A$ is called the **generalised volume** *of the H-S Aeppli class $\{\omega\}_A$.*

Digression

Definition 0.15 *If* ω *is an H-S metric on a compact complex manifold* X *with* $\dim_{\mathbb{C}} X = 3$ *and* $\rho_\omega^{2,\,0}$ *is the* $(2,\,0)$*-torsion form of* ω*, we define the following volume form on* X*:*

$$d\widetilde{V}_\omega := (1 + |\rho_\omega^{2,\,0}|^2_\omega)\, dV_\omega.$$

Its volume depends only on the H-S Aeppli class:

$$\int\limits_X d\widetilde{V}_{\omega_1} = \int\limits_X d\widetilde{V}_{\omega_2} = A, \qquad \text{for all metrics } \omega_1, \omega_2 \in \{\omega\}_A,$$

where $A = A_{\{\omega\}_A} > 0$ is the *generalised volume* of the H-S Aeppli class $\{\omega\}_A$.

Therefore, it seems natural to consider the Monge-Ampère equation:

$$\frac{(\omega + i\partial\bar{\partial}\varphi)^3}{3!} = b\, d\widetilde{V}_\omega,$$

subject to the condition $\omega + i\partial\bar{\partial}\varphi > 0$, where $b > 0$ is a given constant.

By Tosatti-Weinkove '10, $\exists! b > 0$ such that this equation is solvable. Moreover, for that b, the solution $\omega + i\partial\bar{\partial}\varphi > 0$ is unique.

Note that

$$b = \frac{\mathrm{Vol}_{\omega + i\partial\bar{\partial}\varphi}(X)}{A_{\{\omega\}_A}} \in (0,\ 1]$$

since $A_{\{\omega\}_A} = F(\omega + i\partial\bar{\partial}\varphi) + \mathrm{Vol}_{\omega+i\partial\bar{\partial}\varphi}(X) \geq \mathrm{Vol}_{\omega+i\partial\bar{\partial}\varphi}(X)$. We hope that this can shed some light on the mysterious constant b in this context.

(III) Obstruction to the existence of a Kähler metric in a given Hermitian-symplectic Aeppli class

Theorem and Definition 0.16 *(a) The (0, 2)-torsion form* $\rho_\omega^{0,\,2}$ *of any H-S metric* ω *is* E_2**-closed***: it defines an* E_2*-cohomology class*

$$\{\rho_\omega^{0,\,2}\}_{E_2} \in E_2^{0,\,2}(X)$$

on the second page of the Frölicher spectral sequence of X*.*

(b) When $\dim_{\mathbb{C}} X = 3$, $\{\rho_\omega^{0,\,2}\}_{E_2}$ *depends only on the Aeppli class* $\{\omega\}_A$.

(a consequence of $F(\omega) + Vol_\omega(X) = Const$ *when the H-S metric* ω *varies in a given Aeppli class)*

$$\{\rho_\omega^{0,\,2}\}_{E_2} \stackrel{def}{=} \text{the } E_2\text{-}\mathbf{torsion\ class} \text{ of the H-S Aeppli class } \{\omega\}_A.$$

(c) When $\dim_{\mathbb{C}} X = 3$*, the* **vanishing** *of the* E_2*-torsion class:*

$$\{\rho_\omega^{0,\,2}\}_{E_2} = 0 \in E_2^{0,\,2}(X)$$

is a necessary condition for the Aeppli class $\{\omega\}_A$ *to contain a Kähler metric.*

Natural question. *Do there exist compact* 3*-dimensional Hermitian-symplectic manifolds on which all or some* E_2*-torsion classes are non-vanishing?*

(IV) First Cohomological interpretation of the generalised volume

Let X be a compact H-S manifold with $\dim_{\mathbb{C}} X = 3$.

(a) For any H-S metric ω on X, the d-closed real 2-form

$$\widetilde{\omega} = \rho_\omega^{2,\,0} + \omega + \rho_\omega^{0,\,2}$$

is called the **minimal completion** of ω, where $\rho_\omega^{2,\,0}$, resp. $\rho_\omega^{0,\,2}$, is the (2, 0)-torsion form, resp. the (0, 2)-torsion form, of ω. Thus, $d\widetilde{\omega} = 0$.

(b) We have:

$$\int\limits_X \frac{\widetilde{\omega}^3}{3!} = \text{Vol}_\omega(X) + F(\omega) = A_{\{\omega\}_A}.$$

(c) For any Aeppli-cohomologous H-S metrics ω and ω_η:

$$\omega_\eta = \omega + \partial\bar\eta + \bar\partial\eta > 0 \qquad (\text{where } \eta \in C^\infty_{1,\,0}(X,\,\mathbb{C})),$$

the respective minimal completion 2-forms $\widetilde\omega_\eta$ and $\widetilde\omega$ lie in the same De Rham cohomology class.

Thus, $A_{\{\omega\}_A} = \{\widetilde\omega\}^3_{DR}/3!$.

(V) Second Cohomological interpretation of the generalised volume

• Any H-S metric ω on X defines an **E_2-Aeppli class**:

$$\{\omega\}_{E_2,\,A} \in E_{2,\,A}^{1,\,1}(X).$$

(The higher-page Bott-Chern and Aeppli cohomologies were introduced in 2020 by P.-Stelzig-Ugarte.)

• For every $r \in \mathbb{N}$, the notion of page-r-$\partial\bar\partial$-manifold was introduced in 2020 by P.-Stelzig-Ugarte. We have:

$$\textit{page-}0\textit{-}\partial\bar\partial\textit{-manifold} = \partial\bar\partial\textit{-manifold},$$

X is a *page-r-$\partial\bar\partial$-manifold* $\implies$ X is a *page-$(r+1)$-$\partial\bar\partial$-manifold*.

Theorem 0.17 *Let X be a* **page-1-$\partial\bar\partial$-manifold**, *$dim_{\mathbb{C}} X = 3$.*

(a) There is a natural **E_2-Bott-Chern class** $\mathfrak{c}_\omega \in E_{2,\,BC}^{2,\,2}(X)$ *associated with any* E_2**-Aeppli class** $\{\omega\}_{E_2,\,A} \in E_{2,\,A}^{1,\,1}(X)$ *of H-S metrics ω on X.*

(b) Suppose there exists a Hermitian-symplectic metric ω on X whose **E_2-torsion class vanishes** *(i.e. $\{\rho_\omega^{0,\,2}\}_{E_2} = 0 \in E_2^{0,\,2}(X)$).*

Then, the generalised volume $A = F(\omega) + Vol_\omega(X)$ of $\{\omega\}_A$ is given as the following intersection number in cohomology:

$$A = \frac{1}{6}\,\mathfrak{c}_\omega.\{\omega\}_{E_2,\,A}.$$

(VI) Search for critical points of the functional F

Let X be a compact Hermitian-symplectic manifold, $\dim_{\mathbb{C}} X = 3$.

Goal: minimising the map

$$\mathcal{S}_{\{\omega_0\}} \ni \omega \longmapsto F(\omega) = ||\rho_\omega^{2,\,0}||_\omega^2 \in [0,\, +\infty)$$

in a given H-S Aeppli class $\{\omega_0\}_A \in H_A^{1,\,1}(X,\, \mathbb{R})$.

We have seen that, by our results, this is equivalent to maximising the volume within a given H-S Aeppli class: $\{\omega_0\}_A \in H_A^{1,\,1}(X,\, \mathbb{R})$:

$$\mathcal{S}_{\{\omega_0\}} \ni \omega \longmapsto \mathrm{Vol}_\omega(X).$$

Proposition 0.18 *Let X be a compact H-S manifold with $\dim_{\mathbb{C}} X = 3$. Fix an arbitrary Hermitian metric γ and an H-S metric ω on X. Let $A = A_{\{\omega\}_A} > 0$ be the generalised volume of the class $\{\omega\}_A$ and let $c = c_{\omega,\,\gamma} > 0$ be the constant defined by the requirement*

$$\frac{\left(\int\limits_X \omega\wedge\gamma^2/2!\right)^3}{\left(\int\limits_X \gamma^3/3!\right)^2} = \frac{6A}{c}.$$

If there exists a solution $\eta \in C^\infty_{1,\,0}(X,\,\mathbb{C})$ of the Monge-Ampère-type equation

$$(\omega + \partial\bar\eta + \bar\partial\eta)^3 = c\,(\Lambda_\gamma\omega)^3\,\frac{\gamma^3}{3!} \qquad (\star)$$

such that $\omega_\eta := \omega + \partial\bar\eta + \bar\partial\eta > 0$, then ω_η is a **Kähler metric** *lying in the Aeppli cohomology class $\{\omega\}_A$ of ω.*

However, equation ($\star$) is heavily underdetermined and it is hard to see how one could go about solving it. For this reason, we replace it by a family of Monge-Ampère-type equations of the familiar kind after we have stratified the given Hermitian-symplectic Aeppli class $\{\omega\}_A$ by *Bott-Chern subclasses* (or *strata*) in the following way.

(1) We consider a *partition* of $\mathcal{S}_{[\omega]}$ of the shape:

$$\mathcal{S}_{[\omega]} = \cup_{j\in J}\mathcal{D}_{[\omega_j]},$$

where $(\omega_j)_{j\in J}$ is a family of H-S metrics in $\{\omega\}_A$ and

$$\mathcal{D}_{[\omega_j]} := \{\omega' > 0 \mid \omega' - \omega_j \in \mathrm{Im}\,(\partial\bar\partial)\}, \qquad j \in J.$$

(2) For each $j \in J$, we choose an arbitrary Hermitian metric γ_j on X such that $\Lambda_{\gamma_j}\omega_j = 1$ at every point of X. Then, on each Bott-Chern stratum $\mathcal{D}_{[\omega_j]} \subset \mathcal{S}_{[\omega]}$, the Tosatti-Weinkove result in [TW10, Corollary 1] ensures the existence of a *unique* constant $b_j > 0$ such that the equation

$$\frac{(\omega_j + i\partial\bar\partial\varphi)^3}{3!} = b_j A \, \frac{dV_{\gamma_j}}{\int\limits_X dV_{\gamma_j}} \qquad (\star\star\star_j),$$

subject to the extra condition $\omega_j + i\partial\bar\partial\varphi > 0$, is *solvable*, where $A > 0$ is the *generalised volume* of $\{\omega\}_A$. (Hence, A is independent of j.)

In this way, we associate a constant $b_j \in (0, 1]$ with every Bott-Chern stratum $\mathcal{D}_{[\omega_j]}$ of $\mathcal{S}_{[\omega]}$.

(3) The problem of *minimising* the functional F in $\mathcal{S}_{[\omega]}$ (equivalently, *maximising* $\mathcal{S}_{[\omega]} \ni \omega' \mapsto \mathrm{Vol}_{\omega'}(X)$) becomes equivalent to proving that the value 1 is attained by one of the constants b_j.

Proposition 0.19 *If there exists $j \in J$ such that $b_j = 1$, the solution $\omega_j + i\partial\bar\partial\varphi_j$ of equation $(\star\star\star_j)$ is a* **Kähler metric** *lying in the Bott-Chern subclass $\mathcal{D}_{[\omega_j]}$, hence also in the Aeppli class $\{\omega\}_A$.*

(4) The next observation is that, within Bott-Chern subclasses of Hermitian-symplectic metrics that contain a *Gauduchon* metric, the volume remains *constant* and all the metrics are *Gauduchon*. These Bott-Chern subclasses will be called *Gauduchon strata*.

Lemma 0.20 *Let* $dim_{\mathbb{C}} X = 3$. *Suppose that a metric* ω *on* X *is both* **SKT** *and* **Gauduchon**. *Then, for every* $\varphi : X \longrightarrow \mathbb{R}$, *we have*

$$(a)\ \int\limits_X (\omega + i\partial\bar\partial\varphi)^3 = \int\limits_X \omega^3 \quad \textit{and} \quad (b)\ \partial\bar\partial(\omega + i\partial\bar\partial\varphi)^2 = 0.$$

(5) *Volume comparison within a Bott-Chern stratum*

Lemma 0.21 *Let X be a 3-dimensional compact complex manifold. Suppose that ω is an* **SKT non-Gauduchon** *metric on X. Then, the map*

$$\left\{\varphi \in C^\infty(X) \,\Big|\omega + i\partial\bar\partial\varphi > 0\right\} \ni \varphi \longmapsto \int\limits_X \frac{(\omega + i\partial\bar\partial\varphi)^3}{3!} := Vol_{\omega_\varphi}(X)$$

does not achieve any local extremum.

Conclusion. The volume of $\omega_\varphi := \omega + i\partial\bar\partial\varphi > 0$ is *constant* on the *Gauduchon strata* (if any), while it *achieves no local extremum* on the *non-Gauduchon strata*.