Higher-Page Hodge Theory of Compact Complex Manifolds

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Context and motivation

X a compact complex manifold, $\dim_{\mathbb{C}} X = n$

Definition. *X is a* **$\partial\bar\partial$-manifold** *if*

$\forall p, q,\ \forall u \in C^{\infty}_{p,\,q}(X,\,\mathbb{C})$ *s.t.* $du = 0$*, we have equivalences:*

$$u \in Im\, d \iff u \in Im\, \partial \iff u \in Im\, \bar\partial \iff u \in Im\, (\partial\bar\partial).$$

The idea goes back to Deligne-Griffiths-Morgan-Sullivan 1975.

Standard fact. ([DGMS75]) X *is a* **$\partial\bar\partial$-manifold** $\iff$ $\forall k \in \{0, 1, \dots, 2n\}$, *the identity induces an* **isomorphism**

$$H^k_{DR}(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,\,q}_{\bar\partial}(X, \mathbb{C}) \qquad (\textit{Hodge decomposition})$$

in the following sense:

- $\forall (p,\, q)$ *s.t.* $p + q = k$, *every class* $[\alpha^{p,\,q}]_{\bar\partial} \in H^{p,\,q}_{\bar\partial}(X, \mathbb{C})$ *can be represented by a* d**-closed** $(p,\, q)$*-form* $\alpha^{p,\,q}$*;*

- *the linear map*

$$\bigoplus_{p+q=k} H^{p,\,q}_{\bar\partial}(X, \mathbb{C}) \ni \sum_{p+q=k} [\alpha^{p,\,q}]_{\bar\partial} \longmapsto \left\{\sum_{p+q=k} \alpha^{p,\,q}\right\}_{DR} \in H^k_{DR}(X, \mathbb{C})$$

is independent of the choices of d**-closed** *representatives* $\alpha^{p,\,q}$ *of the classes* $[\alpha^{p,\,q}]_{\bar\partial}$ *(i.e.* **well-defined***) and* **bijective**.

(i.e. X is *cohomologically Kähler*)

Standard facts.

- The following implications hold:

X is compact *Kähler* $\implies$ X is *class $\mathcal{C}$* $\implies$ X is a *$\partial\bar\partial$-manifold*

$\implies$ $E_1(X) = E_\infty(X)$ (in the Frölicher spectral sequence – FSS)

If $n \geq 3$, all the implications are strict.

• If X is a $\partial\bar\partial$-manifold, X has the *Hodge symmetry* property:

for all p, q,

(i) every class $[\alpha^{p,\,q}]_{\bar\partial} \in H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ can be represented by a ***d*-closed** $(p,\,q)$-form $\alpha^{p,\,q}$;

(ii) the linear map

$$H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) \ni [\alpha^{p,\,q}]_{\bar\partial} \longmapsto \overline{[\overline{\alpha^{p,\,q}}]_{\bar\partial}} \in \overline{H^{q,\,p}_{\bar\partial}(X,\,\mathbb{C})}$$

is independent of the choices of ***d*-closed** representatives $\alpha^{p,\,q}$ of the classes $[\alpha^{p,\,q}]_{\bar\partial}$ (i.e. **well-defined**) and **bijective**.

Examples.

(1) The twistor space X of any *K3 surface* has $E_1(X) = E_\infty(X)$ but is *not a $\partial\bar{\partial}$-manifold*.

(*no Hodge symmetry* – P. 2011)

(2) Let $X = G/H$, also denoted $I^{(3)}$, be the **Iwasawa manifold**, where

$$G := \left\{ M = \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \; ; \; z_1, z_2, z_3 \in \mathbb{C} \right\} \subset GL_3(\mathbb{C})$$

and $H \subset G$ is its discrete subgroup $\Gamma \subset G$ of matrices with entries $z_1, z_2, z_3 \in \mathbb{Z}[i]$.

$I^{(3)}$ is a compact complex manifold, $\dim_{\mathbb{C}} I^{(3)} = 3$.

There exist C^∞ (1, 0)-forms α, β, γ on X, induced resp. by dz_1, dz_2, $dz_3 - z_1 dz_2$ (look at $M \mapsto M^{-1}dM$) satisfying:

$$\bar\partial\alpha = \bar\partial\beta = \bar\partial\gamma = 0$$

but

$$\partial\alpha = \partial\beta = 0 \quad \text{and} \quad \partial\gamma = -\alpha\wedge\beta \neq 0.$$

Therefore, $E_1(X) \neq E_\infty(X)$. In particular, X is *not a $\partial\bar\partial$-manifold*.

However, $E_2(X) = E_\infty(X)$. This leads to a Hodge theory for X if the $E_1^{p,\,q}(X)$'s are replaced by the $E_2^{p,\,q}(X)$'s.

(exploited in P. 2018: "Non-Kähler Mirror Symmetry of the Iwasawa Manifold")

Main Theorem and Definition. (P.-Stelzig-Ugarte 2020)

Fix $r \in \mathbb{N}^\star$. The following are equivalent:

(1) *(i) $\forall p, q$, every class $\{\alpha^{p,\,q}\}_{E_r} \in E_r^{p,\,q}(X)$ can be represented by a d-**closed** $(p,\,q)$-form $\alpha^{p,\,q}$;*

(ii) $\forall k$, the linear map

$$\bigoplus_{p+q=k} E_r^{p,\,q}(X) \ni \sum_{p+q=k} \{\alpha^{p,\,q}\}_{E_r} \longmapsto \left\{\sum_{p+q=k} \alpha^{p,\,q}\right\}_{DR} \in H^k_{DR}(X,\,\mathbb{C})$$

is **well-defined** *and* **bijective**.

(X is said to have the E_r-Hodge Decomposition property)

(2) $E_r(X) = E_\infty(X)$ *(Frölicher) and the De Rham cohomology of* X *is pure.*

(3) $\forall p, q,\ \forall u \in C^\infty_{p,\,q}(X,\, \mathbb{C})$ *s.t.* $du = 0$, *we have equivalences:*

$$u \in Im\, d \iff u \text{ is } E_r\text{-exact} \iff u \text{ is } \overline{E}_r\text{-exact}$$
$$\iff u \text{ is } E_r\overline{E}_r\text{-exact.}$$

(4) $\forall p, q$, *the canonical linear maps:*

$$E^{p,\,q}_{r,\,BC}(X) \longrightarrow E^{p,\,q}_r(X) \longrightarrow E^{p,\,q}_{r,\,A}(X)$$

are **isomorphisms**.

(5) $\forall p, q$, *the canonical linear map:*

$$E^{p,\,q}_{r,\,BC}(X) \longrightarrow E^{p,\,q}_{r,\,A}(X)$$

is **injective**.

X is said to be a **page-$(r-1)$-$\partial\bar\partial$-manifold** *if X satisfies any of the equivalent properties* (1)-(5).

Observation. (trivial)

(i) X is a page-0-$\partial\bar\partial$-manifold $\iff$ X is a $\partial\bar\partial$-manifold;

(ii) $\{X$ *$\partial\bar\partial$-manifolds*$\} \subset \{X$ *page-1-$\partial\bar\partial$-manifolds*$\} \subset \dots$

$\dots \subset \{X$ *page-r-$\partial\bar\partial$-manifolds*$\} \subset \{X$ *page-$(r+1)$-$\partial\bar\partial$-manifolds*$\} \subset \dots$

Terminology

(a) $\forall p, q$, let

$$H^{p,\,q}_{DR}(X,\,\mathbb{C}) := \Big\{\{\alpha\}_{DR} \in H^{p+q}_{DR}(X,\,\mathbb{C}) \mid \exists \alpha \in C^{\infty}_{p,\,q}(X,\,\mathbb{C}) \text{ representing } \{\alpha\}_{DR}\Big\} \subset H^{p+q}_{DR}(X,\,\mathbb{C}).$$

The De Rham cohomology is *pure* (*pure and full*) if for every k

$$H^{k}_{DR}(X,\,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,\,q}_{DR}(X,\,\mathbb{C}).$$

(i.e. the sum is direct and the $H^{p,\,q}_{DR}(X,\,\mathbb{C})$'s fill out $H^{k}_{DR}(X,\,\mathbb{C})$)

(b) **Recall** (Cordero-Fernandez-Gray-Ugarte 1997)

$$E_r^{p,\,q}(X) = \frac{Z_r^{p,\,q}}{\mathcal{C}_r^{p,\,q}},$$

where

- $Z_r^{p,\,q}$ is the space of E_r-closed $(p,\ q)$-forms (i.e. forms that represent E_r-classes): $\exists\ u_l \in C^\infty_{p+l,\,q-l}(X)$ such that

$$\bar\partial\alpha = 0 \quad \text{and} \quad \begin{aligned} \partial\alpha &= \bar\partial u_1 \\ \partial u_1 &= \bar\partial u_2 \\ &\vdots \\ \partial u_{r-2} &= \bar\partial u_{r-1}. \end{aligned}$$

• $\mathcal{C}_r^{p,\,q}$ is the space of E_r-exact $(p,\ q)$-forms (i.e. forms that represent the zero E_r-class): $\exists\ \zeta \in C^\infty_{p-1,\,q}(X)$ and $\xi \in C^\infty_{p,\,q-1}(X)$ such that

$$\alpha = \partial\zeta + \bar\partial\xi,$$

with ξ arbitrary and ζ satisfying the following tower of $(r-1)$ equations:

$$\begin{aligned} \bar\partial\zeta &= \partial v_{r-3} \\ \bar\partial v_{r-3} &= \partial v_{r-4} \\ &\vdots \\ \bar\partial v_1 &= \partial v_0 \\ \bar\partial v_0 &= 0, \end{aligned}$$

for some forms $v_0, \dots, v_{r-3}$. (When $r = 2$, $\zeta_{r-2} = \zeta_0$ must be $\bar\partial$-closed.)

• **We have** (obvious):

$$\cdots \subset \mathcal{C}_r^{p,\,q}(X) \subset \mathcal{C}_{r+1}^{p,\,q}(X) \subset \cdots \subset \mathcal{Z}_{r+1}^{p,\,q}(X) \subset \mathcal{Z}_r^{p,\,q}(X) \subset \ldots,$$

with $\qquad \{0\} = \mathcal{C}_0^{p,\,q}(X) \subset \mathcal{C}_1^{p,\,q}(X) = (\mathrm{Im}\,\bar\partial)^{p,\,q}$

and $\qquad \mathcal{Z}_1^{p,\,q}(X) = (\ker\bar\partial)^{p,\,q} \subset \mathcal{Z}_0^{p,\,q}(X) = C^\infty_{p,\,q}(X).$

Our definitions

• $\alpha \in C^\infty_{p,\,q}(X)$ is **$E_r\overline{E}_r$-closed** if $\exists$ C^∞ forms $\eta_1, \dots, \eta_{r-1}$ and $\rho_1, \dots, \rho_{r-1}$ such that

$$\begin{aligned} \partial\alpha &= \bar\partial\eta_1 & \bar\partial\alpha &= \partial\rho_1 \\ \partial\eta_1 &= \bar\partial\eta_2 & \bar\partial\rho_1 &= \partial\rho_2 \\ &\vdots \\ \partial\eta_{r-2} &= \bar\partial\eta_{r-1}, & \bar\partial\rho_{r-2} &= \partial\rho_{r-1}. \end{aligned}$$

$(r-1$ equations in each tower$)$

• **Observation:** α $E_r\overline{E}_r$-closed $\implies$ $\partial\bar\partial\alpha = 0$.

• $\alpha \in C^\infty_{p,\,q}(X)$ is **$E_r\overline{E}_r$-exact** if there exist smooth forms ζ, ξ, η such that

$$\alpha = \partial\zeta + \partial\bar\partial\xi + \bar\partial\eta \tag{1}$$

and such that ζ and η further satisfy the following conditions.

$\exists$ C^∞ forms $v_{r-3}, \dots, v_0$ and $u_{r-3}, \dots, u_0$ such that:

$$\begin{aligned} \bar\partial\zeta &= \partial v_{r-3} & \qquad \partial\eta &= \bar\partial u_{r-3} \\ \bar\partial v_{r-3} &= \partial v_{r-4} & \qquad \partial u_{r-3} &= \bar\partial u_{r-4} \\ &\vdots & & \\ \bar\partial v_0 &= 0, & \qquad \partial u_0 &= 0. \end{aligned}$$

($r-1$ equations in each tower)

(c) We also define:

- The **E_r-Bott-Chern** cohomology group of bidegree (p, q) of X:

$$E^{p,\,q}_{r,\,BC}(X) := \frac{\{\alpha \in C^\infty_{p,\,q}(X) \mid d\alpha = 0\}}{\{\alpha \in C^\infty_{p,\,q}(X) \mid \alpha \text{ is } E_r\overline{E}_r\text{-exact}\}}.$$

- The **E_r-Aeppli** cohomology group of bidegree (p, q) of X:

$$E^{p,\,q}_{r,\,A}(X) := \frac{\{\alpha \in C^\infty_{p,\,q}(X) \mid \alpha \text{ is } E_r\overline{E}_r - \text{closed}\}}{\{\alpha \in C^\infty_{p,\,q}(X) \mid \alpha \in \operatorname{Im}\partial + \operatorname{Im}\bar\partial\}}.$$

- **Observation:** $\exists$ canonical maps induced by the identity:

$$E^{p,\,q}_{r,\,BC}(X) \longrightarrow E^{p,\,q}_{r}(X) \longrightarrow E^{p,\,q}_{r,\,A}(X)$$

First class of primary examples (I)

Recall: X is **complex parallelisable** $\stackrel{def}{\Longleftrightarrow}$ $T^{1,0}X$ is **trivial**

These are **not $\partial\bar\partial$-manifolds** unless they are complex tori.

Theorem. If X is a **complex parallelisable nilmanifold**, then X is a **page-1-$\partial\bar\partial$-manifold**.

Sketch of proof.

- X is **complex parallelisable** $\iff$ $X = G/\Gamma$,

with G a **complex Lie group** and Γ a **co-compact discrete subgroup**

(Wang 1954)

• Sakane (1976): if X is **complex parallelisable**, $\exists$ a **canonical isomorphism**

$$H_{\bar\partial}^{p,q}(X, \mathbb{C}) \simeq \Lambda^p(\mathfrak{g}^{1,0}) \otimes H^q(\mathfrak{g}^{0,1})$$

(the Dolbeault cohomology of the Lie algebra $\mathfrak{g}$)

Since

$$\partial|_{\bigwedge^p(\mathfrak{g}^{1,0})} = d|_{\bigwedge^p(\mathfrak{g}^{1,0})}, \quad \bar\partial|_{\bigwedge^p(\mathfrak{g}^{1,0})} = 0 \quad \text{and} \quad \bar\partial|_{\bigwedge^q(\mathfrak{g}^{0,1})} = d|_{\bigwedge^q(\mathfrak{g}^{0,1})},$$

$d_1 = \partial \otimes \mathrm{Id}$, hence $E_2^{p,q}(X) = H^p(\mathfrak{g}^{1,0}) \otimes H^q(\mathfrak{g}^{0,1}) = H^{p,q}(\mathfrak{g}_{\mathbb{C}})$, so $E_2(X) = E_\infty(X)$.

• Nomizu: $H^{p,q}(\mathfrak{g}_{\mathbb{C}}) \simeq H_{DR}^{p,q}(X, \mathbb{C})$ since X is a **nilmanifold** with an invariant complex structure J.

Corollary. *The* **Iwasawa manifold** *is a* **page-1-$\partial\bar{\partial}$-manifold**.

Proposition. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold X_0. For every $t \in B$, we have:*

(i) X_t is a **page-1-$\partial\bar{\partial}$-manifold** *if and only if X_t is* **complex parallelisable** *(i.e. lies in Nakamura's class (i));*

(ii) if X_t lies in one of Nakamura's classes (ii) or (iii), the De Rham cohomology of X_t is **not pure**, *so X_t is* **not** *a* **page-r-$\partial\bar{\partial}$-manifold** *for any $r \in \mathbb{N}$.*

Further primary examples

(II) Two families of **nilmanifolds with abelian complex structures** with members of arbitrarily high dimensions. In some sense, these form the opposite of the first class among nilmanifolds.

Theorem. *Let* $n \geq 3$ *and* G *be the* **nilpotent** *Lie group with* **abelian** *complex structure defined by the structure equations*

$$(Ab1_n) \quad d\omega^1 = 0, \; d\omega^2 = 0, \; d\omega^3 = \omega^2 \wedge \overline{\omega^1}, \ldots, \; d\omega^n = \omega^{n-1} \wedge \overline{\omega^1},$$

or

$$(Ab2_n) \quad d\omega^1 = 0, \ldots, \; d\omega^{n-1} = 0, \; d\omega^n = \omega^1 \wedge \overline{\omega^2} + \omega^3 \wedge \overline{\omega^4} + \cdots + \omega^{n-2} \wedge \overline{\omega^{n-1}}$$ *(only for odd* $n \geq 3$*).*

Then, any nilmanifold $\Gamma \backslash G$ *is a* **page-**1**-**$\partial\bar{\partial}$**-manifold**.

Proposition *Let* $X = (\Gamma\backslash G,\, J)$ *be a complex* **nilmanifold** *of complex dimension 3, different from a torus, endowed with an* **invariant** *complex structure* J.

If there exists $r \in \mathbb{N}^\star$ *such that* X *is a page-*$(r-1)$*-*$\partial\bar\partial$*-manifold, then* J *is equivalent to the* **complex parallelisable** *structure of* $I^{(3)}$ *or to the* **abelian complex structure** $\tilde{J}$ *defined by* $(Ab1_n)$ *in the above Theorem for* $n = 3$.

In both cases $r = 2$*, i.e. both of these manifolds are* **page-**1**-**$\partial\bar\partial$**-manifolds**.

In other words: in complex dimension 3, the only complex nilmanifolds which are page-$(r-1)$-$\partial\bar\partial$ for some $r \in \mathbb{N}^\star$ are, apart from a torus, $I^{(3)}$ and the nilmanifold $\tilde{X} == (\Gamma\backslash G, \tilde{J})$.

Numerical characterisation of page-r-$\partial\bar{\partial}$-manifolds

Let $\quad b(X) = \sum_{k\in\mathbb{Z}} b_k(X), \quad h_{BC}(X) = \sum_{p,q\in\mathbb{Z}} h^{p,q}_{BC}(X)$

and $\quad h_A(X), \quad h_{\partial}(X), \quad h_{\bar{\partial}}(X) \quad$ (defined analogously).

- Angella-Tomassini (2013): there are inequalities:

$$h_{BC}(X) + h_A(X) \overset{(*)}{\geq} h_{\bar{\partial}}(X) + h_{\partial}(X) \overset{(**)}{\geq} 2\, b(X) \tag{2}$$

and X is a $\partial\bar{\partial}$**-manifold** $\iff$ $(*)$ and $(**)$ are both **equalities**.

- Standard fact: $E_1(X) = E_\infty(X)$ $\iff$ $(**)$ is an **equality**.

Our numerical criterion

Theorem. *For every compact complex manifold X and for every $r \in \mathbb{N}^\star$, there is an inequality:*

$$h_{BC}(X) + h_A(X) \geq 2 \left(\sum_{i=1}^{r} e_i(X) - (r-1)b(X) \right),$$

where $e_i := \sum_{p,q \in \mathbb{Z}} \dim E_i^{p,q}(X)$.

Moreover, equality holds for some fixed $r \in \mathbb{N}^\star$ if and only if X is a **page-r-$\partial\bar{\partial}$-manifold***.*

Idea of proof. Consider the bounded double complex

$$A_X := (C^\infty_{p,\,q}(X), \partial, \bar\partial)$$

and apply its decomposition (Stelzig 2020) into bounded **indecomposable** double complexes (**squares**, **even-length zigzags** and **odd-length zigzags**).

Corollary. *For every bidegree* $(p,\,q)$*, we have the inequality:*

$$h^{p,\,q}_{BC} + h^{p,\,q}_{A} \geq h^{p,\,q}_{\bar\partial} + h^{q,\,p}_{\bar\partial}. \tag{3}$$

Moreover, X *is a* **page-1-$\partial\bar\partial$-manifold** *if and only if (3) is an equality for every bidegree* $(p,\,q)$

The inequality was proved in Angella-Tomassini (2013), but the characterisation of the equality is new.

Application: our third class of primary examples (III)

Nakamura manifolds (Nakamura 1975)

Let $G := \mathbb{C} \ltimes_\phi \mathbb{C}^2$, where ϕ is either

$$\phi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} \qquad \text{or} \qquad \phi(z) = \begin{pmatrix} e^{\mathrm{Re}(z)} & 0 \\ 0 & e^{\text{-}\mathrm{Re}(z)} \end{pmatrix}$$

(*complex parallelizable*, resp. *completely solvable* case).

Define X to be the quotient of G by a lattice of the form $\Gamma \ltimes_\phi \Gamma'$ with $\Gamma \subset \mathbb{C}$, $\Gamma' \subset \mathbb{C}^2$ lattices.

The Nakamura manifolds X are among the best known *solvmanifolds*, but are *not nilmanifolds*.

Angella-Kasuya (2017): computed the Hodge, Bott-Chern and Aeppli numbers for certain families of lattices $\Gamma \subset \mathbb{C}$. (These numbers turn out to be independent of $\Gamma' \subset \mathbb{C}^2$). Their calculations yield:

$$h_{BC}(X) = h_{\bar{\partial}}(X).$$

Hence, by our **numerical characterisation**, we get

Corollary. *The complex parallelisable and the completely solvable* **Nakamura manifolds** *considered in Angella-Kasuya (2017) are* **page-**1**-**$\partial\bar{\partial}$**-manifolds**.

Behaviour of page-r-$\partial\bar\partial$-manifolds by geometric operations

In particular, we obtain construction methods for new examples from given ones. These include:

(IV) products of **page-r_i-$\partial\bar\partial$-manifolds**, with possibly different r_i's;

Theorem. *Let X and Y be compact complex manifolds.*

If X is a **page-r-$\partial\bar\partial$-manifold** *and Y is a* **page-r'-$\partial\bar\partial$-manifold**, *the product $X \times Y$ is a* **page-$\tilde r$-$\partial\bar\partial$-manifold**, *where $\tilde r = \max\{r, r'\}$.*

Conversely, if $X \times Y$ is a **page-r-$\partial\bar\partial$-manifold**, *so are both factors X and Y.*

(V) blow-ups of **page-r_1-$\partial\bar{\partial}$-submanifolds** of **page-r_2-$\partial\bar{\partial}$-manifolds**, possibly with $r_1 \neq r_2$;

Theorem. *Let X be a compact complex manifold.*

Let $\widetilde{X}$ be the **blow-up** *of X along a submanifold $Z \subset X$.*

- *If X is* **page-r-$\partial\bar{\partial}$** *and Z is* **page-r'-$\partial\bar{\partial}$**, *then $\widetilde{X}$ is a* **page-$\widetilde{r}$-$\partial\bar{\partial}$-manifold**, *where $\tilde{r} = \max\{r, r'\}$.*

- *Conversely, if $\widetilde{X}$ is* **page r-$\partial\bar{\partial}$**, *so are X and Z.*

- *Moreover, the page-r-$\partial\bar{\partial}$-property of compact complex manifolds is a bimeromorphic invariant if and only if it is stable under passage to submanifolds.*

Idea of proof. Let $\mu : \widetilde{X} \longrightarrow X$

be the blow-up of X along $Z \subset X$. Consider the double complex

$$A_X := (C^\infty_{p,\,q}(X), \partial, \bar\partial).$$

Let $(A_X[i])^{p,q} := A_X^{p-i,q-i}$ be the the shifted double complex. We have the following E_1**-isomorphism**:

$$A_{\widetilde{X}} \simeq_1 A_X \oplus \bigoplus_{i=1}^{\operatorname{codim} Z - 1} A_Z[i].$$

· $A_Z[i]$ is page-r'-$\partial\bar\partial$ $\iff$ A_Z is page-r'-$\partial\bar\partial$ (because the occurring zigzags only get shifted);

· the direct sum is page-$\widetilde{r}$-$\partial\bar\partial$ $\iff$ each summand is page-r-$\partial\bar\partial$ (for possibly different values of r and different from $\widetilde{r}$).

(VI) the **projectivised bundle** $\mathbb{P}(\mathcal{V})$ of any holomorphic vector bundle $\mathcal{V}$ on a **page-r-$\partial\bar\partial$-manifold**;

Theorem. *For any vector bundle $\mathcal{V}$ over X, the projectivised bundle $\mathbb{P}(\mathcal{V})$ is a* **page-r-$\partial\bar\partial$-manifold** *if and only if X is.*

(VII) **small deformations** of a page-1-$\partial\bar\partial$-manifold with fixed Hodge numbers.

Theorem. (consequence of our numerical characterisation + deformation semi-continuity of the Hodge numbers)

If X_0 is a **page-1-$\partial\bar\partial$-manifold**, *then every sufficiently small deformation X_t of X_0 which satisfies*

$$h_{\bar\partial}(X_t) = h_{\bar\partial}(X_0)$$

is again **page-1-$\partial\bar\partial$**.

Serre-type duality results

Theorem. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Fix an arbitrary $r \in \mathbb{N}^\star$.*
For every $p, q \in \{0, \dots, n\}$, the canonical bilinear pairings:

$$E_r^{p,\,q}(X) \times E_r^{n-p,\,n-q}(X) \longrightarrow \mathbb{C}, \qquad (\{\alpha\}_{E_r}, \{\beta\}_{E_r}) \mapsto \int\limits_X \alpha \wedge \beta,$$

and

$$E_{r,\,BC}^{p,\,q}(X) \times E_{r,\,A}^{n-p,\,n-q}(X) \longrightarrow \mathbb{C},$$

$$(\{\alpha\}_{E_{r,\,BC}}, \{\beta\}_{E_{r,\,A}}) \longmapsto \int\limits_X \alpha \wedge \beta,$$

are **well defined** *and* **non-degenerate**.

Idea of proof. Fix an arbitrary Hermitian metric ω on X.

- *Case of $E_r^{\cdot,\,\cdot}(X)$:* use a pseudo-differential Laplacian

$$\widetilde{\Delta}_r^{(\omega)}$$

(introduced in P. 2016 and P. 2019) giving a *Hodge isomorphism*

$$\mathcal{H}_r^{p,\,q} := \ker \widetilde{\Delta}_r^{(\omega)} \simeq E_r^{p,\,q}(X),$$

for every $r \in \mathbb{N}^\star$ and all $p, q \in \{0, \dots, n\}$.

Examples

· When $r=1$, $\widetilde{\Delta}_1^{(\omega)} = \Delta'' = \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial$ (the usual $\bar\partial$-Laplacian, a differential operator);

· When $r=2$, for all $p, q = 0, \dots, n$,

$$\widetilde{\Delta}_2^{(\omega)} := \partial p''\partial^\star + \partial^\star p''\partial + \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial : C^\infty_{p,\,q}(X) \longrightarrow C^\infty_{p,\,q}(X),$$

where $p'' : C^\infty_{p,\,q}(X) \longrightarrow \mathcal{H}^{p,\,q}_{\Delta''}(X) := \ker\Delta''$ is the orthogonal projection w.r.t. the L^2 inner product defined by ω onto the Δ''-harmonic space in the standard 3-space decomposition

$$C^\infty_{p,\,q}(X) = \mathcal{H}^{p,\,q}_{\Delta''}(X) \oplus \operatorname{Im}\bar\partial \oplus \operatorname{Im}\bar\partial^\star.$$

(introduced in P. 2016 as a pseudo-differential operator)

- *Case of* $E_{r,\,BC}^{\cdot,\,\cdot}(X)$ *and* $E_{r,\,A}^{\cdot,\,\cdot}(X)$: we introduced the following

Definition. *Let* $(X,\,\omega)$ *be an* n*-dimensional compact complex Hermitian manifold. Fix* $r\geq 1$ *and a bidegree* $(p,\,q)$*.*

A form $\alpha\in C^\infty_{p,\,q}(X)$ *is* **$E_r^\star$-closed** *w. r. t. the metric* ω $\iff$ $\exists$ *forms* $v_l\in C^\infty_{p-l,\,q+l}(X)$ *with* $l\in\{1,\dots,r-1\}$ *satisfying the following tower of* r *equations:*

$$\begin{aligned}\bar\partial^\star\alpha &= 0\\ \partial^\star\alpha &= \bar\partial^\star v_1\\ \partial^\star v_1 &= \bar\partial^\star v_2\\ &\vdots\\ \partial^\star v_{r-2} &= \bar\partial^\star v_{r-1}.\end{aligned}$$

Example of results.

Proposition. $\forall \alpha \in C^{\infty}_{p,\,q}(X)$, *the following equivalences hold:*

$$(i)\ \ \alpha \in \mathcal{H}^{p,\,q}_r \iff \alpha \ \textit{ is } E_r\textit{-closed and } E_r^{\star}\textit{-closed.}$$

$$(ii)\ \ \alpha \ \textit{ is } E_r\textit{-closed} \iff \star\bar{\alpha} \ \textit{ is } E_r^{\star}\textit{-closed.}$$

Example of application.

The higher-page Bott-Chern and Aeppli cohomologies provide the natural framework for the study of several special types of Hermitian metrics.

Proposition. *Let X be a compact complex manifold with $dim_{\mathbb{C}}X = n$ and let ω be a Hermitian metric on X.*

(i) The metric ω is **strongly Gauduchon (sG)** *if and only if ω^{n-1} is* $E_2\overline{E}_2$**-closed**.

In particular, in this case, ω^{n-1} induces an E_2-Aeppli cohomology class $\{\omega^{n-1}\}_{E_{2,A}} \in E_{2,A}^{n-1,\,n-1}(X)$.

(ii) The metric ω is **Hermitian-symplectic (H-S)** *if and only if ω is* $E_3\overline{E}_3$**-closed**.

In particular, in this case, ω induces an E_3-Aeppli cohomology class $\{\omega\}_{E_{3,A}} \in E_{3,A}^{1,1}(X)$.

When $n=3$, ω is **Hermitian-symplectic (H-S)** *if and only if ω is* $E_2\overline{E}_2$**-closed**.

In particular, in this case, ω induces an E_2-Aeppli cohomology class $\{\omega\}_{E_{2,A}} \in E_{2,A}^{1,1}(X)$.

Applications to the deformation theory and to non-Kähler mirror symmetry

• **Essential deformations**: introduced in (P. 2018) in the special case of the Iwasawa manifold $I^{(3)}$:

the small deformations of $I^{(3)}$ that have a different geometry from $I^{(3)}$ (= the non-complex parallelisable small deformations of $I^{(3)}$

= the small deformations of $I^{(3)}$ that are parametrised by $E_2^{n-1,\,1}$ rather than $E_1^{n-1,\,1}$).

P-Stelzig-Ugarte 2020: define the space of **small essential deformations** for an arbitrary **Calabi-Yau page-1-$\partial\bar\partial$-manifold**.

Theorem. *Let X be a compact* **Calabi-Yau page-1-$\partial\bar\partial$-manifold** *with $dim_{\mathbb{C}} X = n$. Fix a non-vanishing holomorphic $(n, 0)$-form u on X and suppose that*

$$\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u) \in \mathcal{Z}_2^{n-2,\,2} \tag{4}$$

for all $\psi_1(t), \rho_1(s) \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$ such that $\psi_1(t)\lrcorner u, \rho_1(s)\lrcorner u \in \ker d \cup Im\,\partial$.

(i) Then, the **essential Kuranishi family** *of X is* **unobstructed**.

(ii) If, moreover, $\mathcal{Z}_1^{n-1,\,1} = \mathcal{Z}_2^{n-1,\,1}$, the **Kuranishi family** *of X is* **unobstructed**.