Partially Hyperbolic Compact Complex Manifolds

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Starting point

If X is an Oeljeklaus-Toma (O-T) manifold, its universal cover is

$$\widetilde{X} = \mathbb{H}^p \times \mathbb{C}^q,$$

for some p, q such that $p + q = \dim_{\mathbb{C}} X = n$, where

$\mathbb{H}$ is the upper half-plane of $\mathbb{C}$ (a typical hyperbolic manifold)

$\mathbb{C}^q$ is very non-hyperbolic.

(I) Definition (Kasuya-P. 2023)

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Let $p \in \{1, \dots, n-1\}$. Suppose:

$\cdot\, \exists\, E \subset T^{1,\,0}X$ *a* C^∞ *complex vector subbundle,* $rk_{\mathbb{C}} E \geq p$*;*

$\cdot\, \exists\, \omega_E \geq 0, \quad \exists\, \omega_{nE} \geq 0 \qquad C^\infty$ $(1,\,1)$*-forms on* X

such that

(i) $\omega := \omega_E + \omega_{nE} > 0$ *on* X *(i.e.* ω *is a Hermitian metric);*

(ii) $\omega_E(x)(\xi,\,\bar{\xi}) > 0 \quad \forall x \in X,\ \forall \xi \in E_x \subset T^{1,\,0}_x X$*;*

(iii) $\omega_{nE}(x)(\xi,\,\bar{\xi}) = 0 \quad \forall x \in X,\ \forall \xi \in E_x \subset T^{1,\,0}_x X$*;*

(iv) $\Omega := \dfrac{\omega_E^p}{p!}$ *is d-closed and* $\widetilde{d}$*(bounded) on* $(X,\,\omega)$*.*

Then, X *is* **partially p-Kähler hyperbolic** *in the* **E-directions**.

The triple $(E, \Omega, \omega = \omega_E + \omega_{nE})$ is called a partially p-Kähler hyperbolic structure on X.

Observation

(a) If $p \leq n-2$ and $\omega_E > 0$ in all directions, then ω_E is a

Kähler metric

on X. Otherwise, the manifold X need not be Kähler.

(b) If $p = n-1$ and $\omega_E > 0$ in all directions, we can choose $E = T^{1,0}X$ and $\omega_{nE} = 0$. Then, the manifold (X, ω_E) is

balanced hyperbolic

(Marouani-P. 2022).

(II) Let

$$f : \mathbb{C}^q \longrightarrow (X, \omega)$$

be a holomorphic map, $\dim_{\mathbb{C}} X = n$ and $q \in \{1, \ldots, n-1\}$.

Suppose f is non-degenerate at some point $x_0 \in \mathbb{C}^q$:

$$d_{x_0} f : \mathbb{C}^q \longrightarrow T^{1,\,0}_{x_0} X \quad \text{has maximal rank.}$$

Let

$$\Sigma_f := \Big\{ x \in \mathbb{C}^q \;\Big|\; f \text{ is degenerate at } x \Big\} \subset \mathbb{C}^q$$

a proper analytic subset.

Then: $f^\star\omega \geq 0$ on $\mathbb{C}^q$ and $f^\star\omega > 0$ on $\mathbb{C}^q \setminus \Sigma_f$.

($f^\star\omega$ is a degenerate Hermitian metric on X.)

(i) For every $r > 0$, the $(\omega,\ f)$*-volume* of the ball $B_r \subset \mathbb{C}^q$ is

$$\mathrm{Vol}_{\omega,\, f}(B_r) := \int\limits_{B_r} f^\star \omega_q > 0.$$

(ii) For $z \in \mathbb{C}^q$, let $\tau(z) := |z|^2$ be its squared Euclidean norm. At every point $z \in \mathbb{C}^q \setminus \Sigma_f$, we have:

$$\frac{d\tau}{|d\tau|_{f^\star\omega}} \wedge \star_{f^\star\omega}\left(\frac{d\tau}{|d\tau|_{f^\star\omega}}\right) = f^\star \omega_q, \tag{1}$$

where $\star_{f^\star\omega}$ is the Hodge star operator induced by $f^\star\omega$.

Thus, the $(2q-1)$-form

$$d\sigma_{\omega,\,f} := \star_{f^\star\omega}\left(\frac{d\tau}{|d\tau|_{f^\star\omega}}\right)$$

on $\mathbb{C}^q \setminus \Sigma_f$ is the area measure induced by $f^\star\omega$ on the spheres of $\mathbb{C}^q$. This means that its restriction

$$d\sigma_{\omega,\,f,\,t} := \left(\star_{f^\star\omega}\left(\frac{d\tau}{|d\tau|_{f^\star\omega}}\right)\right)_{|S_t} \tag{2}$$

is the area measure induced by the degenerate metric $f^\star\omega$ on the sphere $S_t = \{\tau(z) = t^2\} \subset \mathbb{C}^q$ for every $t > 0$.

In particular, the $(\omega,\ f)$-*area* of the sphere $S_r \subset \mathbb{C}^q$ is

$$A_{\omega,\,f}(S_r) = \int\limits_{S_r} d\sigma_{\omega,\,f,\,r} > 0, \qquad r > 0.$$

Definition (Marouani-P. 2022) *$f : \mathbb{C}^q \longrightarrow X$ has* **subexponential growth** *if the following two conditions are satisfied:*

(i) there exist constants $C_1 > 0$ and $r_0 > 0$ such that

$$\int\limits_{S_t} |d\tau|_{f^\star\omega}\, d\sigma_{\omega,\, f,\, t} \leq C_1 t\; Vol_{\omega,\, f}(B_t), \qquad t > r_0;$$

(ii) for every constant $C > 0$, we have:

$$\limsup_{b\to+\infty}\left(\frac{b}{C} - \log\int\limits_0^b Vol_{\omega,\, f}(B_t)\, dt\right) = +\infty.$$

Definition

$f : \mathbb{C}^q \longrightarrow X$ is **E-horizontal** *if $\forall x \in X$,*

$$Im\left(d_x f : \mathbb{C}^q \longrightarrow T^{1,0}_x X\right) \subset E_x \subset T^{1,0}_x X.$$

Definition (Kasuya-P. 2023)

X is **partially p-hyperbolic** *if $\exists\, E \subset T^{1,0}X$ C^∞ complex vector subbundle, $rk_{\mathbb{C}} E \geq p$, such that*

$\nexists\, E$-horizontal holomorphic map $f : \mathbb{C}^p \longrightarrow X$ with the properties:

- *f is non-degenerate at some point $x_0 \in \mathbb{C}^p$;*
- *f has subexponential growth.*

Theorem A (Kasuya-P. 2023)

X *is partially p-Kähler hyperbolic* $\implies$ X *is partially p-hyperbolic*

Idea of proof. By contradiction: suppose $\exists$ a map $f : \mathbb{C}^p \longrightarrow X$ as above. Then, f lifts to a map $\widetilde{f} : \mathbb{C}^p \longrightarrow \widetilde{X}$ such that $f = \pi_X \circ \widetilde{f}$. Put $\widetilde{\omega} := \pi_X^\star \omega$ on $\widetilde{X}$.

$$\begin{array}{ccc} & & \widetilde{X} \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow \pi_X \\ \mathbb{C}^p & \overset{f}{\longrightarrow} & (X,\, \omega) \end{array}$$

$$\pi_X^\star \Omega = d\widetilde{\Gamma} \quad \text{on } \widetilde{X}, \quad \text{with } \widetilde{\Gamma} \ \ \widetilde{\omega}\text{-bounded}$$

$$\Omega := \frac{\omega_E^p}{p!} \quad \text{on } X$$

- $f^\star \omega = f^\star \omega_E$ on $\mathbb{C}^p$ (since f is E-horizontal)
- $f^\star \Omega = \widetilde{f}^\star(\pi_X^\star \Omega) = d(\widetilde{f}^\star \widetilde{\Gamma})$ on $\mathbb{C}^p$, where $\widetilde{f}^\star \widetilde{\Gamma}$ is $(f^\star \omega)$-bounded

For every $r > 0$, we compute $\mathrm{Vol}_{\omega,\,f}(B_r)$ in two ways:

• Using **Fubini**:

$$\mathrm{Vol}_{\omega,\,f}(B_r) := \int\limits_{B_r} f^\star\omega_p = \int\limits_0^r \Bigg(\int\limits_{S_t} d\mu_{\omega,\,f,\,t}\Bigg)\,dt \overset{\text{Hölder}}{\geq} \dots$$

(depends on $A_{\omega,\,f}(S_t)$, with $t \in [0,\,r]$)

• Using **Stokes** (and the partial p-Kähler hyperbolicity assumption):

$$\mathrm{Vol}_{\omega,\,f}(B_r) = \int\limits_{B_r} f^\star\Omega = \int\limits_{B_r} d(\widetilde{f}^\star\widetilde{\Gamma}) = \int\limits_{S_r} \widetilde{f}^\star\widetilde{\Gamma} \leq \mathrm{Const}\, A_{\omega,\,f}(S_r)$$

since $\widetilde{f}^\star\widetilde{\Gamma}$ is $\widetilde{\omega}$-bounded in $\mathbb{C}^p$.

Then, we get a contradiction via a Gronwall Lemma-type argument.

(III) Examples

O-T manifolds: $X = G/\Lambda$ solvmanifolds

(G is a solvable Lie group, $\Lambda \subset G$ is a lattice)

Specifically: given positive integers s and t, G is the Lie group defined as the semi-direct product

$$G = \mathbb{R}^s \ltimes_\phi (\mathbb{R}^s \oplus \mathbb{C}^t)$$

via the map

$$\phi : \mathbb{R}^s \longrightarrow \mathrm{Aut}(\mathbb{R}^s \oplus \mathbb{C}^t),$$

$$\phi(x) = \mathrm{diag}\left(e^{x_1}, \ldots, e^{x_s}, e^{\psi_1(x)}, \ldots, e^{\psi_t(x)}\right),$$

where $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$ and $\psi_1, \ldots, \psi_t : \mathbb{R}^s \longrightarrow \mathbb{C}$ are functions of the shape

$$\psi_j(x) = \sum_{k=1}^{s} a_{jk}\, x_k, \qquad j = 1, \ldots, t,$$

with constant coefficients $a_{jk} \in \mathbb{C}$, while diag stands for the diagonal matrix whose diagonal entries are those indicated.

In $G = \mathbb{R}^s \ltimes_\phi (\mathbb{R}^s \oplus \mathbb{C}^t)$,

$x = (x_1, \ldots, x_s)$ is the variable in the first copy of $\mathbb{R}^s$;

$y = (y_1, \ldots, y_s)$ is the variable in the second copy of $\mathbb{R}^s$;

$z = (z_1, \ldots, z_t)$ the variable of $\mathbb{C}^t$.

We get C^∞ (1, 0)-forms on G:

$$\begin{aligned} \alpha_j &= dx_j + ie^{-x_j}\, dy_j, \qquad j = 1, \ldots, s, \\ \beta_k &= e^{-\psi_k}\, dz_k, \qquad k = 1, \ldots, t. \end{aligned}$$

They induce:

• an *invariant complex structure* on G;

• C^∞ (1, 0)-forms (denoted by the same symbols) and a complex structure on the solvmanifold $X = G/\Lambda$, for any lattice $\Lambda \subset G$.

A natural *Hermitian metric* on X is

$$\begin{aligned} \omega &= i\alpha_1 \wedge \overline{\alpha}_1 + \cdots + i\alpha_s \wedge \overline{\alpha}_s + i\beta_1 \wedge \overline{\beta}_1 + \cdots + i\beta_t \wedge \overline{\beta}_t > 0 \\ &= \omega_E + \omega_{nE}. \end{aligned}$$

Main observation *On X, we have:*

$$\Omega = \omega_E = d\left(2i\alpha_1 + \cdots + 2i\alpha_s\right).$$

Conclusion *These solvmanifolds (containing all the O-T manifolds) are partially 1-Kähler hyperbolic.*

Recall the construction of **O-T manifolds**

• Let $K \supset \mathbb{Q}$ be a finite extension field of degree $s + 2t$ admitting real embeddings

$$\sigma_1, \ldots, \sigma_s : K \hookrightarrow \mathbb{C}$$

and complex embeddings

$$\sigma_{s+1}, \ldots, \sigma_{s+2t} : K \hookrightarrow \mathbb{C}$$

such that

$$\sigma_{s+i} = \overline{\sigma_{s+i+t}}, \qquad i \in \{1, \ldots, t\}.$$

• Let U be a free subgroup of rank s of the group of units in the ring $\mathcal{O}_K$ (= the algebraic integers satisfying certain conditions).

(i) Associate with (K, U), via suitable functions $\psi_j : \mathbb{R}^s \to \mathbb{C}$ as above, a *lattice*

$$\Lambda \simeq U \ltimes \mathcal{O}_K \subset G.$$

(ii) Associate with Λ the *O-T manifold*

$$X := G/\Lambda.$$

(IV) Ahlfors currents

(1) **Background** Let:

• (X, ω) be a compact complex hermitian manifold, $\dim_{\mathbb{C}} X = n$;

• $f : \mathbb{C}^p \longrightarrow (X, \omega)$ be a holomorphic map, non-degenerate at some point $x_0 \in \mathbb{C}^p$ $(1 \leq p \leq n-1)$;

Consider, for every $r > 0$:

• $[B_r] \geq 0$ the current of integration on the ball $B_r \subset \mathbb{C}^p$;

(bidimension (p, p)-current in $\mathbb{C}^p$, so of bidegree $(0, 0)$)

• the direct-image current $f_\star[B_r] \geq 0$

(a strongly positive bidegree $(n-p, n-p)$-current in X)

The current

$$T_r := \frac{1}{\mathrm{Vol}_{\omega,\, f}(B_r)}\, f_\star[B_r] \geq 0$$

has unit mass w.r.t. ω in X:

$$\int_X T_r \wedge \omega_p = \frac{1}{\mathrm{Vol}_{\omega,\, f}(B_r)} \int_{\mathbb{C}^p} [B_r] \wedge f^\star \omega_p = 1, \qquad \forall\, r > 0.$$

Therefore, $\exists\, (T_{r_\nu})_{\nu \in \mathbb{N}}$ such that $T_{r_\nu} \xrightarrow{\text{weakly}} T \geq 0$ in X.

So, T is a strongly positive bidegree $(n-p,\, n-p)$-current in X.

Observation *T need not be closed.*

Indeed, Stokes yields: $d[B_r] = -[S_r] \neq 0$ for all $r > 0$.

Definition (standard)

T is called an **Ahlfors current** *if* $dT = 0$.

Question

When does an Ahlfors current exist?

De Thélin (2010) gave sufficient conditions.

(2) **Our result:** in the spirit of this work

Theorem (Kasuya-P. 2023)

Suppose $\exists\, f : \mathbb{C}^p \longrightarrow (X,\, \omega)$ *holomorphic map, non-degenerate at some point.*

If

$$\liminf_{r\to+\infty} \frac{A_{\omega,\, f}(S_r)}{Vol_{\omega,\, f}(B_r)} = 0, \qquad (\star)$$

then $\exists\, (T_{r_\nu})_{\nu\in\mathbb{N}}$ *such that* $T_{r_\nu} \xrightarrow{weakly} T \geq 0$ *in* X *with*

$$dT = 0.$$

So, T is an Ahlfors current.

Idea of proof. This follows from the estimate:

$$||\partial T_r|| \leq \frac{1}{\sqrt{2}} \frac{A_{\omega,\, f}(S_r)}{\mathrm{Vol}_{\omega,\, f}(B_r)}, \qquad r > 0,$$

that is proved using integral estimates handled differently to De Thélin's treatment.

The norm $||\ ||$ used on bidegree $(n-p+1,\, n-p)$-currents (such as ∂T_r) on X is defined as follows:

• consider the closed unit ball of C^∞ forms of bidegree $(p-1,\, p)$ with respect to the C^0-norm induced by the metric ω:

$$\mathcal{F}_\omega(p-1,\, p) := \left\{\psi \in C^\infty_{p-1,\, p}(X,\, \mathbb{C}) \mid ||\psi||_{C^0_\omega} := \max_{x\in X} |\psi(x)|_\omega \leq 1\right\}$$

• for every bidegree-$(n-p+1,\, n-p)$-current S on X, set

$$||S|| := \sup_{\psi\in\mathcal{F}_\omega(p-1,\, p)} |\langle S,\, \psi\rangle|.$$

(3) **Link with partial hyperbolicity**

Definition (Kasuya-P. 2023)

Let $E \subset T^{1,0}X$ be a complex vector subbundle with $rk_{\mathbb{C}}E = p \in \{1, \dots, n-1\}$, where $n = dim_{\mathbb{C}}X$.

X is said to be **strongly partially p-hyperbolic** *in the* **E-directions** *if $\nexists\, f : \mathbb{C}^p \longrightarrow X$ holomorphic such that:*

(i) f is non-degenerate at some $x_0 \in \mathbb{C}^p$;

(ii) f is E-horizontal;

(iii) f satisfies the growth condition $(\star)$:

$$\liminf_{r \to +\infty} \frac{A_{\omega,\,f}(S_r)}{Vol_{\omega,\,f}(B_r)} = 0.$$

Theorem (Kasuya-P. 2023)

X *is* **partially** p**-Kähler hyperbolic** *in the* E**-directions**

$\Downarrow$ *(a)*

X *is* **strongly partially** p**-hyperbolic** *in the* E**-directions**

$\Downarrow$ *(b)*

X *is* **partially** p**-hyperbolic** *in the* E**-directions**.

Proof of (a). By contradiction: suppose an E-horizontal holomorphic map $f : \mathbb{C}^p \longrightarrow X$ that is non-degenerate at some point and satisfies the growth condition $(\star)$ existed.

$$f \text{ is } E\text{-horizontal} \implies f^\star\omega = f^\star\omega_E \implies f^\star\omega_p = f^\star\Omega = d(\widetilde{f}^\star\widetilde{\Gamma})$$

on $\mathbb{C}^p$, where

$\widetilde{\Gamma}$ is the $\widetilde{\omega}$-bounded $(2p-1)$-form on the universal cover $\widetilde{X}$ with the property

$$\pi_X^\star\Omega = d\widetilde{\Gamma}$$

given by the partial p-Kähler hyperbolicity hypothesis on X.

We get:

$$1 = \frac{1}{\mathrm{Vol}_{\omega,\,f}(B_r)}\int\limits_{B_r} f^\star\omega_p = \frac{1}{\mathrm{Vol}_{\omega,\,f}(B_r)}\int\limits_{B_r} d(\widetilde{f}^\star\widetilde{\Gamma}) = \frac{1}{\mathrm{Vol}_{\omega,\,f}(B_r)}\int\limits_{S_r} \widetilde{f}^\star\widetilde{\Gamma}$$

$$\leq \frac{C}{\mathrm{Vol}_{\omega,\,f}(B_r)}\int\limits_{S_r} \sigma_{\omega,\,f,\,r} = C\,\frac{A_{\omega,\,f}(S_r)}{\mathrm{Vol}_{\omega,\,f}(B_r)}, \qquad r>0,$$

contradicting the growth hypothesis $(\star)$ on f.

(V) Curvature sign and partial hypebolicity

Recall (Kobayashi 1967)

Let (X, ω) *be a Hermitian manifold.*

If the holomorphic sectional curvature of (X, ω) *is* $\leq -C$*, for some constant* $C > 0$*,*

then

X *is Kobayashi hyperbolic.*

Our case

(a) Brief reminder of a construction from [P.2022]

$(X,\, \omega)$ compact complex Hermitian manifold, $\dim_{\mathbb{C}} X = n$;

The multiplication map:

$$\omega_{n-2} \wedge \cdot : \Lambda^{1,\,1}T^\star X \longrightarrow \Lambda^{n-1,\,n-1}T^\star X$$

is bijective. Hence:

$$\exists\,!\rho_\omega \in C^\infty_{1,\,1}(X,\,\mathbb{R}) \quad \text{such that} \quad i\partial\bar\partial\omega_{n-2} = \omega_{n-2} \wedge \rho_\omega.$$

An explicit computation yields:

$$\star\rho_\omega = \frac{1}{n-1}\,\frac{\omega\wedge i\partial\bar\partial\omega_{n-2}}{\omega_n}\,\omega_{n-1} - i\partial\bar\partial\omega_{n-2}.$$

This throws up the C^∞ function $f_\omega : X \longrightarrow \mathbb{R}$,

$$f_\omega := \frac{\omega\wedge i\partial\bar\partial\omega_{n-2}}{\omega_n}.$$

• the function f_ω plays a role similar in certain respects to the scalar curvature of ω;

• the $(n-1,\, n-1)$-form

$$\star\rho_\omega = \frac{1}{n-1}\,f_\omega\,\omega_{n-1} - i\partial\bar\partial\omega_{n-2}$$

plays a role similar to various curvature forms (e.g. Ricci).

Definition (P. 2022)

The metric ω is said to be **pluriclosed star split** *if*

$$\partial\bar{\partial}(\star\rho_\omega) = 0.$$

Theorem (P. 2022)

Suppose X is **compact**, *$n = dim_{\mathbb{C}}X \geq 3$.*

(i) If ω is pluriclosed star split, then

$$f_\omega > 0 \ \text{on } X \quad \text{or} \quad f_\omega < 0 \ \text{on } X \quad \text{or} \quad f_\omega = 0 \ \text{on } X.$$

(ii) If $f_\omega = Const \neq 0$, then

$$\omega \text{ is pluriclosed star split} \iff \omega \text{ Gauduchon}.$$

(iii) If ω is Gauduchon, then

$$\omega \text{ is pluriclosed star split} \iff f_\omega = Const.$$

(iv) If ω is balanced and pluriclosed star split, then

$$f_\omega = Const \geq 0.$$

Question (P. 2022)

When ω is pluriclosed star split, does the sign of f_ω depend only on the complex structure of X?

- **Yes** if X is a complex nilmanifold of complex dimension 3.

  (follows from Fino-Ugarte 2013)

- **Open** in general.

(b) Link with partial hyperbolicity

Definition $\dim_{\mathbb{C}} X = n$

Let

- $E \subset T^{1,0}X$ *be a* C^∞ *complex vector* *subbundle* *with* $rkE = p \in \{1, \dots, n-1\}$*;*
- Ω *be a* *real* (p, p)*-form on* X*.*

Ω *is* **negative in the** E**-directions** $\overset{def}{\Longleftrightarrow}$ $\forall\, x_0 \in X\ \exists\, C^\infty$ *frame* $\{\xi_1, \dots, \xi_p\}$ *of* E *on a neighbourhood* U *of* x_0 *such that*

$$\Omega(\xi_1, \bar\xi_1, \dots, \xi_p, \bar\xi_p) < 0, \qquad \forall\, x \in U.$$

Theorem (Kasuya-P. 2023)

Let

- X *be a compact complex manifold,* $dim_{\mathbb{C}} X = n$*;*
- $E \subset T^{1,0}X$ *be a* C^∞ *complex vector subbundle with* $rkE = n-1$.

If $\exists\,\omega$ *Hermitian metric on* X *such that*

- $f_\omega > 0$ *on* X*;*
- *the* $(n-1,\, n-1)$*-form* $\star\rho_\omega$ *is negative in the* E*-directions*

then

X *is strongly partially* $(n-1)$*-hyperbolic in the* E*-directions.*

Proof. By contradiction: suppose $\exists\, f : \mathbb{C}^{n-1} \longrightarrow X$ holomorphic, non-degenerate at some point,

E-horizontal and satisfying the growth condition $(\star)$.

Then, f induces an Ahlfors current

$$T = \lim_{r_\nu \to +\infty} T_{r_\nu} \geq 0 \quad \text{bidegree}\,(1,\ 1) - \text{current on } X.$$

We get:

(a) on the one hand, from $dT = 0$ and the formula for $\star\rho_\omega$, we get, via Stokes:

$$\int_X T \wedge \star\rho_\omega = \frac{1}{n-1} \int_X f_\omega \, T \wedge \omega_{n-1} > 0,$$

where the inequality follows from the hypothesis $f_\omega > 0$ and the property $T \geq 0$ (with $T \neq 0$).

(b) on the other hand,

f is E-horizontal and $\star\rho_\omega$ is negative in the E-directions $\implies$

$f^\star(\star\rho_\omega)$ is a negative $(n-1,\, n-1)$-form on $\mathbb{C}^{n-1}$.

Therefore, we get:

$$\int\limits_X T \wedge \star\rho_\omega = \lim_{\nu\to+\infty} \int\limits_X T_{r_\nu} \wedge \star\rho_\omega$$
$$= \lim_{\nu\to+\infty} \frac{1}{\mathrm{Vol}_{\omega,\,f}(B_{r_\nu})} \int\limits_{B_{r_\nu}} f^\star(\star\rho_\omega) \leq 0.$$

This contradicts the inequality obtained under (a). q.e.d.