Balanced Hyperbolic and Divisorially Hyperbolic Compact Complex Manifolds

joint work with **Samir Marouani**

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(I) Background

X compact complex manifold, $n = \dim_{\mathbb{C}} X$

(1) • Kobayashi (1970)

X is Kobayashi hyperbolic $\overset{def}{\Longleftrightarrow}$

the Kobayashi pseudo-distance of X is a distance

• Brody (1978)

X is Brody hyperbolic $\overset{def}{\Longleftrightarrow}$

$\nexists f : \mathbb{C} \longrightarrow X$ non-constant holomorphic map

(*Such a map is called an entire curve.*)

Theorem (Brody 1978) *When X is* **compact**, *one has:*

$$X \text{ is Kobayashi hyperbolic} \iff X \text{ is Brody hyperbolic.}$$

• For a possibly non-compact X, one always has:

$$X \text{ is Kobayashi hyperbolic} \implies X \text{ is Brody hyperbolic}$$

but the converse fails in general.

Question 1 (Marouani-P. 2021) *What is the relevant analogue of the Brody hyperbolicity when entire curves $f : \mathbb{C} \longrightarrow X$ are replaced by divisors $f : \mathbb{C}^{n-1} \longrightarrow X$, where $n = dim_{\mathbb{C}} X$?*

(2) Gromov (1991)

- Let $\pi_X : \widetilde{X} \longrightarrow X$ be the universal cover of X.

- If ω is a Hermitian metric on X, we put $\widetilde{\omega} := \pi_X^\star \omega$ its lift to $\widetilde{X}$.

  (So, $\widetilde{\omega}$ is a Hermitian metric on $\widetilde{X}$.)

- Let α be a C^∞ k-form on X.

α is $\widetilde{d}$(bounded) w.r.t. ω $\stackrel{def}{\Longleftrightarrow}$ $\pi_X^\star \alpha = d\beta$ on $\widetilde{X}$ for some C^∞ $(k-1)$-form β on $\widetilde{X}$ that is bounded w.r.t. $\widetilde{\omega}$.

- X is Kähler hyperbolic $\stackrel{def}{\Longleftrightarrow}$

  $\exists\, \omega$ Kähler metric on X such that ω is $\tilde{d}$(bounded) w.r.t. ω.

Fact (Gromov 1991) *If X is compact, one has:*

$$X \text{ is Kähler hyperbolic} \implies X \text{ is Kobayashi hyperbolic.}$$

However, the converse fails in general.

Question 2 (Marouani-P. 2021) *What is the relevant analogue of the Kähler hyperbolicity when X is not Kähler?*

(3) Fact (Gromov 1991 + Chen-Yang 2017)

If X is compact, one has:

$$X \text{ is Kähler hyperbolic} \implies K_X \text{ is ample } (\implies X \text{ is projective}).$$

Gromov proved that K_X is big. The reinforcement of the result to ample may have been known before Chen-Yang 2017.

Conjecture (Kobayashi)

If X is compact, one expects to have:

$$X \text{ is Kobayashi hyperbolic} \stackrel{?}{\implies} K_X \text{ is ample}$$

$$(\implies X \text{ is projective}).$$

So, the standard notions of hyperbolicity can only occur in the **projective** context.

Question 3 **(Marouani-P. 2021)** *Do any hyperbolicity phenomena occur outside the projective or even outside the Kähler context?*

(4) Definition (Gauduchon 1977)

Let X be a complex manifold with $dim_{\mathbb{C}} X = n$.

A **balanced metric** *on X is any C^∞ positive definite* (1, 1)-*form $\omega > 0$ on X (i.e. any Hermitian metric ω) such that*

$$d\omega^{n-1} = 0.$$

The manifold X is **balanced** $\iff$ $\exists\,\omega$ balanced metric on X.

Examples of **balanced manifolds** include:

(a) all complex parallelisable compact complex manifolds X:

$$T^{1,\,0}X \quad \text{is holomorphically trivial.}$$

Fact (Wang 1954) *A compact complex manifold X is complex parallelisable $\iff$ $X = G/\Gamma$ for some simply connected, connected* **complex** *Lie group G and some discrete subgroup Γ.*

(b) all Calabi-Eckmann manifolds: $X = S^{2n+1} \times S^{2m+1}$ equipped with the Calabi-Eckmann complex structure;

(c) all twistor spaces (Penrose 1976, Atiyah-Hitchin-Singer 1978, Gauduchon 1991);

(d) many nilmanifolds and solvmanifolds: $X = G/\Gamma$ with G a (real) nilpotent or solvable Lie group endowed with an invariant complex structure and Γ a lattice therein.

(II) Our two hyperbolicity notions

(1) Answering Question 2

Definition (Marouani-P. 2021)

Let X be a compact complex manifold with $dim_{\mathbb{C}} X = n$.

X is **balanced hyperbolic** $\stackrel{def}{\Longleftrightarrow}$

$\exists\,\omega$ *balanced metric on X such that ω^{n-1} is $\tilde{d}$(bounded) w.r.t. ω.*

Recall: this means that

$$\pi_X^\star \omega^{n-1} = d\Gamma \quad \text{on } \widetilde{X}$$

for some $\widetilde{\omega}$-bounded C^∞ $(2n-3)$-form Γ on $\widetilde{X}$, where $\pi_X : \widetilde{X} \longrightarrow X$ is the universal cover of X.

Balanced hyperbolic manifolds generalise both

-Gromov's Kähler hyperbolic manifolds;

and

-degenerate balanced manifolds.

Definition (P. 2015 for the name – the notion predates this)

Let X be a compact complex manifold with $dim_{\mathbb{C}} X = n$.

X is **degenerate balanced** $\overset{def}{\Longleftrightarrow}$

$$\exists\,\omega \text{ Hermitian metric on } X \text{ such that } \omega^{n-1} \in Im\,d.$$

• There is no analogous phenomenon in the Kähler case:

if X is *compact*, no C^∞ (1, 1)-form $\omega > 0$ on X can be such that $\omega \in \operatorname{Im} d$.

• **Two known classes** of **degenerate balanced manifolds**:

(a) the connected sums

$$X_k = \sharp_k(S^3 \times S^3)$$

of k copies (with $k \geq 2$) of $S^3 \times S^3$ endowed with the Friedman-Lu-Tian complex structure J_k constructed via conifold transitions, where S^3 is the 3-sphere;

Fu-Li-Yau (2012): $\exists\,\omega$ balanced metric on $(X_k,\, J_k)$.

Moreover, $H^4_{DR}(X_k,\, \mathbb{C}) = \{0\}$, so for any balanced metric ω, $H^4_{DR}(X_k,\, \mathbb{C})$ $\{\omega^2\}_{DR} = 0$, hence ω is *degenerate balanced*.

(b) the Yachou manifolds (1998)

$$X = G/\Gamma$$

arising as the quotient of any **semi-simple** complex Lie group G by a lattice $\Gamma \subset G$.

Obvious fact

Every degenerate balanced manifold is balanced hyperbolic.

(2) Answering Question 1

Observation (trivial)

For any complex Lie group G with $dim_{\mathbb{C}} G = n$ and any lattice $\Gamma \subset G$, there exists a holomorphic map

$$f : \mathbb{C}^{n-1} \longrightarrow X := G/\Gamma$$

that is non-degenerate at some point $x \in \mathbb{C}^{n-1}$.

Reason. There are non-degenerate holomorphic maps:

$$\mathbb{C}^{n-1} \longrightarrow T_e^{1,0} G = \mathfrak{g} \xrightarrow{\exp} G \longrightarrow G/\Gamma,$$

where the exponential map $\exp : \mathfrak{g} \longrightarrow G$ is holomorphic (and immersive at least at $0 \in \mathfrak{g}$ since $d_0 \mathfrak{g} = \mathrm{id}_{\mathfrak{g}}$) since the Lie group G is complex.

However, the Friedman-Lu-Tian and the Yachou manifolds deserve to be called hyperbolic.

Notation. • For any integer $n \geq 2$ and any $r > 0$, let

$$B_r := \{z \in \mathbb{C}^{n-1} \mid |z| < r\}$$

and

$$S_r := \{z \in \mathbb{C}^{n-1} \mid |z| = r\}$$

be the open ball, resp. the sphere, of radius r centred at $0 \in \mathbb{C}^{n-1}$.

For any $(1, 1)$-form $\gamma \geq 0$ on a complex manifold and any positive integer p, we use the notation:

$$\gamma_p := \frac{\gamma^p}{p!}.$$

• For any compact Hermitian manifold (X, ω) with $\dim_{\mathbb{C}} X = n \geq 2$ and any holomorphic map

$$f : \mathbb{C}^{n-1} \longrightarrow X$$

that is *non-degenerate* at some point $x \in \mathbb{C}^{n-1}$ (i.e. its differential map $d_x f : \mathbb{C}^{n-1} \longrightarrow T_{f(x)}X$ at x has maximal rank):

· there exists a proper analytic subset $\Sigma \subset \mathbb{C}^{n-1}$ such that f is *non-degenerate* at every point $z \in \mathbb{C}^{n-1} \setminus \Sigma$;

· $f^\star\omega \geq 0$ on $\mathbb{C}^{n-1}$ and $f^\star\omega > 0$ on $\mathbb{C}^{n-1} \setminus \Sigma$

(i.e. $f^\star\omega$ is a *degenerate metric* on $\mathbb{C}^{n-1}$ and a genuine metric on $\mathbb{C}^{n-1} \setminus \Sigma$)

Definition (Marouani-P. 2021)

(i) For every $r > 0$, the $(\omega,\, f)$-volume of the ball $B_r \subset \mathbb{C}^{n-1}$ is

$$Vol_{\omega,\, f}(B_r) := \int\limits_{B_r} f^\star \omega_{n-1} > 0.$$

(ii) For $z \in \mathbb{C}^{n-1}$, let $\tau(z) := |z|^2$ be its squared Euclidean norm. At every point $z \in \mathbb{C}^{n-1} \setminus \Sigma$, we have:

$$\frac{d\tau}{|d\tau|_{f^\star\omega}} \wedge \star_{f^\star\omega}\left(\frac{d\tau}{|d\tau|_{f^\star\omega}}\right) = f^\star\omega_{n-1}, \tag{1}$$

where $\star_{f^\star\omega}$ is the Hodge star operator induced by $f^\star\omega$.

Thus, the $(2n-3)$*-form*

$$d\sigma_{\omega,\,f} := \star_{f^\star\omega}\left(\frac{d\tau}{|d\tau|_{f^\star\omega}}\right)$$

on $\mathbb{C}^{n-1}\setminus\Sigma$ *is the area measure induced by* $f^\star\omega$ *on the spheres of* $\mathbb{C}^{n-1}$*. This means that its restriction*

$$d\sigma_{\omega,\,f,\,t} := \left(\star_{f^\star\omega}\left(\frac{d\tau}{|d\tau|_{f^\star\omega}}\right)\right)_{|S_t} \tag{2}$$

is the area measure induced by the degenerate metric $f^\star\omega$ *on the sphere* $S_t = \{\tau(z) = t^2\} \subset \mathbb{C}^{n-1}$ *for every* $t > 0$*. In particular, the area of the sphere* $S_r \subset \mathbb{C}^{n-1}$ *w.r.t.* $d\sigma_{\omega,\,f,\,r}$ *is*

$$A_{\omega,\,f}(S_r) = \int\limits_{S_r} d\sigma_{\omega,\,f,\,r} > 0, \qquad r > 0.$$

Definition (Marouani-P. 2021) *Let* (X, ω) *be a compact Hermitian manifold with* $dim_{\mathbb{C}} X = n \geq 2$ *and let*

$$f : \mathbb{C}^{n-1} \longrightarrow X$$

be a holomorphic map, non-degenerate at some point $x \in \mathbb{C}^{n-1}$.

f *has* **subexponential growth** *if the following two conditions are satisfied:*

(i) there exist constants $C_1 > 0$ *and* $r_0 > 0$ *such that*

$$\int\limits_{S_t} |d\tau|_{f^\star\omega}\, d\sigma_{\omega,\, f,\, t} \leq C_1 t\; Vol_{\omega,\, f}(B_t), \qquad t > r_0;$$

(ii) for every constant $C > 0$, we have:

$$\limsup_{b \to +\infty} \left(\frac{b}{C} - \log F(b) \right) = +\infty,$$

where

$$F(b) := \int_0^b Vol_{\omega,\, f}(B_t)\, dt = \int_0^b \left(\int_{B_t} f^\star \omega_{n-1} \right) dt, \qquad b > 0.$$

Observation *Any holomorphic map* $f : \mathbb{C}^{n-1} \longrightarrow (X, \omega)$ *such that*

$$f^\star \omega = \beta := (1/2) \sum_{j=1}^{n-1} i dz_j \wedge d\bar{z}_j$$

(the standard Kähler metric, i.e. the Euclidean metric)

has subexponential growth.

Observation *The following identities hold for all $t > 0$:*

$$\int\limits_{S_t} |d\tau|_{f^\star\omega}\, d\sigma_{\omega,\,f,\,t} = 2\int\limits_{B_t} i\partial\bar\partial\tau \wedge f^\star\omega_{n-2} - \int\limits_{B_t} i(\bar\partial\tau - \partial\tau) \wedge d(f^\star\omega_{n-2})$$

$$= 2\int\limits_{B_t} \Lambda_{f^\star\omega}(i\partial\bar\partial\tau)\, f^\star\omega_{n-1} - \int\limits_{B_t} i(\bar\partial\tau - \partial\tau) \wedge d(f^\star\omega_{n-2}$$

where $\Lambda_{f^\star\omega}$ is the trace w.r.t. $f^\star\omega$ or, equivalently, the pointwise adjoint of the operator of multiplication by $f^\star\omega$, while

$$i\partial\bar\partial\tau = i\partial\bar\partial|z|^2 = \sum_{j=1}^{n-1} i dz_j \wedge d\bar z_j := \beta$$

is the standard metric of $\mathbb{C}^{n-1}$.

Observation (trivial)

The subexponential growth condition on f is independent of the choice of Hermitian metric ω on X.

Proof. Let ω_1 and ω_2 be arbitrary Hermitian metrics on X.

Since X is compact, there exists a constant $A > 0$ such that

$$(1/A)\,\omega_2 \leq \omega_1 \leq A\,\omega_2$$

on X. Hence,

$$(1/A)\,f^{\star}\omega_2 \leq f^{\star}\omega_1 \leq A\,f^{\star}\omega_2$$

on $\mathbb{C}^{n-1}$. □

Standard definition

A holomorphic map $f : \mathbb{C}^{n-1} \to (X, \omega)$ *is of finite order if there exist constants* $C_1, C_2, r_0 > 0$ *such that*

$$Vol_{\omega, f}(B_r) \leq C_1 \, r^{C_2} \quad \textit{for all } r \geq r_0.$$

• By the above proof, f being of finite order does not depend on the choice of Hermitian metric ω on X.

• If f has *finite order*, then f satisfies part (ii) of the *subexponential growth condition*.

Definition (Marouani-P. 2021)

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.

X *is* **divisorially hyperbolic** *if there is no holomorphic map*

$$f : \mathbb{C}^{n-1} \longrightarrow X$$

such that f is **non-degenerate** *at some point* $x \in \mathbb{C}^{n-1}$ *and f has* **subexponential growth**.

Question *What about the case where X is* **non-compact**?

For all compact complex manifolds X, we get the following picture:

$$\begin{array}{ccc}
X \text{ is } \textbf{Kähler hyperbolic} & \Longrightarrow & X \text{ is } \textbf{Brody hyperbolic} \\
\Downarrow & & \Downarrow \\
X \text{ is } \textbf{balanced hyperbolic} & \Longrightarrow & X \text{ is } \textbf{divisorially hyperbolic} \\
\Uparrow & & \\
X \text{ is } \textbf{degenerate balanced} & &
\end{array}$$

The only implication that has yet to be proved is

Theorem (Marouani-P. 2021)

Every **balanced hyperbolic** *compact complex manifold is* **divisorially hyperbolic***.*

Proof. Let X be a balanced hyperbolic compact complex manifold, $\dim_{\mathbb{C}} X = n$. Let ω be a *balanced hyperbolic* metric on X.

Thus, if $\pi_X : \widetilde{X} \longrightarrow X$ is the universal cover of X, we have

$$\pi_X^\star \omega^{n-1} = d\Gamma \quad \text{on } \widetilde{X},$$

where Γ is an $\widetilde{\omega}$-bounded C^∞ $(2n-3)$-form on $\widetilde{X}$ and $\widetilde{\omega} = \pi_X^\star \omega$ is the lift of the metric ω to $\widetilde{X}$.

Suppose there exists a holomorphic map

$$f : \mathbb{C}^{n-1} \longrightarrow X$$

that is non-degenerate at some point $x \in \mathbb{C}^{n-1}$ and has subexponential growth. We will prove that

$$f^{\star}\omega^{n-1} = 0$$

on $\mathbb{C}^{n-1}$, in contradiction to the non-degeneracy assumption.

Since $\mathbb{C}^{n-1}$ is simply connected, there exists a lift $\widetilde{f}$ of f to $\widetilde{X}$, namely a holomorphic map

$$\tilde{f} : \mathbb{C}^{n-1} \longrightarrow \widetilde{X}$$

such that $f = \pi_X \circ \tilde{f}$. In particular, $d_x\tilde{f}$ is injective since $d_x f$ is.

We have:

$$f^\star\omega^{n-1} = \tilde{f}^\star(\pi_X^\star\omega^{n-1}) = d(\tilde{f}^\star\Gamma) \begin{array}{ll} \geq 0 & \text{on } \mathbb{C}^{n-1} \\ > 0 & \text{on } \mathbb{C}^{n-1} \setminus \Sigma, \end{array}$$

where $\Sigma \subset \mathbb{C}^{n-1}$ is the proper analytic subset of all points $z \in \mathbb{C}^{n-1}$ such that $d_z f$ is not of maximal rank.

Claim. *The $(2n-3)$-form $\tilde{f}^\star\Gamma$ is $(f^\star\omega)$-bounded on $\mathbb{C}^{n-1}$.*

Proof of Claim. For any tangent vectors $v_1, \ldots, v_{2n-3}$ in $\mathbb{C}^{n-1}$, we have:

$$\begin{aligned} |(\tilde{f}^\star\Gamma)(v_1, \ldots, v_{2n-3})|^2 &= |\Gamma(\tilde{f}_\star v_1, \ldots, \tilde{f}_\star v_{2n-3})|^2 \\ &\overset{(a)}{\leq} C\, |\tilde{f}_\star v_1|^2_{\widetilde{\omega}} \ldots |\tilde{f}_\star v_{2n-3}|^2_{\widetilde{\omega}} \\ &= C\, |v_1|^2_{\tilde{f}^\star\widetilde{\omega}} \ldots |v_{2n-3}|^2_{\tilde{f}^\star\widetilde{\omega}} \\ &\overset{(b)}{=} C\, |v_1|^2_{f^\star\omega} \ldots |v_{2n-3}|^2_{f^\star\omega}, \end{aligned}$$

where $C > 0$ is a constant independent of the v_j's that exists such that inequality (a) holds thanks to the $\widetilde{\omega}$-*boundedness* of Γ on $\widetilde{X}$, while (b) follows from $\tilde{f}^\star\widetilde{\omega} = f^\star\omega$. □

End of Proof of Theorem.

• On the one hand, we have $d\tau = 2t\, dt$ and

$$\mathrm{Vol}_{\omega,\,f}(B_r) = \int\limits_{B_r} f^\star\omega_{n-1} = \int\limits_0^r \Bigg(\int\limits_{S_t} d\mu_{\omega,\,f,\,t}\Bigg)\, dt = \int_{B_r} d\mu_{\omega,\,f,\,t} \wedge \frac{d\tau}{2t},$$

where $d\mu_{\omega,\,f,\,t}$ is the positive measure on S_t defined by

$$\frac{1}{2t}\, d\mu_{\omega,\,f,\,t} \wedge (d\tau)_{|S_t} = (f^\star\omega_{n-1})_{|S_t}, \qquad t > 0.$$

Thus, the measures $d\mu_{\omega,\,f,\,t}$ and $d\sigma_{\omega,\,f,\,t}$ on S_t are related by

$$\frac{1}{2t}\, d\mu_{\omega,\,f,\,t} = \frac{1}{|d\tau|_{f^\star\omega}}\, d\sigma_{\omega,\,f,\,t}, \qquad t > 0.$$

Now, the Hölder inequality yields:

$$\int_{S_t} \frac{1}{|d\tau|_{f^\star\omega}}\, d\sigma_{\omega,\,f,\,t} \geq \frac{A^2_{\omega,\,f}(S_t)}{\int_{S_t} |d\tau|_{f^\star\omega}\, d\sigma_{\omega,\,f,\,t}}.$$

This leads to:

$$\begin{aligned}\mathrm{Vol}_{\omega,\,f}(B_r) &= \int_0^r \left(\int_{S_t} \frac{1}{2t}\, d\mu_{\omega,\,f,\,t} \right) d\tau = \int_0^r \left(\int_{S_t} \frac{1}{|d\tau|_{f^\star\omega}}\, d\sigma_{\omega,\,f,\,t} \right) d\tau \\ &\geq 2 \int_0^r \frac{A^2_{\omega,\,f}(S_t)}{\int_{S_t} |d\tau|_{f^\star\omega}\, d\sigma_{\omega,\,f,\,t}}\, t\, dt, \qquad r > 0.\end{aligned}$$

• On the other hand, for every $r > 0$, we have:

$$\begin{aligned} \mathrm{Vol}_{\omega,\, f}(B_r) \;&=\; \int\limits_{B_r} f^\star \omega_{n-1} = \int\limits_{B_r} d(\tilde{f}^\star \Gamma) = \int\limits_{S_r} \tilde{f}^\star \Gamma \\ &\overset{(a)}{\leq} \; C \int\limits_{S_r} d\sigma_{\omega,\, f} = C\, A_{\omega,\, f}(S_r), \end{aligned}$$

where $C > 0$ is a constant that exists such that inequality (a) holds thanks to the boundedness of $\tilde{f}^\star \Gamma$.

Putting the above inequalities together, we get for every $r > r_0$:

$$\begin{aligned} \mathrm{Vol}_{\omega,\,f}(B_r) \;&\geq\; \frac{2}{C^2}\int\limits_0^r \mathrm{Vol}_{\omega,\,f}(B_t)\,\frac{t\,\mathrm{Vol}_{\omega,\,f}(B_t)}{\int_{S_t}|d\tau|_{f^\star\omega}\,d\sigma_{\omega,\,f,\,t}}\,dt \\ &\stackrel{(a)}{\geq}\; \frac{2}{C_1\,C^2}\int\limits_{r_0}^r \mathrm{Vol}_{\omega,\,f}(B_t)\,dt \stackrel{(b)}{:=} C_2\,F(r), \end{aligned}$$

where (a) follows from assumption (i) in the subexponential growth condition and (b) is the definition of a function $F : (r_0,\,+\infty) \longrightarrow (0,\,+\infty)$ with $C_2 := 2/(C_1\,C^2)$.

Deriving F, we get for every $r > 0$:

$$F'(r) = \mathrm{Vol}_{\omega,\, f}(B_r) \geq C_2\, F(r).$$

This amounts to

$$\frac{d}{dt}\Big(\log F(t)\Big) \geq C_2, \qquad t > r_0.$$

Integrating this over $t \in [a,\, b]$, with $r_0 < a < b$ arbitrary, we get:

$$-\log F(a) \geq -\log F(b) + C_2\,(b - a), \qquad r_0 < a < b.$$

Now, fix an arbitrary $a > r_0$ and let $b \to +\infty$. Thanks to the *subexponential growth* assumption made on f, there exists a sequence of reals $b_j \to +\infty$ such that the right-hand side of the above inequality for $b = b_j$ tends to $+\infty$ as $j \to +\infty$.

This forces $F(a) = 0$ for every $a > 0$, hence

$$\mathrm{Vol}_{\omega,\, f}(B_r) \ \left(= \int\limits_{B_r} f^\star \omega_{n-1} \right) = 0$$

for every $r > r_0$. This amounts to $f^\star \omega^{n-1} = 0$ on $\mathbb{C}^{n-1}$, contradicting the non-degeneracy assumption on f. $\square$

Examples

(1) Consider the semi-simple complex Lie group $G = SL(2, \mathbb{C})$. Its complex structure is described by three holomorphic (1, 0)-forms α, β, γ that satisfy the structure equations:

$$d\alpha = \beta \wedge \gamma, \qquad d\beta = \gamma \wedge \alpha, \qquad d\gamma = \alpha \wedge \beta.$$

Moreover, the dual of the Lie algebra $\mathfrak{g} = T_eG$ of G is generated, as an $\mathbb{R}$-vector space, by these forms and their conjugates:

$$(T_eG)^\star = \langle \alpha,\ \beta,\ \gamma,\ \overline{\alpha},\ \overline{\beta},\ \overline{\gamma} \rangle.$$

The C^∞ positive definite (1, 1)-form

$$\omega := \frac{i}{2}\alpha\wedge\overline{\alpha} + \frac{i}{2}\beta\wedge\overline{\beta} + \frac{i}{2}\gamma\wedge\overline{\gamma}$$

defines a left-invariant (under the action of G on itself) Hermitian metric on G. We get

$$\omega^2 = \frac{1}{2}\,d(\alpha\wedge d\overline{\alpha} + \beta\wedge d\overline{\beta} + \gamma\wedge d\overline{\gamma}) \in \operatorname{Im} d.$$

So, ω is a *degenerate balanced* metric on G.

Since it is left-invariant under the G-action, ω descends to a *degenerate balanced metric* on the compact quotient $X = G/\Gamma$ of G by any lattice Γ. In particular, this example illustrates Yachou's result in the special case of $G = SL(2, \mathbb{C})$.

Now, consider the holomorphic map

$$f : \mathbb{C}^2 \to G = SL(2, \mathbb{C}), \qquad f(z_1, z_2) = \begin{pmatrix} e^{z_1} & z_2 \\ 0 & e^{-z_1} \end{pmatrix}.$$

This map is non-degenerate at every point $z = (z_1, z_2) \in \mathbb{C}^2$, as can be seen at once.

However, f is *not of subexponential growth*, as can be checked.

Actually, there is no non-degenerate holomorphic map $g : \mathbb{C}^2 \to X = G/\Gamma$ of subexponential growth thanks to X being *degenerate balanced* (hence also *balanced hyperbolic*) and to our above theorem.

(2) Any *complex torus*

$$X = \mathbb{C}^n/\Gamma,$$

where $\Gamma \subset (\mathbb{C}^n, +)$ is any lattice, is *not divisorially hyperbolic*.

Reason. Any Hermitian metric with constant coefficients on $\mathbb{C}^n$ (for example, the Euclidean metric $\beta = (1/2)\sum_j idz_j \wedge d\bar{z}_j$) defines a Kähler metric ω on X:

$$\pi^\star\omega = \beta,$$

where $\pi : \mathbb{C}^n \to X$ is the projection. If $j : \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^n$ is the obvious inclusion $(z_1, \ldots, , z_{n-1}) \mapsto (z_1, \ldots, , z_n)$, the non-degenerate holomorphic map $f = \pi \circ j : \mathbb{C}^{n-1} \to X$ has subexponential growth because $f^\star\omega = j^\star\beta = \beta_0$, where β_0 is the Euclidean metric of $\mathbb{C}^{n-1}$.

(**3**) The *Iwasawa manifold* $X = G/\Gamma$ is *not divisorially hyperbolic*, where $G = (\mathbb{C}^3, \star)$ is the nilpotent complex Lie group (called the Heisenberg group) whose group operation is defined as

$$(\zeta_1, \zeta_2, \zeta_3) \star (z_1, z_2, z_3) = (\zeta_1 + z_1, \zeta_2 + z_2, \zeta_3 + z_3 + \zeta_1 z_2),$$

while the lattice $\Gamma \subset G$ consists of the elements $(z_1, z_2, z_3) \in G$ with $z_1, z_2, z_3 \in \mathbb{Z}[i]$.

Reason. There is an explicit Hermitian metric ω_0 on X that lifts to the Hermitian metric

$$\begin{aligned} \omega &= \pi^\star \omega_0 \\ &= i dz_1 \wedge d\bar{z}_1 + (1 + |z_1|^2)\, i dz_2 \wedge d\bar{z}_2 + i dz_3 \wedge d\bar{z}_3 \\ &- \bar{z}_1\, i dz_3 \wedge d\bar{z}_2 - z_1\, i dz_2 \wedge d\bar{z}_3 \end{aligned}$$

on $G = \mathbb{C}^3$, where $\pi : G \to X$ is the projection.

Considering the non-degenerate holomorphic map $f = \pi \circ j : \mathbb{C}^2 \longrightarrow X$, where $j : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$ is the obvious inclusion $(z_1,\, z_2) \mapsto (z_1,\, z_2,\, 0)$, we get

$$f^\star\omega_0 = j^\star\omega = \omega_{|\mathbb{C}^2} = i dz_1 \wedge d\bar{z}_1 + (1 + |z_1|^2)\, i dz_2 \wedge d\bar{z}_2$$

on $\mathbb{C}^2$. Hence,

$$f^\star\omega_0^2 = 2(1 + |z_1|^2)\, dV_0$$

on $\mathbb{C}^2$, where we put $dV_0 := i dz_1 \wedge d\bar{z}_1 \wedge i dz_2 \wedge d\bar{z}_2$.. Thus, for the ball $B_r \subset \mathbb{C}^2$ of radius r centred at 0, we get

$$\mathrm{Vol}_{\omega_0,\, f}(B_r) = \frac{1}{2}\int\limits_{B_r} f^\star\omega_0^2 = \int\limits_{B_r} (1 + |z_1|^2)\, dV_0 \leq c_2\, r^4(1 + r^2), \qquad r > 0,$$

where $c_2 > 0$ is a constant independent of r. This shows that f is of *finite order*, hence f satisfies property (ii) in the definition of the subexponential growth condition. It also satisfies property (i).

(4) No *Nakamura manifold* $X = G/\Gamma$ is divisorially hyperbolic, where $G = (\mathbb{C}^3, \star)$ is the solvable, non-nilpotent complex Lie group whose group operation is defined as

$$(\zeta_1, \zeta_2, \zeta_3) \star (z_1, z_2, z_3) = (\zeta_1 + z_1, \zeta_2 + e^{-\zeta_1} z_2, \zeta_3 + e^{\zeta_1} z_3),$$

while $\Gamma \subset G$ is a lattice.

(III) Positivity properties of balanced or divisorially hyperbolic manifolds

Question 4 **(Marouani-P. 2021)** *Let X be a compact complex manifold. If X is balanced hyperbolic or merely divisorially hyperbolic, does X have any positivity property (e.g. at the level of K_X)?*

Recall

- **Bott-Chern** cohomology group:

$$H^{p,\,q}_{BC}(X, \mathbb{C}) := \frac{\ker \partial \cap \ker \bar\partial}{\mathrm{Im}\,(\partial\bar\partial)} \qquad \text{(depends on the complex structure)}$$

- **Aeppli** cohomology group:

$$H^{p,\,q}_{A}(X, \mathbb{C}) := \frac{\ker(\partial\bar\partial)}{\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial} \qquad \text{(depends on the complex structure)}$$

We will use the Serre-type duality:

$$H^{1,\,1}_{BC}(X,\, \mathbb{C}) \times H^{n-1,\,n-1}_{A}(X,\, \mathbb{C}) \longrightarrow \mathbb{C},$$

$$(\{u\}_{BC},\, \{v\}_A) \mapsto \{u\}_{BC}.\{v\}_A := \int\limits_X u \wedge v,$$

as well as:

- the *Gauduchon cone* $\mathcal{G}_X$ of X (P. 2015)

$$\mathcal{G}_X := \left\{\{\omega^{n-1}\}_A \in H_A^{n-1,\,n-1}(X,\,\mathbb{R}) \mid \omega \text{ is a Gauduchon metric on } X\right\}$$
$$\subset H_A^{n-1,\,n-1}(X,\,\mathbb{R})$$

-the *strongly Gauduchon (sG) cone* $\mathcal{SG}_X$ of X (P. 2015)

$$\mathcal{SG}_X := \left\{\{\omega^{n-1}\}_A \in H_A^{n-1,\,n-1}(X,\,\mathbb{R}) \mid \omega \text{ is an sG metric on } X\right\}$$
$$\subset H_A^{n-1,\,n-1}(X,\,\mathbb{R}).$$

Recall: a Hermitian metric ω on X is said to be a:

-*Gauduchon metric* (Gauduchon 1077) if

$$\partial\bar\partial\omega^{n-1} = 0$$

-*strongly Gauduchon (sG) metric* (P. 2013) if

$$\partial\omega^{n-1} \in \mathrm{Im}\,\bar\partial.$$

Obviously,

$$\mathcal{SG}_X \subset \mathcal{G}_X.$$

Original observation.

Let X be a compact complex manifold with $dim_{\mathbb{C}} X = n$. The map:

$$P = P_{n-1,\,n-1}^{n-1} : H^2_{DR}(X,\,\mathbb{R}) \longrightarrow H_A^{n-1,\,n-1}(X,\,\mathbb{R})$$
$$\{\alpha\}_{DR} \longmapsto \{(\alpha^{n-1})^{n-1,\,n-1}\}_A,$$

is **well defined** *in the sense that it is independent of the choice of a C^∞ representative α of its De Rham cohomology class, where $(\alpha^{n-1})^{n-1,\,n-1}$ is the component of bidegree $(n-1,\, n-1)$ of the $(2n-2)$-form α^{n-1}.*

Definition (Marouani-P. 2021)

Let $\{\alpha\} \in H^2_{DR}(X, \mathbb{R})$ *be a real De Rham cohomology class (not necessarily of type* $(1, 1)$*).*

$\{\alpha\}$ *is* **divisorially Kähler** $\stackrel{def}{\Longleftrightarrow}$ $P(\{\alpha\}) \in \mathcal{G}_X$

$\{\alpha\}$ *is* **divisorially nef** $\stackrel{def}{\Longleftrightarrow}$ $P(\{\alpha\}) \in \overline{\mathcal{G}}_X$

(the closure of the Gauduchon cone)

A C^∞ *complex line bundle* L *on* X *is* **divisorially nef** $\stackrel{def}{\Longleftrightarrow}$ *its first Chern class* $c_1(L)$ *is divisorially nef.*

Examples of results.

Theorem (Marouani-P. 2021)

Let L be a holomorphic line bundle on an n-dimensional **projective** *manifold X. The following equivalence holds:*

L *is* **divisorially nef** $\quad \Longrightarrow \quad L^{n-1}.D \geq 0$ *for all effective divisors*

where

$$L^{n-1}.D := \int_D \left(\frac{i}{2\pi}\Theta_h(L)\right)^{n-1}$$

and $(i/2\pi)\,\Theta_h(L)$ is the curvature form of L with respect to any Hermitian fibre metric h.

This is the divisorial analogue of the classical nefness property on projective manifolds X:

$$L \text{ is nef } \iff L.C \geq 0 \text{ for every curve } C \subset X.$$

Theorem (Marouani-P. 2021)

A class $\{\alpha\}_{DR} \in H^2_{DR}(X,\, \mathbb{R})$ *is* **divisorially nef**

$\iff$

for every constant $\varepsilon > 0$, *there exists a representative* $\Omega_\varepsilon \in C^\infty_{n-1,\, n-1}(X,\, \mathbb{R})$ *of the class* $P(\{\alpha\}_{DR})$ *such that*

$$\Omega_\varepsilon \geq -\varepsilon\, \omega^{n-1},$$

where $\omega > 0$ *is any pregiven Hermitian metric on* X.

Question 5 **(Marouani-P. 2021)** *Let X be a compact complex manifold.*

X *is* **balanced hyperbolic** *or* **divisorially hyperbolic**

$$\overset{?}{\Longrightarrow}$$

K_X *is* **divisorially nef** *or* **divisorially Kähler**

(IV) Properties of balanced hyperbolic manifolds

(1) A **Hard Lefschetz**-type theorem

Theorem (Marouani-P. 2021)

Let X be a compact complex manifold with $dim_{\mathbb{C}} X = n$.

(i) If ω is a **balanced** *metric on X, the linear map:*

$$\{\omega_{n-1}\}_{DR} \wedge \cdot : H^1_{DR}(X, \mathbb{C}) \longrightarrow H^{2n-1}_{DR}(X, \mathbb{C}),$$

$$\{u\}_{DR} \longmapsto \{\omega_{n-1} \wedge u\}_{DR},$$

is well defined and depends only on the cohomology class $\{\omega_{n-1}\}_{DR} \in H^{2n-2}_{DR}(X, \mathbb{C})$.

(ii) If, moreover, X has the following additional $\partial\bar\partial$-type property: for every form $v \in C^\infty_{1,\,1}(X,\,\mathbb{C})$ such that $dv = 0$, the following implication holds:

$$(\star) \qquad\qquad v \in Im\,\partial \implies v \in Im\,(\partial\bar\partial),$$

the above map is an **isomorphism**.

As a consequence of this discussion, we obtain the following vanishing properties for the cohomology of degenerate balanced manifolds.

Proposition (Marouani-P. 2021) *Let X be a compact* **degenerate balanced** *manifold.*

(i) The Bott-Chern cohomology groups of types $(1,\,0)$ *and* $(0,\,1)$ *of X vanish:* $H^{1,\,0}_{BC}(X,\,\mathbb{C}) = 0$ *and* $H^{0,\,1}_{BC}(X,\,\mathbb{C}) = 0$.

(ii) If, moreover, X satisfies hypothesis $(\star)$, *its De Rham cohomology group of degree* 1 *vanishes:* $H^1_{DR}(X,\,\mathbb{C}) = 0$.

(2) In the L^2 setting of the universal cover $\widetilde{X}$, our main result in degree 1 and its dual degree $2n-1$ is

Theorem (Marouani-P. 2021) *Let X be a compact complex* **balanced hyperbolic** *manifold with $dim_{\mathbb{C}}X = n$.*
Let $\pi : \widetilde{X} \longrightarrow X$ be the universal cover of X and $\widetilde{\omega} := \pi^\star\omega$ the lift to $\widetilde{X}$ of a balanced hyperbolic metric ω on X.

There are no non-zero $\Delta_{\widetilde{\omega}}$-harmonic $L^2_{\widetilde{\omega}}$-forms of pure types and of degrees 1 and $2n-1$ on $\widetilde{X}$:

$$\mathcal{H}^{1,\,0}_{\Delta_{\widetilde{\omega}}}(\widetilde{X},\,\mathbb{C}) = \mathcal{H}^{0,\,1}_{\Delta_{\widetilde{\omega}}}(\widetilde{X},\,\mathbb{C}) = 0 \quad \text{and} \quad \mathcal{H}^{n,\,n-1}_{\Delta_{\widetilde{\omega}}}(\widetilde{X},\,\mathbb{C}) = \mathcal{H}^{n-1,\,n}_{\Delta_{\widetilde{\omega}}}(\widetilde{X},\,\mathbb{C})$$

where $\Delta_{\widetilde{\omega}} := dd^\star_{\widetilde{\omega}} + d^\star_{\widetilde{\omega}}d$ is the d-Laplacian induced by the metric $\widetilde{\omega}$.

In the same L^2 setting of the universal cover $\widetilde{X}$, our main result in degree 2 is

Theorem (Marouani-P. 2021) *Let X be a compact complex* **balanced hyperbolic** *manifold with $\dim_{\mathbb{C}} X = n$.*
Let $\pi : \widetilde{X} \longrightarrow X$ be the universal cover of X and $\widetilde{\omega} := \pi^\star\omega$ the lift to $\widetilde{X}$ of a balanced hyperbolic metric ω on X.
There are no non-zero semi-positive $\Delta_{\widetilde{\tau}}$-harmonic $L^2_{\widetilde{\omega}}$-forms of pure type $(1, 1)$ on $\widetilde{X}$:

$$\left\{\alpha^{1,\,1} \in \mathcal{H}^{1,\,1}_{\Delta_{\widetilde{\tau}}}(\widetilde{X},\,\mathbb{C}) \mid \alpha^{1,\,1} \geq 0\right\} = \{0\},$$

where $\widetilde{\tau} = \widetilde{\tau}_{\widetilde{\omega}} := [\Lambda_{\widetilde{\omega}},\,\partial\widetilde{\omega} \wedge \cdot]$ and $\Delta_{\widetilde{\tau}} := [d + \widetilde{\tau},\,d^\star + \widetilde{\tau}^\star]$.