Non-Kähler Mirror Symmetry of the Iwasawa Manifold

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(I) Standard point of view (Kähler case)

Calabi-Yau (C-Y) threefolds:

compact Kähler manifolds X with $\dim_{\mathbb{C}} X = 3$ such that K_X is *trivial*

Frequent extra assumption (justified by the Bogomolov Decomposition Theorem for C-Y manifolds):

$$h^{1,0}(X) = h^{2,0}(X) = 0$$

This assumption implies that X is **projective**.

(1) **Complex-structure side of the mirror**

Bogomolov-Tian-Todorov Theorem: in this case, the local universal family $(X_t)_{t\in\Delta}$ of deformations of X (= the Kuranishi family of X) is **unobstructed**, i.e.

Δ is **smooth**, so can be viewed as an **open ball**

$$\Delta := \mathrm{Def}(X) \subset H^{0,\,1}(X,\, T^{1,\,0}X).$$

Recall the isomorphisms

$$T_0^{1,0}\Delta \xrightarrow[\simeq]{\rho} H^{0,1}(X, T^{1,0}X) \xrightarrow[\simeq]{T_\Omega} H^{2,1}(X, \mathbb{C}),$$
$$[\theta]_{\bar\partial} \longmapsto [\theta \lrcorner \Omega]_{\bar\partial}$$

where ρ is the **Kodaira-Spencer map** and T is the **Calabi-Yau isomorphism** defined by a fixed holomorphic non-vanishing 3-form

$$\Omega \in H^0(X, K_X) \simeq H^{3,0}(X, \mathbb{C}).$$

Ω is unique up to a multiplicative constant and represents the triviality of K_X (the C-Y structure)

(2) **Metric (Kähler) side of the mirror**

The **Kähler cone** of X:

$$\mathcal{K}_X := \left\{ [\omega] \in H^{1,1}(X, \mathbb{R}) \mid \omega \text{ Kähler metric on } X \right\}$$

the set of all Kähler classes on X

$\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ is an *open convex cone*

The Kähler cone is complexified to

$$\widetilde{\mathcal{K}}_X \subset H^{1,1}(X, \mathbb{C})$$

(the *Kähler moduli space* of X)

to match the *complex moduli space* $\Delta \subset H^{2,1}(X, \mathbb{C})$.

Recall: $H^{1,1}(X, \mathbb{C}) = H^2_{DR}(X, \mathbb{C})$ if we assume $h^{2,0}(X) = 0$, so $\mathcal{K}_X$ is an open convex cone in $H^2_{DR}(X, \mathbb{R})$. Its *complexification* is defined as

$$\widetilde{\mathcal{K}}_X = \mathcal{K}_X \oplus H^2(X, \mathbb{R})/2\pi i\, H^2(X, \mathbb{Z})$$

(1) + (2) **The Mirror Conjecture**

The Kähler C-Y threefolds ought to come in pairs $(X,\, \widetilde{X})$ such that there are local biholomorphisms (= *the mirror maps*)

$$\mathrm{Def}(X) \simeq \widetilde{\mathcal{K}}_{\widetilde{X}} \quad \text{and} \quad \mathrm{Def}(\widetilde{X}) \simeq \widetilde{\mathcal{K}}_{X}.$$

Obvious necessary condition

$$T_0^{1,\,0}\mathrm{Def}(X) \simeq T_{[\widetilde{\omega}_0]}^{1,\,0}\widetilde{\mathcal{K}}_{\widetilde{X}} \text{ and } T_0^{1,\,0}\mathrm{Def}(\widetilde{X}) \simeq T_{[\omega_0]}^{1,\,0}\widetilde{\mathcal{K}}_{X},$$

which means

$$H^{2,\,1}(X) \simeq H^{1,\,1}(\widetilde{X}) \quad \text{and} \quad H^{2,\,1}(\widetilde{X}) \simeq H^{1,\,1}(X),$$

hence

$$h^{2,\,1}(X) = h^{1,\,1}(\widetilde{X}) \quad \text{and} \quad h^{2,\,1}(\widetilde{X}) = h^{1,\,1}(X).$$

Obstruction to an all-Kähler mirror symmetry

There exist Kähler C-Y threefolds X such that

$$h^{2,1}(X) = 0 \qquad \text{(i.e. } X \text{ is } \textit{rigid}\text{, it does not deform).}$$

Hence, if X admits a mirror dual $\widetilde{X}$, the mirror dual cannot be Kähler since $h^{1,1}(\widetilde{X}) = 0$ in this case.

Conclusion: the mirror symmetry cannot hold entirely within the Kähler realm.

Another classical feature of the Kähler (projective) mirror symmetry:

use of *Gromov-Witten invariants* attached to pseudo-holomorphic curves and counting of *rational curves*

But what if there are no curves at all?

(II) Our new point of view (possibly non-Kähler case)

X a compact complex possibly non-Kähler manifold such that K_X is trivial (still called a **Calabi-Yau manifold**) , $\dim_{\mathbb{C}} X = n$

Recall: a **Hermitian metric** on X is any C^∞ positive definite (1, 1)-form $\omega > 0$ on X.

• ω is a **Gauduchon metric** if

$$\partial\bar\partial\omega^{n-1} = 0$$

Gauduchon metrics always exist (Gauduchon 1977).

• ω is a **strongly Gauduchon (sG) metric** (P. 2013) if

$$\partial\omega^{n-1} \text{ is } \bar\partial\text{-exact.}$$

sG metrics need not exist although they do on many manifolds.

Recall: X is an **sGG manifold** (P.-Ugarte 2014) if

every Gauduchon metric on X is strongly Gauduchon.

• every $\partial\bar\partial$**-manifold** is **sGG**;

• the **Iwasawa manifold** and all its small deformations are **sGG** but are not $\partial\bar\partial$**-manifolds**.

Recall:

• if X is a $\partial\bar{\partial}$**-manifold**, the Hodge decomposition and symmetry hold on X;

• if X is a $\partial\bar{\partial}$**-manifold** and if K_X is trivial, the Bogomolov-Tian-Todorov theorem still holds on X (P.2013), i.e.

the Kuranishi family of X is unobstructed.

• Recall the **Bott-Chern cohomology**

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) = \frac{\ker\partial\cap\ker\bar\partial}{\operatorname{Im}\partial\bar\partial}$$

and the **Aeppli cohomology**

$$H^{p,\,q}_{A}(X,\,\mathbb{C}) = \frac{\ker\partial\bar\partial}{\operatorname{Im}\partial+\operatorname{Im}\bar\partial}$$

There is a canonical non-degenerate **duality**

$$H^{1,\,1}_{BC}(X,\,\mathbb{C})\,\times\,H^{n-1,\,n-1}_{A}(X,\,\mathbb{C})\to\mathbb{C},$$
$$([\alpha]_{BC},\qquad[\beta]_A)\mapsto\int\limits_X\alpha\wedge\beta.$$

One of our main tools

The **Gauduchon cone** (P. 2013) of X:

$$\mathcal{G}_X := \left\{[\omega^{n-1}]_A \in H_A^{n-1,\, n-1}(X,\, \mathbb{R}) \right.$$

$$\left. \mid \quad \omega \text{ is a Gauduchon metric on } X \right\}$$

$$\mathcal{G}_X \subset H_A^{n-1,\, n-1}(X,\, \mathbb{R})$$

is an open convex cone in $H_A^{n-1,\, n-1}(X,\, \mathbb{R})$.

- $\mathcal{G}_X$ replaces the Kähler cone $\mathcal{K}_X$ when X is non-Kähler

- $\mathcal{G}_X$ provides a transcendental substitute for cohomology classes of (currents of integration on) curves.

Our testing ground

The **Iwasawa manifold:** is the quotient $X = G/\Gamma$, where

$$G := \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \; ; \; z_1, z_2, z_3 \in \mathbb{C} \right\} \subset GL_3(\mathbb{C})$$

is the **Heisenberg group** and $\Gamma \subset G$ is the subgroup of matrices with entries $z_1, z_2, z_3 \in \mathbb{Z}[i]$.

• The map

$$(z_1, z_2, z_3) \mapsto (z_1, z_2)$$

factors through the action of Γ to a (holomorphically locally trivial) proper holomorphic submersion

$$\pi : X \to B,$$

where the base $B = \mathbb{C}^2/\mathbb{Z}[i] \oplus \mathbb{Z}[i] = \mathbb{C}/\mathbb{Z}[i] \times \mathbb{C}/\mathbb{Z}[i]$ is a two-dimensional Abelian variety (the product of two elliptic curves) and where all the fibres are isomorphic to the Gauss elliptic curve $\mathbb{C}/\mathbb{Z}[i]$.

Consequence 1

There exist no curves normalised by smooth rational curves on X.

Reason: any map from such a curve to any factor $\mathbb{C}/\mathbb{Z}[i]$ would be constant.

(Indeed, thanks to the Riemann-Hurwitz formula, any non-constant map between two smooth curves is genus-decreasing.)

Consequence 2

There exist three holomorphic 1*-forms* $\alpha, \beta, \gamma \in C^{\infty}_{1,\,0}(X,\, \mathbb{C})$ *on* X *such that*

$$d\alpha = d\beta = 0 \quad \textit{and} \quad d\gamma = \partial\gamma = -\alpha \wedge \beta \neq 0.$$

The forms α, β, γ *explicitly determine the whole cohomology of* X*.*

Reason: the $\mathbb{C}^3$-valued holomorphic 1-form on G

$$G \ni M = \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto M^{-1}\, dM = \begin{pmatrix} 0 & dz_1 & dz_3 - z_1\, dz_2 \\ 0 & 0 & dz_2 \\ 0 & 0 & 0 \end{pmatrix}$$

is invariant under the action of Γ.

Hence it descends to a holomorphic 1-form on X giving rise to the (1, 0)-forms α, β, γ on X induced respectively by the forms

$$dz_1,\ dz_2,\ dz_3 - z_1 dz_2$$

of $\mathbb{C}^3$. Thus,

$$\alpha, \beta, \gamma \in C^\infty_{1,\,0}(X,\, \mathbb{C})$$

and

$$\bar\partial\alpha = \bar\partial\beta = \bar\partial\gamma = 0.$$

Note that dz_1, dz_2 are closed and $d(dz_3 - z_1 dz_2) = -dz_1 \wedge dz_2$.

Vertical and horizontal forms

From the exact sequence

$$0 \to \pi^\star \Omega^1_B \to \Omega^1_X \to \Omega^1_{X/B} \to 0,$$

as the map $H^1(\pi^\star \Omega^1_B) = H^1(\mathcal{O}_X) \otimes H^0(\pi^\star \Omega^1_B) \to H^1(\mathcal{O}_X) \otimes H^0(\Omega^1_X) = H^1(\Omega^1_X)$ is injective due to the triviality of Ω^1_B and Ω^1_X,

we get the simple presentation

$$0 \to H^0(\pi^\star \Omega^1_B) \to H^0(\Omega^1_X) \to H^0(\Omega^1_{X/B}) \to 0.$$

Thus, the form γ is a representative of $H^0(\Omega^1_{X/B})$ in $H^0(\Omega^1_X)$. In other words, the forms α and β are *horizontal* (i.e. coming from B), while γ is *vertical* (i.e. lives on the fibres).

Examples of cohomology groups

- **De Rham cohomology**

$$
\begin{aligned}
H^1(X,\mathbb{C}) &= \Big\langle \{\alpha\}_{DR},\ \{\beta\}_{DR}, \{\bar\alpha\}_{DR},\ \{\bar\beta\}_{DR} \Big\rangle = \pi^\star H^1(B,\mathbb{C}),\\
\pi^\star H^2(B,\mathbb{C}) &= \Big\langle \{\alpha\wedge\bar\alpha\}_{DR},\ \{\alpha\wedge\bar\beta\}_{DR}, \{\beta\wedge\bar\alpha\}_{DR},\ \{\beta\wedge\bar\beta\}_{DR} \Big\rangle \simeq H^{1,1}_{BC}(X,\,\mathbb{C})\\
&\simeq \pi^\star H^{1,1}(B,\mathbb{C})\\
H^2(X,\mathbb{C}) &= \pi^\star H^2(B,\mathbb{C}) \oplus \Big\langle \{\gamma\wedge\alpha\}_{DR},\ \{\gamma\wedge\beta\}_{DR} \Big\rangle \oplus \Big\langle \{\bar\gamma\wedge\bar\alpha\}_{DR},\ \{\bar\gamma\wedge\bar\beta\}_{DR} \Big\rangle,\\
\pi^\star H^3(B,\mathbb{C}) &= 0,\\
H^3(X,\mathbb{C}) &= \Big\langle \{\alpha\wedge\beta\wedge\gamma\}_{DR} \Big\rangle \oplus \{\gamma\wedge\pi^\star H^{1,1}(B,\mathbb{C})\}_{DR} \oplus \{\bar\gamma\wedge\pi^\star H^{1,1}(B,\mathbb{C})\}_{DR}\\
&\oplus \Big\langle \{\bar\alpha\wedge\bar\beta\wedge\bar\gamma\}_{DR} \Big\rangle.
\end{aligned}
$$

• Other cohomologies

$$H^{1,0}_{\bar\partial}(X,\mathbb{C}) = \left\langle [\alpha]_{\bar\partial},\, [\beta]_{\bar\partial},\, [\gamma]_{\bar\partial} \right\rangle, \quad H^{0,1}_{\bar\partial}(X,\mathbb{C}) = \left\langle [\overline{\alpha}]_{\bar\partial},\, [\overline{\beta}]_{\bar\partial} \right\rangle = \pi^{\star} H^{0,1}_{\bar\partial}(B,\mathbb{C}),$$

$$H^{3,0}_{\bar\partial}(X,\mathbb{C}) = \left\langle [\alpha\wedge\beta\wedge\gamma]_{\bar\partial} \right\rangle, \quad H^{0,3}_{\bar\partial}(X,\mathbb{C}) = \left\langle [\overline{\alpha}\wedge\overline{\beta}\wedge\overline{\gamma}]_{\bar\partial} \right\rangle,$$

$$\begin{aligned} H^{2,1}_{\bar\partial}(X,\,\mathbb{C}) &= \left\langle [\alpha\wedge\gamma\wedge\overline{\alpha}]_{\bar\partial},\, [\alpha\wedge\gamma\wedge\overline{\beta}]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline{\alpha}]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline{\beta}]_{\bar\partial} \right\rangle \\ &\oplus \left\langle [\alpha\wedge\beta\wedge\overline{\alpha}]_{\bar\partial},\, [\alpha\wedge\beta\wedge\overline{\beta}]_{\bar\partial} \right\rangle \\ &= \left\langle [\alpha\wedge\gamma\wedge\overline{\alpha}]_{\bar\partial},\, [\alpha\wedge\gamma\wedge\overline{\beta}]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline{\alpha}]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline{\beta}]_{\bar\partial} \right\rangle \oplus \pi^{\star} H^{2,1}_{\bar\partial}(B,\,\mathbb{C}), \\ H^{1,2}_{\bar\partial}(X,\,\mathbb{C}) &= \left\langle [\alpha\wedge\overline{\alpha}\wedge\overline{\gamma}]_{\bar\partial},\, [\beta\wedge\overline{\alpha}\wedge\overline{\gamma}]_{\bar\partial},\, [\alpha\wedge\overline{\beta}\wedge\overline{\gamma}]_{\bar\partial},\, [\beta\wedge\overline{\beta}\wedge\overline{\gamma}]_{\bar\partial} \right\rangle \\ &\oplus \left\langle [\gamma\wedge\overline{\alpha}\wedge\overline{\gamma}]_{\bar\partial},\, [\gamma\wedge\overline{\beta}\wedge\overline{\gamma}]_{\bar\partial} \right\rangle. \end{aligned}$$

Known facts about the Iwasawa manifold X:

· X is a compact complex manifold, $\dim_{\mathbb{C}} X = 3$;

· X is a *nilmanifold* since

G is a connected, simply connected, *nilpotent* complex Lie group;

· $E_1(X) \neq E_\infty(X)$ in the Frölicher spectral sequence. In particular, X is not a $\partial\bar\partial$-manifold and very far from being Kähler.

However, $E_2(X) = E_\infty(X)$ in the Frölicher spectral sequence.

· X is *complex parallelisable* (i.e. $T^{1,0}X$ is trivial).

In particular, K_X is trivial, so X is C-Y.

· Nakamura (1975): the Kuranishi family $(X_t)_{t\in\Delta}$ of $X = X_0$ is **unobstructed**,

so Δ is **smooth**, so can be viewed as an **open ball**

$$\Delta := \mathrm{Def}(X) \subset H^{0,\,1}(X,\, T^{1,\,0}X) \simeq H^{2,\,1}(X,\, \mathbb{C}).$$

Moreover, $\dim_{\mathbb{C}}\Delta = 6$ and

$$\Delta = \{t = t_{i\lambda} \in \mathbb{C}^6 \mid |t| < \varepsilon; i = 1, 2, 3; \lambda = 1, 2\}.$$

· the X_t's with $t_{11} = t_{12} = t_{21} = t_{22} = 0$ are *complex paralellisable*;

· the X_t's with $t_{31} = t_{32} = 0$ are *not complex paralellisable*.

• there is no Hodge decomposition of weight 3 on X since $H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$ does not inject canonically into $H^3_{DR}(X,\,\mathbb{C})$.

In fact: $b_3 = 10$

while $h^{3,\,0} = h^{0,\,3} = 1$ and $h^{2,\,1} = h^{1,\,2} = 6$.

Thus $10 = b_3 < h^{3,\,0} + h^{2,\,1} + h^{1,\,2} + h^{0,\,3} = 14$.

So, in a sense, the vector space $H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$ is "too large" to fit into $H^3_{DR}(X,\,\mathbb{C})$.

Main Theorem

The Iwasawa manifold is its own mirror dual in the sense that its Kuranishi family "corresponds" to its Gauduchon cone.

Specifically, we define a **mirror map**

$$\widetilde{\mathcal{M}} : \Delta_{[\gamma]} \to \widetilde{\mathcal{G}}_X$$

that we prove to be a **local biholomorphism** and to define an **isomorphism of variations of Hodge structures (VHS)** parametrised respectively by $\Delta_{[\gamma]}$ and by $\widetilde{\mathcal{G}}_X$.

Terminology:

(i) $(X_t)_{t\in\Delta_{[\gamma]}}$ is the **local universal family of essential deformations** of the Iwasawa manifold $X = X_0$;

(ii) The **complexified Gauduchon cone** is defined as

$$\widetilde{\mathcal{G}}_X := \mathcal{G}_X \oplus H^{2,\,2}_A(X_0,\,\mathbb{R})/2\pi i\, H^{2,\,2}_A(X_0,\,\mathbb{Z}),$$

where $H^{2,\,2}_A(X_0,\,\mathbb{Z})$ is defined beforehand as a **lattice** in $H^{2,\,2}_A(X_0,\,\mathbb{R})$.

Piece of initial evidence for this mirror symmetry

There exists a canonical isomorphism

$$T_0^{1,\,0}\Delta_{[\gamma]} = H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) \simeq H^{2,\,2}_A(X,\,\mathbb{C}) = T^{1,\,0}_{[\omega_0^2]}\widetilde{\mathcal{G}}_X.$$

Notation

$\cdot\ \omega_0 := i\alpha\wedge\overline{\alpha} + i\beta\wedge\overline{\beta} + i\gamma\wedge\overline{\gamma} > 0$

is a Hermitian metric on $X = X_0$ that we show to be **Gauduchon**.

Reason

There are explicit descriptions

$$H^{2,1}_{[\gamma]}(X, \mathbb{C}) = \left\langle [\alpha \wedge \gamma \wedge \overline{\alpha}]_{\bar\partial}, [\alpha \wedge \gamma \wedge \overline{\beta}]_{\bar\partial}, [\beta \wedge \gamma \wedge \overline{\alpha}]_{\bar\partial}, [\beta \wedge \gamma \wedge \overline{\beta}]_{\bar\partial} \right\rangle$$

and

$$H^{2,2}_{A}(X, \mathbb{C}) = \left\langle [\alpha \wedge \gamma \wedge \bar\alpha \wedge \bar\gamma]_A, [\alpha \wedge \gamma \wedge \bar\beta \wedge \bar\gamma]_A, [\beta \wedge \gamma \wedge \bar\alpha \wedge \bar\gamma]_A, [\beta \wedge \gamma \wedge \bar\beta \wedge \bar\gamma]_A \right\rangle$$

that imply the isomorphism $H^{2,1}_{[\gamma]}(X, \mathbb{C}) \simeq H^{2,2}_{A}(X, \mathbb{C})$.

Some steps in the construction of the mirror map

(1) Complex-structure side of the mirror

- **The essential deformations of X**

$H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$ is "pared down" to a 4-dimensional vector subspace

$$\gamma \wedge \pi^{\star} H^{1,\,1}(B,\,\mathbb{C}) = H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) \subset H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$$

that injects canonically into $H^{3}_{DR}(X,\,\mathbb{C})$ and parametrises what we call the *essential deformations* of X.

Specifically, $T_0^{1,\,0}\Delta \xrightarrow[\simeq]{\rho} H^{0,\,1}(X,\,T^{1,\,0}X)$ (Kodaira-Spencer map) and

(i) the map

$$H^{0,\,1}(X,\,T^{1,\,0}X) \xrightarrow{\cdot\lrcorner[\gamma]_{\bar\partial}} H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C}), \quad [\theta]\mapsto[\theta\lrcorner\gamma]_{\bar\partial},$$

is well defined. Its kernel

$$\begin{aligned} H^{0,\,1}_{[\gamma]}(X,\,T^{1,\,0}X) &= \left\{[\theta]\in H^{0,\,1}(X,\,T^{1,\,0}X)\Big/[\theta\lrcorner\gamma]=0\in H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})\right\} \\ &= \left\langle[\overline{\alpha}\otimes\xi_\alpha],\,[\overline{\alpha}\otimes\xi_\beta],\,[\overline{\beta}\otimes\xi_\alpha],\,[\overline{\beta}\otimes\xi_\beta]\right\rangle \end{aligned}$$

is the subspace of

$$H^{0,1}(X,\,T^{1,0}X) = \left\langle [\overline{\alpha}\otimes\xi_\alpha], [\overline{\alpha}\otimes\xi_\beta], [\overline{\alpha}\otimes\xi_\gamma], [\overline{\beta}\otimes\xi_\alpha], [\overline{\beta}\otimes\xi_\beta], [\overline{\beta}\otimes\xi_\gamma] \right\rangle$$

corresponding to the 1^{st}-order deformations of the base torus B (= the *horizontal 1^{st}-order deformations* of X).

We define

$$H^{2,1}_{[\gamma]}(X,\,\mathbb{C}) = \left\langle [\alpha\wedge\gamma\wedge\overline{\alpha}]_{\bar\partial},\, [\alpha\wedge\gamma\wedge\overline{\beta}]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline{\alpha}]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline{\beta}]_{\bar\partial} \right\rangle$$

as the image of $H^{0,1}_{[\gamma]}(X,\,T^{1,0}X)$ under the C-Y isomorphism

$$T_\Omega : H^{0,1}(X,\,T^{1,0}X) \to H^{2,1}_{\bar\partial}(X,\,\mathbb{C}).$$

(ii) Bearing in mind that $\Delta \subset H^{0,\,1}(X,\, T^{1,\,0}X)$ is an open subset, we put

$$\mathrm{EssDef}(X) = \Delta_{[\gamma]} := \Delta \cap H^{0,\,1}_{[\gamma]}(X,\, T^{1,\,0}X).$$

We may say that the family of deformations $(X_t)_{t\in\Delta_{[\gamma]}}$ is "polarised" by the $(1,\,0)$-class $[\gamma]_{\bar\partial} \in H^{1,\,0}_{\bar\partial}(X,\,\mathbb{C})$ by analogy with the standard case of a Kähler class $[\omega]$ on X_0. Recall

the fibres X_t polarised by $[\omega]$, i.e. the fibres X_t for which $[\omega]$ remains of J_t-type $(1,\,1)$, are precisely those corresponding to $[\theta] \in H^{0,\,1}(X_0,\, T^{1,\,0}X_0)$ satisfying the condition $[\theta\lrcorner\omega] = 0$ in $H^{0,\,2}(X_0,\,\mathbb{C})$

(iii) This operation is equivalent to removing from the Kuranishi family $(X_t)_{t\in\Delta}$ the two dimensions corresponding to complex parallelisable deformations X_t of X (that have a similar geometry to that of X, so no geometric information is lost) and we are left with a family

$$(X_t)_{t\in\Delta_{[\gamma]}}$$

of non-complex parallelisable deformations that we call *essential*.

• **Result**

Doing the analogous construction on every *essential deformation* X_t with $t \in \Delta_{[\gamma]}$, we get a Hodge decomposition of weight 3 for every $t \in \Delta_{[\gamma]}$ in the following form:

There exist canonical isomorphisms

$$H^3_{DR}(X,\,\mathbb{C}) \simeq H^{3,\,0}_{\bar\partial}(X_t,\,\mathbb{C}) \oplus H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C}) \oplus H^{1,\,2}_{[\gamma]}(X_t,\,\mathbb{C}) \oplus H^{0,\,3}_{\bar\partial}(X_t,\,\mathbb{C}),$$

and

$$H^{3,\,0}_{\bar\partial}(X_t,\,\mathbb{C}) \simeq \overline{H^{0,\,3}_{\bar\partial}(X_t,\,\mathbb{C})} \quad \text{and} \quad H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) \simeq \overline{H^{1,\,2}_{[\gamma]}(X,\,\mathbb{C})}.$$

Moreover, $\Delta_{[\gamma]} \ni t \mapsto H^{2,1}_{[\gamma]}(X_t, \mathbb{C})$

is a C^∞ vector bundle of rank 4. The vector subbundles of the constant bundle $\mathcal{H}^3 = (\Delta_{[\gamma]} \ni t \mapsto H^3_{DR}(X_t, \mathbb{C}) = H^3(X, \mathbb{C}))$

$$F^2\mathcal{H}^3 := \mathcal{H}^{3,0} \oplus \mathcal{H}^{2,1}_{[\gamma]} \supset F^3\mathcal{H}^3 := \mathcal{H}^{3,0}$$

are *holomorphic* and there is a *flat connection*

$$\nabla : \mathcal{H}^3 \longrightarrow \mathcal{H}^3 \otimes \Omega_{\Delta_{[\gamma]}}$$

(the Gauss-Manin connection) that satisfies the *Griffiths transversality condition*

$$\nabla F^3\mathcal{H}^3 \subset F^2\mathcal{H}^3 \otimes \Omega_{\Delta_{[\gamma]}}.$$

• Link with the Frölicher spectral sequence

There exists a canonical isomorphism

$$H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) \simeq E_2^{2,\,1}(X,\,\mathbb{C})$$

and analogous canonical isomorphisms

$$H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C}) \simeq E_2^{2,\,1}(X_t,\,\mathbb{C}), \qquad t \in \Delta_{[\gamma]}.$$

Recall (known fact):

$$E_2(X_t) = E_\infty(X_t), \qquad t \in \Delta_{[\gamma]},$$

in the Frölicher spectral sequence.

So our Hodge decomposition of weight 3 for the *essential deformations* of the Iwasawa manifold reflects precisely this E_2 degeneration since there exist isomorphisms

$$H^3_{DR}(X, \mathbb{C}) \simeq E_2^{3,0}(X_t, \mathbb{C}) \oplus E_2^{2,1}(X_t, \mathbb{C}) \oplus E_2^{1,2}(X_t, \mathbb{C}) \oplus E_2^{0,3}(X_t, \mathbb{C}).$$

for all $t \in \Delta_{[\gamma]}$.

(2) Metric side of the mirror

- **Construction of two families of Gauduchon metrics**

$\cdot\ (\omega_t)_{t\in\Delta_{[\gamma]}}$ a C^∞ family of Gauduchon metrics on the fibres $(X_t)_{t\in\Delta_{[\gamma]}}$;

$\cdot\ (\omega_t^{1,\,1})_{t\in\Delta}$ a C^∞ family of Gauduchon metrics on the central fibre $X = X_0$ (the Iwasawa manifold);

$\omega_t^{1,\,1}$ is the $(1,\,1)$-component of ω_t w.r.t. the complex structure J_0 of X_0.

• We prove that the Aeppli cohomology groups

$$\Delta_{[\gamma]} \ni t \mapsto H^{2,\,2}_A(X_t,\,\mathbb{C})$$

define a *C^∞ vector bundle* $\mathcal{H}^{2,\,2}_A$ of rank 4 that *injects* as a C^∞ vector subbundle of the constant bundle $\mathcal{H}^4 \to \Delta_{[\gamma]}$ whose fibres are the De Rham cohomology groups $H^4_{DR}(X_t,\,\mathbb{C}) = H^4(X,\,\mathbb{C})$.

This injection is proved by using in a crucial way the

sGG property

(cf. P-Ugarte 2014) of all the small deformations X_t of the Iwasawa manifold $X = X_0$.

We also use the family $(\omega_t)_{t\in\Delta_{[\gamma]}}$ of *Gauduchon metrics* thereon.

Denoting by $\widetilde{H^{2,\,2}_{\omega_t}}$ the image of $H^{2,\,2}_A(X_t,\,\mathbb{C})$ into $H^4(X,\,\mathbb{C})$ under this ω_t-induced injection:

$$H^{2,\,2}_A(X_t,\,\mathbb{C}) \xrightarrow{Q_{\omega_t}} Q_{\omega_t}\bigg(H^{2,\,2}_A(X_t,\,\mathbb{C})\bigg) \subset H^4(X,\,\mathbb{C}),$$

we get a C^∞ vector bundle of rank 4

$$\mathcal{G}_{X_0} \ni \bigg[(\omega_t^{1,\,1})^2\bigg]_A \mapsto \widetilde{H^{2,\,2}_{\omega_t}} \subset H^4(X,\,\mathbb{C})$$

over the subset of the Gauduchon cone

$$\bigg\{\bigg[(\omega_t^{1,\,1})^2\bigg]_A \mid t \in \Delta_{[\gamma]}\bigg\} \subset \mathcal{G}_X.$$

This is to be compared with the VHS induced by the holomorphic family $(B_t)_{t\in\Delta_{[\gamma]}}$ of two-dimensional base tori of the family $(X_t)_{t\in\Delta_{[\gamma]}}$

$$H^2(B,\,\mathbb{C}) \simeq H^{2,\,0}(B_t,\,\mathbb{C})\oplus H^{1,\,1}(B_t,\,\mathbb{C})\oplus H^{0,\,2}(B_t,\,\mathbb{C}), \qquad t\in\Delta_{[\gamma]},$$

thanks to isomorphisms

$$\begin{aligned} H^{1,\,1}(B_t,\,\mathbb{C}) &\simeq H^{2,\,2}_A(X_t,\,\mathbb{C}) \\ &\simeq Q_{\omega_t}(H^{2,\,2}_A(X_t,\,\mathbb{C})) := \widetilde{H^{2,\,2}_{\omega_t}} \subset H^4_{DR}(X,\,\mathbb{C}). \end{aligned}$$

• The positive mirror map

We put

$$\mathcal{M} : \Delta_{[\gamma]} \to \mathcal{G}_X, \qquad t \mapsto \left[(\omega_t^{1,1})^2 \right]_A .$$

This is a kind of "absolute value" of the complex mirror map that we define.