Adiabatic Limit and Deformations of Complex Structures

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Classification of compact complex manifolds

X a compact complex manifold, $n = \dim_{\mathbb{C}} X$

Complex structure: $d = \partial + \bar{\partial}$

Idea

The transcendental methods, introduced for the study of not necessarily algebraic manifolds, are also relevant to the study of **projective manifolds**.

(1) **Metrical point of view:** $\omega > 0$ $(1,1)$**-form** C^∞ **on** X

(**Hermitian metric**, always exists)

Examples : (i) ω is called **Kähler** if $d\omega = 0$

(does not exist in general)

If $\dim_{\mathbb{C}} X \geq 3$, very few manifolds X are Kahler.

(ii) ω is called **Gauduchon** if $\partial\bar\partial\omega^{n-1} = 0$ (always exists)

(iii) ω is called **strongly Gauduchon** if $\partial\omega^{n-1}$ is $\bar\partial$-exact

(P. 2009) (does not exist in general)

(iv) ω is called **balanced** if $d\omega^{n-1} = 0$

(Gauduchon 1977) (does not exist in general)

(2) **Cohomological point of view**

- **De Rham** cohomology group:

$$H^k_{DR}(X, \mathbb{C}) := \frac{\ker d}{\operatorname{Im} d} \qquad \text{(depends only on the differential structure)}$$

- **Dolbeault** cohomology group:

$$H^{p,\,q}_{\bar\partial}(X, \mathbb{C}) := \frac{\ker \bar\partial}{\operatorname{Im} \bar\partial} \qquad \text{(depends on the complex structure)}$$

Topological obstruction to X being Kähler:

$0 \neq \{\omega^k\}_{DR} \in H^{2k}_{DR}(X, \mathbb{C})$, hence $b_{2k} \neq 0$ for all k.

(Betti numbers of X)

(A) First type of operations on compact manifolds

Modifications : $\sigma : \widetilde{X} \to X$ **holomorphic, bimeromorphic**

Examples : (i) X is called **Moishezon** if $\exists\,\sigma : \widetilde{X} \to X$ modification with $\widetilde{X}$ **projective** ;

Recall: $\widetilde{X}$ **projective** $\overset{\text{def}}{\Longleftrightarrow} \exists\, N \in \mathbb{N}^\star$ s.t. $\widetilde{X} \hookrightarrow \mathbb{CP}^N$
(embedding as a closed submanifold)

(ii) X is called **class** $\mathcal{C}$ if $\exists\,\sigma : \widetilde{X} \to X$ modification with $\widetilde{X}$ compact **Kähler**.

Implications (all are strict)

$$\begin{tikzcd}
 & X \textbf{ Kähler} \arrow[dr, Rightarrow] & \\
X \textbf{ projective} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & X \textbf{ class C} \\
 & X \textbf{ Moishezon} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Demailly-Paun (2001) : X is $\mathit{class}\,\mathcal{C} \iff \exists T$ **Kähler current** on X (i.e. $dT = 0$ and $T > 0$).

Moishezon (1967) : if X is **Moishezon** and **non-projective**, then X is **not Kähler**.

(B) Second type of operations on manifolds

Deformations of the **complex structure:** notion of **holomorphic family** $(X_t)_{t\in\Delta}$ of compact complex manifolds

$$\begin{tikzcd} \mathcal{X} \text{ complex manifold} \arrow[d, "\pi"] \\ \Delta \subset \mathbb{C} \text{ disc} \end{tikzcd}$$

X_t $\pi^{-1}(0) = X_0$

0 t

π is a **proper holomorphic submersion**.

Main result

Theorem (P. 2019) *If the fibre* $X_t := \pi^{-1}(t)$ *is* **Moishezon** *for all* $t \in \Delta \setminus \{0\}$, *then the limiting fibre* $X_0 := \pi^{-1}(0)$ *is also* **Moishezon**.

Remarks: (i) This is a deformation closedness result. It is optimal (Hironaka 1962).

(i) The statement is purely algebraic;

(ii) Surprisingly, the proof uses techniques that are

-analytic ;

-non-Kähler (e.g. **$\partial\bar\partial$-manifolds, SG and E_r-sG manifolds**).

(C) Third type of operations on manifolds

Conifold transitions: $\sigma : \widetilde{X} \to X_0$ holomorphic s.t.

$\cdot\, \widetilde{X}$ is **smooth** and X_0 is **singular**, $\dim_{\mathbb{C}} \widetilde{X} = \dim_{\mathbb{C}} X = 3$;

$\cdot\, \exists\, \Lambda = \{x_1, \dots, x_N\} \subset X_0$ s.t. $X_0 \setminus \Lambda$ is **smooth** ;

$\cdot\, \sigma^{-1}(x_i) := C_i$ is a $(-1, -1)$**-curve** : smooth rational curve s.t. $N_{\widetilde{X}} C_i \simeq \mathcal{O}_{C_i}(-1) \oplus \mathcal{O}_{C_i}(-1)$;

$\cdot\, \sigma : \widetilde{X} \setminus \pi^{-1}(\Lambda) \to X_0 \setminus \Lambda$ is **biholomorphic** ;

$\cdot\, \exists\, (X_t)_{t \in \Delta}$ analytic family of complex spaces s.t. X_t is compact **smooth** for all $t \neq 0$.

Typical case : · $\widetilde{X}$ is **Kähler** and **Calabi-Yau** (i.e. $K_{\widetilde{X}}$ is **trivial**) ;

· K_{X_t} is **trivial** ;

· the classes $[E_1], \dots, [E_N]$ **generate** $H_2(\widetilde{X}, \mathbb{Z})$, hence $b_2(X_t) = 0$, hence X_t is **non-Kähler** (and even **non-class** $\mathcal{C}$) for all $t \neq 0$.

However, we have

Fu-Li-Yau (2012) : X_t admits a **balanced metric** for all $t \neq 0$.

Friedman (2017) : X_t is a $\partial\bar{\partial}$**-manifold** for all $t \neq 0$.

$\partial\bar\partial$-manifolds (= cohomologically Kählér manifolds)

Definition (Deligne-Griffiths-Morgan-Sullivan 1976)

A compact complex manifold X is called $\partial\bar\partial$ $\overset{def}{\Longleftrightarrow}$

$\forall u \in C^\infty_{p,\,q}(X,\,\mathbb{C})$ *s.t.* $du = 0$, *the following equivalences hold:*

$$u \in Im\,\partial \iff u \in Im\,\bar\partial \iff u \in Im\,d \iff u \in Im\,(\partial\bar\partial).$$

Fundamental property. If X is a **$\partial\bar\partial$-manifold**, then

$H^k_{DR}(X,\,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ **Hodge decomposition**

$H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) \simeq \overline{H^{q,\,p}_{\bar\partial}(X,\,\mathbb{C})}$ **Hodge symmetry**

Implications (all are strict)

$$\begin{tikzcd}
 & X\ \textbf{balanced} \arrow[dr, Rightarrow] & \\
X\ \textit{class}\ \mathcal{C} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & X\ \textbf{sG} \\
 & \mathbf{X}\ \partial\bar{\partial} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\
 & & \mathbf{E_1(X) = E_\infty(X)}
\end{tikzcd}$$

(Frölicher spectral sequence)

Conjecture (P. 2015)

$$X \text{ is a } \partial\bar\partial\text{-manifold} \overset{?}{\Longrightarrow} X \text{ is a balanced manifold.}$$

Conjecture (P. 2016)

$$\exists \text{ a metric } \omega \text{ on } X \text{ s.t. } \partial\bar\partial\omega = 0 \overset{?}{\Longrightarrow} E_2(X) = E_\infty(X).$$

(Frölicher spectral sequence)

Deformation properties of $\partial\bar\partial$-manifolds

- **Openness (Wu 2007)**

  X_0 *is a* $\partial\bar\partial$*-manifold* $\implies$ X_t *is a* $\partial\bar\partial$*-manifold* $\quad \forall t \sim 0.$

- **Non-openness (Angella-Kasuya 2013)**

  X_t *is a* $\partial\bar\partial$*-manifold* $\quad \forall t \in \Delta \setminus \{0\}$ $\not\Longrightarrow$ X_0 *is a* $\partial\bar\partial$*-manifold.*

- **Deformation limits (P. 2019)**

  X_t *is a* $\partial\bar\partial$*-manifold* $\quad \forall t \in \Delta \setminus \{0\}$

  $\implies$ X_0 *is an* E_r**-sG manifold**,

where $r \in \mathbb{N}^\star$ *is the first page at which the Frölicher spectral sequence of* X_0 *degenerates.*

Recent metrical notions

- E_r**-sG manifolds** for $r = 1, 2, 3$ **(P. 2019)**

X sG manifold $(= E_1$-sG$)$ $\implies$ X E_2-sG $\implies$ X E_3-sG

- **sGG manifolds (P.–Ugarte 2014) :**

every Gauduchon metric is sG

(This is equivalent to a special case of the $\partial\bar\partial$-property.)

Theorem (P. 2011 et P.–Ugarte 2014) *The* **sG** *and* **sGG** *properties are* **deformation open** *and* **stable under modifications**.

Further conjectures

(I) Deformation closedness of the **class C** property

X_t is **class C** *for all* $t \in \Delta \setminus \{0\} \overset{?}{\Longrightarrow} X_0$ *is* **class C**.

(transcendental version of our main result being presented)

This conjecture motivated our work with L. Ugarte where we introduced sGG manifolds.

(II) Analytic Zariski-topology deformation openness of **Kählerianity**

If X_0 *is* **Kähler**, *then conjecturally* $\exists\, \Sigma = \bigcup_{\nu \in \mathbb{N}} \Sigma_\nu \subset \Delta$,

with Σ_ν *proper analytic subset of* Δ *such that*

-X_t *is* **Kähler** *for all* $t \in \Delta \setminus \Sigma$

and

-X_t *is* **class C** *for all* $t \in \Sigma$.

The Frölicher spectral sequence

Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$.

Page 0**:** the **Dolbeault complex**, i.e.

$$\cdots \xrightarrow{d_0} E_0^{p,\,q-1} \xrightarrow{d_0} E_0^{p,\,q} \xrightarrow{d_0} E_0^{p,\,q+1} \xrightarrow{d_0} \dots,$$

with $E_0^{p,\,q} := C^\infty_{p,\,q}(X,\,\mathbb{C})$ (smooth $(p,\,q)$-forms on X) and $d_0 := \bar\partial$. Put

$$E_1^{p,\,q} := \ker d_0^{p,\,q} / \mathrm{Im}\, d_0^{p,\,q-1} = H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}).$$

Page 1**:** the cohomology spaces of page 0, i.e.

$$\cdots \xrightarrow{d_1} E_1^{p-1,\,q} \xrightarrow{d_1} E_1^{p,\,q} \xrightarrow{d_1} E_1^{p+1,\,q} \xrightarrow{d_1} \dots,$$

with differential defined as $d_1([\alpha]_{\bar\partial}) := [\partial\alpha]_{\bar\partial}$.

Page r:

$$\cdots \xrightarrow{d_r} E_r^{p-r,\,q+r-1} \xrightarrow{d_r} E_r^{p,\,q} \xrightarrow{d_r} E_r^{p+r,\,q-r+1} \xrightarrow{d_r} \cdots.$$

So, d_r is of bidegree $(r,\, -r+1)$ for every $r \in \mathbb{N}^\star$. Put

$$E_{r+1}^{p,\,q} := \ker d_r^{p,\,q} / \mathrm{Im}\, d_r^{p-r,\,q+r-1}.$$

Fact (Frölicher 1955): This spectral sequence converges to the **De Rham cohomology** of X, i.e. there are **(non-canonical) isomorphisms**:

$$H^k_{DR}(X,\,\mathbb{C}) \simeq \bigoplus_{p+q=k} E_\infty^{p,\,q}, \qquad k = 0, \dots, 2n,$$

where $E_\infty^{p,\,q} = \cdots = E_{r+2}^{p,\,q} = E_{r+1}^{p,\,q} = E_r^{p,\,q}$ for all p, q and where $r \geq 1$ is the smallest positive integer such that the spectral sequence degenerates at E_r. (We write $E_r(X) = E_\infty(X)$.)

Thus, the degeneration at E_r is a purely **numerical property**:

$$E_r(X) = E_\infty(X) \iff b_k = \sum_{p+q=k} \dim_{\mathbb{C}} E_r^{p,\,q} \quad \forall k = 0, \ldots, 2n.$$

In particular,

$$\sum_{p+q=k} h^{p,\,q} \geq \cdots \geq \sum_{p+q=k} \dim_{\mathbb{C}} E_l^{p,\,q} \geq \sum_{p+q=k} \dim_{\mathbb{C}} E_{l+1}^{p,\,q} \geq \cdots \geq b_k.$$

Hence, the following implications hold:

$$E_1(X) = E_\infty(X) \implies E_2(X) = E_\infty(X) \implies \cdots \implies E_r(X) = E_\infty(X)$$

Relations to other properties

- If X is a **$\partial\bar{\partial}$-manifold**, then $E_1(X) = E_\infty(X)$.

- The converse is false.

  e.g. If $\dim_{\mathbb{C}} X = 2$ (i.e. a complex surface), then

  · $E_1(X) = E_\infty(X)$

  · X is a **$\partial\bar{\partial}$-manifold** $\iff$ X is **Kähler**.

- The property $E_1(X) = E_\infty(X)$ does not imply either the Hodge symmetry or the canonical Hodge decomposition. It only implies the much weaker numerical Hodge decomposition.

The 1^{st} new ingredient in the proof of the main result

$(X,\, \omega)$: a compact complex Hermitian manifold, $\dim_{\mathbb{C}} X = n$

Recall:

- ω is called **Gauduchon** if $\partial\omega^{n-1} \in \ker\bar\partial$ (i.e. E_1-closed);
- ω is called **strongly Gauduchon (sG)** if $\partial\omega^{n-1} \in \operatorname{Im}\bar\partial$

$(\iff \{\partial\omega^{n-1}\}_{E_1} = 0 \in E_1^{n,\,n-1}(X)$, i.e. $\partial\omega^{n-1}$ is $E_1 -$ exact)

Trivial observation

If ω is a **Gauduchon metric**, then $\partial\omega^{n-1}$ is **E_r-closed** for every $r \in \mathbb{N}^\star$.

This means that $\partial\omega^{n-1}$ represents an E_r-cohomology class $\{\partial\omega^{n-1}\}_{E_r} \in E_r^{n,\,n-1}(X)$.

Definition (P. 2019) *Fix any* $r \in \mathbb{N}^\star$. *A Gauduchon metric* ω *on* X *is called an* E_r**-sG metric** *if* $\partial\omega^{n-1}$ *is* E_r**-exact**.

- This means that $\{\partial\omega^{n-1}\}_{E_r} = 0 \in E_r^{n,\,n-1}(X)$, which is equivalent to the existence of forms ζ and ξ such that

$$\partial\omega^{n-1} = \partial\zeta + \bar\partial\xi$$

and such that ζ satisfies the following tower of $(r-1)$ equations:

$$\begin{aligned} \bar\partial\zeta &= \partial v_{r-3} \\ \bar\partial v_{r-3} &= \partial v_{r-4} \\ &\vdots \\ \bar\partial v_0 &= 0. \end{aligned}$$

Definition (P. 2019) *Fix any $r \in \mathbb{N}^\star$. The compact complex manifold X is called an* **E_r-sG manifold** *if there exists an* **E_r-sG metric** *ω on X.*

- The following implications hold:

$$\omega \text{ is } E_1 - sG \implies \omega \text{ is } E_2 - sG \implies \omega \text{ is } E_3 - sG$$
$$X \text{ is } E_1 - sG \implies X \text{ is } E_2 - sG \implies X \text{ is } E_3 - sG.$$

- For bidegree reasons, no new E_r-sG notion is obtained for $r \geq 4$.

- **Obvious equivalence:** ω is E_1-sG $\iff$ ω is sG;

- **L. Ugarte** showed that

· There exist E_2-sG manifolds that are not E_1-sG.

(e.g. all the **Calabi-Eckmann manifolds**, except the **Hopf manifolds**)

· If X is a **Hopf manifold**, then X is not E_r-sG for any $r \in \mathbb{N}^\star$.

· Any possible complex structure on the 6-sphere S^6 (if any) is E_3-sG.

· If the analogous statement could be proved in the E_1-sG case, interesting conclusions would follow for S^6.

The 2^{nd} new ingredient in the proof of the main result

Recall: the $\partial\bar\partial$-property of compact complex manifolds is not deformation closed.

However, we have

Theorem A (P. 2019) *If* X_t *is a* **$\partial\bar\partial$-manifold** *for every* $t \in \Delta \setminus \{0\}$, *then* X_0 *is an* **E_r-sG manifold**, *where* $r \in \mathbb{N}^\star$ *is the smallest positive integer s.t.* $E_r(X_0) = E_\infty(X_0)$.

To be compared with the stronger

Theorem (P. 2009) *Under the same assumptions as above,* X_0 *is an* **sG manifold***(=* **E_1-sG manifold***).*

(given an ad hoc proof in 2009)

The more conceptual proof of the 2019 result relies on the notion of **adiabatic limit** for **complex structures** (P. 2017) inspired by the analogous concept in foliation theory.

Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. For every constant $h \in \mathbb{C}$, let

$$d_h := h\partial + \bar\partial : C^\infty_k(X,\, \mathbb{C}) \longrightarrow C^\infty_{k+1}(X,\, \mathbb{C}), \qquad k \in \{0,\, \dots, 2n\}.$$

Immediate properties.

(1) The pointwise maps

$$\theta_h : \Lambda^{p,\,q}T^\star X \longrightarrow \Lambda^{p,\,q}T^\star X, \qquad u \mapsto \theta_h u := h^p u,$$

are isomorphisms for $h \neq 0$ and the operators d and d_h are related by

$$d_h = \theta_h d \theta_h^{-1}.$$

(2) Hence $d_h^2 = 0$, inducing the d_h-cohomology

$$H^k_{d_h}(X,\,\mathbb{C}) := \ker d_h / \mathrm{Im}\, d_h.$$

(3) We get induced isomorphisms in cohomology:

$$H^k_{DR}(X,\,\mathbb{C}) \xrightarrow{\simeq} H^k_{d_h}(X,\,\mathbb{C}), \qquad \{u\}_d \mapsto \{\theta_h u\}_{d_h},$$

whenever $h \neq 0$.

(5) If ω is a Hermitian metric on X, let

$$\Delta_h : C^\infty_k(X,\,\mathbb{C}) \longrightarrow C^\infty_k(X,\,\mathbb{C}), \qquad \Delta_h := d_h d_h^\star + d_h^\star d_h,$$

be the associated d_h-Laplacian.

It is elliptic, so it induces the Hodge isomorphism

$$\ker \Delta_h \simeq H^k_{d_h}(X,\,\mathbb{C})$$

The Frölicher approximating vector bundle (FAVB)

(P. 2019)

Let $(X_t)_{t\in\Delta}$ be a holomorphic family of compact complex manifolds.

- $d = \partial_t + \bar\partial_t$ for all $t \in \Delta$ since $X_t \simeq X$ (C^∞ diffeomorphism)
- $d_{h,\,t} := h\partial_t + \bar\partial_t$ for $(h,\,t) \in \mathbb{C} \times \Delta$
- $\theta_{h,\,t} : \Lambda^k T^\star X \longrightarrow \Lambda^k T^\star X,$ $u = \sum\limits_{p+q=k} u_t^{p,\,q} \mapsto \theta_{h,\,t} u := \sum\limits_{p+q=k} h^p\, u_t^{p,\,q}$
- For every $(h,\,t) \in \mathbb{C}^\star \times B$, we get an isomorphism in cohomology:

$$\theta_{h,\,t} : H^k_{DR}(X,\,\mathbb{C}) \xrightarrow{\simeq} H^k_{d_{h,\,t}}(X_t,\,\mathbb{C})$$

Proposition (P.2019) *For every* $k = 0, \dots, 2n$, *put*

$$\begin{aligned} \mathcal{A}^k_{h,\,t} &= H^k_{d_{h,\,t}}(X_t,\,\mathbb{C}), \quad (h,\,t) \in \mathbb{C}^\star \times B \\ \mathcal{A}^k_{0,\,t} &= \bigoplus_{p+q=k} E^{p,\,q}_\infty(X_t), \quad (h,\,t) = (0,\,t) \in \{0\} \times B. \end{aligned}$$

Then, the holomorphic trivialisation $(\theta_{h,\,t})_{(h,\,t)\in\mathbb{C}^\star\times B}$ *of the vector bundle* $\mathcal{A}^k \to \mathbb{C}^\star \times B$ *of rank* b_k *extends holomorphically across* $\{0\} \times B$ *to yield a holomorphic vector bundle* $\mathcal{A}^k \to \mathbb{C} \times B$.

Definition (P.2019) *This holomorphic vector bundle is called the* **FAVB** *of the family* $(X_t)_{t\in\Delta}$ *in degree* k.

Proof of Theorem A

- Let γ_0 be an arbitrary Gauduchon metric on X_0. Extend it arbitrarily to a C^∞ family $(\gamma_t)_{t\in\, B}$ of Gauduchon metrics on the X_t's.

- Consider the C^∞ section

$$\sigma(h,\, t) := \begin{cases} \{\partial_t\gamma_t^{n-1}\}_{d_{h,\,t}} \in H^{2n-1}_{d_{h,\,t}}(X_t,\, \mathbb{C}) = \mathcal{A}^{2n-1}_{h,\,t}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad (h,\, t) \in \mathbb{C}^\star \times B, \\ \{\partial_t\gamma_t^{n-1}\}_{E_r(X_t)} \in \displaystyle\bigoplus_{p+q=2n-1} E_r^{p,\,q}(X_t) = \mathcal{A}^{2n-1}_{0,\,t}, \\ \qquad\qquad\qquad\qquad \text{if} \quad (h,\, t) = (0,\, t) \in \{0\} \times B, \end{cases}$$

of the FAVB $\mathcal{A}^{2n+1} \rightarrow \mathbb{C} \times B$.

- X_t is a $\partial\bar\partial$-manifold for all $t \neq 0$, so

$$\sigma(h,\, t) = \{\partial_t \gamma_t^{n-1}\}_{d_{h,\, t}} = 0 \in \mathcal{A}_{h,\, t}^{2n-1}$$

for all $(h,\, t) \in \mathbb{C}^\star \times B^\star$.

Hence, by continuity, σ must be identically zero on $\mathbb{C} \times B$.

In particular,

$$\sigma(0,\, t) = \{\partial_t \gamma_t^{n-1}\}_{E_r(X_t)} = 0 \in \mathcal{A}_{0,\, t}^{2n-1} \qquad \text{for all } t \in B.$$

This means precisely that $\partial_t \gamma_t^{n-1}$ is $E_r(X_t)$-exact for every $t \in B$. In other words, γ_t is an E_r-sG metric on X_t for every $t \in B$, including $t = 0$.

In particular, X_0 is an E_r-sG manifold. q.e.d.