# Classification of compact complex manifolds

$X$ compact complex manifold, $n = \dim_{\mathbb{C}} X$

## (1) **Metrical point of view**

$$\{\omega > 0 \quad / \quad C^{\infty}\ (1,\,1)-\text{form on } X\}$$

$$\Big\downarrow \ \omega \mapsto \omega^{n-1} \ \text{bijection}$$

$$\{\Omega > 0 \quad / \quad C^{\infty}\ (n-1,\,n-1)-\text{form on } X\}$$

(Hermitian metrics, always exist)

| | | | | |
|---|---|---|---|---|
| $d\omega = 0$ | $\Longrightarrow$ | $\exists\ \alpha^{0,\,2} \in C^\infty_{0,\,2}(X,\,\mathbb{C})$ t.q. $d(\overline{\alpha^{0,\,2}} + \omega + \alpha^{0,\,2}) = 0$ | $\Longrightarrow$ | $\partial\bar\partial\omega = 0$ |
| (Kähler) | | (Hermitian-symplectic) | | (SKT) |
| $\Downarrow$ | | | | |
| $d\omega^{n-1} = 0$ | $\Longrightarrow$ | $\partial\omega^{n-1} \in \operatorname{Im}\bar\partial$ | $\Longrightarrow$ | $\partial\bar\partial\omega^{n-1} = 0$ |
| (balanced) | | (strongly Gauduchon (sG)) | | (Gauduchon) |
| (Gauduchon) | | (P. 2009) | | (always exists). |

## (2) **Cohomological point of view**

For $p, q \in \{0, 1, \dots, n\}$, we define :

-**Bott-Chern** cohomology group :

$$H^{p,\,q}_{BC}(X, \mathbb{C}) := \frac{\ker \partial \cap \ker \bar\partial}{\mathrm{Im}(\partial\bar\partial)}$$

(the "finest" cohomology)

-**Aeppli** cohomology group :

$$H^{p,\,q}_{A}(X, \mathbb{C}) := \frac{\ker(\partial\bar\partial)}{\mathrm{Im}\partial + \mathrm{Im}\bar\partial}$$

(the "coarsest" cohomology)

There is a **non-degenerate** canonical duality :

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \times H^{n-p,\,n-q}_{A}(X,\,\mathbb{C}) \to \mathbb{C},$$

$$([\alpha]_{BC},\,[\beta]_A) \longmapsto \int\limits_X \alpha \wedge \beta.$$

(the analogue of the Serre duality)

If $\omega$ is a Hermitian metric on $X$, one defines elliptic operators of order 4 :

$$\Delta_{BC}, \Delta_A : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C})$$

inducing Hodge isomorphisms :

$$\ker \Delta_{BC} \simeq H^{p,\,q}_{BC}(X,\,\mathbb{C})$$

and

$$\ker \Delta_{A} \simeq H^{p,\,q}_{A}(X,\,\mathbb{C}).$$

## (3) **Metrics and cohomology**

**Gauduchon cone of $X$** (P. 2013)

$\mathcal{G}_X = \{[\omega^{n-1}]_A \ / \ \omega$ Gauduchon metric$\}$

$\omega > 0$ $(1,\,1)$-form $C^\infty$ s.t. $\partial\bar\partial\omega^{n-1} = 0$

$\mathcal{G}_X \subset H_A^{n-1,\,n-1}(X,\,\mathbb{R})$, open convex cone.

**Pseudo-effective cone of $X$** (Demailly 1992)

$\mathcal{E}_X = \{[T]_{BC} \ / \ T \geq 0 \ \ (1,\,1) - \text{current}, \ \ dT = 0\}$

$\mathcal{E}_X \subset H_{BC}^{1,\,1}(X,\,\mathbb{R})$, closed convex cone.

**Lamari's positivity criterion** (1998) :

The cones $\mathcal{E}_X$ and $\overline{\mathcal{G}}_X$ are **dual** under the duality :

$$H_{BC}^{1,\,1}(X,\,\mathbb{R}) \times H_A^{n-1,\,n-1}(X,\,\mathbb{R}) \to \mathbb{R}.$$

## Context of applications

## The Abundance Conjecture

$X$ projective (or merely compact Kähler) manifold,

$\dim_{\mathbb{C}} X = n$

$$k(X) = -\infty \stackrel{?}{\Longleftrightarrow} X \text{ is uniruled}$$

($k(X)$ = the Kodaira dimension of $X$)

"$\Longleftarrow$" is trivial

"$\Longrightarrow$" splits as

$$k(X) = -\infty \stackrel{?}{\Longrightarrow} K_X \text{ not psef} \stackrel{?}{\Longrightarrow} X \text{ uniruled}$$

Boucksom, Demailly, Paun et Peternell (2004) :

**Conjecture 1** (BDPP) The pseudo-effective cone

$$\mathcal{E}_X \subset H^{1,1}_{BC}(X, \mathbb{R})$$

is dual to the cone of **moving classes** :

$$\mathcal{M}_X \subset H^{n-1,n-1}_{A}(X, \mathbb{R}),$$

the closure of the cone generated by the classes of $(n-1, n-1)$-currents of the form $\mu_\star(\widetilde{\omega}_1 \wedge \cdots \wedge \widetilde{\omega}_{n-1})$, where $\mu : \widetilde{X} \to X$ is an arbitrary modification and the $\widetilde{\omega}_j$ are arbitrary Kähler metrics on $\widetilde{X}$.

**Example :** if $X$ is projective, the **strongly moving curves** (BDPP) are

$$C = \mu_\star(\widetilde{A}_1 \cap \cdots \cap \widetilde{A}_{n-1}),$$

where the $\widetilde{A}_j$ are very ample divisors in $\widetilde{X}$.

Conjecture 1 serves in [BDPP] for the study of

**Conjecture 2** $X$ compact Kähler. Then :

$K_X$ **non pseudo-effective** $\implies$ $X$ **uniruled**,

where $K_X$ = the canonical bundle of $X$.

BDPP : proved the case where $X$ is **projective**.

# Application I

Resolution of the qualitative part of Demailly's **Transcendental Morse Inequalities Conjecture**

**Theorem** (P. 2014)
*$X$ compact Kähler manifold, $dim_{\mathbb{C}}X = n$,*
*$\{\alpha\}, \{\beta\} \in H^{1,1}_{BC}(X, \mathbb{R})$* **nef** *classes.*

*If $\{\alpha\}^n - n\,\{\alpha\}^{n-1}.\{\beta\} > 0$, then*

*there exists a* **Kähler current** $T \in \{\alpha - \beta\}$.

**Terminology reminder**

-a class $\{\alpha\} \in H^{1,1}_{BC}(X, \mathbb{R})$ is **nef** if

$\forall \varepsilon > 0 \quad \exists \alpha_\varepsilon \in \{\alpha\}$ $C^\infty$ form s.t. $\alpha_\varepsilon \geq -\varepsilon\omega$.

-a (1, 1)- current $T$ is **Kähler** if

$dT = 0$ and $T \geq \varepsilon\omega$ on $X$.

## Effective version

**Theorem** (P. 2014, 2015)
Suppose the classes $\{\alpha\}, \{\beta\}$ are **Kähler**.

*Then, for all* $t \geq 0$*, there exists a real* $(1, 1)$*-current* $T_t \in \{\alpha - t\beta\}$ *such that*

$$(\star) \quad T_t \geq \left(1 - nt\,\frac{\{\alpha\}^{n-1}.\{\beta\}}{\{\alpha\}^n}\right)\alpha \quad \textit{on } X.$$

*Thus, in particular,* $T_1$ *is Kähler if*

$$\{\alpha\}^n - n\,\{\alpha\}^{n-1}.\{\beta\} > 0$$

*and* $T_t$ *is Kähler for all*

$$0 \leq t < \frac{\{\alpha\}^n}{n\,\{\alpha\}^{n-1}.\{\beta\}}.$$

the psef threshold

**Proof :** by Lamari's positivity criterion, we have

$\exists\ T_t$ satisfying $(\star)$ $\iff$

$$\int\limits_X \left[\alpha - t\beta - \left(1 - nt\,\frac{\{\alpha\}^{n-1}.\{\beta\}}{\{\alpha\}^n}\right)\alpha\right]\wedge\gamma^{n-1} \geq 0$$

for every Gauduchon metric $\gamma$ on $X$.

This is equivalent to :

$$nt\,\frac{\{\alpha\}^{n-1}.\{\beta\}}{\{\alpha\}^n}\int\limits_X \alpha\wedge\gamma^{n-1} \geq t\int\limits_X \beta\wedge\gamma^{n-1},$$

hence to :

$$\left(\int\limits_X \alpha\wedge\gamma^{n-1}\right)\left(\int\limits_X \alpha^{n-1}\wedge\beta\right) \geq \frac{\{\alpha\}^n}{n}\int\limits_X \beta\wedge\gamma^{n-1} \qquad (1)$$

for every Gauduchon metric $\gamma$ on $X$.

To prove the last inequality, we consider the Monge-Ampère equation :

$$(MA1) \quad (\alpha + i\partial\bar\partial\varphi)^n = \frac{\{\alpha\}^n}{\int_X \beta \wedge \gamma^{n-1}}\, \beta \wedge \gamma^{n-1}.$$

Yau (1978) : ∃! Kähler metric

$$\widetilde\alpha = \alpha + i\partial\bar\partial\varphi \in \{\alpha\},\ \widetilde\alpha > 0,$$

solution of this equation.

$$(MA1) \iff \det{}_\gamma \widetilde\alpha = \frac{\{\alpha\}^n}{n \int_X \beta \wedge \gamma^{n-1}}\, (\Lambda_\gamma \beta)\, \gamma^n.$$

The l.h.s. term of (1) reads

$$\Big(\int_X \widetilde\alpha \wedge \gamma^{n-1}\Big)\Big(\int_X \widetilde\alpha^{n-1} \wedge \beta\Big) =$$

$$\frac{1}{n^2}\Big(\int_X (\Lambda_\gamma \widetilde\alpha)\, \gamma^n\Big)\Big(\int_X (\Lambda_{\widetilde\alpha} \beta)\, (\det{}_\gamma \widetilde\alpha)\, \gamma^n\Big) =$$

$$\geq \frac{1}{n^2}\Big[\int_X (\Lambda_\gamma \widetilde\alpha \cdot \Lambda_{\widetilde\alpha}\beta)^{\frac12}\, (\det{}_\gamma \widetilde\alpha)^{\frac12}\, \gamma^n\Big]^2$$

(Cauchy-Schwarz)

*Pointwise inequality of traces :*

$$\Lambda_\gamma\widetilde{\alpha}\cdot\Lambda_{\widetilde{\alpha}}\beta \geq \Lambda_\gamma\beta$$

(easily checked in local coordinates)

*Using* $(\star)$, *we get*

$$\left(\int_X \widetilde{\alpha}\wedge\gamma^{n-1}\right)\left(\int_X \widetilde{\alpha}^{n-1}\wedge\beta\right) \geq$$

$$\geq \frac{1}{n^2}\left[\int_X (\Lambda_\gamma\beta)^{\frac{1}{2}}\,(\Lambda_\gamma\beta)^{\frac{1}{2}}\,\gamma^n\right]^2 \frac{\{\alpha\}^n}{n\int_X \beta\wedge\gamma^{n-1}}$$

$$= \frac{\{\alpha\}^n}{n\int_X \beta\wedge\gamma^{n-1}}\left[\int_X \left(\frac{1}{n}\Lambda_\gamma\beta\right)\gamma^n\right]^2$$

$$= \frac{\{\alpha\}^n}{n\int_X \beta\wedge\gamma^{n-1}}\left[\int_X \beta\wedge\gamma^{n-1}\right]^2$$

$$= \frac{\{\alpha\}^n}{n}\int_X \beta\wedge\gamma^{n-1}.$$

This is precisely the desired inequality (1). q.e.d.

# Application II

Partial resolution of the quantitative part of Demailly's **Transcendental Morse Inequalities Conjecture**

$$(\star\star) \qquad \mathrm{Vol}(\{\alpha - \beta\}) \overset{?}{\geq} \{\alpha\}^n - n\,\{\alpha\}^{n-1}.\{\beta\}.$$

**Theorem** (P. 2015)

$$\{\alpha - \beta\}^n \geq \{\alpha\}^n - n\,\{\alpha\}^{n-1}.\{\beta\}.$$

We actually prove rather more.

**Theorem** (P. 2015) *Suppose* $\{\alpha\}, \{\beta\}$ *are Kähler classes.*
*Then, for every* $k \in \{1, 2, \dots, n\}$ *and every smooth positive* $(n-k,\, n-k)$*-form* $\Omega^{n-k,\,n-k} \geq 0$ *such that* $\partial\bar\partial\Omega^{n-k,\,n-k} = 0$*, we have*

$$\{\alpha - \beta\}^k.[\Omega^{n-k,\,n-k}]_A$$

$$\overset{(II_k)}{\geq} \{\alpha^k - k\,\alpha^{k-1} \wedge \beta\}.[\Omega^{n-k,\,n-k}]_A$$

$$\overset{(III_k)}{\geq} \left(1 - \frac{n}{R}\right)\{\alpha\}^k.[\Omega^{n-k,\,n-k}]_A \geq 0,$$

*where* $R := \dfrac{\{\alpha\}^n}{\{\alpha\}^{n-1}.\{\beta\}}$. *(Thus* $R > n$ *by assumption.)*

**Main point : proving inequalities** $(III_k)$

$(III_k) \iff$

$$\frac{n}{k}\Big(\int\limits_X \alpha^k \wedge \Omega^{n-k,\,n-k}\Big)\cdot\Big(\int\limits_X \alpha^{n-1}\wedge\beta\Big) \geq \{\alpha\}^n \int\limits_X \alpha^{k-1}\wedge\beta\wedge\Omega^{n-k,\,n-k}.$$

*Ideally, we would like to replace* $\alpha > 0$ *by a Kähler metric* $\widetilde{\alpha} \in \{\alpha\}$ *solving the Monge-Ampère Hessian equation*

$$(MA\text{-}H) \quad \widetilde{\alpha}^n = C_{n,\,k}\,\widetilde{\alpha}^{k-1}\wedge\beta\wedge\Omega^{n-p,\,n-p},$$

*where* $C_{n,\,k} := \dfrac{\{\alpha\}^n}{\{\alpha\}^{p-1}.\{\beta\}.[\Omega^{n-k,\,n-k}]_A}$.

*Such an* $\widetilde{\alpha}$ *would be a fixed point for the (standard) Monge-Ampère equation*

$$\widetilde{\alpha}^n = C_{n,\,k}\,\alpha^{k-1}\wedge\beta\wedge\Omega^{n-p,\,n-p}.$$

*This kind of equation goes back to Donaldson's J-flow and to work by Chen.*

*It admits a solution under a certain assumption on the class* $\{\alpha\}$*.*

*It was solved by Fang-Lai-Ma 2011.*

**Our case :** *we do not wish to impose any restriction on the class* $\{\alpha\}$*.*

*Hence, we settle for an approximate solution (i.e. up to an* $\varepsilon$*) but allow* $\{\alpha\}$ *to be an arbitrary Kähler class.*

*This leads to the main new idea introduced in [Pop15].*

## The Approximate Fixed Point Technique

*Let* $\mathcal{E}_\alpha := \{T \in \{\alpha\} \;/\; T \geq 0\}$

*be the set of d-closed positive* $(1,\,1)$*-currents in the Kähler class* $\{\alpha\}$.

$\mathcal{E}_\alpha \subset \mathcal{D}'^{1,\,1}(X,\,\mathbb{R})$ *is* **compact** *and* **convex**

*where* $\mathcal{D}'^{1,\,1}(X,\,\mathbb{R})$ *is the locally convex space of real* $(1,\,1)$*-currents on* $X$ *endowed with the weak topology.*

*For every* $\varepsilon > 0$*, we associate with the equation*

$$\widetilde{\alpha}^n = C_{n,\,k}\,\alpha^{k-1} \wedge \beta \wedge \Omega^{n-k,\,n-k}$$

*a* **continuous** *map*

$$R_\varepsilon : \mathcal{E}_\alpha \to \mathcal{E}_\alpha, \quad R_\varepsilon(T) = \alpha_{T,\,\varepsilon},$$

*defined as follows.*

*(i) Blocki-Kolodziej version [BK07] for* **Kähler** *classes of Demailly's regularisation-of-currents theorem [Dem92] :*

$\exists$ $C^\infty$ *d-closed* $(1,\,1)$*-forms* $\omega_\varepsilon \in \{\alpha\} = \{T\}$

$$\textit{such that} \qquad \omega_\varepsilon \geq -\varepsilon\omega$$

$$\textit{and} \qquad \omega_\varepsilon \longrightarrow T \textit{ as } \varepsilon \to 0$$

*in the weak topology of currents.*

*(ii) Set* $\quad u_{T,\,\varepsilon} := (1-\varepsilon)\,\omega_\varepsilon + \varepsilon\omega \geq \varepsilon^2\,\omega > 0.$

*So* $u_{T,\,\varepsilon}$ *is a Kähler metric in the class* $\{\alpha\}$ *and*

$$u_{T,\,\varepsilon} \to T \textit{ as } \varepsilon \to 0$$

*in the weak topology of currents.*

*(iii) Solve the above equation with right-hand term defined by $u_{T,\,\varepsilon}$ instead of $\alpha$ :*

$$\alpha_{T,\,\varepsilon}^n = C_{n,\,k}\; u_{T,\,\varepsilon}^{k-1} \wedge \beta \wedge \Omega^{n-k,\,n-k}.$$

*Put* $R_\varepsilon(T) := \alpha_{T,\,\varepsilon}$.

*Thus the image of $R_\varepsilon$ consists of (smooth) Kähler metrics in $\{\alpha\}$.*

**Upshot :**

*The* **Schauder Fixed Point Theorem** *gives :*

$\exists$ *a current* $T_\varepsilon \in \mathcal{E}_\alpha$ *such that*

$$T_\varepsilon = R_\varepsilon(T_\varepsilon) = \alpha_{T_\varepsilon,\,\varepsilon}.$$

$T_\varepsilon$ *is actually a (smooth) Kähler metric in* $\{\alpha\}$.

**Conclusion.**

$\forall\, \varepsilon > 0$ *we have constructed a Kähler metric* $\widetilde{\alpha}_\varepsilon$ *in the Kähler class* $\{\alpha\}$ *such that*

$$\begin{aligned}\widetilde{\alpha}_\varepsilon^n &= C_{n,\,k}\,[(1-\varepsilon)\,\omega_\varepsilon + \varepsilon\omega]^{p-1} \wedge \beta \wedge \Omega^{n-p,\,n-p}\\ &\geq (1-\varepsilon)^{p-1}\,C_{n,\,k}\,\widetilde{\alpha}_\varepsilon^{p-1} \wedge \beta \wedge \Omega^{n-p,\,n-p}\\ &\quad -O(|\eta_\varepsilon|),\end{aligned}$$

*where the constant* $\eta_\varepsilon \to 0$ *when* $\varepsilon \to 0$.