Deformations of Holomorphic p-Contact Structures

D. Popovici: arXiv:2610.20430
K. Broder and D. Popovici: arXiv:2511.23139
H. Kasuya, D. Popovici, L. Ugarte: arXiv:2502.01447 and 2511.10818

Dan Popovici

Université de Toulouse
France

Conference on Complex Geometry
Institute for Basic Science, Daejeon
9th – 13th November 2026

(I) Preliminaries

Let: $L \longrightarrow X$ be a holomorphic line bundle;

X a compact complex manifold, $\dim_{\mathbb{C}} X = n = 2p+1 = 4l+3$.

Definition (Kasuya-P-Ugarte 2025, Broder-P 2025)

An L-valued **holomorphic p-contact structure** *is a form $\Gamma \in C^{\infty}_{p,0}(X, L)$ such that*

$$(a)\, \bar{\partial}\Gamma = 0 \quad \textit{and} \quad (b)\, \Gamma \wedge \partial\Gamma \neq 0 \quad \textit{everywhere on } X.$$

X: a **holomorphic p-contact manifold**
L: a **p-contact bundle**

Observation

$\partial\Gamma$ is not well defined (in general), but $\Gamma \wedge \partial\Gamma \in C^\infty_{n,\,0}(X,\, L^2)$ is.

$\Gamma = (\Gamma_\alpha)_\alpha$, where $\Gamma_{|U_\alpha} = \Gamma_\alpha \otimes e_\alpha$,
Γ_α is a $\mathbb{C}$-valued $(p,\, 0)$-form on U_α.

We have: $\Gamma_\alpha = g_{\alpha\beta}\, \Gamma_\beta$ on $U_\alpha \cap U_\beta$.

Hence: $\partial\Gamma_\alpha = g_{\alpha\beta}\, \partial\Gamma_\beta + \partial g_{\alpha\beta} \wedge \partial\Gamma_\beta$;

Hence:

$$\begin{aligned} \Gamma_\alpha \wedge \partial\Gamma_\alpha &= g^2_{\alpha\beta}\, \Gamma_\beta \wedge \partial\Gamma_\beta - g_{\alpha\beta}\, \partial g_{\alpha\beta} \wedge \Gamma_\beta \wedge \Gamma_\beta \\ &= g^2_{\alpha\beta}\, \Gamma_\beta \wedge \partial\Gamma_\beta. \end{aligned}$$

For every C^∞ Hermitian fibre metric h on L, if $D_h = D'_h + \bar{\partial}$ is the Chern connection of $(L,\, h)$, we have:

$$\Gamma \wedge \partial\Gamma = \Gamma \wedge D'_h\Gamma \in H^{n,\,0}(X,\, L^2) \simeq H^0(X,\, K_X \otimes L^2) \simeq \mathbb{C}$$

because $K_X \otimes L^2$ must be *trivial*.

Consequence

If $\Gamma \in C^\infty_{p,\,0}(X,\, L)$ *is an L-valued holomorphic p-contact structure on* X*, then:*

$$L^2 \simeq -K_X \qquad \textit{or, equivalently,} \qquad (L^{-1})^2 \simeq K_X.$$

In particular, L^{-1} is a **spin structure** on X.

Any *L-valued holomorphic p-contact structure* $\Gamma_0 \in C^\infty_{p,\,0}(X,\,L)$ gives an identification $H^0(X,\,K_X \otimes L^2) \simeq \mathbb{C}$.

We get a holomorphic, degree-two homogeneous map:

$$\mathbb{C}^{N+1} \simeq H^{p,\,0}_{\bar\partial}(X,\,L) \ni \Gamma \stackrel{T}{\longmapsto} \Gamma \wedge \partial\Gamma \in \mathbb{C}.$$

By the Cauchy inequalities, T is a *homogeneous polynomial* of degree 2, so

$$T \in H^0\Big(\mathbb{P}(H^{p,\,0}_{\bar\partial}(X,\,L)),\,O(2)\Big).$$

Corollary

The set of L-valued holomorphic p-contact structures on X is either $\emptyset$ or

the complement of a degree-two complex hypersurface in $\mathbb{P}\left(H^{p,0}_{\bar{\partial}}(X,\, L)\right)$.

Standard notion

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n = 2p+1$.

A *holomorphic contact structure* on X: $\eta \in C^{\infty}_{1,0}(X,\, L)$ such that

$$\bar{\partial}\eta = 0 \quad \text{and} \quad \eta \wedge (\partial\eta)^p \neq 0 \quad \text{at every point of } X.$$

Intuitively, the $2p+1$ dimensions of X are split into two classes: the direction of η and the $2p$ directions of $(\partial\eta)^p$.

We replace this splitting by the splitting $2p+1 = p + (p+1)$.

If η is a *contact structure*, we have:

$$\eta \wedge (\partial\eta)^p \in H^{n,\,0}_{\bar\partial}\Big(X,\, L^{p+1}\Big), \text{ so}$$

$$L^{p+1} \simeq -K_X.$$

Examples of holomorphic p-contact manifolds (Kasuya-P-Ugarte 2025)

(I) Initial observations

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n = 2p+1$ and p odd.

(1) *If $n = 3$ (and $p = 1$), a form $\Gamma \in C^{\infty}_{1,0}(X, \mathbb{C})$ is a holomorphic contact structure on X if and only if it is a holomorphic 1-contact structure.*

(2) *Let s be the positive integer such that $p = 2s - 1$. For any holomorphic contact structure $\eta \in C^{\infty}_{1,0}(X, \mathbb{C})$ on X, the form*

$$\Gamma := \eta \wedge (\partial\eta)^{s-1} \in C^{\infty}_{p,0}(X, \mathbb{C})$$

is a holomorphic p-contact structure on X.

(II) Examples in dimension 3

• The **Iwasawa manifold** $X = G/\Gamma$:

$G = (\mathbb{C}^3, \star)$ is a nilpotent complex Lie group (= Heisenberg group), where the group operation is defined by

$$(\zeta_1, \zeta_2, \zeta_3) \star (z_1, z_2, z_3) = (\zeta_1 + z_1, \zeta_2 + z_2, \zeta_3 + z_3 + \zeta_1 z_2),$$

$\Gamma \subset G$ is the lattice of $(z_1, z_2, z_3) \in G$ with $z_1, z_2, z_3 \in \mathbb{Z}[i]$.

The cohomology of X is generated by three holomorphic $(1, 0)$-forms α, β, γ on X that satisfy the structure equations:

$$\partial\alpha = \partial\beta = 0 \qquad \text{and} \qquad \partial\gamma = -\alpha \wedge \beta \neq 0 \text{ everywhere on } X.$$

Thus, $\Gamma := \gamma$ defines a **holomorphic contact structure** (= a **holomorphic 1-contact structure)** since

$$\gamma \wedge \partial\gamma = -\alpha \wedge \beta \wedge \gamma \neq 0 \quad \text{at every point of } X.$$

• The **Nakamura manifolds** $X = G/\Gamma$:

$G = (\mathbb{C}^3, \star)$ is a solvable complex Lie group, where the group operation is defined by

$$(\zeta_1, \zeta_2, \zeta_3) \star (z_1, z_2, z_3) = (\zeta_1 + z_1, \zeta_2 + e^{-\zeta_1} z_2, \zeta_3 + e^{\zeta_1} z_3),$$

$\Gamma \subset G$ is a lattice.

The cohomology of X is generated by three holomorphic $(1, 0)$-forms $\varphi_1, \varphi_2, \varphi_3$ on X that satisfy the structure equations:

$$\partial\varphi_1 = 0, \qquad \partial\varphi_2 = \varphi_1 \wedge \varphi_2 \qquad \text{and} \qquad \partial\varphi_3 = -\varphi_1 \wedge \varphi_3.$$

Hence:

• φ_2 and φ_3 define **holomorphic 1-no-contact structures**;

• $\Gamma_1 := \varphi_2 + \varphi_3$ and $\Gamma_2 := \varphi_2 - \varphi_3$ define **holomorphic contact structures** (= **holomorphic 1-contact structures**).

- The manifolds $X = SL(2, \mathbb{C})/\Gamma$:

$SL(2, \mathbb{C})$ is a semi-simple complex Lie group;

$\Gamma \subset SL(2, \mathbb{C})$ is a lattice.

The cohomology of X is generated by three holomorphic $(1, 0)$-forms α, β, γ on X that satisfy the structure equations:

$$\partial\alpha = \beta \wedge \gamma, \qquad \partial\beta = \gamma \wedge \alpha, \qquad \partial\gamma = \alpha \wedge \beta.$$

Hence, each of the forms α, β, γ defines a **holomorphic contact structure** (= a **holomorphic 1-contact structure)** on X.

Proposition (**Kasuya-P-Ugarte 2025**)

Let $X = G/\Gamma$ be the quotient of a nilpotent Lie group of real dimension 6 endowed with a left invariant complex structure.

If X has a holomorphic contact structure, then X is:

- *either the Iwasawa manifold*
- *or the non-complex parallelisable nilmanifold $X = G/\Gamma$, where:*

$G = (\mathbb{C}^3, \star)$ is given by the non-holomorphic group operation:

$$(\zeta_1, \zeta_2, \zeta_3)\star(z_1, z_2, z_3) = \Big(\zeta_1+z_1,\ \zeta_2+z_2+\overline{\zeta}_1\, z_1,\ \zeta_3+z_3-\zeta_1\, z_2-\overline{\zeta}_1\, z_1(\zeta_1+\frac{z_1}{2})$$

the lattice $\Gamma \subset G$ consists of $(z_1, z_2, z_3) \in G$ with $z_1 \in 2\mathbb{Z}[i]$ and $z_2, z_3 \in \mathbb{Z}[i]$;

Hence:

- $X = G/\Gamma$ *is a compact complex* 3*-fold equipped with a basis of* $(1,0)$*-forms* $\varphi_1, \varphi_2, \varphi_3$ *on* X *that satisfy the structure equations:*

$$d\varphi_1 = 0, \qquad d\varphi_2 = \varphi_1 \wedge \overline{\varphi}_1, \qquad d\varphi_3 = \varphi_1 \wedge \varphi_2.$$

- φ_3 *defines a* **holomorphic contact structure** *(=* *a* **holomorphic 1-contact structure)** *on* X *since* $\bar{\partial}\varphi_3 = 0$ *and*

$$\varphi_3 \wedge \partial\varphi_3 = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \neq 0.$$

(III) Examples in arbitrary dimensions $n \equiv 3 \ mod \ 4$

Higher-dimensional Iwasawa-type manifolds $X = G/\Gamma$

Proposition (**Kasuya-P-Ugarte 2025**)

Let $n = 2p+1 = 4l+3$ and let G be the nilpotent n-dimensional complex Lie group equipped with the complex structure defined by either of the following two classes of structure equations involving a basis of holomorphic $(1,0)$-forms $\varphi_1, \dots, \varphi_n$:

(Class I) $$d\varphi_1 = d\varphi_2 = 0,$$
$$d\varphi_3 = \varphi_2 \wedge \varphi_1, \dots, d\varphi_n = \varphi_{n-1} \wedge \varphi_1;$$

(Class II) $$d\varphi_1 = \cdots = d\varphi_{n-1} = 0,$$
$$d\varphi_n = \varphi_1 \wedge \varphi_2 + \varphi_3 \wedge \varphi_4 + \cdots + \varphi_{n-2} \wedge \varphi_{n-1}.$$

Then:

(1) *if* G *is in* **class I**, *given any* $(4l+1)\times(4l+1)$ *invertible upper triangular matrix* $A=(a_{ij})_{3\leq i,j\leq 4l+3}$, *we define:*

- *the* $(1,\,0)$*-forms* $\gamma_u=\sum_{i=u}^{4l+3}a_{ui}\,\varphi_i$, *for every* $u\geq 3$;
- *the* $(p=2l+1,\,0)$*-form* Γ_l *given by*

$$\Gamma_l=\gamma_3\wedge(\gamma_4\wedge\gamma_5+\gamma_5\wedge\gamma_6+\gamma_6\wedge\gamma_7)\wedge\ldots \\ \wedge(\gamma_{4l}\wedge\gamma_{4l+1}+\gamma_{4l+1}\wedge\gamma_{4l+2}+\gamma_{4l+2}\wedge\gamma_{4l+3}).$$

For any co-compact lattice $\Lambda\subset G$, *the* $(p=2l+1,\,0)$*-form* Γ_l *defines a* **holomorphic** p**-contact structure** *on the compact nilmanifold* $X=G/\Lambda$.

Meanwhile, X *does not admit any holomorphic contact structure.*

(2) *if* G *is in* **class II**, *for any co-compact lattice* $\Lambda \subset G$, *the* $(p, 0)$*-form*

$$\Gamma = \varphi_n \wedge (\partial\varphi_n)^l = \varphi_n \wedge \left(\sum_{j=0}^{2l} \varphi_{2j+1} \wedge \varphi_{2j+2} \right)^l$$

defines a **holomorphic** p**-contact structure** *on the compact nilmanifold* $X = G/\Lambda$.

This is induced by the **standard contact structure** φ_n *on* X.

Corollary (**non-deformation openness**)

Let $\pi : \mathcal{X} \longrightarrow B$ *be a proper holomorphic submersion.*

$X_t := \pi^{-1}(t), \quad t \in B$, *is a compact complex manifold.*

Then:

X_0 **holomorphic p-contact** $\not\Rightarrow$ X_t **holomorphic p-contact**

for $t \sim 0$.

Counter-example(Kasuya-P-Ugarte)

X_0 is the *Iwasawa manifold*;
X_t the small deformations in the Nakamura class (ii) or (iii).

Then, no such X_t carries any holomorphic p-contact structure.

We have: $n = 3$, $p = 1$ and

$h^{1,0}_{\bar\partial}(X_0) = 3, \qquad$ but $h^{1,0}_{\bar\partial_t}(X_t) = 2$ for $t \neq 0$.

(II) Holomorphic families of holomorphic p-contact structures

• Let $\pi : \mathcal{X} \longrightarrow B$ be a *proper holomorphic submersion*,

where $\mathcal{X}$ and B are complex manifolds.

(We often take $B \subset \mathbb{C}^N$ to be an open ball centred at 0, for some positive integer N.)

• Let $\mathcal{L} \longrightarrow \mathcal{X}$ be a *holomorphic line bundle* over the total space of the family. For $t \in B$, we denote by

$$L_t := \mathcal{L}_{|X_t} \longrightarrow X_t$$

the restriction of $\mathcal{L}$ to the fibre above t.

It is a holomorphic line bundle on X_t.

Let $X_t := \pi^{-1}(t)$, with $t \in B$, be the fibres.

Suppose that $\dim_{\mathbb{C}} X_t = n = 2p + 1 = 4l + 3$.

Non-jumping assumption

$$(H1) \quad \dim_{\mathbb{C}} H^{p,\,0}_{\bar{\partial}_t}(X_t,\, L_t) \quad \text{is independent of } t \in B. \tag{1}$$

This implies that the coherent sheaf (an $\mathcal{O}_B$-module):

$$\mathcal{H}^{p,\,0}(\mathcal{L}) := \pi_\star \left(\Omega^p_{\mathcal{X}/B} \otimes \mathcal{L} \right) \tag{2}$$

is **locally free**.

The fibres of the holomorphic vector bundle $\mathcal{H}^{p,\,0}(\mathcal{L}) \longrightarrow B$ are:

$$\mathcal{H}^{p,\,0}(\mathcal{L})_t \simeq H^0(X_t,\, \Omega^p_{X_t} \otimes L_t) = H^{p,\,0}_{\bar\partial_t}(X_t,\, L_t), \qquad t \in B.$$

Each cohomology class in $H^{p,\,0}_{\bar\partial_t}(X_t,\, L_t)$ is a singleton, so

$$H^{p,\,0}_{\bar\partial_t}(X_t,\, L_t) = C^\infty_{p,\,0}(X_t,\, L_t) \cap \ker \bar\partial_t, \qquad t \in B.$$

Remark

We don't need to assume that $\mathcal{X}$ is Kähler.

Theorem (P. 2026) (**deformation openness**)

Suppose there exists an L_0-valued holomorphic p-contact structure on X_0:

$$\Gamma_0 \in H^{p,\,0}_{\bar\partial_0}(X_0,\, L_0).$$

Then, after possibly shrinking B about 0, there exists a holomorphic family $(\Gamma_t)_{t\in B}$ of **L_t-valued holomorphic p-contact structures**

$$\Gamma_t \in H^{p,\,0}_{\bar\partial_t}(X_t,\, L_t)$$

on X_t whose member for $t = 0$ is the given Γ_0.

Proof.

By hypothesis, $\Gamma_0 \in \mathcal{H}^{p,\,0}(\mathcal{L})_0$ is a non-zero element in the fibre at $0 \in B$ of the holomorphic vector bundle $\mathcal{H}^{p,\,0}(\mathcal{L})$ over B.

Thus, Γ_0 extends (non-uniquely and non-canonically) to a non-vanishing holomorphic section Γ of this vector bundle on a small neighbourhood of 0 in B (that can be taken to be B after possibly shrinking B):

$$\Gamma \in H^0\Big(B,\, \mathcal{H}^{p,\,0}(\mathcal{L})\Big)$$

such that $\Gamma(0) = \Gamma_0$.

We have a C^∞ (hence also continuous) map:

$$\mathcal{X} \simeq B \times X \ni (t,\, z) \longmapsto \Gamma_t(z) \wedge (\partial_t \Gamma_t)(z),$$

where $t \in B$ is the base variable and

$$z \in X_0 \simeq X \qquad\qquad C^\infty \text{ diffeomorphism}$$

is the fibre (vertical) variable obtained after identifying each fibre X_t with a fixed C^∞ manifold X (by Ehresmann).

From

$$\Gamma_0 \wedge (\partial_0 \Gamma_0) \neq 0, \qquad \text{everywhere on } X_0,$$

by continuity and compactness of the X_t's, we get:

$$\Gamma_t \wedge (\partial_t \Gamma_t) \neq 0, \qquad \text{everywhere on } X_t, \text{ for all } t \sim 0.$$

Corollary

Suppose there exists an L_0-valued holomorphic p-contact structure on X_0:

$$\Gamma_0 \in H^{p,\,0}_{\bar\partial_0}(X_0,\, L_0).$$

Then, after possibly shrinking B about 0, for every p-contact basis $\{\Gamma_1^0, \ldots, \Gamma_r^0\}$ of $\mathcal{H}^{p,\,0}(\mathcal{L})_0 = H^{p,\,0}_{\bar\partial_0}(X_0,\, L_0)$ (always exists!),

there is a holomorphic frame $\Gamma_1, \ldots, \Gamma_r \in H^0(B,\, \mathcal{H}^{p,\,0}(\mathcal{L}))$ such that:

(i) $\Gamma_j(0) = \Gamma_j^0$ for all $j = 1, \ldots, r$;

(ii) $\Gamma_j(t) \in E_t = H^{p,\,0}_{\bar\partial_t}(X_t,\, L_t)$ is an L_t-valued holomorphic **p-contact** *form on X_t for all $t \in B$ and all $j = 1, \ldots, r$.*

(III) Curvature of relative holomorphic p-contact bundles

Proposition (P. 2026)

Suppose that:

(a) B *is* **compact**, $dim_{\mathbb{C}} X = N$;

(b) there exists $\Gamma \in H^0\Big(B,\, \mathcal{H}^{p,\,0}(\mathcal{L})\Big) \setminus \{0\}$.

Then, there exists **no** C^∞ *Hermitian fibre metric* h *on* $\mathcal{H}^{p,\,0}(\mathcal{L})$ *whose Chern curvature form* $i\Theta_h(\mathcal{H}^{p,\,0}(\mathcal{L}))$ *satisfies, for some Gauduchon metric* ρ *on* B*:*

(i) the inequality:

$$\left\{\left(i\Theta_h(\mathcal{H}^{p,\,0}(\mathcal{L}))\wedge\rho^{N-1}\right)u,\,u\right\}_h\leq 0$$

for every $t\in B$ *and every* $u\in\mathcal{H}^{p,\,0}(\mathcal{L})_t$*;*

(ii) and the strict inequality

$$\left\{\left(i\Theta_h(\mathcal{H}^{p,\,0}(\mathcal{L}))\wedge\rho^{N-1}\right)u,\,u\right\}_h< 0$$

for some $t_0\in B$ *and every* $u\in\mathcal{H}^{p,\,0}(\mathcal{L})_{t_0}\setminus\{0\}$*.*

Theorem (P. 2026)

Suppose that:

(a) X_t *is* **compact Kähler** *for every* $t \in B$*;*

(b) there exists $t_0 \in B$ *such that the restriction* $L_{t_0} := \mathcal{L}_{|X_{t_0}}$ *is a* **holomorphic** p**-contact bundle**.

Then, there exists **no** *(singular or otherwise) Hermitian fibre metric* h *on* $\mathcal{L}$ *such that:*

- *the restriction* $h_{t_0} := h_{|L_{t_0}}$ *is non-degenerate;*
- $i\Theta_h(\mathcal{L}) \leq_{n-p,\,\omega} 0$ *everywhere on* $\mathcal{X}$.

Proof.

Follows from a result of Broder-P. (2025) + regularisation in the m-positive setting (Dinew-P. 2025).

Question

Can $\mathcal{L}$ *and/or* $\mathcal{H}^{p,0}(\mathcal{L})$ *be* **(semi-)positive** *in the holomorphic p-contact setting?*

Let us consider the holomorphic vector bundles:

$$E := \mathcal{H}^{p,\,0}(\mathcal{L}) = \pi_\star\Big(\Omega^p_{\mathcal{X}/B} \otimes \mathcal{L}\Big) \longrightarrow B$$

and

$$F := \mathcal{H}^{n,\,0}(\mathcal{L}) = \pi_\star\Big(\Omega^n_{\mathcal{X}/B} \otimes \mathcal{L}^2\Big) \longrightarrow B.$$

F provides a special case of Berndtson's positivity setting.

Our assumption: for every $t \in B$,

$$\exists\, \Gamma_t \in H^{p,\,0}_{\bar\partial_t}\Big(X_t,\, L_t\Big)$$

L_t-valued holomorphic p-contact structure on X_t.

This assumption implies that: $K_{X_t} \otimes L_t^2$ is *trivial* for every $t \in B$.

In particular, we get:

$$H^{n,\,0}(X_t,\, L_t^2) \simeq H^0(X_t,\, K_{X_t} \otimes L_t^2) \simeq H^0(X_t,\, \mathbb{C}) \simeq \mathbb{C}, \qquad t \in B,$$

hence

$$\dim_{\mathbb{C}} H^{n,\,0}(X_t,\, L_t^2) = 1, \qquad t \in B.$$

Therefore, $F := \mathcal{H}^{n,\,0}(\mathcal{L}) = \pi_\star\left(\Omega^n_{\mathcal{X}/B} \otimes \mathcal{L}^2\right) \longrightarrow B$ is *locally free*.

It is a *holomorphic line bundle* whose fibres are:

$$F_t = H^{n,\,0}(X_t,\, L_t^2), \qquad t \in B.$$

There is a standard way of defining a C^∞ Hermitian fibre metric h_F on $F \longrightarrow B$ (and one h_E on $E \longrightarrow B$) from a given C^∞ Hermitian fibre metric h on $\mathcal{L} \longrightarrow \mathcal{X}$.

Definition (standard)

For all $t \in B$ *and* $\Gamma_t, \Omega_t \in F_t = H^{n,\,0}(X_t,\, L_t^2)$, *we set:*

$$\langle \Gamma_t,\, \Omega_t \rangle_{h_{F_t}} := \langle\langle \Gamma_t,\, \Omega_t \rangle\rangle_{F_t,\, h_t^2} := \int\limits_{X_t} i^{n^2}\, \{\Gamma_t,\, \Omega_t\}_{h_t^2}$$

the L^2 **inner product** *induced by* h_t^2 *on* $F_t \longrightarrow X_t$.

Here: $\{\Gamma_t,\, \Omega_t\}_{h_t^2}$ is a $\mathbb{C}$-valued $(n,\, n)$-form on X_t.

Let:

$D_{\mathcal{L}} = D'_{\mathcal{L}} + D''_{\mathcal{L}}$ be the *Chern connection* of $(\mathcal{L},\, h)$;

$D_L = D'_L + D''_L = (D'_{L_t})_{t\in B} + (D''_{L_t})_{t\in B}$ be the *fibrewise Chern connection* of $(L_t)_{t\in B}$.

Definition (P. 2026)

(i) Consider the following **holomorphic quadratic** *bundle map:*

$$S : E \longrightarrow F, \qquad S(u) := u \wedge D'_L u.$$

(ii) Consider the following **holomorphic symmetric bilinear** *bundle map:*

$$A : E \times E \longrightarrow F, \qquad A(u,\, v) := \frac{1}{2}\Big(u \wedge D'_L v + D'_L u \wedge v\Big).$$

(iii) $(E,\, h_E) \longrightarrow B$ *is said to be* **A-twisted Griffiths (semi-)positive** *if for every open subset* $U \subset B$ *and every* p*-contact section* $\Gamma \in C^\infty(U,\, E)$*, we have:*

$$\frac{1}{A(\Gamma,\, \Gamma)}\, A\Big(i\Theta_{h_E}(E)\,\Gamma,\, \Gamma\Big) > 0 \quad (\textit{respectively} \quad \geq 0)$$

as a $\mathbb{C}$*-valued* $(1,\, 1)$*-form on* U*. In this case, we write:*

$$(E,\, h_E) > 0 \ \textit{(A-twisted Griffiths)},$$

respectively $(E,\, h_E) \geq 0$ *(A-twisted Griffiths).*

Observation

For any open subset $U \subset B$ *and any* p**-contact section** $\Gamma \in C^\infty(U,\, E)$,

$$\frac{1}{A(\Gamma,\, \Gamma)}\, A\Big(i\Theta_{h_E}(E)\,\Gamma,\, \Gamma\Big)$$

is a well-defined $\mathbb{C}$*-valued* $(1,\, 1)$*-form on* U*.*

Theorem (P. 2026)

Suppose that for every $t \in B$, there exists an L_t-valued holomorphic p-contact structure on X_t:

$$\Gamma_t \in H^{p,\,0}_{\bar{\partial}_t}\Big(X_t,\, L_t\Big).$$

Then, for any C^∞ Hermitian fibre metric h on $\mathcal{L} \longrightarrow \mathcal{X}$ and any Hermitian metric ω on $\mathcal{X}$, the following implications hold:

$$\begin{aligned} i\Theta_{h_F}(F) \geq 0 \;&\Longrightarrow\; (E,\, h_E) \geq 0 \quad (\textbf{A-twisted Griffiths}) \\ i\Theta_{h_F}(F) > 0 \;&\Longrightarrow\; (E,\, h_E) > 0 \quad (\textbf{A-twisted Griffiths}). \end{aligned}$$

Theorem (Berndtsson 2009)

If ω is **Kähler**, *one has the implications:*

$$\begin{aligned} i\Theta_h(\mathcal{L}) \geq 0 &\implies i\Theta_{h_F}(F) \geq 0 \\ i\Theta_h(\mathcal{L}) > 0 &\implies i\Theta_{h_F}(F) > 0. \end{aligned}$$

(IV) Moduli spaces

Let:

$\pi : \mathcal{X} \longrightarrow B$ be a *proper holomorphic submersion*;

$\mathcal{L} \longrightarrow \mathcal{X}$ be a C^∞ complex line bundle.

Consider

$$\mathcal{H}ol(\mathcal{L}) = \left\{ D'' \textbf{ holomorphic structures on } \mathcal{L} \right\}$$

equipped with the C^∞ topology.

For every $D'' \in \mathcal{H}ol(\mathcal{L})$, consider the holomorphic line bundle:

$$\mathcal{L}_{D''} := (\mathcal{L},\, D'') \longrightarrow \mathcal{X}.$$

Hypothesis $(H_1^{(r)})$

For every fixed $r \in \mathbb{N}$, *we will say that a holomorphic structure* $D'' \in \mathcal{H}ol(\mathcal{L})$ *satisfies* **hypothesis** $(H_1^{(r)})$ *globally on* B *if*

$$(H_1^{(r)}) \qquad dim_{\mathbb{C}} H^{p,0}_{D''_t}(X_t, L_t) = r \quad \text{for all } t \in B.$$

Definition (P. 2026)

For each $r \in \mathbb{N}$, we consider the topological space consisting of

$$\mathcal{C}_p^{(r)}(\mathcal{X},\, B,\, \mathcal{L}) := \Big\{ D'' \in \mathcal{H}ol(\mathcal{L}) \;\Big|\; D'' \text{ satisfies } (H_1^{(r)}) \text{ globally on } B$$
$$\text{and, for every } t_0 \in B, \text{ there exists}$$
$$\text{an open neighbourhood } U \text{ of } t_0,$$
$$\text{with compact closure in } B,$$
$$\text{and a continuous } p\text{-contact section}$$
$$\Gamma \in C^0\Big(U,\, \mathcal{H}^{p,\,0}_{D''}(\mathcal{L})_{|U}\Big)\Big\}$$

equipped with the C^∞ topology.

Lemma (P. 2026)

For each $r \in \mathbb{N}$ and any C^∞ Hermitian fibre metric h on $\mathcal{L}$, the right action:

$$\begin{aligned} \mathcal{C}_p^{(r)}(\mathcal{X},\, B,\, \mathcal{L}) \times U_h(\mathcal{L}) &\longrightarrow \mathcal{C}_p^{(r)}(\mathcal{X},\, B,\, \mathcal{L}), \\ (D'',\, T) &\longmapsto D''_T := T^{-1} \circ D'' \circ T, \end{aligned}$$

is **well defined** *in the following sense:*

(a) if D'' satisfies $(H1^{(r)})$ globally on B, then so does D''_T;

(b) for any open subset $U \subset B$, if $\Gamma \in C^0\left(U,\, \mathcal{H}^{p,\,0}_{D''}(\mathcal{L})\right)$ is a continuous p-contact section, then $T^{-1}\Gamma \in C^0\left(U,\, \mathcal{H}^{p,\,0}_{D''_T}(\mathcal{L})\right)$ is a continuous p-contact section for every $T \in U_h(\mathcal{L})$.

Definition (P. 2026)

For each $r \in \mathbb{N}$, *the quotient set*

$$\mathcal{M}_p^{(r)}(\mathcal{X},\, B,\, \mathcal{L}) := \mathcal{C}_p^{(r)}(\mathcal{X},\, B,\, \mathcal{L})/U_h(\mathcal{L}),$$

equipped with the quotient topology, is called the **r-th moduli space of relative holomorphic p-contact structures** *on* $(\mathcal{X},\, B,\, \mathcal{L})$.

Theorem (P. 2026)

Fix an arbitrary $r \in \mathbb{N}$.

(1) *The moduli space* $\mathcal{M}_p^{(r)}(\mathcal{X},\, B,\, \mathcal{L})$ *is* **Hausdorff**.

(2) *If* B *is* **compact**, *the moduli space* $\mathcal{M}_p^{(r)}(\mathcal{X},\, B,\, \mathcal{L})$ *is* **open** *in the quotient topological space* $\mathcal{P}_p^{(r)}(\mathcal{X},\, B,\, \mathcal{L})/U_h(\mathcal{L})$, *where*

$$\mathcal{P}_p^{(r)}(\mathcal{X},\, B,\, \mathcal{L}) := \left\{ D'' \in \mathcal{H}ol(\mathcal{L}) \mid D'' \text{ satisfies } (H_1^{(r)}) \text{ globally on } B \right\}$$

equipped with the C^∞ *topology.*