m-Positivity and a Monge-Ampère-Type Equation for Forms of Positive Degree

joint work with **Sławomir Dinew (Krakow)**

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General context (Dieu '06, Harvey and Lawson '13, Verbitsky '10, Dinew '22

(X, ω) a complex Hermitian manifold, $\dim_{\mathbb{C}} X = n$;

$m \in \{1, \ldots, n\}$

T be a real current of bidegree $(1, 1)$ on X.

T is **m-semi-positive (resp. m-positive) w.r.t. ω** $\stackrel{def}{\Longleftrightarrow}$

the bidegree-(m, m)-current $T \wedge \omega^{m-1}$ is **strongly semi-positive (resp. strongly positive)** on X.

We denote this property by $T \geq_{m,\omega} 0$ (resp. $T >_{m,\omega} 0$).

When $\varphi : U \longrightarrow \mathbb{R} \cup \{-\infty\}$ is an upper semi-continuous function on an open subset $U \subset X$, we have:

$$\varphi \quad \text{is } \textbf{m}\text{-psh on } U \text{ w.r.t. } \omega \overset{def}{\Longleftrightarrow} T := i\partial\bar{\partial}\varphi \geq_{m,\omega} 0 \text{ on } U.$$

When $m \geq 2$, these notions depend on the choice of ω.

Example. Take:

$$\omega = a\, idz_1 \wedge d\bar{z}_1 + b\, idz_2 \wedge d\bar{z}_2 + c\, idz_3 \wedge d\bar{z}_3$$

and

$$T = p\, idz_1 \wedge d\bar{z}_1 + q\, idz_2 \wedge d\bar{z}_2 + r\, idz_3 \wedge d\bar{z}_3,$$

with $a, b, c > 0$ and $p, q, r \in \mathbb{R}$.

Then,

$$\begin{aligned} T \wedge \omega \;=\; & (pb + qa)\, i dz_1 \wedge d\bar{z}_1 \wedge i dz_2 \wedge d\bar{z}_2 \\ + \; & (pc + ra)\, i dz_1 \wedge d\bar{z}_1 \wedge i dz_3 \wedge d\bar{z}_3 \\ + \; & (qc + rb)\, i dz_2 \wedge d\bar{z}_2 \wedge i dz_3 \wedge d\bar{z}_3, \end{aligned}$$

so we get the equivalence:

$$T \geq_{2,\omega} 0 \iff pb + qa \geq 0, \quad pc + ra \geq 0 \quad \text{and} \quad qc + rb \geq 0.$$

In particular, the current T defined by the coefficients $p = q = 1$ and $r = -1$ satisfies the condition $T \geq_{2,\omega} 0$ when, for example, $a = b = 1$ and $c = 3$, but this T is not 2-semi-positive with respect to ω when $c < a$ or $c < b$.

Lemma. Suppose T is continuous. Then:

$$T \geq_{m,\omega} 0 \iff \lambda_1 + \cdots + \lambda_m \geq 0 \quad \text{at every point of } X,$$

where $\lambda_1 \leq \cdots \leq \lambda_m \leq \cdots \leq \lambda_n$ are the eigenvalues of T w.r.t. ω.

Recall. If T is of class C^2, for every point $x \in X$, there exist local holomorphic coordinates $z_1, \ldots, z_n$ on X centred at x such that

$$\omega(x) = \sum_{j=1}^{n} i dz_j \wedge d\bar{z}_j \qquad \text{and} \qquad T(x) = \sum_{j=1}^{n} \lambda_j(x)\, i dz_j \wedge d\bar{z}_j.$$

Geometric context (Dinew-P 2025)

Definition. *Let* $L \longrightarrow X$ *be a holomorphic line bundle over a complex manifold with* $\dim_{\mathbb{C}} X = n$. *Let* $m \in \{1, \dots, n\}$.

L is m**-semi-positive with a** C^∞ **metric** *(respectively,* m**-semi-positive with a singular metric***)*

$\stackrel{def}{\Longleftrightarrow}$ $\exists\,\omega$ C^∞ *Hermitian metric on* X

$\exists\, h$ C^∞ *(or singular) Hermitian fibre metric on* L

such that

$$i\Theta_h(L) \geq_{m,\omega} 0.$$

First group of results (Dinew-P 2025)

(A) **Vanishing theorems**

Theorem (Dinew-P 2025)

(i) If $i\Theta_h(L) \geq_{q,\,\omega} c\,\omega$ *on* X *for some* $q \in \{1, \dots, n\}$ *and some constant* $c > 0$, *then*

$$H^{n,\,l}_{\bar\partial}(X,\, L) = \{0\} \qquad \textit{for all} \quad l \geq q.$$

(ii) If $i\Theta_h(L) \leq_{n-p,\,\omega} -c\,\omega$ *on* X *for some* $p \in \{0, \dots, n-1\}$ *and some constant* $c > 0$, *then*

$$H^{l,\,0}_{\bar\partial}(X,\, L) = \{0\} \qquad \textit{for all} \quad l \leq p.$$

(B) $\bar\partial$-equation: resolution and L^2-estimates

Theorem (Dinew-P 2025)

(i) If $i\Theta_h(L) \geq_{q,\,\omega} c\,\omega$ *on* X *for some* $q \in \{1, \dots, n\}$ *and some constant* $c > 0$*, then:*

for every $l \geq q$ *and every* $v \in C^\infty_{n,\,l}(X,\,L)$ *such that* $\bar\partial v = 0$,

there exists $u \in C^\infty_{n,\,l-1}(X,\,L)$ *such that* $\bar\partial u = v$ *and*

$$\int\limits_X |u|^2_{\omega,\,h}dV_\omega \leq \frac{1}{c\,l}\int\limits_X |v|^2_{\omega,\,h}dV_\omega.$$

(ii) If $i\Theta_h(L) \leq_{n-q,\,\omega} -c\,\omega$ *for some* $q \in \{1, \dots, n-1\}$ *and some constant* $c > 0$*, then:*

for every $l \leq q$ *and every* $v \in C^\infty_{0,\,l}(X,\,L)$ *such that* $\bar\partial v = 0$,

there exists $u \in C^\infty_{0,\,l-1}(X,\,L)$ *such that* $\bar\partial u = v$ *and*

$$\int\limits_X |u|^2_{\omega,\,h} dV_\omega \leq \frac{1}{c(n-l)} \int\limits_X |v|^2_{\omega,\,h} dV_\omega.$$

The proofs are based on a technique by Demailly:

- **Bochner-Kodaira-Nakano-type identity:**

$$\Delta'' = \Delta'_\tau + [i\Theta_h(L) \wedge \cdot\,,\ \Lambda_\omega] + T_\omega$$

(when ω is general Hermitian)

$$\Delta'' = \Delta' + [i\Theta_h(L) \wedge \cdot\,,\ \Lambda_\omega]$$

(when ω is Kähler)

• **Curvature operator formula (Demailly):**

$$[i\Theta_h(L)\wedge\cdot\,,\,\Lambda_\omega]\,u = \sum_{J,\,K}\left(\sum_{j\in J}\lambda_j + \sum_{k\in K}\lambda_k - \sum_{l=1}^{n}\lambda_l\right) u_{J\overline{K}}\, dz_J\wedge d\bar{z}_K$$

for every form $u = \sum_{J,\,K} u_{J\overline{K}}\, dz_J\wedge d\bar{z}_K$ at a given point $x\in X$ about which local holomorphic coordinates $z_1,\ldots,z_n$ have been chosen to simultaneously diagonalise ω and $i\Theta_h(L)$.

Second group of results (Dinew-P 2025)

Global regularisation theorem (Dinew-P 2025)

Let (X, ω) be a compact complex Hermitian manifold with $\dim_{\mathbb{C}} X = n$. Fix $m \in \{1, \dots, n\}$ and suppose there exists a C^∞ real $(1, 1)$-form χ on X such that $d\chi = 0$ and $\chi >_{m,\omega} 0$.

For any upper semi-continuous L^1_{loc} function $\varphi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ such that $\chi + i\partial\bar\partial\varphi \geq_{m,\omega} 0$ on X, there exists a sequence $(\varphi_j)_{j\geq 1}$ of C^∞ functions $\varphi_j : X \longrightarrow \mathbb{R}$ such that:

(i) $\chi + i\partial\bar\partial\varphi_j \geq_{m,\omega} 0$ *on* X *for every* $j \geq 1$;

(ii) *for every* $x \in X$, $\varphi_j(x) \searrow \varphi(x)$ *as* $j \to \infty$.

Local regularisation theorem (Dinew-P 2025)

Let (X, ω) be a complex Hermitian manifold.

Then, for every:

- *point $p \in X$,*
- *every open neighbourhood $U_p \subset X$ of p,*
- *every m-psh (w.r.t. ω) function $u : U_p \longrightarrow \mathbb{R} \cup \{-\infty\}$,*

there exists a sequence $(u_j)_{j\geq 1}$ of C^∞ strictly m-psh (w.r.t. ω) functions $u_j : U_p \longrightarrow \mathbb{R}$ such that, for every $x \in X$:

$$u_j(x) \searrow u(x)$$

as $j \rightarrow \infty$.

Some ingredients in the proofs.

- Viscosity subsolutions

$\cdot$ $P_m(X,\omega,\chi) := \Big\{\varphi : X \to \mathbb{R}\cup\{-\infty\}$ u.s.c. L^1_{loc} function such that $\chi + i\partial\bar\partial\varphi \geq_{m,\omega} 0\Big\}$.

$\cdot$ $\forall v \in P_m(X,\omega,\chi)\cap C^2$, let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of $\chi + i\partial\bar\partial v$ w.r.t. ω. **Cheng-Xu (2025)** set $F_m[\chi + i\partial\bar\partial v] : X \longrightarrow \mathbb{R}$,

$$F_m[\chi + i\partial\bar\partial v](z) := \Big(\Pi_{|J|=m}\Big[\sum_{j\in J}\lambda_j(z)\Big]\Big)^{1/\binom{n}{m}},$$

at every $z \in X$, where $J = (j_1, \cdots, j_m)$ runs over all ordered m-tuples of indices from $\{1,\cdots,n\}$.

Definition (Cheng-Xu 2025)

Let g be a non-negative continuous function on X. A function $\varphi \in P_m(X, \omega, \chi)$ is said to be a **viscosity subsolution** *of the equation*

$$F_m[\chi + i\partial\bar{\partial}v] = g$$

if for any point $p \in X$ and any C^2 function $\tau : U \to \mathbb{R}$ defined in a neighbourhood U of p such that $\tau \geq \varphi$ on U and $\tau(p) = \varphi(p)$ (any such function τ is called a **testing function for φ at p***), one has $F_m[\chi + i\partial\bar{\partial}\tau](p) \geq g(p)$.*

In this case, we write $F_m[\chi + i\partial\bar{\partial}\varphi] \geq g$ in the viscosity sense.

Theorem (Cheng-Xu 2025)

$$\forall \beta > e \ \ \exists!\, u_j^\beta \in P_m(X,\omega,\chi) \cap C^\infty(X) \textit{ solution to the equation}$$

$$F_m[\chi + i\partial\bar{\partial}u_j^\beta](z) = e^{\beta(u_j^\beta(z) - f_j(z))} F_m^j(z) + \frac{F_m[\chi](z)}{2\beta}.$$

• C^∞ solvability of the Dirichlet problem for the local F_m operator

Theorem (Dinew-P 2025) *Let:*

· (X,ω) *be a complex Hermitian manifold;*

· $p \in X$ *be an arbitrary point;*

· $U_p \subset X$ *a sufficiently small ball in a coordinate chart about* p.

Then, for any constant $\beta > e$, *the Dirichlet problem:*

$$\begin{cases} u_j^\beta \in C^\infty(\overline{U}_p); \qquad i\partial\bar\partial u_j^\beta >_{m,\omega} 0; \\ F_m[i\partial\bar\partial u_j^\beta] = e^{\beta(u_j^\beta - f_j)} + \frac{1}{2\beta} \qquad\qquad \text{in } U_p; \\ u_j^\beta|_{\partial U_p} = f_j|_{\partial U_p} \end{cases}$$

admits a unique solution u_j^β.

Third group of results (Dinew-P 2025)

Definition (Dinew-P 2025)

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ and let $m \in \{1, \dots, n\}$.

(1) *For every real current T of bidegree $(1,\,1)$ such that $dT = 0$ and every Kähler metric (if any) ω on X, the* m^{th} **$[\omega]$-twisted class** *of T is the Bott-Chern class:*

$$[T]_{BC} \wedge [\omega_{m-1}]_{BC} = [T \wedge \omega_{m-1}]_{BC} \in H^{m,\,m}_{BC}(X,\,\mathbb{R}).$$

The Bott-Chern class $[T]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R})$ *is said to be:*

- $[\omega]$-m-**pseudo-effective** *(*$[\omega]$-m-**psef***),*

$$\textit{(}[T]_{BC} \wedge [\omega_{m-1}]_{BC} \geq 0 \textit{ or by } [T \wedge \omega_{m-1}]_{BC} \geq 0,\textit{)}$$

if there exists a real current S *of bidegree* $(m-1, m-1)$ *on* X *such that*

$$T \wedge \omega_{m-1} + i\partial\bar{\partial}S \geq 0 \quad \textit{(strongly)} \tag{1}$$

as an (m, m)*-current on* X*;*

- $[\omega]$-m-**big**,

$$([T]_{BC} \wedge [\omega_{m-1}]_{BC} > 0 \textit{ or by } [T \wedge \omega_{m-1}]_{BC} > 0,)$$

if there exists a real current S of bidegree $(m-1,\, m-1)$ on X such that

$$T \wedge \omega_{m-1} + i\partial\bar{\partial} S \geq \varepsilon\, \gamma^m \;\; \textit{(strongly)}$$

as an $(m,\, m)$-current on X, for some constant $\varepsilon > 0$ and some $(1,\, 1)$-form $\gamma > 0$.

(2) *Suppose that X is* **Kähler**. *For every Kähler class* $[\omega]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R})$,

• *the following subset is called the* $[\omega]$-m-**pseudo-effective** *(*$[\omega]$-m-**psef***)* **cone** *of X :*

$$\mathcal{E}_{[\omega],\, m}(X) := \left\{ [T]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R}) \;\middle|\; [T]_{BC} \wedge [\omega_{m-1}]_{BC} \geq 0 \right\};$$

• *the following subset is called the* $[\omega]$-m-**big cone** *of X :*

$$\mathcal{B}_{[\omega],\, m}(X) := \left\{ [T]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R}) \;\middle|\; [T]_{BC} \wedge [\omega_{m-1}]_{BC} > 0 \right\}.$$

Theorem (Dinew-P 2025)

Let (X, ω) *be a compact Kähler manifold with* $dim_{\mathbb{C}} X = n$*. Fix* $m \in \{1, \dots, n\}$*.*

For every cohomology class $\mathfrak{c} \in H^{1,1}_{BC}(X, \mathbb{R})$*, the following two statements are equivalent:*

(a) There exist a C^∞ *d-closed real* $(1, 1)$*-form* $\alpha \in \mathfrak{c}$ *and a form* $\Omega_0 \in C^\infty_{n-m,\, n-m}(X, \mathbb{R})$ *with the properties:*

$$\Omega_0 > 0 \quad (\textit{weakly}) \qquad \textit{and} \qquad \partial\bar\partial\Omega_0 = 0$$

such that $\alpha \wedge \omega^{m-1} \wedge \Omega_0 > 0$ *everywhere on* X*;*

(b) The cohomology class $-\mathfrak{c} \in H^{1,1}_{BC}(X, \mathbb{R})$ *is* **not** $[\omega]$**-**m**-pseudo-effective**.

Fourth group of results (Dinew-P 2025)

A form-solution Monge-Ampère equation

Definition (Dinew-P 2025)

Let: · $(X,\, \omega)$ *be a compact* **Kähler** *manifold,* $dim_{\mathbb{C}} X = n$*;*

· $m \in \{1, \dots, n\}$*;*

· *a volume form* $dV \in C^\infty_{n,\, n}(X,\, \mathbb{R})$ *with* $dV > 0$*;*

· $\alpha \in C^\infty_{m,\, m}(X,\, \mathbb{R})$ *such that* $d\alpha = 0$ *and* $\alpha > 0$ *(strongly);*

we consider the equation:

$$\left[\star_\omega\left((\alpha + i\partial\bar\partial u) \wedge \omega_{n-m-1}\right)\right]^n = dV \qquad\qquad (\star),$$

whose solutions $u \in C^\infty_{m-1,\, m-1}(X,\, \mathbb{R})$*, if any, are subject to the initial conditions:*

$$\alpha + i\partial\bar\partial u > 0 \;\; \textit{(strongly)} \qquad \textit{and} \qquad u \in \ker \partial^\star_\omega \cap \ker \bar\partial^\star_\omega.$$

Recall (general fact):

Given: $\cdot$ a compact Hermitian manifold (X, ω);
$\cdot$ a form $\zeta \in C^\infty_{r,\, r}(X, \mathbb{C})$;

there exist unique *ω-primitive* forms $\zeta^{(l)}_{prim} \in C^\infty_{r-l,\, r-l}(X, \mathbb{C})$ for $l = 0, \dots, r$ such that

$$\zeta = \zeta^{(0)}_{prim} + \zeta^{(1)}_{prim} \wedge \omega + \cdots + \zeta^{(l)}_{prim} \wedge \omega^l + \cdots + \zeta^{(r)}_{prim}\, \omega^r.$$

We call $\zeta^{(0)}_{prim}, \dots, \zeta^{(r)}_{prim}$ the *ω-primitive coordinates* of $\zeta = (\zeta^{(l)}_{prim})_{0 \leq l \leq r}$.

Theorem (Dinew-P 2025): uniqueness

If there exist $u_1,\, u_2 \in C^\infty_{m-1,\, m-1}(X,\, \mathbb{R})$ *such that*

$$\left[\star_\omega\left((\alpha + i\partial\bar\partial u_1)\wedge\omega_{n-m-1}\right)\right]^n = \left[\star_\omega\left((\alpha + i\partial\bar\partial u_2)\wedge\omega_{n-m-1}\right)\right]^n,$$

satisfying the initial conditions:

$$\alpha + i\partial\bar\partial u_j > 0 \ \textit{(strongly)} \qquad \textit{and} \qquad u_j \in \ker\partial^\star_\omega \cap \ker\bar\partial^\star_\omega, \qquad j = 1, 2,$$

and, furthermore, satisfying the property ("boundary conditions"):

$$(u_1)^{(l)}_{prim} = (u_2)^{(l)}_{prim}, \qquad l = 0, \dots, m-2,$$

then there exists a constant $C \in \mathbb{R}$ *such that*

$$u_1 = u_2 + C\,\omega^{m-1}.$$

Definition (Dinew-P 2025)

We say that equation $(\star)$ *is* **solvable** *if for every* $\alpha \in C^\infty_{m,\,m}(X,\,\mathbb{R})$ *such that* $d\alpha = 0$ *and* $\alpha > 0$ *(strongly) and every volume form* $dV \in C^\infty_{n,\,n}(X,\,\mathbb{R})$ *with* $dV > 0$*, there exists a constant* $c > 0$ *and a form* $u \in C^\infty_{m-1,\,m-1}(X,\,\mathbb{R})$ *such that*

$$\left[\star_\omega\left((\alpha + i\partial\bar\partial u)\wedge\omega_{n-m-1}\right)\right]^n = c\,dV$$

and such that u *satisfies the following initial conditions:*

$\alpha + i\partial\bar\partial u > 0$ *(strongly),* $u \in \ker\partial^\star_\omega \cap \ker\bar\partial^\star_\omega$ *and* $\Lambda^{m-2}_\omega(\Delta''_\omega u) = 0.$

Application

On (X, ω) compact Hermitian, given $m \in \{1, \dots, n-1\}$, we consider the map:

$$\Lambda^{m,\,m}T^\star X \ni \alpha \longmapsto \alpha_\omega := \star_\omega\bigg(\alpha \wedge \omega_{n-m-1}\bigg) \in \Lambda^{1,\,1}T^\star X,$$

where $\star_\omega$ is the Hodge star operator associated with ω.

Set-up:

- ω is **Kähler**;
- $\alpha, \beta \in C^\infty_{m,\,m}(X,\,\mathbb{R})$ are supposed to exist such that:

(i) $d\alpha = d\beta = 0$; (ii) $\alpha > 0$ (strongly) and $\beta \geq C\,\omega_m$ (strongly) for some constant $C > 0$;
(iii) $\Delta''_\omega \alpha_\omega = 0$, where $\Delta''_\omega := \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial$ is the $\bar\partial$-Laplacian.

Theorem (Dinew-P 2025)

Suppose that equation $(\star)$ *is* **solvable**. *Then, if the forms* α *and* β *satisfy the condition:*

$$\frac{1}{(n-m)!}\int\limits_X (\alpha_\omega)^{n-m}\wedge\beta < \frac{1}{n!}\int\limits_X (\alpha_\omega)^n,$$

there exists a d-closed real current T *of bidegree* $(m,\, m)$ *on* X *such that:*

$$(i)\, T \geq \delta\,(\alpha_\omega)_m \quad \textit{for some constant } \delta > 0; \quad (ii)\, T \in \Big[(\alpha_\omega)_m - \beta\Big]_{BC},$$

where $[\,\cdot\,]_{BC}$ *stands for the Bott-Chern cohomology class of the specified form.*

Corollary (Dinew-P 2025)

When $m = n-1$*, the inverse image of the class*

$$[(\star_\omega \alpha)_{n-1} - \beta]_{BC} \in H^{n-1,\, n-1}_{BC}(X,\, \mathbb{R})$$

under the Hard Lefschetz isomorphism

$$H^{1,\,1}_{BC}(X,\, \mathbb{R}) \ni [a]_{BC} \longmapsto [\omega_{n-2} \wedge a]_{BC} \in H^{n-1,\, n-1}_{BC}(X,\, \mathbb{R})$$

is an $[\omega]$-$(n-1)$-**big** *class.*