Non-Kähler Hodge Theory and Deformations of Complex Structures

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To my mother

Contents

Introduction

The main theme of this book is the classification of compact complex manifolds. No such complete classification exists at the moment or may even be achievable in the future, but this text aims at offering a unified panorama of old and especially new developments in this direction.

(I) Differential forms and complex structure

Let X be a complex manifold of complex dimension $n \geq 1$. This means that X is a differentiable (C^∞) manifold equipped with a **holomorphic atlas** with values in $\mathbb{C}^n$, namely with an open cover $(U_\alpha)_\alpha$ and C^∞ maps $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{C}^n$ such that the transition maps $\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$ are *holomorphic*.

Equivalently, a complex manifold is a C^∞-differentiable manifold X equipped with a **complex structure**. This is an *almost complex structure*, namely an endomorphism $J : TX^{\mathbb{R}} \longrightarrow TX^{\mathbb{R}}$ of the real tangent bundle such that $J^2 = -\mathrm{Id}$, which is further required to be *integrable* (in the sense that what is called its *Nijenhuis tensor* N_J vanishes).

Alternatively, the complex structure can be seen as a splitting

$$d = \partial + \bar\partial$$

of the Poincaré differential operator $d : C^\infty_k(X, \mathbb{C}) \longrightarrow C^\infty_{k+1}(X, \mathbb{C})$ of order one acting on the $\mathbb{C}$-valued C^∞ differential forms of any degree $k \in \{0, \dots, 2n\}$ on X into two differential operators of order one:

$$\partial : C^\infty_{p,\,q}(X, \mathbb{C}) \longrightarrow C^\infty_{p+1,\,q}(X, \mathbb{C}) \quad \text{and} \quad \bar\partial : C^\infty_{p,\,q}(X, \mathbb{C}) \longrightarrow C^\infty_{p,\,q+1}(X, \mathbb{C})$$

acting on the $\mathbb{C}$-valued C^∞ differential forms of any bidegree $(p,\, q)$, with $p, q \in \{0, \dots, n\}$, on X.

For any complex structure $d = \partial + \bar\partial$, one has

$$\bar\partial^2 = 0,$$

a property that is equivalent to the *integrability condition*. This further implies that $\partial^2 = 0$ and $\partial\bar\partial + \bar\partial\partial = 0$.

At the local level, if $(z_1, \dots, z_n)$ are local holomorphic coordinates on an open subset $U \subset X$, we have $z_k = x_k + i\, y_k$ for every k and $(x_1, y_1, \dots, x_n, y_n)$ is a system of local C^∞ real coordinates on U. The 1-forms $dz_k := dx_k + idy_k$ are said to be of *bidegree* (or *type*) $(1,\, 0)$, while the 1-forms $d\bar z_k := dx_k - idy_k$ are said to be of *bidegree* (or *type*) $(0,\, 1)$. For any $p, q \in \{0, \dots, n\}$, with $p + q = k \in \{0, \dots, 2n\}$, the differential forms of bidegree (or type) $(p,\, q)$ are those k-forms that are generated (locally on U) by exterior products of p dz_j's and q $d\bar z_k$'s:

$$u = \sum_{|J|=p,\,|K|=q} u_{I\bar J}\, dz_J \wedge d\bar z_K, \tag{1}$$

where the coefficients $u_{I\bar{J}}$ are C^∞ $\mathbb{C}$-valued functions on U, while $J := (1 \leq j_1 < \cdots < j_p \leq n)$ and $K := (1 \leq k_1 < \cdots < k_q \leq n)$ are multi-indices of lengths p, resp. q. One puts $dz_J := dz_{j_1} \wedge \cdots \wedge dz_{j_p}$ and $d\bar{z}_K := d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_q}$.

A C^∞ $(p,\, q)$-form on X is a globally and intrinsically defined object. Its local shape (1) transforms, under a change of local holomorphic coordinates from $(z_1, \ldots, z_n)$ on some open subset $U \subset X$ to $(w_1, \ldots, w_n)$ on some open subset $V \subset X$, according to the usual rules of calculus, starting from the identities:

$$dz_j = \sum_{k=1}^n \frac{\partial z_j}{\partial w_k}\, dw_k \quad \text{and} \quad d\bar{z}_j = \sum_{k=1}^n \frac{\partial \bar{z}_j}{\partial \overline{w}_k}\, d\overline{w}_k \tag{2}$$

on $U \cap V$ for every $j \in \{1, \ldots, n\}$. The vector fields

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j}\right)$$

on U are said to be of type $(1,\, 0)$, while the vector fields

$$\frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\,\frac{\partial}{\partial y_j}\right)$$

on U are said to be of type $(0,\, 1)$. The differential of a C^1 function $f : U \longrightarrow \mathbb{C}$ is the 1-form on U given by

$$df = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\, dx_j + \sum_{j=1}^n \frac{\partial f}{\partial y_j}\, dy_j = \sum_{j=1}^n \frac{\partial f}{\partial z_j}\, dz_j + \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j}\, d\bar{z}_j = \partial f + \bar\partial f, \tag{3}$$

where we put $\partial f := \sum_{j=1}^n \frac{\partial f}{\partial z_j}\, dz_j$ (a $(1,\, 0)$-form) and $\bar\partial f := \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j}\, d\bar{z}_j$ (a $(0,\, 1)$-form). Moreover, a C^∞ function $f : U \longrightarrow \mathbb{C}$ is *holomorphic* if and only if $\frac{\partial f}{\partial \bar{z}_j} = 0$ for all $j \in \{1, \ldots, n\}$, namely if and only if $\bar\partial f = 0$. These are the **Cauchy-Riemann equations**. In particular, this accounts for (2), a special case of (3), since $\partial z_j / \partial \overline{w}_k = 0$ and $\partial \bar{z}_j / \partial w_k = 0$ thanks to the functions z_j being holomorphic.

For an arbitrary C^1 form u of bidegree (or type) $(p,\, q)$ on X, ∂u and $\bar\partial u$ are a $(p+1,\, q)$-form, resp. a $(p,\, q+1)$-form, on X. In local coordinates, they are obtained by applying ∂, resp. $\bar\partial$, to the coefficients of u written locally in the form (1), so we get:

$$\partial u = \sum_{|J|=p,\, |K|=q} \partial u_{I\bar{J}} \wedge dz_J \wedge d\bar{z}_K \quad \text{and} \quad \bar\partial u = \sum_{|J|=p,\, |K|=q} \bar\partial u_{I\bar{J}} \wedge dz_J \wedge d\bar{z}_K, \tag{4}$$

where we have, according to (3),

$$\partial u_{I\bar{J}} = \sum_{j=1}^n \frac{\partial u_{I\bar{J}}}{\partial z_j}\, dz_j \quad \text{and} \quad \bar\partial u_{I\bar{J}} = \sum_{j=1}^n \frac{\partial u_{I\bar{J}}}{\partial \bar{z}_j}\, d\bar{z}_j.$$

We stress again that $(p,\, q)$-forms and k-forms on X (in particular, the 1-form $df = \partial f + \bar\partial f$ for any C^1 function $f : X \to \mathbb{C}$) are globally and intrinsically defined objects on X. Indeed, if $TX^{\mathbb{R}}$ denotes the real tangent bundle and $(TX^{\mathbb{R}})^\star$ the real cotangent bundle of X, the complexified exterior algebra $\Lambda^\bullet(\mathbb{C} \otimes TX)^\star := \mathbb{C} \otimes_{\mathbb{R}} \Lambda^\bullet(TX^{\mathbb{R}})^\star$ splits canonically at every point of X as

$$\Lambda^k(\mathbb{C} \otimes TX)^\star = \sum_{p+q=k} \Lambda^{p,\,q} T^\star X, \qquad 0 \leq k \leq 2n,$$

where the space of $(p,\,q)$-forms is defined pointwise as

$$\Lambda^{p,\,q}T^\star X := \Lambda^p T^\star X \otimes \Lambda^q \overline{T^\star X},$$

where $T^\star X$ is the holomorphic cotangent bundle of X (generated locally by the $(1,\,0)$-forms $dz_1,\dots,dz_n$) and $\overline{T^\star X}$ is the anti-holomorphic cotangent bundle of X (generated locally by the $(0,\,1)$-forms $d\bar{z}_1,\dots,d\bar{z}_n$).

In particular, every k-form α splits uniquely into pure-type forms $\alpha^{p,\,q}$ of respective bidegrees $(p,\,q)$:

$$\alpha = \sum_{p+q=k}\alpha^{p,\,q}.$$

The forms $\alpha^{p,\,q}$ are called the pure-type components of α.

(II) Context

Much of the material discussed in this book lies at the intersection of complex differential geometry, complex algebraic geometry and complex analysis, with significant input from geometric PDE's (mainly the theory of elliptic differential and pseudo-differential operators) and pluripotential theory (mainly the theory of currents).

We now indicate briefly how these fields are involved in our discussion.

(A) Complex algebraic geometry

The main objects of study in algebraic geometry are the projective manifolds. A compact complex manifold X is said to be **projective** if it can be embedded as a closed submanifold into a complex projective space, namely if there exists an integer $N\geq 1$ such that $X\hookrightarrow \mathbb{CP}^N$.

More generally, X is said to be **Moishezon** if it is bimeromorphically equivalent to a projective manifold, namely if there exists a projective manifold $\widetilde{X}$ and a holomorphic bimeromorphic map (called a *modification*) $\mu:\widetilde{X}\longrightarrow X$. Intuitively, Moishezon manifolds are those compact complex manifolds that admit "many" (in a precise sense) divisors (= formal linear combinations with integer coefficients of complex hypersurfaces of X).

In particular, projective manifolds have many subvarieties, which can be used to study the geometry of the ambient manifold. Thus, in this algebraic context, the tools and objects of study are often:

-**subvarieties**: curves, hypersurfaces, etc;

-**holomorphic vector bundles** and their **holomorphic sections**;

These lead to a *positivity theory*, a major theme in algebraic geometry. Intuitively, the more global holomorphic sections a holomorphic vector bundle has, the more "positive" it is. For example, positivity notions for complex line bundles include *ampleness*, *bigness* and *nefness*, all of which lie at the heart of very active current research.

-**coherent sheaves**.

These can be seen as a kind of holomorphic vector bundles with singularities.

(B) Complex analytic and differential geometry

One of the main objects of study in this setting is provided by the **Hermitian metrics** on a given complex manifold X. Such a metric is defined by a C^∞, positive definite, $(1,\,1)$-form ω on X. In local coordinates $(z_1, \dots, z_n)$ on some open subset $U \subset X$, it has the shape

$$\omega = \sum_{j,\,k=1}^{n} \omega_{j\bar{k}}\, i dz_j \wedge d\bar{z}_k,$$

where the $\omega_{j\bar{k}}$'s are $\mathbb{C}$-valued C^∞ functions on U such that the matrix $(\omega_{j\bar{k}})_{1\leq j,\,k\leq n}$ is *positive definite* at every point of U. This is equivalent to requiring all the eigenvalues of the matrix $(\omega_{j\bar{k}})_{1\leq j,\,k\leq n}$ to be *positive* at every point of U and implies the Hermitian property

$$\omega_{j\bar{k}} = \overline{\omega_{k\bar{j}}}, \qquad 1 \leq j, k \leq n.$$

The coefficients $\omega_{j\bar{k}}$ depend on the choice of local holomorphic coordinates $(z_1, \dots, z_n)$, but the pointwise positive definiteness $(\omega_{j\bar{k}})_{1\leq j,\,k\leq n} > 0$ does not.

In other words, a real C^∞ $(1,\,1)$-form ω on X defines a Hermitian metric if and only if $\omega > 0$ (in the sense that its coefficient matrix $(\omega_{j\bar{k}})_{1\leq j,\,k\leq n}$ is positive definite at every point in some, hence any, local holomorphic coordinate system). This is equivalent to saying that ω defines a pointwise (positive definite) inner product $\langle\cdot\,,\,\cdot\rangle_\omega : T^{1,\,0}X \times T^{1,\,0}X \longrightarrow \mathbb{C}$ on the holomorphic tangent bundle of X:

$$\langle\cdot\,,\,\cdot\rangle_{\omega,\,x} : T^{1,\,0}_xX \times T^{1,\,0}_xX \longrightarrow \mathbb{C}, \qquad x \in X,$$

and that the inner product $\langle\cdot\,,\,\cdot\rangle_{\omega,\,x}$ on $T^{1,\,0}_xX$ depends in a C^∞ way on $x \in X$. Explicitly, $T^{1,\,0}X$ is generated by $\partial/\partial z_1, \dots, \partial/\partial z_n$ on U and

$$\left\langle \frac{\partial}{\partial z_j}\,,\,\frac{\partial}{\partial z_k}\right\rangle_\omega = \omega_{j\bar{k}}, \qquad 1 \leq j,\, k \leq n,$$

at every point $x \in X$.

The inner product $\langle\cdot\,,\,\cdot\rangle_\omega$ induces by duality a pointwise inner product, denoted by the same symbol, on the holomorphic cotangent bundle $\Lambda^{1,\,0}T^\star X$ (generated by $dz_1, \dots, dz_n$ on U), given by

$$\langle dz_j\,,\,dz_k\rangle_\omega = \omega^{j\bar{k}}, \qquad 1 \leq j,\, k \leq n,$$

at every point $x \in X$, where the matrix $(\omega^{j\bar{k}})_{1\leq j,\,k\leq n}$ is the transpose of the inverse of $(\omega_{j\bar{k}})_{1\leq j,\,k\leq n}$ at every point. By conjugation, we get an induced inner product on $\Lambda^{0,\,1}T^\star X$ and thus also on $\mathbb{C}T^\star X = \Lambda^{1,\,0}T^\star X \oplus \Lambda^{0,\,1}T^\star X$ by putting $\langle dz_j\,,\,d\bar{z}_k\rangle_\omega = 0$ for all j, k.

More generally, ω induces a pointwise inner product on $\Lambda^{p,\,q}T^\star X$ for every bidegree $(p,\,q)$. From this, we get an L^2 *inner product* on the space of global C^∞ $(p,\,q)$-forms on X, defined by

$$\langle\langle u,\,v\rangle\rangle_\omega := \int\limits_X \langle u(x)\,,\,v(x)\rangle_\omega\, dV_\omega,$$

where $dV_\omega := \omega^n/n!$ is the volume form on X induced by ω.

The L^2 inner product defined by a given Hermitian metric ω induces **formal adjoints** $d^\star = d^\star_\omega : C^\infty_k(X,\,\mathbb{C}) \longrightarrow C^\infty_{k-1}(X,\,\mathbb{C})$, $\partial^\star = \partial^\star_\omega : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p-1,\,q}(X,\,\mathbb{C})$ and $\bar\partial^\star = \bar\partial^\star_\omega : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q-1}(X,\,\mathbb{C})$ of the differential operators d, ∂ and $\bar\partial$.

An important notion is the following

Definition 0.0.1. *A* **Kähler metric** *on a complex manifold X is a Hermitian metric ω on X such that $d\omega = 0$.*

A complex manifold X is said to be a **Kähler manifold** *if a Kähler metric exists on X.*

Being d-closed, any Kähler metric ω defines a De Rham cohomology class $\{\omega\} \in H^2_{DR}(X, \mathbb{R})$, called a *Kähler class*.

Compact Kähler manifolds are few and far between. They are the main object of study of Kähler geometry and they have very good properties, many of which will be presented in this book, often in the more general setting of generalisations of compact Kähler manifolds. Every projective manifold is Kähler, but the converse fails. The following important result characterises projective manifolds within the larger class of compact Kähler manifolds.

Theorem 0.0.2. *(***Kodaira's Embedding Theorem***) A compact complex manifold X is* **projective** *if and only if X carries an* **integral** *Kähler class $\{\omega\} \in H^2(X, \mathbb{Z})$.*

The meaning of a real De Rham cohomology class $\{\omega\} \in H^2_{DR}(X, \mathbb{R})$ of degree 2 being *integral* is that $\{\omega\}$ is the *first Chern class* of a holomorphic line bundle L on X, or equivalently, that $\{\omega\}$ is the cohomology class of the *curvature form* of such a bundle equipped with a Hermitian metric on its fibres.

Unlike projective manifolds, a compact Kähler manifold need not have other submanifolds than its points and itself. In particular, neither complex curves nor complex hypersurfaces need exist on it. Hence, in the transcendental context of compact Kähler manifolds, the objects used for investigation are often analytic generalisations of the algebraic objects used in the study of projective manifolds. Foremost among these are the **(d)-closed positive currents**, which can be regarded as generalisations of complex submanifolds. Indeed, if $Y \subset$ is a p-dimensional complex submanifold of a compact complex n-dimensional manifold X, the *current of integration* $[Y]$ on Y is the d-closed positive current of bidegree $(n-p,\, n-p)$ (equivalently, of bidimension $(p,\, p)$) defined by integrating on Y the restrictions of the smooth $(p,\, p)$-forms on X:

$$C^\infty_{p,\,p}(X,\, \mathbb{C}) \ni \gamma \longmapsto \int\limits_Y \gamma_{|Y} := \langle [Y],\, \gamma \rangle \in \mathbb{C}.$$

The current of integration $[Y]$ on Y makes sense even when Y is a *singular* subvariety of X.

More generally, bidegree $(p,\, p)$-currents T can be viewed as generalisations of $(p,\, p)$-forms. Locally, any such current is of the shape

$$T = \sum_{|J|=|K|=p} T_{J\overline{K}}\, dz_J \wedge d\bar{z}_K,$$

where the $T_{J\overline{K}}$'s are, in general, merely *distributions*. In particular, distributions identify with currents of degree 0. A current T is said to be *closed* (or *d-closed*) if $dT = 0$. In particular, a d-closed bidegree $(p,\, p)$-current T on X defines a De Rham cohomology class $\{T\} \in H^{2p}_{DR}(X, \mathbb{C})$. Locally,

$$dT = \sum_{|J|=|K|=p} dT_{J\overline{K}} \wedge dz_J \wedge d\bar{z}_K,$$

where the differentials $dT_{J\overline{K}}$ (which are currents of degree 1) are computed in the sense of distributions. As we said above, $d[Y] = 0$ for every subvariety $Y \subset X$. When the current T is *positive* (a property that will also be called *semi-positive* in this book), its coefficients $T_{J\overline{K}}$ are even *complex measures*. This brings about the link with *pluripotential theory*.

A transcendental analogue of Moishezon manifolds is provided by the **(Fujiki) class C manifolds**. These are the compact complex manifolds that are bimeromorphically equivalent to compact Kähler manifolds. Specifically, a compact complex manifold X belongs to this class if and only if there exists a compact Kähler manifold $\widetilde{X}$ and a holomorphic bimeromorphic map (called a *modification*) $\mu : \widetilde{X} \longrightarrow X$.

The link with the theory of currents is demonstrated yet again by the following results of Ji-Shiffman and Demailly-Paun characterising Moishezon, respectively *class C manifolds*, by the existence of *singular* analogues of Kähler metrics.

Theorem 0.0.3. *(a) ([JS93]) A compact complex manifold X is a* **Moishezon manifold** *if and only if there exists a* **Kähler current** *T with* **integral class** *$\{T\} \in H^2(X, \mathbb{Z})$ on X.*

(b) ([DP04]) A compact complex manifold X is a **class C manifold** *if and only if there exists a* **Kähler current** *on X.*

A *Kähler current* is a d-closed $(1, 1)$-current T on X such that $T \geq \varepsilon\omega$ on X for some constant $\varepsilon > 0$ and some Hermitian metric ω on X. This is a strong positivity condition on T.

We sum up our discussion so far of the classification of compact complex manifolds in the following implication diagram:

$$\begin{tikzcd}
 & X\ \textbf{Kähler} \arrow[dr, Rightarrow] & \\
X\ \textbf{projective} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & X\ \textbf{class C}. \\
 & X\ \textbf{Moishezon} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

All these implications are strict when $\dim_{\mathbb{C}} X \geq 3$. Examples to this effect will be presented throughout the book. Theorems 0.0.2 and 0.0.3 show that projective and Moishezon manifolds can be characterised by the existence of *integral* objects, so they are in the realm of algebraic geometry, while Kähler and *class $\mathcal{C}$* manifolds are characterised by the existence of *transcendental* objects, so they are in the realm of analytic geometry.

In our study of the classification of compact complex manifolds beyond the Kähler and the *class C* manifolds, we will be mainly pursuing two points of view.

(1) Metric point of view

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$. One way of studying these manifolds is to investigate the various types of special Hermitian metrics they carry. When $n \geq 3$, comparatively few such manifolds are Kähler, hence the need to weaken the Kähler assumption on the metric in order to enlarge the class of manifolds under investigation. The case of complex surfaces involves special phenomena that are well documented in the literature, so we will be mainly interested in the case where $n \geq 3$ and the relatively recent developments it subsumes.

In the next diagram, we give the definitions of six special kinds of Hermitian metrics ω on a given compact complex manifold X together with the various implications among them. Except for the Gauduchon metrics, which always exist on X by [Gau77a], the other five types of metrics need not exist. When they do, they give information about the geometry of the manifold X which then bears the name of the metrics it carries. For example, X is said to be a *Hermitian-symplectic* (H-S), *SKT*, *balanced* or *strongly Gauduchon* (sG) manifold if it carries the stated type of Hermitian metrics.

$$\begin{array}{lclcl} d\omega = 0 & \Longrightarrow & \exists\, \rho^{0,2} \in C^\infty_{0,2}(X,\,\mathbb{C}) \text{ s.t.} & \Longrightarrow & \partial\bar\partial\omega = 0 \\ & & d(\overline{\rho^{0,2}} + \omega + \rho^{0,2}) = 0 & & \\ (\omega \text{ is } \textbf{Kähler}) & & (\omega \text{ is } \textbf{Hermitian-symplectic (H-S)}) & & (\omega \text{ is } \textbf{SKT}) \\ \Downarrow & & & & \\ d\omega^{n-1} = 0 & \Longrightarrow & \exists\, \Omega^{n-2,n} \in C^\infty_{n-2,n}(X,\,\mathbb{C}) \text{ s.t.} & \Longrightarrow & \partial\bar\partial\omega^{n-1} = 0 \\ & & d(\overline{\Omega^{n-2,n}} + \omega^{n-1} + \Omega^{n-2,n}) = 0 & & \\ (\omega \text{ is } \textbf{balanced}) & & (\omega \text{ is } \textbf{strongly Gauduchon (sG)}) & & (\omega \text{ is } \textbf{Gauduchon}). \end{array} \tag{P}$$

Balanced metrics were introduced in [Gau77b] under the name *semi-Kähler* and then discussed again in [Mic83], while strongly Gauduchon (sG) metrics were introduced in [Pop13] by requiring $\partial\omega^{n-1} \in \operatorname{Im}\bar\partial$, a definition that was then proved in [Pop13, Proposition 4.2] to be equivalent to the description on the second line in the above picture (P). In particular, the notion of H-S metric is the analogue in bidegree (1, 1) of the notion of sG metric. Finally, SKT metrics are also called *pluriclosed metrics* in the literature.

Much of the non-Kähler complex geometry centres on special kinds of Hermitian metrics like those we have just mentioned.

(2) Cohomological point of view

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$. Thanks to the integrability property $d^2 = 0$, $\partial^2 = 0$, $\bar\partial^2 = 0$, each of the differential operators $d, \partial, \bar\partial$ induces a complex:

-the **De Rham complex** of X:

$$\cdots \stackrel{d}{\longrightarrow} C^\infty_{k-1}(X,\,\mathbb{C}) \stackrel{d}{\longrightarrow} C^\infty_k(X,\,\mathbb{C}) \stackrel{d}{\longrightarrow} C^\infty_{k+1}(X,\,\mathbb{C}) \stackrel{d}{\longrightarrow} \cdots,$$

giving rise to the **De Rham cohomology** spaces of X:

$$H^k_{DR}(X,\mathbb{C}) := \frac{\ker\,(d : C^\infty_k(X,\,\mathbb{C}) \longrightarrow C^\infty_{k+1}(X,\,\mathbb{C}))}{\operatorname{Im}\,(d : C^\infty_{k-1}(X,\,\mathbb{C}) \longrightarrow C^\infty_k(X,\,\mathbb{C}))}, \qquad k \in \{0,\dots,2n\},$$

depending only on the differential structure of X;

-for every fixed $q \in \{0,\dots,n\}$, the **conjugate Dolbeault complex** of X:

$$\cdots \stackrel{\partial}{\longrightarrow} C^\infty_{p-1,\,q}(X,\,\mathbb{C}) \stackrel{\partial}{\longrightarrow} C^\infty_{p,\,q}(X,\,\mathbb{C}) \stackrel{\partial}{\longrightarrow} C^\infty_{p+1,\,q}(X,\,\mathbb{C}) \stackrel{\partial}{\longrightarrow} \cdots,$$

giving rise to the **conjugate Dolbeault cohomology** spaces of X:

$$H^{p,\,q}_{\partial}(X,\mathbb{C}) := \frac{\ker\,(\partial : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p+1,\,q}(X,\,\mathbb{C}))}{\operatorname{Im}\,(\partial : C^\infty_{p-1,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}))}, \qquad p,q \in \{0,\dots,n\},$$

depending on the complex structure of X;

-for every fixed $p\in\{0,\dots,n\}$, the **Dolbeault complex** of X:

$$\cdots \stackrel{\bar\partial}{\longrightarrow} C^\infty_{p,\,q-1}(X,\,\mathbb{C}) \stackrel{\bar\partial}{\longrightarrow} C^\infty_{p,\,q}(X,\,\mathbb{C}) \stackrel{\bar\partial}{\longrightarrow} C^\infty_{p,\,q+1}(X,\,\mathbb{C}) \stackrel{\bar\partial}{\longrightarrow} \cdots,$$

giving rise to the **Dolbeault cohomology** spaces of X:

$$H^{p,\,q}_{\bar\partial}(X,\mathbb{C}) := \frac{\ker(\bar\partial : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q+1}(X,\,\mathbb{C}))}{\mathrm{Im}\,(\bar\partial : C^\infty_{p,\,q-1}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}))}, \qquad p,q\in\{0,\dots,n\},$$

depending on the complex structure of X.

The compactness of X implies the *finite dimensionality* (as $\mathbb{C}$-vector spaces) of all of the above cohomology spaces whose dimensions are important geometric invariants of a compact complex manifold. Of particular interest are the **Betti numbers**:

$$b_k = b_k(X) := \dim_{\mathbb{C}} H^k_{DR}(X,\mathbb{C}), \qquad k\in\{0,\dots,2n\},$$

depending only on the differential structure of X (so, they are *topological invariants*) and the **Hodge numbers**:

$$h^{p,\,q} = h^{p,\,q}_{\bar\partial}(X) := \dim_{\mathbb{C}} H^{p,\,q}_{\bar\partial}(X,\mathbb{C}), \qquad p,q\in\{0,\dots,n\},$$

depending on the complex structure of X.

Two other cohomologies that play a key role in non-Kähler complex geometry are

-the **Bott-Chern cohomology**, whose spaces are defined as

$$H^{p,\,q}_{BC}(X,\mathbb{C}) := \frac{\ker(\partial : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p+1,\,q}(X,\,\mathbb{C})) \cap \ker(\bar\partial : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q+1}(X,\,\mathbb{C}))}{\mathrm{Im}\,(\partial\bar\partial : C^\infty_{p-1,\,q-1}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}))},$$

and

-the **Aeppli cohomology**, whose spaces are defined as

$$H^{p,\,q}_{A}(X,\mathbb{C}) := \frac{\ker(\partial\bar\partial : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p+1,\,q+1}(X,\,\mathbb{C}))}{\mathrm{Im}\,(\partial : C^\infty_{p-1,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C})) + \mathrm{Im}\,(\bar\partial : C^\infty_{p,\,q-1}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}))},$$

for all $p,q\in\{0,\dots,n\}$.

We denote by $\{\alpha\}_{DR}$, $[\alpha]_\partial$, $[\alpha]_{\bar\partial}$, $[\alpha]_{BC}$, $[\alpha]_A$ the De Rham, conjugate Dolbeault, Dolbeault, Bott-Chern, respectively Aeppli cohomology class of a given form α that represents such a class.

There are well-defined, **canonical** linear maps induced by the identity among these cohomologies:

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \to H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) \to H^{p,\,q}_{A}(X,\,\mathbb{C}), \qquad [\alpha]_{BC} \mapsto [\alpha]_{\bar\partial} \mapsto [\alpha]_A,$$

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \to H^{p,\,q}_{\partial}(X,\,\mathbb{C}) \to H^{p,\,q}_{A}(X,\,\mathbb{C}), \qquad [\alpha]_{BC} \mapsto [\alpha]_{\partial} \mapsto [\alpha]_A,$$

and

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \to H^{p+q}_{DR}(X,\,\mathbb{C}) \to H^{p,\,q}_{A}(X,\,\mathbb{C}), \qquad [\alpha]_{BC} \mapsto \{\alpha\}_{DR} \mapsto [\alpha^{p,\,q}]_A,$$

where, for the last map, $\alpha^{p,\,q}$ denotes the $(p,\,q)$-type component of the $(p+q)$-form $\alpha = \sum_{r+s=p+q}\alpha^{r,\,s}$. By *canonical* we mean that these maps depend only on the complex structure of X, so, in particular, they are independent of the choice of a Hermitian metric. However, these maps need not be either injective or surjective on an arbitrary X. One of the remarkable properties of a class of compact complex manifolds (the so-called **$\partial\bar\partial$-manifolds**) that strictly contains the Kähler class is that all the maps on the first two rows above are *isomorphisms*, while the two maps on the third row are *injective*, respectively *surjective*. In particular, on a compact Kähler manifold X, the Dolbeault, conjugate Dolbeault, Bott-Chern and Aeppli cohomologies are *canonically isomorphic*. For this reason, the subscript can be dropped in that case, so $H^{p,\,q}(X,\,\mathbb{C})$ stands for any (usually Dolbeault in practice) of these cohomology groups of bidegree $(p,\,q)$ on a compact Kähler X.

(a) The Frölicher spectral sequence

Finally, let us mention that one of the main goals of Hodge Theory is to relate the differential structure of a given compact complex manifold X to its complex structure. One way of doing this is to relate the De Rham cohomology of X to its Dolbeault cohomology. This is done by a classical object called the **Frölicher spectral sequence** of X whose basic idea we now set out to explain.

If there is a canonical isomorphism

$$H^k_{DR}(X,\,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) \tag{5}$$

induced by the identity map as discussed above for every $k \in \{0,\dots,2n\}$, the cohomology of X is well behaved. This is what happens when X is a compact Kähler manifold, or more generally a $\partial\bar\partial$-manifold. However, for an arbitrary X, there are no such isomorphisms if only because the inequality

$$b_k(X) \leq \sum_{p+q=k} h^{p,\,q}_{\bar\partial}(X),$$

which always holds for every k, may be strict for some k. Intuitively, in this case some of the Dolbeault cohomology groups $H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ are "too big" to fit inside the De Rham cohomology group $H^{p+q}_{DR}(X,\,\mathbb{C})$. Therefore, they must be refined to fit inside.

The Frölicher spectral sequence of X consists of a finite sequence of complexes, called pages, that successively refine the Dolbeault cohomology of X until it "fits" inside the De Rham cohomology. Specifically, it consists of the following complexes.

Page 0**:** the **Dolbeault complex**, i.e.

$$\dots \stackrel{d_0}{\longrightarrow} E_0^{p,\,q-1}(X) \stackrel{d_0}{\longrightarrow} E_0^{p,\,q}(X) \stackrel{d_0}{\longrightarrow} E_0^{p,\,q+1}(X) \stackrel{d_0}{\longrightarrow} \dots,$$

where $E_0^{p,\,q}(X) := C^\infty_{p,\,q}(X,\,\mathbb{C})$ and $d_0 := \bar\partial$. For every bidegree $(p,\,q)$, put

$$E_1^{p,\,q}(X) := \ker d_0^{p,\,q}/\mathrm{Im}\, d_0^{p,\,q-1} = H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}).$$

This is the Dolbeault cohomology space of X of bidegree $(p,\,q)$.

Page 1**:** the cohomology spaces of page 0, i.e.

$$\dots \stackrel{d_1}{\longrightarrow} E_1^{p-1,\,q}(X) \stackrel{d_1}{\longrightarrow} E_1^{p,\,q}(X) \stackrel{d_1}{\longrightarrow} E_1^{p+1,\,q}(X) \stackrel{d_1}{\longrightarrow} \dots,$$

with differential defined as $d_1([\alpha]_{\bar\partial}) := [\partial\alpha]_{\bar\partial}$.

Then, one continues inductively, defining each page as the cohomology of the previous page.

Page r:

$$\dots \xrightarrow{d_r} E_r^{p-r,\,q+r-1}(X) \xrightarrow{d_r} E_r^{p,\,q}(X) \xrightarrow{d_r} E_r^{p+r,\,q-r+1}(X) \xrightarrow{d_r} \dots.$$

So, d_r is of bidegree $(r,\,-r+1)$ for every $r \in \mathbb{N}^\star$. Put

$$E_{r+1}^{p,\,q}(X) := \ker d_r^{p,\,q}/\mathrm{Im}\, d_r^{p-r,\,q+r-1}.$$

Theorem 0.0.4. *(Frölicher 1955) The Frölicher spectral sequence converges to the* **De Rham cohomology** *of X, i.e. there is an integer $r \geq 1$ such that $E_r^{p,\,q}(X) = E_{r+1}^{p,\,q}(X) = E_{r+2}^{p,\,q}(X) = \dots := E_\infty^{p,\,q}(X)$ for all p, q and there are* **(non-canonical) isomorphisms***:*

$$H^k_{DR}(X,\,\mathbb{C}) \simeq \bigoplus_{p+q=k} E_\infty^{p,\,q}(X), \qquad k = 0, \dots, 2n.$$

If r is the smallest positive integer with the above property, the spectral sequence is said to **degenerate** at page r (or at E_r). We write $E_r(X) = E_\infty(X)$ in this case.

Thus, the degeneration at E_r of the Frölicher spectral sequence is a purely **numerical property**:

$$E_r(X) = E_\infty(X) \iff b_k = \sum_{p+q=k} \dim_{\mathbb{C}} E_r^{p,\,q} \qquad \text{for all } k = 0, \dots, 2n.$$

In particular, for every k and every l the following inequalities hold:

$$\sum_{p+q=k} h_{\bar\partial}^{p,\,q}(X) \geq \dots \geq \sum_{p+q=k} \dim_{\mathbb{C}} E_l^{p,\,q}(X) \geq \sum_{p+q=k} \dim_{\mathbb{C}} E_{l+1}^{p,\,q}(X) \geq \dots \geq b_k(X).$$

Hence, the following implications hold:

$$E_1(X) = E_\infty(X) \implies E_2(X) = E_\infty(X) \implies \dots \implies E_r(X) = E_\infty(X) \implies E_{r+1}(X) = E_\infty(X) \implies \dots$$

In particular, for every given integer $r \geq 1$, we obtain a new class of compact complex manifolds: those X whose Frölicher spectral sequence degenerates at E_r.

(b) $\partial\bar\partial$-manifolds

Note that the Frölicher degeneration at E_1 does not imply the existence of a *canonical* isomorphism (5) for every k. It only implies the existence of non-canonical isomorphisms since finite-dimensional vector spaces of the same dimension are (non-canonically, in general) isomorphic. Nor does the $E_1(X) = E_\infty(X)$ property imply any relation between $H_{\bar\partial}^{p,\,q}(X,\,\mathbb{C})$ and $H_{\bar\partial}^{q,\,p}(X,\,\mathbb{C})$.

There is a stronger property of compact complex manifolds that implies (and is implied by) these cohomological properties. The idea goes back to Deligne-Griffiths-Morgan-Sullivan [DGMS75].

Definition 0.0.5. *A compact complex manifold X is called a* $\partial\bar\partial$**-manifold** *if, for every bidegree $(p,\,q)$ and for every form $u \in C^\infty_{p,\,q}(X,\,\mathbb{C})$ such that $du = 0$, the following equivalences hold:*

$$u \in Im\,\partial \iff u \in Im\,\bar\partial \iff u \in Im\,d \iff u \in Im\,(\partial\bar\partial). \tag{6}$$

It turns out that this property is equivalent to the identity map inducing a *canonical* isomorphism (5) for every k.

Theorem and Definition 0.0.6. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The following statements are equivalent.*

(1) *X is a* **$\partial\bar\partial$-manifold***.*

(2) *For every bidegree (p, q), every Dolbeault cohomology class $[\alpha^{p,q}]_{\bar\partial} \in H^{p,q}_{\bar\partial}(X, \mathbb{C})$ can be represented by a* **d-closed (p, q)-form** *and for every k, the linear map*

$$\bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X, \mathbb{C}) \ni \sum_{p+q=k} [\alpha^{p,q}]_{\bar\partial} \mapsto \left\{ \sum_{p+q=k} \alpha^{p,q} \right\}_{DR} \in H^k_{DR}(X, \mathbb{C})$$

is **well-defined** *by means of d-closed pure-type representatives $\alpha^{p,q}$ of their respective Dolbeault cohomology classes and* **bijective***.*

In this case, X is said to have the **Hodge Decomposition** *property.*

(3) *The Frölicher spectral sequence of X* **degenerates at E_1** *and the De Rham cohomology of X is* **pure***.*

(4) *For all $p, q \in \{0, \dots, n\}$, the canonical linear maps*

$$H^{p,q}_{BC}(X, \mathbb{C}) \longrightarrow H^{p,q}_{\bar\partial}(X, \mathbb{C}) \quad \text{and} \quad H^{p,q}_{\bar\partial}(X, \mathbb{C}) \longrightarrow H^{p,q}_{A}(X, \mathbb{C})$$

are **isomorphisms***.*

(5) *For all $p, q \in \{0, \dots, n\}$, the canonical linear map $H^{p,q}_{BC}(X, \mathbb{C}) \longrightarrow H^{p,q}_{A}(X, \mathbb{C})$ is* **injective***.*

An explanation of the terminology is in order. Well-definedness in (2) means that the map does not depend on the choices of d-closed representatives $\alpha^{p,q}$ of the classes $\{\alpha^{p,q}\}_{\bar\partial}$. Meanwhile, the De Rham cohomology of X is said to be **pure** if, for every k, the vector subspaces $H^{p,q}_{DR}(X, \mathbb{C})$ of $H^k_{DR}(X, \mathbb{C})$, consisting of De Rham cohomology classes representable by pure-type (p, q)-forms with $p + q = k$, are in a *direct sum* and if they *fill out* $H^k_{DR}(X, \mathbb{C})$. Some authors call this property of the De Rham cohomology *pure and full.*

The Hodge Decomposition property, in the strong form defined in (2) of Theorem and Definition 0.0.6, implies the Hodge Symmetry property.

Theorem 0.0.7. *Every* **$\partial\bar\partial$-manifold** *X has the* **Hodge Symmetry** *property in the sense that the following two conditions are satisfied:*

(a) every class $[\alpha^{p,q}]_{\bar\partial} \in H^{p,q}_{\bar\partial}(X, \mathbb{C})$ contains a **d-closed representative** *$\alpha^{p,q}$;*
(b) the linear map

$$H^{p,q}_{\bar\partial}(X, \mathbb{C}) \ni [\alpha^{p,q}]_{\bar\partial} \mapsto \overline{[\overline{\alpha^{p,q}}]_{\bar\partial}} \in \overline{H^{q,p}_{\bar\partial}(X, \mathbb{C})}$$

is **well-defined** *(in the sense that it does not depend on the choice of d-closed representative $\alpha^{p,q}$ of the class $[\alpha^{p,q}]_{\bar\partial}$) and* **bijective***.*

A fundamental result in Hodge Theory is the following

Theorem 0.0.8. *(***$\partial\bar\partial$-lemma***) Every compact Kähler manifold is a $\partial\bar\partial$-manifold.*

For this reason, $\partial\bar\partial$-manifolds are called *cohomologically Kähler manifolds* by some authors. They are precisely those compact complex manifolds that support a Hodge Theory with the same properties as on compact Kähler manifolds, but they need not support any Kähler metric.

We can now continue our implication diagram by adding the last notions presented above.

$$\begin{tikzcd}
 & X\ \textbf{balanced} \arrow[dr, Rightarrow] & \\
X\ \textbf{class}\ \mathcal{C} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & X\ \textbf{strongly Gauduchon} \\
 & \mathbf{X}\ \partial\bar\partial\textbf{-manifold} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\
 & & \mathbf{E_1(X)} = \mathbf{E_\infty(X)}.
\end{tikzcd}$$

All these implications are strict when $\dim_{\mathbb{C}} X \geq 3$. Examples to this effect will be presented throughout the book. That every *class* $\mathcal{C}$ manifold is balanced was proved by Alessandrini and Bassanelli in [AB95] as a consequence of their stronger result to the effect that every modification of a compact balanced manifold is again balanced.

Generalisations of $\partial\bar\partial$-manifolds were found recently by Popovici, Stelzig and Ugarte ([PSU20a], [PSU20b], [PSU20c]). They are called **page r-$\partial\bar\partial$-manifolds**, for a given integer $r \geq 0$. When $r = 0$, these are precisely the $\partial\bar\partial$-manifolds, but the class of page r-$\partial\bar\partial$-manifolds increases as r increases and contains many interesting non-Kähler compact complex manifolds.

(3) Interplay between the metric and the cohomological points of view

Another basic idea of Hodge theory is to interpret the various cohomology spaces as *harmonic spaces*, namely as the kernels of certain elliptic differential operators called Laplacians.

Suppose X is a compact complex manifold on which a Hermitian metric ω has been fixed. Using the L^2 inner product induced by ω on the spaces of C^∞ forms on X, one defines first-order differential operators $d^\star$, $\partial^\star$, $\bar\partial^\star$ as the adjoints of $d, \partial, \bar\partial$ which, in turn, induce **Laplace-Beltrami operators**:

$$\Delta = \Delta_\omega := dd^\star + d^\star d : C^\infty_k(X,\,\mathbb{C}) \to C^\infty_k(X,\,\mathbb{C}),$$

$$\Delta' = \Delta'_\omega := \partial\partial^\star + \partial^\star\partial : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C}),$$

$$\Delta'' = \Delta''_\omega := \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C}),$$

$$\Delta_{BC} := \partial^\star\partial + \bar\partial^\star\bar\partial + (\partial\bar\partial)(\partial\bar\partial)^\star + (\partial\bar\partial)^\star(\partial\bar\partial) + (\partial^\star\bar\partial)^\star(\partial^\star\bar\partial) + (\partial^\star\bar\partial)(\partial^\star\bar\partial)^\star : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C}),$$

$$\Delta_A := (\partial\bar\partial)^\star(\partial\bar\partial) + \partial\partial^\star + \bar\partial\bar\partial^\star + (\partial\bar\partial)(\partial\bar\partial)^\star + (\partial\bar\partial^\star)(\partial\bar\partial^\star)^\star + (\partial\bar\partial^\star)^\star(\partial\bar\partial^\star) : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C}),$$

in every (bi-)degree. Note that Δ, Δ' and Δ'' are of order 2, while Δ_{BC} and Δ_A (called the Bott-Chern, respectively the Aeppli, Laplacian) are of order 4. Each of them is adapted to one type of cohomology on X. They all turn out to be *elliptic* and this, together with the compactness of X, leads to **Hodge isomorphisms**:

$$H^k_{DR}(X,\,\mathbb{C})\simeq\mathcal{H}^k_\Delta(X,\,\mathbb{C}),$$
$$H^{p,\,q}_{\partial}(X,\,\mathbb{C})\simeq\mathcal{H}^{p,\,q}_{\Delta'}(X,\,\mathbb{C}),\qquad H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})\simeq\mathcal{H}^{p,\,q}_{\Delta''}(X,\,\mathbb{C}),$$
$$H^{p,\,q}_{BC}(X,\,\mathbb{C})\simeq\mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\,\mathbb{C}),\qquad H^{p,\,q}_{A}(X,\,\mathbb{C})\simeq\mathcal{H}^{p,\,q}_{\Delta_A}(X,\,\mathbb{C}),$$

where $\mathcal{H}^k_P(X,\,\mathbb{C})$ and $\mathcal{H}^{p,\,q}_P(X,\,\mathbb{C})$ stand for the kernels of P in degree k and bidegree $(p,\,q)$, where $P\in\{\Delta,\Delta',\Delta'',\Delta_{BC},\Delta_A\}$.

In particular, the $\bar\partial$-Laplacian Δ'' yields a Hodge theory for the first page of the Frölicher spectral sequence of X. A Hodge theory for the higher pages (starting from the second one) of the Frölicher spectral sequence was found recently in [Pop16]. Rather surprisingly, the corresponding Laplacian is not a differential operator, but a pseudo-differential operator. It is defined as

$$\widetilde\Delta=\partial p''\partial^\star+\partial^\star p''\partial+\Delta'':C^\infty_{p,\,q}(X,\,\mathbb{C})\longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}),$$

where
$$p'':C^\infty_{p,\,q}(X,\,\mathbb{C})=\ker\Delta''\oplus\mathrm{Im}\,\bar\partial\oplus\mathrm{Im}\,\bar\partial^\star\longrightarrow\ker\Delta''$$
is the *orthogonal projection* onto the kernel of Δ'' induced by the above three-space L^2_ω-orthogonal decomposition of the space of smooth $(p,\,q)$-forms.

A Hodge theory for every page $r\geq 3$ of the Frölicher spectral sequence was then found in [Pop19] and [PSU20b] by means of pseudo-differential Laplacians.

Moreover, higher-page analogues of the Bott-Chern and Aeppli cohomologies were found and given appropriate Hodge theories in [PSU20b].

The way in which metric and cohomological considerations inform each other can also be seen in the following, seemingly purely metric, notion introduced and studied in [Pop15a] and [PU18].

Definition 0.0.9. *([Pop15a], [PU18]) A compact complex manifold X is said to be an* **sGG manifold** *if every Gauduchon metric on X is strongly Gauduchon.*

It turns out that this notion also has an entirely cohomological description as a special case of the $\partial\bar\partial$-property.

Proposition 0.0.10. *([PU18]) Let X be a compact complex manifold with $dim_{\mathbb{C}}X=n$. Then, X is an* **sGG manifold** *if and only if for every d-closed $(n,\,n-1)$-form Γ on X, the following implication holds:*
$$\Gamma\in Im\,\partial\implies\Gamma\in Im\,\bar\partial.$$

This notion even has purely numerical descriptions, one of which being the following

Theorem 0.0.11. *([PU18]) On any compact complex manifold X we have $b_1\leq 2h^{0,\,1}_{\bar\partial}$.*

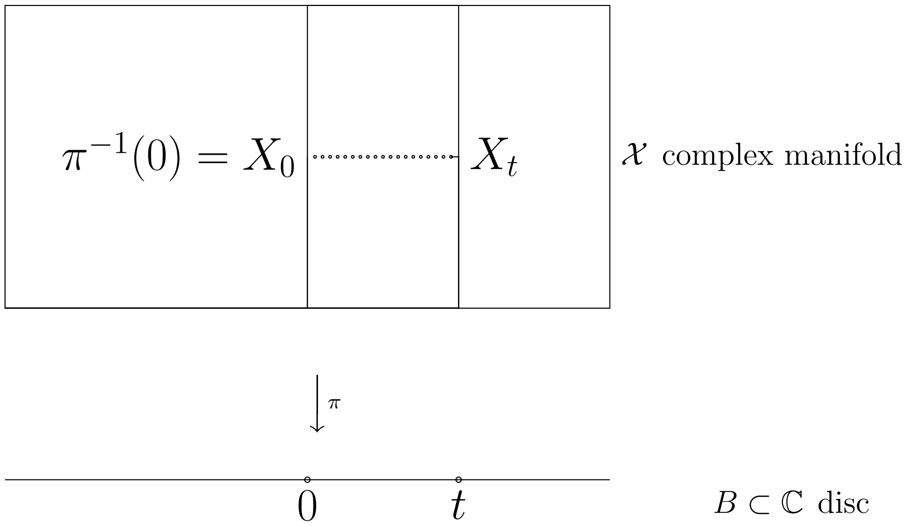
Moreover, X is an **sGG manifold** *if and only if $b_1=2h^{0,\,1}_{\bar\partial}$.*

The class of sGG manifolds lies between the classes of $\partial\bar\partial$- and strongly Gauduchon manifolds and turns out to have good deformation and modification properties.

(4) Deformations of complex structures

Another method of investigating the classification of compact complex manifolds is to study their variation in families. The central notion is the following

Definition 0.0.12. *A* **holomorphic family** *of compact complex manifolds is a proper holomorphic submersion $\pi : \mathcal{X} \longrightarrow B$ between complex manifolds $\mathcal{X}$ and B.*



The manifold $\mathcal{X}$ is called the *total space* of the family, while B, that will often be taken to be a small open disc about the origin in $\mathbb{C}$, is called the *base* of the family. For every $t \in B$, $X_t := \pi^{-1}(t) \subset \mathcal{X}$ is a compact complex manifold, called the fibre over t.

By a classical theorem of Ehresmann's [Ehr47], the differential structure of any fibre X_t remains unchanged under small varations of t. So, any holomorphic family of compact complex manifolds is *locally C^∞ trivial.* It is even *globally C^∞ trivial* if the base B is contractible. Thus, we have C^∞ diffeomorphisms $X_t \simeq X$ for $t \in B$ (if B is e.g. a disc), where X is the C^∞ manifold underlying the fibres X_t.

However, the complex structure J_t of X_t depends on t in general, so the splitting

$$d = \partial_t + \bar\partial_t, \qquad t \in B,$$

depends on t. Thus, the family $(X_t)_{t\in B}$ of compact complex manifolds can be viewed as a single C^∞ manifold X endowed with a family $(J_t)_{t\in B}$ of complex structures varying holomorphically with $t \in B$.

In particular, the De Rham coomology of the fibres X_t is independent of t, so we can identify

$$H^k_{DR}(X_t, \mathbb{C}) \simeq H^k_{DR}(X, \mathbb{C})$$

for every k and every t. However, the Dolbeault, conjugate Dolbeault, Bott-Chern, Aeppli and Frölicher cohomologies depend on t, giving rise to t-dependent vector spaces $H^{p,\,q}_{\bar\partial}(X_t, \mathbb{C})$, $H^{p,\,q}_{\partial}(X_t, \mathbb{C})$, $H^{p,\,q}_{BC}(X_t, \mathbb{C})$, $H^{p,\,q}_{A}(X_t, \mathbb{C})$, $E^{p,\,q}_r(X_t)$ in every bidegree (p, q).

One line of investigation that will be pursued in this book aims at understanding how the various properties of compact complex manifolds that were mentioned above (e.g. projectivity, Moishezon property, Kälerianity, *class $\mathcal{C}$* property, $\partial\bar\partial$-property, etc) vary under deformations of the complex structure. This problem can mainly be considered from two points of view.

The *openness* point of view aims at determining whether a given property that an arbitrary fibre X_{t_0} may have is inherited by all the nearby fibres X_t when t is sufficiently close to t_0. The prototypical example of such a result is the following one by Kodaira and Spencer.

Theorem 0.0.13. *([KS60]) Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds. Suppose the fibre X_{t_0} is* **Kähler** *for some $t_0 \in B$. Then, the fibre X_t is* **Kähler** *for every $t \in B$ sufficiently close to t_0.*

Thus, the Kähler property of compact complex manifolds is open under deformations of the complex structure. So are the *$\partial\bar\partial$-property* (by a result of Wu [WU06], reproved by Angella and Tomassini in [AT13]), the property of the *Frölicher* spectral sequence *degenerating at E_1* (a classical result that follows at once from the Kodaira-Spencer theory) and the *strongly Gauduchon* property ([Pop09a]). However, the *class $\mathcal{C}$* property is not deformation open (by Campana [Cam91a] and Lebrun-Poon [LP92]) and neither is the *balanced* property (by Alessandrini and Bassanelli [AB90]).

The *closedness* point of view aims at determining whether a given property of compact complex manifolds survives in the limit under deformations. In fact, if a fibre, say X_0, has been fixed, it can be viewed as the limit of the nearby fibres X_t when $t \in B$ tends to $0 \in B$. Thus, one can wonder whether the limit fibre X_0 retains a certain property that all the fibres X_t with $t \neq 0$ have.

An example of such a result is the following

Theorem 0.0.14. *([Pop09a], [Pop09b], [Pop10a], [Pop19]) Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds, where $B \subset \mathbb{C}$ is a small disc about the origin. Suppose the fibre X_t is* **Moishezon** *for all $t \in B \setminus \{0\}$. Then, X_0 is again a* **Moishezon** *manifold.*

This result is optimal since an example of Hironaka's [Hir62] shows that the limit fibre X_0 need not be Kähler even if all the other fibres are assumed projective. Note that the statement of Theorem 0.0.14 is purely algebraic, so it falls into the realm of algebraic geometry. However, the proof is analytic and uses non-Kähler techniques and notions, including $\partial\bar\partial$-manifolds, strongly Gauduchon metrics and generalisations thereof introduced in [Pop19], as well as Hodge theory for the higher pages (i.e. starting from the second one) of the Frölicher spectral sequence.

A purely transcendental version of Theorem 0.0.14 is conjectured to hold.

Conjecture 0.0.15. *Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds, where $B \subset \mathbb{C}$ is a small disc about the origin. Suppose the fibre X_t is a* **class** $\mathcal{C}$ *manifold for all $t \in B \setminus \{0\}$. Then, X_0 is again a* **class** $\mathcal{C}$ *manifold.*

A two-stage strategy of attack for this conjecture was proposed in [PU18], where the first stage was implemented. This strategy motivated in part the introduction of sGG manifolds.

To situate these issues in their context, we recall the following major result of Siu's ([Siu98] and [Siu00]), although we will not discuss it in this book.

Theorem 0.0.16. *(***Siu's invariance of the plurigenera***) Let $\pi : \mathcal{X} \longrightarrow \Delta$ be a* **projective** *holomorphic family of compact complex manifolds over the unit disc $\Delta \subset \mathbb{C}$. Then, for every $m \in \mathbb{N}^\star$, the m-genus $\dim_{\mathbb{C}} H^0(X_t,\, mK_{X_t})$ is independent of the fibre $X_t := \pi^{-1}(t)$, $t \in \Delta$.*

By K_{X_t} one means the *canonical line bundle* of the fibre X_t, mK_{X_t} stands for its m^{th} tensor power (in additive notation in the Picard group), while H^0 denotes the space of global holomorphic sections. The invariance of the plurigenera plays a major role in the *Minimal Model Program (MMP)* where projective (or merely compact Kähler) manifolds are to be classified up to birational equivalence. Siu also conjectured the Kähler version of Theorem 0.0.16.

(5) Non-Kähler mirror symmetry

Some of the techniques alluded to above and explained at length throughout this book were used in [Pop18a] to propose a new approach to the *Mirror Symmetry Conjecture* extended to the possibly non-Kähler context.

At the centre of Mirror Symmetry lie the *Calabi-Yau manifolds*. In this book, the term applies to any compact complex manifold X whose canonical line bundle K_X is *trivial*. Many non-Kähler compact complex manifolds, including all the nilmanifolds, satisfy this condition. In the standard Mirror Symmetry theory, additional assumptions, which imply projectiveness, are made on these manifolds. However, in our approach, they need not be Kähler.

The central prediction of Mirror Symmetry is that two kinds of structures, *complex* and *metric*, ought to get exchanged between any Calabi-Yau manifold (of complex dimension 3) and its *mirror dual* manifold, another Calabi-Yau manifold (of complex dimension 3) predicted to correspond to any original such manifold:

$$\bigg\{\textbf{complex structures on } X\bigg\} \longleftrightarrow \bigg\{\textbf{metric structures on } \widetilde{X}\bigg\},$$

where $\widetilde{X}$ is the mirror dual of X. In other words, Calabi-Yau manifolds ought to come in pairs $(X,\, \widetilde{X})$ such that the complex structures on X correspond to the metric structures on $\widetilde{X}$ and vice versa.

We will assume that $n := \dim_{\mathbb{C}}X = \dim_{\mathbb{C}}\widetilde{X} \geq 3$ is arbitrary.

(a) The complex structure side of the mirror

The starting point of the theory on this side of the mirror is the following theorem by Bogomolov, Tian and Todorov to the effect that the Calabi-Yau assumption combined with the Kähler assumption (although the weaker $\partial\bar\partial$-assumption suffices) on a given compact complex manifold X ensures that the complex structure of X can be deformed in all the "available directions". In fact, these directions are parametrised by the cohomology group $H^{0,\,1}_{\bar\partial}(X,\, T^{1,\,0}X)$ of bidegree $(0,\,1)$ with values in the holomorphic tangent bundle $T^{1,\,0}X$ of X. However, for an arbitrary X, not all these deformations need define complex structures (since some of them may not satisfy the integrability condition, so they may only define almost complex structures).

Theorem 0.0.17. *Let X be a* $\partial\bar\partial$**-manifold** *whose canonical bundle K_X is* **trivial**. *Then, the Kuranishi family of X is* **unobstructed**.

The *Kuranishi family* $(X_t)_{t\in B}$ of $X = X_0$ is, intuitively, the family of all possible small deformations of the complex structure of X. In the case of unobstructedness, the base

$$B := \mathrm{Def}(X) \subset H^{0,\,1}_{\bar\partial}(X,\, T^{1,\,0}X)$$

is *smooth* and can be seen as an open ball about 0 in the $\mathbb{C}$-vector space $H^{0,\,1}_{\bar\partial}(X,\, T^{1,\,0}X)$.

In our generalised setting, we often make use of the notion of small *essential* deformations that was introduced in [Pop18a] for the Iwasawa manifold and in [PSU20c] for a larger class of manifolds whose Frölicher spectral sequence does not degenerate at E_1.

(b) The metric side of the mirror

The main tool of investigation on this side of the mirror in the classical approach to Mirror Symmetry is the *Kähler cone* $\mathcal{K}_X$ of a given compact Kähler manifold. It is defined as the set of all the Dolbeault cohomology classes of type (1, 1) that are representable by Kähler metrics. These classes are called *Kähler classes*.

One of the new ideas in our generalised approach to Mirror Symmetry is to replace the Kähler cone by the **Gauduchon cone** $\mathcal{G}_X$ of a given compact complex manifold X. This object was introduced in [Pop15a] as the set of all the Aeppli cohomology classes of type $(n-1,\, n-1)$ that can be represented by the $(n-1)$-st power of a Gauduchon metric:

$$\mathcal{G}_X := \bigg\{[\omega^{n-1}]_A \in H_A^{n-1,\,n-1}(X,\,\mathbb{R}) \mid \omega \text{ is a Gauduchon metric on } X\bigg\} \subset H_A^{n-1,\,n-1}(X,\,\mathbb{R}).$$

It turns out that $\mathcal{G}_X$ is an open convex cone in $H_A^{n-1,\,n-1}(X,\,\mathbb{R})$.

The Gauduchon cone $\mathcal{G}_X$ is never empty (since Gauduchon metrics exist on every X). It serves as a substitute for both the Kähler cone (which is empty when X is non-Kähler) and the various cones of classes of curves on X (which may not exist when X is non-projective).

The well-known duality between curves and divisors in algebraic geometry is replaced in this transcendental context by the duality between the Bott-Chern cohomology of bidegree (1, 1) and the Aeppli cohomology of bidegree $(n-1,\, n-1)$:

$$H_{BC}^{1,\,1}(X,\,\mathbb{C}) \times H_A^{n-1,\,n-1}(X,\,\mathbb{C}) \to \mathbb{C}, \qquad ([\alpha]_{BC},\, [\beta]_A) \mapsto \int\limits_X \alpha\wedge\beta.$$

This pairing is **well-defined** and **non-degenerate**, so the spaces $H_{BC}^{1,\,1}(X,\,\mathbb{C})$ and $H_A^{n-1,\,n-1}(X,\,\mathbb{C})$ are dual to each other.

(a) + (b)

Bringing the two kinds of structures together, we proposed in [Pop18a] a generalisation of the Mirror Symmetry Conjecture that can be loosely formulated as follows.

Conjecture 0.0.18. *The sGG Calabi-Yau compact complex manifolds of complex dimension $n \geq 3$ come in pairs*

$$(X,\, \widetilde{X})$$

such that there exist local biholomorphisms (= the **mirror maps***)*

$$Def(X) \simeq \widetilde{\mathcal{G}}_{\widetilde{X}} \quad et \quad Def(\widetilde{X}) \simeq \widetilde{\mathcal{G}}_X$$

inducing further correspondences between the complex structures on X and the metric structures on $\widetilde{X}$ and vice-versa.

By $\widetilde{\mathcal{G}}_X$ and $\widetilde{\mathcal{G}}_{\widetilde{X}}$ we mean the *complexified* Gauduchon cones of X, respectively $\widetilde{X}$.

Our testing ground

The *Iwasawa manifold* is the 3-dimensional compact complex manifold defined as the quotient $X = G/\Gamma$, where

$$G := \left\{\begin{pmatrix}1 & z_1 & z_3\\ 0 & 1 & z_2\\ 0 & 0 & 1\end{pmatrix} \; ; \; z_1, z_2, z_3 \in \mathbb{C}\right\} \subset GL_3(\mathbb{C})$$

is the **Heisenberg group** and $\Gamma \subset G$ is the subgroup of matrices of the same shape with entries $z_1, z_2, z_3 \in \mathbb{Z}[i]$. This manifold is non-Kähler (and its Frölicher spectral sequence does not even degenerate at E_1, so, in particular, X is not a $\partial\bar\partial$-manifold), but it is an **sGG manifold** with a trivial canonical bundle.

The notion of small **essential deformations** of the Iwasawa manifold was introduced in [Pop18a] (and was later generalised in [PSU20c] to a much larger class of manifolds). The main result of [Pop18a] can be loosely stated as follows.

Theorem 0.0.19. *The Iwasawa manifold is* **its own mirror dual** *in the sense that the* **small essential deformations** *of its complex structure "correspond" to the (Aeppli cohomology classes of its) Gauduchon metrics (i.e. to its own* **Gauduchon cone***).*

(III) Organisation of the material in this book

In chapter 1, after explaining the basics of Hodge Theory (ellipticity of certain differential operators, harmonic theory, duality), we present two points of view on the Frölicher spectral sequence and then we discuss the class of $\partial\bar\partial$-manifolds and several characterisations thereof.

In chapter 2, we present the Kodaira-Spencer deformation theory of smooth families of elliptic differential operators and their geometric applications to families of compact complex manifolds where these operators are mostly Laplacians. The unobstructedness theorem of Bogomolov, Tian and Todorov is also discussed in detail.

In chapter 3, we present the generalisations of several classical Hodge-theoretical notions and results to the context where the Dolbeault cohomology, that forms the first page of the Frölicher spectral sequence, is replaced by the cohomology of the higher pages. These are recent results obtained mainly in [Pop16], [Pop17], [PSU20a], [PSU20b] and [PSU20c]. In particular, the notion of *page-r-$\partial\bar\partial$-manifold*, generalising that of $\partial\bar\partial$-manifold, is presented, together with higher-page analogues of the Bott-Chern and Aeppli cohomologies, an adaptation of the adiabatic limit construction to the case of complex structures and the role it plays in the context of the Frölicher spectral sequence.

In chapter 4, we discuss at length several classes of special Hermitian metrics (Gauduchon, strongly Gauduchon, E_r-sG, balanced, SKT, Hermitian-symplectic) on compact complex manifolds and the roles they play in the classification of compact complex manifolds and in positivity problems in complex geometry.

In chapter 5.1, we present the notion of small deformations of a compact complex manifold that are *co-polarised* by a balanced class. This was introduced in [Pop13] as a generalisation of the classical notion of small deformations polarised by a Kähler class.

In chapter 6, following [Pop18a], we present an extension to the possibly non-Kähler context of the *Mirror Symmetry Conjecture* as an application of some of the ideas and techniques discussed previously in the book. In particular, we show that the Iwasawa manifold is its own mirror dual in this generalised sense. We also introduce the notion of small *essential* deformations of this manifold. This notion was subsequently generalised to the larger class of *page-1-$\partial\bar\partial$-manifolds* in [PSU20c].

In chapter 7, we present two proofs of the fact that any deformation limit of Moishezon manifolds is again Moishezon. This chapter, based on [Pop09a], [Pop09b], [Pop10a] and [Pop19], makes use of strongly Gauduchon and E_r-sG metrics, as well as the class of $\partial\bar\partial$-manifolds, the theory of the

Frölicher spectral sequence, the adiabatic limit construction and the Kodaira-Spencer deformation theory.

Chapter 8 is an appendix gathering standard material on nilmanifolds, solvmanifolds and left-invariant complex structures thereon. These constitute a rich source of examples for many of the issues discussed in the book.

Chapter 1

Cohomology and Metrics

1.1 Bott-Chern and Aeppli Cohomologies

Let X be a compact complex manifold with $n = \dim_{\mathbb{C}} X$. We denote by $C^\infty_{p,q}(X, \mathbb{C}) = C^\infty_{p,q}$, resp. $C^\infty_k(X, \mathbb{C}) = C^\infty_k$, the space of $\mathbb{C}$-valued C^∞ differential forms of bidegree (p, q), resp. of degree k, on X. For every $k \in \{0, \dots, 2n\}$, we have

$$C^\infty_k(X, \mathbb{C}) = \bigoplus_{p+q=k} C^\infty_{p,q}(X, \mathbb{C})$$

and the complex structure of X induces a splitting

$$d = \partial + \bar\partial$$

of the Poincaré differential operator $d : C^\infty_k(X, \mathbb{C}) \to C^\infty_{k+1}(X, \mathbb{C})$ into a part of type $(1, 0)$, $\partial : C^\infty_{p,q}(X, \mathbb{C}) \to C^\infty_{p+1,q}(X, \mathbb{C})$, and a part of type $(0, 1)$, $\bar\partial : C^\infty_{p,q}(X, \mathbb{C}) \to C^\infty_{p,q+1}(X, \mathbb{C})$, defined in every bidegree (p, q). While d depends only on the differential structure of X, ∂ and $\bar\partial$ depend on the complex structure. All three operators d, ∂ and $\bar\partial$ are linear differential operators of order 1.

The following notions are standard and widely used.

Definition 1.1.1. *For every $p, q \in \{0, \dots, n\}$ and every $k \in \{0, \dots, 2n\}$, one defines:*

(i) the **Bott-Chern cohomology group** *of bidegree (or type) (p, q) of X as*

$$H^{p,q}_{BC}(X, \mathbb{C}) = \frac{\ker\partial \cap \ker\bar\partial}{Im\,(\partial\bar\partial)};$$

(ii) the **Dolbeault cohomology group** *of bidegree (or type) (p, q) of X as*

$$H^{p,q}_{\bar\partial}(X, \mathbb{C}) = \frac{\ker\bar\partial}{Im\,\bar\partial};$$

(iii) the **conjugate Dolbeault cohomology group** *of bidegree (or type) (p, q) of X as*

$$H^{p,q}_{\partial}(X, \mathbb{C}) = \frac{\ker\partial}{Im\,\partial};$$

(iv) the **Aeppli cohomology group** *of bidegree (or type) (p, q) of X as*

$$H^{p,\,q}_A(X,\,\mathbb{C}) = \frac{\ker(\partial\bar\partial)}{Im\,\partial + Im\,\bar\partial};$$

(v) the **De Rham cohomology group** *of degree k of X as*

$$H^k_{DR}(X,\,\mathbb{C}) = \frac{\ker d}{Im\,d},$$

where all the kernels and images are considered as $\mathbb{C}$-vector subspaces of $C^\infty_{p,\,q}(X,\,\mathbb{C})$ or $C^\infty_k(X,\,\mathbb{C})$, according to the case.

The first observation is that the Bott-Chern cohomology maps canonically to all the other cohomologies, which map canonically to the Aeppli cohomology. These maps need not be either injective, or surjective, in general.

Lemma 1.1.2. *The following canonical linear maps in cohomology are well defined:*

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \to H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) \to H^{p,\,q}_A(X,\,\mathbb{C}), \qquad [\alpha]_{BC} \mapsto [\alpha]_{\bar\partial} \mapsto [\alpha]_A,$$

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \to H^{p,\,q}_{\partial}(X,\,\mathbb{C}) \to H^{p,\,q}_A(X,\,\mathbb{C}), \qquad [\alpha]_{BC} \mapsto [\alpha]_{\partial} \mapsto [\alpha]_A,$$

and

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \to H^{p+q}_{DR}(X,\,\mathbb{C}) \to H^{p,\,q}_A(X,\,\mathbb{C}), \qquad [\alpha]_{BC} \mapsto \{\alpha\}_{DR} \mapsto [\alpha^{p,\,q}]_A,$$

where, for the last map, $\alpha^{p,\,q}$ denotes the $(p,\,q)$-type component of the $(p+q)$-form $\alpha = \sum_{r+s=k}\alpha^{r,\,s}$.

Proof. The second row of maps can be treated analogously to the first one, so we will only deal with the first and third rows. For any $\alpha \in C^\infty_{p,\,q}(X,\,\mathbb{C})$, the following implications hold:

$$\alpha \in \ker\partial\cap\ker\bar\partial \implies \alpha\in\ker\bar\partial \implies \alpha\in\ker(\partial\bar\partial) \quad \text{and} \quad \alpha\in\ker\partial\cap\ker\bar\partial \implies \alpha\in\ker d,$$

while for any $\alpha \in C^\infty_k(X,\,\mathbb{C})$ ($k = p+q$ here) such that $d\alpha = 0$, we have $\partial\alpha^{r,\,s} + \bar\partial\alpha^{r+1,\,s-1} = 0$, hence $\partial\bar\partial\alpha^{r,\,s} = 0$, for all r, s such that $r+s = k$.

It remains to prove that the above maps are independent of the choices of representatives of the respective cohomology classes. This is equivalent to proving, respectively, the inclusions:

$$\mathrm{Im}\,(\partial\bar\partial) \subset \mathrm{Im}\,\bar\partial \subset \mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial \quad \text{and} \quad \mathrm{Im}\,(\partial\bar\partial)\subset\mathrm{Im}\,d$$

for every $\alpha \in C^\infty_{p,\,q}(X,\,\mathbb{C})$, as well as, in the case of the last map, the following implication for every $\alpha\in C^\infty_k(X,\,\mathbb{C})$: $\alpha\in\mathrm{Im}\,d \implies \alpha^{r,\,s}\in\mathrm{Im}\,\partial+\mathrm{Im}\,\bar\partial$ for all r, s such that $r+s=k$.

These inclusions are obvious. As for the last implication, suppose that $\alpha = d\beta$ for some $\beta\in C^\infty_{k-1}(X,\,\mathbb{C})$. Then, $\alpha^{r,\,s} = \partial\beta^{r-1,\,s} + \bar\partial\beta^{r,\,s-1}\in\mathrm{Im}\,\partial+\mathrm{Im}\,\bar\partial$ for all r, s such that $r+s=k$. □

1.1.1 Basics of Hodge theory

The fundamental fact in Hodge theory is the possibility of realising the various cohomology groups as vector spaces of *harmonic forms*, namely as kernels of Laplace-type elliptic differential operators. The necessary Laplacians are defined using a given Hermitian metric ω on X and depend on ω, as do their kernels, unlike the cohomology groups which are *canonical* (i.e. depend only on the complex structure of X).

Hermitian metrics

We start by recalling the very basic facts about Hermitian metrics (see [Dem97, chapter VI] for further details).

Definition 1.1.3. *A* **Hermitian metric** *ω on X is a family $(\langle\,\cdot,\,\cdot\,\rangle_{\omega(x)})_{x\in X}$, where, for every point $x \in X$,*

$$\langle\,\cdot,\,\cdot\,\rangle_{\omega(x)} : T_x^{1,\,0}X \times T_x^{1,\,0}X \to \mathbb{C}$$

is an inner product on the holomorphic tangent space to X at x, such that the inner products $\langle\,\cdot,\,\cdot\,\rangle_{\omega(x)}$ depend in a C^∞ way on $x \in X$.

By an inner product on a $\mathbb{C}$-vector space we mean a *positive definite sesquilinear map.* It is standard that any Hermitian metric ω on X identifies canonically with a unique C^∞ $(1,\,1)$-form ω (denoted henceforth by the same letter) that is *positive definite* at every point $x \in X$. Some authors call it the Kähler form associated with the Hermitian metric and denote these two objects differently, but we will not use this terminology. In local holomorphic coordinates $z_1,\dots,z_n$ on some open subset $U \subset X$, any such object is of the shape

$$\omega = \sum_{j,k=1}^{n} \omega_{j\bar{k}}\, i dz_j \wedge d\bar{z}_k, \tag{1.1}$$

where the coefficients $\omega_{j\bar{k}} : U \to \mathbb{C}$ are C^∞ functions such that the matrix $(\omega_{j\bar{k}}(x))_{j\bar{k}}$ is *positive definite* (equivalently, its eigenvalues are all *positive*) at every point $x \in X$.

In fact, an equivalent definition for a Hermitian metric ω on X is as a family $(\omega^{(\alpha)})_{\alpha\in\Lambda}$ of locally defined, positive definite C^∞ $(1,\,1)$-forms $\omega^{(\alpha)}$, defined respectively by the analogues of (1.1) on open coordinate subsets $U_\alpha \subset X$ that cover X, such that $\omega^{(\alpha)} = \omega^{(\beta)}$ on $U_\alpha \cap U_\beta$ whenever this intersection is non-empty.

In particular, Hermitian metrics always exist on any given X. Indeed, take any open cover of X by coordinate patches, take any locally defined Hermitian metrics on these patches and glue them together into a global Hermitian metric on X using a partition of unity.

On the other hand, any Hermitian metric ω on X induces, for any bidegree (p,q) (resp. any degree k), a *pointwise inner product* $\langle\,\cdot,\,\cdot\,\rangle = \langle\,\cdot,\,\cdot\,\rangle_\omega$ on the space $C^\infty_{p,\,q}(X,\,\mathbb{C})$ (resp. $C^\infty_k(X,\,\mathbb{C})$). Indeed, the metric on $T_x^{1,\,0}X$ induces the dual metric on $\Lambda^{1,\,0}T_x^\star X$, which in turn induces a metric on $\Lambda^{p,\,0}T_x^\star X$, for every $p \in \{0,\dots,n\}$, in the usual way:

$$\langle dz_{j_1}\wedge\dots dz_{j_p},\, dz_{k_1}\wedge\dots dz_{k_p}\rangle_\omega := \det(\langle dz_{j_l},\, dz_{k_r}\rangle_\omega)_{1\le l,r\le p}.$$

By conjugation, we then get a metric on $\Lambda^{0,\,q}T_x^\star X$, for every $q \in \{0,\dots,n\}$, and finally an induced metric on $\Lambda^{p,\,q}T_x^\star X$, for all $p,q \in \{0,\dots,n\}$. Note that, by construction, $\langle u,\, v\rangle_\omega = 0$ whenever u is of type $(p,\,q)$ and v is of type $(r,\,s)$ with $(p,\,q) \neq (r,\,s)$.

The pointwise inner product induces a *pointwise norm* in the usual way: $|u|^2 = |u|^2_\omega := \langle u,\, u\rangle_\omega$.

Integrating the pointwise inner product leads to

Definition 1.1.4. *For all $p,q \in \{0,\dots,n\}$, the L^2* **inner product** *induced on $C^\infty_{p,\,q}(X,\,\mathbb{C})$ by a Hermitian metric ω on X is defined as*

$$\langle\langle u,\, v\rangle\rangle = \langle\langle u,\, v\rangle\rangle_\omega = \int\limits_X \langle u(x),\, v(x)\rangle_\omega\, dV_\omega(x),$$

where $dV_\omega := \frac{\omega^n}{n!}$ *is the* **volume form** *induced by* ω.

The L^2 *inner product induces an* L^2 **norm** *in the usual way:* $||u||^2 = ||u||^2_\omega := \langle\langle u,\, u\rangle\rangle_\omega$.

Note that dV_ω is a C^∞ positive definite $(n,\, n)$-form on X. In particular, it induces canonically a positive measure on X w.r.t. which the above integration is performed.

Hodge isomorphism for the De Rham and Dolbeault cohomologies

Once a Hermitian metric ω has been fixed on an n-dimensional compact complex manifold X, one defines **formal adjoints** $d^\star = d^\star_\omega : C^\infty_{k+1}(X,\, \mathbb{C}) \to C^\infty_k(X,\, \mathbb{C})$, $\partial^\star = \partial^\star_\omega : C^\infty_{p+1,\, q}(X,\, \mathbb{C}) \to C^\infty_{p,\, q}(X,\, \mathbb{C})$ and $\bar\partial^\star = \bar\partial^\star_\omega : C^\infty_{p,\, q+1}(X,\, \mathbb{C}) \to C^\infty_{p,\, q}(X,\, \mathbb{C})$ w.r.t. ω of the operators $d : C^\infty_k(X,\, \mathbb{C}) \to C^\infty_{k+1}(X,\, \mathbb{C})$, $\partial : C^\infty_{p,\, q}(X,\, \mathbb{C}) \to C^\infty_{p+1,\, q}(X,\, \mathbb{C})$ and $\bar\partial : C^\infty_{p,\, q}(X,\, \mathbb{C}) \to C^\infty_{p,\, q+1}(X,\, \mathbb{C})$ by requiring that the identities

$$\langle\langle du,\, v\rangle\rangle = \langle\langle u,\, d^\star v\rangle\rangle, \quad \langle\langle \partial u,\, v\rangle\rangle = \langle\langle u,\, \partial^\star v\rangle\rangle, \quad \langle\langle \bar\partial u,\, v\rangle\rangle = \langle\langle u,\, \bar\partial^\star v\rangle\rangle$$

hold for all smooth forms u, v of the relevant (bi)degrees. Note that the *compactness* of X is key in these definitions. When X is not compact, one requires the above identities (which are integrations by parts) to hold only for forms u and v such that the intersection of their supports is *compact*.

One then goes on to define the **Laplace-Beltrami operators** corresponding to d, ∂ and resp. $\bar\partial$ by the formulae

$$\begin{aligned}\Delta = \Delta_\omega &:= dd^\star + d^\star d : C^\infty_k(X,\, \mathbb{C}) \to C^\infty_k(X,\, \mathbb{C}),\\ \Delta' = \Delta'_\omega &:= \partial\partial^\star + \partial^\star\partial : C^\infty_{p,\, q}(X,\, \mathbb{C}) \to C^\infty_{p,\, q}(X,\, \mathbb{C}),\\ \Delta'' = \Delta''_\omega &:= \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial : C^\infty_{p,\, q}(X,\, \mathbb{C}) \to C^\infty_{p,\, q}(X,\, \mathbb{C}),\end{aligned}$$

for all $k \in \{0, \dots, 2n\}$ and all $p, q \in \{0, \dots, n\}$. Obviously, all three of them are self-adjoint differential operators of order two.

Note that, in general, $\Delta(C^\infty_{p,q}(X,\, \mathbb{C}))$ is not contained in $C^\infty_{p,q}(X,\, \mathbb{C})$, so Δ does not preserve bidegrees. This is a major source of complications when the metric ω is arbitrary. However, when ω is **Kähler**, the d-Laplacian Δ preserves bidegrees. (See e.g. [Dem97, chapter VI]).

Lemma 1.1.5. *The kernels of the above Laplacians can be described as:*

$$\ker\Delta = \ker d \cap \ker d^\star, \quad \ker\Delta' = \ker\partial \cap \ker\partial^\star, \quad \ker\Delta'' = \ker\bar\partial \cap \ker\bar\partial^\star.$$

Proof. Let us prove the description of $\ker\Delta$, the other two being analogous. Since X is *compact*, for every form u, we have

$$\langle\langle \Delta u,\, u\rangle\rangle = ||du||^2 + ||d^\star u|| \geq 0.$$

Thus, having fixed any form u, we have: $\Delta u = 0$ iff $\langle\langle \Delta u,\, u\rangle\rangle = 0$ iff $du = 0$ and $d^\star u = 0$. $\square$

Let us now recall the following fundamental facts of Hodge theory (see e.g. [Dem97, chapter VI] for the proofs and further details).

Theorem 1.1.6. *Let* $(X,\, \omega)$ *be a* **compact** *Hermitian manifold with* $\dim_{\mathbb{C}} X = n$. *Then:*

(1) *the differential operators* Δ, Δ' *and* Δ'' *are* **elliptic** *(i.e. their principal symbols are injective at every point);*

(2) *the kernels of* Δ, Δ' *and* Δ'' *are* **finite dimensional**, *while their images are* **closed** *and finite codimensional in* $C^\infty_k(X,\, \mathbb{C})$ *(for* Δ*), resp. in* $C^\infty_{p,\, q}(X,\, \mathbb{C})$ *(for* Δ' *and* Δ''*).*

Moreover, for all $k \in \{0, \dots, 2n\}$ and all $p, q \in \{0, \dots, n\}$, the following **orthogonal** *(for the L^2_ω-norm)* **two-space decompositions** *hold:*

$$C^\infty_k(X,\,\mathbb{C}) = \ker\Delta \oplus Im\,\Delta, \quad C^\infty_{p,\,q}(X,\,\mathbb{C}) = \ker\Delta' \oplus Im\,\Delta', \quad C^\infty_{p,\,q}(X,\,\mathbb{C}) = \ker\Delta'' \oplus Im\,\Delta'',$$

where all the kernels and images involved are taken in the respective (bi)degrees.

(3) *furthermore, the following L^2_ω-***orthogonal two-space decompositions** *hold:*

$$Im\,\Delta = Im\,d \oplus Im\,d^\star, \quad Im\,\Delta' = Im\,\partial \oplus Im\,\partial^\star, \quad Im\,\Delta'' = Im\,\bar\partial \oplus Im\,\bar\partial^\star.$$

*Hence, we get the following L^2_ω-***orthogonal three-space decompositions***:*

$$C^\infty_k(X,\,\mathbb{C}) = \ker\Delta \oplus Im\,d \oplus Im\,d^\star,$$

$$C^\infty_{p,\,q}(X,\,\mathbb{C}) = \ker\Delta' \oplus Im\,\partial \oplus Im\,\partial^\star,$$

$$C^\infty_{p,\,q}(X,\,\mathbb{C}) = \ker\Delta'' \oplus Im\,\bar\partial \oplus Im\,\bar\partial^\star,$$

in which we further have:

$$\ker d = \ker\Delta \oplus Im\,d \quad \text{and} \quad \ker d^\star = \ker\Delta \oplus Im\,d^\star,$$

$$\ker\partial = \ker\Delta' \oplus Im\,\partial \quad \text{and} \quad \ker\partial^\star = \ker\Delta' \oplus Im\,\partial^\star,$$

$$\ker\bar\partial = \ker\Delta'' \oplus Im\,\bar\partial \quad \text{and} \quad \ker\bar\partial^\star = \ker\Delta'' \oplus Im\,\bar\partial^\star$$

in all the degrees $k \in \{0, \dots, 2n\}$ and all the bidegrees $(p,\,q)$ with $p, q \in \{0, \dots, n\}$.

Let us only point out that conclusion (2) above follows from **Gårding's estimate** (or the **a priori estimate**, depending on the terminology being used) satisfied by any *elliptic* operator on a *compact* manifold (without boundary). Conclusion (3) further follows from the *integrability* of the operators $d, \partial, \bar\partial$, namely $d^2 = 0$, $\partial^2 = 0$ and $\bar\partial^2 = 0$.

As an immediate consequence, one gets the **Hodge isomorphisms** that display the De Rham, Dolbeault and conjugate Dolbeault cohomology groups as isomorphic to the spaces of **Δ-harmonic,** **Δ''-harmonic** and resp. **Δ'-harmonic spaces** of forms of the same (bi)degrees.

Corollary 1.1.7. *Let $(X,\,\omega)$ be a* **compact** *Hermitian manifold with $dim_{\mathbb{C}}X = n$. Then, for all $k \in \{0, \dots, 2n\}$ and all $p, q \in \{0, \dots, n\}$, the following* **Hodge isomorphisms** *hold:*

$$H^k_{DR}(X,\,\mathbb{C}) \simeq \mathcal{H}^k_\Delta(X,\,\mathbb{C}),$$

$$H^{p,\,q}_{\partial}(X,\,\mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta'}(X,\,\mathbb{C}),$$

$$H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta''}(X,\,\mathbb{C}),$$

where $\mathcal{H}^k_\Delta(X,\,\mathbb{C}) := \ker(\Delta : C^\infty_k(X,\,\mathbb{C}) \to C^\infty_k(X,\,\mathbb{C}))$, $\mathcal{H}^{p,\,q}_{\Delta'}(X,\,\mathbb{C}) := \ker(\Delta' : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C}))$ and $\mathcal{H}^{p,\,q}_{\Delta''}(X,\,\mathbb{C}) := \ker(\Delta'' : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C}))$.

A consequence of this is that a way to relate $H^k_{DR}(X,\,\mathbb{C})$ to $\bigoplus\limits_{p+q=k} H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ is to relate $\mathcal{H}^k_{\Delta}(X,\,\mathbb{C})$ to $\bigoplus\limits_{p+q=k}\mathcal{H}^{p,\,q}_{\Delta''}(X,\,\mathbb{C})$. We will see later on in this chapter that the De Rham and Dolbeault cohomologies need not be isomorphic and we will describe the class of compact complex manifolds, called $\partial\bar\partial$-manifolds, for which there is a *canonical* such isomorphism.

Hodge isomorphism for the Bott-Chern and Aeppli cohomologies

Unlike the De Rham and Dolbeault cohomologies, whose Hodge theories have long been known, the Bott-Chern and Aeppli cohomologies were given a similar treatment much more recently. For this reason, we will spell out most of the details of the proofs in what follows. The main references here are [KS60] and [Sch07]. A brief account also appeared in [Pop15].

The first step is the construction of Laplacians whose kernels will be isomorphic to the Bott-Chern and Aeppli cohomology groups. In the Bott-Chern case, we have

Definition 1.1.8. *(Kodaira-Spencer [KS60, §.6], see also Schweitzer [Sch07, 2.c., p. 9-10])*
The 4*-th order* **Bott-Chern Laplacian** $\Delta_{BC} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C})$ *is defined as*

$$\Delta_{BC} := \partial^\star\partial + \bar\partial^\star\bar\partial + (\partial\bar\partial)(\partial\bar\partial)^\star + (\partial\bar\partial)^\star(\partial\bar\partial) + (\partial^\star\bar\partial)^\star(\partial^\star\bar\partial) + (\partial^\star\bar\partial)(\partial^\star\bar\partial)^\star. \tag{1.2}$$

Note the pattern: Δ_{BC} is a sum of (necessarily non-negative) operators of the shape $A^\star A$. Hence, its kernel is the intersection of the kernels of all its terms, while $\ker(A^\star A) = \ker A$ since $\langle\langle A^\star Au,\,u\rangle\rangle = ||Au||^2$, so $A^\star Au = 0$ iff $Au = 0$. On the other hand, looking ahead to the Hodge isomorphism $\ker\Delta_{BC} \simeq H^{p,\,q}(X,\,\mathbb{C})$ that we wish to get, we need $\ker\Delta_{BC} = \ker\partial \cap \ker\bar\partial \cap \ker(\partial\bar\partial)^\star$, because $\ker(\partial\bar\partial)^\star$ is the orthogonal complement of $\mathrm{Im}\,(\partial\bar\partial)$. This accounts for the first three terms on the right of (1.2). However, their sum is not elliptic, as will be seen shortly, so we add the last three terms on the right of (1.2) to make the sum elliptic. Note that these extra terms do not change the kernel of the sum since each of them ends with either ∂ or $\bar\partial$, which have already featured at the end of the first two terms and have thus already contributed to the kernel of the sum.

The operator Δ_{BC} is obviously self-adjoint and non-negative. Let us now prove its main property.

Proposition 1.1.9. Δ_{BC} *is* **elliptic***.*

Proof. We may assume, without loss of generality, that we are in an open subset of $\mathbb{C}^n$ and that the metric ω is the *standard* one: $\omega = \sum_{j=1}^n idz_j \wedge d\bar z_j$. Indeed, the ellipticity of Δ_{BC} depends solely on its principal part, which remains unchanged if a different metric is chosen. Metric changes affect only the lower order terms.

We will use the following expressions of $\partial^\star$ and $\bar\partial^\star$ in local coordinates w.r.t. the *standard* metric:

$$\begin{aligned} \partial &= \sum_{j=1}^n dz_j \wedge \frac{\partial}{\partial z_j}, \quad \text{hence} \quad \partial^\star = -\sum_{l=1}^n \frac{\partial}{\partial\bar z_l}\left(\frac{\partial}{\partial z_l}\lrcorner\cdot\right) \\ \bar\partial &= \sum_{k=1}^n d\bar z_k \wedge \frac{\partial}{\partial\bar z_k}, \quad \text{hence} \quad \bar\partial^\star = -\sum_{r=1}^n \frac{\partial}{\partial z_r}\left(\frac{\partial}{\partial\bar z_r}\lrcorner\cdot\right), \end{aligned} \tag{1.3}$$

where $\xi\lrcorner\cdot$ is the *contraction* (a zero-th order operator) of differential forms by the vector field ξ. The above formulae follow from the following easy-to-check formulae (for operators of order 1, resp. 0):

$$\left(\frac{\partial}{\partial z_j}\right)^\star = -\frac{\partial}{\partial\bar z_j} \quad \text{and} \quad \left(\frac{\partial}{\partial z_j}\lrcorner\cdot\right)^\star = dz_j\wedge\cdot \tag{1.4}$$

and their conjugates.

Recall that the first-order differential operators $\partial/\partial z_j$ and $\partial/\partial\bar{z}_k$ act non-trivially only on the (function) coefficients of differentials forms, while the zero-th order contraction operators $(\partial/\partial z_j)\lrcorner\cdot$ and $(\partial/\partial\bar{z}_k)\lrcorner\cdot$ act non-trivially only on the dz_j's and the $d\bar{z}_k$'s. For example,

$$\frac{\partial}{\partial z_j}\left(\sum_{I,\,J} u_{I\bar{J}}\, dz_I\wedge d\bar{z}_J\right) = \sum_{I,\,J}\frac{\partial u_{I\bar{J}}}{\partial z_j}\, dz_I\wedge d\bar{z}_J \quad \text{and} \quad \frac{\partial}{\partial z_l}\lrcorner\left(\sum_{I,\,J} u_{I\bar{J}}\, dz_I\wedge d\bar{z}_J\right) = \sum_{I,\,J} u_{I\bar{J}}\,\frac{\partial}{\partial z_l}\lrcorner(dz_I\wedge d\bar{z}_J).$$

So, $\partial/\partial z_j$ and $\partial/\partial\bar{z}_k$ act independently from $(\partial/\partial z_j)\lrcorner\cdot$ and $(\partial/\partial\bar{z}_k)\lrcorner\cdot$. This leads to the formulae:

$$\frac{\partial}{\partial z_j}\left(\frac{\partial}{\partial z_k}\lrcorner u\right) = \frac{\partial}{\partial z_k}\lrcorner\frac{\partial u}{\partial z_j} \quad \text{and} \quad \frac{\partial}{\partial z_j}\left(\frac{\partial}{\partial \bar{z}_k}\lrcorner u\right) = \frac{\partial}{\partial \bar{z}_k}\lrcorner\frac{\partial u}{\partial z_j} \tag{1.5}$$

and their analogues with $\partial/\partial\bar{z}_j$ in place of $\partial/\partial z_j$, for all differential forms u of all bidegrees.

Using formulae (1.3) and (1.5), we see that, for any u, the expressions in local coordinates for the 4-th order terms of $\Delta_{BC}u$ read:

$$\Delta_{BC}u = \partial^\star\partial u + \bar\partial^\star\bar\partial u + T_1 + T_2 + T_3 + T_4,$$

where

$$\begin{aligned}
T_1 &= \bar\partial^\star\partial^\star\partial\bar\partial u = \sum_{j,\,k,\,l,\,r}\frac{\partial}{\partial z_r}\left[\frac{\partial}{\partial\bar{z}_r}\lrcorner\frac{\partial}{\partial\bar{z}_l}\left(\frac{\partial}{\partial z_l}\lrcorner[dz_j\wedge\frac{\partial}{\partial z_j}(d\bar{z}_k\wedge\frac{\partial u}{\partial\bar{z}_k})]\right)\right]\\
&= \sum_{j,\,k,\,l,\,r}\frac{\partial}{\partial z_r}\left[\frac{\partial}{\partial\bar{z}_r}\lrcorner\frac{\partial}{\partial\bar{z}_l}\left(\frac{\partial}{\partial z_l}\lrcorner[dz_j\wedge d\bar{z}_k\wedge\frac{\partial^2 u}{\partial z_j\partial\bar{z}_k}]\right)\right]\\
&= \sum_{j,\,k,\,l,\,r}\frac{\partial}{\partial z_r}\left[\frac{\partial}{\partial\bar{z}_r}\lrcorner\frac{\partial}{\partial\bar{z}_l}\left(\delta_{jl}\, d\bar{z}_k\wedge\frac{\partial^2 u}{\partial z_j\partial\bar{z}_k} + dz_j\wedge d\bar{z}_k\wedge\left(\frac{\partial}{\partial z_l}\lrcorner\frac{\partial^2 u}{\partial z_j\partial\bar{z}_k}\right)\right)\right]\\
&= \sum_{j,\,k,\,l,\,r}\frac{\partial}{\partial z_r}\left[\delta_{jl}\,\delta_{kr}\frac{\partial^3 u}{\partial z_j\partial\bar{z}_k\partial\bar{z}_l} - \delta_{jl}\, d\bar{z}_k\wedge\left(\frac{\partial}{\partial\bar{z}_r}\lrcorner\frac{\partial^3 u}{\partial z_j\partial\bar{z}_k\partial\bar{z}_l}\right)\right]\\
&+ \sum_{j,\,k,\,l,\,r}\frac{\partial}{\partial z_r}\left[-\,\delta_{kr}\, dz_j\wedge\left(\frac{\partial}{\partial z_l}\lrcorner\frac{\partial^3 u}{\partial z_j\partial\bar{z}_k\partial\bar{z}_l}\right) + dz_j\wedge d\bar{z}_k\wedge\left(\frac{\partial}{\partial\bar{z}_r}\lrcorner\frac{\partial}{\partial z_l}\lrcorner\frac{\partial^3 u}{\partial z_j\partial\bar{z}_k\partial\bar{z}_l}\right)\right]\\
&= \sum_{j,\,k}\frac{\partial^4 u}{\partial z_j\partial z_k\partial\bar{z}_j\partial\bar{z}_k} - \sum_{j,\,k,\,r} d\bar{z}_k\wedge\left(\frac{\partial}{\partial\bar{z}_r}\lrcorner\frac{\partial^4 u}{\partial z_j\partial z_r\partial\bar{z}_j\partial\bar{z}_k}\right)\\
&- \sum_{j,\,k,\,l} dz_j\wedge\left(\frac{\partial}{\partial z_l}\lrcorner\frac{\partial^4 u}{\partial z_j\partial z_k\partial\bar{z}_k\partial\bar{z}_l}\right) + \sum_{j,\,k,\,l,\,r} dz_j\wedge d\bar{z}_k\wedge\left(\frac{\partial}{\partial\bar{z}_r}\lrcorner\frac{\partial}{\partial z_l}\lrcorner\frac{\partial^4 u}{\partial z_j\partial z_r\partial\bar{z}_k\partial\bar{z}_l}\right)
\end{aligned} \tag{1.6}$$

and

$$\begin{aligned}
T_2 &= \partial\bar\partial\bar\partial^\star\partial^\star u = \sum_{j,\,k,\,l,\,r} dz_j\wedge\frac{\partial}{\partial z_j}\left[d\bar{z}_k\wedge\frac{\partial}{\partial\bar{z}_k}\left(\frac{\partial}{\partial z_r}\left[\frac{\partial}{\partial\bar{z}_r}\lrcorner\frac{\partial}{\partial\bar{z}_l}\left(\frac{\partial}{\partial z_l}\lrcorner u\right)\right]\right)\right]\\
&= \sum_{j,\,k,\,l,\,r} dz_j\wedge\frac{\partial}{\partial z_j}\left[d\bar{z}_k\wedge\left(\frac{\partial}{\partial\bar{z}_r}\lrcorner\frac{\partial}{\partial z_l}\lrcorner\frac{\partial^3 u}{\partial z_r\partial\bar{z}_k\partial\bar{z}_l}\right)\right]\\
&= \sum_{j,\,k,\,l} dz_j\wedge d\bar{z}_k\wedge\left(\frac{\partial}{\partial\bar{z}_r}\lrcorner\frac{\partial}{\partial z_l}\lrcorner\frac{\partial^4 u}{\partial z_j\partial z_r\partial\bar{z}_k\partial\bar{z}_l}\right)
\end{aligned} \tag{1.7}$$

and

$$\begin{aligned} T_3 &= \bar\partial^\star\partial\partial^\star\bar\partial u = \sum_{j,\,k,\,l,\,r}\frac{\partial}{\partial z_r}\left[\frac{\partial}{\partial\bar z_r}\lrcorner\left(dz_j\wedge\frac{\partial}{\partial z_j}\left[\frac{\partial}{\partial\bar z_l}\left(\frac{\partial}{\partial z_l}\lrcorner(d\bar z_k\wedge\frac{\partial u}{\partial\bar z_k})\right)\right]\right)\right]\\ &= -\sum_{j,\,k,\,l,\,r}\frac{\partial}{\partial\bar z_r}\lrcorner\left[dz_j\wedge d\bar z_k\wedge\left(\frac{\partial}{\partial z_l}\lrcorner\frac{\partial^4 u}{\partial z_r\partial z_j\partial\bar z_l\partial\bar z_k}\right)\right]\\ &= \sum_{j,\,k,\,l} dz_j\wedge\left(\frac{\partial}{\partial z_l}\lrcorner\frac{\partial^4 u}{\partial z_j\partial z_k\partial\bar z_k\partial\bar z_l}\right)-\sum_{j,\,k,\,l} dz_j\wedge d\bar z_k\wedge\left(\frac{\partial}{\partial\bar z_r}\lrcorner\frac{\partial}{\partial z_l}\lrcorner\frac{\partial^4 u}{\partial z_j\partial z_r\partial\bar z_k\partial\bar z_l}\right) \quad (1.8)\end{aligned}$$

and

$$\begin{aligned} T_4 &= \partial^\star\bar\partial\bar\partial^\star\partial u = \sum_{j,\,k,\,l,\,r}\frac{\partial}{\partial\bar z_l}\left[\frac{\partial}{\partial z_l}\lrcorner\left(d\bar z_k\wedge\frac{\partial}{\partial\bar z_k}\left[\frac{\partial}{\partial z_r}\left(\frac{\partial}{\partial\bar z_r}\lrcorner[dz_j\wedge\frac{\partial u}{\partial z_j}]\right)\right]\right)\right]\\ &= -\sum_{j,\,k,\,l,\,r}\frac{\partial}{\partial z_l}\lrcorner\left(d\bar z_k\wedge dz_j\wedge\left(\frac{\partial}{\partial\bar z_r}\lrcorner\frac{\partial^4 u}{\partial z_j\partial z_r\partial\bar z_k\partial\bar z_l}\right)\right)\\ &= \sum_{j,\,k,\,r} d\bar z_k\wedge\left(\frac{\partial}{\partial\bar z_r}\lrcorner\frac{\partial^4 u}{\partial z_j\partial z_r\partial\bar z_j\partial\bar z_k}\right)-\sum_{j,\,k,\,l,\,r} d\bar z_k\wedge dz_j\wedge\left(\frac{\partial}{\partial z_l}\lrcorner\frac{\partial}{\partial\bar z_r}\lrcorner\frac{\partial^4 u}{\partial z_j\partial z_r\partial\bar z_k\partial\bar z_l}\right). \quad (1.9)\end{aligned}$$

Adding up the identities (1.6)–(1.9), we see that all the terms cancel out, except for the first term on the r.h.s. of (1.6). We conclude that the principal part of Δ_{BC} is

$$T_1+T_2+T_3+T_4=\sum_{j,\,k}\frac{\partial^4 u}{\partial z_j\partial z_k\partial\bar z_j\partial\bar z_k}=\frac{1}{16}\sum_{j,\,k}\left(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial y_j^2}\right)\left(\frac{\partial^2}{\partial x_k^2}+\frac{\partial^2}{\partial y_k^2}\right)u.$$

Hence, the principal symbol of Δ_{BC} is

$$\sigma_{\Delta_{BC}}(x;\,(\xi,\,\eta))\,u(x)=\frac{1}{16}\sum_{j,\,k=1}^{n}(\xi_j^2+\eta_j^2)\,(\xi_k^2+\eta_k^2)\,u(x)=\left[\frac{1}{4}\sum_{j=1}^{n}(\xi_j^2+\eta_j^2)\right]^2u(x),$$

for all forms u and all points $(x;\,(\xi,\,\eta))\in{}^{\mathbb{R}}TX$ in the real tangent bundle of X, where we put $(\xi,\,\eta)=\sum\limits_{j=1}^{n}\xi_j(x)\,\frac{\partial}{\partial x_j}+\sum\limits_{j=1}^{n}\eta_j(x)\,\frac{\partial}{\partial y_j}\in{}^{\mathbb{R}}T_xX$.

In particular, $\sigma_{\Delta_{BC}}(x;\,(\xi,\,\eta))$ is injective for all $x\in X$ and all $(\xi,\,\eta)\neq 0$. Consequently, Δ_{BC} is elliptic. □

Thanks to Gårding's estimate for *elliptic* differential operators on *compact* manifolds (see [Dem97, chapter VI, corollary 2.4]), we immediately get the following analogue in the Bott-Chern context of Theorem 1.1.6 and of Corollary 1.1.7.

Corollary 1.1.10. *Let $(X,\,\omega)$ be a* **compact** *Hermitian manifold with $dim_{\mathbb{C}}X=n$. Fix arbitrary $p,q\in\{0,\dots,n\}$.*

*(1) The following L^2_ω-***orthogonal three-space decomposition** *holds:*

$$C^\infty_{p,\,q}(X,\mathbb{C})=\ker\Delta_{BC}\oplus Im\,\partial\bar\partial\oplus(Im\,\partial^\star+Im\,\bar\partial^\star).$$

(2) Moreover

$$\ker\partial\cap\ker\bar\partial=\ker\Delta_{BC}\oplus Im\,\partial\bar\partial,$$

yielding the **Hodge isomorphism**

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\,\mathbb{C}),$$

where $\mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\,\mathbb{C}) = \ker(\Delta_{BC} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C}))$ *is the* **Bott-Chern harmonic space***.*

In particular, $\dim H^{p,\,q}_{BC}(X,\,\mathbb{C}) < +\infty$*.*

(3) We also have

$$\mathit{Im}\,\Delta_{BC} = \mathit{Im}\,\partial\bar\partial \oplus (\mathit{Im}\,\partial^\star + \mathit{Im}\,\bar\partial^\star) \quad \textit{and} \quad \ker(\partial\bar\partial)^\star = \ker\Delta_{BC} \oplus (\mathit{Im}\,\partial^\star + \mathit{Im}\,\bar\partial^\star).$$

Hence

$$\ker\Delta_{BC} = \ker\partial \cap \ker\bar\partial \cap \ker(\partial\bar\partial)^\star.$$

The analogous discussion can be had for the Aeppli cohomology. Following the pattern described after Definition 1.1.8, we put

Definition 1.1.11. *(Schweitzer [Sch07, §.2, 2.c])*

The 4*-th order* **Aeppli Laplacian** $\Delta_A : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C})$ *is defined as*

$$\Delta_A := (\partial\bar\partial)^\star(\partial\bar\partial) + \partial\partial^\star + \bar\partial\bar\partial^\star + (\partial\bar\partial)(\partial\bar\partial)^\star + (\partial\bar\partial^\star)(\partial\bar\partial^\star)^\star + (\partial\bar\partial^\star)^\star(\partial\bar\partial^\star). \tag{1.10}$$

As in the case of the Bott-Chern Laplacian, the first three terms suffice to produce the desired kernel for Δ_A (that will be isomorphic to the Aeppli cohomology group). However, their sum is not elliptic, so we complete it by adding the last three terms which do not change the kernel of the sum, but make it elliptic.

Again as in the case of the Bott-Chern Laplacian, the operator Δ_A is obviously self-adjoint and non-negative. Crucially, we also have

Proposition 1.1.12. Δ_A *is* **elliptic***.*

Proof. It is similar to that of Proposition 1.1.9 and is left to the reader. □

Yet again, Gårding's estimate for *elliptic* differential operators on *compact* manifolds (see [Dem97, chapter VI, corollary 2.4]) leads to the following analogue in the Aeppli context of Corollary 1.1.10.

Corollary 1.1.13. *Let* $(X,\,\omega)$ *be a* **compact** *Hermitian manifold with* $\dim_{\mathbb{C}}X = n$*. Fix arbitrary* $p, q \in \{0, \dots, n\}$*.*

(1) The following L^2_ω**-orthogonal three-space decomposition** *holds:*

$$C^\infty_{p,\,q}(X,\mathbb{C}) = \ker\Delta_A \oplus (\mathit{Im}\,\partial + \mathit{Im}\,\bar\partial) \oplus \mathit{Im}\,(\partial\bar\partial)^\star.$$

(2) Moreover

$$\ker(\partial\bar\partial) = \ker\Delta_A \oplus (\mathit{Im}\,\partial + \mathit{Im}\,\bar\partial),$$

yielding the **Hodge isomorphism**

$$H^{p,\,q}_{A}(X,\,\mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta_A}(X,\,\mathbb{C}),$$

where $\mathcal{H}^{p,\,q}_{\Delta_A}(X,\,\mathbb{C}) = \ker(\Delta_A : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C}))$ *is the* **Aeppli harmonic space***.*

In particular, $\dim H^{p,\,q}_{A}(X,\,\mathbb{C}) < +\infty$*.*

(3) We also have

$$\mathit{Im}\,\Delta_A = (\mathit{Im}\,\partial + \mathit{Im}\,\bar\partial) \oplus \mathit{Im}\,(\partial\bar\partial)^\star \quad \textit{and} \quad \ker\partial^\star \cap \ker\bar\partial^\star = \ker\Delta_A \oplus \mathit{Im}\,(\partial\bar\partial)^\star.$$

Hence

$$\ker\Delta_A = \ker(\partial\bar\partial) \cap \ker\partial^\star \cap \ker\bar\partial^\star.$$

1.1.2 Duality between the Bott-Chern and Aeppli cohomologies of complementary bidegrees

Recall the classical *Poincaré duality* for the De Rham cohomology: for every $k \in \{0, \dots, 2n\}$, the bilinear pairing

$$H^k_{DR}(X,\, \mathbb{C}) \times H^{2n-k}_{DR}(X,\, \mathbb{C}) \to \mathbb{C}, \qquad (\{\alpha\}_{DR},\, \{\beta\}_{DR}) \mapsto \int\limits_X \alpha \wedge \beta,$$

is *well-defined* (i.e. independent of the choices of representatives α and β of their respective cohomology classes) and *non-degenerate* (i.e. for all non-zero classes $\{\alpha\}_{DR} \in H^k_{DR}(X,\, \mathbb{C})$ and $\{\beta\}_{DR} \in H^{2n-k}_{DR}(X,\, \mathbb{C})$, the maps $(\{\alpha\}_{DR},\, \cdot) : H^{2n-k}_{DR}(X,\, \mathbb{C}) \to \mathbb{C}$ and $(\cdot,\, \{\beta\}_{DR}) : H^k_{DR}(X,\, \mathbb{C}) \to \mathbb{C}$ are not identically zero). This means that $H^k_{DR}(X,\, \mathbb{C})$ is the dual of $H^{2n-k}_{DR}(X,\, \mathbb{C})$.

Similarly, the classical *Serre duality* for the Dolbeault cohomology ensures that, for all $p, q \in \{0, \dots, n\}$, the bilinear pairing

$$H^{p,\,q}_{\bar\partial}(X,\, \mathbb{C}) \times H^{n-p,\,n-q}_{\bar\partial}(X,\, \mathbb{C}) \to \mathbb{C}, \qquad ([\alpha]_{\bar\partial},\, [\beta]_{\bar\partial}) \mapsto \int\limits_X \alpha \wedge \beta,$$

is *well-defined* and *non-degenerate*. Thus, $H^{p,\,q}_{\bar\partial}(X,\, \mathbb{C})$ is the dual of $H^{n-p,\,n-q}_{\bar\partial}(X,\, \mathbb{C})$.

We will now derive the analogue in the Bott-Chern-Aeppli context of the Serre duality. The main point is the following

Proposition 1.1.14. *Let $(X,\, \omega)$ be a* **compact** *Hermitian manifold with $\dim_{\mathbb{C}} X = n$. Fix arbitrary $p, q \in \{0, \dots, n\}$. Then, under the Hodge star isomorphism*

$$\star = \star_\omega : C^\infty_{p,\,q}(X,\, \mathbb{C}) \to C^\infty_{n-q,\,n-p}(X,\, \mathbb{C}), \qquad u \wedge \star\bar{v} = \langle u,\, v\rangle_\omega \, dV_\omega, \tag{1.11}$$

the Bott-Chern and Aeppli three-space decompositions are related by the following **isomorphisms***:*

$$\begin{aligned} \star : \mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\, \mathbb{C}) \quad &\longrightarrow \quad \mathcal{H}^{n-q,\,n-p}_{\Delta_A}(X,\, \mathbb{C}) \\ \star : Im\,(\partial\bar\partial) \quad &\longrightarrow \quad Im\,(\partial\bar\partial)^\star \\ \star : (Im\,\partial^\star + Im\,\bar\partial^\star) \quad &\longrightarrow \quad (Im\,\partial + Im\,\bar\partial). \end{aligned}$$

Proof. Thanks to the L^2_ω-orthogonal three-space decompositions given by Corollaries 1.1.10 and 1.1.13, it suffices to prove the inclusions (which, once proved, will be equalities):

$$\begin{aligned} \star(\mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\, \mathbb{C})) \quad &\subset \quad \mathcal{H}^{n-q,\,n-p}_{\Delta_A}(X,\, \mathbb{C}) \\ \star(\mathrm{Im}\,(\partial\bar\partial)) \quad &\subset \quad \mathrm{Im}\,(\partial\bar\partial)^\star \\ \star(\mathrm{Im}\,\partial^\star + \mathrm{Im}\,\bar\partial^\star) \quad &\subset \quad \mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial. \end{aligned}$$

These inclusions follow easily from the following formulae (see [Dem97, chapter VI] or easy verification based on the definition (1.12) of $\star$):

$$\star\star = (-1)^k\,\mathrm{Id} \quad \text{on } k\text{-forms}; \qquad \partial^\star = -\star\bar\partial\star, \qquad \bar\partial^\star = -\star\partial\star, \qquad d^\star = -\star\, d\,\star\,. \tag{1.12}$$

Indeed, for any form u, we have: $\star\partial\bar\partial u = \pm(\star\partial\star)\,(\star\bar\partial\star)\,(\star u) = \pm(\partial\bar\partial)^\star\,(\star u) \in \mathrm{Im}\,(\partial\bar\partial)^\star$. This proves the second of the above inclusions. The third one can be proved analogously. It remains to prove the first one. Thanks to (1.12) and to the descriptions of the kernels of Δ_{BC} and Δ_A in Corollaries 1.1.10 and 1.1.13, we get the following equivalences for every form u:

$$\begin{aligned} u \in \mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\,\mathbb{C}) \quad &\Longleftrightarrow \quad \partial u = 0, \bar\partial u = 0, (\partial\bar\partial)^\star u = 0 \\ &\Longleftrightarrow \quad \bar\partial^\star(\star u) = 0, \partial^\star(\star u) = 0, \partial\bar\partial(\star u) = 0 \iff \star u \in \mathcal{H}^{n-q,\,n-p}_{\Delta_A}(X,\,\mathbb{C}). \end{aligned}$$

□

We are now ready to prove that the Aeppli cohomology is canonically dual to the Bott-Chern cohomology of the complementary bidegree. Note that the next statement depends only on the complex structure of the manifold, no metric is involved.

Theorem 1.1.15. *Let X be a* **compact** *complex manifold with $dim_{\mathbb{C}}X = n$. Then, for all $p, q \in \{0, \dots, n\}$, the bilinear pairing*

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \times H^{n-p,\,n-q}_A(X,\,\mathbb{C}) \to \mathbb{C}, \qquad ([\alpha]_{BC},\,[\beta]_A) \mapsto \int\limits_X \alpha\wedge\beta,$$

is **well-defined** *and* **non-degenerate**. *Thus, $H^{p,\,q}_{BC}(X,\,\mathbb{C})$ is the dual of $H^{n-p,\,n-q}_A(X,\,\mathbb{C})$.*

Proof. If α is changed to another representative $\alpha + \partial\bar\partial u$ of the same Bott-Chern cohomology class, then $\int_X(\alpha+\partial\bar\partial u)\wedge\beta = \int_X \alpha\wedge\beta \pm \int_X u\wedge\partial\bar\partial\beta = \int_X\alpha\wedge\beta$, since $\partial\bar\partial\beta = 0$. Indeed, β represents an Aeppli class.

On the other hand, if β is changed to another representative $\beta + \partial\xi + \bar\partial\zeta$ of the same Aeppli cohomology class, then $\int_X \alpha\wedge(\beta+\partial\xi+\bar\partial\zeta) = \int_X\alpha\wedge\beta \pm \int_X \partial\alpha\wedge\xi \pm \int_X\bar\partial\alpha\wedge\zeta = \int_X\alpha\wedge\beta$, since $\partial\alpha = 0$ and $\bar\partial\alpha = 0$. Indeed, α represents a Bott-Chern class.

We conclude that the bilinear map in the statement is well defined (i.e. independent of the choices of representatives of the cohomology classes involved).

To prove non-degeneracy, we fix an arbitrary Hermitian metric ω on X.

Let $[\alpha]_{BC} \in H^{p,\,q}_{BC}(X,\,\mathbb{C})$ be a non-zero class. Thanks to the Hodge isomorphism for the Bott-Chern cohomology (see (2) of Corollary 1.1.10), this class contains a unique (and necessarily non-zero) Bott-Chern harmonic representative. Let us call it $\alpha \in \mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\,\mathbb{C})\setminus\{0\}$. By Proposition 1.1.14, we must have $\star\alpha \in \mathcal{H}^{n-q,\,n-p}_{\Delta_A}(X,\,\mathbb{C})$. Then, we also have $\star\bar\alpha \in \mathcal{H}^{n-p,\,n-q}_{\Delta_A}(X,\,\mathbb{C})$, as can be immediately checked from the description of $\ker\Delta_A$ given in (3) of Corollary 1.1.13. Thus, $\star\bar\alpha$ defines a class in $H^{n-p,\,n-q}_A(X,\,\mathbb{C})$ and, under the pairing in the statement, we get

$$(\alpha,\,\star\bar\alpha) \mapsto \int\limits_X \alpha\wedge\star\bar\alpha = \int\limits_X |\alpha|^2_\omega\,dV_\omega = ||\alpha||^2_\omega \neq 0.$$

Similarly, let $[\beta]_A \in H^{n-p,\,n-q}_A(X,\,\mathbb{C})$ be a non-zero class and let β be its Aeppli harmonic representative. Then, $\beta\neq 0$ and $\star\bar\beta$ is Bott-Chern harmonic of bidegree $(p,\,q)$. Since

$$(\star\bar\beta,\,\beta) \mapsto \int\limits_X \star\bar\beta\wedge\beta = \int\limits_X |\beta|^2_\omega\,dV_\omega = ||\beta||^2_\omega \neq 0,$$

we are done. □

1.2 The Frölicher spectral sequence

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The *Frölicher spectral sequence (FSS)* of X is an object that relates the complex structure of X to its differential structure at the cohomological level. If no assumption is made on X, the individual Dolbeault cohomology groups $H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ may be "too big" for their direct sum over all p, q with $p+q=k$ to inject into the corresponding De Rham cohomology group $H^k_{DR}(X,\,\mathbb{C})$. In that case, they need to be "pared down" to spaces of smaller dimensions whose direct sum injects. This reduction is made inductively: the first page, denoted by E_1, of the Frölicher spectral sequence consists of the Dolbeault cohomology of X, while every page E_r is defined to be the cohomology of the previous page E_{r-1}. Thus, every E_r is a refinement of E_{r-1}. The process stops (i.e. becomes stationary) after finitely many steps, which means that there exists an $r \in \mathbb{N}^\star$ such that $E_r = E_{r+1} = \dots$. This stationary value is denoted by E_∞. One says that the spectral sequence degenerates at E_r. A key feature is that, for every k, the vector spaces $E^{p,\,q}_\infty$ with $p+q=k$ add up (as a *non-canonical* direct sum, in general) to the De Rham cohomology space $H^k_{DR}(X,\,\mathbb{C})$. For this reason, one says that the Frölicher spectral sequence *converges* to the De Rham cohomology of X. Some authors call it the *Hodge-De Rham spectral sequence*.

A spectral sequence can be associated, in the way that will be described below, with every abstractly defined double complex $A = (A^{p,\,q},\,\partial_1,\,\partial_2)$, where $A = \oplus_{p,\,q\in\mathbb{Z}} A^{p,\,q}$ is a bigraded vector space endowed with endomorphisms $\partial_1 : A^{p,\,q} \to A^{p+1,\,q}$ of bidegree (or type) $(1,\,0)$ and $\partial_2 : A^{p,\,q} \to A^{p,\,q+1}$ of bidegree (or type) $(0,\,1)$, such that the sum $d = \partial_1 + \partial_2$ is *integrable*, namely $d^2 = 0$.

Even more generally, one can associate a spectral sequence with any differential module $(K,\,d)$ equipped with a filtration $\{0\} \subset \dots \subset K_{p+1} \subset K_p \subset \dots \subset K$ by *differential* submodules (i.e. $d(K_p) \subset K_p$ for all p)).

In what follows, we will only be concerned with the double complex $A = (C^\infty_{p,\,q}(X,\,\mathbb{C}),\,\partial,\,\bar\partial)$ on a compact complex manifold X, but many results apply in the more general abstract setting.

There are at least two equivalent points of view on the FSS: the classical, more formal one, that employs the language of filtrations and is widely used in algebraic geometry; and a more recent and concrete one, introduced by Cordero, Fernández, Gray and Ugarte in [CFGU97], that is more suited to our purposes throughout this book. We will describe both of them.

1.2.1 The classical point of view

In this subsection, we will follow, to some extent, the presentation in [Voi02, §.8.3].

Filtration on the spaces of smooth forms

For all non-negative integers $k \leq 2n$ and $p \leq \min\{k,\,n\}$, we let

$$\mathcal{F}^p C^\infty_k(X,\,\mathbb{C}) := \bigoplus_{i\geq p} C^\infty_{i,\,k-i}(X) \subset C^\infty_k(X,\,\mathbb{C})$$

and get a **filtration of** $C^\infty_k(X,\,\mathbb{C})$ for every k:

$$\{0\} \subset \dots \subset \mathcal{F}^{p+1} C^\infty_k(X,\,\mathbb{C}) \subset \mathcal{F}^p C^\infty_k(X,\,\mathbb{C}) \subset \dots \subset C^\infty_k(X,\,\mathbb{C}). \tag{1.13}$$

Definition 1.2.1. *A k-form α lying in $\mathcal{F}^p C^\infty_k(X,\,\mathbb{C})$ is said to be of* **filtration type** $\mathcal{F}^p$.

Note that $d(\mathcal{F}^p C^\infty_k(X,\,\mathbb{C})) \subset \mathcal{F}^p C^\infty_{k+1}(X,\,\mathbb{C})$ for all p, k, so the De Rham complex restricts to $\mathcal{F}^p C^\infty_\bullet(X,\,\mathbb{C})$ for every p. The cohomology of this restricted De Rham complex features on the left in (1.14) below and is a refined version of the filtration (1.15) on the De Rham cohomology.

Filtration on the De Rham cohomology spaces

On the other hand, let

$$F^pH^k_{DR}(X,\,\mathbb{C}) := \frac{\mathcal{F}^pC^\infty_k(X,\,\mathbb{C})\cap\ker d}{\mathcal{F}^pC^\infty_k(X,\,\mathbb{C})\cap\operatorname{Im} d}\subset H^k_{DR}(X,\,\mathbb{C}),$$

be the subspace of De Rham cohomology classes of degree k that are representable by forms in $\mathcal{F}^pC^\infty_k(X,\,\mathbb{C})$. It can be easily seen that $F^pH^k_{DR}(X,\,\mathbb{C})$ coincides with the image of the following canonical linear map induced by the identity:

$$\frac{\ker\left(d:\mathcal{F}^pC^\infty_k(X,\,\mathbb{C})\to\mathcal{F}^pC^\infty_{k+1}(X,\,\mathbb{C})\right)}{\operatorname{Im}\left(d:\mathcal{F}^pC^\infty_{k-1}(X,\,\mathbb{C})\to\mathcal{F}^pC^\infty_k(X,\,\mathbb{C})\right)}\longrightarrow H^k_{DR}(X,\,\mathbb{C}), \tag{1.14}$$

which, for every form $u=\sum\limits_{i\geq p}u^{i,\,k-i}\in\ker d\cap\mathcal{F}^pC^\infty_k(X,\,\mathbb{C})$, maps the class of u modulo $\operatorname{Im}(d:\mathcal{F}^pC^\infty_{k-1}(X,\,\mathbb{C})\to\mathcal{F}^pC^\infty_k(X,\,\mathbb{C}))$ to the De Rham class $\{\sum\limits_{i\geq p}u^{i,\,k-i}\}_{DR}$ of u.

In this way, we get a **filtration** of $H^k_{DR}(X,\,\mathbb{C})$ for every k:

$$\{0\}\subset\cdots\subset F^{p+1}H^k_{DR}(X,\,\mathbb{C})\subset F^pH^k_{DR}(X,\,\mathbb{C})\subset\cdots\subset H^k_{DR}(X,\,\mathbb{C}). \tag{1.15}$$

The successive quotients of the filtration (1.15), namely the vector spaces

$$G_pH^{p+q}_{DR}(X,\,\mathbb{C}) := \frac{F^pH^{p+q}_{DR}(X,\,\mathbb{C})}{F^{p+1}H^{p+q}_{DR}(X,\,\mathbb{C})},$$

are called the **graded modules** associated with the filtration (1.15). The analogous objects $G_pC^\infty_{p+q}(X,\,\mathbb{C})$ for the filtration (1.13) will also be used.

Definition of the Frölicher spectral sequence

The following theorem is a general result that applies to all spectral sequences although we only state it in the Frölicher case.

Theorem 1.2.2. *For every $r\in\mathbb{N}$ and $p,q\in\{0,\dots,n\}$, there is a* **complex** *of $\mathbb{C}$-vector spaces:*

$$\dots\xrightarrow{d_r}E^{p,\,q}_r(X)\xrightarrow{d_r}E^{p+r,\,q-r+1}_r(X)\xrightarrow{d_r}\dots$$

whose morphisms, called **differentials***, are all of type $(r,\,-r+1)$, with the following properties.*

(i) $E^{p,\,q}_0(X)=G_pC^\infty_{p+q}(X,\,\mathbb{C})=\dfrac{\mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C})}{\mathcal{F}^{p+1}C^\infty_{p+q}(X,\,\mathbb{C})}=C^\infty_{p,\,q}(X,\,\mathbb{C})$ *and* $d_0=\bar\partial$*;*

(ii) For every $r\in\mathbb{N}$ and every bidegree $(p,\,q)$, there is a **canonical isomorphism**

$$E^{p,\,q}_{r+1}(X)\simeq\frac{\ker(d_r:E^{p,\,q}_r(X)\to E^{p+r,\,q-r+1}_r(X))}{Im\,(d_r:E^{p-r,\,q+r-1}_r(X)\to E^{p,\,q}_r(X))}.$$

(iii) For r **sufficiently large** *and every bidegree $(p,\,q)$, there is a* **canonical isomorphism**

$$E^{p,\,q}_r(X)\simeq G_pH^{p+q}_{DR}(X,\,\mathbb{C}) := \frac{F^pH^{p+q}_{DR}(X,\,\mathbb{C})}{F^{p+1}H^{p+q}_{DR}(X,\,\mathbb{C})}$$

whose inverse $G_pH^{p+q}_{DR}(X,\,\mathbb{C}) \to E_r^{p,\,q}$ *is induced by the projection*

$$F^pH^{p+q}_{DR}(X,\,\mathbb{C}) \ni \left\{\sum_{i\geq p} u^{i,\,p+q-i}\right\}_{DR} \mapsto \{u^{p,\,q}\}_{E_r} \in E_r^{p,\,q}(X).$$

Notation 1.2.3. *For all* $r \gg 1$ *and all* p, q*, we let*

$$E_\infty^{p,\,q}(X) := E_r^{p,\,q}(X) = E_{r+1}^{p,\,q}(X) = E_{r+2}^{p,\,q}(X) = \dots.$$

Definition 1.2.4. *(i) The sequence of complexes*

$$(E_r^{p,\,q}(X),\, d_r)_{r\in\mathbb{N}}$$

is called the **Frölicher spectral sequence (FSS)** *of the compact complex manifold* X*.*

Alternatively, it is called the **spectral sequence** *associated with the filtration* $F^\bullet H^\bullet_{DR}(X,\,\mathbb{C})$ *of the* **De Rham cohomology** *of* X*.*

(ii) For every $r \in \mathbb{N}$*, the family of complexes* $(E_r^{\bullet,\,\bullet}(X),\, d_r)$ *is called the* r**-th page** *of the Frölicher spectral sequence.*

(iii) The Frölicher spectral sequence of X *is said to* **degenerate at** E_r*, or* **at the** r**-th page***, if* $E_r^{p,\,q}(X) = E_{r+1}^{p,\,q}(X) = E_{r+2}^{p,\,q}(X) = \dots$ *for all* $p, q \in \{0, \dots, n\}$*. In this case, we write*

$$E_r(X) = E_\infty(X).$$

The degeneration property $E_r(X) = E_\infty(X)$ is obviously equivalent to all the differentials d_s vanishing identically for all $s \geq r$.

As for the statement of Theorem 1.2.2, note that (i) says that the zero-th page of the FSS is the Dolbeault complex of X, (ii) says that, for every r, the $(r+1)$-st page is the cohomology of the previous page r-th page (hence the first page E_1 is the Dolbeault cohomology of X), while (iii) gives a **canonical isomorphism**

$$E_\infty^{p,\,q}(X) \simeq G_pH^{p+q}_{DR}(X,\,\mathbb{C})$$

in every bidegree $(p,\,q)$. However, the isomorphism $H^k_{DR}(X,\,\mathbb{C}) \simeq \oplus_{0\leq p\leq k} G_pH^k_{DR}(X,\,\mathbb{C})$ is not canonical, so, in general, we only get **non-canonical isomorphisms**:

$$H^k_{DR}(X,\,\mathbb{C}) \simeq \bigoplus_{p+q=k} E_\infty^{p,\,q}(X), \qquad k \in \{0, \dots, 2n\}. \tag{1.16}$$

Consequently, the degeneration property $E_r(X) = E_\infty(X)$ is equivalent to the existence of *non-necessarily canonical isomorphisms*

$$H^k_{DR}(X,\,\mathbb{C}) \simeq \bigoplus_{p+q=k} E_r^{p,\,q}(X), \qquad k \in \{0, \dots, 2n\}. \tag{1.17}$$

On the other hand, since X is compact, all the spaces $E_1^{p,\,q}(X) = H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ are finite dimensional. Meanwhile, every $E_{r+1}^{p,\,q}(X)$ is a quotient of a subspace of $E_r^{p,\,q}(X)$, so

$$\dots \leq \dim E_{r+1}^{p,\,q}(X) \leq \dim E_r^{p,\,q}(X) \leq \dots \leq \dim E_1^{p,\,q}(X), \qquad p, q \in \{0, \dots, n\}.$$

Together with (1.16), this last fact implies

Corollary 1.2.5. *The following dimension inequalities hold on any compact complex manifold X:*

$$b_k = \sum_{p+q=k} e_\infty^{p,\,q} \leq \cdots \leq \sum_{p+q=k} e_{r+1}^{p,\,q} \leq \sum_{p+q=k} e_r^{p,\,q} \leq \sum_{p+q=k} h_{\bar\partial}^{p,\,q}, \qquad k \in \{0,\dots,2n\},\, r \in \mathbb{N}^\star, \tag{1.18}$$

where the $b_k := \dim H^k_{DR}(X)$ are the **Betti numbers***, the $h_{\bar\partial}^{p,\,q} := \dim H_{\bar\partial}^{p,\,q}(X) = e_1^{p,\,q}$ are the* **Hodge numbers***, and the $e_r^{p,\,q} := \dim E_r^{p,\,q}(X)$.*

Consequently, the degeneration property $E_r(X) = E_\infty(X)$ is *purely numerical*:

Corollary 1.2.6. *Let $r \in \mathbb{N}^\star$. The Frölicher spectral sequence of X degenerates at E_r if and only if*

$$b_k = \sum_{p+q=k} e_r^{p,\,q}$$

for all $k \in \{0,\dots,2n\}$.

Proof of Theorem 1.2.2

Further notation and general definitions, including those of the differentials $(d_r)_{r\geq 1}$, will be introduced in the course of this proof. We will explicitly define the spaces $E_r^{p,\,q}(X)$ and the differentials d_r and will then show that they satisfy the conclusions of Theorem 1.2.2.

For all r, p, q, consider the vector subspace of $\mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C})$:

$$\mathcal{Z}_r^{p,\,q} := \bigg\{\alpha \in \mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C}) \;\mid\; d\alpha \in \mathcal{F}^{p+r}C^\infty_{p+q+1}(X,\,\mathbb{C})\bigg\}.$$

Thus, for a form $\alpha = \sum_{i\geq p}\alpha^{i,\,p+q-i} \in \mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C})$, the condition $d\alpha \in \mathcal{F}^{p+r}C^\infty_{p+q+1}(X,\,\mathbb{C})$ is equivalent to the vanishing of all the pure-type components of $d\alpha$ whose holomorphic degree is $\leq p+r-1$. It is, therefore, a *partial vanishing* condition on $d\alpha$, or a kind of *partial closedness* condition on α, that becomes stronger and stronger as r increases.

The space of *exact* forms of filtration type $\mathcal{F}^p$ with $\mathcal{F}^{p-r}$*-type potentials* is defined as

$$\mathcal{B}_r^{p,\,q} := \mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C}) \cap d\bigg(\mathcal{F}^{p-r}C^\infty_{p+q-1}(X,\,\mathbb{C})\bigg) = d\bigg(\mathcal{Z}_r^{p-r,\,q+r-1}\bigg), \tag{1.19}$$

where the last identity is a useful alternative description of the space $\mathcal{B}_r^{p,\,q}$ that the reader will easily check. Finally, set

$$E_r^{p,\,q}(X) = \frac{\mathcal{Z}_r^{p,\,q}}{\mathcal{B}_{r-1}^{p,\,q} + \mathcal{Z}_{r-1}^{p+1,\,q-1}}. \tag{1.20}$$

We pause briefly to observe a few basic properties of these spaces relative to one another.

Lemma 1.2.7. *The following inclusions hold for all r, p, q:*

(a) $\mathcal{B}_{r-1}^{p,\,q} \subset \mathcal{B}_r^{p,\,q} \subset \mathcal{Z}_r^{p,\,q}$ *and* $\mathcal{Z}_{r-1}^{p+1,\,q-1} \subset \mathcal{Z}_r^{p,\,q}$.

In particular, $\mathcal{B}_{r-1}^{p,\,q} + \mathcal{Z}_{r-1}^{p+1,\,q-1}$ is contained in $\mathcal{Z}_r^{p,\,q}$, so definition (1.20) of $E_r^{p,\,q}(X)$ is meaningful.

(b) $\mathcal{Z}_{r-1}^{p+1,\,q-1} \subset \mathcal{Z}_r^{p,\,q} \subset \mathcal{Z}_{r-1}^{p,\,q}$.

In particular, we infer from (a) and (b) that, in every fixed bidegree $(p,\, q)$, the spaces $\mathcal{Z}_r^{p,\,q}$ become smaller and smaller, while the spaces $\mathcal{B}_r^{p,\,q}$ become larger and larger, as r increases.

(c) $d(\mathcal{Z}_r^{p,\,q}) \subset \mathcal{Z}_r^{p+r,\,q-r+1}$*, hence also* $d(\mathcal{Z}_{r-1}^{p+1,\,q-1}) \subset \mathcal{Z}_{r-1}^{p+r,\,q-r+1}$*;*

(d) $d(\mathcal{B}_{r-1}^{p,\,q}) = \{0\}$*, hence* $d(\mathcal{B}_{r-1}^{p,\,q}) \subset \mathcal{B}_{r-1}^{p+r,\,q-r+1}$

Proof. (a) If $\alpha \in \mathcal{B}_{r-1}^{p,\,q}$, then $\alpha \in \mathcal{F}^p C^\infty_{p+q}(X,\, \mathbb{C})$ and $\alpha = d\beta$ for some $\beta \in \mathcal{F}^{p-r+1} C^\infty_{p+q-1}(X,\, \mathbb{C})$. Since $\mathcal{F}^{p-r+1} C^\infty_{p+q-1}(X,\, \mathbb{C}) \subset \mathcal{F}^{p-r} C^\infty_{p+q-1}(X,\, \mathbb{C})$, we get that $\alpha = d\beta \in \mathcal{B}_r^{p,\,q}$. This proves the first inclusion.

On the other hand, if $\alpha = d\beta \in \mathcal{B}_r^{p,\,q}$, then $\alpha \in \mathcal{F}^p C^\infty_{p+q}(X,\, \mathbb{C})$ and $d\alpha = 0$, hence $\alpha \in \mathcal{Z}_r^{p,\,q}$. This proves the second inclusion.

To prove the third inclusion, we notice that $\alpha \in \mathcal{Z}_{r-1}^{p+1,\,q-1}$ if and only if $\alpha \in \mathcal{F}^{p+1} C^\infty_{p+q}(X,\, \mathbb{C})$ and $d\alpha \in \mathcal{F}^{p+r} C^\infty_{p+q+1}(X,\, \mathbb{C})$. On the other hand, $\alpha \in \mathcal{Z}_r^{p,\,q}$ if and only if $\alpha \in \mathcal{F}^p C^\infty_{p+q}(X,\, \mathbb{C})$ and $d\alpha \in \mathcal{F}^{p+r} C^\infty_{p+q+1}(X,\, \mathbb{C})$. Since $\mathcal{F}^{p+1} C^\infty_{p+q}(X,\, \mathbb{C}) \subset \mathcal{F}^p C^\infty_{p+q}(X,\, \mathbb{C})$, the inclusion is proved.

(b) All three spaces $\mathcal{Z}_{r-1}^{p+1,\,q-1}$, $\mathcal{Z}_r^{p,\,q}$ and $\mathcal{Z}_{r-1}^{p,\,q}$ consist of $(p+q)$-forms α of respective filtration types $\mathcal{F}^{p+1}$, $\mathcal{F}^p$ and $\mathcal{F}^p$, such that $d\alpha$ is of respective filtration types $\mathcal{F}^{p+r}$, $\mathcal{F}^{p+r}$ and $\mathcal{F}^{p+r-1}$. The two stated inclusions follow from the inclusions $\mathcal{F}^{p+1} \subset \mathcal{F}^p$ and $\mathcal{F}^{p+r} \subset \mathcal{F}^{p+r-1}$.

(c) The first inclusion follows at once from $d(\mathcal{Z}_r^{p,\,q}) \subset \mathcal{F}^{p+r} C^\infty_{p+q+1}(X,\, \mathbb{C})$ and $d^2 = 0$.

(d) This is obvious (see last identity in (1.19)). □

Lemma 1.2.7 shows us how to define the *differentials* in the spectral sequence. We let them be induced by d in the following way:

$$d_r : E_r^{p,\,q}(X) \longrightarrow E_r^{p+r,\,q-r+1}(X), \qquad d_r(\{\alpha\}_{E_r}) := \{d\alpha\}_{E_r}. \tag{1.21}$$

Given (1.20), we see that this definition is well posed thanks to (c) of Lemma 1.2.7 and to the inclusion of the first space below in the last:

$$d(\mathcal{B}_{r-1}^{p,\,q} + \mathcal{Z}_{r-1}^{p+1,\,q-1}) = d(\mathcal{Z}_{r-1}^{p+1,\,q-1}) = \mathcal{B}_{r-1}^{p+r,\,q-r+1} \subset \mathcal{B}_{r-1}^{p+r,\,q-r+1} + \mathcal{Z}_{r-1}^{p+r+1,\,q-r}$$

where the first (resp. second) identity follows from (d) of Lemma 1.2.7 (resp. from (1.19)).

To finish the proof of Theorem 1.2.2, we will now check that the objects we have defined satisfy properties (i)–(iii) in the statement.

- *Checking (i).* From the definitions, we get:

$\cdot$ $\mathcal{Z}_0^{p,\,q} = \mathcal{F}^p C^\infty_{p+q}(X,\, \mathbb{C})$, because the condition $d\alpha \in \mathcal{F}^p C^\infty_{p+q+1}(X,\, \mathbb{C})$ is automatically satisfied by all $\alpha \in \mathcal{F}^p C^\infty_{p+q}(X,\, \mathbb{C})$;

$\cdot$ $\mathcal{B}_{-1}^{p,\,q} = \mathcal{F}^p C^\infty_{p+q}(X,\, \mathbb{C}) \cap d(\mathcal{F}^{p+1} C^\infty_{p+q-1}(X,\, \mathbb{C})) = d(\mathcal{F}^{p+1} C^\infty_{p+q-1}(X,\, \mathbb{C}))$;

$\cdot$ $\mathcal{Z}_{-1}^{p+1,\,q-1} = \mathcal{F}^{p+1} C^\infty_{p+q}(X,\, \mathbb{C})$, because the condition $d\alpha \in \mathcal{F}^p C^\infty_{p+q+1}(X,\, \mathbb{C})$ is automatically satisfied by all $\alpha \in \mathcal{F}^{p+1} C^\infty_{p+q}(X,\, \mathbb{C})$.

Consequently, $\mathcal{B}_{-1}^{p,\,q} \subset \mathcal{Z}_{-1}^{p+1,\,q-1}$, hence $\mathcal{B}_{-1}^{p,\,q} + \mathcal{Z}_{-1}^{p+1,\,q-1} = \mathcal{Z}_{-1}^{p+1,\,q-1} = \mathcal{F}^{p+1} C^\infty_{p+q}(X,\, \mathbb{C})$. We get:

$$E_0^{p,\,q}(X) = \frac{\mathcal{F}^p C^\infty_{p+q}(X,\, \mathbb{C})}{\mathcal{F}^{p+1} C^\infty_{p+q}(X,\, \mathbb{C})} = C^\infty_{p,\,q}(X,\, \mathbb{C}).$$

Moreover, the map $d_0 : E_0^{p,\,q}(X) \to E_0^{p,\,q+1}(X)$ acts as follows. For every $\alpha \in C^\infty_{p,\,q}(X,\,\mathbb{C})$,

$$\alpha \mapsto \left(d\alpha = \partial\alpha + \bar\partial\alpha \quad mod\ \mathcal{F}^{p+1}C^\infty_{p+q+1}(X,\,\mathbb{C})\right) = \bar\partial\alpha,$$

because $\partial\alpha \in \mathcal{F}^{p+1}C^\infty_{p+q+1}(X,\,\mathbb{C})$. Therefore, $d_0 = \bar\partial$.

• *Checking (ii).* We will prove that the map $\mathcal{Z}_{r+1}^{p,\,q} \ni \alpha \mapsto \{\alpha\}_{E_r}$ induces a *well-defined* linear *bijection*:

$$\frac{\mathcal{Z}_{r+1}^{p,\,q}}{\mathcal{B}_r^{p,\,q} + \mathcal{Z}_r^{p+1,\,q-1}} \longrightarrow \frac{\ker(d_r : E_r^{p,\,q}(X) \to E_r^{p+r,\,q-r+1}(X))}{\mathrm{Im}\,(d_r : E_r^{p-r,\,q+r-1}(X) \to E_r^{p,\,q}(X))}. \tag{1.22}$$

· The first thing we need to prove in order to show *well-definedness* is the following implication:

$$\alpha \in \mathcal{Z}_{r+1}^{p,\,q} \implies \{\alpha\}_{E_r} \in \ker(d_r : E_r^{p,\,q}(X) \to E_r^{p+r,\,q-r+1}(X)). \tag{1.23}$$

Let $\alpha \in \mathcal{Z}_{r+1}^{p,\,q}$. Then, on the one hand, $d\alpha \in \mathcal{F}^{p+r+1}C^\infty_{p+q+1}(X,\,\mathbb{C})$, so $d\alpha \in \mathcal{Z}_{r-1}^{p+r+1,\,q-r} \subset \mathcal{Z}_{r-1}^{p+r+1,\,q-r} + \mathcal{B}_{r-1}^{p+r,\,q-r+1}$. Hence, $\{d\alpha\}_{E_r} = 0 \in E_r^{p+r,\,q-r+1}$.

On the other hand, since $\mathcal{Z}_{r+1}^{p,\,q} \subset \mathcal{Z}_r^{p,\,q}$ (see (b) of Lemma 1.2.7), α represents a class $\{\alpha\}_{E_r} \in E_r^{p,\,q}(X)$. Then, by (1.21), $d_r(\{\alpha\}_{E_r}) = \{d\alpha\}_{E_r} = 0$, the last identity having been proved just above.

Thus, implication (1.23) is proved.

· The second thing we need to prove in order to show *well-definedness* is the following implication:

$$\alpha \in \mathcal{B}_r^{p,\,q} + \mathcal{Z}_r^{p+1,\,q-1} \implies \{\alpha\}_{E_r} \in \mathrm{Im}\,(d_r : E_r^{p-r,\,q+r-1}(X) \to E_r^{p,\,q}(X)). \tag{1.24}$$

Let $\alpha = \beta + d\gamma$, with $\beta \in \mathcal{Z}_r^{p+1,\,q-1}$ and $d\gamma \in \mathcal{B}_r^{p,\,q}$. The condition on $d\gamma$ means that we can choose $\gamma \in \mathcal{Z}_r^{p-r,\,q+r-1}$ (see (1.19)). Meanwhile, $\mathcal{Z}_r^{p+1,\,q-1} \subset \mathcal{Z}_{r-1}^{p+1,\,q-1} \subset \mathcal{Z}_{r-1}^{p+1,\,q-1} + \mathcal{B}_{r-1}^{p,\,q}$, so the condition on β implies that $\{\beta\}_{E_r} = 0$. Therefore, we get

$$\{\alpha\}_{E_r} = \{\beta\}_{E_r} + \{d\gamma\}_{E_r} = d_r(\{\gamma\}_{E_r}) \in \mathrm{Im}\,(d_r : E_r^{p-r,\,q+r-1}(X) \to E_r^{p,\,q}(X)),$$

where the class $\{\gamma\}_{E_r}$ is a meaningful object because $\gamma \in \mathcal{Z}_r^{p-r,\,q+r-1}$.

This proves implication (1.24).

· We will now prove that the map (1.22) is *injective*.

Let $\alpha \in \mathcal{Z}_{r+1}^{p,\,q}$ such that $\{\alpha\}_{E_r} \in \mathrm{Im}\,(d_r : E_r^{p-r,\,q+r-1}(X) \to E_r^{p,\,q}(X))$. We have to prove that $\alpha \in \mathcal{B}_r^{p,\,q} + \mathcal{Z}_r^{p+1,\,q-1}$.

The hypothesis on $\{\alpha\}_{E_r}$ means that $\{\alpha\}_{E_r} = \{d\beta\}_{E_r}$ for some $\beta \in \mathcal{Z}_r^{p-r,\,q+r-1}$. This is further equivalent to

$$\alpha = d\beta + u + v, \quad \text{with } u \in \mathcal{Z}_{r-1}^{p+1,\,q-1} \quad \text{and} \quad v \in \mathcal{B}_{r-1}^{p,\,q}.$$

Hence, we get: $d\beta \in d(\mathcal{Z}_r^{p-r,\,q+r-1}) = \mathcal{B}_r^{p,\,q}$ (see (1.19) for the last identity). Meanwhile, we have: $v \in \mathcal{B}_{r-1}^{p,\,q} \subset \mathcal{B}_r^{p,\,q}$ (see (a) of Lemma 1.2.7 for the inclusion), hence $d\beta + v \in \mathcal{B}_r^{p,\,q}$. We will now show that $u \in \mathcal{Z}_r^{p+1,\,q-1}$. This will imply that $\alpha = u + (d\beta + v) \in \mathcal{Z}_r^{p+1,\,q-1} + \mathcal{B}_r^{p,\,q}$ and we will be done.

To show that $u \in \mathcal{Z}_r^{p+1,\,q-1}$, we need to show two things:

$$u \in \mathcal{F}^{p+1}C^\infty_{p+q}(X,\,\mathbb{C}) \quad \text{and} \quad du \in \mathcal{F}^{p+r+1}C^\infty_{p+q+1}(X,\,\mathbb{C}).$$

However, we know that $u \in \mathcal{Z}_{r-1}^{p+1,\,q-1}$, hence $u \in \mathcal{F}^{p+1}C^\infty_{p+q}(X,\,\mathbb{C})$. On the other hand, $du = d\alpha$ (because $d(d\beta+v) = 0$, since $d\beta+v \in \mathcal{B}_r^{p,\,q} \subset \mathrm{Im}\,d$) and $d\alpha \in \mathcal{F}^{p+r+1}C^\infty_{p+q+1}(X,\,\mathbb{C})$ because $\alpha \in \mathcal{Z}_{r+1}^{p,\,q}$ (by hypothesis).

· We will now prove that the map (1.22) is *surjective*.

Let $\alpha \in \mathcal{Z}_r^{p,\,q}$ such that $\{\alpha\}_{E_r} \in \ker d_r$. The latter property means that $\{d\alpha\}_{E_r} = 0 \in E_r^{p+r,\,q-r+1}(X)$, which is equivalent to $d\alpha \in \mathcal{B}_{r-1}^{p+r,\,q-r+1} + \mathcal{Z}_{r-1}^{p+r+1,\,q-r}$, and further equivalent to

$$d\alpha = d\beta + \gamma, \quad \text{with} \quad \beta \in \mathcal{Z}_{r-1}^{p+1,\,q-1} \text{ and } \gamma \in \mathcal{Z}_{r-1}^{p+r+1,\,q-r}.$$

On the one hand, this implies that $d(\alpha - \beta) = \gamma \in \mathcal{Z}_{r-1}^{p+r+1,\,q-r} \subset \mathcal{F}^{p+r+1}C^\infty_{p+q+1}(X,\,\mathbb{C})$, which further implies that

$$\alpha - \beta \in \mathcal{Z}_{r+1}^{p,\,q} \tag{1.25}$$

since $\alpha - \beta \in \mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C})$. This last fact follows from $\alpha \in \mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C})$ (because $\alpha \in \mathcal{Z}_r^{p,\,q}$, by assumption) and from $\beta \in \mathcal{F}^{p+1}C^\infty_{p+q}(X,\,\mathbb{C}) \subset \mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C})$.

On the other hand, $\beta \in \mathcal{Z}_{r-1}^{p+1,\,q-1} \subset \mathcal{B}_{r-1}^{p,\,q} + \mathcal{Z}_{r-1}^{p+1,\,q-1}$, hence $\{\beta\}_{E_r} = 0 \in E_r^{p,\,q}(X)$, so

$$\{\alpha - \beta\}_{E_r} = \{\alpha\}_{E_r} \in E_r^{p,\,q}(X). \tag{1.26}$$

We see that (1.25) and (1.25) prove, between them, the surjectivity of the map (1.22).

• *Checking (iii)*. Fix $p, q \in \{0, \dots, n\}$. Let $m := p + q$. For all $s \gg 1$ (at least for $s \geq m$), we have: $\mathcal{F}^sC^\infty_m(X,\,\mathbb{C}) = \{0\}$, $\mathcal{F}^sC^\infty_{m-1}(X,\,\mathbb{C}) = \{0\}$ and $\mathcal{F}^sC^\infty_{m+1}(X,\,\mathbb{C}) = \{0\}$. Therefore,

$$\mathcal{Z}_{s+1}^{p,\,q} = \left\{\alpha \in \mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C}) \;\mid\; d\alpha \in \mathcal{F}^{p+s+1}C^\infty_{m+1}(X,\,\mathbb{C}) = \{0\}\right\} = \mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C}) \cap \ker d.$$

Similarly, we get

$$\mathcal{Z}_s^{p+1,\,q-1} = \mathcal{F}^{p+1}C^\infty_{p+q}(X,\,\mathbb{C}) \cap \ker d.$$

On the other hand, $\mathcal{B}_s^{p,\,q} = d(\mathcal{Z}_s^{p-s,\,q+s-1})$. Now, since $s \geq m = p + q \geq p$, we have $p - s \leq 0$, hence $\mathcal{F}^{p-s}C^\infty_{p+q-1}(X,\,\mathbb{C}) = C^\infty_{p+q-1}(X,\,\mathbb{C})$. Therefore, we get

$$\mathcal{Z}_s^{p-s,\,q+s-1} = \left\{\alpha \in C^\infty_{p+q-1}(X,\,\mathbb{C}) \;\mid\; d\alpha \in \mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C})\right\},$$

hence

$$\mathcal{B}_s^{p,\,q} = d(\mathcal{Z}_s^{p-s,\,q+s-1}) = \mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C}) \cap \operatorname{Im} d, \qquad s \geq m = p + q.$$

Putting together the results of these computations, we infer that, for all $s \geq p + q$, we have:

$$\begin{aligned}
E_{s+1}^{p,\,q}(X) &= \frac{\mathcal{Z}_{s+1}^{p,\,q}}{\mathcal{Z}_s^{p+1,\,q-1} + \mathcal{B}_s^{p,\,q}} = \frac{\mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C}) \cap \ker d}{[\mathcal{F}^pC^\infty_{p+q}(X,\,\mathbb{C}) \cap \operatorname{im} d] + [\mathcal{F}^{p+1}C^\infty_{p+q}(X,\,\mathbb{C}) \cap \ker d]} \\
&\overset{(a)}{\simeq} \frac{\frac{\mathcal{F}^pC^\infty_{p+q}(X,\mathbb{C})\cap\ker d}{\mathcal{F}^pC^\infty_{p+q}(X,\mathbb{C})\cap\operatorname{im} d}}{\frac{[\mathcal{F}^pC^\infty_{p+q}(X,\mathbb{C})\cap\operatorname{im} d]+[\mathcal{F}^{p+1}C^\infty_{p+q}(X,\mathbb{C})\cap\ker d]}{\mathcal{F}^pC^\infty_{p+q}(X,\mathbb{C})\cap\operatorname{im} d}} \overset{(b)}{\simeq} \frac{F^pH^{p+q}_{DR}(X,\,\mathbb{C})}{\frac{\mathcal{F}^{p+1}C^\infty_{p+q}(X,\mathbb{C})\cap\ker d}{\mathcal{F}^{p+1}C^\infty_{p+q}(X,\mathbb{C})\cap\operatorname{im} d}} \\
&= \frac{F^pH^{p+q}_{DR}(X,\,\mathbb{C})}{F^{p+1}H^{p+q}_{DR}(X,\,\mathbb{C})} = G_pH^{p+q}_{DR}(X,\,\mathbb{C}),
\end{aligned}$$

where the canonical isomorphisms (a) and (b) followed respectively from the general canonical isomorphisms:

$$\frac{G}{H} \simeq \frac{G/K}{H/K} \quad \text{and} \quad \frac{A+B}{A} \simeq \frac{B}{A\cap B},$$

that are elementarily known to hold for any modules $K \subset H \subset G$ and A, B.

The proof of Theorem 1.2.2 is complete. $\square$

We end this subsection with the following useful result that gives a necessary and sufficient condition for the spaces $F^pH^k_{DR}(X,\,C)$ in the filtration (1.15) of $H^k_{DR}(X,\,C)$ to coincide with the more refined spaces on the left-hand side of (1.14). We say that d *strictly preserves the filtration* in this case. This result is taken from [Del71, Proposition 1.3.2], but our presentation will mainly follow [SB18, Lemma 1.3].

Proposition 1.2.8. *The Frölicher spectral sequence of X degenerates at E_1 if and only if*

$$F^pH^k_{DR}(X,\,C) = \frac{\ker\,(d:\mathcal{F}^pC^\infty_k(X,\,\mathbb{C})\to\mathcal{F}^pC^\infty_{k+1}(X,\,\mathbb{C}))}{Im\,(d:\mathcal{F}^pC^\infty_{k-1}(X,\,\mathbb{C})\to\mathcal{F}^pC^\infty_k(X,\,\mathbb{C}))}. \tag{1.27}$$

for all $k\in\{0,\dots,2n\}$ and all $p\in\{0,\dots,\min(k,\,n)\}$.

Proof. The latter property in the equivalence we have to prove amounts to having the identity

$$(\star_p) \qquad d(\mathcal{F}^pC^\infty_{k-1}) = \mathcal{F}^pC^\infty_k\cap\operatorname{Im} d$$

for all $k\in\{0,\dots,2n\}$ and all $p\in\mathbb{Z}$. Note that the inclusion "$\subset$" always holds trivially.

"$\implies$" Suppose that $E_1(X)=E_\infty(X)$. We fix an arbitrary $k\in\{0,\dots,2n\}$ and will prove $(\star_p)$ by downward induction on p. It is clear that $(\star_p)$ holds trivially for all $p\geq k+1$ because $\mathcal{F}^pC^\infty_k=\mathcal{F}^pC^\infty_{k-1}=\{0\}$ in that case.

Now, suppose that $(\star_{p+1})$ holds for some p. We will prove that $(\star_p)$ also holds by proving, by upward induction on l, that the following identity holds:

$$(\star_{p,\,l}) \qquad d(\mathcal{F}^pC^\infty_{k-1}) = \mathcal{F}^pC^\infty_k\cap d(\mathcal{F}^{p-l}C^\infty_{k-1})$$

for every l. Again, the inclusion "$\subset$" always holds trivially, while identity $(\star_{p,\,l})$ holds trivially when $l=0$ because $d(\mathcal{F}^pC^\infty_{k-1})\subset\mathcal{F}^pC^\infty_k$.

Having fixed an arbitrary $l\geq 1$, suppose that $(\star_{p,\,l-1})$ holds. We will prove $(\star_{p,\,l})$.

The hypothesis $E_1(X)=E_\infty(X)$ amounts to all the maps d_r vanishing identically for all $r\geq 1$. Let q be such that $k=p-l+q+1$. The vanishing of the map $d_l:E_l^{p-l,\,q}(X)\to E_l^{p,\,q-l+1}(X)$ means that, for every $\alpha\in\mathcal{Z}_l^{p-l,\,q}$ (i.e. for every $\alpha\in\mathcal{F}^{p-l}C^\infty_{k-1}$ such that $d\alpha\in\mathcal{F}^pC^\infty_k$), we can write

$$d\alpha = dw + v$$

for forms $w\in\mathcal{Z}_{l-1}^{p-l+1,\,q-1}$ (i.e. $w\in\mathcal{F}^{p-l+1}C^\infty_{k-1}$ and $dw\in\mathcal{F}^pC^\infty_k$) and $v\in\mathcal{Z}_{l-1}^{p+1,\,q-l}$ (i.e. $v\in\mathcal{F}^{p+1}C^\infty_k$ and $dv\in\mathcal{F}^{p+l}C^\infty_{k+1}$). We get $v=d(\alpha-w)$, hence

$$v\in\mathcal{F}^{p+1}C^\infty_k\cap\operatorname{Im} d = d(\mathcal{F}^{p+1}C^\infty_{k-1}),$$

where the last identity is the inductive hypothesis $(\star_{p+1})$.

Since $d(\mathcal{F}^{p+1}C^\infty_{k-1})\subset d(\mathcal{F}^{p-l+1}C^\infty_{k-1})$ and $w\in\mathcal{F}^{p-l+1}C^\infty_{k-1}$, this implies that

$$d\alpha = dw + v\in\mathcal{F}^pC^\infty_k\cap d(\mathcal{F}^{p-l+1}C^\infty_{k-1}) = d(\mathcal{F}^pC^\infty_{k-1}),$$

where the last identity is the inductive hypothesis $(\star_{p,\,l-1})$.

Since $d\alpha$ was chosen arbitrarily in $\mathcal{F}^pC_k^\infty \cap d(\mathcal{F}^{p-l}C_{k-1}^\infty)$, we have proved the inclusion $\mathcal{F}^pC_k^\infty \cap d(\mathcal{F}^{p-l}C_{k-1}^\infty) \subset d(\mathcal{F}^pC_{k-1}^\infty)$, which is nothing but the non-trivial inclusion in $(\star_{p,\,l})$.

"$\Longleftarrow$" Suppose that $(\star_p)$ holds for all k and p. Then $(\star_{p,\,l})$ holds for all k, p and all $l \in \{0, \dots, p\}$. We need to prove that $E_1(X) = E_\infty(X)$, which is equivalent to proving that all the maps $d_l : E_l^{p-l,\,q}(X) \to E_l^{p,\,q-l+1}(X)$ vanish identically for all $l \geq 1$ and all p, q. This is further equivalent to proving the inclusion:

$$d(\mathcal{Z}_l^{p-l,\,q}) \subset d(\mathcal{Z}_{l-1}^{p-l+1,\,q-1}) + \mathcal{Z}_{l-1}^{p+1,\,q-l} \tag{1.28}$$

for all $l \geq 1$ and all p, q.

To prove this inclusion, let $\alpha \in \mathcal{Z}_l^{p-l,\,q}$. Then, $d\alpha \in \mathcal{F}^pC_k^\infty \cap d(\mathcal{F}^{p-l}C_{k-1}^\infty)$. Hence, by (the non-trivial inclusion of) $(\star_{p,\,l})$, $d\alpha \in d(\mathcal{F}^pC_{k-1}^\infty)$. Thus, there exists $w \in \mathcal{F}^pC_{k-1}^\infty$ such that $d\alpha = dw$.

Now, $w \in \mathcal{Z}_{l-1}^{p-l+1,\,q-1}$ (or, equivalently, $w \in \mathcal{F}^{p-l+1}C_{k-1}^\infty$ and $dw \in \mathcal{F}^pC_k^\infty$). Indeed, we even have: $w \in \mathcal{F}^pC_{k-1}^\infty \subset \mathcal{F}^{p-l+1}C_{k-1}^\infty$ (the inclusion being a consequence of $p \geq p-l+1$, since $l \geq 1$) and $dw = d\alpha \in \mathcal{F}^pC_k^\infty$.

We conclude that

$$d\alpha = dw \in d(\mathcal{Z}_{l-1}^{p-l+1,\,q-1}) \subset d(\mathcal{Z}_{l-1}^{p-l+1,\,q-1}) + \mathcal{Z}_{l-1}^{p+1,\,q-l}.$$

This proves inclusion (1.28) and we are done. □

1.2.2 The Cordero-Fernández-Gray-Ugarte point of view

In this subsection, we will follow the presentation in [CFGU97], but with significant additions and modifications. Although the two points of view on the Frölicher spectral sequence (FSS) discussed in this book are equivalent, the main difference is that the cohomology classes that constitute the spaces $E_r^{p,\,q}(X)$ are represented by *pure-type* forms in the Cordero-Fernández-Gray-Ugarte approach. This is certainly not the case in the classical approach, where these representatives α are only of *filtration type* $\mathcal{F}^p$ such that $d\alpha$ is of *filtration type* $\mathcal{F}^{p+r}$ (see (1.20) and the definition of $\mathcal{Z}_r^{p,\,q}$ in 1.2.1).

Terminology and the main result

The following terminology is taken from [Pop19] and was also used in [PSU20].

Definition 1.2.9. *Fix $r \geq 2$ and $p, q \in \{0, \dots, n\}$.*

*(i) A form $\alpha \in C_{p,\,q}^\infty(X, \mathbb{C})$ is said to be E_r-**closed** if and only if there exist forms $u_l \in C_{p+l,\,q-l}^\infty(X)$ with $l \in \{1, \dots, r-1\}$ satisfying the following tower of r equations:*

$$\begin{aligned}
\bar\partial\alpha &= 0\\
\partial\alpha &= \bar\partial u_1\\
\partial u_1 &= \bar\partial u_2\\
&\vdots\\
\partial u_{r-2} &= \bar\partial u_{r-1}.
\end{aligned}$$

*We say in this case that $\bar\partial\alpha = 0$ and $\partial\alpha$ **runs at least** $(r-1)$ **times**. An $(r-1)$-tuple $(u_1, \dots, u_{r-1})$ of forms with the above property is called a **system of $\bar\partial$-potentials** for $\partial\alpha$.*

(i') If we only have $\bar\partial\alpha = 0$, we say that α is E_1**-closed** *or* $\bar\partial$**-closed**.

(i") We set $\mathcal{X}_r^{p,\,q} := \{\alpha \in C^\infty_{p,\,q}(X,\,\mathbb{C}) \;\mid\; \alpha$ is E_r-closed$\}$.

(ii) Fix $r \geq 2$. A form $\alpha \in C^\infty_{p,\,q}(X,\,\mathbb{C})$ is E_r**-exact** *if and only if there exist forms $\zeta \in C^\infty_{p-1,\,q}(X)$ and $\xi \in C^\infty_{p,\,q-1}(X)$ such that*

$$\alpha = \partial\zeta + \bar\partial\xi,$$

with ξ arbitrary and ζ satisfying the following tower of $(r-1)$ equations:

$$\begin{aligned}\bar\partial\zeta &= \partial v_{r-3}\\ \bar\partial v_{r-3} &= \partial v_{r-4}\\ &\vdots\\ \bar\partial v_1 &= \partial v_0\\ \bar\partial v_0 &= 0,\end{aligned}$$

for some forms $v_0, \dots, v_{r-3}$. (When $r = 2$, $\zeta_{r-2} = \zeta_0$ must be $\bar\partial$-closed.)

We say in this case that $\bar\partial\zeta$ **reaches** 0 **in at most** $(r-1)$ **steps**. *An $(r-2)$-tuple $(v_0, \dots, v_{r-3})$ of forms with the above property is called a* **system of** ∂**-potentials** *for $\bar\partial\zeta$.*

(ii') If $\alpha \in Im\,\bar\partial$, we say that α is E_1**-exact** *or* $\bar\partial$**-exact**.

(ii") We set $\mathcal{Y}_r^{p,\,q} := \{\alpha \in C^\infty_{p,\,q}(X,\,\mathbb{C}) \;\mid\; \alpha$ is E_r-exact$\}$.

Note the obvious inclusions:

$$\dots \subset \mathcal{Y}_r^{p,\,q} \subset \mathcal{Y}_{r+1}^{p,\,q} \subset \dots \subset \mathcal{X}_{r+1}^{p,\,q} \subset \mathcal{X}_r^{p,\,q} \subset \dots,$$

a twofold reason for the dimension of the quotient $\mathcal{X}_r^{p,\,q}/\mathcal{Y}_r^{p,\,q}$ to be non-increasing with r.

The main result of this subsection is the following explicit description of the Frölicher spectral sequence by means of pure-type forms. The reader uninterested in the formal description of the FSS spelt out in §.1.2.2 may wish to adopt the following result as the definition of the FSS.

Theorem 1.2.10. *(Cordero-Fernández-Gray-Ugarte [CFGU97]) Let X be a compact complex manifold with $dim_{\mathbb{C}} X = n$. Fix $r \in \mathbb{N}^\star$ and $p, q \in \{0, \dots, n\}$. Then:*

(1) *the spaces in the Frölicher spectral sequence of X have* **canonical isomorphisms***:*

$$E_r^{p,\,q}(X) \simeq \frac{\mathcal{X}_r^{p,\,q}}{\mathcal{Y}_r^{p,\,q}};$$

(2) *the differentials in the Frölicher spectral sequence of X are given by*

$$d_r : E_r^{p,\,q}(X) \to E_r^{p+r,\,q-r+1}(X), \qquad d_r\bigg(\{\alpha\}_{E_r}\bigg) = (-1)^{r-1}\{\partial u_{r-1}\}_{E_r},$$

for any choice of E_r-closed representatives $\alpha \in C^\infty_{p,\,q}(X,\,\mathbb{C})$ and any choice of $\bar\partial$-potentials $u_1, \dots, u_{r-1}$ of $\partial\alpha$ as in (i) of Definition 1.2.9.

Proof of (1) **of Theorem 1.2.10**

We will keep the notation of §.1.2.2. Starting from definition (1.20) of $E_r^{p,\,q}(X)$, we get canonical isomorphisms:

$$E_r^{p,\,q}(X) \simeq \frac{\mathcal{Z}_r^{p,\,q}/\mathcal{Z}_{r-1}^{p+1,\,q-1}}{(\mathcal{B}_{r-1}^{p,\,q} + \mathcal{Z}_{r-1}^{p+1,\,q-1})/\mathcal{Z}_{r-1}^{p+1,\,q-1}} \simeq \frac{\mathcal{Z}_r^{p,\,q}/\mathcal{Z}_{r-1}^{p+1,\,q-1}}{\mathcal{B}_{r-1}^{p,\,q}/(\mathcal{B}_{r-1}^{p,\,q} \cap \mathcal{Z}_{r-1}^{p+1,\,q-1})}$$

from the general canonical isomorphisms $G/H \simeq (G/K)/(H/K)$ and $(A+B)/A \simeq B/(A \cap B)$ that are elementarily known to hold for any modules $K \subset H \subset G$ and A, B (and were already used in §.1.2.2). So, it suffices to prove the following canonical isomorphisms:

$$(a) \quad \frac{\mathcal{Z}_r^{p,\,q}}{\mathcal{Z}_{r-1}^{p+1,\,q-1}} \simeq \mathcal{X}_r^{p,\,q} \quad \text{and} \quad (b) \quad \frac{\mathcal{B}_{r-1}^{p,\,q}}{\mathcal{B}_{r-1}^{p,\,q} \cap \mathcal{Z}_{r-1}^{p+1,\,q-1}} \simeq \mathcal{Y}_r^{p,\,q}.$$

• *Proof of (a).* Starting from the definition of the spaces $\mathcal{Z}_r^{p,\,q}$, we compute:

$$\begin{aligned}
\frac{\mathcal{Z}_r^{p,\,q}}{\mathcal{Z}_{r-1}^{p+1,\,q-1}} &= \frac{\mathcal{F}^p C^\infty_{p+q} \cap d^{-1}(\mathcal{F}^{p+r} C^\infty_{p+q+1})}{\mathcal{F}^{p+1} C^\infty_{p+q} \cap d^{-1}(\mathcal{F}^{p+r} C^\infty_{p+q+1})} \\
&= \frac{\{\alpha^{p,\,q} + \alpha^{p+1,\,q-1} + \cdots + \alpha^{p+q,\,0} \;\mid\; d(\alpha^{p,\,q} + \alpha^{p+1,\,q-1} + \cdots + \alpha^{p+q,\,0}) \in \mathcal{F}^{p+r} C^\infty_{p+q+1}\}}{\{\alpha^{p+1,\,q-1} + \cdots + \alpha^{p+q,\,0} \;\mid\; d(\alpha^{p+1,\,q-1} + \cdots + \alpha^{p+q,\,0}) \in \mathcal{F}^{p+r} C^\infty_{p+q+1}\}} \\
&\overset{(i)}{\simeq} \{\alpha^{p,\,q} \;\mid\; \exists \alpha^{p+1,\,q-1}, \dots, \alpha^{p+q,\,0} \text{ s. t. } d(\alpha^{p,\,q} + \alpha^{p+1,\,q-1} + \cdots + \alpha^{p+q,\,0}) \in \mathcal{F}^{p+r} C^\infty_{p+q+1}\} \\
&= \mathcal{X}_r^{p,\,q},
\end{aligned}$$

where the isomorphism (i) follows from the fact that the kernel of the following linear surjection induced by the projection onto the $(p,\,q)$-type component:

$$\begin{aligned}
&\{\alpha^{p,\,q} + \alpha^{p+1,\,q-1} + \cdots + \alpha^{p+q,\,0} \;\mid\; d(\alpha^{p,\,q} + \alpha^{p+1,\,q-1} + \cdots + \alpha^{p+q,\,0}) \in \mathcal{F}^{p+r} C^\infty_{p+q+1}\} \\
&\quad\downarrow \\
&\{\alpha^{p,\,q} \;\mid\; \exists \alpha^{p+1,\,q-1}, \dots, \alpha^{p+q,\,0} \text{ s. t. } d(\alpha^{p,\,q} + \alpha^{p+1,\,q-1} + \cdots + \alpha^{p+q,\,0}) \in \mathcal{F}^{p+r} C^\infty_{p+q+1}\}
\end{aligned}$$

is $\{\alpha^{p+1,\,q-1} + \cdots + \alpha^{p+q,\,0} \;\mid\; d(\alpha^{p+1,\,q-1} + \cdots + \alpha^{p+q,\,0}) \in \mathcal{F}^{p+r} C^\infty_{p+q+1}\}$.

• *Proof of (b).* We start by noticing that

$$\begin{aligned}
\mathcal{B}_{r-1}^{p,\,q} \cap \mathcal{Z}_{r-1}^{p+1,\,q-1} &= d(\mathcal{Z}_{r-1}^{p-r+1,\,q+r-2}) \cap \mathcal{Z}_{r-1}^{p+1,\,q-1} \overset{(i)}{=} d(\mathcal{Z}_{r-1}^{p-r+1,\,q+r-2}) \cap \mathcal{F}^{p+1} C^\infty_{p+q}(X,\,\mathbb{C}) \\
&\overset{(ii)}{=} \mathcal{F}^{p+1} C^\infty_{p+q}(X,\,\mathbb{C}) \cap d(\mathcal{F}^{p-r+1} C^\infty_{p+q-1}(X,\,\mathbb{C})),
\end{aligned}$$

where (i) follows from $\mathcal{Z}_{r-1}^{p+1,\,q-1} \subset \mathcal{F}^{p+1} C^\infty_{p+q}(X,\,\mathbb{C})$ and from $d^2 = 0$, while (ii) follows from the inclusions $\mathcal{Z}_{r-1}^{p-r+1,\,q+r-2} \subset \mathcal{F}^{p-r+1} C^\infty_{p+q-1}(X,\,\mathbb{C})$ and $\mathcal{F}^{p+1} C^\infty_{p+q}(X,\,\mathbb{C}) \subset \mathcal{F}^p C^\infty_{p+q}(X,\,\mathbb{C})$.

Therefore, we get:

$$\begin{aligned}\frac{\mathcal{B}_{r-1}^{p,\,q}}{\mathcal{B}_{r-1}^{p,\,q}\cap\mathcal{Z}_{r-1}^{p+1,\,q-1}} &= \frac{\mathcal{F}^pC^\infty_{p+q}\cap d(\mathcal{F}^{p-r+1}C^\infty_{p+q-1})}{\mathcal{F}^{p+1}C^\infty_{p+q}\cap d(\mathcal{F}^{p-r+1}C^\infty_{p+q-1})}\\ &= \frac{\{d\beta\in d(\mathcal{F}^{p-r+1}C^\infty_{p+q-1})\ \mid\ 0=(d\beta)_{p-r+1,\,q+r-1}=\cdots=(d\beta)_{p-1,\,q+1}\}}{\{d\beta\in d(\mathcal{F}^{p-r+1}C^\infty_{p+q-1})\ \mid\ 0=(d\beta)_{p-r+1,\,q+r-1}=\cdots=(d\beta)_{p-1,\,q+1}=(d\beta)_{p,\,q}\}}\\ &\overset{(i)}{\simeq} \{(d\beta)_{p,\,q}\ \mid\ d\beta\in d(\mathcal{F}^{p-r+1}C^\infty_{p+q-1}),0=(d\beta)_{p-r+1,\,q+r-1}=\cdots=(d\beta)_{p-1,\,q+1}\}\\ &= \mathcal{Y}_r^{p,\,q},\end{aligned}$$

where the isomorphism (i) follows from the fact that the kernel of the following linear surjection induced by the projection onto the $(p,\,q)$-type component:

$$\begin{aligned}&\{d\beta\in d(\mathcal{F}^{p-r+1}C^\infty_{p+q-1})\ \mid\ 0=(d\beta)_{p-r+1,\,q+r-1}=\cdots=(d\beta)_{p-1,\,q+1}\}\\ &\quad\downarrow\\ &\{(d\beta)_{p,\,q}\ \mid\ d\beta\in d(\mathcal{F}^{p-r+1}C^\infty_{p+q-1}),0=(d\beta)_{p-r+1,\,q+r-1}=\cdots=(d\beta)_{p-1,\,q+1}\}\end{aligned}$$

is $\{d\beta\in d(\mathcal{F}^{p-r+1}C^\infty_{p+q-1})\ \mid\ 0=(d\beta)_{p-r+1,\,q+r-1}=\cdots=(d\beta)_{p-1,\,q+1}=(d\beta)_{p,\,q}\}$.

Proof of (2) **of Theorem 1.2.10**

The main point is the following

Lemma 1.2.11. *(a) The following linear maps are* **well defined***:*

$$\begin{aligned}\mathcal{X}_r^{p,\,q} &\overset{T_r^{p,\,q}}{\longrightarrow} \frac{\mathcal{Z}_r^{p,\,q}}{\mathcal{Z}_{r-1}^{p+1,\,q-1}} \overset{P_r^{p,\,q}}{\longrightarrow} \frac{\mathcal{Z}_r^{p,\,q}}{\mathcal{Z}_{r-1}^{p+1,\,q-1}+\mathcal{B}_{r-1}^{p,\,q}} = E_r^{p,\,q}(X)\\ \mathcal{X}_r^{p,\,q}\ni\alpha &\mapsto (\alpha-u_1+u_2-\cdots+(-1)^{r-1}\,u_{r-1}\ mod\ \mathcal{Z}_{r-1}^{p+1,\,q-1})\in\mathcal{Z}_r^{p,\,q}/\mathcal{Z}_{r-1}^{p+1,\,q-1}\end{aligned}$$

$$\mathcal{Z}_r^{p,\,q}/\mathcal{Z}_{r-1}^{p+1,\,q-1}\ni(\rho\ mod\ \mathcal{Z}_{r-1}^{p+1,\,q-1})\mapsto(\rho\ mod\ (\mathcal{Z}_{r-1}^{p+1,\,q-1}+\mathcal{B}_{r-1}^{p,\,q})\in\mathcal{Z}_r^{p,\,q}/(\mathcal{Z}_{r-1}^{p+1,\,q-1}+\mathcal{B}_{r-1}^{p,\,q}),$$

where $(u_1,\ldots,u_{r-1})$ is an arbitrary system of $\bar\partial$-potentials for $\partial\alpha$.

(b) The map $T_r^{p,\,q}$ is an **isomorphism** *(and $P_r^{p,\,q}$ is surjective).*

(c) The restriction to $\mathcal{Z}_r^{p,\,q}$ of the projection $C^\infty_{p+q}(X,\,\mathbb{C})\to C^\infty_{p,\,q}(X,\,\mathbb{C})$ onto the $(p,\,q)$-type component is a **surjection** *onto $\mathcal{X}_r^{p,\,q}$:*

$$\pi_r^{p,\,q}:\mathcal{Z}_r^{p,\,q}\longrightarrow\mathcal{X}_r^{p,\,q},\qquad \beta\mapsto\beta^{p,\,q},$$

whose kernel is $\mathcal{Z}_{r-1}^{p+1,\,q-1}$. Moreover, the inverse of $T_r^{p,\,q}$ is induced by this same projection:

$$\mathcal{Z}_r^{p,\,q}/\mathcal{Z}_{r-1}^{p+1,\,q-1}\ni(\beta\ mod\ \mathcal{Z}_{r-1}^{p+1,\,q-1})\overset{(T_r^{p,\,q})^{-1}}{\mapsto}\beta^{p,\,q}\in\mathcal{X}_r^{p,\,q}.$$

(d) The kernel of the surjection $S_r^{p,\,q}:=P_r^{p,\,q}\circ T_r^{p,\,q}:\mathcal{X}_r^{p,\,q}\longrightarrow E_r^{p,\,q}(X)$ is $\mathcal{Y}_r^{p,\,q}$.

Proof. (a) It is obvious that $P_r^{p,\,q}$ is well defined and surjective. To prove that $T_r^{p,\,q}$ is well defined, we need to prove that, for every $\alpha\in\mathcal{X}_r^{p,\,q}$ and for any two systems $(u_1,\dots,u_{r-1})$ and $(v_1,\dots,v_{r-1})$ of $\bar\partial$-potentials for $\partial\alpha$, we have:

$$\rho_1:=\alpha-u_1+u_2-\cdots+(-1)^{r-1}\,u_{r-1},\, \rho_2:=\alpha-v_1+v_2-\cdots+(-1)^{r-1}\,v_{r-1}\in\mathcal{Z}_r^{p,\,q}\quad\text{and}\quad\rho_1-\rho_2\in\mathcal{Z}_{r-1}^{p+1,\,q-1}.$$

From (i) of Definition 1.2.9, we get:

$$\begin{aligned} d\alpha &= \partial\alpha=\bar\partial u_1=du_1-\partial u_1=du_1-\bar\partial u_2=d(u_1-u_2)+\partial u_2\\ &\ \vdots\\ &= d(u_1-u_2+\cdots+(-1)^r\,u_{r-1})+(-1)^{r-1}\,\partial u_{r-1}.\end{aligned}$$

The analogous identity holds for $(v_1,\dots,v_{r-1})$, so we get:

$$d\rho_1=(-1)^{r-1}\,\partial u_{r-1}\quad\text{and}\quad d\rho_2=(-1)^{r-1}\,\partial v_{r-1}.$$

Since $\alpha\in C^\infty_{p,\,q}$ and $u_j,v_j\in C^\infty_{p+j,\,q-j}$ for all $j\in\{1,\dots,r-1\}$, we see that $\rho_1,\rho_2\in\mathcal{F}^pC^\infty_{p+q}$ and $d\rho_1,d\rho_2\in\mathcal{F}^{p+r}C^\infty_{p+q+1}$, which translates to $\rho_1,\rho_2\in\mathcal{Z}_r^{p,\,q}$. This proves the first contention.

On the other hand, from the above identities we also get:

$$d(\rho_1-\rho_2)=(-1)^{r-1}\,\partial(u_{r-1}-v_{r-1})\in\mathcal{F}^{p+r}C^\infty_{p+q+1}.$$

Since $\rho_1-\rho_2\in\mathcal{F}^{p+1}C^\infty_{p+q}$ (because the two occurences of α cancel each other out in the difference), this translates to $\rho_1-\rho_2\in\mathcal{Z}_{r-1}^{p+1,\,q-1}\subset\mathcal{Z}_{r-1}^{p+1,\,q-1}+\mathcal{B}_{r-1}^{p,\,q}$, which proves the second contention.

(b) The map $T_r^{p,\,q}$ is obviously injective since, if $\alpha-u_1+u_2-\cdots+(-1)^{r-1}\,u_{r-1}\in\mathcal{Z}_{r-1}^{p+1,\,q-1}\subset\mathcal{F}^{p+1}C^\infty_{p+q}$, then $\alpha\in\mathcal{F}^{p+1}C^\infty_{p+q}$ (because $u_j\in\mathcal{F}^{p+1}C^\infty_{p+q}$ for all j). However, $\alpha\in C^\infty_{p,\,q}$, so $\alpha=0$.

Let us now prove that $T_r^{p,\,q}$ is surjective. Let $\beta=\beta^{p,\,q}+\beta^{p+1,\,q-1}+\cdots\in\mathcal{Z}_r^{p,\,q}$, where upper indices stand for bidegrees. Then,

$$\mathcal{F}^{p+r}C^\infty_{p+q}\ni d\beta=\bar\partial\beta^{p,\,q}+(\partial\beta^{p,\,q}+\bar\partial\beta^{p+1,\,q-1})+\cdots+(\partial\beta^{p+r-2,\,q-r+2}+\bar\partial\beta^{p+r-1,\,q-r+1})+\mathcal{F}^{p+r}C^\infty_{p+q},$$

where the last occurence of $\mathcal{F}^{p+r}C^\infty_{p+q}$ stands for terms belonging to this space. Hence, $\bar\partial\beta^{p,\,q}+(\partial\beta^{p,\,q}+\bar\partial\beta^{p+1,\,q-1})+\cdots+(\partial\beta^{p+r-2,\,q-r+2}+\bar\partial\beta^{p+r-1,\,q-r+1})=0$, which translates to the vanishing of all the pure-type components of this sum:

$$\begin{aligned}\bar\partial\beta^{p,\,q} &= 0\\ \partial\beta^{p,\,q} &= -\bar\partial\beta^{p+1,\,q-1}\\ \partial\beta^{p+1,\,q-1} &= -\bar\partial\beta^{p+2,\,q-2}\\ &\ \vdots\\ \partial\beta^{p+r-2,\,q-r+2} &= -\bar\partial\beta^{p+r-1,\,q-r+1}.\end{aligned}$$

However, this tower of r equations expresses precisely the fact that $\beta^{p,\,q}\in\mathcal{X}_r^{p,\,q}$ and that $(-\beta^{p+1,\,q-1},\,\beta^{p+2,\,q-2},\dots,(-1)^{r-1}\,\beta^{p+r-1,\,q-r+1})$ is a system of $\bar\partial$-potentials for $\partial\beta^{p,\,q}$. It follows that

$$T_r^{p,\,q}(\beta^{p,\,q})=\beta^{p,\,q}+\beta^{p+1,\,q-1}+\cdots+\beta^{p+r-1,\,q-r+1}\ mod\ \mathcal{Z}_{r-1}^{p+1,\,q-1}=\beta\ mod\ \mathcal{Z}_{r-1}^{p+1,\,q-1}.$$

This proves the surjectivity of $T_r^{p,\,q}$.

(c) To prove this part, it suffices to track the arguments in the above proof of the surjectivity of $T_r^{p,\,q}$ backwards.

Indeed, it was already proved under (b) that, for every $\beta = \beta^{p,\,q} + \beta^{p+1,\,q-1} + \cdots \in \mathcal{Z}_r^{p,\,q}$, we have $\beta^{p,\,q} \in \mathcal{X}_r^{p,\,q}$. To see that $\pi_r^{p,\,q}$ is surjective, let $\beta^{p,\,q} \in \mathcal{X}_r^{p,\,q}$. Then, it satisfies the above tower of r equations for some system $(-\beta^{p+1,\,q-1},\,\beta^{p+2,\,q-2},\dots,(-1)^{r-1}\,\beta^{p+r-1,\,q-r+1})$ of $\bar\partial$-potentials for $\partial\beta^{p,\,q}$. Then, $\beta := \beta^{p,\,q} + \beta^{p+1,\,q-1} + \cdots + \beta^{p+r-1,\,q-r+1} \in \mathcal{Z}_r^{p,\,q}$ because $\beta \in \mathcal{F}^pC^\infty_{p+q}$, and

$$\begin{aligned} d\beta &= \bar\partial\beta^{p,\,q} + (\partial\beta^{p,\,q} + \bar\partial\beta^{p+1,\,q-1}) + \cdots + (\partial\beta^{p+r-2,\,q-r+2} + \bar\partial\beta^{p+r-1,\,q-r+1}) + \partial\beta^{p+r-1,\,q-r+1} \\ &= \partial\beta^{p+r-1,\,q-r+1} \in \mathcal{F}^{p+r}C^\infty_{p+q+1}. \end{aligned}$$

Since $\pi_r^{p,\,q}(\beta) = \beta^{p,\,q}$, the surjectivity of $\pi_r^{p,\,q}$ is proved. The last statement is obvious.

(d) Let $\alpha \in \mathcal{X}_r^{p,\,q}$. We have the following equivalences:

$$\begin{aligned} \alpha \in \ker S_r^{p,\,q} \quad &\iff \quad \alpha - u_1 + u_2 - \cdots + (-1)^{r-1}\,u_{r-1} \in \mathcal{Z}_{r-1}^{p+1,\,q-1} + \mathcal{B}_{r-1}^{p,\,q} \\ &\iff \quad \exists\,\eta \in \mathcal{Z}_{r-1}^{p+1,\,q-1}, \exists\,\gamma \in \mathcal{Z}_{r-1}^{p-r+1,\,q+r-2} \quad \text{such that} \\ &\qquad\quad \alpha = u_1 - u_2 + \cdots + (-1)^r\,u_{r-1} + \eta + d\gamma. \end{aligned}$$

The last identity is further equivalent to:

$$\begin{aligned} \alpha = u_1 - u_2 + \cdots + (-1)^r\,u_{r-1} \;\; &+ \;\; \eta^{p+1,\,q-1} + \cdots + \eta^{p+r-1,\,q-r+1} + \mathcal{F}^{p+r} \\ &+ \;\; \partial\gamma^{p-r+1,\,q+r-2} + \cdots + \partial\gamma^{p+r-2,\,q-r+1} + \mathcal{F}^{p+r} \\ &+ \;\; \bar\partial\gamma^{p-r+1,\,q+r-2} + \cdots + \bar\partial\gamma^{p+r-1,\,q-r} + \mathcal{F}^{p+r}. \end{aligned}$$

We will now equate the terms of equal bidegrees on either side of the above identity. If $r = 1$, in bidegree $(p,\,q)$ we get $\alpha = \bar\partial\gamma^{p-r+1,\,q+r-2}$, so $\alpha \in \mathrm{Im}\,\bar\partial = \mathcal{Y}_1^{p,\,q}$. If $r \geq 2$, the above identity implies:

$$\begin{aligned} \alpha &= \partial\gamma^{p-1,\,q} + \bar\partial\gamma^{p,\,q-1} \\ \bar\partial\gamma^{p-1,\,q} &= -\partial\gamma^{p-2,\,q+1} \\ \bar\partial\gamma^{p-2,\,q+1} &= -\partial\gamma^{p-3,\,q+2} \\ &\vdots \\ \bar\partial\gamma^{p-r+2,\,q+r-3} &= -\partial\gamma^{p-r+1,\,q+r-2} \\ \bar\partial\gamma^{p-r+1,\,q+r-2} &= 0. \end{aligned}$$

This tower of r equations expresses the fact that $\alpha \in \mathcal{Y}_r^{p,\,q}$.

We have thus proved the inclusion $\ker S_r^{p,\,q} \subset \mathcal{Y}_r^{p,\,q}$, which yields the obvious *surjection* below:

$$E_r^{p,\,q}(X) \simeq \frac{\mathcal{X}_r^{p,\,q}}{\ker S_r^{p,\,q}} \twoheadrightarrow \frac{\mathcal{X}_r^{p,\,q}}{\mathcal{Y}_r^{p,\,q}} \simeq E_r^{p,\,q}(X),$$

where the first isomorphism follows from the surjectivity of $S_r^{p,\,q}$ (a composition of two surjections), while the last isomorphism was proved in (1) of Theorem 1.2.10. Since all these vector spaces are finite dimensional, we get $\dim\ker S_r^{p,\,q} = \dim\mathcal{Y}_r^{p,\,q}$, hence $\ker S_r^{p,\,q} = \mathcal{Y}_r^{p,\,q}$ since we already had one inclusion. $\square$

End of proof of (2) *of Theorem 1.2.10.* The maps $S_r^{p,\,q}$ factor through $\mathcal{Y}_r^{p,\,q}$ to isomorphisms $\widehat{S_r^{p,\,q}}$ and we get:

$$\frac{\mathcal{X}_r^{p,\,q}}{\mathcal{Y}_r^{p,\,q}} \xrightarrow[\simeq]{\widehat{S_r^{p,\,q}}} E_r^{p,\,q}(X) \xrightarrow{d_r} E_r^{p+r,\,q-r+1}(X) \xrightarrow[\simeq]{(\widehat{S_r^{p+r,\,q-r+1}})^{-1}} \frac{\mathcal{X}_r^{p+r,\,q-r+1}}{\mathcal{Y}_r^{p+r,\,q-r+1}},$$

where d_r is the differential defined in the classical way in (1.21). From the way in which each of these maps acts, that we made explicit above, we get that the composition

$$(\widehat{S_r^{p+r,\,q-r+1}})^{-1} \circ d_r \circ \widehat{S_r^{p,\,q}} : \frac{\mathcal{X}_r^{p,\,q}}{\mathcal{Y}_r^{p,\,q}} \longrightarrow \frac{\mathcal{X}_r^{p+r,\,q-r+1}}{\mathcal{Y}_r^{p+r,\,q-r+1}}$$

acts, for $\alpha \in \mathcal{X}_r^{p,\,q}$, as follows:

$$\alpha \mod \mathcal{Y}_r^{p,\,q} \mapsto \bigg(d(\alpha - u_1 + u_2 - \cdots + (-1)^{r-1}\,u_{r-1})\bigg)^{p+r,\,q-r+1} \mod \mathcal{Y}_r^{p+r,\,q-r+1}.$$

Since the $(p+r,\, q-r+1)$-type component of $d(\alpha - u_1 + u_2 - \cdots + (-1)^{r-1}\,u_{r-1})$ is $(-1)^{r-1}\,\partial u_{r-1}$, we get that the above composition of maps acts, for $\alpha \in \mathcal{X}_r^{p,\,q}$, as follows:

$$\frac{\mathcal{X}_r^{p,\,q}}{\mathcal{Y}_r^{p,\,q}} \ni (\alpha \mod \mathcal{Y}_r^{p,\,q}) \mapsto ((-1)^{r-1}\,\partial u_{r-1} \mod \mathcal{Y}_r^{p+r,\,q-r+1}) \in \frac{\mathcal{X}_r^{p+r,\,q-r+1}}{\mathcal{Y}_r^{p+r,\,q-r+1}}.$$

This completes the proof of (2) of Theorem 1.2.10. $\square$

An immediate consequence of Theorem 1.2.10 is that the maps d_1 are defined by ∂ in the Dolbeault cohomology:

Corollary 1.2.12. *The first page of the Frölicher spectral sequence consists of the linear maps:*

$$E_1^{p,\,q}(X) = H_{\bar\partial}^{p,\,q}(X,\,\mathbb{C}) \xrightarrow{d_1} E_1^{p+1,\,q}(X) = H_{\bar\partial}^{p+1,\,q}(X,\,\mathbb{C}), \qquad d_1([\alpha]_{\bar\partial}) = [\partial\alpha]_{\bar\partial}.$$

This fact has an interesting consequence on the geometry of the manifold X.

Definition 1.2.13. *Let $p \in \{0, \dots, n\}$. A holomorphic p-form on X is a form $\alpha \in C^\infty_{p,\,0}(X,\,\mathbb{C})$ such that $\bar\partial\alpha = 0$.*

A key observation is the following

Proposition 1.2.14. *Let X be an n-dimensional compact complex manifold whose Frölicher spectral sequence degenerates at E_1.*

*Then, for every $p \in \{0, \dots, n\}$, every holomorphic p-form on X is d-**closed**.*

Proof. Let $\alpha \in C^\infty_{p,\,0}(X,\,\mathbb{C})$ be a holomorphic p-form. Since $\bar\partial\alpha = 0$, α represents a Dolbeault cohomology class $[\alpha]_{\bar\partial} \in H_{\bar\partial}^{p,\,0}(X,\,\mathbb{C}) = E_1^{p,\,0}(X)$.

Now, since $E_1(X) = E_\infty(X)$, all the maps d_1 vanish identically. Hence, $d_1([\alpha]_{\bar\partial}) = [\partial\alpha]_{\bar\partial} = 0$ (see Corollary 1.2.12), which means that the $(p+1,\,0)$-form $\partial\alpha$ is $\bar\partial$-exact. Thus, $\partial\alpha = \bar\partial\beta$ for some $(p+1,\,-1)$-form β. For bidegree reasons, we must have $\beta = 0$, hence $\partial\alpha = 0$. Then also $d\alpha = 0$. $\square$

One way in which this result can be used is to conclude that $E_1(X) \neq E_\infty(X)$ for a specific compact complex manifold X on which a non-d-closed holomorphic p-form has been found.

1.2.3 The notion of purity for the De Rham cohomology

In this subsection, we discuss a property that the De Rham cohomology may or may not have and its links with the Frölicher spectral sequence. We will follow the presentation of [PSU20, §.3.1].

We start with a statement that is immediate to prove.

Lemma 1.2.15. *The following relations hold:*

$$\begin{aligned} C^\infty_k(X,\,\mathbb{C}) &= \mathcal{F}^pC^\infty_k(X,\,\mathbb{C})\oplus\overline{\mathcal{F}^{k-p+1}C^\infty_k(X,\,\mathbb{C})} && \text{for all}\quad 0\leq p\leq\min\{k,\,n\}; && (1.29)\\ C^\infty_{p,\,q}(X) &= \mathcal{F}^pC^\infty_k(X,\,\mathbb{C})\cap\overline{\mathcal{F}^qC^\infty_k(X,\,\mathbb{C})} && \text{for all}\quad p,q\ \text{such that}\ p+q=k. && (1.30)\end{aligned}$$

Proof. Left to the reader. □

Now, for all $p,q\in\{0,\dots,n\}$, let us consider the following space of De Rham cohomology classes of degree $p+q$ that are representable by pure-type $(p,\,q)$-forms:

$$H^{p,\,q}_{DR}(X):=\left\{\mathfrak{c}\in H^{p+q}_{DR}(X,\,\mathbb{C})\ \mid\ \exists\alpha\in C^\infty_{p,\,q}(X)\cap\mathfrak{c}\right\}\subset H^{p+q}_{DR}(X,\,\mathbb{C}).$$

This definition makes it obvious that the analogue of the Hodge symmetry for the spaces $H^{p,\,q}_{DR}(X)$ always holds. In other words, the conjugation induces an isomorphism

$$H^{p,\,q}_{DR}(X)\ni\{\alpha\}_{DR}\mapsto\overline{\{\bar\alpha\}}_{DR}\in\overline{H^{q,\,p}_{DR}(X)}\qquad\text{for all}\quad 0\leq p,q\leq n. \tag{1.31}$$

The following analogue in cohomology of identity (1.30), resp. of one of the inclusions defining the filtration (1.13) of $C^\infty_k(X,\,\mathbb{C})$, can be immediately proved to hold.

Lemma 1.2.16. *The following relations hold:*

$$\begin{aligned} H^{p,\,q}_{DR}(X) &= F^pH^k_{DR}(X,\,\mathbb{C})\cap\overline{F^qH^k_{DR}(X,\,\mathbb{C})} && \text{for all}\quad p,q\ \text{such that}\ p+q=k; && (1.32)\\ H^{i,\,k-i}_{DR}(X) &\subset F^pH^k_{DR}(X,\,\mathbb{C}) && \text{for all}\quad i\geq p\ \text{and all}\quad p\leq k. && (1.33)\end{aligned}$$

Proof. Everything is obvious, except perhaps the inclusion "$\supset$" in (1.32) which can be proved as follows. Let $\{\alpha\}_{DR}=\{\beta\}_{DR}\in F^pH^k_{DR}(X,\,\mathbb{C})\cap\overline{F^qH^k_{DR}(X,\,\mathbb{C})}$ with $\alpha=\sum_{i\geq p}\alpha^{i,\,k-i}\in\mathcal{F}^pC^\infty_k(X,\,\mathbb{C})$ and $\beta=\sum_{s\leq p}\beta^{s,\,k-s}\in\overline{\mathcal{F}^qC^\infty_k(X,\,\mathbb{C})}$. Since α and β are De Rham-cohomologous, there exists a form $\sigma\in C^\infty_{k-1}(X,\,\mathbb{C})$ such that $\alpha-\beta=d\sigma$. This identity implies, after equating the terms with a holomorphic degree $>p$ on either side, the second identity below:

$$\alpha-\alpha^{p,\,q}=\sum_{i>p}\alpha^{i,\,k-i}=d(\sum_{j\geq p}\sigma^{j,\,k-1-j})-\bar\partial\sigma^{p,\,q-1},$$

which, in turn, implies that $\{\alpha\}_{DR}=\{\alpha^{p,\,q}-\bar\partial\sigma^{p,\,q-1}\}_{DR}$. Since $\alpha^{p,\,q}-\bar\partial\sigma^{p,\,q-1}$ is a $(p,\,q)$-form, we get $\{\alpha\}_{DR}\in H^{p,\,q}_{DR}(X)$ and we are done. □

Note that, with no assumption on X, the subspaces $H^{i,\,k-i}_{DR}(X)$ may have non-zero mutual intersections inside $H^k_{DR}(X,\,\mathbb{C})$, so they may not sit in a direct sum.

Let us now introduce the following

Definition 1.2.17. *Let X be an n-dimensional compact complex manifold. The De Rham cohomology of X is said to be* **pure** *if*

$$H^k_{DR}(X,\,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,\,q}_{DR}(X) \qquad \textit{for all} \quad k\in\{0,\dots,2n\}.$$

Note that the above definition requires all the subspaces $H^{p,\,q}_{DR}(X)$ of $H^k_{DR}(X,\,\mathbb{C})$ with $p+q=k$ to form a direct sum and to add up to the total space $H^k_{DR}(X,\,\mathbb{C})$.

Note on terminology 1.2.18. *Some authors call this property* **complex-$\mathcal{C}^\infty$-pure-and-full** *in degree k (cf. [LZ09]). It was remarked in [AT11] that the* **complex-$\mathcal{C}^\infty$-full** *property* **in degree** *k (i.e. the sum of the $H^{p,\,q}_{DR}(X)$'s is not necessarily direct but it fills out $H^k_{DR}(X,\,\mathbb{C})$) implies the* **complex-$\mathcal{C}^\infty$-pure** *property* **in degree** *$(2n-k)$ (i.e. the sum of the $H^{p,\,q}_{DR}(X)$'s is direct but it may not fill out $H^{2n-k}_{DR}(X,\,\mathbb{C})$). We will show further down that the converse is also true, i.e. the* **complex-$\mathcal{C}^\infty$-pure** *property* **in degree** *k implies the* **complex-$\mathcal{C}^\infty$-full** *property* **in degree** *$(2n-k)$. Therefore, compact complex manifolds satisfying either the* **complex-$\mathcal{C}^\infty$-full** *property* **in every degree** *k or the* **complex-$\mathcal{C}^\infty$-pure** *property* **in every degree** *k are of* **pure** *De Rham cohomology in the sense of our Definition 1.2.17. In algebraic geometry, this property is referred to by saying that "the Hodge filtration induces a pure Hodge structure on the de Rham cohomology".*

Proposition 1.2.19. *Suppose X is an n-dimensional compact complex manifold whose De Rham cohomology is* **pure**. *Then*

$$F^pH^k_{DR}(X,\,\mathbb{C}) = \bigoplus_{i\geq p} H^{i,\,k-i}_{DR}(X) \qquad \textit{for all} \quad p\leq k. \tag{1.34}$$

In particular, the spaces $E^{p,\,q}_\infty(X)$ in the Frölicher spectral sequence of X are given by

$$E^{p,\,q}_\infty(X) \simeq H^{p,\,q}_{DR}(X) \qquad \textit{for all} \quad p,q\in\{0,\dots,n\}, \tag{1.35}$$

where $\simeq$ stands for the canonical isomorphism induced by the identity.

Proof. Inclusion "$\supset$" in (1.34) follows at once from (1.33) and from the De Rham purity assumption.

To prove inclusion "$\subset$" in (1.34), let $\{\alpha\}_{DR}\in F^pH^k_{DR}(X,\,\mathbb{C})$ with $\alpha = \sum\limits_{r\geq p}\alpha^{r,\,k-r}\in\ker d$. Since $F^pH^k_{DR}(X,\,\mathbb{C})\subset H^k_{DR}(X,\,\mathbb{C}) = \oplus_{p+q=k}H^{p,\,q}_{DR}(X)$ (the last identity being due to the purity assumption), there exist pure-type d-closed forms $\beta^{r,\,k-r}$ such that $\{\alpha\}_{DR} = \{\sum_{0\leq r\leq k}\beta^{r,\,k-r}\}_{DR}$. Hence, there exists a $(k-1)$-form σ such that $\alpha - \sum_{0\leq r\leq k}\beta^{r,\,k-r} = -d\sigma$, which amounts to

$$\alpha^{r,\,k-r} - \beta^{r,\,k-r} + \partial\sigma^{r-1,\,k-r} + \bar\partial\sigma^{r,\,k-r-1} = 0, \qquad r\in\{0,\dots,k\},$$

with the understanding that $\alpha^{r,\,k-r}=0$ whenever $r<p$.

Therefore, $\beta^{r,\,k-r} = \partial\sigma^{r-1,\,k-r} + \bar\partial\sigma^{r,\,k-r-1}$ whenever $r<p$. Since every $\beta^{r,\,k-r}$ is d-closed (hence also ∂- and $\bar\partial$-closed), we infer that $\sigma^{r-1,\,k-r}$ and $\sigma^{r,\,k-r-1}$ are $\partial\bar\partial$-closed for every $r<p$. Hence

$$\begin{aligned}\sum_{r=0}^k\beta^{r,\,k-r} - d\Big(\sum_{r<p}\sigma^{r,\,k-r-1}\Big) &= \sum_{r<p}(\beta^{r,\,k-r} - \partial\sigma^{r-1,\,k-r} - \bar\partial\sigma^{r,\,k-r-1}) - \partial\sigma^{p-1,\,k-p} + \sum_{r\geq p}^k\beta^{r,\,k-r}\\ &= \sum_{r\geq p}\beta^{r,\,k-r} - \partial\sigma^{p-1,\,k-p}.\end{aligned} \tag{1.36}$$

Note that from the identity $\beta^{p-1,\,k-p+1} = \partial\sigma^{p-2,\,k-p+1} + \bar\partial\sigma^{p-1,\,k-p}$ and the d-closedness of $\beta^{p-1,\,k-p+1}$ we infer that $\partial\sigma^{p-1,\,k-p} \in \ker d$.

Thus, (1.36) shows that the k-form $\sum_{r\geq p}\beta^{r,\,k-r} - \partial\sigma^{p-1,\,k-p} \in \mathcal{F}^pC^\infty_k(X,\,\mathbb{C})$, whose all pure-type components are d-closed, is De Rham-cohomologous to $\sum_{0\leq r\leq k}\beta^{r,\,k-r}$, hence to α. Consequently, we have

$$\{\alpha\}_{DR} = \left\{\sum_{r\geq p}\beta^{r,\,k-r} - \partial\sigma^{p-1,\,k-p}\right\}_{DR} \in \bigoplus_{i\geq p} H^{i,\,k-i}_{DR}(X).$$

The proof of (1.34) is complete.

Identity (1.35) follows at once from (1.34) and from (iii) of Theorem 1.2.2. □

We end this discussion by noticing an immediate consequence of a standard fact.

Observation 1.2.20. *Let X be a compact complex manifold X with $\dim_{\mathbb{C}}X = n$. Then, the following inequality holds between the dimensions of the Bott-Chern cohomology space $H^{p,\,q}_{BC}(X)$ of X and of $H^{p,\,q}_{DR}(X)$:*

$$h^{p,\,q}_{BC} \geq h^{p,\,q}_{DR} \quad \textit{for all } 0 \leq p, q \leq n. \tag{1.37}$$

Moreover, if the De Rham cohomology of X is **pure**, *then*

$$\sum_{p+q=k} h^{p,\,q}_{BC} \geq b_k \quad \textit{for all } 0 \leq k \leq 2n, \tag{1.38}$$

where $b_k := \dim_{\mathbb{C}}H^k_{DR}(X,\,\mathbb{C})$ is the k-th Betti number of X.

Proof. It is standard (and immediate to check) that, for every bidegree $(p,\,q)$, the canonical map $H^{p,\,q}_{BC}(X) \ni \{\alpha\}_{BC} \mapsto \{\alpha\}_{DR} \in H^{p+q}_{DR}(X,\,\mathbb{C})$ is well defined (i.e. independent of the choice of representative α of the Bott-Chern class $\{\alpha\}_{BC}$). This map need not be either injective, or surjective, but its image is, obviously, $H^{p,\,q}_{DR}(X)$. Hence inequality (1.37).

Inequality (1.38) follows at once from (1.37) and from the De Rham purity assumption. □

1.3 $\partial\bar\partial$-Manifolds

Let X be a compact complex manifold with $\dim_{\mathbb{C}}X = n$. The following notion goes back to Deligne-Griffiths-Morgan-Sullivan [DGMS75]. The name was given in [Pop14, Definition 1.6].

Definition 1.3.1. *A compact complex manifold X is said to be a $\partial\bar\partial$*-**manifold** *if for any d-closed pure-type form u on X, the following exactness properties are equivalent:*

$$u \textit{ is } d\textit{-exact} \iff u \textit{ is } \partial\textit{-exact} \iff u \textit{ is } \bar\partial\textit{-exact} \iff u \textit{ is } \partial\bar\partial\textit{-exact}.$$

The main interest in $\partial\bar\partial$-manifolds stems from the fact that they support the same nice Hodge theory as compact Kähler manifolds do. For this reason, some authors call them **cohomologically Kähler manifolds**. They are actually characterised by these Hodge-theoretical properties, as we will see in the next subsection.

1.3.1 Cohomological characterisations of $\partial\bar\partial$-manifolds

We will discuss four such characterisations: in terms of the Dolbeault cohomology; in terms of De Rham purity and the FSS; in terms of the Bott-Chern and Aeppli cohomologies; in purely numerical terms involving the Bott-Chern and Aeppli cohomologies.

(I) Characterisation in terms of the Dolbeault cohomology

The first of these characterisations will show that $\partial\bar\partial$-manifolds are precisely those compact complex manifolds that *canonically* support the Hodge decomposition. In particular, the Frölicher spectral sequence of any $\partial\bar\partial$-manifold will be seen to degenerate at E_1.

Theorem 1.3.2. *A compact complex n-dimensional manifold X is a $\partial\bar\partial$-**manifold** if and only if the **identity induces an isomorphism** between $\oplus_{p+q=k}H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ and $H^k_{DR}(X,\,\mathbb{C})$, for every $k\in\{0,\dots,2n\}$, in the following sense:*

*(a) for every bidegree $(p,\,q)$ with $p+q=k$, every Dolbeault cohomology class $[\alpha^{p,\,q}]_{\bar\partial}\in H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ contains a d-**closed representative** $\alpha^{p,\,q}$;*
(b) the linear map

$$\bigoplus_{p+q=k}H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})\ni\sum_{p+q=k}[\alpha^{p,\,q}]_{\bar\partial}\mapsto\bigg\{\sum_{p+q=k}\alpha^{p,\,q}\bigg\}_{DR}\in H^k_{DR}(X,\,\mathbb{C})\tag{1.39}$$

*is **well-defined** by means of d-closed reprsentatives (in the sense that it does not depend on the choices of d-closed representatives $\alpha^{p,\,q}$ of the Dolbeault classes $[\alpha^{p,\,q}]_{\bar\partial}$) and **bijective**.*

The above latter property of manifolds has a name:

Definition 1.3.3. *If the identity induces an isomorphism $\oplus_{p+q=k}H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})\simeq H^k_{DR}(X,\,\mathbb{C})$ in the sense of Theorem 1.3.2 for every $k\in\{0,\dots,2n\}$, we say that the manifold X has the **Hodge Decomposition** property.*

We saw earlier in (1.17) that the property $E_1(X)=E_\infty(X)$ is equivalent to a weaker, non-canonical, ersatz Hodge decomposition property. Note that whenever the identity induces a well-defined (not necessarily injective) linear map $H^{p,\,q}_{\bar\partial}(X)\longrightarrow H^k_{DR}(X,\,\mathbb{C})$ by means of d-closed representatives of the Dolbeault classes in $H^{p,\,q}_{\bar\partial}(X)$, the image of this map is $H^{p,\,q}_{DR}(X)$. Indeed, one inclusion is obvious. The reverse inclusion follows from the trivial observation that any d-closed $(p,\,q)$-form is $\bar\partial$-closed, so it defines a Dolbeault cohomology class.

Proof of Theorem 1.3.2.

" $\Longrightarrow$ " Suppose that X is a $\partial\bar\partial$-manifold.

(a) Let $[\alpha^{p,\,q}]_{\bar\partial}\in H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ be an arbitrary class and let $\alpha^{p,\,q}$ be an arbitrary representative of it. Then, $\bar\partial\alpha^{p,\,q}=0$. We need to prove the existence of a $(p,\,q-1)$-form β such that $d(\alpha-\bar\partial\beta)=0$, which is equivalent to $\partial\alpha=\partial\bar\partial\beta$. Such a β exists if and only if $\partial\alpha$ is $\partial\bar\partial$-exact.

Now, $\partial\alpha$ is d-closed, ∂-exact and of pure type $(p+1,\,q)$, so by the $\partial\bar\partial$-assumption on X, $\partial\alpha$ must also be $\partial\bar\partial$-exact.

(b) To prove well-definedness, fix a bidegree (p, q) and let $\alpha_1^{p,q}$ and $\alpha_2^{p,q}$ be d-closed representatives of the same Dolbeault cohomology class. This means that $\alpha_1^{p,q} - \alpha_2^{p,q}$ is $\bar\partial$-exact. Since it is also d-closed and of pure type, it must be d-exact, by the $\partial\bar\partial$-assumption on X. Thus, $\{\alpha_1\}_{DR} = \{\alpha_2\}_{DR}$, which is what we had to prove.

We will now prove that, for every bidegree (p, q) with $p+q = k$, the identity induces an *injection*

$$H^{p,q}_{\bar\partial}(X, \mathbb{C}) \hookrightarrow H^k_{DR}(X, \mathbb{C}), \qquad [\alpha^{p,q}]_{\bar\partial} \mapsto \{\alpha^{p,q}\}_{DR},$$

by means of d-closed representatives $\alpha^{p,q}$ of their respective Dolbeault cohomology classes.

Indeed, let $\alpha^{p,q}$ be a d-closed representative of its Dolbeault cohomology class such that $\{\alpha^{p,q}\}_{DR} = 0$. Then, $\alpha^{p,q}$ is a d-exact pure-type form, so by the $\partial\bar\partial$-assumption on X, $\alpha^{p,q}$ must also be $\bar\partial$-exact. Thus, $[\alpha^{p,q}]_{\bar\partial} = 0$. This proves the injectivity of the above map $H^{p,q}_{\bar\partial}(X, \mathbb{C}) \longrightarrow H^k_{DR}(X, \mathbb{C})$.

Next, we prove that for any fixed k and any distinct bidegrees $(p, q) \neq (r, s)$ with $p+q = r+s = k$, the images in $H^k_{DR}(X, \mathbb{C})$ of $H^{p,q}_{\bar\partial}(X, \mathbb{C})$ and $H^{r,s}_{\bar\partial}(X, \mathbb{C})$ under the above canonical injections induced by the identity meet only at 0. Suppose to the contrary that there exist d-closed and non-$\bar\partial$-exact forms $\alpha^{p,q}$ and $\alpha^{r,s}$ of the shown bidegrees such that $\{\alpha^{p,q}\}_{DR} = \{\alpha^{r,s}\}_{DR}$. Then, $\alpha^{p,q} - \alpha^{r,s} = d\beta$ for some form β, so

$$\alpha^{p,q} = \partial\beta^{p-1,q} + \bar\partial\beta^{p,q-1} \qquad \text{and} \qquad \alpha^{r,s} = -\partial\beta^{r-1,s} - \bar\partial\beta^{r,s-1}.$$

Then, $\alpha^{p,q} - \bar\partial\beta^{p,q-1} = \partial\beta^{p-1,q}$. Meanwhile, the pure-type form $\partial\beta^{p-1,q}$ is d-closed and ∂-exact, so by the $\partial\bar\partial$-assumption on X, it is also $\bar\partial$-exact. Hence, $\alpha^{p,q} - \bar\partial\beta^{p,q-1}$ is $\bar\partial$-exact and then so is $\alpha^{p,q}$. This contradicts the hypothesis on $\alpha^{p,q}$.

We conclude from the discussions of the last two points that the map (1.39) is injective. In particular,

$$\sum_{p+q=k} h^{p,q} \leq b_k, \qquad k \in \{0, \dots, 2n\}.$$

On the other hand, we know from Corollary 1.2.5 that the reverse inequality holds on any compact complex manifold X (whether it is $\partial\bar\partial$ or not). This fact forces the equalities $\sum_{p+q=k} h^{p,q} = b_k$ for $k \in \{0, \dots, 2n\}$, so the map (1.39) must be bijective.

The proof of the Hodge Decomposition property of X is now complete.

"$\Longleftarrow$" Suppose that X has the Hodge Decomposition property.

Fix a d-closed (p, q)-form α for an arbitrary bidegree (p, q). Put $k = p + q$. We will expand on the arguments in the proof of the implication (iii) $\Longrightarrow$ (i) of Proposition (5.17) of [DGMS75].

• Let us first prove the equivalence:

$$\alpha \in \operatorname{Im} \bar\partial \overset{(i)}{\Longleftrightarrow} \alpha \in \operatorname{Im} d.$$

Since α is d-closed and of pure type, α is also $\bar\partial$-closed. Thus, α defines classes in both $H^{p,q}_{\bar\partial}(X)$ and $H^{p,q}_{DR}(X)$. Meanwhile, X has the Hodge Decomposition property, so the identity induces a linear injection $H^{p,q}_{\bar\partial}(X) \hookrightarrow H^{p+q}_{DR}(X)$ whose image is $H^{p,q}_{DR}(X)$. Therefore, $[\alpha]_{\bar\partial} = 0$ implies $\{\alpha\}_{DR} = 0$ (because the image of 0 under a linear map is 0) and $\{\alpha\}_{DR} = 0$ implies $[\alpha]_{\bar\partial} = 0$ (by injectivity of this map). This proves the above equivalence.

Since the above equivalence has been proved in every bidegree, by conjugation we also get the equivalence $u \in \operatorname{Im} \partial \Leftrightarrow u \in \operatorname{Im} d$ in every bidegree for every d-closed pure-type form u.

- Let us now prove the equivalence:

$$\alpha \in \mathrm{Im}\,(\partial\bar\partial) \overset{(ii)}{\Longleftrightarrow} \alpha \in \mathrm{Im}\,\bar\partial.$$

The implication " $\Longrightarrow$ " being trivial, we will prove the implication "$\Longleftarrow$". Suppose there exists $\beta \in C^\infty_{p,\,q-1}(X)$ such that $\alpha = \bar\partial\beta$. Then $\alpha - d\beta = -\partial\beta \in C^\infty_{p+1,\,q-1}(X)$, hence

$$\oplus_{l\leq p} H^{l,\,k-l}_{DR}(X) = \overline{F^q H^k_{DR}(X,\,\mathbb{C})} \in \{\alpha\}_{DR} = \{-\partial\beta\}_{DR} \in F^{p+1}H^k_{DR}(X,\,\mathbb{C}) = \oplus_{i\geq p+1} H^{i,\,k-i}_{DR}(X),$$

where the De Rham purity assumption was used. Since $\oplus_{i\geq p+1} H^{i,\,k-i}_{DR}(X) \cap \oplus_{l\leq p} H^{l,\,k-l}_{DR}(X) = \{0\}$, we get $\{\alpha\}_{DR} = 0$, a fact that also follows from the assumption on α and the equivalence (i).

Recall that $F^pH^k_{DR}(X,\,\mathbb{C})$ is the image of the map

$$\frac{\ker(d : \mathcal{F}^pC^\infty_k(X,\,\mathbb{C}) \to \mathcal{F}^pC^\infty_{k+1}(X,\,\mathbb{C}))}{\mathrm{Im}\,(d : \mathcal{F}^pC^\infty_{k-1}(X,\,\mathbb{C}) \to \mathcal{F}^pC^\infty_k(X,\,\mathbb{C}))} \to H^k_{DR}(X,\,\mathbb{C})$$

and that this map is injective for all p, k if and only if the Frölicher spectral sequence of X degenerates at E_1. Indeed, the map is injective if and only if

$$\mathrm{Im}\,(d : \mathcal{F}^pC^\infty_{k-1}(X,\,\mathbb{C}) \to \mathcal{F}^pC^\infty_k(X,\,\mathbb{C})) = \mathcal{F}^pC^\infty_k(X,\,\mathbb{C}) \cap \mathrm{Im}\,d,$$

which Proposition 1.2.8 ensures to be true if and only if $E_1(X) = E_\infty(X)$.

In our case, $E_1(X) = E_\infty(X)$ (because X has the Hodge Decomposition property). Using the injectivity of the above map, the vanishing of the class $\{\alpha\}_{DR} \in F^pH^k_{DR}(X,\,\mathbb{C})$ implies that $\alpha = du$ for some form $u \in \mathcal{F}^pC^\infty_{k-1}(X,\,\mathbb{C})$. From the analogous argument for $\overline{F^qH^k_{DR}(X,\,\mathbb{C})}$ we infer that the vanishing of the class $\{\alpha\}_{DR} \in \overline{F^qH^k_{DR}(X,\,\mathbb{C})}$ implies that $\alpha = dv$ for some form $v \in \overline{\mathcal{F}^qC^\infty_{k-1}(X,\,\mathbb{C})}$.

In particular, $u - v \in \ker d \cap (\mathcal{F}^pC^\infty_{k-1}(X,\,\mathbb{C}) + \overline{\mathcal{F}^qC^\infty_{k-1}(X,\,\mathbb{C})})$, hence $u - v$ defines a class

$$\{u - v\}_{DR} \in F^pH^{k-1}_{DR}(X,\,\mathbb{C}) + \overline{F^qH^{k-1}_{DR}(X,\,\mathbb{C})}.$$

Therefore, there exist forms $u_1 \in \mathcal{F}^pC^\infty_{k-1}(X,\,\mathbb{C}) \cap \ker d$ and $v_1 \in \overline{\mathcal{F}^qC^\infty_{k-1}(X,\,\mathbb{C})} \cap \ker d$ such that

$$u - v = u_1 - v_1 + d\eta$$

for some form $\eta \in C^\infty_{k-2}(X,\,\mathbb{C})$. Equating the terms of bidegree $(p-1,\,q)$, we get $v^{p-1,\,q} = v_1^{p-1,\,q} - \partial\eta^{p-2,\,q} - \bar\partial\eta^{p-1,\,q-1}$. Hence,

$$\alpha = dv = \partial v^{p-1,\,q} = \partial v_1^{p-1,\,q} - \partial\bar\partial\eta^{p-1,\,q-1}.$$

Now, since $v_1 \in \overline{\mathcal{F}^qC^\infty_{k-1}(X,\,\mathbb{C})} \cap \ker d$, we get $0 = dv_1 = \partial v_1^{p-1,\,q} +$ *forms of holomorphic degrees* $\geq q+1$. Thus, $\partial v_1^{p-1,\,q} = 0$, so $\alpha = \partial\bar\partial\eta^{p-1,\,q-1} \in \mathrm{Im}\,(\partial\bar\partial)$. □

(II) Characterisation in terms of De Rham purity and the FSS

We have already seen that $E_1(X) = E_\infty(X)$ whenever X is a $\partial\bar\partial$-manifold. We will now see by how much the converse fails.

Theorem 1.3.4. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Then, the following two conditions are equivalent:*

(i) X is a $\partial\bar\partial$**-manifold**;

(ii) the Frölicher spectral sequence of X **degenerates at** E_1 *and the De Rham cohomology of X is* **pure**.

Let us recall the following standard fact. When X is a compact complex surface (i.e. $\dim_{\mathbb{C}} X = 2$), the Frölicher spectral sequence of X always degenerates at E_1 (see [BHPV04, Theorem IV-2.8]), but the De Rham cohomology of X is pure if (and only if) the first Betti number b_1 of X is even (see [BHPV04, Proposition IV-2.9]).

Proof of Theorem 1.3.4. We know from Theorem 1.3.2 that X is a $\partial\bar\partial$-manifold if and only if X has the Hodge Decomposition property.

$(i) \implies (ii)$ We have already seen that the Hodge Decomposition property implies $E_1(X) = E_\infty(X)$ and that the image of each $H^{p,\,q}_{\bar\partial}(X)$ in $H^{p+q}_{DR}(X,\,\mathbb{C})$ under the map induced by the identity is $H^{p,\,q}_{DR}(X)$. We get (ii).

$(ii) \implies (i)$ Since the De Rham cohomology of X is supposed pure, we know from Proposition 1.2.19 that $E^{p,\,q}_\infty(X) \simeq H^{p,\,q}_{DR}(X)$ (isomorphism induced by the identity) for all bidegrees $(p,\,q)$. On the other hand, $E^{p,\,q}_\infty(X) = E^{p,\,q}_r(X)$ for all bidegrees $(p,\,q)$ since we are assuming that $E_r(X) = E_\infty(X)$. Together with the De Rham purity assumption, these facts imply that X has the Hodge Decomposition property. □

Definition 1.3.5. *Fix $p, q \in \{0, \dots, n\}$. We say that the* **conjugation induces an isomorphism** *between $H^{p,\,q}_{\bar\partial}(X)$ and the conjugate of $H^{q,\,p}_{\bar\partial}(X)$ if the following two conditions are satisfied:*

(a) every class $[\alpha^{p,\,q}]_{\bar\partial} \in H^{p,\,q}_{\bar\partial}(X)$ contains a d-**closed representative** $\alpha^{p,\,q}$;
(b) the linear map

$$H^{p,\,q}_{\bar\partial}(X) \ni [\alpha^{p,\,q}]_{\bar\partial} \mapsto \overline{[\overline{\alpha^{p,\,q}}]_{\bar\partial}} \in \overline{H^{q,\,p}_{\bar\partial}(X)}$$

is **well-defined** *(in the sense that it does not depend on the choice of d-closed representative $\alpha^{p,\,q}$ of the class $\{\alpha^{p,\,q}\}_{E_r}$) and* **bijective**.

Moreover, if the conjugation induces an isomorphism $H^{p,\,q}_{\bar\partial}(X) \simeq \overline{H^{q,\,p}_{\bar\partial}(X)}$ for every $p, q \in \{0, \dots, n\}$, we say that the manifold X has the **Hodge Symmetry** *property.*

We shall now see that the Hodge Decomposition property implies the Hodge Symmetry property.

Corollary 1.3.6. *Any $\partial\bar\partial$-manifold has the Hodge Symmetry property.*

Proof. We have already noticed in (1.31) that the conjugation (trivially) induces an isomorphism between any space $H^{p,\,q}_{DR}(X)$ and the conjugate of $H^{q,\,p}_{DR}(X)$. Meanwhile, we have seen that the $\partial\bar\partial$-assumption implies that the identity induces an isomorphism between any space $H^{p,\,q}_{\bar\partial}(X)$ and $H^{p,\,q}_{DR}(X)$. Hence, the conjugation induces an isomorphism between any space $H^{p,\,q}_{\bar\partial}(X)$ and the conjugate of $H^{q,\,p}_{\bar\partial}(X)$. □

(III) Characterisation in terms of the Bott-Chern and Aeppli cohomologies

Most of the material in this part is to be found, explicitly or implicitly, in [DGMS75], while the presentation is taken from [Pop14, §.4.3] and owes some key observations to [Wu06]. We will often refer to the canonical linear maps induced by the identity map in Lemma 1.1.2.

The main result is the following

Theorem 1.3.7. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The following statements are equivalent.*

*(i) X is a $\partial\bar\partial$-**manifold**;*

*(ii) For every bidegree $(p,\,q)$, the canonical linear map $H^{p,\,q}_{BC}(X,\,\mathbb{C}) \longrightarrow H^{p,\,q}_{A}(X,\,\mathbb{C})$ is **injective**;*

*(iii) For every bidegree $(p,\,q)$, the canonical linear map $H^{p,\,q}_{BC}(X,\,\mathbb{C}) \longrightarrow H^{p,\,q}_{A}(X,\,\mathbb{C})$ is **surjective**.*

We will actually prove rather more, a bidegree-sensitive statement. For a fixed bidegree $(p,\,q)$, we will say that the $\partial\bar\partial$-property holds in $C^\infty_{p,\,q}(X,\,\mathbb{C})$ if all the exactness properties of Definition 1.3.1 are equivalent for every d-closed form $u\in C^\infty_{p,\,q}(X,\,\mathbb{C})$. We start by introducing some ad hoc terminology.

Definition 1.3.8. *For a given $k\in\{0,1,\dots,2n\}$, a given compact complex manifold X (with $\dim_{\mathbb{C}} X = n$) is said to satisfy property:*

(A_k) *if the natural map $H^{p,\,q}_{BC}(X,\,\mathbb{C}) \longrightarrow H^{p,\,q}_{A}(X,\,\mathbb{C})$ is **injective** for all p,q such that $p+q=k$.*

This property is clearly equivalent to property

(A'_k) $\ker\partial\cap\ker\bar\partial\cap(Im\,\partial+Im\,\bar\partial) = Im\,(\partial\bar\partial)$, *for all p,q s.t. $p+q=k$.*

(B_k) *if the natural map $H^{p,\,q}_{BC}(X,\,\mathbb{C}) \longrightarrow H^{p,\,q}_{A}(X,\,\mathbb{C})$ is **surjective** for all p,q such that $p+q=k$.*

This property is clearly equivalent to property

(B'_k) $Im\,\partial+Im\,\bar\partial+(\ker\partial\cap\ker\bar\partial) = \ker(\partial\bar\partial)$, *for all p,q s.t. $p+q=k$.*

(C_k) *if the natural maps $H^{p,\,q}_{BC}(X,\,\mathbb{C}) \longrightarrow H^{p,\,q}_{\partial}(X,\,\mathbb{C})$ and $H^{p,\,q}_{BC}(X,\,\mathbb{C}) \longrightarrow H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ are **injective** for all p,q such that $p+q=k$.*

This property is clearly equivalent to the simultaneous occurence of

$(C'_k)(i)$ $Im\,\partial\cap\ker\bar\partial = Im\,(\partial\bar\partial)$ *and* $(C'_k)(ii)$ $Im\,\bar\partial\cap\ker\partial = Im\,(\partial\bar\partial)$, *for all p,q such that $p+q=k$.*

(D'_k) *if* (i) $Im\,\bar\partial+\ker\partial = \ker(\partial\bar\partial)$ *and* (ii) $Im\,\partial+\ker\bar\partial = \ker(\partial\bar\partial)$ *for all p,q such that $p+q=k$.*

(L_k) *if the $\partial\bar\partial$-property holds in every space of forms $C^\infty_{p,\,q}(X,\,\mathbb{C})$ with $p+q=k$.*

It is obvious that (L_k) implies each of the other properties listed above and that (L_k) is implied by the simultaneous occurence of these other properties. As already pointed out, the following equivalences are obvious

$$(A_k) \iff (A'_k), \quad (B_k) \iff (B'_k), \quad (C_k) \iff (C'_k).$$

The inclusions $\supset$ in (A'_k), $\subset$ in (B'_k), $\supset$ in $(C'_k)(i),(ii)$ and $\subset$ in $(D'_k)(i),(ii)$ are obvious. The following statement is implicit in [DGMS] and, obviously, implies Theorem 1.3.7.

Proposition 1.3.9. *(contained in Lemma 5.15 of [DGMS75]) Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. For every $k \in \{1,\dots,2n\}$, the following equivalences hold:*

$$(L_k) \iff (A_k) \iff (C_k) \iff (D'_{k-1}) \iff (B_{k-1}).$$

Proof. Fix an arbitrary $k \in \{1,\dots,2n\}$. In view of what we have already noticed, it suffices to prove the equivalences

$$(A'_k) \iff (C'_k) \iff (D'_{k-1}) \iff (B'_{k-1}).$$

Proof of $(A'_k) \implies (C'_k)$. Let $u \in C^\infty_{p,q}(X,\mathbb{C})$ (where $p+q=k$) such that $\bar\partial u = 0$ and $u = \partial v$ for some $(p-1,q)$-form v. Then $u = \partial v + 0 \in \ker\partial \cap \ker\bar\partial \cap (\operatorname{Im}\partial + \operatorname{Im}\bar\partial)$. Then (A'_k) forces $u \in \operatorname{Im}(\partial\bar\partial)$. This proves (i) of (C'_k). The proof of (ii) of (C'_k) is similar with ∂ and $\bar\partial$ reversed.

Proof of $(C'_k) \implies (A'_k)$. Let $u \in C^\infty_{p,q}(X,\mathbb{C})$ (where $p+q=k$) such that $\partial u = 0$, $\bar\partial u = 0$ and $u = \partial v + \bar\partial w$ for some $(p-1,q)$-form v and some $(p,q-1)$-form w. Then we have:

· $\operatorname{Im}\partial \ni \partial v = u - \bar\partial w \in \ker\bar\partial$, hence $\partial v \in \operatorname{Im}\partial \cap \ker\bar\partial = \operatorname{Im}(\partial\bar\partial)$, the last identity of subspaces being given by the hypothesis $(C'_k)(i)$. Thus $\partial v \in \operatorname{Im}(\partial\bar\partial)$.

· $\operatorname{Im}\bar\partial \ni \bar\partial w = u - \partial v \in \ker\partial$, hence $\bar\partial w \in \operatorname{Im}\bar\partial \cap \ker\partial = \operatorname{Im}(\partial\bar\partial)$, the last identity of subspaces being given by the hypothesis $(C'_k)(ii)$. Thus $\bar\partial w \in \operatorname{Im}(\partial\bar\partial)$.

It is now clear that $u = \partial v + \bar\partial w \in \operatorname{Im}(\partial\bar\partial)$. This proves (A'_k).

Proof of $(C'_k) \implies (D'_{k-1})$. Let $u \in C^\infty_{r,s}(X,\mathbb{C})$ (where $r+s=k-1$) such that $\partial\bar\partial u = 0$. Then:

· ∂u is a k-form of type $(r+1,s)$ and $\partial u \in \ker\bar\partial \cap \operatorname{Im}\partial = \operatorname{Im}(\partial\bar\partial)$, the last identity of subspaces being given by the hypothesis $(C'_k)(i)$. Hence $\partial u = \partial\bar\partial\zeta$ for some $(r,s-1)$-form ζ. This amounts to $\partial(u - \bar\partial\zeta) = 0$ or again to $u - \bar\partial\zeta \in \ker\partial$.

We get $u = \bar\partial\zeta + (u - \bar\partial\zeta) \in \operatorname{Im}\bar\partial + \ker\partial$. This proves $(D'_{k-1})(i)$.

· $\bar\partial u$ is a k-form of type $(r,s+1)$ and $\bar\partial u \in \ker\partial \cap \operatorname{Im}\bar\partial = \operatorname{Im}(\partial\bar\partial)$, the last identity of subspaces being given by the hypothesis $(C'_k)(ii)$. Hence $\bar\partial u = \partial\bar\partial w$ for some $(r-1,s)$-form w. This amounts to $\bar\partial(u + \partial w) = 0$ or again to $u + \partial w \in \ker\bar\partial$.

We get $u = -\partial w + (u + \partial w) \in \operatorname{Im}\partial + \ker\bar\partial$. This proves $(D'_{k-1})(ii)$.

Proof of $(D'_{k-1}) \implies (C'_k)$. Let $u \in C^\infty_{p,q}(X,\mathbb{C})$ (where $p+q=k$) such that $\bar\partial u = 0$ and $u = \partial v$ for some $(k-1)$-form v of type $(p-1,q)$. Then $0 = \bar\partial u = -\partial\bar\partial v$, hence $v \in \ker(\partial\bar\partial)$. Since $\ker(\partial\bar\partial) = \operatorname{Im}\bar\partial + \ker\partial$ for $(k-1)$-forms by $(D'_{k-1})(i)$, we can find a $(p-1,q-1)$-form w and a $(p-1,q)$-form ζ such that

$$v = \bar\partial w + \zeta \quad \text{and} \quad \partial\zeta = 0.$$

Applying ∂, we get: $u = \partial v = \partial\bar\partial w$. Thus $u \in \mathrm{Im}\,(\partial\bar\partial)$. This proves $(C'_k)(i)$.

Reversing the roles of ∂ and $\bar\partial$, we get $(C'_k)(ii)$ in a similar way from $(D'_{k-1})(ii)$.

Proof of $(D'_{k-1}) \implies (B'_{k-1})$. Let $u \in C^\infty_{r,\,s}(X,\,\mathbb{C})$ (where $r+s=k-1$) such that $\partial\bar\partial u = 0$. Thanks to $(D'_{k-1})(ii)$, we can find an $(r-1,\,s)$-form v and an $(r,\,s)$-form w such that

$$u = \partial v + w \quad \text{and} \quad w \in \ker\bar\partial.$$

Now since $\bar\partial w = 0$, we also have $\partial\bar\partial w = 0$. Hence by $(D'_{k-1})(i)$ we can write

$$w = \bar\partial\zeta + \rho \quad \text{with} \quad \rho \in \ker\partial$$

for some $(r,\,s-1)$-form ζ and some $(r,\,s)$-form ρ. We get: $\rho = w - \bar\partial\zeta$ and since $w, \bar\partial\zeta \in \ker\bar\partial$, we finally get $\rho \in \ker\bar\partial$. Given the choice of ρ, this implies that $\rho \in \ker\partial\cap\ker\bar\partial$.

Putting the bits together, we have

$$u = \partial v + \bar\partial\zeta + \rho \in \mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial + (\ker\partial\cap\ker\bar\partial).$$

This proves (B'_{k-1}).

Proof of $(B'_{k-1}) \implies (D'_{k-1})$. This implication is trivial because $\mathrm{Im}\,\partial + (\ker\partial\cap\ker\bar\partial) \subset \ker\partial$ and $\mathrm{Im}\,\bar\partial + (\ker\partial\cap\ker\bar\partial) \subset \ker\bar\partial$.

Proposition 1.3.9 is proved. $\square$

Proof of Theorem 1.3.7. It follows trivially from Proposition 1.3.9 since X being a $\partial\bar\partial$-manifold is equivalent to property (L_k) of Definition 1.3.8 holding for all k. $\square$

(IV) Numerical characterisation

The material in this part is due to Angella and Tomassini [AT13]. The dimensions of the Bott-Chern, resp. Aeppli, cohomology spaces $H^{p,\,q}_{BC}(X)$ and $H^{p,\,q}_A(X)$ will be denoted by $h^{p,\,q}_{BC}$, resp. $h^{p,\,q}_A$. Similarly, a lower case letter will stand for the dimension of the vector space denoted by the same capital letter. The main result is the following

Theorem 1.3.10. *([AT13]) Let X be a compact complex manifold with $\dim_{\mathbb{C}}X = n$. Then, for every $k\in\{0,\dots,2n\}$, the following inequality holds:*

$$\sum_{p+q=k}(h^{p,\,q}_{BC} + h^{p,\,q}_A) \geq 2b_k. \tag{1.40}$$

Moreover, X is a $\partial\bar\partial$-manifold if and only if equality occurs in (1.40) for every $k\in\{0,\dots,2n\}$.

The proof proceeds in several straightforward steps, the first of which consists in considering the following vector spaces introduced by Varouchas in [Var86]:

$$A^{\bullet,\,\bullet} := \frac{\mathrm{Im}\,\bar\partial\cap\mathrm{Im}\,\partial}{\mathrm{Im}\,\partial\bar\partial}, \qquad B^{\bullet,\,\bullet} := \frac{\ker\bar\partial\cap\mathrm{Im}\,\partial}{\mathrm{Im}\,\partial\bar\partial}, \qquad C^{\bullet,\,\bullet} := \frac{\ker\partial\bar\partial}{\ker\bar\partial + \mathrm{Im}\,\partial},$$

and

$$D^{\bullet,\,\bullet} := \frac{\mathrm{Im}\,\bar\partial\cap\ker\partial}{\mathrm{Im}\,\partial\bar\partial}, \qquad E^{\bullet,\,\bullet} := \frac{\ker\partial\bar\partial}{\ker\partial + \mathrm{Im}\,\bar\partial}, \qquad F^{\bullet,\,\bullet} := \frac{\ker\partial\bar\partial}{\ker\bar\partial + \ker\partial}.$$

These vector spaces are easily seen to be finite-dimensional by comparisons with $H^{\bullet,\bullet}_{BC}(X)$ and $H^{\bullet,\bullet}_{A}(X)$ which are known to be finite-dimensional. (See §.1.1.1.)

Lemma 1.3.11. *The identity map induces the following* **well-defined** *maps in cohomology and the following sequences are* **exact** *for every bidegree* $(p,\,q)$*:*

$$0 \longrightarrow D^{p,\,q} \hookrightarrow H^{p,\,q}_{BC}(X) \xrightarrow{T^{p,\,q}_{BC}} H^{p,\,q}_{\partial}(X) \xrightarrow{T^{p,\,q}_{E}} E^{p,\,q} \twoheadrightarrow F^{p,\,q} \longrightarrow 0. \tag{1.41}$$

$$0 \longrightarrow A^{p,\,q} \hookrightarrow B^{p,\,q} \xrightarrow{T^{p,\,q}_{B}} H^{p,\,q}_{\bar\partial}(X) \xrightarrow{T^{p,\,q}_{A}} H^{p,\,q}_{A}(X) \twoheadrightarrow C^{p,\,q} \longrightarrow 0, \tag{1.42}$$

Proof. It consists in straigtforward verifications that are left to the reader. They also appear in [Var86, section 3.1.]. □

The first properties of the above vector spaces are summed up in the following

Lemma 1.3.12. *Let X be a compact complex n-dimensional manifold.*

(i) The following identities hold in every bidegree $(p,\,q)$*:*

$$A^{p,\,q} = \overline{A^{q,\,p}} \text{ and } F^{p,\,q} = \overline{F^{q,\,p}}; \qquad B^{p,\,q} = \overline{D^{q,\,p}} \text{ and } C^{p,\,q} = \overline{E^{q,\,p}}.$$

Moreover, the following inclusions and surjections (defined by the identity map on forms) hold:

$$A^{p,\,q} \subset D^{p,\,q} \subset H^{p,\,q}_{BC}(X) \quad \text{and} \quad H^{p,\,q}_{A}(X) \twoheadrightarrow C^{p,\,q} \twoheadrightarrow F^{p,\,q}$$

in every bidegree $(p,\,q)$*. Likewise, the conjugated relations hold:*

$$A^{p,\,q} \subset B^{p,\,q} \subset H^{p,\,q}_{BC}(X) \quad \text{and} \quad H^{p,\,q}_{A}(X) \twoheadrightarrow E^{p,\,q} \twoheadrightarrow F^{p,\,q}.$$

(ii) The following canonical bilinear pairings are **well defined** *and* **non-degenerate***, hence define* **dualities***:*

$$A^{p,\,q} \times F^{n-p,\,n-q} \longrightarrow \mathbb{C}, \quad B^{p,\,q} \times E^{n-p,\,n-q} \longrightarrow \mathbb{C} \quad \text{and} \quad C^{p,\,q} \times D^{n-p,\,n-q} \longrightarrow \mathbb{C},$$

for every bidegree $(p,\,q)$, where in each of the three cases every pair $(\{\alpha\},\,\{\beta\})$ of classes of respective bidegrees $(p,\,q)$ and $(n-p,\,n-q)$ is mapped to $\int_X \alpha\wedge\beta \in \mathbb{C}$.

In particular, the exact sequences (1.41) and (1.42) are **dual** *to each other in complementary bidegrees (i.e. if (1.41) is considered in bidegree $(p,\,q)$, its dual is (1.42) considered in bidegree $(n-p,\,n-q)$).*

(iii) Consequently, we get the following dimension identities:

$$a^{p,\,q} = a^{q,\,p} = f^{n-p,\,n-q} = f^{n-q,\,n-p} \quad \text{and} \quad b^{p,\,q} = d^{q,\,p} = e^{n-p,\,n-q} = c^{n-q,\,n-p},$$

for every bidegree $(p,\,q)$*.*

(iv) The linear maps $\bar\partial : C^{p,\,q} \to D^{p,\,q+1}$ and $\partial : E^{p,\,q} \to B^{p+1,\,q}$ are **well defined** *and* **bijective** *for every bidegree* $(p,\,q)$*.*

Consequently, we get the following dimension identities:

$$c^{p,\,q} = d^{p,\,q+1} \quad \text{and} \quad e^{p,\,q} = b^{p+1,\,q}.$$

Proof. Again, the proof consists in straigtforward verifications that are left to the reader. The duality statements can be proved in the same way as the duality between the Bott-Chern and Aeppli cohomologies was proved in §.1.1.1. □

From these considerations, we go on to get the following result whose set of relations (1.44) proves the first statement in Theorem 1.3.10.

Corollary 1.3.13. *For every bidegree $(p,\, q)$, the following inequality holds:*

$$h^{p,\,q}_{BC} + h^{p,\,q}_{A} \geq h^{p,\,q}_{\bar\partial} + h^{q,\,p}_{\bar\partial}. \tag{1.43}$$

In particular, for every $k \in \{0, \dots, 2n\}$, we have:

$$h^{k}_{BC} + h^{k}_{A} = 2h^{k}_{\bar\partial} + a^k + f^k \geq 2h^{k}_{\bar\partial} \geq 2b_k, \tag{1.44}$$

where $h^{k}_{BC} := \sum_{p+q=k} h^{p,\,q}_{BC}$, $a^k := \sum_{p+q=k} a^{p,\,q}$ and the numbers h^k_A, b^k, c^k, d^k, e^k, f^k are defined analogously, while b_k is the k-th Betti number of X.

Proof. From the exact sequence (1.41) in bidegree $(p,\, q)$, resp. the exact sequence (1.42) in bidegree $(q,\, p)$, we infer the identities:

$$h^{p,\,q}_{BC} = d^{p,\,q} + h^{p,\,q}_{\bar\partial} - e^{p,\,q} + f^{p,\,q} \quad \text{and} \quad h^{q,\,p}_{A} = a^{q,\,p} - b^{q,\,p} + h^{q,\,p}_{\bar\partial} + c^{q,\,p}.$$

Since $h^{p,\,q}_{A} = h^{q,\,p}_{A}$, summing up the above two identities and using the numerical relations obtained under (iii) of Lemma 1.3.12 to cancel the terms reoccuring with opposite signs, we get

$$h^{p,\,q}_{BC} + h^{p,\,q}_{A} = (h^{p,\,q}_{\bar\partial} + h^{q,\,p}_{\bar\partial}) + (a^{p,\,q} + f^{p,\,q}).$$

The contention follows since $a^{p,\,q}, f^{p,\,q} \geq 0$, while $h^{k}_{\bar\partial} \geq b_k$ by Corollary 1.2.5. □

The next result proves one implication of the second statement in Theorem 1.3.10.

Corollary 1.3.14. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.*

If X is a $\partial\bar\partial$**-manifold**, *then*

$$h^{k}_{BC} + h^{k}_{A} = 2b_k \tag{1.45}$$

for every $k \in \{0, \dots, 2n\}$.

Proof. If X is a $\partial\bar\partial$-manifold, then $E_1(X) = E_\infty(X)$, hence $h^{k}_{\bar\partial} = b_k$ for all k. (See Theorem 1.3.4 and Corollary 1.2.6.)

On the other hand, if X is a $\partial\bar\partial$-manifold, then $a^{p,\,q} = 0$ for all p, q, as follows immediately from the definition of $A^{p,\,q}$ and from Definition 1.3.1. Then, by (iii) of Lemma 1.3.12, $f^{p,\,q} = 0$ for all p, q. Hence, $a^k = f^k = 0$ for all k.

The contention now follows from (1.44). □

We will now split the proof of the remaining implication of the second statement in Theorem 1.3.10 into three lemmas which are motivated by the (immediate) observation that X is a $\partial\bar\partial$-manifold if and only if the canonical linear map $\sum_{p+q=k} H^{p,\,q}_{BC}(X,\, \mathbb{C}) \longrightarrow H^k_{DR}(X,\, \mathbb{C})$ is an isomorphism for all k. The goal is to establish these isomorphisms under hypothesis (1.40).

Lemma 1.3.15. *If $h^{k}_{BC} + h^{k}_{A} = 2b_k$ for all $k \in \{0, \dots, 2n\}$, then $E_1(X) = E_\infty(X)$ and $a^k = f^k = 0$ for all k.*

Proof. We infer from (1.44) that, if $h^k_{BC} + h^k_A = 2b_k$ for all k, then $a^k = f^k = 0$ and $h^k_{\bar\partial} = b_k$ for all k. Meanwhile, Corollary 1.2.6 ensures that $E_1(X) = E_\infty(X)$ if and only if $h^k_{\bar\partial} = b_k$ for all k. □

The next lemma is slightly more substantial.

Lemma 1.3.16. *Fix an arbitrary $k \in \{0, \dots, 2n\}$. If $a^{k+1} = 0$, then the identity-induced canonical linear map*

$$\bigoplus_{p+q=k} H^{p,\,q}_{BC}(X,\,\mathbb{C}) \longrightarrow H^k_{DR}(X,\,\mathbb{C})$$

is **surjective**.

Proof. Proving the claim is equivalent to proving that every class in $H^k_{DR}(X,\,\mathbb{C})$ admits a representative whose all pure-type components are d-closed.

Let $\{\alpha\}_{DR} \in H^k_{DR}(X,\,\mathbb{C})$ be an arbitrary class and let $\alpha = \sum_{l=0}^k (-1)^l\,\alpha^{k-l,\,l}$ be an arbitrary (necessarily d-closed) representative of it, where $\alpha^{k-l,\,l}$ is of pure type $(k-l,\,l)$ for every l. The property $d\alpha = 0$ translates to

$$\partial\alpha^{k,\,0} = 0, \quad \bar\partial\alpha^{0,\,k} = 0 \quad \text{and} \quad \bar\partial\alpha^{k-l,\,l} - \partial\alpha^{k-l-1,\,l+1} = 0 \quad \text{for all} \quad l \in \{0, \dots, k-1\}.$$

Thus, we get the first identity below, while the hypothesis $a^{k+1} = 0$ implies $a^{k-l,\,l+1} = 0$ for all l, hence the last identity below:

$$\bar\partial\alpha^{k-l,\,l} = \partial\alpha^{k-l-1,\,l+1} \in \operatorname{Im}\bar\partial \cap \operatorname{Im}\partial = \operatorname{Im}(\partial\bar\partial).$$

Therefore, for every $l \in \{0, \dots, k-1\}$, there exists $\eta^{k-l-1,\,l} \in C^\infty_{k-l-1,\,l}(X,\,\mathbb{C})$ such that

$$\bar\partial\alpha^{k-l,\,l} = \partial\alpha^{k-l-1,\,l+1} = \partial\bar\partial\eta^{k-l-1,\,l}.$$

Put

$$\eta := \sum_{l=0}^{k-1} (-1)^l\,\eta^{k-l-1,\,l}.$$

We obviously have $\{\alpha\}_{DR} = \{\alpha + d\eta\}_{DR}$, while

$$\alpha + d\eta = (\alpha^{k,\,0} + \partial\eta^{k-1,\,0}) + \sum_{l=1}^{k-1}(-1)^l\,(\eta^{k-l,\,l} + \partial\eta^{k-l-1,\,l} - \bar\partial\alpha^{k-l,\,l-1}) + (-1)^k\,(\alpha^{0,\,k} - \bar\partial\eta^{0,\,k-1}),$$

where the parantheses on the right enclose pure-type forms that are easily seen to be both ∂- and $\bar\partial$-closed, hence d-closed.

Thus, the class $\{\alpha\}_{DR}$ is represented the form $\alpha + d\eta$ whose all pure-type components are d-closed. □

The last of the three lemmas is straightforward.

Lemma 1.3.17. *If $h^k_{BC} \geq b_k$ and $h^k_{BC} + h^k_A = 2b_k$ for all $k \in \{0, \dots, 2n\}$, then $h^k_{BC} = b_k$ for all $k \in \{0, \dots, 2n\}$.*

Proof. The Bott-Chern/Aeppli duality (see Theorem 1.1.15), resp. the Poincaré duality, implies the first, resp. the last, identity below:

$$b_k \leq h^k_{BC} = h^{2n-k}_A = 2b_{2n-k} - h^{2n-k}_{BC} \leq b_{2n-k} = b_k, \qquad k \in \{0, \dots, 2n\}.$$

Consequently, $h^k_{BC} = b_k$ for all k. □

End of proof of Theorem 1.3.10. Suppose that $h^k_{BC} + h^k_A = 2b_k$ for all $k \in \{0, \dots, 2n\}$. Then, by Lemma 1.3.15, $a^k = 0$ for all k. Hence, by Lemma 1.3.15, the map $\sum_{p+q=k} H^{p,q}_{BC}(X, \mathbb{C}) \longrightarrow H^k_{DR}(X, \mathbb{C})$ is surjective for all k. In particular, $h^k_{BC} \geq b_k$ for all k.

The last identity and the hypothesis imply, thanks to Lemma 1.3.17, that $h^k_{BC} = b_k$ for all k. Hence, the surjective linear map $\sum_{p+q=k} H^{p,q}_{BC}(X, \mathbb{C}) \longrightarrow H^k_{DR}(X, \mathbb{C})$ is an isomorphism, which amounts to X being a $\partial\bar\partial$-manifold. □

1.3.2 The $\partial\bar\partial$-lemma on compact Kähler manifolds

Compact Kähler manifolds are the object of study of Kähler geometry and constitute fundamental examples of $\partial\bar\partial$-manifolds. However, there exist many non-Kähler $\partial\bar\partial$-manifolds, as we will see further down.

Definition 1.3.18. *(i) A* **Kähler metric** *on a complex manifold X is a Hermitian metric ω such that $d\omega = 0$.*

(ii) A complex manifold X is said to be a **Kähler manifold** *if there exists a Kähler metric on X.*

Hermitian commutation relations

Before discussing the so-called *$\partial\bar\partial$-lemma on compact Kähler manifolds*, let us briefly review the generalisation to the Hermitian context of the **Kähler commutation relations**. This generalisation was established by Demailly in [Dem84] (see also [Dem97, VII, §.1), but originates in Griffiths's work [Gri69] and is also much related to §.1 of Chapter 1 in Ohsawa's work [Ohs82]. We refer the reader to the original sources for the proofs.

Proposition 1.3.19. *([Dem84], see also [Dem97, VII, §.1]) Let (X, ω) be a compact complex Hermitian manifold. Then, the following* **Hermitian commutation relations** *hold in every bidegree:*

$$\begin{aligned} &(i)\ (\partial + \tau)^\star = i\,[\Lambda,\, \bar\partial]; \qquad (ii)\ (\bar\partial + \bar\tau)^\star = -i\,[\Lambda,\, \partial]; \\ &(iii)\ \partial + \tau = -i\,[\bar\partial^\star,\, L]; \qquad (iv)\ \bar\partial + \bar\tau = i\,[\partial^\star,\, L], \end{aligned} \tag{1.46}$$

where the upper symbol $\star$ stands for the formal adjoint w.r.t. the L^2 inner product induced by ω, $L = L_\omega := \omega \wedge \cdot$ is the Lefschetz operator of multiplication by ω, $\Lambda = \Lambda_\omega := L^\star$ and $\tau = \tau_\omega := [\Lambda,\, \partial\omega \wedge \cdot]$ is the torsion operator (of order zero and type $(1,\, 0)$) associated with the metric ω.

Again following [Dem97, VII, §.1], the commutation relations (1) immediately induce via the Jacobi identity the Bochner-Kodaira-Nakano-type identity

$$\Delta'' = \Delta' + [\partial,\, \tau^\star] - [\bar\partial,\, \bar\tau^\star] \tag{1.47}$$

relating the $\bar\partial$-Laplacian $\Delta'' = [\bar\partial,\, \bar\partial^\star] = \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial$ and the ∂-Laplacian $\Delta' = [\partial,\, \partial^\star] = \partial\partial^\star + \partial^\star\partial$. This, in turn, induces the following Bochner-Kodaira-Nakano-type identity (cf. [Dem84]) in which the first-order terms have been absorbed in the twisted Laplace-type operator $\Delta'_\tau := [\partial + \tau,\, (\partial + \tau)^\star]$:

$$\Delta'' = \Delta'_\tau + T_\omega, \tag{1.48}$$

where $T_\omega := \Big[\Lambda,\, [\Lambda,\, \frac{i}{2}\,\partial\bar\partial\omega]\Big] - [\partial\omega\wedge\cdot,\, (\partial\omega\wedge\cdot)^\star]$ is a zeroth order operator of type $(0,\,0)$ associated with the torsion of ω. Formula (1.48) is obtained from (1.47) via the following identities (cf. [Dem84] or [Dem97, VII, §.1]) which have an interest of their own:

$$\begin{aligned} &(i)\ [L,\,\tau] = 3\,\partial\omega\wedge\cdot, \qquad (ii)\ [\Lambda,\,\tau] = -2i\,\bar\tau^\star, \\ &(iii)\ [\partial,\,\bar\tau^\star] = -[\partial,\,\bar\partial^\star] = [\tau,\,\bar\partial^\star], \qquad (iv)\ -[\bar\partial,\,\bar\tau^\star] = [\tau,\,(\partial+\tau)^\star] + T_\omega. \end{aligned} \tag{1.49}$$

Note that (iii) yields, in particular, that ∂ and $\bar\partial^\star + \bar\tau^\star$ anti-commute, hence by conjugation, $\bar\partial$ and $\partial^\star + \tau^\star$ anti-commute, i.e.

$$[\partial,\,\bar\partial^\star + \bar\tau^\star] = 0 \quad\text{and}\quad [\bar\partial,\,\partial^\star + \tau^\star] = 0. \tag{1.50}$$

The Kähler case

If the metric ω is Kähler, then $\partial\omega = 0$, so $\tau = 0$ and (1.47) and (1.48) reduce to

$$\Delta'' = \Delta' = \frac{1}{2}\,\Delta \tag{1.51}$$

as operators $\Delta', \Delta'', \Delta : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C})$ in every bidegree $(p,\,q)$. Note that, in particular, Δ preserves bidegrees when the metric ω is Kähler.

Also note that, thanks to ω being Kähler, (4.83) reduces to

$$[\partial,\,\bar\partial^\star] = 0 \quad\text{and}\quad [\bar\partial,\,\partial^\star] = 0, \tag{1.52}$$

meaning that ∂ anti-commutes with $\bar\partial^\star$ and $\bar\partial$ anti-commutes with $\partial^\star$.

We can now prove the following fundamental fact that underlies the theory of $\partial\bar\partial$-manifolds.

Theorem 1.3.20. *(the* $\partial\bar\partial$**-lemma***) Every compact* **Kähler** *manifold is a* $\partial\bar\partial$**-manifold**.

Proof. Let $(X,\,\omega)$ be a compact Hermitian manifold. Fix a bidegree $(p,\,q)$ and let $u \in C^\infty_{p,\,q}(X,\,\mathbb{C})$ such that $du = 0$. Thus, $\partial u = 0$ and $\bar\partial u = 0$.

Notice that the following implications always hold, even if the metric ω is not Kähler:

$$u \in \operatorname{Im}(\partial\bar\partial) \implies u \in \operatorname{Im}\partial \quad\text{and}\quad u \in \operatorname{Im}\bar\partial \quad\text{and}\quad u \in \operatorname{Im} d.$$

(Fot the last one, note that, if $u = \partial\bar\partial v$, then $u = d(\bar\partial v)$.)

Similarly, the following equivalences always hold, even if the metric ω is not Kähler:

$$\begin{aligned} u \in \operatorname{Im}\bar\partial &\iff u \perp \mathcal{H}^{p,\,q}_{\Delta''}(X,\,\mathbb{C}) \\ u \in \operatorname{Im}\partial &\iff u \perp \mathcal{H}^{p,\,q}_{\Delta'}(X,\,\mathbb{C}) \\ u \in \operatorname{Im} d &\iff u \perp \mathcal{H}^{p+q}_{\Delta}(X,\,\mathbb{C}). \end{aligned}$$

Now, suppose that ω is *Kähler*. Thanks to (1.51), the above equivalences imply the following equivalences:

$$u \in \operatorname{Im}\bar\partial \iff u \in \operatorname{Im}\partial \iff u \in \operatorname{Im} d,$$

so it remains to prove the implication:

$$u \in \mathrm{Im}\,\bar\partial \implies u \in \mathrm{Im}\,(\partial\bar\partial).$$

To prove this implication, suppose that $u = \bar\partial v$ for some $v \in C^\infty_{p,\,q-1}(X,\,\mathbb{C})$. Thanks to the L^2_ω-orthogonal 3-space decomposition

$$C^\infty_{p,\,q-1}(X,\,\mathbb{C}) = \ker\Delta' \oplus \mathrm{Im}\,\partial \oplus \mathrm{Im}\,\partial^\star,$$

the form v splits uniquely as $v = w + \partial\alpha + \partial^\star\beta$, with $w \in \ker\Delta'$. We get

$$u = \bar\partial v = \bar\partial w + \bar\partial\partial\alpha + \bar\partial\partial^\star\beta. \tag{1.53}$$

Now, (1.51) implies the first identity in

$$w \in \ker\Delta' = \ker\Delta'' = \ker\bar\partial \cap \ker\bar\partial^\star.$$

In particular, $\bar\partial w = 0$. On the other hand, from (1.52) we get: $\bar\partial\partial^\star\beta = -\partial^\star\bar\partial\beta$.

In the light of the last two pieces of information, from (1.53) we get:

$$\ker\partial \ni u + \partial\bar\partial\alpha = \partial^\star(-\bar\partial\beta) \in \mathrm{Im}\,\partial^\star$$

Since $\ker\partial \perp \mathrm{Im}\,\partial^\star$, we get $u + \partial\bar\partial\alpha = 0$, hence $u = -\bar\partial\partial\alpha \in \mathrm{Im}\,(\partial\bar\partial)$. $\square$

1.3.3 A counter-example: the Iwasawa manifold

We have seen so far that, for a compact complex manifold X, the following implications hold:

$$X \text{ is Kähler} \implies X \text{ is } \partial\bar\partial \implies E_1(X) = E_\infty(X).$$

We will see later on that, when $\dim_{\mathbb{C}} X \geq 3$, all these implications are strict, but let us now point out an example where even the weakest of these properties is not satisfied. Historically, this was the first example of a compact complex manifold X for which $E_1(X) \neq E_\infty(X)$.

Definition 1.3.21. *The* **Iwasawa manifold** $X = G/\Gamma$*, denoted sometimes by* $I^{(3)}$*, is the compact complex manifold of complex dimension* 3 *defined as the quotient of the Heisenberg group*

$$G := \left\{\begin{pmatrix}1 & z_1 & z_3\\ 0 & 1 & z_2\\ 0 & 0 & 1\end{pmatrix}\ ;\ z_1, z_2, z_3 \in \mathbb{C}\right\} \subset GL_3(\mathbb{C})$$

by its discrete subgroup $\Gamma \subset G$ *of matrices with entries* $z_1, z_2, z_3 \in \mathbb{Z}[i]$*.*

The *Heisenberg group* G is a simply connected, connected *complex* Lie group whose complex manifold structure is inherited from $\mathbb{C}^3$ via the obvious diffeomorphism $G \simeq \mathbb{C}^3$ and whose group operation is the multiplication of matrices

$$\begin{pmatrix}1 & z_1 & z_3\\ 0 & 1 & z_2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & w_1 & w_3\\ 0 & 1 & w_2\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & z_1 + w_1 & z_3 + z_1w_2 + w_3\\ 0 & 1 & z_2 + w_2\\ 0 & 0 & 1\end{pmatrix}.$$

Since the holomorphic 1-form on G

$$G \ni M \mapsto M^{-1}\, dM$$

is invariant under the action of Γ, it descends to a holomorphic 1-form on X. An elementary calculation shows that

$$\text{if} \quad M = \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{then} \quad M^{-1}\, dM = \begin{pmatrix} 0 & dz_1 & dz_3 - z_1\, dz_2 \\ 0 & 0 & dz_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We get *holomorphic* 1-forms on the Iwasawa manifold X induced by the following forms on $\mathbb{C}^3$:

$$\alpha = \varphi_1 := dz_1, \quad \beta = \varphi_2 := dz_2, \quad \gamma = \varphi_3 := dz_3 - z_1 dz_2. \tag{1.54}$$

Denoting the induced forms by the same symbols $\varphi_1, \varphi_2, \varphi_3$ (or α, β, resp. γ), it is obvious that

$$d\varphi_1 = d\varphi_2 = 0 \quad \text{while} \quad d\varphi_3 = -\varphi_1 \wedge \varphi_2 \neq 0 \quad \text{on} \quad X. \tag{1.55}$$

Since the holomorphic 1-form φ_3 on X is *not d-closed*, we conclude the following fact via Proposition 1.2.14.

Proposition 1.3.22. *The Frölicher spectral sequence of the Iwasawa manifold does not degenerate at E_1.*

In particular, the Iwasawa manifold is not a $\partial\bar\partial$-manifold, hence not a Kähler manifold.

Due to the key role played by the Iwasawa manifold in non-Kähler geometry, we will now give a rundown of its basic properties that will be fleshed out throughtout the book.

The map $(z_1, z_2, z_3) \mapsto (z_1, z_2)$ factors through the action of Γ to a (holomorphically locally trivial) proper holomorphic submersion

$$\pi : X \to B,$$

where the base $B = \mathbb{C}^2/\mathbb{Z}[i] \oplus \mathbb{Z}[i] = \mathbb{C}/\mathbb{Z}[i] \times \mathbb{C}/\mathbb{Z}[i]$ is a two-dimensional Abelian variety (the product of two elliptic curves) and where all the fibres are isomorphic to the Gauss elliptic curve $\mathbb{C}/\mathbb{Z}[i]$. This description displays the non-existence on X of curves normalised by smooth rational curves, as any map from such a curve to any factor $\mathbb{C}/\mathbb{Z}[i]$ would be constant. Indeed, thanks to the Riemann-Hurwitz formula, any non-constant map between two smooth curves is genus-decreasing.)

From the exact sequence

$$0 \to \pi^\star\Omega^1_B \to \Omega^1_X \to \Omega^1_{X/B} \to 0,$$

as the map $H^1(\pi^\star\Omega^1_B) = H^1(\mathcal{O}_X) \otimes H^0(\pi^\star\Omega^1_B) \to H^1(\mathcal{O}_X) \otimes H^0(\Omega^1_X) = H^1(\Omega^1_X)$ is injective due to the triviality of Ω^1_B and Ω^1_X, we get the simple presentation

$$0 \to H^0(\pi^\star\Omega^1_B) \to H^0(\Omega^1_X) \to H^0(\Omega^1_{X/B}) \to 0.$$

Thus, the form γ (also denoted by φ_3) is a representative of $H^0(\Omega^1_{X/B})$ in $H^0(\Omega^1_X)$. In other words, the forms α and β are *horizontal* (i.e. coming from B), while γ is *vertical* (i.e. lives on the fibres).

It follows that the De Rham cohomology of the Iwasawa manifold X reads

$$\begin{aligned}
H^1_{DR}(X,\mathbb{C}) &= \left\langle \{\alpha\}_{DR},\, \{\beta\}_{DR}, \{\bar\alpha\}_{DR},\, \{\bar\beta\}_{DR} \right\rangle = \pi^\star H^1_{DR}(B,\mathbb{C}),\\
\pi^\star H^2_{DR}(B,\mathbb{C}) &= \left\langle \{\alpha\wedge\bar\alpha\}_{DR},\, \{\alpha\wedge\bar\beta\}_{DR}, \{\beta\wedge\bar\alpha\}_{DR},\, \{\beta\wedge\bar\beta\}_{DR} \right\rangle \simeq H^{1,1}_{BC}(X,\,\mathbb{C}) \simeq \pi^\star H^{1,1}(B,\mathbb{C})\\
H^2_{DR}(X,\mathbb{C}) &= \pi^\star H^2_{DR}(B,\mathbb{C}) \oplus \left\langle \{\gamma\wedge\alpha\}_{DR},\, \{\gamma\wedge\beta\}_{DR} \right\rangle \oplus \left\langle \{\bar\gamma\wedge\bar\alpha\}_{DR},\, \{\bar\gamma\wedge\bar\beta\}_{DR} \right\rangle,\\
\pi^\star H^3_{DR}(B,\mathbb{C}) &= 0,\\
H^3_{DR}(X,\mathbb{C}) &= \left\langle \{\alpha\wedge\beta\wedge\gamma\}_{DR} \right\rangle \oplus \{\gamma\wedge\pi^\star H^{1,1}(B,\mathbb{C})\}_{DR} \oplus \{\bar\gamma\wedge\pi^\star H^{1,1}(B,\mathbb{C})\}_{DR}\\
&\quad \oplus \left\langle \{\bar\alpha\wedge\bar\beta\wedge\bar\gamma\}_{DR} \right\rangle\\
H^4_{DR}(X,\mathbb{C}) &= \left\langle \{\alpha\wedge\beta\wedge\gamma\wedge\bar\alpha\}_{DR},\, \{\alpha\wedge\beta\wedge\gamma\wedge\bar\beta\}_{DR} \right\rangle\\
&\quad \oplus \left\langle \{\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma\}_{DR},\, \{\alpha\wedge\gamma\wedge\bar\beta\wedge\bar\gamma\}_{DR},\, \{\beta\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma\}_{DR},\, \{\beta\wedge\gamma\wedge\bar\beta\wedge\bar\gamma\}_{DR} \right\rangle\\
&\quad \oplus \left\langle \{\alpha\wedge\bar\alpha\wedge\bar\beta\wedge\bar\gamma\}_{DR},\, \{\beta\wedge\bar\alpha\wedge\bar\beta\wedge\bar\gamma\}_{DR} \right\rangle. \qquad (1.56)
\end{aligned}$$

Furthermore, thanks to (1.55), the triple Massey product of the De Rham cohomology classes $\{\alpha\}, \{\beta\}, \{\beta\} \in H^1_{DR}(X,\,\mathbb{C})$ is

$$\langle\alpha,\beta,\beta\rangle = \{\beta\wedge\gamma\}_{DR} \in H^2_{DR}(X,\mathbb{C})/\{\alpha\}\cup H^1_{DR}(X,\,\mathbb{C}) + \{\beta\}\cup H^1_{DR}(X,\mathbb{C}).$$

Thus, $\langle\alpha,\beta,\beta\rangle \neq 0$ thanks to (1.56) and to $\{\alpha\}\cup H^1_{DR}(X,\,\mathbb{C}) + \{\beta\}\cup H^1_{DR}(X,\mathbb{C}) = \pi^\star H^2_{DR}(B,\mathbb{C})$. Therefore, we get the following strenghtening of a part of Proposition 1.3.22.

Proposition 1.3.23. *No complex structure on the C^∞ manifold underlying the Iwasawa manifold (in particular, no deformation of X) is Kähler or even $\partial\bar\partial$.*

It is known ([Nak75], [Sch07], [Ang11]) that the forms α, β, γ generate the entire cohomology of X. For example, we shall need the following descriptions in terms of generators of the following cohomology groups:

$$\begin{aligned}
H^{1,0}_{\bar\partial}(X,\mathbb{C}) &= \left\langle [\alpha]_{\bar\partial},\, [\beta]_{\bar\partial},\, [\gamma]_{\bar\partial} \right\rangle, \quad H^{0,1}_{\bar\partial}(X,\mathbb{C}) = \left\langle [\overline\alpha]_{\bar\partial},\, [\overline\beta]_{\bar\partial} \right\rangle = \pi^\star H^{0,1}_{\bar\partial}(B,\mathbb{C}),\\
H^{1,1}_{\bar\partial}(X,\,\mathbb{C}) &= \left\langle [\alpha\wedge\overline\alpha]_{\bar\partial},\, [\alpha\wedge\overline\beta]_{\bar\partial},\, [\beta\wedge\overline\alpha]_{\bar\partial},\, [\beta\wedge\overline\beta]_{\bar\partial},\, [\gamma\wedge\overline\alpha]_{\bar\partial},\, [\gamma\wedge\overline\beta]_{\bar\partial} \right\rangle,\\
H^{3,0}_{\bar\partial}(X,\mathbb{C}) &= \left\langle [\alpha\wedge\beta\wedge\gamma]_{\bar\partial} \right\rangle, \quad H^{0,3}_{\bar\partial}(X,\mathbb{C}) = \left\langle [\overline\alpha\wedge\overline\beta\wedge\overline\gamma]_{\bar\partial} \right\rangle,\\
H^{2,1}_{\bar\partial}(X,\,\mathbb{C}) &= \left\langle [\alpha\wedge\gamma\wedge\overline\alpha]_{\bar\partial},\, [\alpha\wedge\gamma\wedge\overline\beta]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline\alpha]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline\beta]_{\bar\partial} \right\rangle \oplus \left\langle [\alpha\wedge\beta\wedge\overline\alpha]_{\bar\partial},\, [\alpha\wedge\beta\wedge\overline\beta]_{\bar\partial} \right\rangle\\
&= [\gamma\wedge\pi^\star H^{1,1}_{\bar\partial}(B,\,\mathbb{C})]_{\bar\partial} \oplus \pi^\star H^{2,1}_{\bar\partial}(B,\,\mathbb{C}), \qquad (1.57)\\
H^{1,2}_{\bar\partial}(X,\,\mathbb{C}) &= \left\langle [\alpha\wedge\overline\alpha\wedge\overline\gamma]_{\bar\partial},\, [\beta\wedge\overline\alpha\wedge\overline\gamma]_{\bar\partial},\, [\alpha\wedge\overline\beta\wedge\overline\gamma]_{\bar\partial},\, [\beta\wedge\overline\beta\wedge\overline\gamma]_{\bar\partial} \right\rangle \oplus \left\langle [\gamma\wedge\overline\alpha\wedge\overline\gamma]_{\bar\partial},\, [\gamma\wedge\overline\beta\wedge\overline\gamma]_{\bar\partial} \right\rangle.
\end{aligned}$$

On the other hand, since G is a connected, simply connected, *nilpotent* complex Lie group, X is what is called a *nilmanifold*. Furthermore, X is a *complex parallelisable* compact complex manifold (i.e. its holomorphic tangent bundle $T^{1,\,0}X$ is trivial – see §.4.5.3, including Definition 4.5.29 and Theorem 4.5.30, as well as §.8, for further details). In particular, its canonical bundle K_X is trivial, so X is a *Calabi-Yau manifold* in the generalised sense that will be adopted throughout this book.

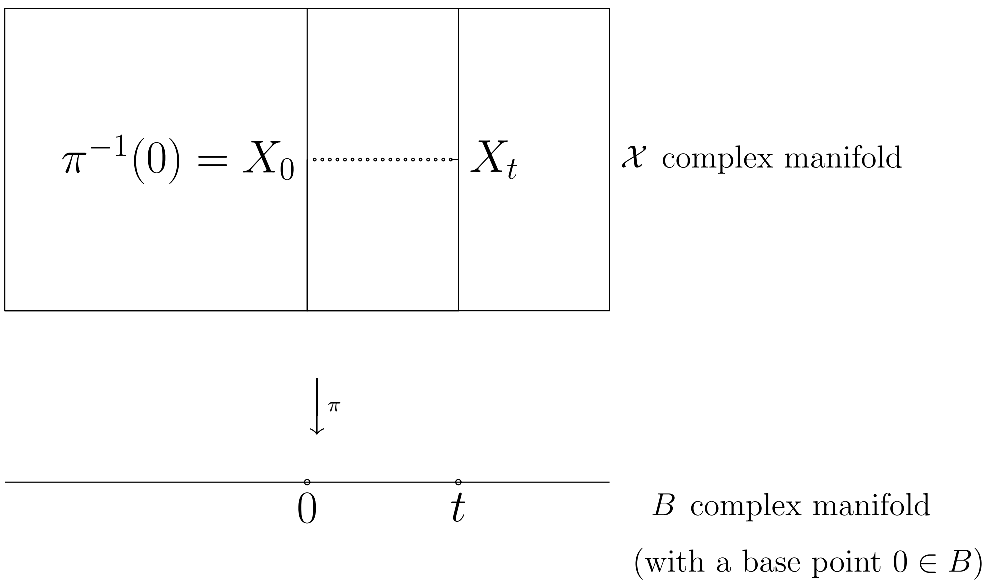
Chapter 2

Kodaira-Spencer Deformation Theory

The main object of study in this chapter and throughout much of this book is the following concept.

Definition 2.0.1. *A* **holomorphic family** *of compact complex manifolds is a* **proper holomorphic submersion** $\pi : \mathcal{X} \longrightarrow B$ *between complex manifolds* $\mathcal{X}$ *and* B.

In this context, $\mathcal{X}$ is called the *total space*, while B is called the *base*, of the family. Above every point $t \in B$ there lies a compact complex manifold $X_t := \pi^{-1}(t) \subset \mathcal{X}$, called the *fibre* above t. Of course, the smoothness of X_t follows from π being submersive, while the compactness of X_t follows from π being proper. Thus, we have a family $(X_t)_{t\in B}$ of equidimensional compact complex manifolds parametrised by the points of the base B. We usually let $m = \dim_{\mathbb{C}} B$ and $n = \dim_{\mathbb{C}} X_t$ for $t \in B$.



A common situation occurs when the base B is an open ball about the origin in some $\mathbb{C}^m$ or, more generally, when a base point $0 \in B$ has been fixed. We can then take the fibre above $0 \in B$ as a reference fibre and view the fibres X_t for $t \in B$ sufficiently close to 0 as *small deformations* of X_0. If t is allowed to lie anywhere in B, the family $\pi : \mathcal{X} \longrightarrow B$ can be seen as a family of *holomorphic deformations* of X_0.

On the other hand, if the proper submersion $\pi : \mathcal{X} \longrightarrow B$ is only assumed to be C^∞ and $\mathcal{X}$ and

B are only assumed to be C^∞ (not necessarily complex analytic) manifolds, we have a C^∞ *family of compact differentiable manifolds* or a C^∞ *family of deformations* of X_0.

2.1 Ehresmann's theorem

The fundamental fact of life in deformation theory is the following

Theorem 2.1.1. *(Ehresmann [Ehr47]) (i) Every holomorphic family of compact complex manifolds is* **locally** C^∞ **trivial** *in the following sense.*

There exists a C^∞ manifold X such that every point $t_0 \in B$ has an open neighbourhood $U \subset B$ for which there exists a C^∞ **diffeomorphism**

$$T : \mathcal{X}_U \longrightarrow X \times U \quad \textit{such that} \quad pr_2 \circ T = \pi,$$

where $\mathcal{X}_U = \pi^{-1}(U) \subset \mathcal{X}$ and $pr_2 : X \times U \longrightarrow U$ is the projection on the second factor.

(ii) If the base B is **contractible**, *the family is even* **globally** C^∞ **trivial** *in the sense that there exists a C^∞ manifold X and a C^∞* **diffeomorphism**

$$T : \mathcal{X} \longrightarrow X \times B \quad \textit{such that} \quad pr_2 \circ T = \pi,$$

where $pr_2 : X \times B \longrightarrow B$ is the projection on the second factor.

(iii) Suppose that the base B of the family is an open ball about the origin in some $\mathbb{C}^m$.

The local trivialisation $T = (T_0, \pi) : \mathcal{X} \longrightarrow X_0 \times B$ of (i), obtained after possibly replacing B by a neighbourhood U of $0 \in B$, can be chosen such that the fibres of the map $T_0 : \mathcal{X} \longrightarrow X_0$ are **complex submanifolds** *of $\mathcal{X}$.*

For the proof, we will follow [Dem96, I.§.10] and [Voi02, §.9.1.1], but let us first make a few

Comments.

(1) The fact that the submanifolds $T_0^{-1}(x) \subset \mathcal{X}$, with $x \in X_0$, are **complex** (i.e. **holomorphic**) says that the family $(J_t)_{t\in B}$ of complex structures of the fibres $(X_t)_{t\in B}$ varies **holomorphically** with $t \in B$. Meanwhile, (i) implies that, at least locally, all the fibres X_t are C^∞-diffeomorphic to a fixed C^∞ manifold X:

$$X_t \overset{C^\infty}{\simeq} X, \quad t \in U.$$

Thus, Theorem 2.1.1 says that locally, and also globally if the base B is contractible, giving a holomorphic family $\pi : \mathcal{X} \longrightarrow B$ of compact complex manifolds is equivalent to giving a C^∞ manifold X equipped with a holomorphic family $(J_t)_{t\in B}$ of complex structures.

(2) The proof, which will be seen to be an easy consequence of the classical *Tubular Neighbourhood Theorem*, will show that parts (i) and (ii) of Theorem 2.1.1 remain true when the family $\pi : \mathcal{X} \longrightarrow B$ is only supposed to be C^∞.

(3) Giving a C^∞ trivialisation $T = (T_0, \pi) : \mathcal{X} \longrightarrow X_0 \times B$ as in Theorem 2.1.1 (after possibly shrinking B about its base point 0 and taking the reference C^∞ manifold X to be the C^∞ manifold

underlying X_0 – a posteriori, it will be the same C^∞ manifold underlying any fibre X_t sufficiently close to X_0) is equivalent to giving its first component

$$T_0 : \mathcal{X} \longrightarrow X_0.$$

Moreover, up to composing T_0 with $(T_{0|X_0})^{-1}$, we may assume that

$$T_{0|X_0} = \mathrm{Id}_{X_0}.$$

In other words, we may assume that T_0 is a *retraction* of $\mathcal{X}$ onto X_0.

(4) Part (iii) of Theorem 2.1.1 implies that, for every $x \in X_0$, the T_0-fibre through x is a *complex submanifold* of $\mathcal{X}$ that is *biholomorphic* to B via

$$\pi_{|T_0^{-1}(x)} : T_0^{-1}(x) \stackrel{\simeq}{\longrightarrow} B.$$

However, the complex submanifolds $(T_0^{-1}(x))_{x\in X_0}$ do not vary holomorphically with $x \in X_0$ because the map $T_0 : \mathcal{X} \longrightarrow X_0$ is not holomorphic, hence the fibres X_t are not bimeromorphically equivalent, in general. Nevertheless, the C^∞-diffeomorphisms

$$T_{0|X_t} : X_t \longrightarrow X_0, \qquad t \in B,$$

enable one to view the complex structures of the fibres X_t as complex structures J_t on X_0 (or, equivalently, on the C^∞ manifold X that underlies all the fibres X_t) that vary in a *holomorphic* way with $t \in B$.

(5) Since, in general, the complex structure J_t of X_t varies (albeit holomorphically) with $t \in B$, the Dolbeault, E_r, Bott-Chern and Aeppli cohomology spaces of the fibres X_t, as well as their dimensions, change with t.

However, since it depends only on the differential structure of the fibres, which is locally constant, the De Rham cohomology of X_t is *locally constant* in the sense that we can identify:

$$H^k_{DR}(X_t, \mathbb{C}) = H^k_{DR}(X, \mathbb{C}), \qquad k \in \{0, \dots, 2n\},$$

for all t in a small enough neighbourhood of any given point $t_0 \in B$.

Proof of Theorem 2.1.1.

The arguments for the proofs of (i) and (iii), resp. (ii), are those given in [Voi02, §.9.1.1], resp. [Dem96, I.§.10].

(i) We may assume that $t_0 = 0$. To find a local trivialisation $T = (T_0, \pi) : \mathcal{X}_U \longrightarrow X_0 \times U$, we need to find a submersion $T_0 : \mathcal{X}_U \longrightarrow X_0$ (i.e. a way of projecting some $\mathcal{X}_U$ onto X_0) with an extra property. If $\mathcal{X}_U$ were the total space of a bundle and X_0 one of its fibres, we could choose T_0 to be the bundle projection. However, we are almost there since the Tubular Neighbourhood Theorem identifies a small neighbourhood of X_0 in the ambient manifold $\mathcal{X}$ with a small neighbourhood of the said X_0 in its normal bundle in $\mathcal{X}$. So, all we have to do is to apply this classical theorem.

Since the manifolds $\mathcal{X}$ and X are real analytic (and even more), the *real analytic Tubular Neighbourhood Theorem* yields:

-an open neighbourhood W of X_0 in the total space of its *normal bundle* $N_{\mathcal{X}}X_0$ in $\mathcal{X}$ (i.e. an open neighbourhood W of the zero section of $N_{\mathcal{X}}X_0$);

-an open neighbourhood V of X_0 in $\mathcal{X}$;

-a *real analytic diffeomorphism* $\psi : W \stackrel{\simeq}{\longrightarrow} V$.

When the family $\pi : \mathcal{X} \longrightarrow B$ is only C^∞, we apply the usual *Tubular Neighbourhood Theorem* that yields a C^∞ diffeomorphism $\psi : W \stackrel{\simeq}{\longrightarrow} V$.

• Let us recall briefly the outline of the proof of this standard theorem. Fix a *real analytic* metric ω on $\mathcal{X}$ in the neighbourhood of X_0 and consider the induced *geodesic map*:

$$N_{\mathcal{X}}X_0 \supset W \ni (x,\, u) \stackrel{\psi}{\mapsto} \gamma_u(1) \in V \subset \mathcal{X},$$

where W is an open neighbourhood of X_0 in $N_{\mathcal{X}}X_0$, $x \in X_0$ and

$$u \in (N_{\mathcal{X}}X_0)_x \stackrel{\omega}{\simeq} \{u \in T_x\mathcal{X} \;\; | \;\; u \perp T_xX_0\},$$

while $\gamma_u(1)$ is the endpoint of the unique *geodesic* γ_u starting at x with tangent vector u. The orthogonality condition $u \perp T_xX_0$ is w.r.t. the metric ω and so is the identification between the fibre of the normal bundle at x and the orthogonal complement of the tangent space to X_0 at x in the tangent space to $\mathcal{X}$ at x.

The map ψ is the identity map on X_0 (i.e. when $u = 0$) and its differential at every point of X_0 is an isomorphism. Hence, by the (real analytic) Local Inversion Theorem, ψ maps an open neighbourhood W of X_0 in $N_{\mathcal{X}}X_0$ (shrink the original W if necessary) onto an open neighbourhood V of X_0 in $\mathcal{X}$ (again, shrink the original V if necessary).

• We now go back to the proof of Theorem 2.1.1. Denote by $\sigma : N_{\mathcal{X}}X_0 \longrightarrow X_0$ the normal bundle projection map and consider the composition $T_0 := \sigma \circ \psi^{-1} : V \longrightarrow X_0$ defined by

$$V \stackrel{\psi^{-1}}{\longrightarrow} W \stackrel{\sigma}{\to} X_0.$$

Thus, T_0 is a *real analytic retraction* of V onto X_0.

Consequently, since π is also a submersion, the differential of the map $(T_0,\, \pi) : V \longrightarrow X_0 \times B$ is *invertible* along X_0. By the (real analytic) Local Inversion Theorem and the *compactness* of X_0, there exists an open neighbourhood V' of X_0 in V such that

$$(T_0,\, \pi)_{|V'} : V' \longrightarrow X_0 \times B$$

is a (real analytic) *embedding*.

Now, π is *proper* and $\pi^{-1}(0) = X_0 \subset V'$, so there exists an open neighbourhood $U \subset B$ of 0 such that $V'' := \pi^{-1}(U) \subset V'$. Hence, $(T_0,\, \pi)(V'') = X_0 \times U$, and therefore the map

$$T := (T_0,\, \pi)_{|V''} : V'' = \pi^{-1}(U) \longrightarrow X_0 \times U$$

is a *(real analytic) diffeomorphism* satisfying the condition $pr_2 \circ T = \pi$. This proves (i).

(iii) For every $x \in X_0$, the map

$$\psi_x : W_x := (N_{\mathcal{X}}X_0)_x \cap W \longrightarrow \mathcal{X}$$

is *real analytic*, hence it admits a power series expansion in any arbitrarily chosen $\mathbb{R}$-linear coordinates on the *complex* vector space $(N_{\mathcal{X}}X_0)_x$. Let

$$\psi_x^h : W_x \longrightarrow \mathcal{X}$$

be the *holomorphic* map obtained by retaining the *holomorphic part* of this power series expansion of ψ_x.

Now, ψ_x^h is independent of the choice of coordinates and varies in a C^∞ way with $x \in X_0$, so we get a C^∞ map

$$\psi^h : W \longrightarrow \mathcal{X}$$

which is *holomorphic* on the fibres of σ (i.e. the fibres of the normal bundle $N_\mathcal{X} X_0$).

Moreover, the differential $d_x(\pi \circ \psi)$ of $\pi \circ \psi$ at every point $x \in X_0$ is $\mathbb{C}$-*linear* (and, of course, surjective) because the differential $d_x\psi$ of ψ induces the canonical isomorphism between $(N_N X_0)_x$ and $(N_\mathcal{X} X_0)_x$. Therefore, the differential $d_x(\pi \circ \psi)$ of $\pi \circ \psi$ at every point $x \in X_0$ equals the differential $d_x(\pi \circ \psi^h)$ of $\pi \circ \psi^h$ at x. So, the latter differential must be *surjective.* Therefore, by the Local Inversion Theorem, ψ^h is a local diffeomorphism from some neighbourhood W' of X_0 in $N_\mathcal{X} X_0$ to $\mathcal{X}$.

It remains to put

$$T := (\sigma \circ (\psi^h)^{-1},\, \pi) : \mathcal{X} \longrightarrow X_0 \times B$$

to get the desired trivialisation required to prove (iii).

(ii) Suppose that B is *contractible* and fix a base point $0 \in B$. Then, there exists a C^∞ *homotopy*:

$$H : B \times [0,\, 1] \longrightarrow B \quad \text{such that} \quad H(\bullet,\, 0) = \mathrm{Id}_B \quad \text{and} \quad H(\bullet,\, 1) = (B \longrightarrow \{0\}),$$

where $(B \longrightarrow \{0\})$ stands for the constant map sending all the points in B to $0 \in B$.

Let us now consider the C^∞ manifold

$$\widetilde{\mathcal{X}} := \bigg\{(x,\, t,\, s) \in \mathcal{X} \times B \times [0,\, 1] \;\mid\; \pi(x) = H(t,\, s)\bigg\}.$$

Note that we can view $\mathcal{X}$ as the set of pairs $(x,\, t)$ such that $t \in B$ and $x \in X_t$. The condition $x \in X_t$ is obviously equivalent to $\pi(x) = t$. Similarly, we can view $\widetilde{\mathcal{X}}$ as the set of triples $(x,\, t,\, s)$ such that $(x,\, H(t,\, s)) \in \mathcal{X}$, or equivalently such that $x \in X_{H(t,\, s)}$. In particular, if we put

$$\widetilde{\mathcal{X}}_{|B\times\{s\}} := \{(x,\, t,\, s') \in \widetilde{\mathcal{X}} \;\mid\; s' = s\}, \qquad \text{for all} \quad s \in [0,\, 1],$$

we get $\widetilde{\mathcal{X}}_{|B\times\{0\}} = \mathcal{X}$ and $\widetilde{\mathcal{X}}_{|B\times\{1\}} = X_0 \times B$.

Thus, proving (ii) reduces to finding a C^∞ diffeomorphism $T : \widetilde{\mathcal{X}}_{|B\times\{0\}} \longrightarrow \widetilde{\mathcal{X}}_{|B\times\{1\}}$ such that $pr_2 \circ T = \pi$.

Now, the map $\tilde{\pi} := pr_2 \times pr_3 : \widetilde{\mathcal{X}} \longrightarrow B \times [0,\, 1]$ is still a C^∞ *submersion*, as can easily be checked. Therefore, the vector field $\partial/\partial s$ on $B \times [0,\, 1]$ lifts under $\tilde{\pi}$ to a vector field ξ on $\widetilde{\mathcal{X}}$ in the sense that

$$\tilde{\pi}_\star \xi = \frac{\partial}{\partial s}.$$

(For the fact that currents and vector fields have well-defined pullbacks or lifts under *submersions*, thanks to the possibility of integrating differential forms along the (necessarily smooth) fibres of a submersion and to a generalised Fubini theorem involving fibrewise integrations, the reader is referred to [Dem97, I§.2.15].)

Let φ_s be the *flow* of the vector field $\xi \in C^\infty(\widetilde{\mathcal{X}},\, T\widetilde{\mathcal{X}})$. Then, for every $s \in [0,\, 1]$,

$$\varphi_s : \widetilde{\mathcal{X}}_{|B\times\{0\}} \longrightarrow \widetilde{\mathcal{X}}_{|B\times\{s\}}, \qquad \varphi_s(x,\, t,\, 0) = (\rho_s(x),\, t,\, s),$$

is an *isomorphism* that commutes with the projection on B, where ρ_s is a smooth function of x.

Therefore, we can choose $T = \varphi_1 : \widetilde{\mathcal{X}}_{|B\times\{0\}} = \mathcal{X} \longrightarrow \widetilde{\mathcal{X}}_{|B\times\{1\}} = X_0 \times B$ and we are done. □

2.2 The Kodaira-Spencer map

Let $\pi : \mathcal{X} \longrightarrow B$ be a *holomorphic family* of compact complex manifolds. Fix an arbitrary base point $0 \in B$. The differential map

$$d\pi : T^{1,\,0}\mathcal{X} \longrightarrow \pi^\star(T^{1,\,0}B)$$

is a morphism of *holomorphic vector bundles* over $\mathcal{X}$. Since $X_0 = \pi^{-1}(0) \subset \mathcal{X}$, we have

$$T^{1,\,0}X_0 = \ker\bigg((d\pi)_{|X_0}\bigg),$$

so we get an *exact sequence* of *holomorphic vector bundles* over X_0:

$$0 \longrightarrow T^{1,\,0}X_0 \longrightarrow T^{1,\,0}\mathcal{X}_{|X_0} \xrightarrow{d\pi} \pi^\star(T^{1,\,0}B)_{|X_0} \longrightarrow 0. \tag{2.1}$$

Meanwhile, $\pi^\star(T^{1,\,0}B)_{|X_0} = X_0 \times T_0^{1,\,0}B$ is the *trivial* holomorphic vector bundle over X_0 of fibre $T_0^{1,\,0}B$. Therefore, the exact sequence (2.1) defines an extension of the holomorphic vector bundle $T^{1,0}X_0$ by the trivial holomorphic vector bundle of fibre $T_0^{1,\,0}B$. This extension is equivalent to the *connecting morphism*

$$\rho : T_0^{1,\,0}B = H^0(X_0,\, \pi^\star(T^{1,\,0}B)_{|X_0}) \longrightarrow H^1(X_0,\, \mathcal{O}_{X_0}(T^{1,\,0}X_0)) \simeq H^{0,\,1}(X_0,\, T^{1,\,0}X_0)$$

that is part of the long exact sequence associated with (2.1).

Definition 2.2.1. *The linear map $\rho : T_0^{1,\,0}B \longrightarrow H^{0,\,1}(X_0,\, T^{1,\,0}X_0)$ is called the* **Kodaira-Spencer map at** 0 *of the family $\pi : \mathcal{X} \longrightarrow B$.*

The main interest in the Kodaira-Spencer map stems from the following loosely stated principle that will be made precise in the next two subsections.

Fact 2.2.2. *The Kodaira-Spencer map at 0 can be seen as the* **differential** *at $t = 0$ of the map*

$$B \ni t \mapsto J_t, \tag{2.2}$$

where J_t is the complex structure of the fibre X_t.

In other words, the Kodaira-Spencer map is the classifying map for the 1-*st order deformations* of (the complex structure of) X_0.

2.2.1 The formal approach

The presentation in this subsection will follow that of [Voi02, §.9.1.2], as has the one above in §.2.2.

• Let $X := X_0$ and let $x \in X$ be an arbitrarily fixed point. We start by reinterpreting J_t at x as a form $\alpha_t \in \Omega^{0,1}_{X,x} \otimes T^{1,0}_x X = \mathrm{Hom}(T^{0,1}_x X, T^{1,0}_x X)$.

Let $T = (T_0, \pi) : \mathcal{X} \xrightarrow{\simeq} X_0 \times B$ be a local C^∞ trivialisation of the family near 0 given by Ehresmann's Theorem 2.1.1. (Shrink B about 0 if necessary.) For every $t \in B$, we have a C^∞ diffeomorphism

$$T_t : X_t \xrightarrow{\simeq} X_0 := X, \quad x_t \mapsto x,$$

induced by T. We fix a point $x \in X$ and let $x_t := T_t^{-1}(x) \in X_t$ for all $t \in B$. The differential at x_t of this diffeomorphism defines an isomorphism of $\mathbb{C}$-vector spaces

$$(dT_t)_{x_t} : T^{1,0}_{x_t} X_t \xrightarrow{\simeq} T^{1,0}_x X, \quad t \in B,$$

which associates with the complex structure J_t on $T_{x_t}X_t$ a unique complex structure I_t on T_xX in the obvious, bijective, way. In this way, we get a family

$$B \ni t \mapsto I_t$$

of complex structures on T_xX.

Now, giving I_0 is equivalent to giving the $\mathbb{C}$-vector subspace $T^{1,0}_x X \subset \mathbb{C}T_xX$ and is still equivalent to giving the direct sum decomposition

$$\mathbb{C}T_xX = T^{1,0}_x X \oplus T^{0,1}_x X.$$

Moreover, once I_0 has been fixed, one can specify any other complex structure on T_xX (in particular, every other I_t) in terms of I_0 in the following way. For every $t \in B$ close to 0, the $\mathbb{C}$-vector subspace $(T^{1,0}_x X)_t \subset \mathbb{C}T_xX$ defining I_t is actually defined by a form

$$\alpha_t \in \Omega^{0,1}_{X,x} \otimes T^{1,0}_x X = \mathrm{Hom}(T^{0,1}_x X, T^{1,0}_x X), \quad T^{0,1}_x X \ni u \mapsto \alpha_t(u) \in T^{1,0}_x X,$$

which takes every point u of $T^{0,1}_x X$ to its $(T^{0,1}_x X)_t$-*parallel projection* onto $T^{1,0}_x X$.

Indeed, if we have specified the $\mathbb{C}$-vector subspace $(T^{1,0}_x X)_t \subset \mathbb{C}T_xX$, we let $(T^{0,1}_x X)_t \subset \mathbb{C}T_xX$ be its complex conjugate w.r.t. the real structure of $\mathbb{C}T_xX$. Then, the linear map $\alpha_t : T^{0,1}_x X \longrightarrow T^{1,0}_x X$ is defined as the composition:

$$T^{0,1}_x X \xrightarrow[\simeq]{(a)} (T^{1,0}_x X)_t \xrightarrow{(b)} T^{1,0}_x X,$$

where the isomorphism (a) is the inverse of the composition of maps:

$$(T^{1,0}_x X)_t \hookrightarrow (T^{1,0}_x X)_t \oplus (T^{0,1}_x X)_t = \mathbb{C}T_xX = T^{1,0}_x X \oplus T^{0,1}_x X \twoheadrightarrow T^{0,1}_x X,$$

(so, the isomorphism (a) is the $(T^{1,0}_x X)$-*parallel* projection onto $(T^{1,0}_x X)_t$ restricted to $T^{0,1}_x X$), while the map (b) is the projection onto $T^{1,0}_x X$ restricted to $(T^{0,1}_x X)_t$.

Conversely, if we have specified a linear map $\alpha_t : T^{0,1}_x X \longrightarrow T^{1,0}_x X$, we associate with it a complex structure I_t on T_xX by declaring that the vectors of type (0, 1) for I_t are

$$(T^{0,1}_x X)_t := \{u - \alpha_t(u) \mid u \in T^{0,1}_x X\}. \tag{2.3}$$

(The vectors $u \in T^{0,1}_x X$ are, of course, the vectors of type (0, 1) for I_0.)

We conclude that, once we have fixed a point $x \in X$ and a complex structure I_0 on T_xX, giving a family of complex structures $(I_t)_{t\in B}$ on T_xX whose member for $t=0$ is I_0 is equivalent to giving a family of forms $\alpha_t \in \Omega^{0,1}_{X,x} \otimes T^{1,0}_x X = \mathrm{Hom}(T^{0,1}_x X,\, T^{1,0}_x X)$ with $t \in B$ such that $\alpha_0 = 0$.

• If we now allow the point $x \in X$ to move, we conclude that giving a holomorphic family $(J_t)_{t\in B}$ of complex structures on the fibres $(X_t)_{t\in B}$ whose member for $t=0$ is a pregiven J_0 (or, equivalently, a holomorphic family $(I_t)_{t\in B}$ of complex structures on the fixed manifold $X := X_0$ whose member for $t=0$ is a pregiven J_0) is equivalent to giving a *holomorphic* family $(\alpha_t)_{t\in B}$ of smooth $T^{1,0}X$-valued $(0,\,1)$-forms such that $\alpha_0 = 0$.

In other words, we have proved the following

Proposition 2.2.3. *Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds. Fix an arbitrary base point $0 \in B$ and, for every $\in B$, denote by J_t the complex structure of $X_t := \pi^{-1}(t)$.*

Then, the map $B \ni t \mapsto J_t$ is equivalent to the holomorphic map

$$B \ni t \overset{\alpha}{\mapsto} \alpha_t \in C^\infty_{0,1}(X,\, T^{1,0}X), \tag{2.4}$$

where $X := X_0$ and the forms α_t are uniquely associated with the complex structures J_t as explained above.

• We are now in a position to make Fact 2.2.2 precise by proving that the Kodaira-Spencer map at 0 is intimately related to the differential of the above map α at $t=0$. This provides the first reason we will discuss for thinking of the Kodaira-Spencer map at 0 as the 1*-st order variation* of the complex structure of X_0 (identified with the form α_0) as it deforms to the complex structures of the nearby fibres X_t (identified respectively with the forms α_t when t remains close to 0).

Theorem 2.2.4. *In the situation of Proposition 2.2.3, the map*

$$T^{1,0}_0 B \ni u \mapsto (d_0\alpha)(u) \in C^\infty_{0,1}(X,\, T^{1,0}X)$$

*takes values in the space of $\bar\partial$-***closed** *forms in $C^\infty_{0,1}(X,\, T^{1,0}X)$.*

Moreover, for every $u \in T^{1,0}_0 B$, the following identity holds:

$$[(d_0\alpha)(u)]_{\bar\partial} = \rho(u) \in H^{0,1}(X,\, T^{1,0}X), \tag{2.5}$$

where $\rho : T^{1,0}_0 B \longrightarrow H^{0,1}(X,\, T^{1,0}X)$ is the Kodaira-Spencer map at 0 of the family $\pi : \mathcal{X} \longrightarrow B$.

The $\bar\partial$-closedness conclusion of Theorem 2.2.4 is w.r.t. the *canonical* $(0,\,1)$-connection $\bar\partial$ of the *holomorphic* vector bundle $T^{1,0}X$ (which is the holomorphic tangent bundle of X_0). As usual, $[\ \ ]_{\bar\partial}$ denotes a $\bar\partial$-cohomology class for the canonical $\bar\partial$ of $T^{1,0}X$, while $H^{0,1}(X,\, T^{1,0}X)$ is the corresponding $\bar\partial$-cohomology space at the level of $C^\infty_{0,1}(X,\, T^{1,0}X)$.

Proof of Theorem 2.2.4. We will divide the proof into two steps.

• *Step* 1*: reinterpretation of the Kodaira-Spencer map.*

The main observation is that part (iii) of Ehresmann's Theorem 2.1.1 enables us to view the quotient bundle $\pi^\star(T^{1,0}B)$ of $T^{1,0}\mathcal{X}$ as a **subbundle** (albeit only C^∞) and thus, in particular, to get a C^∞ **splitting** of the exact sequence (2.1).

Indeed, after possibly shrinking B about 0, there exists a C^∞ trivialisation $T = (T_0,\, \pi) : \mathcal{X} \xrightarrow{\simeq} X_0 \times B$ such that, for every $x \in X_0$, $T_0^{-1}(x)$ is a *complex (i.e. holomorphic) submanifold* of $\mathcal{X}$. Therefore, the *holomorphic* tangent bundles of the complex submanifolds $T_0^{-1}(x)$ form a C^∞ *complex* *subbundle* $T^{1,0}_{\mathcal{X}/B}$ of $T^{1,0}_{\mathcal{X}}$. (Note that this subbundle is only C^∞ because the complex submanifolds $T_0^{-1}(x) \subset \mathcal{X}$ vary only in a C^∞ way with $x \in X_0$.) Moreover, we have a vector bundle isomorphism:

$$d\pi : T^{1,0}_{\mathcal{X}/B} \xrightarrow{\simeq} \pi^\star(T^{1,0}B)$$

whose inverse induces a C^∞ *morphism* of vector bundles $\sigma := j \circ (d\pi)^{-1} : \pi^\star(T^{1,0}B) \longrightarrow T^{1,0}_{\mathcal{X}}$ that makes the following diagram commutative:

$$\begin{tikzcd} \pi^\star(T^{1,0}B) \arrow[r, "\sigma"] \arrow[d, "(d\pi)^{-1}"', "\simeq"] & T^{1,0}_{\mathcal{X}} \\ T^{1,0}_{\mathcal{X}/B} \arrow[ur, "j"] & \end{tikzcd}$$

where j is the inclusion map. Note that σ defines a C^∞ *splitting* of the short exact sequence

$$0 \longrightarrow T^{1,0}_{\mathcal{X}/B} \xrightarrow{j} T^{1,0}_{\mathcal{X}} \longrightarrow \pi^\star(T^{1,0}B) \longrightarrow 0,$$

which, by restriction to X_0, defines a C^∞ *splitting* of the short exact sequence (2.1).

In particular, σ defines a C^∞ vector bundle section:

$$\sigma \in C^\infty(\mathcal{X},\, \mathrm{Hom}\,(\pi^\star(T^{1,0}B),\, T^{1,0}_{\mathcal{X}})).$$

Since the morphism vector bundle $\mathrm{Hom}\,(\pi^\star(T^{1,0}B),\, T^{1,0}_{\mathcal{X}})$ is *holomorphic*, it is equipped with a canonical $\bar\partial$ operator. When applied to σ, it induces a vector $(0,\,1)$-form

$$\bar\partial\sigma \in C^\infty_{0,1}(\mathcal{X},\, \mathrm{Hom}\,(\pi^\star(T^{1,0}B),\, T^{1,0}_{\mathcal{X}})).$$

With this construction understood, the definition of ρ as the *connecting morphism* induced by the short exact sequence (2.1) implies the following expression for ρ in terms of σ:

$$T^{1,0}_0B \ni v \mapsto \rho(v) = [(\bar\partial\sigma)(v)]_{\bar\partial}.$$

- *Step* 2*: computations in local coordinates.*

Thanks to *Step* 1, we are reduced to proving the formula

$$(\bar\partial\sigma)(v) = (d_0\alpha)(v), \qquad \text{for all} \quad v \in T^{1,0}_0B. \tag{2.6}$$

This is a local identity, so it lends itself to being proved by a computation in local coordinates. Let $n = \dim_{\mathbb{C}} X_t$ and $m = \dim_{\mathbb{C}}$.

Let $(t_1, \dots, t_m)$ be a system of local holomorphic coordinates centred at 0 on B and let $(z_1, \dots, z_n)$ be locally defined holomorphic functions on $\mathcal{X}$ such that $(z_1, \dots, z_n,\, \pi^\star t_1, \dots, \pi^\star t_m)$ is a local holomorphic coordinate system on $\mathcal{X}$. W.r.t. these coordinates, the map $\pi : \mathcal{X} \longrightarrow B$ is given by

$$(z_1, \dots, z_n,\, \pi^\star t_1, \dots, \pi^\star t_m) \overset{\pi}{\mapsto} (t_1, \dots, t_m).$$

Meanwhile, the *horizontal projection* $T_0 : \mathcal{X} \longrightarrow X_0 = X$ has the following shape in these coordinates:

$$T_0 = (T_0^{(1)}(z_1, \dots, z_n,\, t_1, \dots, t_m), \dots, T_0^{(n)}(z_1, \dots, z_n,\, t_1, \dots, t_m)),$$

where the $T_0^{(j)}$'s are C^∞ functions of $(z_1, \dots, z_n,\, t_1, \dots, t_m)$ that are *holomorphic* in the t_i's.

Thus, if we put $x := T_0(z_1, \dots, z_n, t_1, \dots, t_m)$, $(T^{0,1}_x X)_t$ is generated by

$$(dT_0)_x(\frac{\partial}{\partial \bar{z}_i}) = \frac{\partial T_0}{\partial \bar{z}_i}(x) = \sum_j \frac{\partial T_0^{(j)}}{\partial \bar{z}_i}(x)\,\frac{\partial}{\partial z_j} + \sum_j \frac{\partial \overline{T_0^{(j)}}}{\partial \bar{z}_i}(x)\,\frac{\partial}{\partial \bar{z}_j} \in T^{1,0}_x X \oplus T^{0,1}_x X,$$

because $\partial T_0/\partial \bar{t}_i = 0$ for all i. Hence, at $x := T_0(z_1, \dots, z_n, t_1, \dots, t_m)$, we get:

$$T^{0,1}_x X \ni u := \sum_j \frac{\partial \overline{T_0^{(j)}}}{\partial \bar{z}_i}(x)\,\frac{\partial}{\partial \bar{z}_j} \overset{\alpha_t}{\longmapsto} \alpha_t(u) = -\sum_j \frac{\partial T_0^{(j)}}{\partial \bar{z}_i}(x)\,\frac{\partial}{\partial z_j} \in T^{1,0}_x X$$

because, as we saw above, the vectors of type $(0,\,1)$ for $I_{t,\,x}$ are $(dT_0)_x(\partial/\partial \bar{z}_i)$ and we know, from the definition of α_t, that they are given by $u - \alpha_t(u)$ when u ranges over $T^{0,1}_x X$. (See (2.3).)

Since $T_{0|X_0} = \mathrm{Id}_{X_0}$, we have $(\partial \overline{T_0^{(j)}}/\partial \bar{z}_i)(z_1, \dots, z_n, 0, \dots, 0) = \delta_{ij}$ (the Kronecker symbol). Hence, the above formula for $\alpha_t(u)$ yields:

$$\alpha_t(\frac{\partial}{\partial \bar{z}_i}) = -\sum_j \frac{\partial T_0^{(j)}}{\partial \bar{z}_i}(x)\,\frac{\partial}{\partial z_j} \quad \text{at} \quad (z_1, \dots, z_n, 0, \dots, 0).$$

Differentiating this identity w.r.t. t_k, we find at $(z_1, \dots, z_n, 0, \dots, 0)$:

$$\frac{\partial \alpha_t}{\partial t_k}_{|t=0}\left(\frac{\partial}{\partial \bar{z}_i}\right) = -\frac{\partial}{\partial \bar{z}_i}\left(\sum_j \frac{\partial T_0^{(j)}}{\partial t_k}_{|t=0}\frac{\partial}{\partial z_j}\right) \quad \text{for all} \quad k \in \{1, \dots, m\}. \tag{2.7}$$

This computes one of the terms in (2.6). To compute the other term, we notice that the vector field $\sigma(\partial/\partial t_k)$ is the unique vector field ξ of type $(1,\,0)$ on $\mathcal{X}$ with the following two properties:

$$(dT_0)(\xi) = 0 \quad \text{and} \quad (d\pi)(\xi) = \frac{\partial}{\partial t_k} \in C^\infty(B,\, T^{1,0}B).$$

(In other words, $\sigma(\partial/\partial t_k)$ is the unique *horizontal* vector field of type $(1,\,0)$ on $\mathcal{X}$ which is a *lift* under π of the $(1,\,0)$-vector field $\partial/\partial t_k$ on B.)

Now, $(dT_0)(\frac{\partial}{\partial t_k}) = \sum\limits_j \frac{\partial T_0^{(j)}}{\partial t_k}\frac{\partial}{\partial z_j}$ and $dT_0 = \mathrm{Id}$ on X_0. Hence, we find at $(z_1, \dots, z_n, 0, \dots, 0)$:

$$\sigma(\frac{\partial}{\partial t_k}) = \frac{\partial}{\partial t_k} - \sum_j \frac{\partial T_0^{(j)}}{\partial t_k}\,\frac{\partial}{\partial z_j} \quad \text{for all} \quad k \in \{1, \dots, m\}. \tag{2.8}$$

Putting together (2.7) and (2.8), we get:

$$\frac{\partial \alpha_t}{\partial t_k}_{|t=0}\left(\frac{\partial}{\partial \bar{z}_i}\right) = \frac{\partial}{\partial \bar{z}_i}\left(\sigma(\frac{\partial}{\partial t_k})_{|t=0}\right) \quad \text{for all } k \in \{1, \dots, m\} \text{ and all } i \in \{1, \dots, n\}.$$

This is equivalent to

$$(d_0\alpha)\left(\frac{\partial}{\partial t_k}_{|t=0}\right)\left(\frac{\partial}{\partial \bar{z}_i}\right) = (\bar\partial\sigma)\left(\frac{\partial}{\partial \bar{z}_i}\right)\left(\frac{\partial}{\partial t_k}_{|t=0}\right) \quad \text{for all } k \in \{1, \dots, m\} \text{ and all } i \in \{1, \dots, n\},$$

which is further equivalent to

$$(d_0\alpha)\bigg(\frac{\partial}{\partial t_k}_{|t=0}\bigg) = (\bar\partial\sigma)\bigg(\frac{\partial}{\partial t_k}_{|t=0}\bigg) \quad \text{for all } k\in\{1,\dots,m\}.$$

This proves (2.6) and we are done. □

2.2.2 The analytic approach

The presentation in this subsection will follow that of [Kod86, §.9.1.2]. The context and the notation are the same as throughout this chapter.

The complex structure J_0 (resp. J_t) of X_0 (resp. X_t) is equivalent to the corresponding $\bar\partial$-operator, $\bar\partial_0$ (resp. $\bar\partial_t$). The point of view adopted here consists in working directly with $\bar\partial_t$ and studying its variation with t; in other words, viewing the family $\pi : \mathcal{X} \longrightarrow B$ as a holomorphic family $(\bar\partial_t)_{t\in B}$ of complex structures (or, equivalently, of $\bar\partial$-operators) on $X_0 := X$.

Expression of $\bar\partial_t$ in terms of $\bar\partial_0$

Let $U_j \subset X_0$ be a coordinate patch and let $(\zeta_j^\alpha(z,\,t))_{1\leq\alpha\leq n}$ be a local J_t*-holomorphic coordinate system* on U_j, where $t=(t_1,\dots,t_m)\in B$ are local holomorphic coordinates centred at 0 on B. On the other hand, let $(z_1,\dots,z_n)$ be a local J_0*-holomorphic coordinate system* centred at an arbitrarily given point $x\in X_0$. Thus,

$$\zeta_j^\alpha(z,\,t) = \zeta_j^\alpha(z_1,\dots,z_n,\,t_1,\dots,t_m), \qquad \alpha\in\{1,\dots,n\}$$

are C^∞ functions of the variables $z_1,\dots,z_n,\,t_1,\dots,t_m$ and

$$(\zeta_j^1(z_1,\dots,z_n,\,t_1,\dots,t_m),\dots,\zeta_j^n(z_1,\dots,z_n,\,t_1,\dots,t_m),\,t_1,\dots,t_m)$$

is a system of local *holomorphic* coordinates on $\mathcal{X}$.

In particular, for $t=0$, $(\zeta_j^1(z,\,0),\dots,\zeta_j^n(z,\,0))$ and $(z_1,\dots,z_n)$ are two systems of local J_0-*holomorphic* coordinates on X_0. Therefore, the functions $\zeta_j^\alpha(z,\,0)$ are *holomorphic* functions of $z_1,\dots,z_n$ and

$$\det\left(\frac{\partial\zeta_j^\alpha(z,\,0)}{\partial z_\lambda}\right)_{1\leq\alpha,\,\lambda\leq n} \neq 0.$$

(The last statement follows from $0\neq d\zeta_j^1(z,\,0)\wedge\cdots\wedge d\zeta_j^n(z,\,0) = \det(\;)\,dz_1\wedge\cdots\wedge dz_n$.) By continuity w.r.t. t, we get

$$\det\left(\frac{\partial\zeta_j^\alpha(z,\,t)}{\partial z_\lambda}\right)_{1\leq\alpha,\,\lambda\leq n} \neq 0 \quad \text{for all} \quad t\in B,$$

after possibly shrinking B about 0.

Lemma 2.2.5. *For every $\alpha\in\{1,\dots,n\}$ and every $t\in B$ sufficiently close to 0, there exists a unique n-tuple $(\psi_j^1(z,\,t),\dots,\psi_j^n(z,\,t))$ of J_0-(0, 1)-forms such that the latter identity below holds:*

$$\bar\partial_0\zeta_j^\alpha(z,\,t) = \sum_{\nu=1}^n \frac{\partial\zeta_j^\alpha(z,\,t)}{\partial\bar z_\nu}\,d\bar z_\nu = \sum_{\lambda=1}^n \psi_j^\lambda(z,\,t)\,\frac{\partial\zeta_j^\alpha(z,\,t)}{\partial z_\lambda}. \tag{2.9}$$

Proof. Note that (2.9) features identities of J_0-(0, 1)-forms and that, when $t \neq 0$, $\zeta_j^\alpha(z, t)$ need not be J_0-holomorphic, so, in general, $\bar\partial_0\zeta_j^\alpha(z, t) \neq 0$.

The system of the latter identities in (2.9) when α is allowed to vary from 1 to n can be put in matrix form as

$$\begin{pmatrix} \bar\partial_0\zeta_j^1(z, t) \\ \vdots \\ \bar\partial_0\zeta_j^n(z, t) \end{pmatrix} = \left(\frac{\partial\zeta_j^\alpha(z, t)}{\partial z_\lambda}\right)_{1\leq\alpha,\,\lambda\leq n} \begin{pmatrix} \psi_j^1(z, t) \\ \vdots \\ \psi_j^n(z, t) \end{pmatrix}.$$

Since the $(n \times n)$-matrix $(\partial\zeta_j^\alpha(z, t)/\partial z_\lambda)_{1\leq\alpha,\,\lambda\leq n}$ is invertible, this system has a unique solution $(\psi_j^1(z, t), \dots, \psi_j^n(z, t))$. $\square$

Let us now write every J_0-(0, 1)-form $\psi_j^\lambda(z, t)$ in terms of the J_0-holomorphic coordinates $(z_1, \dots, z_n)$:

$$\psi_j^\lambda(z, t) := \sum_{\nu=1}^n \psi_{j\nu}^\lambda(z, t)\, d\bar z_\nu, \qquad \text{for all} \quad \lambda \in \{1, \dots, n\},$$

where the $\psi_{j\nu}^\lambda(z, t)$ are C^∞ functions on $U_j \times B$. We will now make the following key observation.

Lemma and Definition 2.2.6. *Let $X = \cup_j U_j$ be an open covering of X_0 by coordinate patches with the above properties. For all j, k, the following* **gluing** *property holds:*

$$\sum_{\lambda=1}^n \psi_j^\lambda(z, t)\,\frac{\partial}{\partial z_\lambda} = \sum_{\lambda=1}^n \psi_k^\lambda(z, t)\,\frac{\partial}{\partial z_\lambda} \qquad \text{on} \quad (U_j \times B) \cap (U_k \times B). \tag{2.10}$$

In other words, after possibly shrinking B about 0, for every $t \in B$, we get a globally defined C^∞ $T^{1,0}X_0$**-valued** J_0**-(0, 1)-form** *on X_0 that we denote by*

$$\psi(t) = \psi(z, t) = \sum_{\lambda=1}^n \psi^\lambda(z, t)\,\frac{\partial}{\partial z_\lambda} \in C^\infty_{0,1}(X_0, T^{1,0}X_0).$$

Proof. The coefficients of the $\psi_j^\lambda(z, t)$ on the right-hand side of (2.9) can be expressed as

$$\frac{\partial\zeta_j^\alpha(z, t)}{\partial z_\lambda} = \sum_{\beta=1}^n \frac{\partial\zeta_j^\alpha}{\partial\zeta_k^\beta}\,\frac{\partial\zeta_k^\beta(z, t)}{\partial z_\lambda}, \qquad \alpha, \lambda, j, k, \tag{2.11}$$

because $\zeta_j^\alpha(z, t)$ is a holomorphic function of $\zeta_k^1, \dots, \zeta_k^n, t_1, \dots, t_m$:

$$\zeta_j^\alpha = f_{jk}^\alpha(\zeta_k^1, \dots, \zeta_k^n, t_1, \dots, t_m), \qquad \alpha, j.$$

Applying $\bar\partial_0$ to both sides of the above identity, we get:

$$\bar\partial_0\zeta_j^\alpha(z, t) = \sum_{\beta=1}^n \frac{\partial\zeta_j^\alpha}{\partial\zeta_k^\beta}\,\bar\partial_0\zeta_k^\beta(z, t), \qquad \alpha, j, k. \tag{2.12}$$

Indeed, to be even more explicit, for every λ we have:

$$\frac{\partial\zeta_j^\alpha(z, t)}{\partial\bar z_\lambda} = \sum_{\beta=1}^n \frac{\partial\zeta_j^\alpha}{\partial\zeta_k^\beta}\,\frac{\partial\zeta_k^\beta(z, t)}{\partial\bar z_\lambda}.$$

So, it suffices to multiply both sides by $d\bar{z}_\lambda$ and sum over λ to get (2.12).

Now, on the one hand, we get:

$$\sum_{\lambda=1}^{n}\frac{\partial\zeta_j^\alpha(z,\,t)}{\partial z_\lambda}\,\psi_j^\lambda(z,\,t)\overset{(a)}{=}\bar\partial_0\zeta_j^\alpha(z,\,t)\overset{(b)}{=}\sum_{\beta=1}^{n}\frac{\partial\zeta_j^\alpha}{\partial\zeta_k^\beta}\sum_{\lambda=1}^{n}\psi_k^\lambda(z,\,t)\,\frac{\partial\zeta_k^\beta(z,\,t)}{\partial z_\lambda},$$

where (a) follows from (2.9) and (b) follows from (2.12) and again (2.9) with indices $(k,\,\beta)$ in place of $(j,\,\alpha)$. On the other hand, the left-hand side quantity above can be written in the following way thanks to (2.11):

$$\sum_{\lambda=1}^{n}\frac{\partial\zeta_j^\alpha(z,\,t)}{\partial z_\lambda}\,\psi_j^\lambda(z,\,t)=\sum_{\lambda=1}^{n}\left(\sum_{\beta=1}^{n}\frac{\partial\zeta_j^\alpha}{\partial\zeta_k^\beta}\,\frac{\partial\zeta_k^\beta(z,\,t)}{\partial z_\lambda}\right)\psi_j^\lambda(z,\,t)=\sum_{\beta=1}^{n}\frac{\partial\zeta_j^\alpha}{\partial\zeta_k^\beta}\sum_{\lambda=1}^{n}\psi_j^\lambda(z,\,t)\,\frac{\partial\zeta_k^\beta(z,\,t)}{\partial z_\lambda}.$$

Using the fact that $\det(\partial\zeta_k^\beta(z,\,t)/\partial z_\lambda)_{\beta,\,\lambda}\neq 0$ for all $t\in B$ sufficiently close to 0 and comparing the two expressions we got for the left-hand side quantity, we get:

$$\psi_j^\lambda(z,\,t)=\psi_k^\lambda(z,\,t)\quad\text{ for all }\quad(z,\,t)\in U_j\cap U_k\ \text{ and all }\ \lambda\in\{1,\dots,n\}.$$

The proof is complete. □

The vector-valued form $\psi(t)\in C^\infty_{0,1}(X_0,\,T^{1,0}X_0)$ defined in formula (2.10) will be our main object of study in the remainder of this subsection. When the vector fields $\partial/\partial z_\lambda$ are considered as 1-st order differential operators, so can $\psi(t)$, which will then take $\mathbb{C}$-valued smooth functions on $X_0=X$ to $\mathbb{C}$-valued J_0-(0, 1)-forms. The following immediate observation shows that what we have proved so far is that the vector-valued form $\psi(t)$, viewed as a differential operator as explained above, measures the discrepancy between $\bar\partial_t$ and $\bar\partial_0$.

Observation 2.2.7. *Formula (2.9) is equivalent to*

$$(\bar\partial_0-\psi(t))\,\zeta_j^\alpha(z,\,t)=0\qquad\textit{for all }\alpha\in\{1,\dots,n\}.\tag{2.13}$$

Since the $\zeta_j^\alpha(z,\,t)$, with $\alpha\in\{1,\dots,n\}$, are the coordinates defining the complex structure J_t on U_j, what this actually says is the following

Theorem 2.2.8. *After possibly shrinking B about 0, for every $t\in B$ and for every locally defined $\mathbb{C}$-valued C^∞ function f on $X:=X_0$, the following equivalence holds:*

$$f\ \textit{ is }J_t-\textit{holomorphic}\iff(\bar\partial_0-\psi(t))\,f\equiv 0.\tag{2.14}$$

Proof. It follows immediately from (2.13) by writing $f=f(\zeta_j^1(z,\,t),\dots,\zeta_j^n(z,\,t))$. Indeed, we have

$$\begin{aligned}(\bar\partial_0-\psi(t))\,f(\zeta_j^1(z,\,t),\dots,\zeta_j^n(z,\,t)) &= \sum_{\alpha=1}^{n}\frac{\partial f}{\partial\zeta_j^\alpha}(\bar\partial_0-\psi(t))\,\zeta_j^\alpha(z,\,t)+\sum_{\alpha=1}^{n}\frac{\partial f}{\partial\bar\zeta_j^\alpha}(\bar\partial_0-\psi(t))\,\bar\zeta_j^\alpha(z,\,t)\\ &\overset{(a)}{=}\sum_{\alpha=1}^{n}\sum_{\nu=1}^{n}\frac{\partial f}{\partial\bar\zeta_j^\alpha}\left(\frac{\partial\bar\zeta_j^\alpha(z,\,t)}{\partial\bar z_\nu}-\sum_{\mu=1}^{n}\psi_{\bar\nu}^{\mu}\,\frac{\partial\bar\zeta_j^\alpha}{\partial z_\mu}\right)d\bar z_\nu\\ &= \sum_{\alpha,\,\nu,\,\lambda}\left(\delta_\nu^\lambda-\sum_{\mu}\psi_{\bar\nu}^{\mu}\,\bar\psi_{\bar\mu}^{\lambda}\right)\overline{\left(\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}\right)}\,d\bar z_\nu\,\frac{\partial f}{\partial\bar\zeta_j^\alpha},\end{aligned}$$

where the first sum on the r.h.s. of the first line vanishes because $(\bar\partial_0 - \psi(t))\,\zeta_j^\alpha(z,\,t) = 0$ for all α by (2.13), while to get (a), we expanded each J_0-$(0,\,1)$-form $\psi^\mu(z,\,t)$ as $\psi^\mu(z,\,t) = \sum_{\nu=1}^{n} \psi^\mu_{\bar\nu}(z,\,t)\,d\bar z^\nu$. Since

$$\det\left(\delta^\lambda_\nu - \sum_\mu \psi^\mu_{\bar\nu}\,\bar\psi^\lambda_{\bar\mu}\right)_{\lambda,\,\nu} \neq 0,$$

we conclude that $(\bar\partial_0 - \psi(t))\,f = 0$ if and only if $\frac{\partial f}{\partial\bar\zeta_j^\alpha} = 0$ for all $\alpha \in \{1, \dots, n\}$. The latter condition being equivalent to f being J_t-holomorphic on U_j, we are done. $\square$

The upshot of this discussion is that the vector-valued $(0,\,1)$-form $\psi(t)$, which can also be viewed as a 1-st order differential operator of bidegree $(0,\,1)$, is the analogue in this approach of (and, indeed, equivalent to) the vector-valued $(0,\,1)$-form $-\alpha_t$ of Proposition 2.2.3.

Conclusion 2.2.9. *For every $t \in B$ sufficiently close to 0, the deformation X_t of X_0, or equivalently the deformation J_t of the complex structure J_0, is equivalent to the vector-valued $(0,\,1)$-form*

$$\psi(t) = \psi(z,\,t) = \sum_{\lambda=1}^{n} \psi^\lambda(z,\,t)\,\frac{\partial}{\partial z_\lambda} \in C^\infty_{0,\,1}(X_0,\,T^{1,\,0}X_0)$$

and again equivalent to the 1-st order differential operator

$$\bar\partial_t \simeq \bar\partial_0 - \psi(t)$$

which can be identified to the $\bar\partial$-operator of the complex structure J_t thanks to property (2.14).

Moreover, $\psi(0) = 0$.

We draw the reader's attention to the following minor, but key, fact. The $\bar\partial$-operator $\bar\partial_t$ of the complex structure J_t takes functions f to J_t-$(0,\,1)$-forms $\bar\partial_t f$ (and J_t-$(p,\,q)$-forms u to J_t-$(p,\,q+1)$-forms $\bar\partial_t u$). By contrast, the operator $\bar\partial_0 - \psi(t)$ takes functions f to J_0-$(0,\,1)$-forms (and J_0-$(p,\,q)$-forms to J_0-$(p,\,q+1)$-forms). Hence, the operators $\bar\partial_t$ and $\bar\partial_0 - \psi(t)$ do not coincide in general (unless $t = 0$). They are only *equivalent*, namely they determine each other, as the notation of Conclusion 2.2.9 indicates.

1-st order deformations of X_0

The main result of this subsection is the following analogue of Theorem 2.2.4 . Like its predecessor, it makes precise in the language of this subsection the principle that was loosely stated as Fact 2.2.2.

Theorem 2.2.10. *Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds and let $\psi(t) = \psi(z,\,t) \in C^\infty_{0,\,1}(X_0,\,T^{1,\,0}X_0)$ be the object defined in Lemma and Definition 2.2.6.*

Then, for every holomorphic vector field $\frac{\partial}{\partial t} \in \Gamma(U,\,T^{1,\,0}B)$ on some small open neighbourhood $U \subset B$ of 0, the following two statements hold.

*(a) The $T^{1,\,0}X_0$-valued J_0-$(0,\,1)$-form $\frac{\partial\psi(t)}{\partial t}_{|t=0}$ is $\bar\partial_0$-**closed**, hence it defines a cohomology class*

$$\left\{\frac{\partial\psi(t)}{\partial t}_{|t=0}\right\}_{\bar\partial_0} \in H^{0,\,1}(X_0,\,T^{1,\,0}X_0);$$

(b) The following identity holds:

$$\rho\left(\frac{\partial}{\partial t}_{|t=0}\right) = -\left\{\frac{\partial\psi(t)}{\partial t}_{|t=0}\right\}_{\bar\partial_0},$$

where $\rho : T^{1,\,0}_0 B \longrightarrow H^{0,\,1}(X_0,\,T^{1,\,0}X_0)$ is the Kodaira-Spencer map at 0 of the family $\pi : \mathcal{X} \longrightarrow B$.

Preliminaries to the proof of Theorem 2.2.10

We start by defining a pairing between $T^{1,0}X_0$-valued J_0-$(0, \bullet)$-forms that will play a key role in what follows.

Definition 2.2.11. *Let $p, q \in \{0, \dots, n\}$ and let $\varphi \in C^\infty_{0,p}(X_0, T^{1,0}X_0)$ and $\psi \in C^\infty_{0,q}(X_0, T^{1,0}X_0)$. The* **bracket** *$[\varphi, \psi] \in C^\infty_{0,p+q}(X_0, T^{1,0}X_0)$ of φ and ψ is defined in the following way.*

If $U \subset X_0$ is an open coordinate patch equipped with a system $(z_1, \dots z_n)$ of local J_0-holomorphic coordinates on X_0 in which φ and ψ are of the shape:

$$\varphi = \sum_{\lambda=1}^{n} \varphi^\lambda \frac{\partial}{\partial z_\lambda} \quad \textit{and} \quad \psi = \sum_{\lambda=1}^{n} \psi^\lambda \frac{\partial}{\partial z_\lambda},$$

with the φ^λ's, resp. the ψ^λ's, $\mathbb{C}$-valued $(0, p)$-forms, resp. $\mathbb{C}$-valued $(0, q)$-forms, we put

$$[\varphi, \psi] := \sum_{\lambda,\, \mu=1}^{n} \left(\varphi^\mu \wedge \frac{\partial \psi^\lambda}{\partial z_\mu} - (-1)^{pq}\, \psi^\mu \wedge \frac{\partial \varphi^\lambda}{\partial z_\mu} \right) \frac{\partial}{\partial z_\lambda}.$$

As usual, when $\frac{\partial}{\partial z_\mu}$ acts as a 1-st order differential operator on forms (e.g. on ψ^λ or φ^λ), it acts on their (function) coefficients only. We now list the basic properties of this bracket.

Lemma 2.2.12. *(i) $[\varphi, \psi]$ is independent of the choice of local coordinates $(z_1, \dots z_n)$.*

(ii) For all $p, q, r \in \{0, \dots, n\}$ and all vector-valued forms $\varphi \in C^\infty_{0,p}(X_0, T^{1,0}X_0)$, $\psi \in C^\infty_{0,q}(X_0, T^{1,0}X_0)$, $\tau \in C^\infty_{0,r}(X_0, T^{1,0}X_0)$, the following identities hold:

(a) $[\varphi, \psi] = -(-1)^{pq}\, [\psi, \varphi]$; *(anti-commutation)*

(b) $\bar\partial_0[\varphi, \psi] = [\bar\partial_0\varphi, \psi] + (-1)^p\, [\varphi, \bar\partial_0\psi]$; *(Leibniz rule)*

(c) $(-1)^{pr}\, [[\varphi, \psi], \tau] + (-1)^{qp}\, [[\psi, \tau], \varphi] + (-1)^{rq}\, [[\tau, \varphi], \psi] = 0.$ *(Jacobi identity)*

Proof. It is straightforward and is left to the reader. □

In what follows, we may assume without loss of generality that $m = 1$, so $(t_1, \dots, t_m) = t$. Otherwise, we can choose local coordinates $t_1, \dots, t_m$ near 0 on B such that $t = t_k$ for some k.

The next result expresses the key *integrability condition* for $\bar\partial_0 - \psi(t)$ (namely the property $(\bar\partial_0 - \psi(t))^2 = 0$ that makes it into a *complex*, rather than just an *almost complex, structure*) as a condition on $\psi(t)$ via the bracket. In other words, this provides a complement or a converse to Conclusion 2.2.9 in that it specifies which vector-valued forms $\psi(t) \in C^\infty_{0,1}(X_0, T^{1,0}X_0)$ can be realised as a small deformation J_t of the complex structure J_0 of a given X_0.

Lemma 2.2.13. *Let X_0 be a compact complex manifold and let $\psi(t) \in C^\infty_{0,1}(X_0, T^{1,0}X_0)$. We denote by $\bar\partial_0$ the $\bar\partial$-operator of X_0 and also the canonical $(0, 1)$-connection it induces on the holomorphic tangent bundle $T^{1,0}X_0$.*

Then, the following equivalence holds:

$$(\bar\partial_0 - \psi(t))^2 = 0 \iff \bar\partial_0\psi(t) = \frac{1}{2}\, [\psi(t), \psi(t)]. \tag{2.15}$$

The condition on the r.h.s. of equivalence (2.15) will be called the *integrability condition* imposed on $\psi(t)$. The parameter t in Lemma 2.2.13 does not play any role and could have been omitted, but we kept it for the sake of notation continuity.

Proof of Lemma 2.2.13. The statement is of a local nature, so we fix local holomorphic coordinates $z_1, \dots z_n$ on X_0 and write

$$\psi(t) = \sum_{\lambda=1}^{n} \psi^\lambda \, \frac{\partial}{\partial z_\lambda},$$

where the ψ^λ's are $\mathbb{C}$-valued $(0,\,1)$-forms.

Fix an arbitrary bidegree $(p,\,q)$ and an arbitrary form $u \in C^\infty_{p,\,q}(X_0,\,\mathbb{C})$. We get:

$$\begin{aligned}(\bar\partial_0 - \psi(t))^2\, u &= \bar\partial_0^2 u - \bar\partial_0(\psi(t)u) - \psi(t)(\bar\partial_0 u) + (\psi(t)\wedge\psi(t))\, u \\ &= -\bar\partial_0\bigg(\sum_{\lambda=1}^{n}\psi^\lambda\wedge\frac{\partial u}{\partial z_\lambda}\bigg) - \sum_{\lambda=1}^{n}\psi^\lambda\wedge\frac{\partial}{\partial z_\lambda}(\bar\partial_0 u) + \sum_{\lambda,\,\mu=1}^{n}\psi^\lambda\wedge\frac{\partial}{\partial z_\lambda}\bigg(\psi^\mu\wedge\frac{\partial u}{\partial z_\mu}\bigg).\end{aligned}$$

Since

$$-\bar\partial_0\bigg(\sum_{\lambda=1}^{n}\psi^\lambda\wedge\frac{\partial u}{\partial z_\lambda}\bigg) = -\sum_{\lambda=1}^{n}\bar\partial_0\psi^\lambda\wedge\frac{\partial u}{\partial z_\lambda} + \sum_{\lambda=1}^{n}\psi^\lambda\wedge\bar\partial_0\bigg(\frac{\partial u}{\partial z_\lambda}\bigg) = -(\bar\partial_0\psi(t))\, u + \sum_{\lambda=1}^{n}\psi^\lambda\wedge\bar\partial_0\bigg(\frac{\partial u}{\partial z_\lambda}\bigg)$$

and since $\bar\partial_0(\frac{\partial u}{\partial z_\lambda}) = \frac{\partial}{\partial z_\lambda}(\bar\partial_0 u)$ for every λ (see below), the above formula reduces to

$$\begin{aligned}(\bar\partial_0 - \psi(t))^2\, u &= -(\bar\partial_0\psi(t))\, u + \sum_{\lambda,\,\mu=1}^{n}\psi^\lambda\wedge\frac{\partial\psi^\mu}{\partial z_\lambda}\wedge\frac{\partial u}{\partial z_\mu} + \sum_{\lambda,\,\mu=1}^{n}\psi^\lambda\wedge\psi^\mu\wedge\frac{\partial^2 u}{\partial z_\lambda\partial z_\mu} \\ &= -(\bar\partial_0\psi(t))\, u + \frac{1}{2}\,[\psi(t),\,\psi(t)]\, u,\end{aligned}$$

where the last identity follows from $\sum\limits_{\lambda,\,\mu=1}^{n}\psi^\lambda\wedge\psi^\mu\wedge\frac{\partial^2 u}{\partial z_\lambda\partial z_\mu} = 0$ due to the fact that $\psi^\lambda\wedge\psi^\mu = -\psi^\mu\wedge\psi^\lambda$ for all $\lambda,\,\mu$.

This proves equivalence (2.15).

It remains to check that $\bar\partial_0(\frac{\partial u}{\partial z_\lambda}) = \frac{\partial}{\partial z_\lambda}(\bar\partial_0 u)$ for every $\mathbb{C}$-valued form u and every λ. Let

$$u = \sum_{|I|=p,\,|J|=q} u_{I\bar J}\, dz_I\wedge d\bar z_J.$$

We have:

$$\begin{aligned}\bar\partial_0\bigg(\frac{\partial u}{\partial z_\lambda}\bigg) &= \sum_{I,\,J}\bar\partial_0\bigg(\frac{\partial u_{I\bar J}}{\partial z_\lambda}\bigg)\wedge dz_I\wedge d\bar z_J = \sum_{I,\,J}\sum_k\frac{\partial}{\partial\bar z_k}\bigg(\frac{\partial u_{I\bar J}}{\partial z_\lambda}\bigg)\, d\bar z_k\wedge dz_I\wedge d\bar z_J \\ &= \frac{\partial}{\partial z_\lambda}\bigg(\sum_{I,\,J}\bar\partial_0 u_{I\bar J}\wedge dz_I\wedge d\bar z_J\bigg) = \frac{\partial}{\partial z_\lambda}(\bar\partial_0 u)\end{aligned}$$

and we are done. □

Let us now see yet another way of proving that the vector-valued form $\psi(t)$ associated with a deformation of complex structures satisfies the *integrability condition* (2.15).

Lemma 2.2.14. *Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds and, for every $t \in B$ sufficiently close to 0, let $\psi(t) \in C^\infty_{0,1}(X_0, T^{1,0}X_0)$ be the vector-valued form of Lemma and Definition 2.2.6. Then, $\psi(t)$ satisfies the* **integrability condition***:*

$$\bar\partial_0\psi(t) = \frac{1}{2}[\psi(t), \psi(t)], \qquad t \in B. \tag{2.16}$$

Proof. Applying $\bar\partial_0$ to (2.9) (with the variable t dropped to lighten the notation), we get:

$$0 = \bar\partial_0^2\zeta_j^\alpha = \bar\partial_0\bigg(\sum_{\lambda=1}^n\psi_j^\lambda\,\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}\bigg) = \sum_{\lambda=1}^n\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}\,\bar\partial_0\psi_j^\lambda - \sum_{\lambda=1}^n\psi_j^\lambda\wedge\bar\partial_0\bigg(\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}\bigg),$$

which yields

$$\sum_{\lambda=1}^n\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}\,\bar\partial_0\psi_j^\lambda = \sum_{\mu=1}^n\psi_j^\mu\wedge\bar\partial_0\bigg(\frac{\partial\zeta_j^\alpha}{\partial z_\mu}\bigg) = \sum_{\mu,\,\nu=1}^n\frac{\partial^2\zeta_j^\alpha}{\partial z_\mu\,\partial\bar z_\nu}\,\psi_j^\mu\wedge d\bar z_\nu. \tag{2.17}$$

Meanwhile, from (2.9) we also get:

$$\frac{\partial\zeta_j^\alpha}{\partial\bar z_\nu} = \sum_{\lambda=1}^n\psi_{j\bar\nu}^\lambda\,\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}, \qquad \nu\in\{1,\dots,n\},$$

so (2.17) becomes

$$\begin{aligned}\sum_{\lambda=1}^n\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}\,\bar\partial_0\psi_j^\lambda &= \sum_{\lambda,\,\mu=1}^n\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}\,\psi_j^\mu\wedge\sum_{\nu=1}^n\frac{\partial\psi_{j\bar\nu}^\lambda}{\partial z_\mu}\,d\bar z_\nu + \sum_{\lambda,\,\mu=1}^n\frac{\partial^2\zeta_j^\alpha}{\partial z_\lambda\,\partial z_\mu}\,\psi_j^\mu\wedge\sum_{\nu=1}^n\psi_{j\bar\nu}^\lambda\,d\bar z_\nu\\ &= \sum_{\lambda,\,\mu=1}^n\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}\,\psi_j^\mu\wedge\frac{\partial\psi_j^\lambda}{\partial z_\mu} + \sum_{\lambda,\,\mu=1}^n\frac{\partial^2\zeta_j^\alpha}{\partial z_\lambda\,\partial z_\mu}\,\psi_j^\mu\wedge\psi_j^\lambda = \sum_{\lambda,\,\mu=1}^n\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}\,\psi_j^\mu\wedge\frac{\partial\psi_j^\lambda}{\partial z_\mu},\end{aligned}$$

where the last identity follows from the vanishing of the latter sum of its l.h.s. due to the anti-commutation of 1-forms $\psi_j^\mu\wedge\psi_j^\lambda = -\psi_j^\lambda\wedge\psi_j^\mu$ for all λ and μ.

We have thus got:

$$\sum_{\lambda=1}^n\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}\,\bar\partial_0\psi_j^\lambda = \sum_{\lambda=1}^n\bigg(\sum_{\mu=1}^n\psi_j^\mu\wedge\frac{\partial\psi_j^\lambda}{\partial z_\mu}\bigg)\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}, \qquad \text{for all}\quad j,\alpha.$$

Let us now restore the variable t. Since $\det\left(\frac{\partial\zeta_j^\alpha(z,t)}{\partial z_\lambda}\right)_{\alpha,\,\lambda}\neq 0$ for all $t\in B$ close to 0, we get:

$$\bar\partial_0\psi_j^\lambda(t) = \sum_{\mu=1}^n\psi_j^\mu(t)\wedge\frac{\partial\psi_j^\lambda(t)}{\partial z_\mu}, \qquad \text{for all}\quad j,\lambda,$$

which amounts to the integrability condition $\bar\partial_0\psi(t) = \frac{1}{2}[\psi(t),\,\psi(t)]$ holding for $t\in B$ close to 0. $\square$

Proof of Theorem 2.2.10

(a) Taking $(\partial/\partial t)_{|t=0}$ in the integrability condition and using the commutation of $(\partial/\partial t)$ with $\bar\partial_0$, we get:

$$\bar\partial_0\bigg(\frac{\partial\psi(t)}{\partial t}_{|t=0}\bigg) = \frac{1}{2}\bigg[\frac{\partial\psi(t)}{\partial t}_{|t=0},\,\psi(0)\bigg] + \frac{1}{2}\bigg[\psi(0),\,\frac{\partial\psi(t)}{\partial t}_{|t=0}\bigg] = 0$$

because $\psi(0)=0$. Thus, $(\partial\psi(t)/\partial t)_{|t=0}$ is $\bar\partial_0$-closed.

(b) Recall the notation:

$$\zeta_j^\alpha(z,\,t) := f_{jk}^\alpha(\zeta_k^1(z,\,t),\dots,\zeta_k^n(z,\,t),\,t_1,\dots,t_m) \quad \text{on} \quad (U_j\times B)\cap(U_k\times B) \tag{2.18}$$

that we will now shorten to $\zeta_j^\alpha = f_{jk}^\alpha(\zeta_k,\,t)$ for all α, j, k. So, f_{jk}^α is the *change of coordinates* map that also encodes the change of complex structure through its dependence on t.

The cohomology class $\rho(\frac{\partial}{\partial t}_{|t=0}) \in H^{0,\,1}(X_0,\,T^{1,\,0}X_0) \simeq H^1(X_0,\,\mathcal{O}_{X_0}(T^{1,\,0}X_0))$ can be identified with the Čech cohomology class of a 1-*cocycle* $\{\theta_{jk}\}\in\mathcal{Z}^1(\{U_j\},\,\mathcal{O}_{X_0}(T^{1,\,0}X_0))$, where

$$\theta_{jk} := \sum_{\alpha=1}^n\frac{\partial f_{jk}^\alpha(\zeta_k,\,t)}{\partial t}_{|t=0}\frac{\partial}{\partial\zeta_j^\alpha}, \qquad j,k.$$

Recall that we are assuming, without loss of generality, that $m=1$, so $(t_1,\dots,t_m)=t$. Taking $(\partial/\partial t)_{|t=0}$ in (2.18), we get:

$$\frac{\partial\zeta_j^\alpha}{\partial t}_{|t=0} = \sum_{\beta=1}^n\frac{\partial\zeta_j^\alpha}{\partial\zeta_k^\beta}\frac{\partial\zeta_k^\beta}{\partial t}_{|t=0} + \frac{\partial f_{jk}^\alpha(\zeta_k,\,t)}{\partial t}_{|t=0}.$$

If we put:

$$\xi_j := \sum_{\alpha=1}^n\frac{\partial\zeta_j^\alpha}{\partial t}_{|t=0}\frac{\partial}{\partial\zeta_j^\alpha}\in C^\infty(U_j,\,T^{1,\,0}X_0), \quad \text{for all} \quad j,$$

and if we multiply the preceding identity by $\partial/\partial\zeta_j^\alpha$ and then sum over $\alpha\in\{1,\dots,n\}$, we get:

$$\begin{aligned}\xi_j &= \sum_{\alpha,\,\beta=1}^n\frac{\partial\zeta_j^\alpha}{\partial\zeta_k^\beta}\frac{\partial\zeta_k^\beta}{\partial t}_{|t=0}\frac{\partial}{\partial\zeta_j^\alpha} + \sum_{\alpha=1}^n\frac{\partial f_{jk}^\alpha(\zeta_k,\,t)}{\partial t}_{|t=0}\frac{\partial}{\partial\zeta_j^\alpha}\\ &= \sum_{\beta=1}^n\frac{\partial\zeta_k^\beta}{\partial t}_{|t=0}\bigg(\sum_{\alpha=1}^n\frac{\partial\zeta_j^\alpha}{\partial\zeta_k^\beta}\frac{\partial}{\partial\zeta_j^\alpha}\bigg) + \theta_{jk}\\ &= \sum_{\beta=1}^n\frac{\partial\zeta_k^\beta}{\partial t}_{|t=0}\frac{\partial}{\partial\zeta_k^\beta} + \theta_{jk} = \xi_k + \theta_{jk}.\end{aligned}$$

In other words, we have:

$$\theta_{jk} = \xi_j - \xi_k \quad \text{for all} \quad j,\,k, \qquad \text{or equivalently} \qquad \delta\{\xi_j\} = -\{\theta_{jk}\}, \tag{2.19}$$

where δ is the Čech differential.

On the other hand, taking $(\partial/\partial t)_{|t=0}$ in (2.9), we get

$$\bar\partial_0\bigg(\frac{\partial\zeta_j^\alpha}{\partial t}_{|t=0}\bigg) = \sum_{\lambda=1}^n \frac{\partial\psi_j^\lambda}{\partial t}_{|t=0}\,\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}(z,\,0)$$

because $\psi_j^\lambda(z,\,0)=0$. (Recall that $\psi(0)=0$.) Using this identity after we have applied $\bar\partial_0$ to the definition of ξ_j, we get:

$$\begin{aligned}\bar\partial_0\xi_j &= \sum_{\alpha=1}^n \bar\partial_0\bigg(\frac{\partial\zeta_j^\alpha}{\partial t}_{|t=0}\bigg)\frac{\partial}{\partial\zeta_j^\alpha} = \sum_{\alpha,\,\lambda=1}^n \frac{\partial\psi_j^\lambda}{\partial t}_{|t=0}\,\frac{\partial\zeta_j^\alpha}{\partial z_\lambda}(z,\,0)\,\frac{\partial}{\partial\zeta_j^\alpha}\\ &= \sum_{\lambda=1}^n \frac{\partial\psi_j^\lambda}{\partial t}_{|t=0}\,\frac{\partial}{\partial z_\lambda} = \frac{\partial}{\partial t}\bigg(\sum_{\lambda=1}^n \psi_j^\lambda\frac{\partial}{\partial z_\lambda}\bigg)_{|t=0} = \frac{\partial\psi}{\partial t}_{|t=0}.\end{aligned}$$

Thus, we have got:

$$\bar\partial_0\xi_j = \frac{\partial\psi}{\partial t}_{|t=0} \qquad \text{for all} \quad j. \tag{2.20}$$

From the fact that $\rho(\frac{\partial}{\partial t}_{|t=0}) \in H^{0,\,1}(X_0,\,T^{1,\,0}X_0) \simeq H^1(X_0,\,\mathcal{O}_{X_0}(T^{1,\,0}X_0))$ identifies with the Čech cohomology class of $\{\theta_{jk}\}$ and from (2.19) and (2.20), we get

$$\rho\bigg(\frac{\partial}{\partial t}_{|t=0}\bigg) = -\bigg\{\frac{\partial\psi(t)}{\partial t}_{|t=0}\bigg\}_{\bar\partial_0},$$

which is what we had to prove.

Indeed, (2.20) means that the Čech 0-cochain $\{\xi_j\}\in\mathcal{C}^0(\{U_j\},\,\mathcal{C}^\infty_{T^{1,\,0}X_0})$, where $\mathcal{C}^\infty_{T^{1,\,0}X_0}$ is the sheaf of germs of C^∞ sections of the vector bundle $T^{1,\,0}X_0$, when viewed in the long exact sequence of cohomology groups

$$0\longrightarrow H^0(X_0,\,\mathcal{O}_{X_0}(T^{1,\,0}X_0))\longrightarrow C^\infty_{0,\,0}(X_0,\,T^{1,\,0}X_0)\xrightarrow{\bar\partial_0}\mathcal{Z}^{0,\,1}_{\bar\partial_0}(T^{1,\,0}X_0)\xrightarrow{\delta^\star} H^1(X_0,\,\mathcal{O}_{X_0}(T^{1,\,0}X_0))\longrightarrow\dots$$

induced by the short exact sequence of sheaves

$$0\longrightarrow\mathcal{O}_{X_0}(T^{1,\,0}X_0)\longrightarrow\mathcal{C}^\infty_{T^{1,\,0}X_0}\xrightarrow{\bar\partial_0}\bar\partial_0\mathcal{C}^\infty_{T^{1,\,0}X_0}\longrightarrow 0,$$

has the property that $\bar\partial_0\{\xi_j\}=\frac{\partial\psi}{\partial t}_{|t=0}$, which implies that $\delta^\star(\frac{\partial\psi}{\partial t}_{|t=0})=\rho(\frac{\partial}{\partial t}_{|t=0})$. $\square$

We have proved, in two different ways, that the Kodaira-Spencer map at a point $0\in B$ can be viewed as classifying the 1-st order deformations of the (complex structure of) X_0. (See Theorems 2.2.4 and 2.2.10.) For this reason, we introduce the following piece of notation.

Notation 2.2.15. *Let $\pi:\mathcal{X}\longrightarrow B$ be a holomorphic family of compact complex manifolds. We fix a reference point $0\in B$ and let $\rho: T_0^{1,\,0}B\longrightarrow H^{0,\,1}(X_0,\,T^{1,\,0}X_0)$ be the Kodaira-Spencer map at 0. For every holomorphic vector field $\frac{\partial}{\partial t}\in\Gamma(U,\,T^{1,\,0}B)$ on some small open neighbourhood $U\subset B$ of 0, we put*

$$\frac{\partial X_t}{\partial t}_{|t=0} := \rho\bigg(\frac{\partial}{\partial t}_{|t=0}\bigg).$$

2.3 The Kodaira-Nirenberg-Spencer existence theorem

The point of view taken in this section is a kind of converse to that of the previous section. Specifically, we will address the following

Question 2.3.1. *Let X be a compact complex manifold and let $\theta \in H^{0,1}(X,\, T^{1,0}X)$.*

When does there exist a holomorphic family of compact complex manifolds $\pi : \mathcal{X} \longrightarrow B$, with B a small open disc about 0 in $\mathbb{C}$, such that

$$\pi^{-1}(0) = X \quad \text{and} \quad \frac{\partial X_t}{\partial t}_{|t=0} = -\theta?$$

The problem being local, it suffices to consider a small neighbourhood of 0 in the base B, while supposing B to be 1-dimensional places no restriction at all on the context. The gist of the question is: when can X be deformed in the *direction* of the given $-\theta$? A posteriori, $-\theta$ will be the tangent vector at 0 to B.

The presentation in this section will follow [Kod86, §.5.3] and [KNS58].

2.3.1 Obstructions to deforming a given complex structure

We start by describing a procedure for the construction of small holomorphic deformations $(X_t)_{t\in B}$ of a given $X := X_0$ when this is possible. The study will also reveal the *obstructions* to being able to do this. We will denote by $\bar\partial := \bar\partial_0$ the $\bar\partial$-operator of the complex structure of $X = X_0$.

Let $m := \dim_{\mathbb{C}} H^{0,1}(X,\, T^{1,0}X)$ and let $B \subset \mathbb{C}^m$ be a small open ball that we will allow ourselves to shrink about 0 as much as necessary. In view of Conclusion 2.2.9 and Lemmas 2.2.13 and 2.2.14, we need to construct vector-valued forms $\psi(t) \in C^\infty_{0,1}(X,\, T^{1,0}X)$ depending *holomorphically* on $t \in B$ and satisfying the *integrability condition* (2.16) for all $t \in B$ close to 0, such that $\psi(0) = 0$.

Thus, we need to construct $\psi(t)$ as a *convergent* power series

$$\psi(t) = \psi_1(t) + \sum_{\nu=2}^{+\infty} \psi_\nu(t), \tag{2.21}$$

where, for every $\nu \in \mathbb{N}^\star$, the vector-valued form

$$\psi_\nu(t) = \sum_{\nu_1+\cdots+\nu_m=\nu} \psi_{\nu_1\dots\nu_m}\, t_1^{\nu_1}\dots t_m^{\nu_m} \in C^\infty_{0,1}(X,\, T^{1,0}X)$$

is a *homogeneous polynomial* of degree ν in the variables $t = (t_1, \dots, t_m) \in B \subset \mathbb{C}^m$. In particular, we are looking to construct vector-valued forms $\psi_{\nu_1\dots\nu_m} \in C^\infty_{0,1}(X,\, T^{1,0}X)$ for $(\nu_1, \dots, \nu_m) \in \mathbb{N}^m$.

The integrability condition (2.16) is equivalent to the following system of equations:

$$\begin{aligned} &(\text{Eq.}\,1) \qquad \bar\partial\psi_1(t) = 0 \\ &(\text{Eq.}\,\nu) \qquad \bar\partial\psi_\nu(t) = \frac{1}{2}\sum_{\mu=1}^{\nu-1}[\psi_\mu(t),\, \psi_{\nu-\mu}(t)], \qquad \text{with} \qquad \nu \geq 2, \end{aligned} \tag{2.22}$$

that must be satisfied for all $t \in B$ sufficiently close to 0. Note that, for every $\nu \geq 1$, the terms featuring in (Eq. ν) are homogeneous polynomials of degree ν in $t = (t_1, \dots, t_m) \in B$. This is an inductively defined system of equations in that, for every $\nu \geq 2$, the right-hand side term of (Eq. ν) is determined by the solutions ψ_λ of the previous equations (Eq. λ) with $\lambda \leq \nu - 1$.

Suppose, furthermore, that a vector-valued form $\theta \in H^{0,1}(X,\, T^{1,0}X)$ has been given beforehand and that we are looking to deform $X = X_0$ in the *direction of* $-\theta$. To make a choice, suppose that $(\partial/\partial t)_{|t=0} = (\partial/\partial t_k)_{|t=0}$ for some $k \in \{1,\dots,m\}$. Then, Theorem 2.2.10 imposes the following extra condition on $\psi(t)$:

$$\left\{\frac{\partial\psi(t)}{\partial t_k}_{|t=0}\right\}_{\bar\partial} = \theta, \qquad \text{or equivalently} \qquad \{\psi_{0\dots1\dots0}\}_{\bar\partial} = \theta, \tag{2.23}$$

with 1 in the k-th slot in $\psi_{0\dots1\dots0}$.

Construction of $\psi_1(t)$

Note that $\bar\partial\psi_{0\dots1\dots0} = 0$ (with 1 in the k-th slot) for all $k \in \{1,\dots,m\}$, because $\bar\partial\psi_1(t) = 0$ for all t close to 0, by (Eq. 1). Since, ideally, we would like to reach every $\theta \in H^{0,1}(X,\, T^{1,0}X)$ (i.e. to deform X in all possible directions), we let

$$\{\beta_1,\dots,\beta_m\}$$

be a collection of m $\bar\partial$-closed vector-valued forms $\beta_\lambda \in C^\infty_{0,1}(X,\, T^{1,0}X)$ such that the set of their cohomology classes

$$\left\{\{\beta_1\}_{\bar\partial},\dots,\{\beta_m\}_{\bar\partial}\right\}$$

is a *basis* of $H^{0,1}(X,\, T^{1,0}X)$, and we let

$$\psi_1(t) = \beta_1\, t_1 + \cdots + \beta_m\, t_m \in C^\infty_{0,1}(X,\, T^{1,0}X) \cap \ker\bar\partial$$

for a priori arbitrary complex variables $t_1,\dots,t_m \in \mathbb{C}$ such that $(t_1,\dots,t_m)$ is as close as will be necessary to $0 \in \mathbb{C}^m$.

In other words, we choose $\psi_{0\dots1\dots0} = \beta_\lambda$ (with 1 in the λ-th slot) for every $\lambda \in \{1,\dots,m\}$. In this way, $\psi_1(t)$ satisfies (Eq. 1) for all $t = (t_1,\dots,t_m) \in \mathbb{C}^m$ and $\psi(t)$ can be made to satisfy condition (2.23) for any pregiven choice of $\theta \in H^{0,1}(X,\, T^{1,0}X)$ after the $\psi_\nu(t)$'s with $\nu \geq 2$ have been constructed.

Construction of $(\psi_\nu(t))_{\nu\geq 2}$

We have to solve the equations (Eq. ν), for all $\nu \geq 2$, once $\psi_1(t)$ has been chosen as explained above.

Now, for a given $\nu \geq 2$, equation (Eq. ν) is *solvable* if and only if its right-hand side term is *$\bar\partial$-exact*. This need not always be the case, but the weaker statement below always holds.

Lemma 2.3.2. *For every $\nu \geq 2$, the vector-valued form on the right-hand side of equation (Eq. ν) is $\bar\partial$*-**closed**.

Proof. We will run an induction on $\nu \geq 2$. Equation (Eq. 1) is now known to be satisfied.

If $\nu = 2$, the r.h.s. term of equation (Eq. ν) is $(1/2)\,[\psi_1(t),\, \psi_1(t)]$. Using (b) of Lemma 2.2.12 (the Leibniz rule for the bracket), we get:

$$\bar\partial[\psi_1(t),\, \psi_1(t)] = [\bar\partial\psi_1(t),\, \psi_1(t)] - [\psi_1(t),\, \bar\partial\psi_1(t)] = 0,$$

because $\bar\partial\psi_1(t) = 0$ thanks to equation (Eq. 1).

Fix an arbitrary $\nu \geq 2$ and suppose that the right-hand side term of (Eq. l) is $\bar{\partial}$-closed for all $l \in \{2, \dots, \nu\}$. We will prove that the right-hand side term of (Eq. $\nu+1$) is $\bar{\partial}$-closed. (Actually, we implicitly suppose that the form $\psi_l(t)$ has been constructed for every $l \in \{1, \dots, \nu\}$, since otherwise the right-hand side term of (Eq. $\nu+1$) does not exist and, thus, there is nothing to prove. This means that we also suppose the equation (Eq. l) to be solvable for all $l \in \{2, \dots, \nu\}$, namely that its right-hand side term is even $\bar{\partial}$-exact.)

We start with a general remark. For every $\nu \in \mathbb{N}^\star$ and every $t \in \mathbb{C}^m$, we consider the following expression of degree $\leq \nu$ in t:

$$\sigma_\nu(t) := \psi_1(t) + \cdots + \psi_\nu(t) \in C^\infty_{0,1}(X, T^{1,0}X).$$

Then, the integrability condition (2.16), which we already know is equivalent to the system of equations (Eq. ν) with $\nu \in \mathbb{N}^\star$, is further equivalent to the following condition:

$$\forall\, \nu \geq 1, \quad \bar{\partial}\sigma_\nu(t) = \frac{1}{2}\,[\sigma_\nu(t),\, \sigma_\nu(t)] + \mathcal{O}(t^{\nu+1}) \tag{2.24}$$

(because $\bar{\partial}\sigma_\nu(t)$ is the part of degree $\leq \nu$ in t of $\bar{\partial}\psi(t)$ and $[\sigma_\nu(t),\, \sigma_\nu(t)]$ contains, without being reduced to, all the terms of degree $\leq \nu$ in t of $[\psi(t),\, \psi(t)]$). This is further equivalent to

$$\forall\, \nu \geq 1, \quad \bar{\partial}\sigma_\nu(t) - \frac{1}{2}\,[\sigma_\nu(t),\, \sigma_\nu(t)] = \varphi_{\nu+1}(t) + \mathcal{O}(t^{\nu+2}) \tag{2.25}$$

for some *homogeneous* expression $\varphi_{\nu+1}(t)$ of *degree* $\nu+1$ *in* t (which is, obviously, uniquely determined by this requirement). Note that $\bar{\partial}\sigma_\nu(t)$ is the part of degree $\leq \nu$ in t of $-(1/2)\,[\sigma_\nu(t),\, \sigma_\nu(t)]$, while $\varphi_{\nu+1}(t)$ is the homogeneous part of degree $\nu+1$ in t of $-(1/2)\,[\sigma_\nu(t),\, \sigma_\nu(t)]$.

Applying $\bar{\partial}$ and noticing that $\bar{\partial}(-(1/2)\,[\sigma_\nu(t),\, \sigma_\nu(t)]) = [\sigma_\nu(t),\, \bar{\partial}\sigma_\nu(t)]$ (thanks to (a) and (b) of Lemma 2.2.12), we get:

$$\begin{aligned}
& \bar{\partial}\varphi_{\nu+1}(t) && = && [\sigma_\nu(t),\, \bar{\partial}\sigma_\nu(t)] + \mathcal{O}(t^{\nu+2}), \quad \forall\, \nu \geq 1 \\
\overset{(a)}{\iff} \; & \bar{\partial}\varphi_{\nu+1}(t) && = && \frac{1}{2}\Big[\sigma_\nu(t),\, [\sigma_\nu(t),\, \sigma_\nu(t)]\Big] + \mathcal{O}(t^{\nu+2}), \quad \forall\, \nu \geq 1 \\
\overset{(b)}{\iff} \; & \bar{\partial}\varphi_{\nu+1}(t) && = && \frac{1}{2}\Big[\psi(t),\, [\psi(t),\, \psi(t)]\Big] + \mathcal{O}(t^{\nu+2}), \quad \forall\, \nu \geq 1 \\
\overset{(c)}{\iff} \; & \bar{\partial}\varphi_{\nu+1}(t) && = && \mathcal{O}(t^{\nu+2}), \quad \forall\, \nu \geq 1 \\
\overset{(d)}{\iff} \; & \bar{\partial}\varphi_{\nu+1}(t) && = && 0, \quad \forall\, \nu \geq 1,
\end{aligned} \tag{2.26}$$

where (a) follows from (2.24) and $[\sigma_\nu(t),\, \mathcal{O}(t^{\nu+1})] \in \mathcal{O}(t^{\nu+2})$ (since all the terms in $\sigma_\nu(t)$ are of degree ≥ 1 in t); (b) follows from the fact that $[\psi(t),\, [\psi(t),\, \psi(t)]]$ has the same terms of degree $\leq \nu+1$ in t as $[\sigma_\nu(t),\, [\sigma_\nu(t),\, \sigma_\nu(t)]]$; (c) follows from the fact that $[\psi(t),\, [\psi(t),\, \psi(t)]] = 0$ by the *Jacobi identity* (see (c) of Lemma 2.2.12); (d) follows from $\bar{\partial}\varphi_{\nu+1}(t)$ being homogeneous of degree $\nu+1$ in t.

After this general remark, let us now return to the induction process in the proof of Lemma 2.3.2. Recall that we are supposing the forms $\psi_1(t), \dots, \psi_\nu(t)$ to have been constructed as solutions of equations (Eq. 1), ..., (Eq. ν). We are looking to prove that

$$\frac{1}{2}\sum_{\mu=1}^{\nu}[\psi_\mu(t),\, \psi_{\nu+1-\mu}(t)] \in \ker\bar{\partial}.$$

Since this vector-valued form is the homogeneous part of degree $\nu+1$ of $(1/2)\,[\psi(t),\,\psi(t)]$, which coincides with the homogeneous part of degree $\nu+1$ of $(1/2)\,[\sigma_\nu(t),\,\sigma_\nu(t)]$, which further coincides, thanks to (2.25), with $-\varphi_{\nu+1}(t)$, we are done thanks to (2.26).

The proof of Lemma 2.3.2 is complete. □

Conclusion 2.3.3. *All the* **obstructions** *to solving the equations* $(Eq.\,\nu)_{\nu\in\mathbb{N}^\star}$ *lie in* $H^{0,\,2}(X,\,T^{1,\,0}X)$.

Proof. Since $\psi_\nu(t)\in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$ for all $\nu\in\mathbb{N}^\star$, Lemma 2.3.2 yields:

$$\frac{1}{2}\sum_{\mu=1}^{\nu-1}[\psi_\mu(t),\,\psi_{\nu-\mu}(t)]\in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)\cap\ker\bar\partial,\qquad \nu\geq 2.$$

Therefore, the right-hand side terms of equations (Eq. ν) define cohomology classes

$$\left\{\frac{1}{2}\sum_{\mu=1}^{\nu-1}[\psi_\mu(t),\,\psi_{\nu-\mu}(t)]\right\}_{\bar\partial}\in H^{0,\,2}(X,\,T^{1,\,0}X),\qquad \nu\geq 2.$$

Obviously, these classes vanish in $H^{0,\,2}(X,\,T^{1,\,0}X)$ if and only if the right-hand side terms of equations $(\mathrm{Eq.}\,\nu)_{\nu\geq 2}$ are $\bar\partial$-exact.

Meanwhile, for every $\nu\geq 2$, equation (Eq. ν) is solvable if and only if its right-hand side term is $\bar\partial$-exact. □

This answers the *qualitative* part of Question 2.3.1 and potentially gives a *sufficient* (but, as will be seen in the next section, *unnecessary*) condition for the existence of small holomorphic deformations of a given compact complex manifold X in all the available *directions*. However, we have yet to determine whether the solutions $(\psi_\nu)_{\nu\geq 1}$ of equations $(\mathrm{Eq.}\,\nu)_{\nu\geq 1}$ (which, as we have seen above, exist if all the obstructions vanish, for example if $H^{0,\,2}(X,\,T^{1,\,0}X)=0$) can be chosen in such a way that the power series (2.21) converges absolutely (w.r.t. an appropriate norm) on a small neighbourhood of $0\in B$ (i.e. whether it defines genuine small holomorphic deformations of the complex structure of X). In other words, we have yet to determine whether the above candidate for a sufficient condition is indeed one such condition.

Note also that, for every $\nu\geq 2$, even if equation (Eq. ν) is solvable, the solution is not unique, since we can add any element in $\ker\bar\partial$ to any solution to get another solution. Finding appropriate choices of solutions will be key.

2.3.2 Convergence of the power series defining small deformations

To lift the suspense, let us say right away that the *qualitative obstructions* found in the previous subsection are the only obstructions to deforming the complex structure of X. In other words, if all the equations $(\mathrm{Eq.}\,\nu)_{\nu\geq 2}$ are solvable, their solutions $(\psi_\nu)_{\nu\geq 2}$ can always be chosen such that the power series (2.21) converges absolutely. This is the content of the following important **existence theorem of Kodaira-Nirenberg-Spencer**.

Theorem 2.3.4. *([KNS58]) Let X be a compact complex manifold such that $H^{0,\,2}(X,\,T^{1,\,0}X)=0$.*

Then, there exists a holomorphic family $\pi:\mathcal{X}\longrightarrow B\subset\mathbb{C}^m$ of compact complex manifolds, where $m:=dim_{\mathbb{C}}H^{0,\,1}(X,\,T^{1,\,0}X)$ and B is a small open ball about the origin in $\mathbb{C}^m$, such that:

(i) $\pi^{-1}(0)=X$*;*

(ii) the Kodaira-Spencer map at 0

$$\rho : T_0^{1,\,0}B \longrightarrow H^{0,\,1}(X,\,T^{1,\,0}X), \qquad \frac{\partial}{\partial t}_{|t=0} \mapsto \frac{\partial X_t}{\partial t}_{\;|t=0},$$

is an **isomorphism**.

In other words, if the space $H^{0,\,2}(X,\,T^{1,\,0}X)$ that contains all the qualitative obstructions to locally deforming X *vanishes*, then X can, indeed, be deformed in all the available directions (parametrised by $H^{0,\,1}(X,\,T^{1,\,0}X)$).

Preliminaries to the proof of Theorem 2.3.4

We will prove the absolute convergence of the power series (2.21) w.r.t. a family of *Hölder norms*, for the definition of which we need to fix an (arbitrary) Hermitian metric ω on X. As in the case of $\mathbb{C}$-valued forms, ω induces a pointwise and an L^2 inner product on the spaces $C^\infty_{0,\,q}(X,\,T^{1,\,0}X)$ of smooth $T^{1,\,0}X$-valued $(0,\,q)$-forms on X. The L^2 inner product then induces a *formal adjoint* $\bar\partial^\star = \bar\partial^\star_\omega : C^\infty_{0,\,q}(X,\,T^{1,\,0}X) \longrightarrow C^\infty_{0,\,q-1}(X,\,T^{1,\,0}X)$ of the canonical $(0,\,1)$-connection $\bar\partial : C^\infty_{0,\,q-1}(X,\,T^{1,\,0}X) \longrightarrow C^\infty_{0,\,q}(X,\,T^{1,\,0}X)$ of the holomorphic vector bundle $T^{1,\,0}X$, which induces, in turn, a *Laplace-type* differential operator of order two

$$\Delta'' = \Delta''_\omega := \bar\partial\bar\partial^\star_\omega + \bar\partial^\star_\omega\bar\partial : C^\infty_{0,\,q}(X,\,T^{1,\,0}X) \longrightarrow C^\infty_{0,\,q}(X,\,T^{1,\,0}X), \qquad q \in \{0,\dots,n\},$$

where $n := \dim_{\mathbb{C}} X$.

As in the scalar case, Δ'' is *elliptic* and $\bar\partial$ is *integrable* (i.e. $\bar\partial^2 = 0$), so by standard elliptic theory we get an L^2_ω-orthogonal 3-*space decomposition*:

$$C^\infty_{0,\,q}(X,\,T^{1,\,0}X) = \mathcal{H}^{0,\,q}_{\Delta''}(X,\,T^{1,\,0}X) \oplus \bar\partial C^\infty_{0,\,q-1}(X,\,T^{1,\,0}X) \oplus \bar\partial^\star C^\infty_{0,\,q+1}(X,\,T^{1,\,0}X)$$

in which $\ker\bar\partial = \mathcal{H}^{0,\,q}_{\Delta''}(X,\,T^{1,\,0}X) \oplus \bar\partial C^\infty_{0,\,q-1}(X,\,T^{1,\,0}X)$, where $\mathcal{H}^{0,\,q}_{\Delta''}(X,\,T^{1,\,0}X) := \ker\Delta''$ is the *harmonic space* of bidegree $(0,\,q)$.

- **Definition of the Hölder norm**

(a) *Case of functions in open subsets of* $\mathbb{R}^{2n}$ *(the local situation)*

Let $U \subset \mathbb{R}^{2n}$ be an open subset, let $x = (x_1,\dots,x_{2n})$ be the real coordinates in $\mathbb{R}^{2n}$ and let $f : U \longrightarrow \mathbb{C}$ be a C^∞ function. For every multi-index $h = (h_1,\dots h_{2n}) \in \mathbb{N}^{2n}$, we consider the differential operator of order $|h| := h_1 + \dots h_{2n}$:

$$D^h := \left(\frac{\partial}{\partial x_1}\right)^{h_1} \dots \left(\frac{\partial}{\partial x_{2n}}\right)^{h_{2n}}.$$

Definition 2.3.5. *For every $k \in \mathbb{N}$ and every $\alpha \in (0,\,1)$, the* **Hölder norm** $|f|^U_{k+\alpha}$ *of* $f : U \longrightarrow \mathbb{C}$ *is defined as*

$$|f|^U_{k+\alpha} := \sum_{|h|=0}^{k} \sum_{D_h} \sup_{x\in U} \left|D^h f(x)\right| + \sum_{D_k} \sup_{x,\,y\in U} \frac{|D^k f(x) - D^k f(y)|}{|x-y|^\alpha}.$$

Note that the first double sum on the right-hand side of the expression of $|f|^U_{k+\alpha}$ is the C^k *norm* $|f|_{C^k}$ *of* f, while the second sum will be used later on to get *equicontinuity* and apply the Ascoli Theorem.

(b) *Case of global C^∞ sections of $\Lambda^{0,\,q}T^\star X \otimes T^{1,\,0}X$ (the global situation)*

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ and let $X = \cup_j U_j$ be a *finite* open covering of X by coordinate patches. For every j, let $z_j = (z_j^1, \dots, z_j^n)$ be a system of *holomorphic* coordinates on U_j. Putting $z_j^\nu := x_j^{2\nu-1} + i\, x_j^{2\nu}$, we get *real coordinates* $x_j = (x_j^1, \dots, x_j^{2n})$ on U_j, for every j.

Now, for every vector-valued form $\varphi \in C^\infty_{0,\,q}(X,\, T^{1,\,0}X)$, we write

$$\varphi_{|U_j} = \frac{1}{q!} \sum_{\substack{\lambda \\ \nu_1,\dots,\,\nu_q}} \varphi^\lambda_{j\bar\nu_1\dots\bar\nu_q}(x_j)\, d\bar{z}_j^{\nu_1} \wedge \dots \wedge d\bar{z}_j^{\nu_q} \otimes \frac{\partial}{\partial z_j^\lambda},$$

where the $\varphi^\lambda_{j\bar\nu_1\dots\bar\nu_q}(x_j)$ are C^∞ functions on U_j viewed as an open subset of $\mathbb{R}^{2n}$.

Definition 2.3.6. *For every $k \in \mathbb{N}$, every $\alpha \in (0,\, 1)$ and every $\varphi \in C^\infty_{0,\,q}(X,\, T^{1,\,0}X)$, the* **Hölder norm** $|\varphi|_{k+\alpha}$ *of φ is defined as*

$$|\varphi|_{k+\alpha} := \max_j \max_{\substack{\lambda \\ \nu_1,\dots,\,\nu_q}} \left|\varphi^\lambda_{j\bar\nu_1\dots\bar\nu_q}\right|^{U_j}_{k+\alpha}.$$

Note that $|\varphi|_{k+\alpha}$ depends on the choices of a covering $(U_j)_j$ of X and of coordinates $(z_j)_j$ thereon. However, the induced topology on $C^\infty_{0,\,q}(X,\, T^{1,\,0}X)$ does not depend on these choices. We now fix once and for all a choice of $(U_j)_j$ and choices of $(z_j)_j$. For the sake of comparison, recall that the C^0 **norm** of $\varphi \in C^\infty_{0,\,q}(X,\, T^{1,\,0}X)$ on X is defined as:

$$|\varphi|_0 := \max_j \max_{\substack{\lambda \\ \nu_1,\dots,\,\nu_q}} \sup_{x_j \in U_j} \left|\varphi^\lambda_{j\bar\nu_1\dots\bar\nu_q}(x_j)\right|.$$

• Fundamental elliptic theory results

We will only remind the reader of the standard **a priori estimate** for the *elliptic* differential operator $\Delta'' : C^\infty_{0,\,q}(X,\, T^{1,\,0}X) \longrightarrow C^\infty_{0,\,q}(X,\, T^{1,\,0}X)$ on a given *compact Hermitian manifold* $(X,\, \omega)$. This estimate is not peculiar to Δ'', but is satisfied by every *elliptic* operator on a *compact* manifold.

Theorem 2.3.7. *For every $q \in \{0, \dots, n\}$, every integer $k \geq 2$ and every $\alpha \in (0,\, 1)$, there exists a constant $C_{k,\,\alpha} > 0$, depending only on k and α, such that the following* **a priori estimate** *holds:*

$$|\varphi|_{k+\alpha} \leq C_{k,\,\alpha}\,(|\Delta''\varphi|_{k-2+\alpha} + |\varphi|_0) \tag{2.27}$$

for every $\varphi \in C^\infty_{0,\,q}(X,\, T^{1,\,0}X)$.

Proof. Since this is standard elliptic theory fare, we leave the reader peruse the proof in, for example, the appendix to [Kod86]. $\square$

However, we will spell out the details of the proof of the next result, which is a corollary of Theorem 2.3.7 and will play a key role in what follows. Recall first that the restriction of

$$\mathcal{H}^{0,q}_{\Delta''}(X,\,T^{1,0}X)\oplus\operatorname{Im}\Delta''=C^\infty_{0,\,q}(X,\,T^{1,0}X)\stackrel{\Delta''}{\longrightarrow}C^\infty_{0,\,q}(X,\,T^{1,0}X)=\mathcal{H}^{0,q}_{\Delta''}(X,\,T^{1,0}X)\oplus\operatorname{Im}\Delta''$$

to $\operatorname{Im}\Delta''$ defines an *isomorphism*:

$$\Delta''_{|\operatorname{Im}\Delta''}:\operatorname{Im}\Delta''\longrightarrow\operatorname{Im}\Delta''$$

whose inverse, when extended by 0 across $\mathcal{H}^{0,q}_{\Delta''}(X,\,T^{1,0}X):=\ker\Delta''$, is called the **Green operator**:

$$G:C^\infty_{0,\,q}(X,\,T^{1,0}X)\longrightarrow C^\infty_{0,\,q}(X,\,T^{1,0}X),\qquad G_{|\operatorname{Im}\Delta''}:=\left(\Delta''_{|\operatorname{Im}\Delta''}\right)^{-1}\quad\text{and}\quad G_{|\ker\Delta''}:=0.$$

To simplify the notation, we will denote the Green operator of Δ'' by $G:=\Delta''^{-1}$.

Lemma 2.3.8. *For every $q\in\{0,\dots,n\}$, every integer $k\geq 2$ and every $\alpha\in(0,\,1)$, there exists a constant $C_1:=C_1(k,\,\alpha)>0$, depending only on k and α, such that the following estimate holds:*

$$|\Delta''^{-1}\psi|_{k+\alpha}\leq C_1\,|\psi|_{k-2+\alpha}\tag{2.28}$$

for every $\psi\in C^\infty_{0,\,q}(X,\,T^{1,0}X)$ such that $\psi\perp\ker\Delta''$.

The *orthogonality* (with respect to the L^2 inner product induced by ω) constraint placed on ψ is, obviously, equivalent to the requirement $\psi\in\operatorname{Im}\Delta''$. Thus, estimate (2.28) improves on the general *a priori estimate* (2.27) in this special case in that the C^0-norm term on the right is no longer necessary.

Proof of Lemma 2.3.8. Since the restriction of Δ'' to $\operatorname{Im}\Delta''$ is bijective onto $\operatorname{Im}\Delta''$, every $\psi\in\operatorname{Im}\Delta''$ is determined by $\varphi:=\Delta''^{-1}\psi$.

The *a priori estimate* (2.27) applied to $\varphi=\Delta''^{-1}\psi$ yields:

$$|\Delta''^{-1}\psi|_{k+\alpha}\leq C_{k,\,\alpha}\,(|\psi|_{k-2+\alpha}+|\Delta''^{-1}\psi|_0)$$

for every $\psi\in\operatorname{Im}\Delta''\subset C^\infty_{0,\,q}(X,\,T^{1,0}X)$. Thus, to prove (2.28), it suffices to prove that there exists a constant $C_2:=C_2(k,\,\alpha)>0$, depending only on k and α, such that the following estimate holds:

$$|\Delta''^{-1}\psi|_0\leq C_2\,|\psi|_{k-2+\alpha}\tag{2.29}$$

for every $\psi\in\operatorname{Im}\Delta''\subset C^\infty_{0,\,q}(X,\,T^{1,0}X)$ such that $\psi\perp\ker\Delta''$.

We will prove this fact by *contradiction*. Suppose there exists no constant $C_2>0$ such that inequality (2.29) holds for all $\psi\in\operatorname{Im}\Delta''\subset C^\infty_{0,\,q}(X,\,T^{1,0}X)$ with $\psi\perp\ker\Delta''$. Then, for every $m\in\mathbb{N}^\star$, there exists $\psi_m\in\operatorname{Im}\Delta''\subset C^\infty_{0,\,q}(X,\,T^{1,0}X)$ such that

$$(a)\quad|\Delta''^{-1}\psi_m|_0=1\quad\text{ and }\quad(b)\quad|\psi_m|_{k-2+\alpha}<\frac{1}{m}\tag{2.30}$$

(or simply $|\Delta''^{-1}\psi_m|_0=1$ for all $m\in\mathbb{N}^\star$ and $\lim\limits_{m\to+\infty}|\psi_m|_{k-2+\alpha}=0$).

Then, by the general *a priori estimate* (2.27) applied to $\varphi_m=\Delta''^{-1}\psi_m$, we get

$$|\Delta''^{-1}\psi_m|_{k+\alpha}\leq C_{k,\,\alpha}\,(|\psi_m|_{k-2+\alpha}+|\Delta''^{-1}\psi_m|_0)<\left(\frac{1}{m}+1\right)C_{k,\,\alpha}<2C_{k,\,\alpha},\qquad m\in\mathbb{N}^\star.$$

Given the definition of the Hölder norm $|\ |_{k+\alpha}$, these inequalities imply that the coefficients of $\Delta''^{-1}\psi_m$ have *uniformly bounded* and *uniformly equicontinuous* derivatives of every order $l \leq k$ in every U_j. Therefore, *Ascoli's theorem* implies the existence of a subsequence $(\psi_{l_m})_m$ of $(\psi_m)_m$ and the existence of $\widetilde{\varphi} \in C^\infty_{0,\,q}(X,\,T^{1,\,0}X)$ such that

$$\Delta''^{-1}\psi_{l_m} \xrightarrow[m\to+\infty]{uniformly} \widetilde{\varphi} \quad \text{and} \quad D^h(\Delta''^{-1}\psi_{l_m}) \xrightarrow[m\to+\infty]{uniformly} D^h\widetilde{\varphi} \quad \text{for all } h \text{ such that } |h| \leq k.$$

Now, since $k \geq 2$ and Δ'' is of order 2, this and (a) of (2.30) further imply that

$$|\widetilde{\varphi}|_0 = 1 \quad \text{and} \quad \psi_{l_m} = \Delta''(\Delta''^{-1}\psi_{l_m}) \xrightarrow[m\to+\infty]{uniformly} \Delta''\widetilde{\varphi}. \tag{2.31}$$

Meanwhile, since $k-2 \geq 0$, from (b) of (2.30) we get:

$$|\psi_{l_m}|_0 \leq |\psi_{l_m}|_{k-2+\alpha} \xrightarrow[m\to+\infty]{} 0, \quad \text{hence} \quad \psi_{l_m} \xrightarrow[m\to+\infty]{uniformly} 0. \tag{2.32}$$

From (2.31) and (2.32), we get $\Delta''\widetilde{\varphi} = 0$, so $\widetilde{\varphi} \in \ker\Delta''$.

On the other hand, we will show that $\widetilde{\varphi} \in \operatorname{Im}\Delta''$. Then, we can conclude that $\widetilde{\varphi} \in \ker\Delta'' \cap \operatorname{Im}\Delta'' = \{0\}$, where the last identity follows from $\ker\Delta'' \perp \operatorname{Im}\Delta''$. Consequently, $\widetilde{\varphi} = 0$, which contradicts the property $|\widetilde{\varphi}|_0 = 1$. (See (2.31).)

Now, proving that $\widetilde{\varphi} \in \operatorname{Im}\Delta''$ is equivalent to proving that $\widetilde{\varphi} \perp \ker\Delta''$. Let $u \in \ker\Delta''$, arbitrary. Since $\Delta''^{-1} : \operatorname{Im}\Delta'' \to \operatorname{Im}\Delta''$ is bijective, for every m, there is a unique v_m such that $\Delta''^{-1}\psi_{l_m} = \Delta'' v_m$. We get:

$$\langle\langle\Delta''^{-1}\psi_{l_m},\,u\rangle\rangle = \langle\langle\Delta'' v_m,\,u\rangle\rangle = \langle\langle v_m,\,\Delta'' u\rangle\rangle = 0, \quad \text{for every} \quad m \in \mathbb{N}^\star.$$

On the other hand, we know that $\Delta''^{-1}\psi_{l_m}$ converges, as $m \to +\infty$, to $\widetilde{\varphi}$ uniformly, hence also in the L^2 topology since X is compact and the metric ω is smooth. Therefore, $\langle\langle\Delta''^{-1}\psi_{l_m},\,u\rangle\rangle$ converges to $\langle\langle\widetilde{\varphi},\,u\rangle\rangle$. Since $\langle\langle\Delta''^{-1}\psi_{l_m},\,u\rangle\rangle = 0$ for all m, we get $\langle\langle\widetilde{\varphi},\,u\rangle\rangle = 0$. Since $u \in \ker\Delta''$ is arbitrary, we get $\widetilde{\varphi} \perp \ker\Delta''$ and we are done. □

• Minimal L^2-norm solutions of $\bar\partial$ equations

If $\rho \in \bar\partial(C^\infty_{0,\,q}(X,\,T^{1,\,0}X))$, it is obvious that the equation $\bar\partial\varphi = \rho$ has solutions φ. However, the solution is unique only up to $\ker\bar\partial$. Since the orthogonal complement of $\ker\bar\partial$ in $C^\infty_{0,\,q}(X,\,T^{1,\,0}X)$ is $\operatorname{Im}\bar\partial^\star$ (see the L^2-orthogonal decomposition recalled earlier in this subsection), the solution of minimal L^2 norm is the unique solution that lies in $\operatorname{Im}\bar\partial^\star$.

The following result gives a formula and a Hölder-norm estimate for the (unique) minimal L^2-norm solution of this equation.

Lemma 2.3.9. *Let $(X,\,\omega)$ be an n-dimensional compact Hermitian manifold. Fix $q \in \{0,\dots,n\}$.*

For every $\rho \in \bar\partial(C^\infty_{0,\,q}(X,\,T^{1,\,0}X))$, the **minimal L^2_ω-norm solution** *of the equation*

$$\bar\partial\varphi = \rho \tag{2.33}$$

is given by the following **Neumann formula***:*

$$\varphi = \bar\partial^\star\Delta''^{-1}\rho. \tag{2.34}$$

Moreover, for every integer $k \geq 1$ and every $\alpha \in (0,\,1)$, there exists a constant $c_{k,\,\alpha} > 0$ **independent of** *ρ such that the minimal L^2_ω-norm solution φ of equation (2.33) satisfies the following estimate:*

$$|\varphi|_{k+\alpha} \leq c_{k,\,\alpha}\,|\rho|_{k-1+\alpha}. \tag{2.35}$$

Proof. Since $\bar\partial^\star\Delta''^{-1}\rho \in \operatorname{Im}\bar\partial^\star$, it will be the minimal L^2-norm solution of equation (2.33) if it is a solution at all. To prove this last fact, note that $\bar\partial\Delta''^{-1}\rho = 0$ since $\Delta''^{-1}\rho \in \operatorname{Im}\bar\partial \subset \ker\bar\partial$. (Note that $\bar\partial$ commutes with Δ'', hence also with Δ''^{-1}.) Therefore, we get:

$$\bar\partial(\bar\partial^\star\Delta''^{-1}\rho) = (\bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial)\,(\Delta''^{-1}\rho) = \Delta''\Delta''^{-1}\rho = \rho,$$

proving that $\bar\partial^\star\Delta''^{-1}\rho$ is, indeed, a solution of equation (2.33).

Now, for $\varphi = \bar\partial^\star\Delta''^{-1}\rho$, we have

$$|\varphi|_{k+\alpha} = |\bar\partial^\star\Delta''^{-1}\rho|_{k+\alpha} \overset{(a)}{\leq} K_1\,|\Delta''^{-1}\rho|_{k+1+\alpha} \overset{(b)}{\leq} K_1\,C_1\,|\rho|_{k-1+\alpha},$$

where $K_1 > 0$ is a constant independent of ρ that trivially exists such that inequality (a) is satisfied because $\bar\partial^\star$ is a differential operator of *order* 1, while inequality (b) follows from the key a priori estimate (2.28) since $\rho \in \operatorname{Im}\bar\partial$, so $\rho \perp \ker\Delta''$.

Estimate (2.35) follows by taking $c_{k,\,\alpha} := K_1C_1$. □

- **An auxiliary scalar power series**

One of the key ideas in the proof of the convergence of the power series (2.21) of vector-valued forms is to compare it with an elementary scalar power series. Even if the $T^{1,\,0}X$-valued forms are of bidegree $(0,\,1)$ in our case, we introduce the following piece of notation (cf. [KNS58]) in the more general case of the bidegree $(0,\,q)$.

Notation 2.3.10. *Fix $k \in \mathbb{N}^\star$ and $\alpha \in (0,\,1)$. For any formal power series*

$$\psi(t) = \sum_{\nu\geq 1}\psi_\nu(t) = \sum_{\nu\geq 1}\sum_{\nu_1+\dots+\nu_m=\nu}\psi_{\nu_1\dots\nu_m}\,t_1^{\nu_1}\dots t_m^{\nu_m}$$

with **vector-valued form coefficients** *$\psi_{\nu_1\dots\nu_m} \in C^\infty_{0,\,q}(X,\,T^{1,\,0}X)$ and any formal power series*

$$a(t) = \sum_{\nu\geq 1}a_\nu(t) = \sum_{\nu\geq 1}\sum_{\nu_1+\dots+\nu_m=\nu}a_{\nu_1\dots\nu_m}\,t_1^{\nu_1}\dots t_m^{\nu_m}$$

with **constant numerical coefficients** *$a_{\nu_1\dots\nu_m} \geq 0$, we give the following meaning to the symbol $\ll$:*

$$|\psi|_{k+\alpha}(t) \ll a(t) \iff |\psi_{\nu_1\dots\nu_m}|_{k+\alpha} \leq a_{\nu_1\dots\nu_m} \quad \textit{for all} \quad \nu_1,\dots,\nu_m \in \mathbb{N}.$$

Similarly, if $b(t)$ is another formal power series with constant numerical coefficients $b_{\nu_1\dots\nu_m} \geq 0$, we write $b(t) \ll a(t)$ to mean that $b_{\nu_1\dots\nu_m} \leq a_{\nu_1\dots\nu_m}$ for all $\nu_1,\dots,\nu_m \in \mathbb{N}$.

The following elementary observation will play a key role in the proof of Theorem 2.3.4.

Lemma 2.3.11. *Consider the fomal power series $f(s) := \sum\limits_{n\geq 1}\frac{s^n}{n^2}$ with $s \in \mathbb{C}$. Then,*

$$f(s)^2 \ll 16s\,f(s).$$

Proof. We have:

$$\begin{aligned} f(s)^2 &= \left(\sum_{l\geq 1}\frac{s^l}{l^2}\right)\left(\sum_{r\geq 1}\frac{s^r}{r^2}\right) = \left(\sum_{l,\,r\geq 1}\frac{s^{l+r}}{l^2\,r^2}\right) \stackrel{(a)}{=} \sum_{n\geq 1}\sum_{l=1}^{n}\frac{s^{n+1}}{l^2\,(n+1-l)^2} \\ &= s\sum_{n\geq 1}s^n\sum_{l=1}^{n}\frac{1}{l^2\,(n+1-l)^2}, \end{aligned}$$

where identity (a) follows by putting $l+r := n+1\geq 2$.

Now, if $l\leq\frac{n}{2}$, then $n+1-l\geq\frac{n}{2}+1>\frac{n}{2}$. Thus, either $l\geq\frac{n}{2}$ or $n+1-l>\frac{n}{2}$. Hence

$$l\,(n+1-l)\geq\frac{n}{2}\,l\quad\text{if } l\leq\frac{n}{2};\qquad\text{and}\qquad l\,(n+1-l)\geq\frac{n}{2}\,(n+1-l)\quad\text{if } l\geq\frac{n}{2}.$$

Therefore, we get:

$$\begin{aligned} \sum_{l=1}^{n}\frac{1}{l^2\,(n+1-l)^2} &= \sum_{l=1}^{[\frac{n}{2}]}\frac{1}{l^2\,(n+1-l)^2} + \sum_{l=[\frac{n}{2}]+1}^{n}\frac{1}{l^2\,(n+1-l)^2} \\ &\leq \sum_{l=1}^{[\frac{n}{2}]}\frac{4}{n^2 l^2} + \sum_{l=[\frac{n}{2}]+1}^{n}\frac{4}{n^2\,(n+1-l)^2} \leq \frac{4}{n^2}\sum_{l=1}^{+\infty}\frac{1}{l^2} + \frac{4}{n^2}\sum_{l=1}^{+\infty}\frac{1}{l^2} \\ &\leq 2\,\frac{4}{n^2}\,\frac{\pi^2}{6} < \frac{16}{n^2}, \end{aligned}$$

which proves the contention. $\square$

Proof of Theorem 2.3.4

Once $\psi_1(t)=\beta_1\,t_1+\cdots+\beta_m\,t_m\in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)\cap\ker\bar\partial$ has been chosen arbitrarily, for arbitrarily fixed representatives $\beta_1,\ldots,\beta_m$ of a basis $(\{\beta_1\}_{\bar\partial},\ldots,\{\beta_m\}_{\bar\partial})$ of $H^{0,\,1}(X,\,T^{1,\,0}X)$ (see explanations in §.2.3.1), we will define $\psi_\nu(t)$ to be the *minimal L^2_ω-norm solution* of equation (Eq. ν) for every $\nu\geq 2$. We will then go on to prove the absolute convergence in all the Hölder norms $|\ \ |_{k+\alpha}$, with $k\geq 2$ and $\alpha\in(0,\,1)$, of the power series (2.21) for this choice of $\psi_\nu(t)$'s and for all t in a sufficiently small (but depending on k) neighbourhood of 0 in B. That the C^k solution $\psi(z,\,t)$ (which is also *holomorphic* in t) obtained in this way is also C^∞ will be seen by a different argument in the end.

Thanks to the Neumann formula (2.34), the (unique) minimal L^2_ω-norm solution of equation (Eq. 2) is

$$\psi_2(t)=\bar\partial^\star\Delta''^{-1}\left(\frac{1}{2}\,[\psi_1(t),\,\psi_1(t)]\right). \tag{2.36}$$

Then, we continue by induction on $\nu\geq 2$: once $\psi_1(t),\ldots,\psi_\nu(t)$ have been constructed as the minimal L^2_ω-norm solutions of equations (Eq. 2), ..., (Eq. ν) respectively, we define $\psi_{\nu+1}(t)$ to be the minimal L^2_ω-norm solution of equation (Eq. $\nu+1$). Thanks to the Neumann formula (2.34), this is

$$\psi_{\nu+1}(t)=\bar\partial^\star\Delta''^{-1}\left(\frac{1}{2}\sum_{\mu=1}^{\nu}[\psi_\mu(t),\,\psi_{\nu+1-\mu}(t)]\right) = -\bar\partial^\star\Delta''^{-1}\varphi_{\nu+1}(t), \tag{2.37}$$

where the last identity follows from the definition and the expression proved for $\varphi_{\nu+1}(t)$ in the proof of Lemma 2.3.2.

We now form the power series $\psi(t)$ (see 2.21) with these choices of $\psi_\nu(t)$'s. We will need the

General Remark 2.3.12. *Let X be a compact complex manifold. Fix an integer $k \geq 2$ and $0 < \alpha < 1$. Then, there exists a* **constant** $C' = C'_{k,\alpha} > 0$ *such that*

$$|[\varphi,\, \psi]|_{k-1+\alpha} \leq C'\, |\varphi|_{k+\alpha}\, |\psi|_{k+\alpha} \quad \textit{for all } \varphi, \psi \in C^\infty_{0,1}(X,\, T^{1,0}X). \tag{2.38}$$

Proof. This follows trivially from the fact that φ and ψ contain vector fields $\partial/\partial z_j$ which, when used as 1-st order differential operators in the bracket $[\varphi,\, \psi]$, add one partial derivative to the $(k-1)$ partial derivatives already controlled by the Hölder norm $|\ |_{k-1+\alpha}$. □

We now come to the main estimate for the power series $\psi(t)$.

Claim 2.3.13. *Fix an integer $k \geq 2$ and $0 < \alpha < 1$. Then, there exist* **constants** $A = A_{k,\alpha} > 0$ *and $B = B_{k,\alpha} > 0$ such that*

$$|\psi|_{k+\alpha}(t) \ll \frac{A}{B}\, f(B\,(t_1 + \cdots + t_m)). \tag{2.39}$$

Proof of Claim 2.3.13. Recall that $\sigma_1(t) = \psi_1(t) = \beta_1\, t_1 + \cdots + \beta_m\, t_m$. Thus, $\sigma_1(t)$ satisfies the estimate

$$|\sigma_1|_{k+\alpha}(t) \ll \frac{A}{B}\, f(B\,(t_1 + \cdots + t_m)) \tag{2.40}$$

if $A > 0$ is *large enough*, for any $B > 0$. Indeed, since $\sigma_1(t)$ is homogeneous of degree 1 in t, only the term $B\,(t_1 + \cdots + t_m)$ of $f(B\,(t_1 + \cdots + t_m))$ is relevant on the right of (2.40) and the two occurrences of B cancel each other out. We choose $A > 0$ large enough for (2.40) to hold.

Next, we pick any positive constant B such that

$$B > 64\, c_{k,\alpha} C' A, \tag{2.41}$$

where $c_{k,\alpha} > 0$ is the constant occuring in the general estimate (2.35) of Lemma 2.3.9, $C' > 0$ is the constant occuring in the general estimate (2.38) and $A > 0$ is constant chosen to satisfy (2.40).

With these choices of constants A and B, we will prove that

$$|\sigma_\nu|_{k+\alpha}(t) \ll \frac{A}{B}\, f(B\,(t_1 + \cdots + t_m)), \tag{2.42}$$

for every $\nu \geq 1$, by induction on ν. This will prove (2.39). Recall that we denote $\sigma_\nu(t) := \psi_1(t) + \cdots + \psi_\nu(t)$ for every $\nu \geq 1$.

The case $\nu = 1$ was settled in (2.40). Suppose that we have proved (2.42) for ν. We will now prove it for $\nu + 1$.

The first, second and third comparisons below are respectively implied by estimates (2.38), (2.42) and Lemma 2.3.11:

$$\begin{aligned} |[\sigma_\nu,\, \sigma_\nu]|_{k-1+\alpha}(t) \ &\ll\ C'\, |\sigma_\nu|_{k+\alpha}(t)\, |\sigma_\nu|_{k+\alpha}(t) \ll C'\, \frac{A^2}{B^2}\, f^2(B\,(t_1 + \cdots + t_m)) \\ &\ll\ 16\, C'\, \frac{A^2}{B}\, (t_1 + \cdots + t_m)\, f(B\,(t_1 + \cdots + t_m)). \end{aligned}$$

Since $2\,\varphi_{\nu+1}(t)$ is the homogeneous part of degree $\nu+1$ in $t = (t_1, \ldots, t_m)$ of $-[\sigma_\nu(t),\, \sigma_\nu(t)]$ (cf. proof of Lemma 2.3.2), the homogeneous part of degree $\nu+1$ in t of the above estimate reads:

$$\begin{aligned} |2\,\varphi_{\nu+1}|_{k-1+\alpha}(t) \ &\ll\ 16\, C'\, \frac{A^2}{B}\, (t_1 + \cdots + t_m)\, \frac{B^\nu\, (t_1 + \cdots + t_m)^\nu}{\nu^2} \\ &=\ \frac{16\, C' A^2 B^{\nu-1}}{\nu^2}\, (t_1 + \cdots + t_m)^{\nu+1}. \end{aligned} \tag{2.43}$$

Therefore, we get:

$$\begin{aligned}
|\psi_{\nu+1}|_{k+\alpha}(t) &= |\bar\partial^\star \Delta''^{-1}\varphi_{\nu+1}|_{k+\alpha}(t) \overset{(a)}{\ll} c_{k,\,\alpha}\,|\varphi_{\nu+1}|_{k-1+\alpha}(t) \\
&\overset{(b)}{\ll} c_{k,\,\alpha}\,\frac{8\,C'A^2B^{\nu-1}}{\nu^2}\,(t_1+\cdots+t_m)^{\nu+1} \overset{(c)}{\ll} \frac{AB^\nu}{8\nu^2}\,(t_1+\cdots+t_m)^{\nu+1} \\
&\overset{(d)}{\ll} \frac{AB^\nu}{(\nu+1)^2}\,(t_1+\cdots+t_m)^{\nu+1}, \qquad (2.44)
\end{aligned}$$

where (a) follows from the general estimate (2.35) of Lemma 2.3.9, (b) follows from (2.43), (c) follows from the choice (2.41) of B, while (d) follows from the trivial inequality $(1/8\nu^2) < 1/(\nu+1)^2$ that holds for all $\nu \geq 1$.

Since the last quantity on the right-hand side of (2.44) is precisely the homogeneous part of degree $\nu+1$ in t of $(A/B)\,f(B\,(t_1+\cdots+t_m))$, this proves (2.42) for $\nu+1$ and finishes the inductive process.

Claim 2.3.13 is proved. □

End of proof of Theorem 2.3.4.

Let us consider the formal power series with constant numerical coefficients:

$$a(t) := (A/B)\,f(B\,(t_1+\cdots+t_m)),$$

where $f(s) := \sum_{\nu\geq 1}(s^\nu/\nu^2)$ is the formal power series defined in Lemma 2.3.11. Since the radius of convergence of $f(s)$ is 1, for every fixed integer $k \geq 2$, the power series $a(t)$ *converges absolutely* for all $t \in B_{\varepsilon_0(k)} \subset \mathbb{C}^m$, where $B_{\varepsilon_0(k)}$ is the open ball centred at the origin of radius $\varepsilon_0(k)$ in $\mathbb{C}^m$. We may choose any $\varepsilon_0(k) > 0$ such that

$$\varepsilon_0(k) < \frac{1}{m\,B_k},$$

where $B_k := B > 64\,c_{k,\alpha}C'A$ is the constant (necessarily dependent on k) chosen in (2.41).

Consequently, Claim 2.3.13 implies that for every fixed integer $k \geq 2$, $\psi(t) \in C^k_{0,\,1}(X,\,T^{1,\,0}X)$ for all $t \in B_{\varepsilon_0(k)}$. However, B_k might tend to $+\infty$ when k tends to $+\infty$, in which case $\varepsilon_0(k)0$ tends to 0 and the ball $B_{\varepsilon_0(k)}$ on which $\psi(t)$ is of class C^k shrinks to a point. In this case, we cannot infer that $\psi(t)$ is of class C^∞ for t in some open ball centred at $0 \in B$. So, we need an extra argument to pass from C^k for all $k \in \mathbb{N}$ to C^∞.

Before spelling it out, note that, in local coordinates $(z,\,t)$, we have

$$\psi(t) = \psi(z,\,t) = \sum_{\alpha,\,\mu=1}^n \psi^\alpha_{\bar\mu}(z,\,t)\,d\bar z_\mu \otimes \frac{\partial}{\partial z_\alpha},$$

where the coefficients $\psi^\alpha_{\bar\mu}(z,\,t)$ are C^k functions of $(z,\,t)$ which are even *holomorphic* in t (by construction as a convergent power series).

Now, $\psi(z,\,t)$ is C^∞ in $(z,\,t)$ if and only if $\psi^\alpha_{\bar\mu}(z,\,t)$ is C^∞ in $(z,\,t)$ for all α and all μ. Recall that $\psi_\nu(t) \in \mathrm{Im}\,\bar\partial^\star \subset \ker\bar\partial^\star$ for all $\nu \geq 2$, because it is the solution of minimal L^2 norm of a $\bar\partial$ equation. This implies that $\Delta''\psi_\nu(t) = \bar\partial^\star\bar\partial\psi_\nu(t)$ for all $\nu \geq 2$, hence

$$\Delta''\bigg(\sum_{\nu=2}^{+\infty}\psi_\nu(t)\bigg) = \bar\partial^\star\bar\partial\bigg(\sum_{\nu=1}^{+\infty}\psi_\nu(t)\bigg) = \bar\partial^\star\bar\partial\psi(t),$$

where the first identity follows from $\bar\partial\psi_1(t) = 0$.

Meanwhile, we know that $\bar\partial\psi(t) = (1/2)\,[\psi(t),\,\psi(t)]$ (the *integrability condition*). Hence:

$$\Delta''\bigg(\sum_{\nu=2}^{+\infty}\psi_\nu(t)\bigg) = \bar\partial^\star\bar\partial\psi(t) = \frac{1}{2}\,\bar\partial^\star[\psi(t),\,\psi(t)].$$

Since $\partial\psi(t)/\partial\bar t_\lambda = 0$ for all $\lambda\in\{1,\dots,m\}$ (because $\psi(t)$ is *holomorphic* in $t=(t_1,\dots,t_m)$), we infer that

$$\sum_{\lambda=1}^{m}\frac{\partial^2\psi(t)}{\partial t_\lambda\,\partial\bar t_\lambda} + \Delta''\psi(t) - \frac{1}{2}\,\bar\partial^\star[\psi(t),\,\psi(t)] - \bar\partial\bar\partial^\star\psi_1(t) = 0. \tag{2.45}$$

Now, this 2-nd order PDE, for which $\psi(t)$ is a solution, is *elliptic* for t in a small enough neighbourhood of 0 in B. To see this, note that the sum of the first two terms on the l.h.s. of (2.45) constitutes a Laplacian in the $(z,\,t)$ variables, which is an elliptic operator, while the principal part of the third term $-\frac{1}{2}\,\bar\partial^\star[\psi(t),\,\psi(t)]$ is

$$-\frac{1}{2}\sum_{\lambda=1}^{n}\sum_{\beta,\gamma=1}^{n}\omega^{\bar\gamma\beta}\sum_{\mu=1}^{n}\bigg[\psi^\mu(t)\,\frac{\partial^2\psi^\lambda_{\bar\gamma}(t)}{\partial z_\beta\,\partial z_\mu} - \psi^\mu_{\bar\gamma}(t)\sum_{\nu=1}^{n}\frac{\partial^2\psi^\lambda_{\bar\nu}(t)}{\partial z_\beta\,\partial z_\mu}\bigg]\frac{\partial}{\partial z_\lambda}.$$

Now, recall that $\psi(0)=0$. Hence, the coefficients $\psi^\mu(t)$ and $\psi^\mu_{\bar\gamma}(t)$ tend to 0 when $|t|\to 0$. Therefore, when $|t|$ is small enough, the coefficients of the principal part of $-\frac{1}{2}\,\bar\partial^\star[\psi(t),\,\psi(t)]$ are small and can be absorbed into the coefficients of the principal part of Δ''.

We can now conclude: since $\psi(z,\,t)$ is a solution of an *elliptic* PDE for all t such that $|t|<\varepsilon^\star$, for some $\varepsilon^\star>0$ small enough, $\psi(z,\,t)$ depends in a C^∞ way on $(z,\,t)\in X\times B_{\varepsilon^\star}$. Since, as has been explained, it is also *holomorphic* in t, the proof of Theorem 2.3.4 is complete. □

2.4 The Bogomolov-Tian-Todorov theorem

Let X be a compact complex manifold with $\dim_{\mathbb{C}}X = n$. We saw in §.2.3 that, if the vector space $H^{0,\,2}(X,\,T^{1,\,0}X)$ that contains all the possible qualitative obstructions to locally deforming the complex structure of X vanishes, then we can indeed deform this complex structure in all the available directions. These directions are parametrised by the vector space $H^{0,\,1}(X,\,T^{1,\,0}X)$. So, there are no quantitative obstructions.

The main result of §.2.3 throws up the natural question of what happens if the vector space $H^{0,\,2}(X,\,T^{1,\,0}X)$ does not vanish. Can we still deform the complex structure of X in all the available directions, at least under certain assumptions on X? In this section we will pin down a pair of assumptions on X under which this happens, namely there are no obstructions to deforming such an X. As in the previous case of §.2.3, these assumptions constitute only a sufficient condition for the unobstructedness of the small deformations of X in all the available directions.

In the classical Bogomolov-Tian-Todorov theorem, the first of the above mentioned hypotheses on X was its Kählerianity. We will see that this assumption can be weakened to the requirement that X be only a *$\partial\bar\partial$-manifold*. The second hypothesis that is often made on X requires it to also belong to another important class of compact complex manifolds that we now discuss briefly.

2.4.1 Calabi-Yau manifolds

For many authors, Calabi-Yau (C-Y) manifolds are at least supposed to be Kähler and often even projective. We will not make any Kählerianity assumption on them, but will strip them to their one key feature.

Definition 2.4.1. *A compact complex manifold X is said to be a* **Calabi-Yau manifold** *if its canonical bundle K_X is* **trivial**.

Let $n = \dim_{\mathbb{C}} X$. Recall that the *canonical bundle* of X is the holomorphic line bundle of $(n,\,0)$-forms on X:

$$K_X := \Lambda^{n,\,0}T^\star X = \det(\Lambda^{1,\,0}T^\star X) = -\det(T^{1,\,0}X).$$

Thus, if $(z_1,\dots,z_n)$ is a system of local holomorphic coordinates on X, the holomorphic n-form $dz_1\wedge\cdots\wedge dz_n$ defines a local holomorphic frame of K_X. As with any holomorphic line bundle, the triviality is equivalent to the existence of a *non-vanishing* global holomorphic section:

$$\begin{aligned} K_X \quad \text{is trivial} \qquad & \iff \quad \exists\, u\in H^0(X,\,K_X)\simeq H^{n,\,0}_{\bar\partial}(X,\,\mathbb{C}) \quad \text{such that} \quad u(x)\neq 0\ \forall\, x\in X \\ & \iff \quad \exists\, u\in C^\infty_{n,\,0}(X,\,\mathbb{C}) \quad \text{such that} \quad \bar\partial u = 0 \quad \text{and} \quad u(x)\neq 0\ \forall\, x\in X. \end{aligned}$$

When K_X is trivial, the Hodge number $h^{n,\,0}_{\bar\partial}=1$, so the non-vanishing holomorphic n-form u on X is *unique up to a multiplicative constant*. Such a form will be called a *Calabi-Yau form*. Note that $H^{n,\,0}_{\bar\partial}(X,\,\mathbb{C}) = C^\infty_{n,\,0}(X,\,\mathbb{C})\cap\ker\bar\partial$ since, for bidegree reasons, the only $\bar\partial$-exact $(n,\,0)$-form is zero. So, every $u\in C^\infty_{n,\,0}(X,\,\mathbb{C})\cap\ker\bar\partial$ identifies with $[u]_{\bar\partial}\in H^{n,\,0}_{\bar\partial}(X,\,\mathbb{C})\simeq H^0(X,\,K_X)$.

Before moving on, let us mention the following elementary observation that will prove useful in computations in local coordinates.

Lemma 2.4.2. *Let X be an n-dimensional Calabi-Yau manifold. Fix a form $u\in C^\infty_{n,\,0}(X,\,\mathbb{C})\cap\ker\bar\partial$ such that $u(z)\neq 0$ for every $z\in X$. Then, for every $x\in X$, there exist local holomorphic coordinates $z_1,\dots,z_n$ centred at x on an open neighbourhood U of x such that*

$$u_{|U} = dz_1\wedge\cdots\wedge dz_n.$$

Proof. Let $w_1,\dots,w_n$ be arbitrary local holomorphic coordinates centred at x on an open neighbourhood $\tilde U$ of x. Let $U = D(x_1,\,R_1)\times\cdots\times D(x_n,\,R_n)\subset\tilde U$ be an open polydisc about $x=(x_1,\dots,x_n)$. Then $u_{|U} = f\,dw_1\wedge\cdots\wedge dw_n$ for some non-vanishing holomorphic function $f: U\to\mathbb{C}$. Define new local holomorphic coordinates $z_1,\dots,z_n$ centred at x by $z_2:=w_2,\dots z_n:=w_n$ and

$$z_1(w_1,\,w_2,\dots,w_n) = \int\limits_{x_1}^{w_1} f(t,\;w_2,\dots,w_n)\,dt, \qquad (w_1,\,w_2,\dots,w_n)\in U,$$

where the integral can be taken along any path in $D(x_1,\,R_1)$ from x_1 to w_1 because the disc is (obviously) simply connected. We get:

$$\frac{\partial z_1}{\partial w_1} = f \quad \text{and} \quad dz_1\wedge\cdots\wedge dz_n = \det\left(\frac{\partial z_j}{\partial w_k}\right)_{1\leq j,\,k\leq n} dw_1\wedge\cdots\wedge dw_n = f\,dw_1\wedge\cdots\wedge dw_n = u_{|U}.$$

Since f is non-vanishing, $z_1,\dots,z_n$ are indeed coordinates. The proof is complete. $\square$

Lemma and Definition 2.4.3. *Suppose that K_X is* **trivial** *and let u be a* **Calabi-Yau form** *on X. Then, for every $q = 0, \dots, n$, u defines an isomorphism (that will be called the* **Calabi-Yau isomorphism***):*

$$T_u \ : \ C^\infty_{0,\,q}(X,\, T^{1,\,0}X) \xrightarrow{\cdot\lrcorner u} C^\infty_{n-1,\,q}(X,\, \mathbb{C}) \tag{2.46}$$

mapping any $\theta \in C^\infty_{0,\,q}(X,\, T^{1,\,0}X)$ to $T_u(\theta) := \theta\lrcorner u$, where the operation denoted by $\cdot\lrcorner$ combines the contraction of u by the $(1,\,0)$-vector field component of θ with the exterior multiplication by the $(0,\,q)$-form component.

Proof. Once we have fixed a system $(z_1, \dots, z_n)$ of local holomorphic coordinates on some open subset $U \subset X$, we can write:

$$\theta = \sum_{\substack{|J|=q\\ 1\leq j\leq n}} \theta^j_{\bar{J}}\, d\bar{z}_J \otimes \frac{\partial}{\partial z_j} \in C^\infty_{0,\,q}(U,\, T^{1,\,0}X) \quad \text{and} \quad u = f\, dz_1 \wedge \dots \wedge dz_n,$$

for some C^∞ functions θ^j_J and some *non-vanishing* C^∞ function f on U. We have:

$$T_u(\theta) = \theta\lrcorner u = \sum_{\substack{|J|=q\\ 1\leq j\leq n}} (-1)^{j-1}\, f\theta^j_{\bar{J}}\, d\bar{z}_J \wedge dz_1 \wedge \dots \wedge \widehat{dz_j} \wedge \dots \wedge dz_n,$$

where the symbol $\widehat{\ }$ indicates a missing factor.

The map T_u is injective and surjective because f does not vanish. $\square$

We shall now compare, in the case $q = 1$, the images under the operation $\cdot\lrcorner u$ of the kernel and the image of $\bar\partial$ with the analogous subspaces of $C^\infty_{n-1,\,1}(X,\, \mathbb{C})$, with the result that the Calabi-Yau isomorphism is extended in cohomology.

Lemma and Definition 2.4.4. *Suppose that K_X is* **trivial** *and let u be a* **Calabi-Yau form** *on X. Then, when $q = 1$, the isomorphism T_u of (2.46) satisfies:*

$$T_u(\ker\bar\partial) = \ker\bar\partial \quad \textit{and} \quad T_u(Im\,\bar\partial) = Im\,\bar\partial. \tag{2.47}$$

Hence T_u induces an isomorphism in cohomology

$$T_{[u]} \ : \ H^{0,\,1}(X,\, T^{1,\,0}X) \xrightarrow{\cdot\lrcorner[u]} H^{n-1,\,1}(X,\, \mathbb{C}) \tag{2.48}$$

defined by $T_{[u]}([\theta]) = [\theta\lrcorner u]$ for all $[\theta] \in H^{0,\,1}(X,\, T^{1,\,0}X)$.

The isomorphism $T_{[u]}$ will be called the **Calabi-Yau isomorphism in cohomology***.*

Proof. It relies on the following formulae that the reader can easily check in local coordinates:

$$\bar\partial(\theta\lrcorner u) = (\bar\partial\theta)\lrcorner u + \theta\lrcorner(\bar\partial u) = (\bar\partial\theta)\lrcorner u, \;\; \bar\partial(\xi\lrcorner u) = (\bar\partial\xi)\lrcorner u - \xi\lrcorner(\bar\partial u) = (\bar\partial\xi)\lrcorner u \tag{2.49}$$

for all $\theta \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$ and all $\xi \in C^\infty(X,\, T^{1,\,0}X)$. Note, however, that the analogous identities for ∂ fail. These formulae imply the inclusions:

$$T_u(\ker\bar\partial) \subset \ker\bar\partial \quad \text{and} \quad T_u(\mathrm{Im}\,\bar\partial) \subset \mathrm{Im}\,\bar\partial.$$

To prove the reverse inclusion of the former equality in (2.47), suppose that $\theta\lrcorner u \in \ker\bar\partial$ for some $\theta \in C^\infty_{0,1}(X, T^{1,0}X)$. By (2.49), this means that $(\bar\partial\theta)\lrcorner u = 0$, which is equivalent to $\bar\partial\theta = 0$ since the map T_u of (2.46) is an isomorphism.

To prove the reverse inclusion of the latter equality in (2.47), let $\theta \in C^\infty_{0,1}(X, T^{1,0}X)$ such that $\theta\lrcorner u = \bar\partial v$ for some $(n-1,0)$-form v. If we let

$$v = \sum_j v_j\, dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$$

with respect to local holomorphic coordinates $z_1, \dots, z_n$ on some open subset $U \subset X$ and use the notation in the proof of Lemma and Definition 2.4.3, the identity $\theta\lrcorner u = \bar\partial v$ reads:

$$\sum_{j,k}(-1)^{j-1} f\theta^j_{\bar k}\, d\bar z_k \wedge dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n = \sum_{j,k}\frac{\partial v_j}{\partial \bar z_k}\, d\bar z_k \wedge dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n,$$

which is equivalent to $\theta^j_{\bar k} = \frac{\partial}{\partial \bar z_k}((-1)^{j-1}\,\frac{v_j}{f})$ for all j, k since f is holomorphic without zeroes. Setting $\xi_j := (-1)^{j-1}\,\frac{v_j}{f}$ for all j, we get the local representative of a global vector field

$$\xi := \sum_j \xi_j\,\frac{\partial}{\partial z_j} \in C^\infty(X, T^{1,0}X)$$

satisfying $\theta = \bar\partial\xi$ on X. Hence $\theta \in \operatorname{Im}\bar\partial$. We have thus proved that $\operatorname{Im}\bar\partial \subset T_u(\operatorname{Im}\bar\partial)$, hence the latter identity in (2.47). □

We shall now see that the isomorphisms T_u of (2.46) and their inverses enable one to extend the Lie bracket of $T^{1,0}X$ to $\mathbb{C}$-valued $(n-1, q)$-forms.

Definition 2.4.5. *([Tia87, p. 631]) Suppose that K_X is* **trivial** *and let u be a* **Calabi-Yau form** *on X. For all $q_1, q_2 \in \{0, \dots, n\}$, define the following bracket:*

$$[\cdot,\,\cdot] : C^\infty_{n-1,\,q_1}(X, \mathbb{C}) \times C^\infty_{n-1,\,q_2}(X, \mathbb{C}) \longrightarrow C^\infty_{n-1,\,q_1+q_2}(X, \mathbb{C}),$$

$$[\zeta_1,\,\zeta_2] := T_u\Big[T_u^{-1}\zeta_1,\,T_u^{-1}\zeta_2\Big], \tag{2.50}$$

where the operation $[\ ,\ ]$ on the right-hand side of (2.50) combines the Lie bracket of the $T^{1,0}X$-parts of $T_u^{-1}\zeta_1 \in C^\infty_{0,\,q_1}(X, T^{1,0}X)$ and $T_u^{-1}\zeta_2 \in C^\infty_{0,\,q_2}(X, T^{1,0}X)$ with the wedge product of their $(0, q_1)$– and respectively $(0, q_2)$-form parts.

In other words, putting $\zeta_1 := \Phi_1\lrcorner u$ and $\zeta_2 := \Phi_2\lrcorner u$ with (uniquely determined) $\Phi_1 \in C^\infty_{0,\,q_1}(X, T^{1,0}X)$ and $\Phi_2 \in C^\infty_{0,\,q_2}(X, T^{1,0}X)$, we define:

$$[\Phi_1\lrcorner u,\,\Phi_2\lrcorner u] := [\Phi_1,\,\Phi_2]\lrcorner u. \tag{2.51}$$

The main technical ingredient in [Tia87] and [Tod89] was the following general observation, the so-called **Tian-Todorov lemma** (cf. Lemma 3.1. in [Tia87], Lemma 1.2.4. in [Tod89]).

Lemma 2.4.6. *Let X be a compact complex manifold ($n = \dim_{\mathbb{C}} X$) such that K_X is* **trivial**. *Then, for any forms $\zeta_1, \zeta_2 \in C^\infty_{n-1,1}(X, \mathbb{C})$ such that $\partial\zeta_1 = \partial\zeta_2 = 0$, we have*

$$[\zeta_1,\,\zeta_2]\in Im\,\partial.$$

More precisely, the identity

$$[\theta_1\lrcorner u,\,\theta_2\lrcorner u]=-\partial(\theta_1\lrcorner(\theta_2\lrcorner u)) \tag{2.52}$$

holds for $\theta_1,\theta_2\in C^\infty_{0,1}(X,\,T^{1,0}X)$ *whenever* $\partial(\theta_1\lrcorner u)=\partial(\theta_2\lrcorner u)=0$.

Proof. We will make a computation in local coordinates $z_1,\dots,z_n$ chosen such that $u=dz_1\wedge\cdots\wedge dz_n$ on some open subset $U\subset X$. Such coordinates exist by Lemma 2.4.2.

To lighten the notation, we will sometimes put $\widehat{dz_\lambda}:=dz_1\wedge\cdots\wedge\widehat{dz_\lambda}\wedge\cdots\wedge dz_n$ for every λ. Similarly, we sometimes put $\widehat{dz_\lambda\wedge dz_\mu}:=dz_1\wedge\cdots\wedge\widehat{dz_\lambda}\wedge\cdots\wedge\widehat{dz_\mu}\wedge\cdots\wedge dz_n$ when $\lambda<\mu$.

Let

$$\theta_{1|U}=\varphi=\sum_{\lambda=1}^n\varphi^\lambda\,\frac{\partial}{\partial z_\lambda}\quad\text{ and }\quad\theta_{2|U}=\psi=\sum_{\mu=1}^n\psi^\mu\,\frac{\partial}{\partial z_\mu},$$

with $\varphi^\lambda,\psi^\mu\in C^\infty_{0,1}(U,\,\mathbb{C})$. From Definition 2.2.11, we get:

$$[\varphi\lrcorner u,\,\psi\lrcorner u]=[\varphi,\,\psi]\lrcorner u=\sum_{\lambda=1}^n(-1)^{\lambda-1}\left[\sum_{\mu=1}^n\left(\varphi^\mu\wedge\frac{\partial\psi^\lambda}{\partial z_\mu}+\psi^\mu\wedge\frac{\partial\varphi^\lambda}{\partial z_\mu}\right)\right]\wedge\widehat{dz_\lambda}. \tag{2.53}$$

On the other hand, we have: $\psi\lrcorner u=\sum\limits_{\mu=1}^n(-1)^{\mu-1}\,\psi^\mu\wedge dz_1\wedge\cdots\wedge\widehat{dz_\mu}\wedge\cdots\wedge dz_n$. Hence,

$$\begin{aligned}\varphi\lrcorner(\psi\lrcorner u)&=\sum_{\lambda,\,\mu=1}^n(-1)^\mu\,\varphi^\lambda\wedge\psi^\mu\wedge\frac{\partial}{\partial z_\lambda}\lrcorner(dz_1\wedge\cdots\wedge\widehat{dz_\mu}\wedge\cdots\wedge dz_n)\\&=\sum_{\lambda<\mu}(-1)^{\lambda+\mu-1}\,\varphi^\lambda\wedge\psi^\mu\wedge dz_1\wedge\cdots\wedge\widehat{dz_\lambda}\wedge\cdots\wedge\widehat{dz_\mu}\wedge\cdots\wedge dz_n\\&+\sum_{\lambda>\mu}(-1)^{\lambda+\mu}\,\varphi^\lambda\wedge\psi^\mu\wedge dz_1\wedge\cdots\wedge\widehat{dz_\mu}\wedge\cdots\wedge\widehat{dz_\lambda}\wedge\cdots\wedge dz_n.\end{aligned}$$

By permuting the indices λ and μ in the last sum, we get:

$$\varphi\lrcorner(\psi\lrcorner u)=\sum_{\lambda<\mu}(-1)^{\lambda+\mu-1}\,(\varphi^\lambda\wedge\psi^\mu-\varphi^\mu\wedge\psi^\lambda)\wedge dz_1\wedge\cdots\wedge\widehat{dz_\lambda}\wedge\cdots\wedge\widehat{dz_\mu}\wedge\cdots\wedge dz_n.$$

Applying ∂, we get:

$$\begin{aligned}\partial(\varphi\lrcorner(\psi\lrcorner u))&=\sum_{\lambda<\mu}(-1)^{\lambda+\mu-1}\left(\sum_{\rho=1}^n\frac{\partial(\varphi^\lambda\wedge\psi^\mu-\varphi^\mu\wedge\psi^\lambda)}{\partial z_\rho}\wedge dz_\rho\right)\wedge\widehat{dz_\lambda\wedge dz_\mu}\\&=\sum_{\lambda<\mu}(-1)^\mu\,\frac{\partial(\varphi^\lambda\wedge\psi^\mu-\varphi^\mu\wedge\psi^\lambda)}{\partial z_\lambda}\wedge\widehat{dz_\mu}+\sum_{\lambda<\mu}(-1)^{\lambda-1}\,\frac{\partial(\varphi^\lambda\wedge\psi^\mu-\varphi^\mu\wedge\psi^\lambda)}{\partial z_\mu}\wedge\widehat{dz_\lambda}.\end{aligned}$$

After permuting the indices λ and μ in the last sum, we get:

$$\begin{aligned}\partial(\varphi\lrcorner(\psi\lrcorner u))&=\sum_\mu^n(-1)^{\mu-1}\left(\sum_{\lambda>\mu}\frac{\partial(\varphi^\mu\wedge\psi^\lambda-\varphi^\lambda\wedge\psi^\mu)}{\partial z_\lambda}-\sum_{\lambda<\mu}\frac{\partial(\varphi^\lambda\wedge\psi^\mu-\varphi^\mu\wedge\psi^\lambda)}{\partial z_\lambda}\right)\wedge\widehat{dz_\mu}\\&=\sum_\mu^n(-1)^{\mu-1}\left[\varphi^\mu\wedge\sum_{\lambda>\mu}\frac{\partial\psi^\lambda}{\partial z_\lambda}-\sum_{\lambda>\mu}\psi^\lambda\wedge\frac{\partial\varphi^\mu}{\partial z_\lambda}+\psi^\mu\wedge\sum_{\lambda>\mu}\frac{\partial\varphi^\lambda}{\partial z_\lambda}-\sum_{\lambda>\mu}\varphi^\lambda\wedge\frac{\partial\psi^\mu}{\partial z_\lambda}\right.\\&+\left.\psi^\mu\wedge\sum_{\lambda<\mu}\frac{\partial\varphi^\lambda}{\partial z_\lambda}-\sum_{\lambda<\mu}\varphi^\lambda\wedge\frac{\partial\psi^\mu}{\partial z_\lambda}+\varphi^\mu\wedge\sum_{\lambda<\mu}\frac{\partial\psi^\lambda}{\partial z_\lambda}-\sum_{\lambda<\mu}\psi^\lambda\wedge\frac{\partial\varphi^\mu}{\partial z_\lambda}\right]\wedge\widehat{dz_\mu}.\end{aligned}$$

Collecting terms yields the first identity below, while the second one follows by permuting the indices λ and μ:

$$\begin{aligned}\partial(\varphi\lrcorner(\psi\lrcorner u)) &= \sum_{\mu=1}^{n}(-1)^{\mu-1}\left[\varphi^\mu\wedge\sum_{\lambda\neq\mu}\frac{\partial\psi^\lambda}{\partial z_\lambda}-\sum_{\lambda\neq\mu}\psi^\lambda\wedge\frac{\partial\varphi^\mu}{\partial z_\lambda}+\psi^\mu\wedge\sum_{\lambda\neq\mu}\frac{\partial\varphi^\lambda}{\partial z_\lambda}-\sum_{\lambda\neq\mu}\varphi^\lambda\wedge\frac{\partial\psi^\mu}{\partial z_\lambda}\right]\wedge\widehat{dz_\mu}\\ &= \sum_{\lambda=1}^{n}(-1)^{\lambda-1}\left[\sum_{\mu=1}^{n}\left(\varphi^\lambda\wedge\frac{\partial\psi^\mu}{\partial z_\mu}-\psi^\mu\wedge\frac{\partial\varphi^\lambda}{\partial z_\mu}+\psi^\lambda\wedge\frac{\partial\varphi^\mu}{\partial z_\mu}-\varphi^\mu\wedge\frac{\partial\psi^\lambda}{\partial z_\mu}\right)\right]\wedge\widehat{dz_\lambda}.\end{aligned}$$

Note that the last sum runs a priori over the indices $\mu\neq\lambda$. However, the paranthesis term of the sum $\sum_\mu$ corresponding to $\mu=\lambda$ vanishes in an obvious way (pairwise cancellations of the terms).

Recalling (2.53), we see that the sums featuring in the second and fourth terms in the paranthesis under $\sum_{\mu=1}^n$ in the above expression for $\partial(\varphi\lrcorner(\psi\lrcorner u))$ yield $-[\varphi\lrcorner u,\,\psi\lrcorner u]$. Therefore, we get:

$$\begin{aligned}\partial(\varphi\lrcorner(\psi\lrcorner u)) &= -[\varphi\lrcorner u,\,\psi\lrcorner u]\\ &+ \sum_{\lambda=1}^{n}(-1)^{\lambda-1}\,\varphi^\lambda\wedge\sum_{\mu=1}^{n}\frac{\partial\psi^\mu}{\partial z_\mu}\wedge\widehat{dz_\lambda}+\sum_{\lambda=1}^{n}(-1)^{\lambda-1}\,\psi^\lambda\wedge\sum_{\mu=1}^{n}\frac{\partial\varphi^\mu}{\partial z_\mu}\wedge\widehat{dz_\lambda}.\end{aligned}\tag{2.54}$$

Let us now express the hypothesis $\partial(\varphi\lrcorner u)=\partial(\psi\lrcorner u)=0$ in local coordinates.

Applying ∂ in the expression we had above for $\psi\lrcorner u$, we get:

$$\partial(\psi\lrcorner u)=\sum_{\mu=1}^{n}(-1)^{\mu-1}\left(\sum_{l=1}^{n}\frac{\partial\psi^\mu}{\partial z_l}\,dz_l\right)\wedge dz_1\wedge\cdots\wedge\widehat{dz_\mu}\wedge\cdots\wedge dz_n=\left(\sum_{\mu=1}^{n}\frac{\partial\psi^\mu}{\partial z_\mu}\right)\wedge dz_1\wedge\cdots\wedge dz_n.$$

Thus, the hypothesis $\partial(\psi\lrcorner u)=0$ is equivalent to $\sum_{\mu=1}^n(\partial\psi^\mu/\partial z_\mu)=0$. (Recall that this sum is a $\mathbb{C}$-valued $(0,\,1)$-form.) Similarly, the hypothesis $\partial(\varphi\lrcorner u)=0$ is equivalent to $\sum_{\mu=1}^n(\partial\varphi^\mu/\partial z_\mu)=0$.

We conclude that (2.54) reduces to

$$\partial(\varphi\lrcorner(\psi\lrcorner u))=-[\varphi\lrcorner u,\,\psi\lrcorner u]$$

unde the assumption $\partial(\varphi\lrcorner u)=\partial(\psi\lrcorner u)=0$. This proves (2.52) and completes the proof of Lemma 2.4.6. □

□

2.4.2 The unobstructedness theorem for Calabi-Yau $\partial\bar\partial$-manifolds

We are now in a position to state and prove the main result of this section. It is an occasion for witnessing the key role played by *$\partial\bar\partial$-manifolds* which do the same job in this context (see e.g. [Pop13, Theorem 1.2]) as the stronger compact *Kähler* manifolds of the original theorem (see [Tia87] and [Tod89]).

By the *Kuranishi family* of X being *unobstructed*, we will mean that (the complex structure of) X can be locally deformed in all the available directions. In other words, the conclusion of Theorem 2.3.4 holds. Equivalently, there is a holomorphic family of compact complex manifolds $\pi:\mathcal{X}\longrightarrow B$ whose central fibre X_0 is the given X and whose base B is an open ball centred at the origin in $H^{0,\,1}(X,\,T^{1,\,0}X)$. This family of small deformations of X is called the *Kuranishi family* of X. The *Kuranishi family* is a general object that exists for every compact complex manifold X (see [Kur62]), even when the small deformations of X are *obstructed*, but the base B need not be smooth. In general, B is only an *analytic subset* of $H^{0,\,1}(X,\,T^{1,\,0}X)$. Unobstructedness of the Kuranishi family is thus equivalent to B being *smooth* and an *open ball* in $H^{0,\,1}(X,\,T^{1,\,0}X)$. The Kuranishi family being an *infinitesimal* notion, the size of B is irrelevant, so B can be shrunk at will about 0.

Theorem 2.4.7. *(Bogomolov-Tian-Todorov theorem for Calabi-Yau $\partial\bar\partial$-manifolds) Let X be a* **$\partial\bar\partial$-manifold** *whose canonical bundle K_X is* **trivial**. *Then, the Kuranishi family of X is* **unobstructed**.

Proof. This proof is taken from [Pop13].

Let $[\eta] \in H^{0,1}(X, T^{1,0}X)$ be an arbitrary nonzero class. Pick any d-closed representative w_1 of the class $[\eta]\lrcorner[u] \in H^{n-1,1}(X, \mathbb{C})$. Such a d-closed representative exists by (a) of Theorem 1.3.2 thanks to the $\partial\bar\partial$ assumption on X. This is virtually the only modification of the proof compared to the Kähler case where the Δ''-harmonic representative of the class $[\eta]\lrcorner[u]$ was chosen. Since $\Delta' = \Delta''$ in the Kähler case, Δ''-harmonic forms are also ∂-closed, hence d-closed, but this no longer holds in the non-Kähler case.

Since T_u is an isomorphism, there is a unique $\Phi_1 \in C^\infty_{0,1}(X, T^{1,0}X)$ such that $\Phi_1\lrcorner u = w_1$. Now $\bar\partial w_1 = 0$, so the former equality in (2.47) implies that $\bar\partial\Phi_1 = 0$. Moreover, since $[\Phi_1\lrcorner u] = [w_1]$, (2.48) implies that $[\Phi_1] = [\eta] \in H^{0,1}(X, T^{1,0}X)$ and this is the original class we started off with. However, Φ_1 need not be the Δ''-harmonic representative of the class $[\eta]$ in the non-Kaehler case (in contrast to the Kähler case of [Tia87] and [Tod89]). Meanwhile, by the choice of w_1, we have

$$\partial(\Phi_1\lrcorner u) = 0,$$

so the Tian-Todorov Lemma 2.4.6 applied to $\zeta_1 = \zeta_2 = \Phi_1\lrcorner u$ yields $[\Phi_1\lrcorner u,\, \Phi_1\lrcorner u] \in \operatorname{Im}\partial$. On the other hand, $[\Phi_1\lrcorner u,\, \Phi_1\lrcorner u] \in \ker\bar\partial$ by Lemma 2.3.2. By the $\partial\bar\partial$ property of X applied to the $(n-1,\, 2)$-form $1/2\,[\Phi_1\lrcorner u,\, \Phi_1\lrcorner u]$, there exists $\psi_2 \in C^\infty_{n-2,1}(X, \mathbb{C})$ such that

$$\bar\partial\partial\psi_2 = \frac{1}{2}\,[\Phi_1\lrcorner u,\, \Phi_1\lrcorner u].$$

We can choose ψ_2 of minimal L^2-norm with this property (i.e. $\psi_2 \in \operatorname{Im}(\partial\bar\partial)^\star$, see the orthogonal three-space decomposition for the Aeppli cohomology in (1) and (2) of Corollary 1.1.13). Put $w_2 := \partial\psi_2 \in C^\infty_{n-1,1}(X, \mathbb{C})$. Since T_u is an isomorphism, there is a unique $\Phi_2 \in C^\infty_{0,1}(X, T^{1,0}X)$ such that $\Phi_2\lrcorner u = w_2$. Implicitly, $\partial(\Phi_2\lrcorner u) = 0$. Moreover, using (2.49), we get

$$(\bar\partial\Phi_2)\lrcorner u = \bar\partial(\Phi_2\lrcorner u) = \frac{1}{2}\,[\Phi_1\lrcorner u,\, \Phi_1\lrcorner u] = \frac{1}{2}\,[\Phi_1,\, \Phi_1]\lrcorner u,$$

where the last identity follows from (2.51). Hence

$$\text{(Eq. 1)} \qquad \bar\partial\Phi_2 = \frac{1}{2}\,[\Phi_1,\, \Phi_1].$$

We can now continue inductively. Suppose we have constructed $\Phi_1, \dots, \Phi_{N-1} \in C^\infty_{0,1}(X, T^{1,0}X)$ such that

$$\partial(\Phi_k\lrcorner u) = 0 \quad \text{ and } \quad \bar\partial(\Phi_k\lrcorner u) = \frac{1}{2}\sum_{l=1}^{k-1}[\Phi_l\lrcorner u,\, \Phi_{k-l}\lrcorner u], \qquad 1 \le k \le N-1.$$

By formulae (2.49), (2.51) and since T_u is an isomorphism, the latter identity above is equivalent to

$$\text{(Eq. } (k-1)) \qquad \bar\partial\Phi_k = \frac{1}{2}\sum_{l=1}^{k-1}[\Phi_l,\, \Phi_{k-l}], \qquad 1 \le k \le N-1.$$

Then, again by Lemma 2.3.2, we have

$$\frac{1}{2}\sum_{l=1}^{N-1}[\Phi_l\lrcorner u,\, \Phi_{N-l}\lrcorner u] \in \ker\bar\partial.$$

On the other hand, since $\Phi_1\lrcorner u, \dots, \Phi_{N-1}\lrcorner u \in \ker\partial$, the Tian-Todorov Lemma 2.4.6 gives

$$[\Phi_l\lrcorner u,\, \Phi_{N-l}\lrcorner u] \in \operatorname{Im}\partial \quad \text{for all} \quad l = 1, \dots, N-1.$$

Thanks to the last two relations, the $\partial\bar\partial$ property of X implies the existence of a form $\psi_N \in C^\infty_{n-2,\,1}(X,\,\mathbb{C})$ such that

$$\bar\partial\partial\psi_N = \frac{1}{2}\sum_{l=1}^{N-1}[\Phi_l\lrcorner u,\, \Phi_{N-l}\lrcorner u].$$

We can choose ψ_N of minimal L^2-norm with this property (i.e. $\psi_N \in \operatorname{Im}(\partial\bar\partial)^\star$). Letting $w_N := \partial\psi_N \in C^\infty_{n-1,\,1}$, there exists a unique $\Phi_N \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$ such that $\Phi_N\lrcorner u = w_N$. Implicitly

$$\partial(\Phi_N\lrcorner u) = 0.$$

We also have $\bar\partial(\Phi_N\lrcorner u) = \frac{1}{2}\sum\limits_{l=1}^{N-1}[\Phi_l\lrcorner u,\, \Phi_{N-l}\lrcorner u]$ by construction. By formulae (2.49), (2.51) and since T_u is an isomorphism, this amounts to

$$(\text{Eq.}\,(N-1)) \qquad \bar\partial\Phi_N = \frac{1}{2}\sum_{l=1}^{N-1}[\Phi_l,\, \Phi_{N-l}].$$

We have thus shown inductively that the equation (Eq. k) is solvable for every $k \in \mathbb{N}^\star$. As we saw in §.2.3.2, this implies the convergence of the power series $\Phi(t) := \Phi_1\, t + \Phi_2\, t^2 + \cdots + \Phi_N\, t^N + \dots$ in all the Hölder norms $|\ \ |_{k+\alpha}$, with $k \geq 2$ and $\alpha \in (0,\,1)$, for all $t \in \mathbb{C}$ such that $|t| < \varepsilon_k$, because the ψ_ν's have been chosen of minimal L^2 norms with their respective properties. This produces a form $\Phi(t) \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$ that varies *holomorphically* with $t \in D(0,\, \varepsilon^\star) \subset \mathbb{C}$ for some $\varepsilon^\star > 0$ and defines a complex structure $\bar\partial_t$ on X that identifies with $\bar\partial - \Phi(t)$ and is the deformation of the original complex structure $\bar\partial$ of X in the direction of the originally given $[\eta] \in H^{0,\,1}(X,\, T^{1,\,0}X)$. The proof is complete. $\square$

We end this section by noticing that the full force of the $\partial\bar\partial$ assumption is not needed in Theorem 2.4.7, but only a special case thereof, since only two applications in very particular situations have been made of it.

First, we needed any Dolbeault cohomology class $[\alpha] \in H^{n-1,\,1}(X,\,\mathbb{C})$ (denoted by $[\eta]\lrcorner[u]$ in the proof) to be representable by a d-closed form. The proof of Lemma **??** shows this to be equivalent to requiring that any ∂-exact $(n,\,1)$-form $\partial\alpha$ for which $\bar\partial\alpha = 0$ be $\partial\bar\partial$-exact. This is equivalent to requiring the following linear map (which is always well defined)

$$A_1 \ :\ H^{n-1,\,1}_{\bar\partial}(X,\,\mathbb{C}) \longrightarrow H^{n,\,1}_{BC}(X,\,\mathbb{C}), \qquad [\alpha]_{\bar\partial} \mapsto [\partial\alpha]_{BC} \tag{2.55}$$

to vanish identically, where the subscript BC indicates a Bott-Chern cohomology group. By duality, the vanishing of A_1 is equivalent to the vanishing of its dual map

$$A_1^\star \ :\ H^{0,\,n-1}_{A}(X,\,\mathbb{C}) \longrightarrow H^{1,\,n-1}_{\bar\partial}(X,\,\mathbb{C}), \qquad [u]_A \mapsto [\partial u]_{\bar\partial}, \tag{2.56}$$

where the subscript A indicates an Aeppli cohomology group.

The other special case of the $\partial\bar\partial$ lemma needed in the proof of Theorem 2.4.7 was the requirement that any ∂-exact and d-closed $(n-1,\,2)$-form β (denoted by $[\Phi_1\lrcorner u,\, \Phi_1\lrcorner u]$ in the proof) be $\partial\bar\partial$-exact. This is equivalent to requiring the following linear map (which is always well defined)

$$B \;:\; H^{n-1,\,2}_{BC}(X,\,\mathbb{C}) \longrightarrow H^{n-1,\,2}_{\partial}(X,\,\mathbb{C}), \qquad [\beta]_{BC} \mapsto [\beta]_{\partial} \tag{2.57}$$

to be injective. From the exact sequence

$$H^{n-2,\,2}_{A}(X,\,\mathbb{C}) \xrightarrow{A_2} H^{n-1,\,2}_{BC}(X,\,\mathbb{C}) \xrightarrow{B} H^{n-1,\,2}_{\partial}(X,\,\mathbb{C}),$$

we infer that B being injective is equivalent to the linear map A_2 vanishing identically, where

$$A_2 \;:\; H^{n-2,\,2}_{A}(X,\,\mathbb{C}) \longrightarrow H^{n-1,\,2}_{BC}(X,\,\mathbb{C}), \qquad [v]_A \mapsto [\partial v]_{BC}. \tag{2.58}$$

This discussion can be summed up as follows.

Observation 2.4.8. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ whose canonical bundle K_X is trivial such that the linear maps A_1 and A_2 defined in (2.55) and (2.58) vanish identically. Then the Kuranishi family of X is unobstructed.*

2.4.3 Two examples of unobstructedness for C-Y non-$\partial\bar\partial$ manifolds

We saw in §.1.3.3 that the 3-dimensional Iwasawa manifold, that we will often denote by $I^{(3)}$, is not a $\partial\bar\partial$-manifold. It does not even have the weaker *Frölicher degeneration at* E_1 property. (See Proposition 1.3.22.) We also saw in Proposition 1.3.23 that no small deformation of $I^{(3)}$ is a $\partial\bar\partial$-manifold. However, we will see later that Nakamura divided the small deformations of $I^{(3)}$ into three classes, (i), (ii) and (iii), in [Nak75] and that those in class (iii) have the property $E_1(X) = E_\infty(X)$. (See e.g. tables 2.2 and 2.3 in [Ang14].)

On the other hand, we also saw in §.1.3.3 that $I^{(3)}$ is a *complex parallelisable* manifold, meaning that its holomorphic tangent bundle is *trivial*. This implies that its canonical bundle is also *trivial*, making $I^{(3)}$ a *Calabi-Yau manifold* in the sense of Definition 2.4.1.

Thus, as a *Calabi-Yau non-$\partial\bar\partial$-manifold*, $I^{(3)}$ does not satisfy the hypotheses of Theorem 2.4.7, so we cannot use that theorem to determine the (un)obstructedness status of the Kuranishi family of $I^{(3)}$.

Nevertheless, following [Nak75], we will now see that a simple computation proves the *unobstructedness* of the Kuranishi family of $I^{(3)}$. A striking feature is that the general process described in §.2.3.1 becomes *finite* in the case of $I^{(3)}$, namely the power series (2.21) contains only finitely many (actually, only two) terms. So, no convergence issue is involved.

Unobstructedness of the Kuranishi family of the Iwasawa manifold $I^{(3)}$

Let $X = I^{(3)}$ be the *Iwasawa manifold*. It is a *nilmanifold* since the Heisenberg group G defining it is *nilpotent*. Let $\varphi_1 = dz_1$, $\varphi_2 = dz_2$, $\varphi_3 = dz_3 - z_1 dz_2$ be the holomorphic 1-forms on X defined in (1.54). They are linearly independent at every point of X. Since φ_1 and φ_2 are d-closed while φ_3 is not d-closed, the number r introduced in Definition 2.4.12 below equals 2 in the case of $I^{(3)}$. Hence, by Kodaira's theorem 4.5.37, the $\mathbb{C}$-vector space $H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})$ has complex dimension 2 and is spanned by the Dolbeault cohomology classes $[\overline\varphi_1]_{\bar\partial}$ and $[\overline\varphi_2]_{\bar\partial}$. Let $\theta_1, \theta_2, \theta_3 \in H^0(X,\,\Omega^1_X)$ be the holomorphic vector fields dual to $\varphi_1, \varphi_2, \varphi_3$. They are given by

$$\theta_1 = \frac{\partial}{\partial z_1}, \;\; \theta_2 = \frac{\partial}{\partial z_2} + z_1\frac{\partial}{\partial z_3}, \;\; \theta_3 = \frac{\partial}{\partial z_3} \tag{2.59}$$

and satisfy the relations

$$[\theta_1,\,\theta_2] = -[\theta_2,\,\theta_1] = \theta_3, \qquad [\theta_1,\,\theta_3] = [\theta_2,\,\theta_3] = 0, \tag{2.60}$$

i.e. $[\theta_i,\,\theta_j] = 0$ whenever $\{i,\,j\} \neq \{1,\,2\}$.

Since the holomorphic tangent bundle $T^{1,0}X$ is trivial and spanned by $\theta_1, \theta_2, \theta_3$, the cohomology group $H^{0,1}(X,\,T^{1,0}X)$ of $T^{1,0}X$-valued $(0,\,1)$-forms on X is a $\mathbb{C}$-vector space of dimension 6 spanned by the classes of $\theta_i\,\overline{\varphi}_\lambda$:

$$H^{0,1}(X,\,T^{1,0}X) = \bigoplus_{1\leq i\leq 3,\,1\leq\lambda\leq 2} \mathbb{C}\{\theta_i\,\overline{\varphi}_\lambda\}, \qquad \dim_{\mathbb{C}} H^{0,1}(X,\,T^{1,0}X) = 6. \tag{2.61}$$

This will be seen to imply that the Kuranishi family of $X = I^{(3)}$ is a 6-parameter family.

Proposition 2.4.9. *The Kuranishi family of the* 3*-dimensional nilmanifold* $I^{(3)}$ *is* **unobstructed**.

Proof. For $X = I^{(3)}$, we get:

$$[\theta_i\overline{\varphi}_\lambda,\,\theta_k\overline{\varphi}_\nu] = [\theta_i,\,\theta_k]\,\overline{\varphi}_\lambda\wedge\overline{\varphi}_\nu, \qquad i,k,\lambda,\nu = 1,2,3, \tag{2.62}$$

with $[\theta_i,\,\theta_k]$ given in (4.122).

We have seen in (4.124) that the classes $\{\theta_i\,\overline{\varphi_\lambda}\}$, with $1\leq i\leq 3$, $1\leq\lambda\leq 2$, form a basis of $H^{0,1}(X,\,T^{1,0}X)$. Consequently the Kuranishi family of X can be described by 6 parameters $t = (t_{i\lambda})_{1\leq i\leq 3,\,1\leq\lambda\leq 2}$. By (4.112), the $T^{1,0}X$-valued $(0,\,1)$-forms $\theta_i\,\overline{\varphi_\lambda}$ are Δ''-harmonic when $1\leq\lambda\leq 2$. Thus, to construct the vector $(0,\,1)$-forms $\psi(t)\in C^\infty(X,\,\Lambda^{0,1}T^\star X\otimes T^{1,0}X)$ that describe the Kuranishi family of $X = \mathbb{C}^3/\Gamma$, we start off by setting

$$\psi_1(t) := \sum_{i=1}^{3}\sum_{\lambda=1}^{2} t_{i\lambda}\theta_i\overline{\varphi}_\lambda, \qquad t = (t_{i\lambda})_{1\leq i\leq 3,\,1\leq\lambda\leq 2}, \tag{2.63}$$

for which we see that

$$\frac{1}{2}[\psi_1(t),\,\psi_1(t)] = \frac{1}{2}\sum_{i,j=1,2,3}\,\sum_{\lambda,\mu=1,2} t_{i\lambda}t_{j\mu}[\theta_i,\,\theta_j]\,\overline{\varphi}_\lambda\wedge\overline{\varphi}_\mu.$$

By (4.122), this translates to

$$\begin{aligned}\frac{1}{2}[\psi_1(t),\,\psi_1(t)] = \frac{1}{2}(t_{11}t_{22}\theta_3\,\overline{\varphi}_1\wedge\overline{\varphi}_2 \quad &+ \quad t_{12}t_{21}\theta_3\,\overline{\varphi}_2\wedge\overline{\varphi}_1 \\ &- \quad t_{21}t_{12}\theta_3\,\overline{\varphi}_1\wedge\overline{\varphi}_2 - t_{22}t_{11}\theta_3\,\overline{\varphi}_2\wedge\overline{\varphi}_1).\end{aligned}$$

Since $\overline{\varphi}_1\wedge\overline{\varphi}_2 = -\overline{\varphi}_2\wedge\overline{\varphi}_1$, we get

$$\frac{1}{2}[\psi_1(t),\,\psi_1(t)] = (t_{11}t_{22} - t_{12}t_{21})\,\theta_3\,\overline{\varphi}_1\wedge\overline{\varphi}_2. \tag{2.64}$$

On the other hand, for the choice (4.125) we see that

$$\bar\partial\psi_1(t) = d\psi_1(t) = \sum_{i=1}^{3}\sum_{\lambda=1}^{2} t_{i\lambda}\,\theta_i\,d\overline{\varphi}_\lambda = 0 \tag{2.65}$$

since $d\overline{\varphi}_1 = d\overline{\varphi}_2 = 0$. Now setting

$$\psi_2(t) := -(t_{11}t_{22} - t_{12}t_{21})\, \theta_3\overline{\varphi}_3, \tag{2.66}$$

and using (1.55) and (4.126), we find

$$\begin{aligned} \bar\partial\psi_2(t) &= d\psi_2(t) = (t_{11}t_{22} - t_{12}t_{21})\, \theta_3\, (-d\,\overline{\varphi}_3) \\ &= (t_{11}t_{22} - t_{12}t_{21})\, \theta_3\, \overline{\varphi}_1 \wedge \overline{\varphi}_2 = \frac{1}{2}\, [\psi_1(t),\, \psi_1(t)]. \end{aligned} \tag{2.67}$$

In particular, $[\psi_1(t),\, \psi_1(t)]$ is seen to be $\bar\partial$-exact here (although it need not be so in the case of an arbitrary manifold), but the solution $\psi_2(t)$ of equation (4.129) need not be of minimal L^2-norm (unlike the $\psi_2(t)$ defined in the case of a general manifold). In other words, in the special case of the Iwasawa manifold, a solution $\psi_2(t)$ of (4.129) is easily observed and we are spared the application of the general formulae. This readily yields the desired $\psi(t)$ by setting

$$\psi(t) := \psi_1(t) + \psi_2(t) = \sum_{i=1}^{3}\sum_{\lambda=1}^{2} t_{i\lambda}\, \theta_i\, \overline{\varphi}_\lambda - (t_{11}t_{22} - t_{12}t_{21})\, \theta_3\, \overline{\varphi}_3, \tag{2.68}$$

for which we find

$$\frac{1}{2}[\psi(t),\, \psi(t)] = \sum_{j,,k=1}^{2} \frac{1}{2}[\psi_j(t),\, \psi_k(t)] = \frac{1}{2}[\psi_1(t),\, \psi_1(t)]. \tag{2.69}$$

Indeed, $[\psi_j(t),\, \psi_k(t)] = 0$ for all $(i,\, j) \neq (1,\, 1)$ since these terms involve only brackets of the shape $[\theta_3,\, \theta_i] = 0$ and $[\theta_i,\, \theta_3] = 0$ which vanish by (4.122).

On the other hand, combining (4.127) and (4.129), we get

$$\bar\partial\psi(t) = \bar\partial\psi_1(t) + \bar\partial\psi_2(t) = \bar\partial\psi_2(t) = \frac{1}{2}[\psi_1(t),\, \psi_1(t)]. \tag{2.70}$$

Then (4.131) and (4.132) yield

$$\bar\partial\psi(t) = \frac{1}{2}[\psi(t),\, \psi(t)], \tag{2.71}$$

showing that $\psi(t)$ defined in (4.130) satisfies the integrability condition (2.15).

Thus, this $T^{1,0}X$-valued $(0,\, 1)$-form $\psi(t)$ defines a *locally complete* complex analytic family of deformations X_t of $X = I^{(3)}$ depending on 6 *effective* parameters $t = (t_{i\lambda})_{1\leq i\leq 3,\, 1\leq\lambda\leq 2}$.

Unobstructedness of the Kuranishi family of the 5-dimensional Iwasawa manifold $I^{(5)}$

Let us now consider the nilmanifold $X = I^{(5)}$ of complex dimension 5 whose complex structure is described by five holomorphic $(1,\, 0)$-forms $\varphi_1, \dots, \varphi_5$ satisfying the equations

$$d\varphi_1 = d\varphi_2 = 0, \quad d\varphi_3 = \varphi_1 \wedge \varphi_2, \quad d\varphi_4 = \varphi_1 \wedge \varphi_3, \quad d\varphi_5 = \varphi_2 \wedge \varphi_3. \tag{2.72}$$

If $\theta_1, \dots,\, \theta_5$ form the dual basis of $(1,\, 0)$-vector fields, then $[\theta_i,\, \theta_j] = 0$ except in the following cases:

$$[\theta_1,\, \theta_2] = -\theta_3, \quad [\theta_1,\, \theta_3] = -\theta_4, \quad [\theta_2,\, \theta_3] = -\theta_5, \tag{2.73}$$

hence also $\quad [\theta_2,\, \theta_1] = \theta_3, \quad [\theta_3,\, \theta_1] = \theta_4, \quad [\theta_3,\, \theta_2] = \theta_5.$

In particular, $H^{0,1}(X, T^{1,0}X) = \langle[\overline{\varphi}_1 \otimes \theta_i], [\overline{\varphi}_2 \otimes \theta_i] \mid i = 1, \ldots, 5\rangle$, so $\dim_{\mathbb{C}} H^{0,1}(X, T^{1,0}X) = 10$.

This manifold is the 5-dimensional analogue of the Iwasawa manifold, hence the notation $I^{(5)}$. We will now see that $I^{(5)}$ shares with $I^{(3)}$ the unobstructedness property of the Kuranishi family. The following fact was observed in [Rol11].

Proposition 2.4.10. *The Kuranishi family of the* 5*-dimensional nilmanifold described above is* **unobstructed**.

Proof. Consider any $\psi_1(t) := \sum_{i=1}^{5}\sum_{\lambda=1}^{2} t_{i\lambda}\, \theta_i\, \overline{\varphi}_\lambda$ with arbitrary coefficients $t_{i\lambda} \in \mathbb{C}$ such that $|t|$ is close to 0. Then

$$[\psi_1(t),\, \psi_1(t)] = \sum_{i,\,j=1}^{5}\sum_{\lambda,\,\mu=1}^{2} t_{i\lambda}\, t_{j\mu}\, [\theta_i,\, \theta_j]\, \overline{\varphi}_\lambda \wedge \overline{\varphi}_\mu.$$

Since $[\theta_i,\, \theta_j] = 0$ except when $(i,\, j) \in \{(1,\, 2),\, (2,\, 1),\, (1,\, 3),\, (3,\, 1), (2,\, 3),\, (3,\, 2)\}$, we get

$$\begin{aligned}
[\psi_1(t),\, \psi_1(t)] &= [-(t_{11}t_{22} - t_{21}t_{12})\, \theta_3 + (t_{12}t_{21} - t_{22}t_{11})\, \theta_3 - (t_{11}t_{32} - t_{31}t_{12})\, \theta_4 + (t_{12}t_{31} - t_{32}t_{11})\, \theta_4 \\
&- (t_{21}t_{32} - t_{31}t_{22})\, \theta_5 + (t_{22}t_{31} - t_{32}t_{21})\, \theta_5]\, \overline{\varphi}_1 \wedge \overline{\varphi}_2 \\
&= 2\, [D_{21}(t)\, \theta_3 + D_{31}(t)\, \theta_4 + D_{32}(t)\, \theta_5]\; \overline{\varphi}_1 \wedge \overline{\varphi}_2,
\end{aligned}$$

where we set $D_{ji} := D_{ji}^{12}$ and

$$D_{ji}^{\lambda\mu} := \begin{vmatrix} t_{i\mu} & t_{j\lambda} \\ t_{j\mu} & t_{i\lambda} \end{vmatrix}$$

.

Since $\overline{\varphi}_1 \wedge \overline{\varphi}_2 = \bar\partial\overline{\varphi}_3$, equation (Eq. 2) reads

$$\bar\partial\psi_2(t) = (D_{21}(t)\, \theta_3 + D_{31}(t)\, \theta_4 + D_{32}(t)\, \theta_5)\; \bar\partial\overline{\varphi}_3,$$

so an obvious solution is $\psi_2(t) = (D_{21}(t)\, \theta_3 + D_{31}(t)\, \theta_4 + D_{32}(t)\, \theta_5)\; \overline{\varphi}_3$.

We now go on to compute

$$\begin{aligned}
[\psi_1(t),\, \psi_2(t)] &= \sum_{i=1}^{5}\sum_{\lambda=1}^{2} t_{i\lambda}\, D_{21}(t)\, [\theta_i,\, \theta_3]\, \overline{\varphi}_\lambda \wedge \overline{\varphi}_3 \\
&+ \sum_{i=1}^{5}\sum_{\lambda=1}^{2} t_{i\lambda}\, D_{31}(t)\, [\theta_i,\, \theta_4]\, \overline{\varphi}_\lambda \wedge \overline{\varphi}_3 + \sum_{i=1}^{5}\sum_{\lambda=1}^{2} t_{i\lambda}\, D_{32}(t)\, [\theta_i,\, \theta_5]\, \overline{\varphi}_\lambda \wedge \overline{\varphi}_3
\end{aligned}$$

All the terms on the second line above vanish since $[\theta_i,\, \theta_4] = [\theta_i,\, \theta_5] = 0$ for all i, and so do the terms with $i \notin \{1,\, 2\}$ on the first line (see (6.83)), so using (6.83) we get

$$[\psi_1(t),\, \psi_2(t)] = -\sum_{\lambda=1}^{2} t_{1\lambda}\, D_{21}(t)\, \theta_4\, \overline{\varphi}_\lambda \wedge \overline{\varphi}_3 - \sum_{\lambda=1}^{2} t_{2\lambda}\, D_{21}(t)\, \theta_5\, \overline{\varphi}_\lambda \wedge \overline{\varphi}_3.$$

We infer that equation (Eq. 3) $\bar\partial\psi_3(t) = [\psi_1(t),\, \psi_2(t)]$ has the obvious solution

$$\psi_3(t) = -D_{21}(t)\, [(t_{11}\, \theta_4 + t_{21}\, \theta_5)\, \overline{\varphi}_4 + (t_{12}\, \theta_4 + t_{22}\, \theta_5)\, \overline{\varphi}_5].$$

To study equation (Eq. 4), namely

$$\bar\partial\psi_4(t) = [\psi_1(t),\, \psi_3(t)] + \frac{1}{2}\,[\psi_2(t),\, \psi_2(t)],$$

we notice that $[\psi_1(t),\, \psi_3(t)] = [\psi_2(t),\, \psi_2(t)] = 0$ because $[\theta_i,\, \theta_4] = [\theta_i,\, \theta_5] = 0$ for all i and $[\theta_3,\, \theta_3] = 0$. Consequently, equation (Eq. 4) is the trivial equation $\bar\partial\psi_4(t) = 0$ admitting the trivial solution $\psi_4(t) = 0$.

We conclude that the Kuranishi family of X is unobstructed and the deformations of its complex structure in any pregiven direction $\psi_1(t) := \sum\limits_{i=1}^{5}\sum\limits_{\lambda=1}^{2} t_{i\lambda}\,\theta_i\,\overline{\varphi}_\lambda$ are defined by the finite sum

$$\psi(t) = \sum_{i=1}^{5}\sum_{\lambda=1}^{2} t_{i\lambda}\,\theta_i\,\overline{\varphi}_\lambda + (D_{21}(t)\,\theta_3 + D_{31}(t)\,\theta_4 + D_{32}(t)\,\theta_5)\;\overline{\varphi}_3 - D_{21}(t)\,[(t_{11}\,\theta_4 + t_{21}\,\theta_5)\,\overline{\varphi}_4 + (t_{12}\,\theta_4 + t_{22}\,\theta_5)\,\overline{\varphi}_5].$$

So, no convergence issues are involved. □

Further details on complex parallelisable manifolds

We take this opportunity to spell out some general facts that were either used above or will be used in the sequel.

Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. Since there are no non-zero $\bar\partial$-exact (1, 0)-forms on X (for obvious bidegree reasons), we have

$$H^{1,\,0}_{\bar\partial}(X,\,\mathbb{C}) = \{u \in C^\infty(X,\,\Lambda^{1,\,0}T^\star X)\ ;\ \bar\partial u = 0\},$$

i.e. $H^{1,\,0}_{\bar\partial}(X,\,\mathbb{C})$ consists of holomorphic 1-forms on X. Putting $h^{1,\,0}_{\bar\partial}(X) := \dim_{\mathbb{C}} H^{1,\,0}_{\bar\partial}(X,\,\mathbb{C})$, we have the trivial

Observation 2.4.11. *If X is complex parallelisable, then $h^{1,\,0}_{\bar\partial}(X) = n$.*

Proof. By the *complex parallelisable* hypothesis on X, the rank-n analytic sheaf Ω^1_X is trivial, hence it is generated by n holomorphic 1-forms $\varphi_1, \dots, \varphi_n \in H^{1,\,0}_{\bar\partial}(X,\,\mathbb{C})$ that are linearly independent at every point of X. In particular, $\{\varphi_1, \dots, \varphi_n\}$ is a basis of $H^{1,\,0}_{\bar\partial}(X,\,\mathbb{C}) \simeq H^0(X,\,\Omega^1_X)$. □

Suppose now that X is compact *complex parallelisable*. Let $\theta_1, \dots, \theta_n \in H^0(X,\,T^{1,\,0}X)$ be n holomorphic vector fields that are linearly independent at every point of X, chosen to be dual to the holomorphic (1, 0)-forms $\varphi_1, \dots, \varphi_n \in H^{1,\,0}(X,\,\mathbb{C})$ considered in the above proof. For every smooth function $g : X \to \mathbb{C}$, we have

$$\partial g = \sum_{\lambda=1}^{n}(\theta_\lambda g)\,\varphi_\lambda, \qquad \bar\partial g = \sum_{\lambda=1}^{n}(\bar\theta_\lambda g)\,\overline{\varphi}_\lambda, \tag{2.74}$$

i.e. the familiar formalism induced by local holomorphic coordinates finds a global analogue on a compact *complex parallelisable* manifold in a formalism where θ_λ replaces $\partial/\partial z_\lambda$ and φ_λ replaces dz_λ. Thus any (0, 1)-form φ on X has a unique decomposition

$$\varphi = \sum_{\lambda=1}^{n} f_\lambda \overline{\varphi}_\lambda$$

with $f_1, \dots, f_n : X \to \mathbb{C}$ functions on X. Thus there is an implicit L^2 inner product on $C^\infty(X,\, \Lambda^{0,\,1}T^\star X)$ defined as follows (no Hermitian metric is needed on X): for any $\varphi = \sum\limits_{\lambda=1}^{n} f_\lambda \overline{\varphi}_\lambda, \psi = \sum\limits_{\lambda=1}^{n} g_\lambda \overline{\varphi}_\lambda \in C^\infty(X,\, \Lambda^{0,\,1}T^\star X)$, set

$$\langle\langle\varphi,\, \psi\rangle\rangle := \int\limits_X \left(\sum_{\lambda=1}^{n} f_\lambda\, \bar{g}_\lambda\right) i^{n^2} \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \overline{\varphi}_1 \wedge \cdots \wedge \overline{\varphi}_n. \tag{2.75}$$

It is clear that $dV := i^{n^2} \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \overline{\varphi}_1 \wedge \cdots \wedge \overline{\varphi}_n > 0$ is a global volume form on X and that the above L^2 inner product is independent of the choices made. We can define the formal adjoint $\bar\partial^\star$ of $\bar\partial$ w.r.t. this L^2 inner product in the usual way: for any smooth $(0,\, 1)$-form φ, define $\bar\partial^\star\varphi$ to be the unique smooth function on X satisfying

$$\langle\langle\bar\partial^\star\varphi,\, g\rangle\rangle = \langle\langle\varphi,\, \bar\partial g\rangle\rangle$$

for any smooth function g on X. A trivial calculation using Stokes's theorem gives

$$\bar\partial^\star\varphi = -\sum_{\lambda=1}^{n} \theta_\lambda f_\lambda \tag{2.76}$$

for any smooth $(0,\, 1)$-form $\varphi = \sum\limits_{\lambda=1}^{n} f_\lambda \overline{\varphi}_\lambda$ on X. Thus we see that

$$\bar\partial^\star\overline{\varphi}_\nu = 0, \qquad \nu = 1, \dots, n, \tag{2.77}$$

because $\overline{\varphi}_\nu = \sum\limits_{\lambda=1}^{n} \delta_{\nu\lambda}\overline{\varphi}_\lambda$ and $\theta_\lambda\delta_{\nu\lambda} = 0$ (since the $\delta_{\nu\lambda}$ are constants).

Now, let us introduce the following

Definition 2.4.12. *Let X be a* **complex parallelisable** *compact complex* **nilmanifold** *with $n = dim_{\mathbb{C}}X = n$.*

We denote by $r \in \{0, 1, \dots, n\}$ the maximal number of d-**closed** *holomorphic 1-forms on X that are linearly independent at every point of X.*

We will see in a moment that r is an *invariant* of X. In our case, even if X is not a nilmanifold, r is the number of d-closed forms among $\varphi_1, \dots, \varphi_n$. After a possible reordering, we can suppose that $\varphi_1, \dots, \varphi_r$ are d-closed and $\varphi_{r+1}, \dots, \varphi_n$ are not d-closed. Then we have

$$\partial\varphi_1 = \cdots = \partial\varphi_r = 0 \quad \text{or equivalently} \quad \bar\partial\overline{\varphi}_1 = \cdots = \bar\partial\overline{\varphi}_r = 0. \tag{2.78}$$

Thus the $\bar\partial$-closed $(0,\, 1)$-forms $\overline{\varphi}_1, \dots, \overline{\varphi}_r$ define Dolbeault $(0,\, 1)$-cohomology classes in $H^{0,\,1}(X,\, \mathbb{C})$.

We can define the $\bar\partial$-Laplacian on forms of X in the usual way:

$$\Delta'' := \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial.$$

The corresponding harmonic space of $(0,\, 1)$-forms $\mathcal{H}^{0,\,1}_{\Delta''}(X,\, \mathbb{C}) := \ker\Delta'' = \ker\bar\partial \cap \ker\bar\partial^\star$ satisfies the Hodge isomorphism $\mathcal{H}^{0,\,1}_{\Delta''}(X,\, \mathbb{C}) \simeq H^{0,\,1}(X,\, \mathbb{C})$. Notice that (4.110) and (4.111) give

$$\Delta''\overline{\varphi}_\nu = 0, \qquad \nu = 1, \dots, r, \tag{2.79}$$

i.e. the forms $\overline{\varphi}_1, \dots, \overline{\varphi}_r$ are Δ''-harmonic. On the other hand, $\overline{\varphi}_{r+1}, \dots, \overline{\varphi}_n$ are not Δ''-harmonic. Thus, the number r satisfies:

$$r \leq h^{0,1}(X). \tag{2.80}$$

Suppose now that the compact *complex parallelisable* X is a *nilmanifold.*

Fact 2.4.13. *(see e.g. [Nak75] or [CFGU00, p.5405-5406]) If X is a compact* **complex parallelisable nilmanifold***, the holomorphic 1-forms $\varphi_1, \dots, \varphi_n$ that form a basis of $H^{1,0}(X, \mathbb{C})$ can be chosen such that*

$$d\varphi_\mu = \sum_{1\leq\lambda<\nu\leq n} c_{\mu\lambda\nu}\,\varphi_\lambda \wedge \varphi_\nu, \qquad 1 \leq \mu \leq n, \tag{2.81}$$

with constant coefficients $c_{\mu\lambda\nu} \in \mathbb{C}$ satisfying

$$c_{\mu\lambda\nu} = 0 \quad \textit{whenever} \quad \mu \leq \lambda \ \textit{or}\ \mu \leq \nu.$$

Taking this standard fact (which in [Nak75] follows from the existence of a *Chevalley decomposition* of the nilpotent Lie algebra $\mathfrak{g}$) for granted, we now spell out the details of the proof of the following result of Kodaira's along the lines of [Nak75, Theorem 3, p. 100].

Theorem 2.4.14. *If X is a compact* **complex parallelisable nilmanifold***, then $h^{0,1}_{\bar\partial}(X) = r$.*

Moreover, the Δ''-harmonic $(0, 1)$-forms $\overline\varphi_1, \dots, \overline\varphi_r$ form a basis of the harmonic space $\mathcal{H}^{0,1}_{\Delta''}(X, \mathbb{C})$. Equivalently, the Dolbeault $(0, 1)$-cohomology classes $[\overline\varphi_1]_{\bar\partial}, \dots, [\overline\varphi_r]_{\bar\partial}$ form a basis of $H^{0,1}_{\bar\partial}(X, \mathbb{C})$.

Proof. The only thing that needs proving is that the linearly independent forms $\overline\varphi_1, \dots, \overline\varphi_r \in \mathcal{H}^{0,1}_{\Delta''}(X, \mathbb{C})$ generate $\mathcal{H}^{0,1}_{\Delta''}(X, \mathbb{C})$. Pick an arbitrary C^∞ $(0, 1)$-form φ on X and write

$$\varphi = \sum_{\lambda=1}^{n} f_\lambda\,\overline\varphi_\lambda$$

with C^∞ functions $f_1, \dots, f_n$ on X. Using formula (4.107) for $\bar\partial$ and the obvious identities $\bar\partial\,\overline\varphi_\lambda = d\,\overline\varphi_\lambda$, $\lambda = 1, \dots, n$, due to φ_λ being holomorphic, we get:

$$\begin{aligned}
\bar\partial\varphi &= \sum_{\lambda,\,\nu=1}^{n} (\bar\theta_\nu f_\lambda)\,\overline\varphi_\nu \wedge \overline\varphi_\lambda + \sum_{\mu=1}^{n} f_\mu\, d\,\overline\varphi_\mu \\
&= \sum_{\lambda,\,\nu=1}^{n} (\bar\theta_\nu f_\lambda)\,\overline\varphi_\nu \wedge \overline\varphi_\lambda + \sum_{\mu=1}^{n} f_\mu \sum_{1\leq\nu<\lambda\leq n} \overline{c_{\mu\nu\lambda}}\,\overline\varphi_\nu \wedge \overline\varphi_\lambda \\
&= \sum_{1\leq\nu<\lambda\leq n} \left(\bar\theta_\nu f_\lambda - \bar\theta_\lambda f_\nu + \sum_{\mu=1}^{n} \overline{c_{\mu\nu\lambda}}\, f_\mu\right)\overline\varphi_\nu \wedge \overline\varphi_\lambda,
\end{aligned} \tag{2.82}$$

where the second line above follows from the conjugate of (4.114).

Now φ is Δ''-harmonic if and only if

(i) $\bar\partial\varphi = 0 \Longleftrightarrow \bar\theta_\nu f_\lambda - \bar\theta_\lambda f_\nu + \sum_{\mu=1}^{n} \overline{c_{\mu\nu\lambda}}\, f_\mu = 0 \qquad$ for $1 \leq \nu < \lambda \leq n$ (cf. (4.115));

and

(ii) $\bar\partial^\star\varphi = 0 \Longleftrightarrow \sum\limits_{\lambda=1}^n \theta_\lambda\, f_\lambda = 0$ (cf. (4.109)).

Suppose that φ is Δ''-harmonic. Then the above (i) reads:

$$\bar\theta_\lambda f_\nu = \sum_{\mu=1}^n \overline{c_{\mu\nu\lambda}}\, f_\mu + \bar\theta_\nu f_\lambda, \quad 1\leq \nu<\lambda\leq n.$$

Summing over $\lambda = 1,\dots,n$ and using formula (4.107) for $\bar\partial$, we get

$$\bar\partial f_\nu = \sum_{\lambda=1}^n (\bar\theta_\lambda f_\nu)\,\overline{\varphi}_\lambda = \sum_{\lambda,\,\mu=1}^n \overline{c_{\mu\nu\lambda}}\, f_\mu\overline{\varphi}_\lambda + \sum_{\lambda=1}^n (\bar\theta_\nu f_\lambda)\,\overline{\varphi}_\lambda, \quad \nu = 1,\dots,n,$$

with the understanding that $c_{\mu\nu\lambda}=0$ if $\nu\geq\lambda$. Now $\Delta'' f_\nu = \bar\partial^\star\bar\partial f_\nu$ since f_ν is a function. Taking $\bar\partial^\star$ on either side above and using formula (4.109) for $\bar\partial^\star$, we get

$$\begin{aligned}\Delta'' f_\nu &= -\sum_{\lambda,\,\mu=1}^n \theta_\lambda\,(\overline{c_{\mu\nu\lambda}}\, f_\mu) - \sum_{\lambda=1}^n \theta_\lambda\,(\bar\theta_\nu f_\lambda)\\ &= -\sum_{\lambda,\,\mu=1}^n \overline{c_{\mu\nu\lambda}}\,\theta_\lambda f_\mu, \quad \text{for all } \nu = 1,\dots,n, \end{aligned} \tag{2.83}$$

because $\theta_\lambda\,(\overline{c_{\mu\nu\lambda}} f_\mu) = \overline{c_{\mu\nu\lambda}}\,\theta_\lambda f_\mu$ due to $\overline{c_{\mu\nu\lambda}}$ being constant, while $\sum\limits_{\lambda=1}^n \theta_\lambda f_\lambda = 0$ due to φ being Δ''-harmonic (cf. (ii) or (4.109)).

Taking now $\nu=n$ in (4.116), we get $\Delta'' f_n = 0$ since $c_{\mu n\lambda}=0$ for all μ,λ by Fact 4.5.36 and the obvious inequality $\mu\leq\nu=n$. Thus the compactness of X and the ellipticity of Δ'' yield

$$f_n \quad \text{is constant on } X \text{ if } \Delta''\varphi = 0. \tag{2.84}$$

Taking now $\nu = n-1$ in (4.116) and using the fact that $\theta_\lambda f_n = 0$ for all λ (since f_n is constant by (4.117)), we get

$$\Delta'' f_{n-1} = -\sum_{\lambda=1}^n \left(\sum_{\mu=1}^{n-1} \overline{c_{\mu n-1\lambda}}\,\theta_\lambda f_\mu\right) = 0 \quad \text{on } X,$$

since $c_{\mu n-1\lambda} = 0$ for all $\mu = 1,\dots,n-1$ and $\lambda = 1,\dots,n$ by Fact 4.5.36 and the obvious inequality $\mu\leq\nu=n-1$. Hence we get

$$f_{n-1} \quad \text{is constant on } X \text{ if } \Delta''\varphi = 0. \tag{2.85}$$

We can now continue by decreasing induction on ν. Taking $\nu = n-2$ in (4.116) and using the fact that $\theta_\lambda f_n = \theta_\lambda f_{n-1} = 0$ for all λ (since f_n is constant by (4.117) and f_{n-1} is constant by (4.118)), we get

$$\Delta'' f_{n-2} = -\sum_{\lambda=1}^n \left(\sum_{\mu=1}^{n-2} \overline{c_{\mu n-2\lambda}}\,\theta_\lambda f_\mu\right) = 0 \quad \text{on } X,$$

since $c_{\mu n-2\lambda} = 0$ for all $\mu = 1,\dots,n-2$ and $\lambda = 1,\dots,n$ by Fact 4.5.36 and the obvious inequality $\mu\leq\nu=n-2$. Hence we get

$$f_{n-2} \quad \text{is constant on } X \text{ if } \Delta''\varphi = 0. \tag{2.86}$$

Running a decreasing induction on ν, we get

$$f_\nu := C_\nu \quad \text{is constant on } X \text{ for all } \nu = 1, \dots, n \text{ if } \Delta''\varphi = 0. \tag{2.87}$$

We conclude that whenever $\Delta''\varphi = 0$ we have

$$\varphi = \sum_{\nu=1}^{n} C_\nu \, \overline{\varphi}_\nu \quad \text{with } C_\nu \text{ constant for all } \nu = 1, \dots, n.$$

On the other hand, since $\Delta''\varphi = 0$, we must have $\bar\partial\varphi = 0$ which amounts to $\sum_{\nu=1}^{n} C_\nu \, \bar\partial\overline{\varphi}_\nu = 0$. However, we know that $\bar\partial\overline{\varphi}_\nu = 0$ for all $\nu \in \{1, \dots, r\}$ (cf. (4.111)), hence $\sum_{\nu=r+1}^{n} C_\nu \, \bar\partial\overline{\varphi}_\nu = 0$. Now the forms

$$\bar\partial\overline{\varphi}_\nu = d\overline{\varphi}_\nu = \sum_{\lambda,\mu} \overline{c_{\nu\lambda\mu}} \, \overline{\varphi}_\lambda \wedge \overline{\varphi}_\mu, \quad \nu = 1, \dots, n,$$

are linearly independent because $\overline{\varphi}_1, \dots, \overline{\varphi}_n$ are linearly independent at every point of X. Hence $C_\nu = 0$ for all $\nu = r+1, \dots, n$. We get

$$\varphi = \sum_{\nu=1}^{r} C_\nu \, \overline{\varphi}_\nu \quad \text{with } C_\nu \text{ constant for all } \nu = 1, \dots, r.$$

Since φ has been chosen arbitrary in $\mathcal{H}^{0,1}_{\Delta''}(X, \mathbb{C})$, we have proved that the linearly independent forms $\overline{\varphi}_1, \dots, \overline{\varphi}_r \in \mathcal{H}^{0,1}_{\Delta''}(X, \mathbb{C})$ generate $\mathcal{H}^{0,1}_{\Delta''}(X, \mathbb{C})$. The proof of Kodaira's theorem 4.5.37 is complete. □

When applying Observation 4.5.35 and Kodaira's Theorem 4.5.37 to the Iwasawa manifold (for which $r = 2$), we get the following standard fact.

Observation 2.4.15. *For the Iwasawa manifold, we have:*

$$h^{1,0} = 3 \quad \textit{and} \quad h^{0,1} = 2.$$

Since, on the other hand, the first Betti number is $b_1 = 4$ (cf. e.g. [Ang14]), we see that $b_1 < h^{1,0} + h^{0,1}$. Thus, we obtain again the following fact that was already observed in Proposition 1.3.22: the Frölicher spectral sequence of the Iwasawa manifold does not degenerate at E_1.

2.5 Smooth families of elliptic differential operators

In this section, we discuss the fundamental results of Kodaira and Spencer on the behaviour of the eigenvalues, the eigenspaces, the kernels, the orthogonal projections thereon and the Green operators of elliptic differential operators in smooth families thereof. In particular, the semi-continuity results and their geometric applications pervade the theory of deformations of complex structures. Our presentation will closely follow [KS60] and [Kod86, Chapter 7].

2.5.1 Statements of the main results

We first fix the notation for this subsection. Let

$$\pi : B \longrightarrow X$$

be a C^∞ *complex* vector bundle over a *compact* oriented differentiable manifold X. Let $L(B) := C^\infty(X, B)$ be the $\mathbb{C}$-vector space of global C^∞ sections of B and let $E : L(B) \longrightarrow L(B)$ be an *elliptic, self-adjoint* linear partial differential operator of *even order* m on B. We will usually denote by $\psi \in L(B)$ an arbitrary smooth global section of B.

We also fix a *Riemannian* metric g on X and a *Hermitian* metric h on the fibres on B. Together, g and h induce a *pointwise inner product* $\langle\ ,\ \rangle = \langle\ ,\ \rangle_{g,h}$ and an L^2 *inner product* $\langle\langle\ ,\ \rangle\rangle = \langle\langle\ ,\ \rangle\rangle_{g,h}$ on the space $L(B)$ of global C^∞ sections of B.

The discussion in §.1.1.1 yields the following information via the Gårding estimate (or the a priori estimate). The $\mathbb{C}$-vector space $\mathbb{F} := \ker E \subset L(B)$ is *finite dimensional*, since E is elliptic and X is compact. Since E is also self-adjoint, there is an $L^2_{g,h}$-*orthogonal two-space decomposition*:

$$L(B) = \ker E \oplus \operatorname{Im} E.$$

We denote by

$$F : L(B) \longrightarrow \ker E = \mathbb{F}$$

the *orthogonal projection* onto F. There is an operator $G = E^{-1} : L(B) \longrightarrow \operatorname{Im} E \subset L(B)$, called the *Green operator* of E, such that

$$EG(\psi) = GE(\psi) = \psi - F\psi, \quad \text{for all } \psi \in L(B).$$

Thus, the restriction $E_{|\operatorname{Im} E} : E \longrightarrow E$ is *bijective* and G is the extension by 0 over $\ker E$ of the inverse of this restriction.

We now recall the following standard fundamental result for the reader's convenience. For a *self-adjoint elliptic* operator on a *compact* manifold, it affirms the existence of a *countable orthonormal basis* of *eigenvectors* and the fact that the spectrum is *real*, *discrete* and has $+\infty$ as its only accumulation point.

Theorem 2.5.1. *There exists a countable set of sections $\{e_h \mid h \in \mathbb{N}^\star\} \subset L(B)$, such that:*

(i) $Ee_h = \lambda_h\, e_h$, *with* $\lambda_h \in \mathbb{R}$, *for every* $h \in \mathbb{N}^\star$;

(ii) $\{e_h \mid h \in \mathbb{N}^\star\}$ *is an* **orthonormal basis** *of* $L(B)$, *namely:*

(a) $\langle\langle e_h,\, e_k\rangle\rangle = \delta_{hk}$ *for all* $h, k \in \mathbb{N}^\star$;

(b) for every $\psi \in L(B)$, *we have*

$$\psi = \sum_{h=1}^{+\infty} \langle\langle \psi,\, e_h\rangle\rangle\, e_h,$$

where the series **converges** *in the* L^2 *norm* $||\ ||$.

(iii) $\lambda_1 \leq \lambda_2 \leq \dots \lambda_h \leq \lambda_{h+1} \leq \dots$ *and* $\lim_{h\to+\infty} \lambda_h = +\infty$.

Proof. See, for example, [Gil84, Lemma 1.6.3]. □

Henceforth, we shall denote by $\lambda_1 \leq \lambda_2 \leq \dots \lambda_h \leq \lambda_{h+1} \leq \dots$ the ordered eigenvalues of a given E as above. As a consequence of the above, we get the following formulae.

Corollary 2.5.2. *(i) For every $\psi \in L(B)$, we have*

$$E\psi = \sum_{h=1}^{+\infty} \lambda_h \langle\langle \psi,\, e_h \rangle\rangle\, e_h.$$

(ii) Suppose that $\dots \leq \lambda_q < \lambda_{q+1} = \dots = \lambda_p = 0 < \lambda_{p+1} \leq \dots$ for some $q \leq p$. Then, $\{e_{q+1}, \dots, e_p\}$ is an **orthonormal basis** *of* $\ker E$ *and we have*

$$\begin{aligned} F\psi &= \sum_{h=q+1}^{p} \langle\langle \psi,\, e_h \rangle\rangle\, e_h \\ G\psi &= \sum_{h \notin \{q+1,\dots,p\}} \frac{1}{\lambda_h} \langle\langle \psi,\, e_h \rangle\rangle\, e_h \end{aligned}$$

We will now introduce the analogues of a family of manifolds for vector bundles, sections thereof and differential operators.

Definition 2.5.3. *Let X be a compact oriented differentiable manifold X and let $\Delta \subset \mathbb{R}^N$ be a small open subset, for some integer $N \geq 1$.*

(i) We say that $(B_t)_{t\in\Delta}$ is a C^∞ **family of C^∞ complex vector bundles** *$B_t \longrightarrow X$ over X (or that B_t* **varies** *C^∞ with $t \in \Delta$) if there exists a C^∞ complex vector bundle $\pi : \mathcal{B} \longrightarrow X \times \Delta$ such that*

$$B_t = \pi^{-1}(X \times \{t\}) = \mathcal{B}_{|X\times\{t\}}, \qquad t \in \Delta.$$

(ii) Let $(B_t)_{t\in\Delta}$ be a C^∞ family of C^∞ complex vector bundles $B_t \longrightarrow X$ as in (i).

(a) For every $t \in \Delta$, let $\psi_t \in L(B_t) = C^\infty(X,\, B_t)$ be a smooth global section of B_t.

We say that $(\psi_t)_{t\in\Delta}$ is a C^∞ **family of sections** *(or that ψ_t* **varies** *C^∞ with $t \in \Delta$) if there exists a C^∞ section $\widetilde{\psi} \in C^\infty(X \times \Delta,\, \mathcal{B})$ of $\mathcal{B}$ such that*

$$\psi_t = \widetilde{\psi}_{|X\times\{t\}}, \qquad t \in \Delta.$$

(b) For every $t \in \Delta$, let $E_t : L(B_t) \longrightarrow L(B_t)$ be a linear operator on B_t.

We say that $(E_t)_{t\in\Delta}$ is a C^∞ **family of linear operators** *(or that E_t* **varies** *C^∞ with $t \in \Delta$) if for every C^∞ family $(\psi_t)_{t\in\Delta}$ of sections $\psi_t \in L(B_t)$, the family $(E_t\psi_t)_{t\in\Delta}$ is a C^∞ family of sections.*

(c) For every $t \in \Delta$, let h_t be a Hermitian metric on B_t in the sense that $h_t = \langle\ ,\ \rangle_t = (\langle\ ,\ \rangle_{t,x})_{x\in X}$ is a family of positive definite inner products on the fibres $(B_t)_x$ of B_t that vary in a C^∞ way with the point $x \in X$.

We say that $(h_t)_{t\in\Delta}$ is a C^∞ **family of (fibre) metrics** *(or that h_t* **varies** *C^∞ with $t \in \Delta$) if there exists a Hermitian metric h on the vector bundle $\mathcal{B}$ such that*

$$h_t = h_{|X\times\{t\}}, \qquad t \in \Delta.$$

Part (ii)(a) can be reworded in a more concrete, but equivalent way, in a local trivialisation.

Observation 2.5.4. *In the context of (ii)(a) of the above Definition 2.5.3, a family* $(\psi_t)_{t\in\Delta}$ *of sections* $\psi_t \in L(B_t) = C^\infty(X,\, B_t)$ *is a* C^∞ **family of sections** *(or* C^∞*, for short) if and only if* $\psi_t^1, \dots, \psi_t^r$ *are* C^∞ **functions** *of* $(x,\, t) \in X \times \Delta$*, where* $\psi_t^1, \dots, \psi_t^r$ *are the fibre coordinates of* ψ_t *and* $r := rk\, B_t$ *for all* $t \in \Delta$.

Part (ii)(b) of Definition 2.5.3 can be made more concrete in local coordinates as follows.

Observation 2.5.5. *In the context of Definition 2.5.3, for every* $\psi_t \in L(B_t)$ *write:*

$$(E_t\psi_t)(x) = \sum_{\mu=0}^{m} E_\mu(x,\, t,\, D_j)\, \psi_t(x), \qquad t \in \Delta,$$

where every $E_\mu(x,\, t,\, D_j)$ *is a homogeneous polynomial of degree* μ *in the partial derivatives* D_j.

Then, $(E_t)_{t\in\Delta}$ *is a* C^∞ **family of linear operators** *if and only if the coefficients of the polynomials* $E_\mu(x,\, t,\, D_j)$ *are all* C^∞ **functions** *of* $(x,\, t) \in X \times \Delta$.

In the context of (ii)(c) of Definition 2.5.3, once a Riemannian metric g has been fixed on X, a C^∞ family $(h_t)_{t\in\Delta}$ of (fibre) metrics on the vector bundles $(B_t)_{t\in\Delta}$ induces a C^∞ family $(\langle\langle\ ,\ \rangle\rangle_t)_{t\in\Delta}$ of L^2 **inner products** on the sections of the B_t's and the associated C^∞ family $(||\ ,\ ||_t)_{t\in\Delta}$ of L^2 **norms**.

The Kodaira-Spencer fundamental theorems ([KS60]) on families of elliptic operators

Let:

- $(B_t,\, h_t)_{t\in\Delta}$ *be a* C^∞ *family of Hermitian* C^∞ *complex vector bundles on a compact Riemannian manifold* $(X,\, g)$*;*
- $(E_t,\, h_t)_{t\in\Delta}$ *be a* C^∞ *family of self-adjoint* **elliptic** *linear differential operators* $E_t : L(B_t) \longrightarrow L(B_t)$ *of even order* m*;*
- $(\lambda_h(t))_{h\in\mathbb{N}^\star}$ *be, for every fixed* $t \in \Delta$*, the eigenvalues of* E_t *and let* $(e_h(t))_{h\in\mathbb{N}^\star}$ *be the corresponding eigensections* $e_h(t) \in L(B_t)$ *such that:*

$$\cdot\ E_t\, e_h(t) = \lambda_h(t)\, e_h(t), \qquad h \in \mathbb{N}^\star,\ \ t \in \Delta;$$

$$\cdot\ (e_h(t))_{h\in\mathbb{N}^\star} \ \ \textit{is an orthonormal basis of } L(B_t), \quad t \in \Delta;$$

$$\cdot\ \lambda_1(t) \leq \dots \leq \lambda_h(t) \leq \dots \quad \textit{and} \quad \lim_{h\to+\infty} \lambda_h(t) = +\infty.$$

Then, the following statements hold.

Theorem A *For every* $h \in \mathbb{N}^\star$*, the function* $\Delta \ni t \mapsto \lambda_h(t)$ *is* **continuous**.

Theorem B *The function*

$$\Delta \ni t \mapsto dim\ \ker E_t$$

is **upper-semicontinuous**.

Theorem C *If* $\dim \ker E_t$ *is* **independent of** $t \in \Delta$, *then* $(F_t)_{t\in\Delta}$ *is a* C^∞ **family of linear operators**, *where* $F_t : L(B_t) \longrightarrow \ker E_t$ *is the orthogonal projection w.r.t. the* L^2 *inner product* $\langle\langle\ ,\ \rangle\rangle_t$, *for every* $t \in \Delta$.

Theorem D *If* $\dim \ker E_t$ *is* **independent of** $t \in \Delta$, *then* $(G_t)_{t\in\Delta}$ *is a* C^∞ **family of linear operators**, *where* $G_t := E_t^{-1}$ *is the Green operator of* E_t *for every* $t \in \Delta$.

Before moving on to the proofs of these theorems, let us point out the

Observation 2.5.6. *Having fixed an arbitrary* $h \in \mathbb{N}^\star$, *the function* $\Delta \ni t \mapsto \lambda_h(t)$ **need not be differentiable**.

Example. Notice that the eigenvalues of the operator $E_t := \begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix}$ are $\lambda_1(t) = 1 + \sqrt{t}$ and $\lambda_1(t) = 1 - \sqrt{t}$, which are not differentiable functions of t. $\Box$

2.5.2 Preliminary steps in the proofs of Theorems A, B, C, D

Step 1. First, we have the following analogue for vector bundles of Ehresmann's theorem. We assume that $\Delta \subset \mathbb{R}^N$ is a small open ball about 0 and $\Delta_\varepsilon \subset \Delta$ is the ball of radius ε about 0.

Theorem 2.5.7. *Let* $(B_t)_{t\in\Delta}$ *be a* C^∞ **family of** C^∞ **complex vector bundles** $B_t \longrightarrow X$ *and let* $\pi : \mathcal{B} \longrightarrow X \times \Delta$ *be the corresponding* C^∞ *complex vector bundle such that* $B_t = \pi^{-1}(X \times \{t\}) = \mathcal{B}_{|X\times\{t\}}$ *for all* $t \in \Delta$.

Then, there exist $\varepsilon > 0$ *and a* C^∞ **diffeomorphism** $T : B_0 \times \Delta_\varepsilon \longrightarrow \mathcal{B}$ *of* $\mathbb{C}$*-vector bundles over* $X \times \Delta_\varepsilon$ *that respects the fibres, namely the restriction of* T *to every fibre* $\pi_0^{-1}(x,\, t)$ *of the vector bundle* $\pi_0 : B_0 \times \Delta_\varepsilon \longrightarrow X \times \Delta_\varepsilon$ *defines a* $\mathbb{C}$*-vector space* **isomorphism**

$$T_{|\pi_0^{-1}(x,\,t)} : \pi_0^{-1}(x,\, t) \xrightarrow{\simeq} \pi^{-1}(x,\, t)$$

onto the corresponding fibre of $\mathcal{B}$ *for every* $(x,\, t) \in X \times \Delta_\varepsilon$.

Proof. This situation is easier to handle than the one in Ehresmann's theorem since the vector bundle $\pi : \mathcal{B} \longrightarrow X \times \Delta$ is given. The details are left to the reader and are also spelt out in [Kod86, Lemma 7.1]. $\Box$

Consequently, the vector bundles B_t are C^∞ *isomorphic* to B_0 for all $t \in \Delta$ sufficiently close to 0. Therefore, we may assume without loss of generality that all the B_t's coincide with a fixed C^∞ vector bundle $B \longrightarrow X$, after possibly shrinking Δ about 0. In particular, henceforth, we place ourselves in the situation

$$E_t : L(B) \longrightarrow L(B), \qquad t \in \Delta.$$

However, the Hermitian fibre metric h_t on B depends on $t \in \Delta$ and so do $\langle\ ,\ \rangle_t$, $\langle\langle\ ,\ \rangle\rangle_t$ and $||\ ||_t$, but they are mutually equivalent by uniform multiplicative constants.

Notation 2.5.8. *For every* $k \in \mathbb{N}^\star$, *let* $||\psi||_k$ *stand for the* k**-th Sobolev norm** *of a given section* $\psi \in L(B)$.

After possibly shrinking Δ about 0, we may assume that $||\psi||_k$ is independent of $t \in \Delta$. Note that the meaning of $||\psi||_0$ is either as the Sobolev norm W^0 or as the L^2 norm induced by the Hermitian

L^2 inner product $\langle\langle\ ,\ \rangle\rangle_t$ at $t=0$. However, there is no risk of confusion since it can easily be seen that the L^2 norms $||\ \ ||_0$ and $||\ \ ||_t$, induced respectively by the L^2 inner products $\langle\langle\ ,\ \rangle\rangle_0$ and $\langle\langle\ ,\ \rangle\rangle_t$, are *uniformly equivalent* in the following sense. There exists a constant $K_0 > 1$ *independent of* $t \in \Delta$ such that

$$\frac{1}{K_0}\,||\psi||_0 \leq ||\psi||_t \leq K_0\,||\psi||_0, \qquad \psi \in L(B). \tag{2.88}$$

Step 2. The following technical result will be crucial.

Theorem 2.5.9. *Let $(E_t)_{t\in\Delta}$ be a C^∞ family of* **elliptic**, *not necessarily self-adjoint, differential operators of* **even order** *m. Suppose that $E_t : L(B) \longrightarrow L(B)$ is* **bijective** *for all $t \in \Delta$.*

If there exists a constant $c > 0$ **independent** *of $t \in \Delta$ such that*

$$||E_t\psi||_0 \geq c\,||\psi||_0 \quad \textit{for all } \psi \in L(B), \tag{2.89}$$

the inverse operator E_t^{-1} varies in a C^∞ way with $t \in \Delta$.

The key point about hypothesis (2.89) is the uniformity of the constant c. A constant depending on t with this property always exists thanks to E_t being elliptic and to X being compact, as follows from the *a priori estimate* by arguments for the Sobolev norms similar to those used in the proof of (2.29) with Hölder norms.

The main ingredients in the proof of Theorem 2.5.9.

We start by introducing the notion of C^r family $(\psi_t)_{t\in\Delta}$ of C^∞ sections $\psi_t \in L(B)$. This will mean that the sections ψ_t depend in a C^r way on t, but surprisingly, this property is not the complete analogue of the notion of C^∞ family $(\psi_t)_{t\in\Delta}$ of C^∞ sections $\psi_t \in L(B)$.

Definition 2.5.10. *For every $t \in \Delta$, let $\psi_t \in L(B) = C^\infty(X,\,B)$ be a smooth global section of B.*

We say that $(\psi_t)_{t\in\Delta}$ is a C^r **family of C^∞ sections** *(or that ψ_t* **varies** *C^r with $t \in \Delta$) if $D^l\psi^1(x,\,t),\dots,D^l\psi^r(x,\,t)$ are C^r* **functions** *of $(x,\,t)$ for every $l=(l_1,\dots,l_n)$, where*

$$D^l := \frac{\partial^{l_1+\cdots+l_n}}{\partial x_1^{l_1}\dots\partial x_n^{l_n}}$$

and $\psi_t = (\psi^1(x,\,t),\dots,\psi^r(x,\,t))$ are the local fibre coordinates of ψ_t in a local trivialisation of B.

In other words, it does not suffice to require each component $\psi^j(x,\,t)$ of ψ_t to be C^r in $(x,\,t)$, but we need all their derivatives of arbitrary orders w.r.t. the variables $x=(x_1,\dots,x_n)$ to be C^r in $(x,\,t)$. We saw in Observation 2.5.4 that, when $r=+\infty$, this extra requirement is unnecessary.

However, in the case of families of operators, the C^r regularity is defined in analogous fashion to the C^∞ regularity.

Definition 2.5.11. *For every $t \in \Delta$, let $E_t : L(B_t) \longrightarrow L(B_t)$ be a linear operator on B_t.*

We say that $(E_t)_{t\in\Delta}$ is a C^r **family of linear operators** *(or that E_t* **varies** *C^r with $t \in \Delta$) if for every C^r family $(\psi_t)_{t\in\Delta}$ of C^∞ sections $\psi_t \in L(B_t)$, the family $(E_t\psi_t)_{t\in\Delta}$ is a C^r family of C^∞ sections.*

From the definitions, we immediately get the following equivalence:

$$E_t^{-1} \text{ varies } C^\infty \text{ with } t \in \Delta \iff E_t^{-1} \text{ varies } C^r \text{ with } t \in \Delta \text{ for every } r \in \mathbb{N}.$$

The other main ingredient in the proof of Theorem 2.5.9 is the following **a priori estimate** w.r.t. **Sobolev norms** in families of *elliptic* operators. This is to be compared with its Hölder norm analogue for just one elliptic operator that we saw in Theorem 2.3.7.

Theorem 2.5.12. *Let $(E_t)_{t\in\Delta}$ be a C^∞ family of* **elliptic** *linear differential operators $E_t : L(B) \longrightarrow L(B)$ of even order m.*

Then, for every $k \in \mathbb{N}$, there exists a constant $c_k > 0$ **independent of** *$t \in \Delta$ such that the following* **uniform a priori estimate** *holds:*

$$||\psi||^2_{k+m} \leq c_k \, (||E_t\psi||^2_k + ||\psi||^2_0) \tag{2.90}$$

for every $\psi \in L(B)$ and every $t \in \Delta$.

Proof. This is a standard result whose proof can be found, e.g. in [Kod86, Lemma 7.3]. □

We will apply this result in the following way to the proof of Theorem 2.5.9. For every $t \in \Delta$, we have:

$$||\psi||_0 \leq \frac{1}{c}\,||E_t\psi||_0 \leq \frac{1}{c}\,||E_t\psi||_k, \qquad \psi \in L(B),$$

where the first inequality is hypothesis (2.89) of Theorem 2.5.9, while the second inequality is trivial. Together with the *uniform a priori estimate* (2.90), this yields:

$$||\psi||_{k+m} \leq c'_k \, ||E_t\psi||_k, \qquad \psi \in L(B), \;\; t \in \Delta, \tag{2.91}$$

where $c'_k > 0$ is a constant independent of $t \in \Delta$ and of $\psi \in L(B)$.

On the other hand, for every point x in a chart domain, the **Sobolev inequality** yields:

$$|D^l\psi^\lambda(x)| \leq c_{k+m-l,\,l}\,||\psi||_{k+m}, \quad \psi = (\psi^1, \dots, \psi^r) \in L(B), \quad l \text{ such that } k+m-l > \frac{n}{2},$$

where $n := \dim_{\mathbb{R}} X$. Together with (2.91), this leads to

$$|D^l\psi^\lambda(x)| \leq c'_{k,\,l}\,||E_t\psi||_k, \quad \psi = (\psi^1, \dots, \psi^r) \in L(B), \quad k > l - m + \frac{n}{2}, \;\; t \in \Delta, \;\; x \in X, \tag{2.92}$$

for some constant $c'_{k,\,l} > 0$ independent of $t \in \Delta$.

Inequality (2.92) will be the main tool in the proof of Theorem 2.5.9.

Proof of Theorem 2.5.9 by induction.

We will prove the following implication by induction on $r \in \mathbb{N}$:

$(\varphi_t)_{t\in\Delta}$ is a C^r family of sections $\varphi_t \in L(B) \implies$

$$(\psi_t := E_t^{-1}\varphi_t)_{t\in\Delta} \text{ is a } C^r \text{ family of sections } \psi_t \in L(B). \tag{2.93}$$

• *The case $r = 0$.* Suppose that $(\varphi_t = E_t\psi_t)_{t\in\Delta}$ varies in a C^0 way with $t \in \Delta$. We have to prove that, for every D^l and every $\lambda \in \{1, \dots, r\}$, $D^l\psi^\lambda(x, t)$ is *continuous* in (x, t).

Since $D^l\psi^\lambda(x,\,t)$ is continuous in x, it suffices to prove that, for every $s\in\Delta$, we have the convergence:

$$D^l\psi^\lambda(x,\,t) \xrightarrow[t\longrightarrow s]{} D^l\psi^\lambda(x,\,s) \quad \textbf{uniformly} \text{ w.r.t. } x.$$

To see this, fix any $s\in\Delta$ and any $l\in\mathbb{N}$. Choose an integer k such that $k>l-m+\frac{n}{2}$ and set $c':=c'_{k,\,l}$, the constant of (2.92). Then, (2.92) applied to $\psi_t-\psi_s\in L(B)$ yields the first inequality below:

$$\begin{aligned}|D^l\psi^\lambda(x,\,t)-D^l\psi^\lambda(x,\,s)| &\leq c'\,||E_t(\psi_t-\psi_s)||_k \leq c'\,||E_t\psi_t-E_s\psi_s||_k+c'\,||E_s\psi_s-E_t\psi_s||_k\\ &= c'\,||\varphi_t-\varphi_s||_k+c'\,||E_t\psi_s-E_s\psi_s||_k.\end{aligned} \tag{2.94}$$

Now, $||\varphi_t-\varphi_s||_k$ converges to 0 as $t\to s$, because this k-th Sobolev norm involves the L^2 norms of all the derivatives up to order k w.r.t. the variables $x=(x_1,\dots,x_n)$ and because $D^l\varphi^\lambda(x,\,t)$ is continuous in $(x,\,t)$ for every λ and every l, by the C^0 assumption on the family $(\varphi_t)_{t\in\Delta}$. Thus, $|D^l\varphi^\lambda(x,\,t)-D^l\varphi^\lambda(x,\,s)|$ converges uniformly w.r.t. x to 0 when $t\to s$, for every l.

Meanwhile, $||E_t\psi_s-E_s\psi_s||_k$ converges to 0 as $t\to s$, because the coefficients of E_t are C^∞ functions of $(x,\,t)$.

Thus, we infer from (2.94) that $D^l\psi^\lambda(x,\,t)$ converges uniformly w.r.t. x to $D^l\psi^\lambda(x,\,s)$ when $t\to s$. This completes the proof of the case $r=0$.

• *The case* $r=1$. Suppose that $(\varphi_t=E_t\psi_t)_{t\in\Delta}$ varies in a C^1 way with $t\in\Delta$. We have to prove that $(\psi_t=E_t^{-1}\varphi_t)_{t\in\Delta}$ varies in a C^1 way with $t\in\Delta$.

Let $t=(t_1,\dots,t_N)\in\mathbb{R}^N$. For every k, define the *derivative of* φ_t *w.r.t.* t_k by

$$\frac{\partial\varphi_t}{\partial t_k}=\left(\frac{\partial\varphi^1(x,\,t)}{\partial t_k},\dots,\frac{\partial\varphi^r(x,\,t)}{\partial t_k}\right)\in L(B),$$

where $\varphi_t(x)=(\varphi^1(x,\,t),\dots,\varphi^r(x,\,t))$ in a local trivialisation of B. Note that $\partial\varphi_t/\partial t_k\in L(B)$ because the transition matrices of B do not depend on t.

Suppose we have proved that ψ_t is differentiable in $t\in\Delta$. Then, differentiating the identity $\varphi_t=E_t\psi_t$ w.r.t. t_k, we get:

$$\frac{\partial\varphi_t}{\partial t_k}=E_t\,\frac{\partial\psi_t}{\partial t_k}+\frac{\partial E_t}{\partial t_k}\,\psi_t,$$

hence

$$\frac{\partial\psi_t}{\partial t_k}=E_t^{-1}\left(\frac{\partial\varphi_t}{\partial t_k}-\frac{\partial E_t}{\partial t_k}\,\psi_t\right):=\eta_{k,\,t},\qquad k\in\{1,\dots,N\},$$

where $\partial E_t/\partial t_k$ is the differential operator obtained from E_t by differentiating w.r.t. t_k the coefficients of E_t viewed as polynomials of $D_1,\dots,D_n$.

Now, φ_t varies in a C^1 way with $t\in\Delta$. Hence, $\frac{\partial\varphi_t}{\partial t_k}$ varies in a C^0 way with $t\in\Delta$ and $\psi_t=E_t^{-1}\varphi_t$ varies in a C^0 way with $t\in\Delta$ by the case $r=0$. Consequently, $\eta_{k,\,t}$ varies in a C^0 way with $t\in\Delta$.

Therefore, to prove that ψ_t varies in a C^1 way with $t\in\Delta$, we have to prove that, for every D^l, $D^l\psi^\lambda(x,\,t)$ is differentiable in $t\in\Delta$ and

$$\frac{\partial}{\partial t_k}D^l\psi^\lambda(x,\,t)=D^l\eta^\lambda_{k,\,t}(x),\qquad \lambda\in\{1,\dots,r\},\;\; k\in\{1,\dots,N\},$$

where $\eta_{k,\,t}(x)=(\eta^1_{k,\,t}(x),\dots,\eta^r_{k,\,t}(x))$ in a local trivialisation of B.

The proof of this fact is straightforward and is left to the reader, as is the case $r\geq 2$. For the details, the reader is referred to [Kod86, Theorem 7.5]. □

Step 3. Henceforth, we shall assume that each operator E_t is **self-adjoint** (and, of course, also **elliptic**).

Lemma 2.5.13. *Let $(a_h)_{h\in\mathbb{N}}$ be an arbitrary sequence of complex numbers. Fix any $t \in \Delta$. The following statements are equivalent.*

(i) $(a_h)_{h\in\mathbb{N}}$ represents an element of $L(B)$, in the sense that there exists $\psi \in L(B)$ such that $\psi = \sum\limits_{h=1}^{+\infty} a_h\, e_h(t)$.

(ii) For every $l \in \mathbb{N}^\star$, $\sum\limits_{h=1}^{+\infty} |\lambda_h(t)|^{2l}\, |a_h|^2 < +\infty$.

Proof. It is standard and straightforward and is left to the reader. □

Note that the expression in (ii) of Lemma 2.5.13 is the squared L^2 norm (induced by h_t) of $E_t^l\psi$:

$$||E_t^l\psi||_t^2 = \sum_{h=1}^{+\infty} |\lambda_h(t)|^{2l}\, |a_h|^2, \qquad l \in \mathbb{N}^\star,$$

where $E_t^l := E_t \circ \cdots \circ E_t$ (l times) and $\psi = \sum\limits_{h=1}^{+\infty} a_h\, e_h(t) \in L(B)$. Thus, the implication $(i) \implies (ii)$ follows immediately from $E_t^l : L(B) \longrightarrow L(B)$, for every l.

Since it will be needed later, we now state and prove a key ingredient for the proof of the above Lemma 2.5.13 that we have not elaborated on.

Lemma 2.5.14. *Fix $t \in \Delta$. For every $k \in \mathbb{N}^\star$, there exists a constant $c_k'' > 0$ such that, for every $\psi = \sum\limits_{h=1}^{+\infty} a_h\, e_h(t) \in L(B)$, the following inequality holds:*

$$||\psi||_k^2 \leq c_k'' \sum_{h=1}^{+\infty} \left(1 + \sum_{l=1}^{k} |\lambda_h(t)|^{2l}\right) |a_h|^2. \tag{2.95}$$

Proof. We will apply the *uniform a priori estimate* (2.90) repeatedly. Since the estimate remains true when the constant c_k is replaced by a larger one, we may assume that $c_k \geq 1$. When $k = 0$, for all $\psi \in L(B)$ and all $t \in \Delta$, we get:

$$||\psi||_m^2 \leq c_0\, (||E_t\psi||_0^2 + ||\psi||_0^2).$$

When $k = m$, (2.90) implies the first inequality below, while the second one follows from the above inequality and from $c_0 \geq 1$:

$$||\psi||_{2m}^2 \leq c_m\, (||E_t\psi||_m^2 + ||\psi||_0^2) \leq c_0\, c_m\, (||E_t^2\psi||_0^2 + ||E_t\psi||_0^2 + ||\psi||_0^2).$$

Continuing inductively, for every $q \in \mathbb{N}^\star$, we get:

$$||\psi||_{qm}^2 \leq c_0\, c_m\, \dots\, c_{(q-1)m}\, (\sum_{l=1}^{q} ||E_t^l\psi||_0^2 + ||\psi||_0^2).$$

Now, fix an arbitrary $k \in \mathbb{N}^\star$. Then, there exists a unique $q \in \mathbb{N}$ such that $(q-1)m < k \leq qm$. Since $m \geq 2$, $q \leq k$. Putting $\hat{c}_k := c_0\, c_m\, \dots\, c_{(q-1)m} > 0$ (a constant independent of $t \in \Delta$) for this q and using (2.88), we get:

$$||\psi||_k^2 \leq ||\psi||_{qm}^2 \leq \hat{c}_k\,(\sum_{l=1}^{k} ||E_t^l\psi||_0^2 + ||\psi||_0^2) \leq \hat{c}_k\, K_0^2\,(\sum_{l=1}^{k} ||E_t^l\psi||_t^2 + ||\psi||_t^2).$$

Since $||E_t^l\psi||_t^2 = \sum_{h=1}^{+\infty} |\lambda_h(t)|^{2l}\,|a_h|^2$ and $||\psi||_t^2 = \sum_{h=1}^{+\infty} |a_h|^2$, (2.95) follows by taking $c_k'' := \hat{c}_k\, K_0^2$. $\square$

The main technique for the proofs of Theorems A, B, C, D consists in considering, for every $\zeta \in \mathbb{C}$, the elliptic differential operator

$$E_t(\zeta) := E_t - \zeta : L(B) \longrightarrow L(B), \qquad t \in \Delta,$$

to which the following simple but critical observation and several of the above preliminary results will be applied.

Observation 2.5.15. *If $\zeta \notin Spec\,(E_t) := \{\lambda_1(t),\, \lambda_2(t), \dots\}$, then $E_t(\zeta) : L(B) \longrightarrow L(B)$ is* **bijective**.

Proof. Since $\ker E_t(\zeta)$ is the eigenspace of E_t corresponding to the "eigenvalue" ζ, we have $\ker E_t(\zeta) = \{0\}$ if ζ is not an actual eigenvalue of E_t. Therefore, $E_t(\zeta)$ is *injective* if $\zeta \notin \mathrm{Spec}\,(E_t)$.

To prove surjectivity, pick an arbitrary $\varphi = \sum_{h=1}^{+\infty} b_h e_h(t) \in L(B)$, with $b_h \in \mathbb{C}$ for every h. Note that, for any $\psi = \sum_{h=1}^{+\infty} a_h e_h(t) \in L(B)$, we have $E_t(\zeta)\,\psi = \sum_{h=1}^{+\infty} (\lambda_h(t) - \zeta)\, a_h e_h(t)$. We want a $\psi = \sum_{h=1}^{+\infty} a_h e_h(t) \in L(B)$ (i.e. we want a sequence of complex numbers $(a_h)_{h\in\mathbb{N}^\star}$) such that $E_t(\zeta)\,\psi = \varphi$, which is equivalent to having

$$a_h = \frac{b_h}{\lambda_h(t) - \zeta}, \qquad h \in \mathbb{N}^\star.$$

So, we have no choice but to define the a_h's by this formula. Note that they are well defined since $\lambda_h(t) - \zeta \neq 0$, for every $h \in \mathbb{N}^\star$, thanks to the assumption $\zeta \notin \mathrm{Spec}\,(E_t)$.

We will now use Lemma 2.5.13 to show that the sequence $(a_h)_{h\in\mathbb{N}^\star}$ represents indeed an element $\psi = \sum_{h=1}^{+\infty} a_h e_h(t) \in L(B)$.

Since $\varphi = \sum_{h=1}^{+\infty} b_h e_h(t) \in L(B)$, the implication "(i) $\implies$ (ii)" of Lemma 2.5.13 ensures that $\sum_{h=1}^{+\infty} |\lambda_h(t)|^{2l}\,|b_h|^2 < +\infty$ for every $l \in \mathbb{N}^\star$. We infer that

$$\sum_{h=1}^{+\infty} |\lambda_h(t)|^{2l}\,|a_h|^2 = \sum_{h=1}^{+\infty} \frac{|\lambda_h(t)|^{2l}\,|b_h|^2}{|\lambda_h(t) - \zeta|^2} \leq C \sum_{h=1}^{+\infty} |\lambda_h(t)|^{2l}\,|b_h|^2 < +\infty, \qquad l \in \mathbb{N}^\star,$$

because

$$\frac{1}{|\lambda_h(t) - \zeta|^2} \leq C, \qquad h \in \mathbb{N}^\star,$$

for some constant $C > 0$ independent of h since $\lim\limits_{h\to+\infty}\lambda_h(t) = +\infty$, Spec (E_t) is *discrete* with no finite accumulation point and $\zeta \notin$ Spec (E_t).

Therefore, the implication "(ii) $\implies$ (i)" of Lemma 2.5.13 ensures that the sequence $(a_h)_{h\in\mathbb{N}^\star}$ represents indeed an element $\psi = \sum\limits_{h=1}^{+\infty} a_h e_h(t) \in L(B)$. □

We now come to a key technical result saying that the hypothesis of Theorem 2.5.9 is **uniformly** satisfied by the operators $E_t(\zeta) : L(B) \longrightarrow L(B)$ when $t \in \Delta$ and $\zeta \notin$ Spec (E_t) vary very little.

Lemma 2.5.16. *Let $t_0 \in \Delta$ and let $\zeta_0 \in \mathbb{C} \setminus$ Spec (E_{t_0}). Then, there exist constants $\delta, c > 0$ such that, for all $t \in \Delta$ with $|t - t_0| < \delta$ and all $\zeta \in \mathbb{C}$ with $|\zeta - \zeta_0| < \delta$, the following inequality holds:*

$$||E_t(\zeta)\,\psi||_0 \geq c\,||\psi||_0 \tag{2.96}$$

for all $\psi \in L(B)$.

Proof. We will proceed by contradiction. Suppose that for every $q \in \mathbb{N}^\star$, there exist $t_q \in \Delta$, $\zeta_q \in \mathbb{C}$ and $\psi_q \in L(B)$ such that

$$|t_q - t_0| < \frac{1}{q}, \;\; |\zeta_q - \zeta_0| < \frac{1}{q}, \;\; ||E_{t_q}(\zeta_q)\,\psi_q||_0 < \frac{1}{q}, \;\text{ and } ||\psi_q||_0 = 1.$$

We wish to understand the variation of the difference $E_{t_q}(\zeta_q)\,\psi_q - E_{t_0}(\zeta_0)\,\psi_q$ as $q \to +\infty$. Recall that m is the order of each E_t, hence also the order of each $E_t(\zeta)$.

The *uniform family version* of the *a priori estimate* (2.90) for *elliptic* operators, applied with $k = 0$, yields the first inequality below for a constant $c_0 > 0$ independent of q:

$$||\psi_q||^2_m \leq c_0\left(||E_{t_q}(\zeta_q)\,\psi_q||^2_0 + ||\psi_q||^2_0\right) < c_0\left(\frac{1}{q^2} + 1\right) \leq 2c_0, \qquad q \in \mathbb{N}^\star. \tag{2.97}$$

We get

$$||E_{t_q}(\zeta_q)\,\psi_q - E_{t_0}(\zeta_0)\,\psi_q||_0 = ||(E_{t_q} - E_{t_0})\,\psi_q - (\zeta_q - \zeta_0)\,\psi_q||_0 \leq ||(E_{t_q} - E_{t_0})\,\psi_q||_0 + \frac{1}{q}$$

and the last quantity tends to 0 as $q \to +\infty$. Indeed, the coefficients of E_t are C^∞ functions of $(x,\, t)$, $\lim_{q\to+\infty} t_q = t_0$, the order of each operator $E_{t_q} - E_{t_0}$ is m and $||\psi_q||_m$ stays bounded, as $q \to +\infty$, thanks to (2.97).

Now, $||E_{t_q}(\zeta_q)\,\psi_q||_0 < \frac{1}{q} \xrightarrow[q\to+\infty]{} 0$. So, together with the above conclusion, this implies that

$$\lim_{q\to+\infty} ||E_{t_0}(\zeta_0)\,\psi_q||_0 = 0. \tag{2.98}$$

On the other hand, for every $\psi = \sum\limits_{h=1}^{+\infty} a_h e_h(t_0) \in L(B)$, we have:

$$||E_{t_0}(\zeta_0)\,\psi||_{t_0} = \left|\left|\sum_{h=1}^{+\infty}(\lambda_h(t_0) - \zeta_0)\,a_h e_h(t_0)\right|\right|_{t_0} \geq \mu_0\,||\psi||_{t_0},$$

where $\mu_0 := \min\limits_{h\in\mathbb{N}^\star} |(\lambda_h(t_0) - \zeta_0| > 0$. This implies the second inequality below:

$$K_0\,||E_{t_0}(\zeta_0)\,\psi_q||_0 \geq ||E_{t_0}(\zeta_0)\,\psi_q||_{t_0} \geq \mu_0\,||\psi_q||_{t_0} \geq \frac{\mu_0}{K_0}\,||\psi_q||_0 = \frac{\mu_0}{K_0} > 0, \qquad q \in \mathbb{N}^\star,$$

where the first and third inequalities follow from the fact that the L^2 norms $|| \, ||_0$ and $|| \, ||_{t_0}$ are *uniformly equivalent* (by means of positive constants independent of q that we denoted by K_0 and $1/K_0$ in (2.88)). Consequently,

$$||E_{t_0}(\zeta_0)\,\psi_q||_0 \geq \frac{\mu_0}{K_0^2} > 0, \qquad q \in \mathbb{N}^\star,$$

which contradicts (2.98). □

Step 4. The next goal is to express the **spectral projection operators** by a **Cauchy integral formula**. Let

$$W := \left\{(t,\,\zeta) \in \Delta \times \mathbb{C} \mid \zeta \notin \operatorname{Spec} E_t\right\} \subset \Delta \times \mathbb{C}.$$

Lemma 2.5.16 implies that W is **open** in $\Delta \times \mathbb{C}$ (because (2.96) implies that $\ker E_t(\zeta) = \{0\}$, which amounts to $\zeta \notin \operatorname{Spec} E_t$).

Meanwhile, $E_t(\zeta) : L(B) \longrightarrow L(B)$ is *bijective* for all $(t,\,\zeta) \in W$. Let

$$G_t(\zeta) := E_t(\zeta)^{-1} : L(B) \longrightarrow L(B), \qquad (t,\,\zeta) \in W,$$

be its inverse. From Theorem 2.5.9 and Lemma 2.5.16, we get the following crucial piece of information which is the culmination of the technical work we have been doing in this subsection.

Conclusion 2.5.17. $G_t(\zeta)$ **varies in a** C^∞ **way** *with* $(t,\,\zeta) \in W$.

Now, fix an arbitrary $t_0 \in \Delta$ and pick a **Jordan curve** C (i.e. a closed simple curve C in the complex plane) such that

$$C \cap \operatorname{Spec} E_{t_0} = \emptyset. \tag{2.99}$$

Such a curve exists because $\operatorname{Spec} E_{t_0} \subset \mathbb{R}$ is discrete. As is well known, C divides the plane $\mathbb{C}$ into two disjoint regions: the *interior* of C, denoted by $\operatorname{int}(C)$, and the *exterior* of C, denoted by $\operatorname{ext}(C)$.

Property (2.99) means that $\{t_0\} \times C \subset W$. Since W is open in $\Delta \times C$, there exists $\delta > 0$ such that $[t_0 - \delta,\, t_0 + \delta] \times C \subset W$. For any $t \in (t_0 - \delta,\, t_0 + \delta)$, we put:

$$\mathbb{F}_t(C) := \bigoplus_{\lambda(t) \in \operatorname{int}(C)} \mathcal{H}_{\lambda(t)}(E_t) \subset L(B),$$

where $\mathcal{H}_{\lambda(t)}(E_t)$ is the $\lambda(t)$-eigenspace of E_t. Note that, by *ellipticity* of E_t and *compactness* of X, the $\mathbb{C}$-vector space $\mathbb{F}_t(C)$ is *finite dimensional*. Furthermore, for any $t \in (t_0 - \delta,\, t_0 + \delta)$, we let

$$F_t(C) : L(B) \longrightarrow \mathbb{F}_t(C)$$

be the $L^2_{h_t}$-*orthogonal projection* onto $\mathbb{F}_t(C)$.

The following simple **Cauchy integral formula** for **orthogonal spectral projectors** will play a key role in the sequel.

Lemma 2.5.18. *The orthogonal projector* $F_t(C) : L(B) \longrightarrow \mathbb{F}_t(C)$ *satisfies the following formula:*

$$F_t(C)\,\psi = -\frac{1}{2\pi i} \int\limits_{\zeta \in C} G_t(\zeta)\,\psi\, d\zeta, \qquad \psi \in L(B),\;\; t \in (t_0 - \delta,\, t_0 + \delta). \tag{2.100}$$

Proof. Let $\psi = \sum_{h=1}^{+\infty} a_h e_h(t) \in L(B)$. Since $E_t(\zeta)\,\psi = \sum_{h=1}^{+\infty} (\lambda_h(t) - \zeta)\, a_h e_h(t)$ and $G_t(\zeta) = E_t(\zeta)^{-1}$, we get:

$$-G_t(\zeta)\,\psi = \sum_{h=1}^{+\infty} \frac{a_h}{\zeta - \lambda_h(t)}\, e_h(t), \qquad \zeta \in C.$$

Therefore,

$$-\frac{1}{2\pi i}\int\limits_{\zeta\in C} G_t(\zeta)\,\psi\, d\zeta = \sum_{h=1}^{+\infty}\left(\frac{1}{2\pi i}\int\limits_{\zeta\in C}\frac{1}{\zeta - \lambda_h(t)}\,d\zeta\right) a_h e_h(t) = \sum_{\lambda_h(t)\in \mathrm{int}\,(C)} a_h e_h(t) = F_t(C)\,\psi,$$

where the following elementary fact on the winding number of a Jordan curve around a point in the complex plane has been used:

$$\frac{1}{2\pi i}\int\limits_{\zeta\in C}\frac{1}{\zeta-\lambda_h(t)}\,d\zeta = 1 \quad \text{if} \quad \lambda_h(t)\in \mathrm{int}\,(C) \quad \text{and} \quad \frac{1}{2\pi i}\int\limits_{\zeta\in C}\frac{1}{\zeta-\lambda_h(t)}\,d\zeta = 0 \quad \text{if} \quad \lambda_h(t)\in \mathrm{ext}\,(C).$$

□

Corollary 2.5.19. *For any $t_0 \in \Delta$ and any Jordan curve $C \subset \mathbb{C}$ such that*

$$C \cap \mathit{Spec}\, E_{t_0} = \emptyset,$$

the orthogonal projector $F_t(C)$ varies in a C^∞ way with $t \in (t_0 - \delta,\, t_0 + \delta)$ if $\delta > 0$ is **small enough**.

Proof. Let $(\psi_t)_{t\in(t_0-\delta,\, t_0+\delta)}$ be a C^∞ family of sections $\psi_t \in L(B)$. Then, by formula (2.100), we have

$$F_t(C)\,\psi_t = -\frac{1}{2\pi i}\int\limits_{\zeta\in C} G_t(\zeta)\,\psi_t\,d\zeta, \qquad t \in (t_0-\delta,\, t_0+\delta). \tag{2.101}$$

Since $G_t(\zeta)$ varies in a C^∞ way with $(t,\,\zeta)\in W$ (by Conclusion 2.5.17) and ψ_t varies in a C^∞ way with $t \in (t_0-\delta,\, t_0+\delta)$, we conclude that $F_t(C)\,\psi_t$ varies in a C^∞ way with $t\in(t_0-\delta,\, t_0+\delta)$. □

The following consequence is the cornerstone of much of what follows.

Corollary 2.5.20. *For any $t_0 \in \Delta$ and any Jordan curve $C \subset \mathbb{C}$ such that*

$$C \cap \mathit{Spec}\, E_{t_0} = \emptyset,$$

the number $\dim_{\mathbb{C}}\mathbb{F}_t(C)$ of eigenvalues, counted with multiplicities, of E_t lying in $\mathrm{int}\,(C)$ is **independent** *of $t \in (t_0-\delta,\, t_0+\delta)$ if $\delta > 0$ is* **small enough**.

Proof. Let $d := \dim_{\mathbb{C}}\mathbb{F}_{t_0}(C)$ and let $\{e_1, \dots, e_d\}$ be a basis of $\mathbb{F}_{t_0}(C)$. There are two inequalities to prove.

• The inequality $\dim_{\mathbb{C}}\mathbb{F}_t(C) \geq \dim_{\mathbb{C}}\mathbb{F}_{t_0}(C)$ for all $t\in(t_0-\delta,\, t_0+\delta)$ (if $\delta>0$ is small enough) is immediate to prove. Indeed, since $F_t(C) : L(B) \longrightarrow \mathbb{F}_t(C)$ varies in a C^∞ way with $t\in(t_0-\delta,\, t_0+\delta)$ (by Corollary 2.5.19), $F_t(C)\,e_j$ varies in a C^∞ way with $t\in(t_0-\delta,\, t_0+\delta)$, for every $j\in\{1,\dots,d\}$. Meanwhile, the property of linear independence is stable under small continuous deformations.

Therefore, since the $e_j = F_{t_0}(C)\,e_j$, with $j\in\{1,\dots,d\}$, are linearly independent and since the $F_t(C)\,e_j$ vary continuously (even in a C^∞ way) with t, the $F_t(C)\,e_j$, with $j\in\{1,\dots,d\}$, remain linearly independent elements of $\mathbb{F}_t(C)$ for all t sufficiently close to t_0. Thus, $\dim_{\mathbb{C}}\mathbb{F}_t(C) \geq d$ for all t close enough to t_0.

• We will prove the reverse inequality $\dim_{\mathbb{C}}\mathbb{F}_t(C) \leq \dim_{\mathbb{C}}\mathbb{F}_{t_0}(C)$ for all $t \in (t_0 - \delta,\, t_0 + \delta)$ (if $\delta > 0$ is small enough) by contradiction. Suppose there exists a sequence $(t_q)_{q\in\mathbb{N}^\star}$ of points $t_q \in \Delta$ such that

$$|t_q - t_0| < \frac{1}{q} \quad \text{and} \quad \dim_{\mathbb{C}}\mathbb{F}_{t_q}(C) \geq d+1$$

for all $q \in \mathbb{N}^\star$. Then, there exist at least $d+1$ eigenvalues (counted with multiplicities) $\lambda_{h_\nu}(t_q) := \lambda_\nu^{(q)}$ of E_{t_q} that lie in the interior of C, namely

$$\lambda_\nu^{(q)} \in \operatorname{Spec} E_{t_q} \cap \operatorname{int}(C), \qquad \nu \in \{1, \dots, d+1\}.$$

Let $e_\nu^{(q)} := e_{h_\nu}(t_q)$ be the corresponding eigenvectors of E_{t_q} lying in the orthonormal basis of $L(B)$ consisting of such eigenvectors that was chosen earlier.

The *Sobolev inequality* together with (2.95) yields:

$$\left|D^l e_\nu^{(q)\,\lambda}(x)\right|^2 \leq (c_{k-l,\,l})^2\, c''_k \left(1 + \sum_{\alpha=1}^k |\lambda_\nu^{(q)}|^{2\alpha}\right), \tag{2.102}$$

for all $\nu \in \{1, \dots, d+1\}$, all points x in a given chart domain U, all integers $k > m + 1 + \frac{n}{2}$ and all multi-indices l such that $|l| \leq m+1$, where $e_\nu^{(q)} := (e_\nu^{(q)\,1}, \dots, e_\nu^{(q)\,r})$ are the fibre coordinates of $e_\nu^{(q)}$.

Note that the right-hand side of (2.102) is *bounded* as $q \to +\infty$, since $\lambda_\nu^{(q)} \in \operatorname{int}(C)$ for all q. Therefore, the sequence $(D^l e_\nu^{(q)\,\lambda}(x))_{q\in\mathbb{N}^\star}$ is *uniformly bounded* when $x \in U$, for all l such that $|l| \leq m+1$. This implies that the sequence $(D^l e_\nu^{(q)\,\lambda})_{q\in\mathbb{N}^\star}$ of functions is *equicontinuous* on U, for all l such that $|l| \leq m$.

Now, the sequence $(D^l e_\nu^{(q)\,\lambda})_{q\in\mathbb{N}^\star}$ of functions being *uniformly bounded* and *equicontinuous*, it admits a *uniformly convergent subsequence*, by Ascoli's Theorem. Thus, we may assume that the sequence $(D^l e_\nu^{(q)\,\lambda})_{q\in\mathbb{N}^\star}$ of functions *converges uniformly* on U for all l such that $|l| \leq m$.

Let

$$e_\nu^\lambda := \lim_{q\to+\infty} e_\nu^{(q)\,\lambda}, \qquad \nu \in \{1, \dots, d+1\},\ \ \lambda \in \{1, \dots, r\}.$$

Then, every e_ν^λ is of class C^m and we have:

$$D^l e_\nu^{(q)\,\lambda} \xrightarrow[q\to+\infty]{} D^l e_\nu^\lambda \quad \nu \in \{1, \dots, d+1\},\ \ \lambda \in \{1, \dots, r\},\ \ |l| \leq m.$$

In other words, we get convergences:

$$e_\nu^{(q)\,\lambda} \xrightarrow[q\to+\infty]{} e_\nu = (e_\nu^1, \dots, e_\nu^r), \qquad \nu \in \{1, \dots, d+1\},$$

to C^m sections $e_\nu = (e_\nu^1, \dots, e_\nu^r)$ of B. The sections $\{e_1, \dots, e_{d+1}\}$ are the sections of B we set out to produce. They satisfy the following properties:

· $\langle\langle e_\nu^{(q)},\, e_\mu^{(q)}\rangle\rangle_{t_q} = \delta_{\nu\mu}$ for all q, hence $\langle\langle e_\nu,\, e_\mu\rangle\rangle_{t_0} = \delta_{\nu\mu}$ for all ν, μ;

· $E_{t_q} e_\nu^{(q)} \xrightarrow[q\to+\infty]{} E_{t_0} e_\nu$ because E_{t_q} has C^∞ coefficients and is of order m, for every q;

· $E_{t_q} e_\nu^{(q)} = \lambda_\nu^{(q)}\, e_\nu^{(q)}$ for all q, hence $\lim_{q\to+\infty} \lambda_\nu^{(q)}$ exists. We denote this limit by λ_ν.

We get: $E_{t_0} e_\nu = \lambda_\nu\, e_\nu$ for all $\nu \in \{1, \dots, d+1\}$, hence $\dim_{\mathbb{C}}\mathbb{F}_{t_0}(C) \geq d+1$, which contradicts the definition of d as $d := \dim_{\mathbb{C}}\mathbb{F}_{t_0}(C)$. $\square$

2.5.3 The details of the proofs of Theorems A, B, C, D

The main ingredient will be Corollary 2.5.20, of which Theorems B and C are almost immediate consequences.

Proof of Theorem A.

It is a standard fact that the spectra of the operators $(E_t)_{t\in\Delta}$ are *uniformly bounded below.* (In the geometric applications that will be given in the next section, the operators E_t will even be non-negative, so their spectra will be uniformly bounded below by 0.) Thus, there exists a constant $\beta\in\mathbb{R}$ independent of $t\in\Delta$ such that

$$\lambda_1(t)>\beta \quad \text{for all } t\in\Delta.$$

Fix a point $t_0\in\Delta$. We will prove by induction on $h\in\mathbb{N}^\star$ that

$$\lim_{t\to t_0}\lambda_h(t)=\lambda_h(t_0).$$

Case $h=1$. Let $\varepsilon>0$ be so small that $\beta<\lambda_1(t_0)-\varepsilon$ (and arbitrary with this property). Let $C_\varepsilon\subset\mathbb{C}$ be the circle of radius ε centred at $\lambda_1(t_0)$ in the complex plane and let $C\subset\mathbb{C}$ be a Jordan curve meeting the real axis at only two points: β and $\lambda:=\lambda_1(t_0)-\varepsilon$, such that $C\cap C_\varepsilon=\{\lambda\}$.

Thus, $\mathbb{F}_{t_0}(C)=\{0\}$ (i.e. there are no eigenvalues of E_{t_0} in $\text{int}\,(C)$).

On the one hand, Corollary 2.5.20 applied to C yields a small $\delta>0$ such that

$$\mathbb{F}_t(C)=\{0\} \quad \text{for all } t\in(t_0-\delta,\,t_0+\delta).$$

This translates to $\lambda_1(t)\in\text{ext}\,(C)\cap\mathbb{R}$. Since $\lambda_1(t)>\beta$, we get $\lambda_1(t)>\lambda$. Thus, we have:

$$\lambda_1(t)>\lambda_1(t_0)-\varepsilon, \qquad t\in(t_0-\delta,\,t_0+\delta). \tag{2.103}$$

On the other hand, Corollary 2.5.20 applied to C_ε yields: $\dim\mathbb{F}_t(C_\varepsilon)=\dim\mathbb{F}_{t_0}(C_\varepsilon)\geq 1$ for all $t\in(t_0-\delta,\,t_0+\delta)$, where the last inequality follows from $\lambda_1(t_0)\in\text{int}\,(C_\varepsilon)$. (Shrink the previous $\delta>0$ if necessary.) Therefore, for every $t\in(t_0-\delta,\,t_0+\delta)$, there exists $\lambda_h(t)\in\text{int}\,(C_\varepsilon)$. This implies that

$$\lambda_1(t)\leq\lambda_h(t)<\lambda_1(t_0)+\varepsilon, \qquad t\in(t_0-\delta,\,t_0+\delta). \tag{2.104}$$

Putting (2.103) and (2.104) together, we get:

$$\lambda_1(t_0)-\varepsilon<\lambda_1(t)<\lambda_1(t_0)+\varepsilon, \qquad t\in(t_0-\delta,\,t_0+\delta).$$

Since the small $\varepsilon>0$ was arbitrary, this proves that $\lim\limits_{t\to t_0}\lambda_1(t)=\lambda_1(t_0)$.

Case $h\geq 2$. Suppose we have proved that

$$\lim_{t\to t_0}\lambda_k(t)=\lambda_k(t_0) \quad \text{for all } k\in\{1,\dots,h-1\}. \tag{2.105}$$

We will now prove that $\lim\limits_{t\to t_0}\lambda_h(t)=\lambda_h(t_0)$.

(a) Suppose that $\lambda_1(t_0)=\cdots=\lambda_h(t_0)$. Fix an arbitrary $\varepsilon>0$ and let $C,C_\varepsilon\subset\mathbb{C}$ be Jordan curves as in the case $h=1$. Then, Corollary 2.5.20 applied to C_ε yields:

$$\dim\mathbb{F}_t(C_\varepsilon)=\dim\mathbb{F}_{t_0}(C_\varepsilon)\geq h, \qquad t\in(t_0-\delta,\,t_0+\delta)$$

if $\delta > 0$ is small enough. Hence, $\lambda_1(t), \dots, \lambda_h(t) \in \text{int}\,(C_\varepsilon)$ for all $t \in (t_0 - \delta,\, t_0 + \delta)$. (Note that (2.105) has also been used here.) In particular,

$$|\lambda_h(t) - \lambda_h(t_0)| < \varepsilon, \qquad t \in (t_0 - \delta,\, t_0 + \delta).$$

This proves that $\lim\limits_{t \to t_0} \lambda_h(t) = \lambda_h(t_0)$.

(b) Let l be the (unique) integer such that $2 \leq l \leq h$ and

$$\lambda_1(t_0) \leq \lambda_{l-1}(t_0) < \lambda_l(t_0) = \lambda_{l+1}(t_0) = \cdots = \lambda_h(t_0).$$

Let $\varepsilon > 0$ be so small that $\lambda_{l-1}(t_0) < \lambda_h(t_0) - \varepsilon$ (and arbitrary with this property). Let $C_{h,\varepsilon} \subset \mathbb{C}$ be the circle of radius ε centred at $\lambda_h(t_0)$ in the complex plane and let $C_h \subset \mathbb{C}$ be a Jordan curve meeting the real axis at only two points: $\lambda := \lambda_1(t_0) - \varepsilon$ and $\mu := \lambda_h(t_0) - \varepsilon$, such that $C_h \cap C_{h,\varepsilon} = \{\mu\}$.

Thus, $\dim \mathbb{F}_{t_0}(C_{h,\varepsilon}) \geq h - l + 1$ (because E_{t_0} has at least the eigenvalues $\lambda_l(t_0), \lambda_{l+1}(t_0), \dots, \lambda_h(t_0)$ in $\text{int}\,(C_{h,\varepsilon})$). Therefore, Corollary 2.5.20 applied to $C_{h,\varepsilon}$ yields:

$$\dim \mathbb{F}_t(C_{h,\varepsilon}) = \dim \mathbb{F}_{t_0}(C_{h,\varepsilon}) \geq h - l + 1, \qquad t \in (t_0 - \delta,\, t_0 + \delta)$$

if $\delta > 0$ is small enough. Together with (2.105), this implies that

$$\lambda_h(t) < \lambda_h(t_0) + \varepsilon, \qquad t \in (t_0 - \delta,\, t_0 + \delta). \tag{2.106}$$

On the other hand, $\dim \mathbb{F}_{t_0}(C_h) = l - 1$ (because the only eigenvalues of E_{t_0} lying in $\text{int}\,(C_h)$ are $\lambda_1(t_0), \dots, \lambda_{l-1}(t_0)$.) Therefore, Corollary 2.5.20 applied to C_h yields:

$$\dim \mathbb{F}_t(C_h) = \dim \mathbb{F}_{t_0}(C_h) = l - 1, \qquad t \in (t_0 - \delta,\, t_0 + \delta).$$

(Shrink the previous $\delta > 0$ if necessary.) This amounts to exactly $l - 1$ eigenvalues of E_t lying in $\text{int}\, C_h$. Meanwhile, from (2.105), we get:

$$\lim_{t \to t_0} \lambda_k(t) = \lambda_k(t_0) \in \text{int}\,(C_h), \quad k \in \{1, \dots, l-1\}.$$

Hence, $\lambda_1(t), \dots, \lambda_{l-1}(t) \in \text{int}\,(C_h)$ for all $t \in (t_0 - \delta,\, t_0 + \delta)$. We conclude that the *only* eigenvalues of E_t lying in $\text{int}\,(C_h)$ are $\lambda_1(t), \dots, \lambda_{l-1}(t)$ whenever $t \in (t_0 - \delta,\, t_0 + \delta)$. This implies that

$$\lambda_k(t) \in \text{ext}\,(C_h), \qquad k \geq l,$$

whenever $t \in (t_0 - \delta,\, t_0 + \delta)$. Hence,

$$\mu = \lambda_h(t_0) - \varepsilon < \lambda_l(t) \leq \cdots \leq \lambda_h(t) \leq \dots, \qquad t \in (t_0 - \delta,\, t_0 + \delta). \tag{2.107}$$

Putting together (2.106) and (2.107), we get:

$$\lambda_h(t_0) - \varepsilon < \lambda_h(t) < \lambda_h(t_0) + \varepsilon, \qquad t \in (t_0 - \delta,\, t_0 + \delta).$$

This proves that $\lim\limits_{t \to t_0} \lambda_h(t) = \lambda_h(t_0)$. $\square$

Proof of Theorem B.

Recall that we set $\mathbb{F}_t := \ker E_t = \mathcal{H}_0(E_t)$ for all $t \in \Delta$. Fix $t_0 \in \Delta$. We have to prove that

$$\exists\ \delta > 0 \quad \text{such that} \quad \dim \mathbb{F}_t \leq \dim \mathbb{F}_{t_0} \quad \forall t \in (t_0 - \delta,\, t_0 + \delta).$$

For any $\varepsilon > 0$, let $C_\varepsilon := C(0,\, \varepsilon) \subset \mathbb{C}$ be the circle of radius ε centred at the origin in the complex plane. Since Spec E_{t_0} is discrete, $\mathbb{F}_{t_0} = \mathbb{F}_{t_0}(C_\varepsilon)$ (i.e. 0 is the only eigenvalue of E_{t_0} lying in int (C_ε)) if ε is small enough.

Corollary 2.5.20 applied to C_ε yields:

$$\dim\mathbb{F}_t(C_\varepsilon) = \dim\mathbb{F}_{t_0}(C_\varepsilon), \qquad t \in (t_0 - \delta,\, t_0 + \delta),$$

if $\delta > 0$ is small enough. Since $\dim\mathbb{F}_{t_0}(C_\varepsilon) = \dim\ker E_{t_0}$ and since $\mathbb{F}_t = \ker E_t \subset \mathbb{F}_t(C_\varepsilon)$ for all t, we infer that

$$\dim \mathbb{F}_t \leq \dim \mathbb{F}_t(C_\varepsilon) = \dim\mathbb{F}_{t_0}(C_\varepsilon) = \dim \mathbb{F}_{t_0} \quad \forall t \in (t_0 - \delta,\, t_0 + \delta).$$

The proof is complete. □

Proof of Theorem C.

As in the proof of Theorem B, fix $t_0 \in \Delta$ and choose $\varepsilon > 0$ so small that $\mathbb{F}_{t_0} = \mathbb{F}_{t_0}(C_\varepsilon)$, where $C_\varepsilon := C(0,\, \varepsilon) \subset \mathbb{C}$. Since the following conditions are fulfilled:

(i) $\dim\mathbb{F}_t(C_\varepsilon) = \dim\mathbb{F}_{t_0}(C_\varepsilon)$, $\quad t \in (t_0 - \delta,\, t_0 + \delta)$, (by Corollary 2.5.20 applied to C_ε);

(ii) $\mathbb{F}_t \subset \mathbb{F}_t(C_\varepsilon)$ for all $t \in \Delta$ and $\mathbb{F}_{t_0} = \mathbb{F}_{t_0}(C_\varepsilon)$ (by the choice of ε);

(iii) $\dim\mathbb{F}_t = \dim\mathbb{F}_{t_0}$ for all $t \in \Delta$ (by hypothesis),

we deduce that $\mathbb{F}_t = \mathbb{F}_t(C_\varepsilon)$ for all $t \in (t_0 - \delta,\, t_0 + \delta)$. Hence, $F_t = F_t(C_\varepsilon)$ (equality of operators) for all $t \in (t_0 - \delta,\, t_0 + \delta)$.

Now, we know from Corollary 2.5.19 that the operators $F_t(C_\varepsilon)$ vary in a C^∞ way with $t \in (t_0 - \delta,\, t_0 + \delta)$. (Shrink $\delta > 0$ if necessary.) Hence, the operators F_t vary in a C^∞ way with $t \in (t_0 - \delta,\, t_0 + \delta)$.

Since $t_0 \in \Delta$ was chosen arbitrarily, we conclude that the operators F_t vary in a C^∞ way with $t \in \Delta$. □

Proof of Theorem D.

We will use the following elementary general observation, an application of the Residue Theorem.

Lemma 2.5.21. *Let $C \subset \mathbb{C}$ be a circle centred at the origin in the complex plane. Then, for every point $z \in \mathbb{C} \setminus C$, the following formula holds:*

$$\frac{1}{2\pi i} \int\limits_{\zeta \in C} \frac{1}{\zeta(z - \zeta)}\, d\zeta = \begin{cases} 0, & \textit{if } z \in \textit{int}\,(C) \\ \frac{1}{z}, & \textit{if } z \in \textit{ext}\,(C). \end{cases} \tag{2.108}$$

Proof. Let $\tilde{C} \subset \mathbb{C}$ be a circle centred at the origin of radius strictly greater than the radius of C.

(a) *Case where $z \in$ int(C).* Consider the holomorphic function

$$\text{int}\,(\tilde{C}) \setminus \{0,\, z\} \ni \zeta \overset{g}{\longmapsto} \frac{1}{\zeta(z - \zeta)} \in \mathbb{C}.$$

Both poles z and 0 lie in the interior of C, so the Residue Theorem yields:

$$\frac{1}{2\pi i}\int\limits_{\zeta\in C}\frac{1}{\zeta(z-\zeta)}\,d\zeta = \mathrm{Res}_0\, g + \mathrm{Res}_z\, g = \frac{1}{z} - \frac{1}{z} = 0.$$

(b) *Case where $z \in ext(C)$.* We can choose the circle $\tilde{C} \subset \mathbb{C}$ so close to C that z lies in its exterior (but, of course, we still have $\mathrm{int}\,(C) \subsetneq \mathrm{int}\,(\tilde{C})$.) Consider the holomorphic function

$$\mathrm{int}\,(\tilde{C}) \setminus \{0\} \ni \zeta \stackrel{g}{\longmapsto} \frac{1}{\zeta(z-\zeta)} \in \mathbb{C}.$$

The only pole 0 lies in the interior of C, so the Residue Theorem yields:

$$\frac{1}{2\pi i}\int\limits_{\zeta\in C}\frac{1}{\zeta(z-\zeta)}\,d\zeta = \mathrm{Res}_0\, g = \frac{1}{z}.$$

The proof of Lemma 2.5.21 is complete. □

The main idea in the proof of Theorem D is to derive an operator analogue of the case $z \in \mathrm{ext}\,(C)$ of the elementary formula (2.108). The role of the variable $z \in \mathbb{C}$ will be played by any of the elliptic self-adjoint operators E_t, while the role of $1/z$ will be played by its Green operator $G_t := E_t^{-1}$. Similarly, the role of $1/(z-\zeta)$ will be played by the operator $G_t(\zeta) = E_t(\zeta)^{-1}$, a genuine inverse since recall that $E_t(\zeta) = E_t - \zeta : L(B) \longrightarrow L(B)$ is bijective for all $(t,\, \zeta) \in W$. Since E_t is invertible only on the orthogonal complement of its kernel, the role of the condition $z \in \mathrm{ext}\,(C)$ will be played by the condition $\mathrm{Spec}\, E_t \setminus \{0\} \subset \mathrm{ext}\,(C)$. Thus, we derive the following analogue for the Green operator of the Cauchy formula-type expression obtained in Lemma 2.5.18 for the orthogonal projector $F_t(C)$.

Lemma 2.5.22. *Under the assumptions of Theorem D, fix an arbitrary point $t_0 \in \Delta$. Then, the following formula holds:*

$$G_t\,\psi = \frac{1}{2\pi i}\int\limits_{\zeta\in C}\frac{1}{\zeta}\,G_t(\zeta)\,\psi\,d\zeta, \qquad \psi\in L(B), \quad t\in(t_0-\delta,\, t_0+\delta), \tag{2.109}$$

for any circle C centred at the origin in the complex plane such that

$$Spec\, E_{t_0} \setminus \{0\} \subset ext\,(C)$$

and for every $\delta > 0$ small enough.

Proof. We have seen earlier (cf. Corollary 2.5.2) that, for every $\psi = \sum\limits_{h=1}^{+\infty} a_h\, e_h(t) \in L(B)$, we have

$$G_t\,\psi = \sum_{\lambda_h(t)\neq 0}\frac{a_h}{\lambda_h(t)}\,e_h(t). \tag{2.110}$$

Fix $t_0 \in \Delta$ and let $C = C(0,\, \varepsilon) \subset \mathbb{C}$ be a circle such that $\mathrm{Spec}\, E_{t_0} \setminus \{0\} \subset \mathrm{ext}\,(C)$. Choose $\delta > 0$ so small that $(t_0-\delta,\, t_0+\delta) \times C \subset W$. (Recall that W is *open*.) For every $t \in (t_0-\delta,\, t_0+\delta)$, define the linear operator $G_t(C) : L(B) \longrightarrow L(B)$ by the right-hand side of (2.109), namely

$$G_t(C)\,\psi := \frac{1}{2\pi i}\int\limits_{\zeta\in C}\frac{1}{\zeta}\,G_t(\zeta)\,\psi\,d\zeta. \tag{2.111}$$

We will prove that $G_t = G_t(C)$ for all $t \in (t_0-\delta,\, t_0+\delta)$ by working out the expression of $G_t(C)\,\psi$ in terms of the Fourier series expansion of ψ and observing in the end that it coincides with the analogous expression (2.110) for $G_t\psi$.

We have

$$G_t(\zeta)\,\psi = G_t(\zeta)\left(\sum_{h=1}^{+\infty} a_h\, e_h(t)\right) = \sum_{h=1}^{+\infty} \frac{a_h}{\lambda_h(t)-\zeta}\, e_h(t).$$

Hence,

$$G_t(C)\,\psi \;=\; \frac{1}{2\pi i}\int\limits_{\zeta\in C}\sum_{h=1}^{+\infty}\frac{a_h}{\zeta\,(\lambda_h(t)-\zeta)}\,e_h(t)\,d\zeta = \sum_{h=1}^{+\infty}\left(\frac{1}{2\pi i}\int\limits_{\zeta\in C}\frac{1}{\zeta\,(\lambda_h(t)-\zeta)}\,d\zeta\right) a_h\,e_h(t).$$

Now, by the elementary Lemma 2.5.21 applied with $z=\lambda_h(t)$, the paranthesis coefficient of $a_h\,e_h(t)$ in the last sum above equals 0 if $\lambda_h(t)\in \mathrm{int}\,(C)$, while it equals $1/\lambda_h(t)$ if $\lambda_h(t)\in \mathrm{ext}\,(C)$. (Note that Lemma 2.5.21 is applicable in our case because $\mathrm{Spec}\,E_t \cap C = \emptyset$ for every $t\in(t_0-\delta,\, t_0+\delta)$, by our choice of δ.) Therefore, we get:

$$G_t(C)\,\psi = \sum_{\lambda_h(t)\in \mathrm{ext}\,(C)} \frac{a_h}{\lambda_h(t)}\, e_h(t).$$

It is now time to use the hypothesis: $\dim\ker E_{t_0} = \dim\ker E_t$ for all $t\in\Delta$, of Theorem D. As in the proof of Theorem C, it implies that $\ker E_t = \mathbb{F}_t(C)$ for all $t\in(t_0-\delta,\, t_0+\delta)$. (Shrink the previous $\delta>0$ if necessary.) This means that, for every $t\in(t_0-\delta,\, t_0+\delta)$, 0 is the only eigenvalue of E_t lying in the interior of C. In other words, for every such t and for every $h\in\mathbb{N}^\star$, we have the equivalence:

$$\lambda_h(t)\neq 0 \iff \lambda_h(t)\in \mathrm{ext}\,(C).$$

(Recall that $\mathrm{Spec}\,E_t\cap C=\emptyset$ for every $t\in(t_0-\delta,\, t_0+\delta)$.)

Therefore, we finally get

$$G_t(C)\,\psi = \sum_{\lambda_h(t)\neq 0} \frac{a_h}{\lambda_h(t)}\, e_h(t)$$

for every $t\in(t_0-\delta,\, t_0+\delta)$ and every $\psi = \sum\limits_{h=1}^{+\infty} a_h\,e_h(t)\in L(B)$, which is precisely formula (2.110) that we had for $G_t\psi$. $\square$

End of proof of Theorem D. We saw in Conclusion 2.5.17 that $G_t(\zeta)$ varies in a C^∞ way with $(t,\,\zeta)\in W$. This implies, thanks to (2.109), that G_t varies in a C^∞ way with $t\in(t_0-\delta,\, t_0+\delta)$.

Since $t_0\in\Delta$ was chosen arbitrarily, we conclude that the operators G_t vary in a C^∞ way with $t\in\Delta$. $\square$

2.6 Deformation openness results

In this section, we apply the results of section 2.5 to derive information about the behaviour of three properties of compact complex manifolds studied in chapter 1 (Kählerianity, the $\partial\bar\partial$-property and the Frölicher degeneration at E_1) under holomorphic deformations of the complex structure.

The main sources for this subsection are again [KS60] and [Kod86]. We will mention the extra ones as they come along.

The scene switches back to a holomorphic family $\pi : \mathcal{X} \longrightarrow B$ of compact complex manifolds $X_t := \pi^{-1}(t)$, with $\dim_{\mathbb{C}} X_t = n$ for all $t \in B$. It will be sufficient to restrict attention to the case where the base B is an open disc about the origin in $\mathbb{C}$, i.e. $m = 1$. We will denote by X the C^∞ manifold that underlies all the fibres X_t. So, there are C^∞ *diffeomorphisms:*

$$X_t \simeq X, \qquad t \in B.$$

(See Ehresmann's Theorem 2.1.1.) We still denote by J_t the complex structure of X_t, for every $t \in B$. When a Hermitian metric ω_t has been fixed on X_t, for every $t \in B$, such that the family $(\omega_t)_{t\in B}$ varies in a C^∞ way with $t \in$, the abstract self-adjoint elliptic operators E_t considered in section 2.5 will often be the $\bar\partial_t$-, the Bott-Chern or the Aeppli Laplacians Δ''_t, $\Delta_{BC,\,t}$, $\Delta_{A,\,t}$ of the fibres $(X_t,\,\omega_t)$.

Two points of view will be adopted. The following terminology was used in [Pop14].

Definition 2.6.1. (i) *A given property* (P) *of a compact complex manifold is said to be* **open** *under holomorphic deformations if for every holomorphic family of compact complex manifolds* $(X_t)_{t\in B}$ *and for every* $t_0 \in B$*, the following implication holds:*

$$X_{t_0} \text{ has property } (P) \implies X_t \text{ has property } (P) \text{ for all } t \in B \text{ sufficiently close to } t_0.$$

(ii) *A given property* (P) *of a compact complex manifold is said to be* **closed** *under holomorphic deformations if for every holomorphic family of compact complex manifolds* $(X_t)_{t\in B}$ *and for every* $t_0 \in B$*, the following implication holds:*

$$X_t \text{ has property } (P) \text{ for all } t \in B\setminus\{t_0\} \implies X_{t_0} \text{ has property } (P).$$

It is obvious that if a property (P) is both open and closed, then all the fibres of a family whose base B is connected satisfy (P) whenever one of them satisfies (P).

In this section, we will deal with *deformation openness properties.* Deformation closedness properties will be investigated in chapter 7.

2.6.1 Deformation behaviour of the cohomology dimensions

Recall that, unlike the De Rham cohomology, which is the same for all the fibres:

$$H^k_{DR}(X_t,\,\mathbb{C}) = H^k_{DR}(X,\,\mathbb{C}), \qquad t \in B,$$

the Dolbeault, Bott-Chern and Aeppli cohomologies of every X_t depend on J_t. So, our first task will be to probe these dependencies.

Theorem 2.6.2. *Let* $\pi : \mathcal{X} \longrightarrow B$ *be a holomorphic family of compact complex manifolds* $X_t := \pi^{-1}(t)$*, with* $\dim_{\mathbb{C}} X_t = n$ *for all* $t \in B$*. Fix an arbitrary bidegree* $(p,\,q)$*.*

(i) The functions: $B \ni t \longmapsto h^{p,\,q}(t) := \dim_{\mathbb{C}} H^{p,\,q}_{\bar\partial}(X_t,\,\mathbb{C})$,

$$B \ni t \longmapsto h^{p,\,q}_{BC}(t) := \dim_{\mathbb{C}} H^{p,\,q}_{BC}(X_t,\,\mathbb{C}),$$

$$B \ni t \longmapsto h^{p,\,q}_{A}(t) := \dim_{\mathbb{C}} H^{p,\,q}_{A}(X_t,\,\mathbb{C}),$$

are **upper-semicontinuous***.*

(ii) If the Hodge number $h^{p,\,q}(t)$ *is* **independent of** $t \in B$*, then the map*

$$B \ni t \longmapsto H^{p,\,q}_{\bar\partial}(X_t,\,\mathbb{C})$$

defines a C^∞ **vector bundle** *on B.*

The analogous statement holds for $h^{p,\,q}_{BC}(t)$ and $h^{p,\,q}_{A}(t)$.

Proof. Let $(\omega_t)_{t\in B}$ be a C^∞ family of Hermitian metrics on the respective fibres $(X_t)_{t\in B}$. By Corollaries 1.1.7, 1.1.10 and 1.1.13, we have Hodge isomorphisms:

$$H^{p,\,q}_{\bar\partial}(X_t,\,\mathbb{C})\simeq\mathcal{H}^{p,\,q}_{\Delta''}(X_t,\,\mathbb{C}),$$

$$H^{p,\,q}_{BC}(X_t,\,\mathbb{C})\simeq\mathcal{H}^{p,\,q}_{\Delta_{BC}}(X_t,\,\mathbb{C}),$$

$$H^{p,\,q}_{A}(X_t,\,\mathbb{C})\simeq\mathcal{H}^{p,\,q}_{\Delta_{A}}(X_t,\,\mathbb{C}),$$

where $\mathcal{H}^{p,\,q}_{\Delta''}(X_t,\,\mathbb{C}):=\ker(\Delta''_t:C^\infty_{p,\,q}(X_t,\,\mathbb{C})\to C^\infty_{p,\,q}(X_t,\,\mathbb{C}))$, $\mathcal{H}^{p,\,q}_{\Delta_{BC}}(X_t,\,\mathbb{C})=\ker(\Delta_{BC,\,t}:C^\infty_{p,\,q}(X_t,\,\mathbb{C})\to C^\infty_{p,\,q}(X_t,\,\mathbb{C}))$ and $\mathcal{H}^{p,\,q}_{\Delta_{A}}(X_t,\,\mathbb{C})=\ker(\Delta_{A,\,t}:C^\infty_{p,\,q}(X_t,\,\mathbb{C})\to C^\infty_{p,\,q}(X_t,\,\mathbb{C}))$.

Since $(\Delta''_t)_{t\in B}$, $(\Delta_{BC,\,t})_{t\in B}$ and $(\Delta_{A,\,t})_{t\in B}$ are C^∞ families of *elliptic, self-adjoint* differential operators of *even* orders, Theorem B of section 2.5 implies contention (i).

To prove (ii), consider the $L^2_{\omega_t}$*-orthogonal projections*:

$$\pi^{p,\,q}_t:C^\infty_{p,\,q}(X_t,\,\mathbb{C})\longrightarrow\mathcal{H}^{p,\,q}_{\Delta''}(X_t,\,\mathbb{C}),$$

$$\pi^{p,\,q}_{BC,\,t}:C^\infty_{p,\,q}(X_t,\,\mathbb{C})\longrightarrow\mathcal{H}^{p,\,q}_{\Delta_{BC}}(X_t,\,\mathbb{C}),$$

$$\pi^{p,\,q}_{A,\,t}:C^\infty_{p,\,q}(X_t,\,\mathbb{C})\longrightarrow\mathcal{H}^{p,\,q}_{\Delta_{A}}(X_t,\,\mathbb{C}).$$

By Theorem C of section 2.5, each of these orthogonal projectors depends in a C^∞ way on $t\in B$ under the corresponding dimension invariance hypothesis. $\square$

The first main consequence of the upper-semicontinuity of the Hodge numbers under deformations is the **deformation openness** of the **Frölicher degeneration property at** E_1.

Theorem 2.6.3. *Let $\pi:\mathcal{X}\longrightarrow B$ be a holomorphic family of compact complex manifolds $X_t:=\pi^{-1}(t)$, with $t\in B$. Fix an arbitrary reference point $0\in B$.*

If the **Frölicher spectral sequence of X_0 degenerates at** *E_1, then, for all $t\in B$ sufficiently close to 0, we have:*

(a) the **Frölicher spectral sequence of X_t degenerates at** *E_1;*

(b) $h^{p,\,q}(t)=h^{p,\,q}(0)$ for every bidegree $(p,\,q)$.

Proof. By Corollary 1.2.6, the hypothesis $E_1(X_0)=E_\infty(X_0)$ is equivalent to the numerical identities:

$$b_k=\sum_{p+q=k}h^{p,\,q}(0),\qquad k\in\{0,1,\dots,2n\},\tag{2.112}$$

where $b_k:=\dim_{\mathbb{C}}H^k_{DR}(X,\,\mathbb{C})$ is the k-th Betti number of the fibres. For every $t\in B$ sufficiently close to 0, we get:

$$b_k\overset{(i)}{\leq}\sum_{p+q=k}h^{p,\,q}(t)\overset{(ii)}{\leq}\sum_{p+q=k}h^{p,\,q}(0)\overset{(iii)}{=}b_k,\tag{2.113}$$

where (i) is the dimension inequality (1.18) that is valid on any manifold, (ii) is the upper-semicontinuity property of Theorem 2.6.2, while (iii) is (2.112).

Thus, inequalities (i) and (ii) must be equalities for every $t\in B$ sufficiently close to 0. Now, (i) being an equality for every degree k is equivalent to $E_1(X_t)=E_\infty(X_t)$, by Corollary 1.2.6, while (ii) being an equality for every degree k is equivalent to $h^{p,\,q}(t)=h^{p,\,q}(0)$ for every bidegree $(p,\,q)$. □

The second main consequence of the upper-semicontinuity of the Hodge numbers under deformations is the **deformation openness** of the $\partial\bar\partial$**-property** of compact complex manifolds. This fact was first proved by Wu in [Wu06] and was reproved by Angella and Tomassini in [AT13] as a consequence of their *numerical characterisation* of $\partial\bar\partial$-manifolds (see Theorem 1.3.10) and of the *upper-semicontinuity* of the Bott-Chern and Aeppli cohomology dimensions. We present below the proof in [AT13].

Theorem 2.6.4. *Let $\pi:\mathcal{X}\longrightarrow B$ be a holomorphic family of compact complex manifolds $X_t:=\pi^{-1}(t)$, with $t\in B$. Fix an arbitrary reference point $0\in B$.*

*If the fibre X_0 is a $\partial\bar\partial$-**manifold**, then, for all $t\in B$ sufficiently close to 0, we have:*

*(a) the fibre X_t is a $\partial\bar\partial$-**manifold**;*

(b) $h^{p,\,q}_{BC}(t)=h^{p,\,q}_{BC}(0)$ and $h^{p,\,q}_{A}(t)=h^{p,\,q}_{A}(0)$ for every bidegree $(p,\,q)$.

Proof. By Theorem 1.3.10, the $\partial\bar\partial$ assumption on X_0 is equivalent to the identities:

$$\sum_{p+q=k}(h^{p,\,q}_{BC}(0)+h^{p,\,q}_{A}(0))=2b_k,\qquad k\in\{0,1,\dots,2n\}.$$

Meanwhile, the upper-semicontinuity properties of Theorem 2.6.2 yield:

$$h^{p,\,q}_{BC}(0)\geq h^{p,\,q}_{BC}(t)\quad\text{and}\quad h^{p,\,q}_{A}(0)\geq h^{p,\,q}_{A}(t)$$

for all bidegrees $(p,\,q)$ and all $t\in B$ sufficiently close to 0. Finally, by (1.40), we always have the inequalities:

$$\sum_{p+q=k}(h^{p,\,q}_{BC}(t)+h^{p,\,q}_{A}(t))\geq 2b_k,\qquad t\in B,\quad k\in\{0,1,\dots,2n\}.$$

Putting together all these pieces of information, we get:

$$2b_k\overset{(i)}{\leq}\sum_{p+q=k}(h^{p,\,q}_{BC}(t)+h^{p,\,q}_{A}(t))\overset{(ii)}{\leq}\sum_{p+q=k}(h^{p,\,q}_{BC}(0)+h^{p,\,q}_{A}(0))=2b_k,\qquad k\in\{0,1,\dots,2n\},$$

for all $t\in B$ sufficiently close to 0. Hence, both of the above inequalities must be equalities.

In particular, inequalities (i) being equalities for all $k\in\{0,1,\dots,2n\}$ and all $t\in B$ sufficiently close to 0 amounts to X_t being a $\partial\bar\partial$-manifold for all $t\in B$ sufficiently close to 0, thanks again to Theorem 1.3.10. This proves (a).

Meanwhile, inequalities (ii) being equalities for all bidegrees $(p,\,q)$ and all $t\in B$ sufficiently close to 0 proves (b). □

2.6.2 Deformation openness of the Kähler property

We are now ready to present the Kodaira-Spencer theorem (see [KS60, Theorem 15]) saying that if a fibre X_0 in a holomorphic family of compact complex manifolds is **Kähler**, then all the nearby fibres X_t are also **Kähler**. Moreover, any Kähler metric ω_0 on X can be deformed to a C^∞ family of Kähler metrics ω_t on the nearby fibres X_t. Historically, this important result kick-started the theory of deformation openness and closedness of various properties of compact complex manifolds.

Let us start with a very simple but crucial observation.

Lemma 2.6.5. *Let ω be a Hermitian metric on a compact complex manifold X. The equivalence holds:*

$$\omega \text{ is Kähler} \iff \Delta_{BC}\omega = 0,$$

where $\Delta_{BC} : C^\infty_{1,1}(X, \mathbb{C}) \longrightarrow C^\infty_{1,1}(X, \mathbb{C})$ is the Bott-Chern Laplacian induced by ω.

Proof. We know from Corollary 1.1.10 that $\ker\Delta_{BC} = \ker\partial \cap \ker\bar\partial \cap \ker(\partial\bar\partial)^\star$. So, one implication of the above equivalence is obvious: if $\Delta_{BC}\omega = 0$, then $\partial\omega = 0$, which means that ω is Kähler.

Suppose now that ω is Kähler, namely $d\omega = 0$. This implies $\partial\omega = 0$ and $\bar\partial\omega = 0$. To prove that $(\partial\bar\partial)^\star\omega = 0$, we will use the standard formulae (see (1.12)):

$$\star\star = (-1)^k \,\mathrm{Id} \quad \text{on k-forms}; \qquad \partial^\star = -\star\bar\partial\star, \qquad \bar\partial^\star = -\star\partial\star$$

and the standard formula

$$\star\omega = \frac{\omega^{n-1}}{(n-1)!}, \tag{2.114}$$

where $\star = \star_\omega$ is the Hodge star operator induced by the Hermitain metric ω (see (1.12)).

We get the equivalences:

$$(\partial\bar\partial)^\star\omega = 0 \iff \star\partial\bar\partial(\star\omega) = 0 \iff \partial\bar\partial\frac{\omega^{n-1}}{(n-1)!} = 0,$$

where the second one uses the fact that $\star$ is an isomorphism. Now, the last identity holds since $\bar\partial\omega^{n-1} = (n-1)\,\omega^{n-2}\wedge\bar\partial\omega = 0$. Indeed, $\bar\partial\omega = 0$ by the Kähler assumption on ω. $\square$

We can now state and prove the main result of this subsection.

Theorem 2.6.6. *Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds $X_t := \pi^{-1}(t)$, with $t \in B$. Fix an arbitrary reference point $0 \in B$.*

(a) If the fibre X_0 is a **Kähler manifold***, then the fibre X_t is a* **Kähler manifold** *for all $t \in B$ sufficiently close to 0.*

(b) Moreover, given any **Kähler metric** *ω_0 on X_0, there exists a small neighbourhood U of 0 in B and a C^∞ family $(\omega_t)_{t\in U}$ of* **Kähler metrics** *on the respective fibres X_t whose member for $t = 0$ is ω_0.*

Proof. Since (b) implies (a), we will prove (b). Let ω_0 be a Kähler metric on X_0. In particular, ω_0 is a smooth J_0-type $(1, 1)$-form on X_0, hence a smooth 2-form on X (the C^∞ manifold underlying the fibres X_t for $t \in B$ close to 0.) For every $t \in B$, let ω_t be the J_t-type $(1, 1)$-component of the 2-form ω_0. Clearly, the member for $t = 0$ of the family of forms $(\omega_t)_{t\in B}$ is ω_0. Moreover, the ω_t's vary in a C^∞ way with t because they are the J_t-type $(1,\,1)$-components of a fixed 2-form and the J_t's depend in a (at least) C^∞ way on t.

Now, ω_0 is *positive definite* because it is a metric on X_0. By continuity w.r.t. t, ω_t remains *positive definite* for all $t \in U$ if the neighbourhood U of 0 in B is small enough. Hence, ω_t is a Hermitian metric on X_t for every $t \in U$, so $(\omega_t)_{t\in U}$ is a C^∞ family of Hermitian metrics on the respective fibres X_t, whose member for $t = 0$ is the original Kähler metric ω_0.

We have to change the metrics ω_t with $t \in U \setminus \{0\}$ to make them Kähler. Lemma 2.6.5 tells us that this amounts to making the ω_t's *Bott-Chern harmonic* w.r.t. themselves (i.e. for the Bott-Chern Laplacians induced by the ω_t's).

Let us therefore consider the *$L^2_{\omega_t}$-orthogonal projectors:*

$$F_t : C^\infty_{1,\,1}(X_t,\,\mathbb{C}) \longrightarrow \mathcal{H}^{1,\,1}_{\Delta_{BC}}(X_t,\,\mathbb{C}), \qquad t \in U,$$

onto the kernels of the Bott-Chern Laplacians $\Delta_{BC,\,t}$ induced by the ω_t's in J_t-bidegree $(1,\,1)$.

The crucial piece of information that we need at this point is provided by conclusion (b) of Theorem 2.6.4. Since X_0 is a $\partial\bar\partial$-manifold (because it is even Kähler, by hypothesis), the dimension $h^{1,\,1}_{BC}(t)$ of $\mathcal{H}^{1,\,1}_{\Delta_{BC}}(X_t,\,\mathbb{C})$ (= the dimension of $H^{1,\,1}_{BC}(X_t,\,\mathbb{C})$, thanks to the Hodge isomorphism) is *independent of* $t \in U$ if the neighbourhood U of 0 in B is *small enough*. Therefore, by Theorem C of section 2.5, *F_t varies in a C^∞ way with $t \in U$.*

Now, put

$$\widetilde{\omega}_t := \frac{1}{2}\,(F_t\omega_t + \overline{F_t\omega_t}), \qquad t \in U.$$

The J_t-type $(1,\,1)$-forms $\widetilde{\omega}_t$ have the following properties:

(i) $\widetilde{\omega}_t$ is a *real* form (i.e. it equals its conjugate) for every $t \in U$;

(ii) $\widetilde{\omega}_t$ varies in a C^∞ way with $t \in U$, because F_t and ω_t do;

(iii) $\widetilde{\omega}_0 = \omega_0$ because $F_0\omega_0 = \omega_0$ (recall that ω_0 is *Kähler* on X_0 and Lemma 2.6.5 applies) and ω_0 is real;

(iv) $\widetilde{\omega}_t$ is *positive definite* on X_t for all $t \in U$ (shrink U about 0 if necessary), because $\widetilde{\omega}_0$ is and $\widetilde{\omega}_t$ varies (at least) continuously with $t \in U$;

(v) $\widetilde{\omega}_t \in \ker\partial_t$ for all $t \in U$, because $F_t\omega_t \in \mathcal{H}^{1,\,1}_{\Delta_{BC}}(X_t,\,\mathbb{C}) = \ker\partial_t \cap \ker\bar\partial_t \cap \ker(\partial_t\bar\partial_t)^\star \subset \ker\partial_t \cap \ker\bar\partial_t$.

(Note that the Bott-Chern harmonic space $\mathcal{H}^{1,\,1}_{\Delta_{BC}}(X_t,\,\mathbb{C})$ in (v) is defined by the Hermitian metric ω_t, rather than $\widetilde{\omega}_t$.)

Properties (i)-(v) amount to saying that $(\widetilde{\omega}_t)_{t\in U}$ is a C^∞ family of *Kähler metrics* on the respective fibres X_t, whose member for $t = 0$ is the originally given Kähler metric ω_0 on X_0. □

2.6.3 Non-deformation openness of the *class $\mathcal{C}$* property

In this short subsection, we point out, without going into the details of the proofs, a property of compact complex manifolds that is not open under deformations of the complex structure. We take this opportunity to introduce two other well-known classes of compact complex manifolds.

A (Fujiki) *class $\mathcal{C}$* manifold is a compact complex manifold that is bimeromorphic to a compact Kähler manifold.

Definition 2.6.7. *A compact complex manifold X is said to be a* **(Fujiki)** *class* $\mathcal{C}$ **manifold** *if there exists a proper holomorphic bimeromorphic map (called a* **modification***)*

$$\mu : \widetilde{X} \to X$$

from a compact **Kähler** *manifold $\widetilde{X}$.*

Fujiki introduced *class* $\mathcal{C}$ manifolds X as meromorphic images of compact Kähler manifolds in [Fuj78], while Varouchas gave them the above nice characterisation in [Var86]. It is a result of Demailly and Paun that *class* $\mathcal{C}$ manifolds are characterised by the existence of a Kähler current.

Theorem 2.6.8. *(Demailly-Paun [DP04]) A compact complex manifold X is of class $\mathcal{C}$ if and only if there exists a Kähler current T on X.*

Meanwhile, a *Moishezon manifold* is a compact complex manifold that is bimeromorphic to a projective manifold.

Definition 2.6.9. *A compact complex manifold X is said to be a* **Moishezon manifold** *if there exists a proper holomorphic bimeromorphic map*

$$\mu : \widetilde{X} \to X$$

from a **projective** *manifold $\widetilde{X}$.*

Thus Moishezon manifolds are to projective manifolds what *class* $\mathcal{C}$ manifolds are to compact Kähler manifolds.

The special case of integral cohomology classes is relevant in characterisations of some of the above classes of manifolds. Recall that the De Rham cohomology 2-class $\{\omega\} \in H^2_{DR}(X, \mathbb{R})$ (resp. $\{T\} \in H^2_{DR}(X, \mathbb{R})$) defined by a C^∞ d-closed real $(1, 1)$-form ω (resp. by a d-closed real $(1, 1)$-current T) is said to be *integral* if it is the first Chern class of a holomorphic line bundle $L \to X$ or, equivalently, if ω (resp. T) is the curvature form (resp. curvature current) $\frac{i}{\pi}\Theta_h(L)$ of a holomorphic line bundle $(L, h) \to X$ endowed with a C^∞ (resp. singular) Hermitian fibre metric h.

There are neat characterisations of projective and Moishezon manifolds mirroring the general case of arbitrary (i.e. possibly transcendental) classes that occur on Kähler and *class* $\mathcal{C}$ manifolds.

Theorem 2.6.10. *(Kodaira's Embedding Theorem) A compact complex manifold X is projective if and only if there exists a Kähler metric ω on X whose De Rham cohomology class $\{\omega\} \in H^2_{DR}(X, \mathbb{R})$ is integral.*

Thus projective manifolds are *integral class* special cases of compact Kähler manifolds. Likewise, Moishezon manifolds are *integral class* special cases of *class* $\mathcal{C}$ manifolds as the following characterisation shows.

Theorem 2.6.11. *(Ji-Shiffman [JS93]) A compact complex manifold X is Moishezon if and only if there exists a Kähler current T on X whose De Rham cohomology class $\{T\} \in H^2_{DR}(X, \mathbb{R})$ is integral.*

Finally, let us mention the following

Theorem 2.6.12. *Every* **class $\mathcal{C}$ manifold** *is a* **$\partial\bar\partial$-manifold**.

Proof. Let X be *class* $\mathcal{C}$ manifold. There exists a composition $\mu : \widetilde{X} \longrightarrow X$ of finitely many blow-ups with smooth centres such that $\widetilde{X}$ is compact Kähler. Then, by Theorem 1.3.20, $\widetilde{X}$ is a $\partial\bar\partial$-manifold, so its contraction X is again a $\partial\bar\partial$-manifold by (4) of Theorem 3.3.33 proved in chapter 3. $\square$

The relations among these properties of a compact complex manifold X are summed up in the following diagram (skew arrows indicate implications):

$(\star)$

$$\begin{tikzcd}
 & X \text{ Kähler} \arrow[dr, Rightarrow] & & & \\
X \text{ projective} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & X \textit{ class } \mathcal{C} \arrow[r, Rightarrow] & \begin{array}{l} X \text{ is a} \\ \partial\bar\partial\text{-manifold} \end{array} \arrow[r, Rightarrow] & E_1(X) = E_\infty(X). \\
 & X \text{ Moishezon} \arrow[ur, Rightarrow] & & &
\end{tikzcd}$$

This introduction was a prelude to the following result of Campana's (see [Cam91a] and also Lebrun-Poon [LP92]).

Theorem 2.6.13. *The* **class** $\mathcal{C}$ **property** *of compact complex manifolds is* **not open** *under holomorphic deformations.*

Sketch of proof. In [Cam91a] and [LP92], families of twistor spaces $(X_t)_{t\in B}$ are pointed out in which the central fibre X_0 is *Moishezon*. In particular, X_0 is also *class* $\mathcal{C}$. The fibres are *twistor spaces* of a special kind constructed by Lebrun in [Leb91]. As with all twistor spaces, $\dim_{\mathbb{C}} X_t = 3$ for all $t \in B$.

Now, it is a standard fact that the Moishezon property is characterised by the maximality of the *algebraic dimension* $a(X)$ of the manifold. In fact, $a(X) \leq \dim_{\mathbb{C}} X$ for any compact complex manifold X and X is Moishezon if and only if $a(X) = \dim_{\mathbb{C}} X$.

Thus, in the examples alluded to above, $a(X_0) = 3$. Meanwhile, the nearby fibres X_t can be chosen to be decidedly non-Moishezon, namely $a(X_t) = 0$ for $t \neq 0$ close to 0.

On the other hand, a result of Campana [Cam91b] says that the Moishezon and *class* $\mathcal{C}$ properties are equivalent for twistor spaces. So, X_t is not a *class* $\mathcal{C}$ manifold for any $t \neq 0$ close to 0. $\square$

Note that the above proof also shows the following

Corollary 2.6.14. *The* **Moishezon property** *of compact complex manifolds is* **not open** *under holomorphic deformations.*

This last fact is hardly surprising since a property tied up with integral classes is not naturally expected to be deformation open.

A by-product of the Campana-Lebrun-Poon proof sketched above and of the *deformation openness* of the $\partial\bar\partial$-property seen in Theorem 2.6.4, we get what was probably the first example of a $\partial\bar\partial$-manifold that is not of *class* $\mathcal{C}$, observed in [Pop14, Observation 4.10].

Observation 2.6.15. *There exist twistor spaces that are* $\partial\bar\partial$**-manifolds***, but are* **not** *of* **class** $\mathcal{C}$*.*

Proof. We place ourselves in the setting of the Campana-Lebrun-Poon Theorem 2.6.13 and its proof. Being Moishezon, the central fibre X_0 is a $\partial\bar\partial$-manifold. Therefore, by Theorem 2.6.4, the nearby fibres X_t are still $\partial\bar\partial$-manifolds for all $t \in B$ sufficiently close to 0. However, no X_t with $t \neq 0$ close to 0 is a *class* $\mathcal{C}$ manifold, by the proof of Theorem 2.6.13.

Thus, every X_t with $t \neq 0$ close to 0 is an example of a $\partial\bar\partial$-manifold that is not of *class* $\mathcal{C}$. □

Chapter 3

Higher-Page Hodge Theory of Compact Complex Manifolds

The phrase "higher-page" refers to the pages E_r with $r \geq 2$ of the Frölicher spectral sequence (FSS) of a compact complex manifold. The main thrust of this chapter is to extend some basic results in Hodge Theory presented in chapters 1 and 2 to these higher pages of the FSS.

3.1 Pseudo-differential Laplacian and Hodge isomorphism for the second page of the FSS

The material in this section is taken from [Pop16]. Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Fix an arbitrary Hermitian metric ω on X. As usual, consider the formal adjoints $\partial^\star, \bar\partial^\star$ of ∂, resp. $\bar\partial$ w.r.t. the L^2 inner product defined by ω and the usual Laplace-Beltrami operators $\Delta', \Delta'' : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C})$ defined as $\Delta' = \partial\partial^\star + \partial^\star\partial$ and $\Delta'' = \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial$. We saw in Theorem 1.1.6 that they are elliptic, self-adjoint and non-negative differential operators of order 2 that induce 3-space L^2_ω-orthogonal decompositions:

$$C^\infty_{p,\,q}(X,\,\mathbb{C}) = \ker\Delta' \oplus \operatorname{Im}\partial \oplus \operatorname{Im}\partial^\star \quad \text{and} \quad C^\infty_{p,\,q}(X,\,\mathbb{C}) = \ker\Delta'' \oplus \operatorname{Im}\bar\partial \oplus \operatorname{Im}\bar\partial^\star \tag{3.1}$$

where the harmonic spaces $\ker\Delta' := \mathcal{H}^{p,\,q}_{\Delta'}(X,\,\mathbb{C})$, $\ker\Delta'' := \mathcal{H}^{p,\,q}_{\Delta''}(X,\,\mathbb{C})$ are *finite dimensional*, while

$$\ker\partial = \ker\Delta' \oplus \operatorname{Im}\partial \quad \text{and} \quad \ker\bar\partial = \ker\Delta'' \oplus \operatorname{Im}\bar\partial. \tag{3.2}$$

We denote by

$$p' = p'_{p,\,q} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow \ker\Delta' \quad \text{and} \quad p'' = p''_{p,\,q} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow \ker\Delta'' \tag{3.3}$$

the *orthogonal projections* defined by the orthogonal splittings (3.1) onto the Δ'-harmonic, resp. the Δ''-harmonic spaces in bidegree $(p,\,q)$. Similarly, let

$$p'_\perp : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow \operatorname{Im}\Delta' = \operatorname{Im}\partial \oplus \operatorname{Im}\partial^\star \quad \text{and} \quad p''_\perp : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow \operatorname{Im}\Delta'' = \operatorname{Im}\bar\partial \oplus \operatorname{Im}\bar\partial^\star \tag{3.4}$$

denote the orthogonal projections onto $(\ker\Delta')^\perp = \operatorname{Im}\Delta'$, resp. onto $(\ker\Delta'')^\perp = \operatorname{Im}\Delta''$. Note that the operators $p', p'', p'_\perp, p''_\perp$ are not differential operators and depend on the metric ω. They clearly satisfy the properties:

$$p' = (p')^\star = (p')^2, \quad p'' = (p'')^\star = (p'')^2, \quad p'_\perp = (p'_\perp)^\star = (p'_\perp)^2, \quad p''_\perp = (p''_\perp)^\star = (p''_\perp)^2. \tag{3.5}$$

We start by giving a metric interpretation of the spaces $E_2^{p,\,q}(X)$ on the second page of the FSS of X.

Proposition 3.1.1. *For every p, $q = 0, 1, \dots, n$, define the ω-dependent $\mathbb{C}$-vector space*

$$\widetilde{H}^{p,\,q}(X,\,\mathbb{C}) := \ker(p'' \circ \partial) \cap \ker\bar\partial \Big/ \Big(Im\,\bar\partial + Im\,(\partial_{|\ker\bar\partial})\Big) \tag{3.6}$$

in which all the kernels and images involved are understood as subspaces of $C^\infty_{p,\,q}(X,\,\mathbb{C})$. For every C^∞ $(p,\,q)$-form $\alpha \in \ker(p''\circ\partial)\cap\ker\bar\partial$, let $\widetilde{[\alpha]} \in \widetilde{H}^{p,\,q}(X,\,\mathbb{C})$ denote the class of α modulo $Im\,\bar\partial + Im\,(\partial_{|\ker\bar\partial})$. Then, for every p, q, the following linear map:

$$T = T^{p,\,q} : \widetilde{H}^{p,\,q}(X,\,\mathbb{C}) \longrightarrow E_2^{p,\,q}(X), \qquad \widetilde{[\alpha]} \longmapsto \{\alpha\}_{E_2}, \tag{3.7}$$

is **well defined** *and an* **isomorphism***.*

Proof. First note that the inclusion $\operatorname{Im}\bar\partial + \operatorname{Im}(\partial_{|\ker\bar\partial}) \subset \ker(p''\circ\partial)\cap\ker\bar\partial$ does hold, so the space $\widetilde{H}^{p,\,q}(X,\,\mathbb{C})$ is meaningful. Indeed, $\operatorname{Im}\bar\partial \subset \ker\bar\partial$ trivially and $\operatorname{Im}\bar\partial \subset \ker(p''\circ\partial)$ because for every form u, $p''(\partial\bar\partial u) = -p''\bar\partial\partial u = 0$ since $\operatorname{Im}\bar\partial$ is orthogonal onto $\ker\Delta''$ (see (3.1)), so $p''\bar\partial = 0$. Thus $\operatorname{Im}\bar\partial \subset \ker(p''\circ\partial)\cap\ker\bar\partial$. Moreover, $\operatorname{Im}(\partial_{|\ker\bar\partial}) \subset \ker(p''\circ\partial)$ because $\partial^2 = 0$ and $\operatorname{Im}(\partial_{|\ker\bar\partial}) \subset \ker\bar\partial$ because for any form $v \in \ker\bar\partial$, we have $\bar\partial(\partial v) = -\partial(\bar\partial v) = 0$. Thus $\operatorname{Im}(\partial_{|\ker\bar\partial}) \subset \ker(p''\circ\partial)\cap\ker\bar\partial$.

Then note that for any $\widetilde{[\alpha]} \in \widetilde{H}^{p,\,q}(X,\,\mathbb{C})$, we do have $\{\alpha\}_{\bar\partial} \in \ker d_1$, so the d_1-class $[[\alpha]_{\bar\partial}]_{d_1} = \{\alpha\}_{E_2}$ is a meaningful element of $E_2^{p,\,q}(X)$. Indeed, $d_1(\{\alpha\}_{\bar\partial}) = \{\partial\alpha\}_{\bar\partial}$, $\partial\alpha \in \ker\bar\partial = \ker\Delta'' \oplus \operatorname{Im}\bar\partial$ (because $\alpha \in \ker\bar\partial$ and (3.2) holds) and $p''(\partial\alpha) = 0$ (because $\alpha \in \ker(p''\circ\partial)$). The last two relations amount to $\partial\alpha \in \operatorname{Im}\bar\partial$. This is equivalent to $\{\partial\alpha\}_{\bar\partial} = 0$, i.e. to $d_1(\{\alpha\}_{\bar\partial}) = 0$.

To complete the proof of the well-definedness of T, it remains to show that $\{\alpha\}_{E_2}$ does not depend on the choice of representative α of the class $\widetilde{[\alpha]}$, i.e. that the zero element of $\widetilde{H}^{p,\,q}(X,\,\mathbb{C})$ is mapped by T to the zero element of $E_2^{p,\,q}(X)$. To prove this, let $\alpha \in \ker(p''\circ\partial)\cap\ker\bar\partial$ be a $(p,\,q)$-form such that $\alpha = \bar\partial u + \partial v$ with $v \in \ker\bar\partial$. We want to show that $\{\alpha\}_{E_2} = 0 \in E_2^{p,\,q}(X)$, i.e. that $\{\alpha\}_{\bar\partial} = d_1(\{\beta\}_{\bar\partial})$ or equivalently that $\{\alpha\}_{\bar\partial} = \{\partial\beta\}_{\bar\partial}$ for some $\beta \in C^\infty_{p-1,\,q}(X,\,\mathbb{C})$ such that $\bar\partial\beta = 0$. This is equivalent to showing that $\alpha = \partial\beta + \bar\partial\gamma$ for some $\beta \in C^\infty_{p-1,\,q}(X,\,\mathbb{C})$ such that $\bar\partial\beta = 0$ and some $\gamma \in C^\infty_{p,\,q-1}(X,\,\mathbb{C})$. We can choose $\beta := v$ and $\gamma := u$.

To prove that T is injective, let $\alpha \in \ker(p''\circ\partial)\cap\ker\bar\partial$ be a $(p,\,q)$-form s.t. $T(\widetilde{[\alpha]}) = \{\alpha\}_{E_2} = 0$. Then $\{\alpha\}_{\bar\partial} = \{\partial\beta\}_{\bar\partial}$ for some $\beta \in C^\infty_{p-1,\,q}(X,\,\mathbb{C})$ such that $\bar\partial\beta = 0$. Hence $\alpha = \partial\beta + \bar\partial\gamma$ for some $\gamma \in C^\infty_{p,\,q-1}(X,\,\mathbb{C})$. Thus, $\alpha \in \operatorname{Im}\bar\partial + \operatorname{Im}(\partial_{|\ker\bar\partial})$, so $\widetilde{[\alpha]} = 0$.

To prove that T is surjective, let $\{\alpha\}_{E_2} \in E_2^{p,\,q}(X)$. Then $\bar\partial\alpha = 0$ (i.e. $\alpha \in \ker\bar\partial$) and $d_1(\{\alpha\}_{\bar\partial}) = \{\partial\alpha\}_{\bar\partial} = 0$ (i.e. $\partial\alpha \in \operatorname{Im}\bar\partial$, which is equivalent, since we already have $\partial\alpha \in \ker\bar\partial = \ker\Delta'' \oplus \operatorname{Im}\bar\partial$, to $p''(\partial\alpha) = 0$, i.e. to $\alpha \in \ker(p''\circ\partial_{|\ker\bar\partial})$). Thus, $\alpha \in \ker(p''\circ\partial)\cap\ker\bar\partial$. It is clear that $\{\alpha\}_{E_2} = T(\widetilde{[\alpha]})$ by definition of T. $\square$

The isomorphism (3.7) naturally prompts the introduction of a Laplace-type operator which, surprisingly, is not a differential operator. It will be the main tool of investigation in this section.

Definition 3.1.2. *([Pop16, Definition 3.2.]) Let (X,ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}}X = n$. For every $p,q\in\{0,1,\dots,n\}$, we define the operator $\widetilde\Delta : C^\infty_{p,q}(X,\mathbb{C}) \longrightarrow C^\infty_{p,q}(X,\mathbb{C})$ by*

$$\widetilde\Delta := \partial p''\partial^\star + \partial^\star p''\partial + \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial. \tag{3.8}$$

We call $\widetilde\Delta$ the **pseudo-differential Laplacian** *associated with the Hermitian metric ω for the second page of the FSS.*

In other words, we have:

$$\widetilde\Delta = \Delta'_{p''} + \Delta'', \quad \text{where } \Delta'_{p''} := \partial p''\partial^\star + \partial^\star p''\partial : C^\infty_{p,q}(X,\mathbb{C}) \longrightarrow C^\infty_{p,q}(X,\mathbb{C}). \tag{3.9}$$

Thus, $\widetilde\Delta$ is the sum of a pseudo-differential regularising operator ($\Delta'_{p''}$) and an elliptic differential operator of order two (the classical $\bar\partial$-Laplacian Δ'').

Clearly, $\widetilde\Delta$ is a non-negative self-adjoint operator whose kernel is $\ker\widetilde\Delta = \ker\Delta'_{p''}\cap\ker\Delta''$ and

$$\ker\Delta'_{p''} = \ker(p''\circ\partial)\cap\ker(p''\circ\partial^\star) \supset \ker\partial\cap\ker\partial^\star = \ker\Delta' \tag{3.10}$$

because $\langle\langle\Delta'_{p''}u,\,u\rangle\rangle = ||p''\partial u||^2 + ||p''\partial^\star u||^2$. Actually, if we put $\Delta'_{p''_\perp} := \partial p''_\perp\partial^\star + \partial^\star p''_\perp\partial$, then $0\leq\Delta'_{p''}\leq\Delta' = \Delta'_{p''} + \Delta'_{p''_\perp}$ since

$$\begin{aligned}\langle\langle\Delta' u,\,u\rangle\rangle = ||\partial u||^2 + ||\partial^\star u||^2 &= ||p''\partial u||^2 + ||p''\partial^\star u||^2 + ||p''_\perp\partial u||^2 + ||p''_\perp\partial^\star u||^2\\ &= \langle\langle\Delta'_{p''}u,\,u\rangle\rangle + \langle\langle\Delta'_{p''_\perp}u,\,u\rangle\rangle\end{aligned} \tag{3.11}$$

for any form u. Indeed, for example, $\partial u = p''\partial u + p''_\perp\partial u$ and $p''\partial u\perp p''_\perp\partial u$, while $\langle\langle\partial^\star p''\partial u,\,u\rangle\rangle = \langle\langle p''\partial u,\,\partial u\rangle\rangle = \langle\langle p''\partial u,\,p''\partial u\rangle\rangle = ||p''\partial u||^2$.

We now pause briefly to notice some of the properties of the pseudo-differential Laplacian $\widetilde\Delta$.

Lemma 3.1.3. *(i) If the metric ω is Kähler, then $\Delta'_{p''}=0$, so $\widetilde\Delta = \Delta''$.*

(ii) For every $p,q=0,1,\dots,n$, let $(\psi_j^{p,q})_{1\leq j\leq h^{p,q}}$ be an arbitrary orthonormal basis of the Δ''-harmonic space $\mathcal{H}^{p,q}_{\Delta''}(X,\mathbb{C})\subset C^\infty_{p,q}(X,\mathbb{C})$. Then $\Delta'_{p''}$ is given by the formula

$$\Delta'_{p''}u = \sum_{j=1}^{h^{p-1,q}}\langle\langle u,\,\partial\psi_j^{p-1,q}\rangle\rangle\,\partial\psi_j^{p-1,q} + \sum_{j=1}^{h^{p+1,q}}\langle\langle u,\,\partial^\star\psi_j^{p+1,q}\rangle\rangle\,\partial^\star\psi_j^{p+1,q}, \qquad u\in C^\infty_{p,q}(X,\mathbb{C}). \tag{3.12}$$

(iii) For all p,q, $\widetilde\Delta : C^\infty_{p,q}(X,\mathbb{C})\longrightarrow C^\infty_{p,q}(X,\mathbb{C})$ behaves like an elliptic self-adjoint differential operator in the sense that $\ker\widetilde\Delta$ is finite-dimensional, $Im\,\widetilde\Delta$ is closed and finite codimensional in $C^\infty_{p,q}(X,\mathbb{C})$, there is an orthogonal (for the L^2 inner product induced by ω) 2-space decomposition

$$C^\infty_{p,q}(X,\mathbb{C}) = \ker\widetilde\Delta\bigoplus Im\,\widetilde\Delta \tag{3.13}$$

giving rise to an orthogonal 3-space decomposition

$$C^\infty_{p,q}(X,\mathbb{C}) = \ker\widetilde\Delta\bigoplus\bigg(Im\,\bar\partial + Im\,(\partial_{|\ker\bar\partial})\bigg)\bigoplus\bigg(Im\,(\partial^\star\circ p'') + Im\,\bar\partial^\star\bigg) \tag{3.14}$$

in which $\ker\widetilde\Delta\oplus(\mathit{Im}\,\bar\partial+\mathit{Im}\,(\partial_{|\ker\bar\partial}))=\ker(p''\circ\partial)\cap\ker\bar\partial$, $\ker\widetilde\Delta\oplus(\mathit{Im}\,(\partial^\star\circ p'')+\mathit{Im}\,\bar\partial^\star)=\ker\bar\partial^\star\cap\ker(p''\circ\partial^\star)$ *and* $(\mathit{Im}\,\bar\partial+\mathit{Im}\,(\partial_{|\ker\bar\partial}))\oplus(\mathit{Im}\,(\partial^\star\circ p'')+\mathit{Im}\,\bar\partial^\star)=\mathit{Im}\,\widetilde\Delta$.
Moreover, $\widetilde\Delta$ *has a compact resolvent which is a pseudo-differential operator* G *of order* -2*, the Green's operator of* $\widetilde\Delta$*, hence the spectrum of* $\widetilde\Delta$ *is discrete and consists of non-negative eigenvalues that tend to* $+\infty$*.*

Proof. (i) If ω is Kähler, $\Delta'=\Delta''$, hence $p'=p''$. Since $\ker\Delta'$ is orthogonal to both $\mathrm{Im}\,\partial$ and $\mathrm{Im}\,\partial^\star$, $p'\circ\partial=0$ and $p'\circ\partial^\star=0$. Thus $p''\circ\partial=0$ and $p''\circ\partial^\star=0$, so $\Delta'_{p''}=0$.

(ii) Since $\ker\Delta''$ is finite-dimensional, $p'':C^\infty_{p,\,q}(X,\,\mathbb{C})\longrightarrow\ker\Delta''$ is a regularising operator of finite rank defined by the C^∞ kernel $\sum\limits_{j=1}^{h^{p,\,q}}\psi_j^{p,\,q}(x)\otimes(\psi_j^{p,\,q})^\star(y)$. Consequently, for every $u\in C^\infty_{p,\,q}(X,\,\mathbb{C})$,

$$(p''u)(x)=\int\limits_X\sum_{j=1}^{h^{p,\,q}}\psi_j^{p,\,q}(x)\,\langle u(y),\,\psi_j^{p,\,q}(y)\rangle\,dV_\omega(y),\quad\text{i.e.}\quad p''u=\sum_{j=1}^{h^{p,\,q}}\langle\langle u,\,\psi_j^{p,\,q}\rangle\rangle\,\psi_j^{p,\,q}.\tag{3.15}$$

Taking successively $u=\partial^\star v$ with $v\in C^\infty_{p+1,\,q}(X,\,\mathbb{C})$ and $u=\partial w$ with $w\in C^\infty_{p-1,\,q}(X,\,\mathbb{C})$, we get

$$p''\partial^\star v=\sum_{j=1}^{h^{p,\,q}}\langle\langle v,\,\partial\psi_j^{p,\,q}\rangle\rangle\,\psi_j^{p,\,q}\quad\text{and}\quad p''\partial w=\sum_{j=1}^{h^{p,\,q}}\langle\langle w,\,\partial^\star\psi_j^{p,\,q}\rangle\rangle\,\psi_j^{p,\,q}.$$

Formula (3.12) follows at once from these identities.

(iii) Since $\ker\widetilde\Delta\subset\ker\Delta''$ and the latter kernel is finite-dimensional thanks to Δ'' being elliptic, $\ker\widetilde\Delta$ is finite-dimensional.

The operator $\widetilde\Delta$ is elliptic pseudo-differential as the sum of an elliptic differential operator and a regularising one, so the elliptic theory applies to it. But we can also argue starting from the obvious inequality $\widetilde\Delta\geq\Delta''\geq0$ (which follows from $\langle\langle\Delta'_{p''}u,\,u\rangle\rangle\geq0$ for all u) and combining it with the Gårding inequality for the elliptic differential operator Δ''. We get constants $\delta_1,\delta_2>0$ such that

$$\delta_2\,||u||_1^2\leq\langle\langle\Delta''u,\,u\rangle\rangle+\delta_1\,||u||^2\leq\langle\langle\widetilde\Delta u,\,u\rangle\rangle+\delta_1\,||u||^2,\quad u\in C^\infty_{p,\,q}(X,\,\mathbb{C}),\tag{3.16}$$

where $||\;||_1$ denotes the Sobolev norm W^1 and $||\;||$ denotes the $L^2=W^0$ norm. Since $\langle\langle\widetilde\Delta u,\,u\rangle\rangle\leq\frac{1}{2}\,||\widetilde\Delta u||^2+\frac{1}{2}\,||u||^2$, we get

$$\delta_2\,||u||_1^2\leq\frac{1}{2}\,||\widetilde\Delta u||^2+(\delta_1+\frac{1}{2})\,||u||^2,\quad u\in C^\infty_{p,\,q}(X,\,\mathbb{C}).\tag{3.17}$$

This suffices to prove that $\mathrm{Im}\,\widetilde\Delta$ is closed by the usual method using the Rellich Lemma (see e.g. [Dem96, 3.10, p. 18-19]). From closedness of $\mathrm{Im}\,\widetilde\Delta$ and self-adjointness of $\widetilde\Delta$ we get (3.13).

Now (3.14) is easily deduced from (3.13) as follows. It is clear that

$$\mathrm{Im}\,\widetilde\Delta\subset\mathrm{Im}\,(\partial\circ p'')+\mathrm{Im}\,(\partial^\star\circ p'')+\mathrm{Im}\,\bar\partial+\mathrm{Im}\,\bar\partial^\star.$$

Since $\mathrm{Im}\,(\partial\circ p'')=\mathrm{Im}\,(\partial_{|\ker\Delta''})$ and $\ker\Delta''\subset\ker\bar\partial$, we get $\mathrm{Im}\,(\partial\circ p'')\subset\mathrm{Im}\,(\partial_{|\ker\bar\partial})$, hence

$$\mathrm{Im}\,\widetilde\Delta\subset\bigg(\mathrm{Im}\,\bar\partial+\mathrm{Im}\,(\partial_{|\ker\bar\partial})\bigg)\oplus\bigg(\mathrm{Im}\,(\partial^\star\circ p'')+\mathrm{Im}\,\bar\partial^\star\bigg).\tag{3.18}$$

Indeed, we can easily check that the middle sum on the r.h.s. of (3.18) is orthogonal. We have $\mathrm{Im}\,\bar\partial\perp\mathrm{Im}\,\bar\partial^\star$ since $\bar\partial^2=0$ and $\mathrm{Im}\,\bar\partial\perp\mathrm{Im}\,(\partial^\star\circ p'')$ since $\langle\langle\bar\partial u,\,\partial^\star p''v\rangle\rangle=\langle\langle\partial\bar\partial u,\,p''v\rangle\rangle=0$ for all

u, v because $\partial\bar\partial u \in \mathrm{Im}\,\bar\partial \perp \ker\Delta'' \ni p''v$. Similarly, $\mathrm{Im}\,(\partial_{|\ker\bar\partial}) \perp \mathrm{Im}\,(\partial^\star\circ p'')$ since $\partial^2 = 0$ and $\mathrm{Im}\,(\partial_{|\ker\bar\partial}) \perp \mathrm{Im}\,\bar\partial^\star$ since $\langle\langle\partial u,\, \bar\partial^\star v\rangle\rangle = \langle\langle\bar\partial\partial u,\, v\rangle\rangle = 0$ for all $u \in \ker\bar\partial$ and all v.

Now, putting together (3.13) and (3.18), we get

$$C^\infty_{p,\,q}(X,\,\mathbb{C}) \subset \ker\widetilde\Delta \bigoplus \bigg(\mathrm{Im}\,\bar\partial + \mathrm{Im}\,(\partial_{|\ker\bar\partial})\bigg) \bigoplus \bigg(\mathrm{Im}\,(\partial^\star\circ p'') + \mathrm{Im}\,\bar\partial^\star\bigg)$$

in which the inclusion must be an equality because all the three mutually orthogonal spaces on the r.h.s. are contained in $C^\infty_{p,\,q}(X,\,\mathbb{C})$. This proves (3.14) and also that the inclusion in (3.18) is an equality.

The first of the three 2-space decompositions stated after (3.14) will be proved as (3.22) in the proof of the next Theorem 3.1.4, while the second one can be proved analogously. The third one is (3.18) that was seen above to be an equality.

The last two statements about the Green's operator and the spectrum are proved in the usual way using the elliptic theory. $\square$

We now get the **Hodge isomorphism** for the **second page of the Frölicher spectral sequence** that we have been aiming at. It is the main result of this section.

Theorem 3.1.4. *([Pop16, Theorem 3.4]) Let $(X,\,\omega)$ be a compact Hermitian manifold with $\dim_{\mathbb{C}}X = n$. For every $p, q \in \{0, 1, \dots, n\}$, let $\widetilde{\mathcal{H}}^{p,\,q}_{\widetilde\Delta}(X,\,\mathbb{C})$ stand for the kernel of $\widetilde\Delta$ acting on $(p,\,q)$-forms. Then the map*

$$S = S^{p,\,q} : \widetilde{\mathcal{H}}^{p,\,q}_{\widetilde\Delta}(X,\,\mathbb{C}) \longrightarrow \widetilde{H}^{p,\,q}(X,\,\mathbb{C}), \qquad \alpha \longmapsto \widetilde{[\alpha]}, \tag{3.19}$$

is an **isomorphism**. *In particular, its composition with the isomorphism $T : \widetilde{H}^{p,\,q}(X,\,\mathbb{C}) \longrightarrow E_2^{p,\,q}(X)$ defined in (3.7) yields the* **Hodge isomorphism:**

$$T\circ S = T^{p,\,q}\circ S^{p,\,q} : \widetilde{\mathcal{H}}^{p,\,q}_{\widetilde\Delta}(X,\,\mathbb{C}) \longrightarrow E_2^{p,\,q}(X), \qquad \alpha \longmapsto \{\alpha\}_{E_2}. \tag{3.20}$$

Thus, every class $\{\alpha\}_{E_2} \in E_2^{p,\,q}(X)$ contains a unique $\widetilde\Delta$-harmonic representative α.

Proof. Thanks to (3.10), we have

$$\widetilde{\mathcal{H}}^{p,\,q}_{\widetilde\Delta}(X,\,\mathbb{C}) = \ker(p''\circ\partial)\cap\ker(p''\circ\partial^\star)\cap\ker\bar\partial\cap\ker\bar\partial^\star \subset \ker(p''\circ\partial)\cap\ker\bar\partial. \tag{3.21}$$

In particular, every form $\alpha \in \widetilde{\mathcal{H}}^{p,\,q}_{\widetilde\Delta}(X,\,\mathbb{C})$ defines a class $\widetilde{[\alpha]} \in \widetilde{H}^{p,\,q}(X,\,\mathbb{C})$, so the map $S^{p,\,q}$ is well defined. We now prove the following orthogonal decomposition

$$\ker(p''\circ\partial)\cap\ker\bar\partial = \ker\widetilde\Delta \bigoplus \bigg(\mathrm{Im}\,\bar\partial + \mathrm{Im}\,(\partial_{|\ker\bar\partial})\bigg), \tag{3.22}$$

where $\ker\widetilde\Delta = \widetilde{\mathcal{H}}^{p,\,q}_{\widetilde\Delta}(X,\,\mathbb{C})$ is given by (3.21). It is clear that (3.22) implies that S is an isomorphism.

Thanks to the 3-space orthogonal decomposition (3.14), proving (3.22) is equivalent to proving

$$\ker(p''\circ\partial)\cap\ker\bar\partial = \bigg(\mathrm{Im}\,(\partial^\star\circ p'') + \mathrm{Im}\,\bar\partial^\star\bigg)^\perp. \tag{3.23}$$

Now, the r.h.s. term in (3.23) is the intersection of $(\mathrm{Im}\,(\partial^\star\circ p''))^\perp = \ker\,(\partial^\star\circ p'')^\star = \ker\,(p''\circ\partial)$ with $(\mathrm{Im}\,\bar\partial^\star)^\perp = \ker\bar\partial$. This proves (3.23), hence also (3.22). $\square$.

3.2 Serre-type duality for the higher pages of the FSS

The material in this section is taken from [PSU20b]. Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. As usual, for every $r \in \mathbb{N}$, we let $E_r^{p,\,q}(X)$ stand for the space of bidegree $(p,\,q)$ featuring on the r^{th} page of the Frölicher spectral sequence of X.

In this section, we extend the classical **Serre duality** (that holds, as recalled in §.1.1.2, for the Dolbeault cohomology, or equivalently for the *first page* of the FSS) to all the pages. For the sake of perspicuity, we will first treat the case $r = 2$ and then the more technically involved case $r \geq 3$.

3.2.1 Serre-type duality for the second page of the FSS

The main ingredient in the proof of the next statement is the Hodge theory for the E_2-cohomology described in §.3.1 and based on the construction of the pseudo-differential Laplacian $\widetilde{\Delta}$.

Theorem 3.2.1. *For every $p, q \in \{0, \dots, n\}$, the canonical bilinear pairing*

$$E_2^{p,\,q}(X) \times E_2^{n-p,\,n-q}(X) \longrightarrow \mathbb{C}, \qquad \bigg(\{\alpha\}_{E_2},\, \{\beta\}_{E_2}\bigg) \mapsto \int\limits_X \alpha \wedge \beta, \tag{3.24}$$

is **well defined** *(i.e. independent of the choices of representatives of the cohomology classes involved) and* **non-degenerate**.

Proof. • To prove *well-definedness*, let $\{\alpha\}_{E_2} \in E_2^{p,\,q}(X)$ and $\{\beta\}_{E_2} \in E_2^{n-p,\,n-q}(X)$ be arbitrary classes in which we choose arbitrary representatives α, β. Thus, $\bar\partial\alpha = 0$, $\partial\alpha \in \operatorname{Im}\bar\partial$ (since $\{\alpha\}_{\bar\partial} \in \ker d_1$) and β has the analogous properties. In particular, $\partial\beta = \bar\partial v$ for some $(n-p+1,\, n-q-1)$-form v. Any other representative of the class $\{\alpha\}_{E_2}$ is of the shape $\alpha + \partial\eta + \bar\partial\zeta$ for some $(p-1,\,q)$-form $\eta \in \ker\bar\partial$ and some $(p,\,q-1)$-form ζ. (Indeed, $\{\partial\eta\}_{\bar\partial} = d_1(\{\eta\}_{\bar\partial})$.) We have:

$$\begin{aligned}
\int\limits_X (\alpha + \partial\eta + \bar\partial\zeta) \wedge \beta &= \int\limits_X \alpha \wedge \beta + (-1)^{p+q} \int\limits_X \eta \wedge \partial\beta + (-1)^{p+q} \int\limits_X \zeta \wedge \bar\partial\beta \qquad \text{(by Stokes)} \\
&= \int\limits_X \alpha \wedge \beta + (-1)^{p+q} \int\limits_X \eta \wedge \bar\partial v \qquad (\text{since } \partial\beta = \bar\partial v \;\; \text{and } \bar\partial\beta = 0) \\
&= \int\limits_X \alpha \wedge \beta + \int\limits_X \bar\partial\eta \wedge v = \int\limits_X \alpha \wedge \beta \qquad (\text{by Stokes and } \bar\partial\eta = 0).
\end{aligned}$$

Similarly, the integral $\int_X \alpha \wedge \beta$ does not change if β is replaced by $\beta + \partial a + \bar\partial b$ with $a \in \ker\bar\partial$.

• To prove *non-degeneracy* for the pairing (3.24), we fix an arbitrary Hermitian metric ω on X and use the pseudo-differential Laplacian associated with ω introduced in Definition 3.1.2, as well as the Hodge Isomorphism Theorem 3.1.4.

Claim 3.2.2. *For every $\alpha \in C^\infty_{p,\,q}(X)$, the equivalence holds: $\widetilde{\Delta}\alpha = 0 \iff \widetilde{\Delta}(\star\bar\alpha) = 0$, where $\star = \star_\omega$ is the Hodge-star operator associated with ω.*

Suppose for a moment that this claim has been proved. To prove non-degeneracy for the pairing (3.24), let $\{\alpha\}_{E_2} \in E_2^{p,\,q}(X)$ be an arbitrary non-zero class whose unique $\widetilde{\Delta}$-harmonic representative is denoted by α. So, $\alpha \neq 0$ and $\star\bar\alpha \in \mathcal{H}^{n-p,\,n-q}_{\widetilde{\Delta}}(X) \setminus \{0\}$. In particular, $\star\bar\alpha$ represents an element in

$E_2^{n-p,\,n-q}(X)$ and the pair $(\{\alpha\}_{E_2},\,\{\star\bar\alpha\}_{E_2})$ maps under (3.24) to $\int_X\alpha\wedge\star\bar\alpha=\int_X|\alpha|^2_\omega\,dV_\omega=||\alpha||^2_{L^2_\omega}\neq 0$. Since p,q and α were arbitrary, we conclude that the pairing (3.24) is non-degenerate.

- *Proof of Claim 3.2.2.* Since $\widetilde\Delta$ is a sum of non-negative operators of the shape $A^\star A$, we have

$$\ker\widetilde\Delta=\ker(p''\partial)\cap\ker(p''\partial^\star)\cap\ker\bar\partial\cap\ker\bar\partial^\star.$$

Thus, the third orthogonal 3-space decomposition under (3) of Theorem 1.1.6 yields the following equivalence:

$$\alpha\in\ker\widetilde\Delta\iff(i)\ \partial\alpha\in\operatorname{Im}\bar\partial\oplus\operatorname{Im}\bar\partial^\star,\ \ (ii)\ \partial^\star\alpha\in\operatorname{Im}\bar\partial\oplus\operatorname{Im}\bar\partial^\star\quad\text{and}\quad(iii)\ \alpha\in\ker\bar\partial\cap\ker\bar\partial^\star.$$

Let $\alpha\in\ker\widetilde\Delta$. Since $\star:\Lambda^{p,\,q}T^\star X\longrightarrow\Lambda^{n-q,\,n-p}T^\star X$ is an isomorphism, the well-known identities $\star\star=(-1)^{p+q}$ on $(p,\,q)$-forms, $\partial^\star=-\star\bar\partial\star$ and $\bar\partial^\star=-\star\partial\star$ yield:

$$\bar\partial\alpha=0\iff\partial\bar\alpha=0\iff\bar\partial^\star(\star\bar\alpha)=0\quad\text{and}\quad\bar\partial^\star\alpha=0\iff\partial^\star\bar\alpha=0\iff\bar\partial(\star\bar\alpha)=0.$$

Thus, α satisfies condition (iii) if and only if $\star\bar\alpha$ satisfies condition (iii).

Meanwhile, α satisfies condition (ii) if and only if there exist forms ξ,η such that $\partial^\star\alpha=\bar\partial\xi+\bar\partial^\star\eta$. The last identity is equivalent to

$$\bar\partial^\star\bar\alpha=\partial\bar\xi+\partial^\star\bar\eta\iff-(\star\star)\partial(\star\bar\alpha)=\pm\star\partial\star(\star\bar\xi)\pm\star(-\star\bar\partial\star\bar\eta)\iff\partial(\star\bar\alpha)=\pm\bar\partial^\star(\star\bar\xi)\pm\bar\partial(\star\bar\eta).$$

Thus, α satisfies condition (ii) if and only if $\star\bar\alpha$ satisfies condition (i).

Similarly, α satisfies condition (i) if and only if there exist forms u,v such that $\partial\alpha=\bar\partial u+\bar\partial^\star v$. The last identity is equivalent to

$$\bar\partial\bar\alpha=\partial\bar u+\partial^\star\bar v\iff-\star\bar\partial\star(\star\bar\alpha)=-\star\partial(\star\star\bar u)-\star\partial^\star(\star\star\bar v)\iff\partial^\star(\star\bar\alpha)=\bar\partial^\star(\star\bar u)+\bar\partial(\star\bar v).$$

Thus, α satisfies condition (i) if and only if $\star\bar\alpha$ satisfies condition (ii).

This completes the proof of Claim 3.2.2 and implicitly that of Theorem 3.2.1. □.

3.2.2 Serre-type duality for the pages $r\geq 3$ of the FSS

In this subsection, we prove the following analogue of Theorem 3.2.1 for every $r\geq 3$.

Theorem 3.2.3. *For every $r\in\mathbb{N}^\star$ and all $p,q\in\{0,\dots,n\}$, the canonical bilinear pairing*

$$E_r^{p,\,q}(X)\times E_r^{n-p,\,n-q}(X)\longrightarrow\mathbb{C},\qquad\Big(\{\alpha\}_{E_r},\,\{\beta\}_{E_r}\Big)\mapsto\int\limits_X\alpha\wedge\beta,\tag{3.25}$$

is **well defined** *(i.e. independent of the choices of representatives of the cohomology classes involved) and* **non-degenerate**.

This result is the sum of Corollaries 3.2.5 and 3.2.10 that will be proved separately.

In order to prove Theorem 3.2.3 (whose only case $r\geq 3$ still needs a proof), we will construct elliptic pseudo-differential operators $\widetilde\Delta^{(\omega)}_{(r)}$ associated with any given Hermitian metric ω on X whose kernels are isomorphic to the spaces $E_r^{p,\,q}(X)$ in every bidegree $(p,\,q)$. This extends to arbitrary $r\in\mathbb{N}^\star$ the construction performed in Definition 3.1.2 for $r=2$. We then apply this construction to prove the existence of a (non-degenerate) duality between every space $E_r^{p,\,q}(X)$ and the space $E_r^{n-p,\,n-q}(X)$.

Let X be an arbitrary compact complex n-dimensional manifold. Fix $r\in\mathbb{N}$ and a bidegree $(p,\,q)$ with $p,q\in\{0,\dots,n\}$. We will use the terminology of Definition 1.2.9, except that the $\mathbb{C}$-vector space of C^∞ E_r-closed (resp. E_r-exact) $(p,\,q)$-forms will now be denoted by $\mathcal{Z}_r^{p,\,q}(X)$ (resp. $\mathcal{C}_r^{p,\,q}(X)$), instead of $\mathcal{X}_r^{p,\,q}$ (resp. $\mathcal{Y}_r^{p,\,q}(X)$). Of course, $\mathcal{C}_r^{p,\,q}(X)\subset\mathcal{Z}_r^{p,\,q}(X)$ and $E_r^{p,\,q}(X)=\mathcal{Z}_r^{p,\,q}(X)/\mathcal{C}_r^{p,\,q}(X)$.

The following statement is implicit in the results of §.1.2.2.

Proposition 3.2.4. *Let X be an arbitrary compact complex n-dimensional manifold. Fix $r\in\mathbb{N}$ and a bidegree $(p,\,q)$ with $p,q\in\{0,\dots,n\}$.*

*(i) A smooth $\mathbb{C}$-valued $(p,\,q)$-form α on X represents an E_r-cohomology class, denoted by $\{\alpha\}_{E_r}\in E_r^{p,\,q}(X)$, on the r^{th} page of the FSS if and only if α is E_r-**closed**.*

*(ii) A smooth $\mathbb{C}$-valued $(p,\,q)$-form α on X represents the zero E_r-cohomology class, i.e. $\{\alpha\}_{E_r}=0\in E_r^{p,\,q}(X)$, on the r^{th} page of the FSS if and only if α is E_r-**exact**.*

The immediate consequence that we notice is the well-definedness of the pairing that parallels on any page of the Frölicher spectral sequence the classical Serre duality.

Corollary 3.2.5. *Let X be a compact complex manifold with $dim_{\mathbb{C}}X=n$. For every $r\in\mathbb{N}^\star$ and every $p,q\in\{0,\dots,n\}$, the canonical bilinear pairing*

$$E_r^{p,\,q}(X)\times E_r^{n-p,\,n-q}(X)\longrightarrow\mathbb{C},\qquad(\{\alpha\}_{E_r},\,\{\beta\}_{E_r})\mapsto\int\limits_X\alpha\wedge\beta,$$

is **well defined** *(i.e. independent of the choices of representatives of the E_r-classes involved).*

Proof. By symmetry, it suffices to prove that $\int_X\alpha\wedge\beta=0$ whenever $\alpha\in C^\infty_{p,\,q}(X)$ is E_r-exact and $\beta\in C^\infty_{n-p,\,n-q}(X)$ is E_r-closed. By Definition 1.2.9 and Proposition 3.2.4, these conditions are equivalent to

$$\bar\partial\beta=0,\quad\partial\beta=\bar\partial u_1,\quad\partial u_1=\bar\partial u_2,\dots\quad,\partial u_{r-2}=\bar\partial u_{r-1},$$

for some forms u_j and to: $\alpha=\partial\zeta+\bar\partial\xi$ for some form ζ satisfying:

$$\bar\partial\zeta=\partial v_{r-3},\quad\bar\partial v_{r-3}=\partial v_{r-4},\dots\quad,\bar\partial v_1=\partial v_0,\bar\partial v_0=0$$

for some forms v_k. We get:

$$\int\limits_X\alpha\wedge\beta=\int\limits_X\partial\zeta\wedge\beta+\int\limits_X\bar\partial\xi\wedge\beta.$$

Every integral on the r.h.s. above is seen to vanish by repeated integration by parts. Specifically, $\int_X\bar\partial\xi\wedge\beta=\pm\int\limits_X\xi\wedge\bar\partial\beta=0$ since $\bar\partial\beta=0$, while for every $l\in\{1,\dots,r-2\}$ we have:

$$\begin{aligned}\int\limits_X\partial\zeta\wedge\beta &= \pm\int\limits_X\zeta\wedge\partial\beta=\pm\int\limits_X\zeta\wedge\bar\partial u_1=\pm\int\limits_X\bar\partial\zeta\wedge u_1=\pm\int\limits_X\partial v_{r-3}\wedge u_1\\ &= \pm\int\limits_X v_{r-3}\wedge\partial u_1=\pm\int\limits_X v_{r-3}\wedge\bar\partial u_2=\pm\int\limits_X\bar\partial v_{r-3}\wedge u_2=\pm\int\limits_X\partial v_{r-4}\wedge u_2\\ &\;\;\vdots\\ &= \pm\int\limits_X v_0\wedge\partial u_{r-2}=\pm\int\limits_X v_0\wedge\bar\partial u_{r-1}=\pm\int\limits_X\bar\partial v_0\wedge u_{r-1}=0,\end{aligned}$$

since $\bar\partial v_0 = 0$. $\square$

We will now prove that the above pairing is also non-degenerate, thus defining a Serre-type duality on every page of the Frölicher spectral sequence. Much of the following discussion is fairly technical and appeared in [Pop19, §.2.2 and Appendix], so we will only recall the bare bones. Further details are given in §.3.2.3.

Let us fix an arbitrary Hermitian metric ω on X. For every bidegree $(p,\,q)$, *ω-harmonic spaces* (also called *E_r-harmonic spaces*):

$$\cdots \subset \mathcal{H}_{r+1}^{p,\,q} \subset \mathcal{H}_r^{p,\,q} \subset \cdots \subset \mathcal{H}_1^{p,\,q} \subset C^\infty_{p,\,q}(X)$$

were inductively constructed in [Pop17, §.3.2, especially Definition 3.3. and Corollary 3.4. — see Definition 3.5.11 below] such that every subspace $\mathcal{H}_r^{p,\,q} = \mathcal{H}_r^{p,\,q}(X,\,\omega)$ is isomorphic to the corresponding space $E_r^{p,\,q}(X)$ on the r^{th} page of the Frölicher spectral sequence.

Moreover, these spaces fit into the inductive construction described in the next

Proposition 3.2.6. *Let $(X,\,\omega)$ be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$.*

(i) For every bidegree $(p,\,q)$, the space $C^\infty_{p,\,q}(X)$ splits successively into mutually L^2_ω-orthogonal subspaces as follows:

$$\begin{array}{ccc}
C^\infty_{p,\,q}(X) = Im\, d_0 \;\oplus & \underbrace{\mathcal{H}_1^{p,\,q}} & \oplus\; Im\, d_0^\star \\
& \| & \\
& \overbrace{Im\, d_1^{(\omega)} \;\oplus\; \underbrace{\mathcal{H}_2^{p,\,q}} \;\oplus\; Im\,(d_1^{(\omega)})^\star} & \\
& \| & \\
& \vdots & \\
& \| & \\
& \overbrace{Im\, d_{r-1}^{(\omega)} \;\oplus\; \underbrace{\mathcal{H}_r^{p,\,q}} \;\oplus\; Im\,(d_{r-1}^{(\omega)})^\star} & \\
& \| & \\
& \overbrace{Im\, d_r^{(\omega)} \;\oplus\; \underbrace{\mathcal{H}_{r+1}^{p,\,q}} \;\oplus\; Im\,(d_r^{(\omega)})^\star} & \\
& \| & \\
& \vdots &
\end{array}$$

where, for $r \in \mathbb{N}^\star$, the operators $d_r^{(\omega)}$ are defined as

$$d_r^{(\omega)} = d_r^{(\omega)p,\,q} = p_r \partial D_{r-1} p_r : \mathcal{H}_r^{p,\,q} \longrightarrow \mathcal{H}_r^{p+r,\,q-r+1} \tag{3.26}$$

using the L^2_ω-orthogonal projections $p_r = p_r^{p,\,q} : C^\infty_{p,\,q}(X) \longrightarrow \mathcal{H}_r^{p,\,q}$ onto the ω-harmonic spaces $\mathcal{H}_r^{p,\,q}$ and where we inductively define

$$D_{r-1} := ((\widetilde\Delta^{(1)})^{-1}\bar\partial^\star\partial)\dots((\widetilde\Delta^{(r-1)})^{-1}\bar\partial^\star\partial) \quad \textit{and} \quad D_0 = Id.$$

(So, $p_1 = p''$.) See (iii) below for the inductive definition of the pseudo-differential Laplacians $\widetilde\Delta^{(r)}$.

Thus, the triples $(p_r,\, d_r^{(\omega)},\, \mathcal{H}_{r+1}^{p,\,q})$ *are defined by induction on* $r \in \mathbb{N}^\star$*: once the triple* $(p_{r-1},\, d_{r-1}^{(\omega)},\, \mathcal{H}_r^{p,\,q})$ *has been constructed for all the bidegrees* $(p,\,q)$*, it induces* p_r*, which induces* $d_r^{(\omega)}$*, which induces* $\mathcal{H}_{r+1}^{p,\,q}$ *defined as the* L^2_ω*-orthogonal complement of* $Im\, d_r^{(\omega)}$ *in* $\ker\, d_r^{(\omega)}$*.*

The operators $d_r^{(\omega)}$ *can also be considered to be defined on the whole spaces of smooth forms:*

$$d_r^{(\omega)} = p_r\partial D_{r-1}p_r : C^\infty_{p,\,q}(X) \longrightarrow C^\infty_{p+r,\,q-r+1}(X).$$

(ii) The above definition of $d_r^{(\omega)}$ *follows from the requirement that the following diagram be commutative:*

$$\begin{array}{ccc} E_r^{p,\,q}(X) & \xrightarrow{\quad d_r \quad} & E_r^{p+r,\,q-r+1}(X) \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ \mathcal{H}_r^{p,\,q} & \xrightarrow{d_r^{(\omega)}=p_r\partial D_{r-1}p_r} & \mathcal{H}_r^{p+r,\,q-r+1}, \end{array}$$

where the maps $d_r : E_r^{p,\,q}(X) \longrightarrow E_r^{p+r,\,q-r+1}(X)$ *are the differentials on the* r^{th} *page of the Frölicher spectral sequence. Thus, the maps* $d_r^{(\omega)}$ *are the metric realisations, at the level of the harmonic spaces, of the canonical maps* d_r*.*

(iii) For every $r \in \mathbb{N}^\star$*, the adjoint of* $d_r^{(\omega)}$ *is*

$$(d_r^{(\omega)})^\star = p_r D_{r-1}^\star \partial^\star p_r : \mathcal{H}_r^{p+r,\,q-r+1} \longrightarrow \mathcal{H}_r^{p,\,q}.$$

It induces the "Laplacian"

$$\widetilde{\Delta}^{(\omega)}_{(r+1)} = d_r^{(\omega)}\,(d_r^{(\omega)})^\star + (d_r^{(\omega)})^\star\, d_r^{(\omega)} : \mathcal{H}_r^{p,\,q} \longrightarrow \mathcal{H}_r^{p,\,q}$$

given by the explicit formula

$$\widetilde{\Delta}^{(\omega)}_{(r+1)} = p_r\,[(\partial D_{r-1}p_r)\,(\partial D_{r-1}p_r)^\star + (p_r\partial D_{r-1})^\star\,(p_r\partial D_{r-1}) + \widetilde{\Delta}^{(r)}]\,p_r,$$

which is the restriction and co-restriction to $\mathcal{H}_r^{p,\,q}$ *of the pseudo-differential Laplacian*

$$\widetilde{\Delta}^{(r+1)} \;\; := \;\; (\partial D_{r-1}p_r)\,(\partial D_{r-1}p_r)^\star + (p_r\partial D_{r-1})^\star\,(p_r\partial D_{r-1}) + \widetilde{\Delta}^{(r)} : C^\infty_{p,\,q}(X) \longrightarrow C^\infty_{p,\,q}(X).$$

(iv) For every $r \in \mathbb{N}^\star$ *and every bidegree* $(p,\,q)$*, the following orthogonal* 3*-space decomposition holds:*

$$\mathcal{H}_r^{p,\,q} = Im\, d_r^{(\omega)} \oplus \mathcal{H}_{r+1}^{p,\,q} \oplus Im\,(d_r^{(\omega)})^\star,$$

where $\ker d_r^{(\omega)} = Im\, d_r^{(\omega)} \oplus \mathcal{H}_{r+1}^{p,\,q}$*. In particular, this confirms that* $\mathcal{H}_{r+1}^{p,\,q}$ *is the orthogonal complement for the* L^2_ω*-inner product of* $Im\, d_r^{(\omega)}$ *in* $\ker d_r^{(\omega)}$*. Moreover,*

$$\mathcal{H}_{r+1}^{p,\,q} = \ker \widetilde{\Delta}^{(\omega)}_{(r+1)} = \ker d_r^{(\omega)} \cap \ker (d_r^{(\omega)})^\star \simeq E_{r+1}^{p,\,q}(X),$$

for every $r \in \mathbb{N}$ *and all* $p, q \in \{0, \ldots, n\}$*.*

Proof. The verification of the details of these statements was done in [Pop19, §.2.2 and Appendix]. □

We saw in (i) of Proposition 3.2.4 how the E_r-closedness property of a differential form is characterised in explicit terms. We will now define by analogy the property of $E_r^\star$-closedness when a Hermitian metric has been fixed.

Definition 3.2.7. *Let $(X,\,\omega)$ be an n-dimensional compact complex Hermitian manifold. Fix $r\geq 1$ and a bidegree $(p,\,q)$. A form $\alpha\in C^\infty_{p,\,q}(X)$ is said to be* $E_r^\star$**-closed** *with respect to the metric ω if and only if there exist forms $v_l\in C^\infty_{p-l,\,q+l}(X)$ with $l\in\{1,\dots,r-1\}$ satisfying the following tower of r equations:*

$$\begin{aligned}\bar\partial^\star\alpha &= 0\\ \partial^\star\alpha &= \bar\partial^\star v_1\\ \partial^\star v_1 &= \bar\partial^\star v_2\\ &\vdots\\ \partial^\star v_{r-2} &= \bar\partial^\star v_{r-1}.\end{aligned}$$

We say in this case that $\bar\partial^\star\alpha=0$ and $\partial^\star\alpha$ **runs at least $(r-1)$ times**.

We can now use the E_r-closedness and $E_r^\star$-closedness properties to characterise the $\mathcal{H}_r$-harmonicity property defined above.

Proposition 3.2.8. *Let $(X,\,\omega)$ be an n-dimensional compact complex Hermitian manifold. Fix $r\geq 1$ and a bidegree $(p,\,q)$. For any form $\alpha\in C^\infty_{p,\,q}(X)$, the following equivalence holds:*

$$\alpha\in\mathcal{H}_r^{p,\,q} \iff \alpha \text{ is } E_r\text{-closed and } E_r^\star\text{-closed.}$$

Proof. We know from Proposition 3.2.6 that $\alpha\in\mathcal{H}_{r+1}^{p,\,q}$ if and only if $\alpha\in\mathcal{H}_r^{p,\,q}$ and $\alpha\in\ker d_r^{(\omega)}\cap\ker(d_r^{(\omega)})^\star$. Now, for $\alpha\in\mathcal{H}_r^{p,\,q}$, the definition of $d_r^{(\omega)}$ shows that $\alpha\in\ker d_r^{(\omega)}$ if and only if $\alpha\in\ker(p_r\partial D_{r-1})$ and this last fact is equivalent to α being E_{r+1}-closed. Similarly, for $\alpha\in\mathcal{H}_r^{p,\,q}$, the definition of $(d_r^{(\omega)})^\star$ shows that $\alpha\in\ker(d_r^{(\omega)})^\star$ if and only if $\alpha\in\ker(\partial D_{r-1}p_r)^\star$ and this last fact is equivalent to α being $E_{r+1}^\star$-closed. □

Corollary 3.2.9. *In the setting of Proposition 3.2.8, the following equivalence holds:*

$$\alpha \text{ is } E_r\text{-closed} \iff \star\bar\alpha \text{ is } E_r^\star\text{-closed.}$$

Proof. We know from (i) of Proposition 3.2.4 that α is E_r-closed if and only if there exist forms $u_l\in C^\infty_{p+l,\,q-l}(X)$ for $l=1,\dots,r-1$ such that

$$(-\star\partial\star)\star\bar\alpha=0,\;\;(-\star\bar\partial\star)\star\bar\alpha=(-\star\partial\star)\star\bar u_1,\dots,(-\star\bar\partial\star)\star\bar u_{r-2}=(-\star\partial\star)\star\bar u_{r-1}.$$

Indeed, we have transformed the E_r-closedness condition of (i) in Proposition 3.2.4 by conjugating and applying the Hodge star operator several times. Since $-\star\partial\star=\bar\partial^\star$ and $-\star\bar\partial\star=\partial^\star$, the above conditions are equivalent to $\star\bar\alpha$ being $E_r^\star$-closed (with the forms $\star\bar u_l$ playing the part of the forms v_l). □

An immediate consequence of this discussion is the analogue on every page E_r of the Frölicher spectral sequence of the classical **Serre duality**. The well-definedness was proved in Corollary 3.2.5. The case $r=1$ is the Serre duality, while the case $r=2$ was proved in Theorem 3.2.1.

Corollary 3.2.10. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. For every $r \in \mathbb{N}^\star$ and every $p, q \in \{0, \dots, n\}$, the canonical bilinear pairing*

$$E_r^{p,\,q}(X) \times E_r^{n-p,\,n-q}(X) \longrightarrow \mathbb{C}, \qquad (\{\alpha\}_{E_r},\, \{\beta\}_{E_r}) \mapsto \int\limits_X \alpha \wedge \beta,$$

is **non-degenerate**.

Proof. Let $\{\alpha\}_{E_r} \in E_r^{p,\,q}(X) \setminus \{0\}$. If we fix an arbitrary Hermitian metric ω on X, we know from Proposition 3.2.6 that the associated harmonic space $\mathcal{H}_r^{p,\,q}$ is isomorphic to $E_r^{p,\,q}(X)$ and that the class $\{\alpha\}_{E_r}$ contains a (unique) representative α lying in $\mathcal{H}_r^{p,\,q}$. By Proposition 3.2.8, this is equivalent to α being both E_r-closed and $E_r^\star$-closed, while by Corollary 3.2.9, this is further equivalent to $\star\bar\alpha$ being both $E_r^\star$-closed and E_r-closed, hence to $\star\bar\alpha$ lying in $\mathcal{H}_r^{n-p,\,n-q}$.

In particular, $\star\bar\alpha$ represents a non-zero class $\{\star\bar\alpha\}_{E_r} \in E_r^{n-p,\,n-q}(X)$. We have

$$(\{\alpha\}_{E_r},\, \{\star\bar\alpha\}_{E_r}) \mapsto \int\limits_X \alpha \wedge \star\bar\alpha = ||\alpha||^2 > 0,$$

where $||\ ||$ stands for the L^2_ω-norm. This shows that for every non-zero class $\{\alpha\}_{E_r} \in E_r^{p,\,q}(X)$, the map $(\{\alpha\}_{E_r},\, \cdot) : E_r^{n-p,\,n-q}(X) \longrightarrow \mathbb{C}$ does not vanish identically, proving the non-degeneracy of the pairing. $\square$

3.2.3 Appendix to §.3.2.2

We now give some further technical details for the discussion in §.3.2.2 and refer to the appendix of [Pop19] for yet another round of technical details of the inductive construction of a Hodge theory for the pages E_r, with $r \geq 3$, of the Frölicher spectral sequence.

Let X be an n-dimensional compact complex manifold. We fix an arbitrary Hermitian metric ω on X. As seen in §.3.2.2, for every bidegree $(p,\, q)$, the *ω-harmonic spaces* (also called *E_r-harmonic spaces*)

$$\cdots \subset \mathcal{H}_{r+1}^{p,\,q} \subset \mathcal{H}_r^{p,\,q} \subset \cdots \subset \mathcal{H}_1^{p,\,q} \subset C^\infty_{p,\,q}(X)$$

are such that every $\mathcal{H}_r^{p,\,q}$ is isomorphic to the corresponding space $E_r^{p,\,q}(X)$ featuring on the r^{th} page of the Frölicher spectral sequence of X.

Moreover, the pseudo-differential "Laplacians" $\widetilde\Delta^{(r+1)} : \mathcal{H}_r^{p,\,q} \longrightarrow \mathcal{H}_r^{p,\,q}$ are such that

$$\ker \widetilde\Delta^{(r)} = \mathcal{H}_r^{p,\,q}, \qquad r \in \mathbb{N}^\star,$$

where $\widetilde\Delta^{(1)} = \Delta'' = \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial$ is the usual $\bar\partial$-Laplacian.

The conclusion of the construction in the appendix to [Pop19] was the following statement. It gives a 3-space orthogonal decomposition of each space $C^\infty_{p,\,q}(X)$, for every fixed $r \in \mathbb{N}^\star$, that parallels the standard decomposition $C^\infty_{p,\,q}(X) = \ker \Delta'' \oplus \operatorname{Im} \bar\partial \oplus \operatorname{Im} \bar\partial^\star$ for $r = 1$.

Proposition 3.2.11. *(Corollary 4.6 in [Pop19]) Let $(X,\, \omega)$ be a compact complex n-dimensional Hermitian manifold. For every $r \in \mathbb{N}^\star$, put $D_{r-1} := ((\widetilde\Delta^{(1)})^{-1}\bar\partial^\star\partial)\dots((\widetilde\Delta^{(r-1)})^{-1}\bar\partial^\star\partial)$ and $D_0 = Id$.*

(i) For all $r \in \mathbb{N}^\star$ and all $(p,\, q)$, the kernel of $\widetilde\Delta^{(r+1)} : C^\infty_{p,\,q}(X) \longrightarrow C^\infty_{p,\,q}(X)$ is given by

$$\begin{aligned}\ker \widetilde\Delta^{(r+1)} &= \Big(\ker(p_r\partial D_{r-1}) \cap \ker(\partial D_{r-1}p_r)^\star\Big) \cap \Big(\ker(p_{r-1}\partial D_{r-2}) \cap \ker(\partial D_{r-2}p_{r-1})^\star\Big)\\ &\ \vdots\\ &\cap\ \Big(\ker(p_1\partial) \cap \ker(\partial p_1)^\star\Big) \cap \Big(\ker\bar\partial \cap \ker\bar\partial^\star\Big).\end{aligned}$$

(ii) For all $r\in\mathbb{N}^\star$ and all $(p,\,q)$, the following orthogonal 3-space decomposition (in which the sums inside the big parentheses need not be orthogonal or even direct) holds:

$$\begin{aligned} C^\infty_{p,\,q}(X) &= \ker\widetilde{\Delta}^{(r+1)}\oplus\bigg(Im\,\bar\partial + Im\,(\partial p_1) + Im\,(\partial D_1 p_2) + \cdots + Im\,(\partial D_{r-1}p_r)\bigg) \\ &\oplus \bigg(Im\,\bar\partial^\star + Im\,(p_1\partial)^\star + Im\,(p_2\partial D_1)^\star + \cdots + Im\,(p_r\partial D_{r-1})^\star\bigg), \end{aligned} \tag{3.27}$$

where $\ker\widetilde{\Delta}^{(r+1)}\oplus(Im\,\bar\partial+Im\,(\partial p_1)+Im\,(\partial D_1p_2)+\cdots+Im\,(\partial D_{r-1}p_r)) = \ker\bar\partial\cap\ker(p_1\partial)\cap\ker(p_2\partial D_1)\cap\cdots\cap\ker(p_r\partial D_{r-1})$ *and* $\ker\widetilde{\Delta}^{(r+1)}\oplus(Im\,\bar\partial^\star + Im\,(p_1\partial)^\star + Im\,(p_2\partial D_1)^\star + \cdots + Im\,(p_r\partial D_{r-1})^\star) = \ker\bar\partial^\star\cap\ker(\partial p_1)^\star\cap\ker(\partial D_1p_2)^\star\cap\cdots\cap\ker(\partial D_{r-1}p_r)^\star$.

For each $r\in\mathbb{N}^\star$, $p_r = p_r^{p,\,q}$ stands for the L^2_ω-orthogonal projection onto $\mathcal{H}^{p,\,q}_r$.

We will now cast the 3-space decomposition (3.27) in the terms used in the present chapter. Based on the terminology introduced in Definition 1.2.9, we defined the following vector spaces for every $r\in\mathbb{N}^\star$ and every bidegree $(p,\,q)$:

$$\begin{aligned} \mathcal{E}^{p,\,q}_{\partial,\,r} &:= \{\alpha\in C^\infty_{p,\,q}(X)\ |\ \partial\alpha \text{ reaches 0 in at most r steps}\}, \\ \mathcal{E}^{p,\,q}_{\bar\partial,\,r} &:= \{\beta\in C^\infty_{p,\,q}(X)\ |\ \bar\partial\beta \text{ reaches 0 in at most r steps}\}. \end{aligned}$$

When a Hermitian metric ω has been fixed on X and the adjoint operators $\partial^\star$ and $\bar\partial^\star$ with respect to ω have been considered, we define the analogous subspaces $\mathcal{E}^{p,\,q}_{\partial^\star,\,r}$ and $\mathcal{E}^{p,\,q}_{\bar\partial^\star,\,r}$ of $C^\infty_{p,\,q}(X)$ by replacing ∂ with $\partial^\star$ and $\bar\partial$ with $\bar\partial^\star$ in the definitions of $\mathcal{E}^{p,\,q}_{\partial,\,r}$ and $\mathcal{E}^{p,\,q}_{\bar\partial,\,r}$.

Part (ii) of Proposition 3.2.11 can be reworded as follows.

Proposition 3.2.12. *Let $(X,\,\omega)$ be a compact complex n-dimensional Hermitian manifold. For every $r\in\mathbb{N}^\star$ and for all $p,q\in\{0,\dots,n\}$, the following orthogonal 3-space decomposition (in which the sums inside the big parantheses need not be orthogonal or even direct) holds:*

$$C^\infty_{p,\,q}(X) = \mathcal{H}^{p,\,q}_r\oplus\bigg(Im\,\bar\partial + \partial(\mathcal{E}^{p-1,\,q}_{\bar\partial,\,r-1})\bigg)\oplus\bigg(\partial^\star(\mathcal{E}^{p+1,\,q}_{\bar\partial^\star,\,r-1}) + Im\,\bar\partial^\star\bigg), \tag{3.28}$$

where $\mathcal{H}^{p,\,q}_r$ is the E_r-harmonic space induced by ω (see §3.2.2 and earlier in this appendix) and the next two big parantheses are the spaces of E_r-exact $(p,\,q)$-forms, respectively $E_r^\star$-exact $(p,\,q)$-forms:

$$Im\,\bar\partial + \partial(\mathcal{E}^{p-1,\,q}_{\bar\partial,\,r-1}) = \mathcal{C}^{p,\,q}_r \quad \text{and} \quad \partial^\star(\mathcal{E}^{p+1,\,q}_{\bar\partial^\star,\,r-1}) + Im\,\bar\partial^\star = {}^\star\mathcal{C}^{p,\,q}_r.$$

Moreover, we have

$$\begin{aligned} \mathcal{Z}^{p,\,q}_r &= \mathcal{H}^{p,\,q}_r\oplus\bigg(Im\,\bar\partial + \partial(\mathcal{E}^{p-1,\,q}_{\bar\partial,\,r-1})\bigg) = \mathcal{H}^{p,\,q}_r\oplus\mathcal{C}^{p,\,q}_r, \\ {}^\star\mathcal{Z}^{p,\,q}_r &= \mathcal{H}^{p,\,q}_r\oplus\bigg(\partial^\star(\mathcal{E}^{p+1,\,q}_{\bar\partial^\star,\,r-1}) + Im\,\bar\partial^\star\bigg) = \mathcal{H}^{p,\,q}_r\oplus{}^\star\mathcal{C}^{p,\,q}_r \end{aligned}$$

where $\mathcal{Z}^{p,\,q}_r$ and ${}^\star\mathcal{Z}^{p,\,q}_r$ are the spaces of smooth E_r-closed, resp. $E_r^\star$-closed, $(p,\,q)$-forms.

3.3 Page-r-$\partial\bar\partial$-manifolds

The material in this section is taken from [PSU20a].

3.3.1 Definition of page-r-$\partial\bar\partial$-manifolds

Recall that X is a fixed n-dimensional compact complex manifold and $E_r^{p,\,q}(X)$ stands for the space of bidegree $(p,\,q)$ on the r-th page of the Frölicher spectral sequence of X.

Definition 3.3.1. *Fix $r \in \mathbb{N}^\star$ and $k \in \{0,\dots,2n\}$. We say that the* **identity induces an isomorphism** *between $\oplus_{p+q=k}E_r^{p,\,q}(X)$ and $H^k_{DR}(X,\,\mathbb{C})$ if the following two conditions are satisfied:*

*(a) for every bidegree $(p,\,q)$ with $p+q=k$, every class $\{\alpha^{p,\,q}\}_{E_r} \in E_r^{p,\,q}(X)$ contains a d-***closed representative of pure type** *$\alpha^{p,\,q} \in C^\infty_{p,\,q}(X)$;*

(b) the linear map

$$\bigoplus_{p+q=k} E_r^{p,\,q}(X) \ni \sum_{p+q=k}\{\alpha^{p,\,q}\}_{E_r} \mapsto \left\{\sum_{p+q=k}\alpha^{p,\,q}\right\}_{DR} \in H^k_{DR}(X,\,\mathbb{C})$$

is **well-defined** *(in the sense that it does not depend on the choices of d-closed representatives $\alpha^{p,\,q}$ of the classes $\{\alpha^{p,\,q}\}_{E_r}$) and* **bijective***.*

Moreover, if, for a fixed $r \in \mathbb{N}^\star$, the identity induces an isomorphism $\oplus_{p+q=k}E_r^{p,\,q}(X) \simeq H^k_{DR}(X,\,\mathbb{C})$ for every $k \in \{0,\dots,2n\}$, we say that the manifold X has the **E_r-Hodge Decomposition** *property.*

Note that whenever the identity induces a well-defined (not necessarily injective) linear map $E_r^{p,\,q}(X) \longrightarrow H^k_{DR}(X,\,\mathbb{C})$, the image of this map is $H^{p,\,q}_{DR}(X)$. Indeed, one inclusion is obvious. The reverse inclusion follows from the observation that any d-closed $(p,\,q)$-form defines an E_r-cohomology class (i.e. it is E_r-closed in the terminology of [Pop19]). Further note that whenever X has the E_r-Hodge Decomposition property, the Frölicher spectral sequence of X degenerates at E_r (at the latest).

Definition 3.3.2. *Fix $r \in \mathbb{N}^\star$ and $p,q \in \{0,\dots,n\}$. We say that the* **conjugation induces an isomorphism** *between $E_r^{p,\,q}(X)$ and the conjugate of $E_r^{q,\,p}(X)$ if the following two conditions are satisfied:*

*(a) every class $\{\alpha^{p,\,q}\}_{E_r} \in E_r^{p,\,q}(X)$ contains a d-***closed representative of pure type** *$\alpha^{p,\,q} \in C^\infty_{p,\,q}(X)$;*

(b) the linear map

$$E_r^{p,\,q}(X) \ni \{\alpha^{p,\,q}\}_{E_r} \mapsto \overline{\{\overline{\alpha^{p,\,q}}\}_{E_r}} \in \overline{E_r^{q,\,p}(X)}$$

is **well-defined** *(in the sense that it does not depend on the choice of d-closed representative $\alpha^{p,\,q}$ of the class $\{\alpha^{p,\,q}\}_{E_r}$) and* **bijective***.*

Moreover, if, for a fixed $r \in \mathbb{N}^\star$, the conjugation induces an isomorphism $E_r^{p,\,q}(X) \simeq \overline{E_r^{q,\,p}(X)}$ for every $p,q \in \{0,\dots,n\}$, we say that the manifold X has the **E_r-Hodge Symmetry** *property.*

We shall now see that the E_r-Hodge Decomposition property implies the E_r-Hodge Symmetry property. This follows from the following characterisation of the former property.

Theorem and Definition 3.3.3. *Let X be a compact complex manifold with $dim_{\mathbb{C}}X = n$. Fix an arbitrary $r \in \mathbb{N}^\star$. Then, the following two conditions are equivalent:*

(i) X has the **E_r-Hodge Decomposition** *property;*

(ii) the Frölicher spectral sequence of X **degenerates at** E_r *(we will denote this by $E_r(X) = E_\infty(X)$) and the De Rham cohomology of X is* **pure**.

A compact complex manifold X that satisfies any of the equivalent conditions (i) and (ii) is said to be a **page-$(r-1)$-$\partial\bar\partial$-manifold**.

Proof. $(i) \implies (ii)$ We have already noticed that the E_r-Hodge Decomposition property implies $E_r(X) = E_\infty(X)$ and that the image of each $E_r^{p,\,q}(X)$ in $H^{p+q}_{DR}(X,\,\mathbb{C})$ under the map induced by the identity is $H^{p,\,q}_{DR}(X)$. We get (ii).

$(ii) \implies (i)$ Since the De Rham cohomology of X is supposed pure, we know from Proposition 1.2.19 that $E_\infty^{p,\,q}(X) \simeq H^{p,\,q}_{DR}(X)$ (isomorphism induced by the identity) for all bidegrees $(p,\,q)$. On the other hand, $E_\infty^{p,\,q}(X) = E_r^{p,\,q}(X)$ for all bidegrees $(p,\,q)$ since we are assuming that $E_r(X) = E_\infty(X)$. Combined with the De Rham purity assumption, these facts imply that X has the E_r-Hodge Decomposition property. □

Corollary 3.3.4. *Any page-$(r-1)$-$\partial\bar\partial$-manifold has the E_r-Hodge Symmetry property.*

Proof. We have already noticed in (1.31) that the conjugation (trivially) induces an isomorphism between any space $H^{p,\,q}_{DR}(X)$ and the conjugate of $H^{q,\,p}_{DR}(X)$. Meanwhile, we have seen that the page-$(r-1)$-$\partial\bar\partial$-assumption implies that the identity induces an isomorphism between any space $E_r^{p,\,q}(X)$ and $H^{p,\,q}_{DR}(X)$. Hence, the conjugation induces an isomorphism between any space $E_r^{p,\,q}(X)$ and the conjugate of $E_r^{q,\,p}(X)$. □

Another obvious consequence of (ii) of Theorem and Definition 3.3.3 is that the page-r-$\partial\bar\partial$-property becomes weaker and weaker as r increases.

Corollary 3.3.5. *Let X be a compact complex manifold. Then, for every $r \in \mathbb{N}^\star$, the following implication holds:*

$$X \text{ is a page-}r\text{-}\partial\bar\partial\text{-manifold} \implies X \text{ is a page-}(r+1)\text{-}\partial\bar\partial\text{-manifold}.$$

Indeed, the purity of the De Rham cohomology is independent of r, while the property $E_r(X) = E_\infty(X)$ obviously implies $E_{r+1}(X) = E_\infty(X)$ for every $r \in \mathbb{N}$.

3.3.2 Characterisation in terms of squares and zigzags

The goal of this subsection is to relate the page-r-$\partial\bar\partial$-property to structural results about double complexes. Specifically, we work here with arbitrary double complexes, i.e. bigraded vector spaces $A = \bigoplus_{p,q\in\mathbb{Z}} A^{p,q}$ with endomorphisms ∂_1, ∂_2 of bidegrees $(1,0)$, resp. $(0,1)$, satisfying $d^2 = 0$ for $d = \partial_1 + \partial_2$. This degree of generality has the advantage of emphasising which aspects of the theory are purely algebraic. Even if one is only interested in the complex $A_X := (C^\infty_{p,q}(X), \partial, \bar\partial)$ of $\mathbb{C}$-valued forms on a complex manifold X, in the more general setting one can consider certain finite-dimensional subcomplexes on an equal footing.

There are now two Frölicher-style spectral sequences, starting from column, i.e. (∂_2-), resp. row, i.e. (∂_1-), cohomology and converging to the total (De Rham) cohomology of $(A,\,d)$. We denote them by

$${}_iE_r^{p,q}(A) \Longrightarrow (H^{p+q}_{DR}(A), F_i) \quad i = 1, 2.$$

In the case $A = A_X$, the case $i = 1$ is the Frölicher spectral sequence and $i = 2$ its conjugate.

The following is a minor extension to general double complexes of the definition (based on its second characterisation) of the page-$(r-1)$-$\partial\bar{\partial}$-property of manifolds.

Definition 3.3.6. *A double complex A is said to satisfy the* **page-$(r-1)$-$\partial_1\partial_2$-property** *if both Frölicher spectral sequences degenerate at page r and the De Rham cohomology is pure.*

Just as before, one can also see that this property is equivalent to the statement that for $i = 1, 2$, every ${}_iE_r^{p,q}(X)$-class contains a $(\partial_1 + \partial_2)$-closed representative and the corresponding map

$$\bigoplus_{p+q=k} {}_iE_r^{p,q}(A) \longrightarrow H_{DR}^k(A)$$

induced by the identity is well-defined and bijective.

The following observation will motivate the subsequent considerations.

Observation 3.3.7. *The Frölicher spectral sequences, as well as H_{DR}, H_A and H_{BC}, are compatible with direct sums. In particular, if $A = B \oplus C$, then A satisfies the page-r-$\partial_1\partial_2$-property if and only if B and C do.*

Recall that a (nonzero) double complex A is called **indecomposable** if there exists no nontrivial decomposition $A = B \oplus C$ into subcomplexes B, C, while A is called **bounded** if $A^{p,q} \neq 0$ for only finitely many bidegrees (p, q).

Theorem 3.3.8. *([Ste20, KQ19]) For every bounded double complex over a field K, there exists an isomorphism*

$$A \cong \bigoplus_C C^{\oplus \operatorname{mult}_C(A)},$$

where C runs over a set of representatives for the isomorphism classes of bounded **indecomposable** *double complexes and $\operatorname{mult}_C(A)$ are cardinal numbers uniquely determined by A.*

Moreover, each bounded **indecomposable** *double complex is isomorphic to a complex of one of the following types:*

- **square***: a double complex generated by a single pure-(p,q)-type element a in a given bidegree with no further relations:*

$$\begin{array}{ccc} \langle \partial_2 a \rangle & \longrightarrow & \langle \partial_2\partial_1 a \rangle \\ \uparrow & & \uparrow \\ \langle a \rangle & \longrightarrow & \langle \partial_1 a \rangle. \end{array}$$

- **even-length zigzag of type 1 and length $2l$.** *This is a complex generated by elements $a_1, \ldots a_l$ and their differentials such that $\partial_2 a_1 = 0$ and $\partial_1 a_1 = -\partial_2 a_2$, $\partial_1 a_2 = -\partial_2 a_3$, ..., $\partial_1 a_{l-1} = -\partial_2 a_l$, $\partial_1 a_l \neq 0$. It is of the shape:*

$$\begin{array}{ccccccc}
\langle a_1 \rangle & \longrightarrow & \langle \partial_1 a_1 \rangle & & & & \\
 & & \uparrow & & & & \\
 & & \langle a_2 \rangle & \longrightarrow & & & \\
 & & & & ... & & \\
 & & & & & & \\
 & & & & \uparrow & & \\
 & & & & \langle a_l \rangle & \longrightarrow & \langle \partial_1 a_l \rangle .
\end{array}$$

Here, as in all the following examples, the length of a zigzag is the number of its vertices.

- **even-length zigzag of type** 2 **and length** $2l$. *This is a complex of the shape:*

$$\begin{array}{ccccc}
\langle \partial_2 a_1 \rangle & & & & \\
\uparrow & & & & \\
\langle a_1 \rangle & \longrightarrow & \langle \partial_1 a_1 \rangle & & \\
 & & \uparrow & & \\
 & & \langle a_2 \rangle & \longrightarrow & \\
 & & & & ... \\
 & & & & \\
 & & & & \uparrow \\
 & & & & \langle a_l \rangle .
\end{array}$$

- **odd-length zigzag of type** M **and length** $2l+1$. *This is a complex generated by elements* $a_1, ..., a_{l+1}$ *such that* $\partial_1 a_i = -\partial_2 a_{i+1}$, $\partial_2 a_1 = 0$ *and* $\partial_1 a_{l+1} = 0$. *It has the shape:*

$$\begin{array}{ccccc}
\langle a_1 \rangle & \longrightarrow & \langle \partial_1 a_1 \rangle & & \\
 & & \uparrow & & \\
 & & \langle a_2 \rangle & \longrightarrow & \\
 & & & & ... \\
 & & & & \\
 & & & & \uparrow \\
 & & & & \langle a_{l+1} \rangle .
\end{array}$$

The special case where $l = 0$ *is also called a* **dot**.

- **odd-length zigzag of type** L **and length** $2l+1$ $(l > 0)$. *This is a complex generated by elements* $a_1, ..., a_l$ *such that both* $\partial_2 a_1 \neq 0 \neq \partial_1 a_l$ *and* $\partial_1 a_i = -\partial_2 a_{i+1}$. *It has the shape:*

$$\begin{tikzcd} \langle \partial_2 a_1 \rangle & \\ \langle a_1 \rangle \arrow[u] \arrow[r] & \langle \partial_1 a_1 \rangle \end{tikzcd}$$

$$\dots$$

$$\begin{tikzcd} \langle \partial_2 a_l \rangle & \\ \langle a_l \rangle \arrow[u] \arrow[r] & \langle \partial_1 a_l \rangle. \end{tikzcd}$$

It is a useful exercise to work out which indecomposable complexes satisfy the page-r-$\partial_1\partial_2$-property. Doing it and combining it with Observation 3.3.7, one gets

Theorem 3.3.9. *Let A be a bounded double complex over a field K. The following are equivalent:*

1. *A satisfies the page-r-$\partial_1\partial_2$-property.*
2. *There exists an isomorphism between A and a direct sum of squares, even-length zigzags of length $\leq 2r$ and odd-length zigzags of length one (i.e. dots).*

Proof. It follows at once from [Ste20, Thm C, Prop. 6, Cor. 7] as pointed out above. ☐

Remark 3.3.10. *This theorem also gives a quick alternative proof to Prop. 3.3.22 (equivalence of page-0-$\partial_1\partial_2$ with the usual $\partial_1\partial_2$-property) of the next subsection.*

Indeed, the page-0-$\partial_1\partial_2$-property means that there is a decomposition of A into squares and dots. Obviously, both satisfy the usual $\partial_1\partial_2$-property. Conversely, in any zigzag of length ≥ 2, there is a closed element ('form') of pure type, which is ∂_1- or ∂_2-exact, but no nonzero element in a zigzag is $\partial_1\partial_2$-exact. Hence, if A satisfies the usual $\partial_1\partial_2$-property, in any decomposition of A into elementary complexes only squares and length-one zigzags can occur.

Definition 3.3.11. *A map $A \longrightarrow B$ of double complexes is an E_r-**isomorphism** if ${}_iE_r(f)$ is an isomorphism for $i \in \{1, 2\}$.*

One writes $A \simeq_r B$ if there exist such an E_r-isomorphism. The usefulness of this notion stems from the following

Lemma 3.3.12. *([Ste20, Prop. 12]) If H is a linear functor from the category of double complexes to the category of vector spaces which maps squares and even-length zigzags of length $\leq 2r$ to 0, then $H(f)$ is an isomorphism for any E_r-isomorphism f.*

Lemma 3.3.13. *([Ste20, Prop. 11]) For two double complexes A, B one has $A \simeq_1 B$ if and only if 'the same zigzags occur in A and B', i.e. $\mathrm{mult}_Z(A) = \mathrm{mult}_Z(B)$ for all zigzags Z.*

Example 3.3.14. *Examples of functors H satisfying the hypotheses of Lemma 3.3.12 are provided by H_{DR}, $H_{BC}^{p,q}$, $E_r^{p,q}$ or $H_A^{p,q}$.*

By their explicit description above, one sees that an indecomposable double complex C is determined up to isomorphism by its shape $S(C) = \{(p,q) \in \mathbb{Z}^2 \mid C^{p,q} \neq 0\}$. Abusing notation slightly, it is sometimes convenient to write $\mathrm{mult}_S(A)$ instead of $\mathrm{mult}_C(A)$, when $S = S(C)$.

We will need the following duality results in the special case $A = A_X$, which follow from the real structure and the Serre duality.

Lemma 3.3.15. *([Ste20, Ch. 4]) Let $A = A_X$ for a compact complex manifold X and define the conjugate complex by $\bar{A}^{p,q} = \overline{A^{p,q}}$ and the dual complex DA by $DA^{p,q} = \mathrm{Hom}(A^{n-p,n-q}, \mathbb{C})$, for all p, q.*

Then, conjugation $\omega \mapsto \overline{\omega}$ and integration $\omega \mapsto \int_X \omega \wedge \_$ define an isomorphism, resp. an E_1-isomorphism: $A \cong \bar{A}$, resp. $A \to DA$.

In particular, the set of zigzags occuring in A_X is symmetric under reflection along the diagonal and the antidiagonal. More precisely, for any zigzag shape S, $\mathrm{mult}_S(A) = \mathrm{mult}_{rS}(A) = \mathrm{mult}_{dS}(A)$, where $rS = \{(p,q) \in \mathbb{Z}^2 \mid (q,p) \in S\}$ and $dS := \{(p,q) \in \mathbb{Z}^2 \mid (n-p, n-q) \in S\}$.

As a consequence, we obtain

Proposition 3.3.16. *Fix arbitrary integers $0 \leq k \leq 2n$.*

A compact complex manifold X of dimension n satisfies the complex-$\mathcal{C}^\infty$-pure property in degree k if and only if it satisfies the complex-$\mathcal{C}^\infty$-full property in degree $2n - k$

Proof. Let Z be a zigzag with $H^k_{DR}(Z) \neq 0$. The sum of the subspaces $H^{p,q}_{DR}(Z)$ with $p + q = k$ is not direct if and only if Z is of odd length and of type L. Meanwhile, the sum of the subspaces $H^{p,q}_{DR}(Z)$ with $p + q = k$ is strictly contained in $H^k_{DR}(Z)$ if and only if Z is of odd length > 1 (i.e. not a dot) and of type M. (See [Ste20, Prop. 6, Cor. 7] for both of these statements.).

Hence, X is complex-$\mathcal{C}^\infty$-pure in degree k if and only if $\mathrm{mult}_Z(A_X) = 0$ for all odd zigzags Z of type L with $H^k_{DR}(Z) \neq 0$ and X is complex $\mathcal{C}^\infty$-full in degree k if and only if $\mathrm{mult}_Z(A_X) = 0$ for all odd zigzags Z of type M and length > 1 with $H^k_{DR}(Z) \neq 0$.

The result then follows from Lemma 3.3.15 and Lemma 3.3.13 since zigzags of type L and those of type M and length greater than 1 are exchanged when forming the dual complex.

□

Corollary 3.3.17. *For a compact complex manifold X, the following statements are equivalent:*

1. *X satisfies the complex-$\mathcal{C}^\infty$-pure property in all degrees;*

2. *X satisfies the complex-$\mathcal{C}^\infty$-full property in all degrees;*

3. *The De Rham cohomology of X is pure (in the sense of Definition 1.2.17).*

3.3.3 Numerical characterisation of page-r-$\partial\bar{\partial}$-manifolds and applications

Let X be a compact connected complex manifold. Let $b(X) = \sum_{k \in \mathbb{Z}} b_k(X)$, $h_{BC}(X) = \sum_{p,q \in \mathbb{Z}} h^{p,q}_{BC}(X)$ and define $h_A(X)$, $h_\partial(X)$ and $h_{\bar{\partial}}(X)$ analogously. Angella and Tomassini showed in [AT13] that there are inequalities:

$$h_{BC}(X) + h_A(X) \overset{(*)}{\geq} h_{\bar{\partial}}(X) + h_\partial(X) \overset{(**)}{\geq} 2\, b(X) \tag{3.29}$$

and that both of these inequalities are equalities if and only if X is a $\partial\bar{\partial}$-manifold.

It is a standard fact about spectral sequences that equality in $(**)$ is equivalent to the degeneration at E_1 of the Frölicher spectral sequence (and its conjugate). One application of our methods is a generalisation of inequality $(*)$ and a characterisation of the equality case in terms of our new classes of manifolds introduced in this paper. The following general statement is new.

Theorem 3.3.18. *For every compact complex manifold X and for every $r \in \mathbb{N}^\star$, there is an inequality:*

$$h_{BC}(X) + h_A(X) \geq 2\left(\sum_{i=1}^{r} e_i(X) - (r-1)b(X)\right),$$

where $e_i := \sum_{p,q\in\mathbb{Z}} \dim E_i^{p,q}(X)$. Moreover, equality holds for some fixed $r \in \mathbb{N}^\star$ if and only if X is a page-r-$\partial\bar{\partial}$-manifold.

Remark 3.3.19. *Since $h_{BC}^{p,q}(X) = h_A^{n-p,n-q}(X)$ by duality, one gets $h_{BC}(X) = h_A(X)$ and conjugation yields $h_\partial(X) = h_{\bar{\partial}}$. Therefore, one can replace (3.29) by the equivalent inequalities $h_{BC}(X) \geq h_{\bar{\partial}}(X) \geq b(X)$ and have the same characterisations for the equality cases.*

For a concrete (non-nilmanifold) instance where this characterisation can be applied, consider $G := \mathbb{C} \ltimes_\phi \mathbb{C}^2$, where ϕ is either

$$\phi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} \qquad \text{or} \qquad \phi(z) = \begin{pmatrix} e^{\mathrm{Re}(z)} & 0 \\ 0 & e^{-\mathrm{Re}(z)} \end{pmatrix}$$

(*complex parallelizable*, resp. *completely solvable* case) and define X to be the quotient of G by a lattice of the form $\Gamma \ltimes_\phi \Gamma'$ with $\Gamma \subset \mathbb{C}$, $\Gamma' \subset \mathbb{C}^2$ lattices. These were studied in [Nak75] and are called *Nakamura manifolds*. They are among the best known *solvmanifolds*, but are *not nilmanifolds*. In [AK17], Angella and Kasuya computed the Hodge, Bott-Chern and Aeppli numbers for certain families of lattices Γ. (These numbers turn out to be independent of Γ'). In particular, their calculations yield the equality $h_{BC}(X) = h_{\bar{\partial}}(X)$. Hence, by Theorem 3.3.18, we get

Corollary 3.3.20. *The complex parallelisable and completely solvable* **Nakamura manifolds** *considered in [AK17] are* **page-1-$\partial\bar{\partial}$-manifolds**.

Using the upper-semicontinuity of h_{BC} and h_A in families of manifolds, we infer the stability of page-1-$\partial\bar{\partial}$-manifolds with fixed Hodge numbers under small deformations of the complex structure from Theorem 3.3.18. The analogous statement for $r \geq 2$ and constant e_i with $i \leq r$ also holds.

Corollary 3.3.21. *If X_0 is a* **page-1-$\partial\bar{\partial}$-manifold**, *then every sufficiently small deformation X_t of X_0 which satisfies $h_{\bar{\partial}}(X_t) = h_{\bar{\partial}}(X_0)$ is again* **page-1-$\partial\bar{\partial}$**.

If one drops the condition on constant Hodge numbers, one cannot say much in general. In fact, as we will see, the Iwasawa manifold is page-1-$\partial\bar{\partial}$, but any small deformation with different Hodge numbers is not.

In order to prove Theorem 3.3.18, we will work with abstract bounded double complexes rather than double complexes of forms and prove the following (more general) statement.

For a bounded double complex A with finite-dimensional cohomology, let ${}_1e_r(A)$, resp. ${}_2e_r(A)$, be the total dimension of the r-th page of the row, resp. column, spectral sequence. There is always an inequality:

$$h_{BC}(A) + h_A(A) \geq \sum_{i=1}^{r} ({}_1e_i(A) + {}_2e_i(A)) - 2(r-1)b(A)$$

and the equality is equivalent to the page-r-$\partial_1\partial_2$ property for A.

Proof of Theorem 3.3.18. Let us compute the quantities

$$LHS := h_A + h_{BC} \quad \text{and} \quad RHS_r := \sum_{i=1}^{r} ({}_1e_i + {}_2e_i)) - 2(r-1)b$$

individually for every possible indecomposable double complex, i.e. for squares and all zigzags. The result then follows from the additivity of both quantities under direct sums.

Before spelling out the details, let us say that LHS counts all the zigzags, weighted by their length, and counts the dots twice. When $r = 1$, RHS_1 counts all the zigzags twice. For an arbitrary r, the count on the right becomes slightly more involved.

For a square S, one has $h_A(S) = h_{BC}(S) = {}_1e_i(S) = {}_1e_i(S) = b(S) = 0$, so $LHS(S) = RHS_r(S) = 0$ for any r.

For a dot D, one has $h_A(D) = h_{BC}(D) = {}_1e_i(D) = {}_2e_i(D) = b(S) = 1$, so $LHS(D) = RHS_r(D)$ for any r.

For any other zigzag Z of length $l \geq 2$, one has $LHS(Z) = l$, while RHS depends on the parity of the length l. For l odd, one has ${}_1e_i(Z) = {}_2e_i(Z) = 1$ for all i and $b(Z) = 1$, so $RHS_r(Z) = 2$. In particular, the inequality is always strict for odd $l > 2$. If $l = 2k$ is even, then $b(Z) = 0$ and

$${}_1e_i(Z) + {}_2e_i(Z) = \begin{cases} 2 & \text{for } i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$RHS_r(Z) = \min\{2r, l\}.$$

Thus, we get the inequality. Moreover, equality holds if and only if $l \leq 2r$. This completes the proof. □

3.3.4 Examples of page-r-$\partial\bar{\partial}$-manifolds and counterexamples

We shall organise our examples in several classes, each flagged by a specific heading.

(I) Case $r = 0$ and low dimensions

The first observation is the following rewording of (5.21) in [DGMS75]. See also Theorem 1.3.2.

Proposition 3.3.22. *For any compact complex manifold X, the following equivalence holds:*

$$X \textit{ is a } \partial\bar{\partial}\textbf{-manifold} \iff X \textit{ is a } \textbf{page-0-}\partial\bar{\partial}\textbf{-manifold}.$$

In dimensions one and two, it follows from well-known results that the only possible examples of page-r-$\partial\bar{\partial}$-manifolds are Kähler:

Observation 3.3.23. *Any compact complex curve is Kähler, hence a $\partial\bar{\partial}$-manifold. A compact complex surface is a page-r-$\partial\bar{\partial}$-manifold (for some r) if any only if it is Kähler.*

Proof. It is standard that the Frölicher spectral sequence of any compact complex surface degenerates at E_1. It is equally standard that H^k_{DR} is always pure for $k = 0, 2, 4$, while it follows from the Buchdahl-Lamari results (see [Buc99] and [Lam99]) that H^1_{DR} (and hence H^3_{DR}) is pure iff the surface is Kähler. $\square$

(II) Case of the Iwasawa manifold and its small deformations

Recall that the Iwasawa manifold $I^{(3)}$ is the nilmanifold of complex dimension 3 obtained as the quotient of the Heisenberg group of 3×3 upper triangular matrices with entries in $\mathbb{C}$ by the subgroup of those matrices with entries in $\mathbb{Z}[i]$.

It is well known that the Iwasawa manifold is not a $\partial\bar\partial$-manifold. In fact, its Frölicher spectral sequence is known to satisfy $E_1 \neq E_2 = E_\infty$. On the other hand, it is known that the De Rham cohomology of the Iwasawa manifold can be generated in every degree by De Rham classes of (d-closed) pure-type forms. (See e.g. [Ang14].) Together with Cor. 3.3.17 this yields

Proposition 3.3.24. *The* **Iwasawa manifold** *is a* **page-1-$\partial\bar\partial$-manifold**.

However, the situation is more complex for the small deformations of the Iwasawa manifold, all of which are already known to not be $\partial\bar\partial$-manifolds. The following result shows, in particular, that unlike the $\partial\bar\partial$-property, the page-1-$\partial\bar\partial$-property is *not deformation open.*

Proposition 3.3.25. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold X_0. For every $t \in B$, we have:*

(i) X_t is a **page-1-$\partial\bar\partial$-manifold** *if and only if X_t is* **complex parallelisable** *(i.e. lies in Nakamura's class (i));*

(ii) if X_t lies in one of Nakamura's classes (ii) or (iii), the De Rham cohomology of X_t is **not pure**, *so X_t is* **not** *a* **page-r-$\partial\bar\partial$-manifold** *for any $r \in \mathbb{N}$.*

Proof. That deformations in Nakamura's class (i) are page-1-$\partial\bar\partial$-manifolds can be proved in the same way as the Iwasawa manifold was proved to have this property in Proposition 3.3.24. This fact also follows from the far more general Proposition 3.3.28 since all the small deformations X_t of X_0 are nilmanifolds.

To show (ii), we will actually prove a slightly more general result. Calculations of Angella [Ang14] show that the hypotheses of the next Lemma are satisfied in this case. $\square$

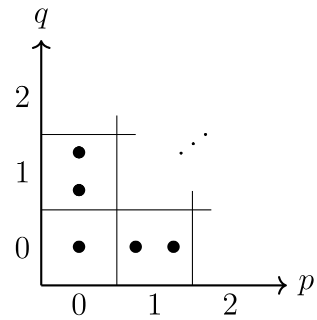
Lemma 3.3.26. *Let X be a compact complex manifold with $b_1 = 4$, $h^{1,0}_{\bar\partial} = h^{0,1}_{\bar\partial} = 2$ and $h^{1,0}_A = 3$. Then, either $H^1_{DR}(X,\mathbb{C})$ or $H^2_{DR}(X,\mathbb{C})$ is not pure.*

Proof. The proof is combinatorial. We first give a proof using the notation of [Ste20] and then a more pictorial proof that is hopefully clearer without having read all of [Ste20]. We will exploit the fact that the De Rham, Dolbeault and Aeppli cohomologies of indecomposable complexes are computable. This is spelt out in detail in [Ste20]. Summarised briefly, an even-length zigzag has a nonzero differential in the Frölicher spectral sequence or its conjugate, but has no De Rham cohomology. Meanwhile, odd-length zigzags have no differentials in the Frölicher spectral sequence, but have a nonzero De Rham cohomology and $h^{p,q}_A$ counts the zigzags that have a nonzero component in degree (p, q) with possibly outgoing but no incoming arrows.

- *The formal proof.* Denote by $A = (C^\infty_{p,q}(X), \partial, \bar\partial)$ the double complex of $\mathbb{C}$-valued forms on X. We investigate for which zigzag shapes S with $(1, 0) \in S$ or $(0, 1) \in S$, one can have $\mathrm{mult}_S(A) \neq 0$.

Assume $H^1_{DR}(X,\mathbb{C})$ is pure. This is equivalent to $\operatorname{mult}_{S_1^{p,q}}(A)=0$ unless $p+q=1$, i.e. $(p,q)=(1,0)$ or $(p,q)=(0,1)$, in which case $\operatorname{mult}_{S_1^{1,0}}(A)=\operatorname{mult}_{S_1^{0,1}}(A)=2$. Since $h^{1,0}_{\bar\partial}+h^{0,1}_{\bar\partial}=b_1$, there are no differentials in the Frölicher spectral sequences starting or ending in $(1,0)$ or $(0,1)$, i.e., one has $\operatorname{mult}_S(A)=0$ for all even-length zigzag shapes such that $(1,0)\in S$ or $(0,1)\in S$. The only possible zigzag shapes containing $(1,0)$ or $(0,1)$ that are left are $S_2^{1,2}, S_2^{2,1}$ and $S_2^{2,2}$. One of the latter two has to occur since otherwise we would have $h^{1,0}_A=2$. But this means that $H^2_{DR}(X,\mathbb{C})$ is not pure!

• *The pictorial proof.* Choose any decomposition of the double complex A into squares and zigzags and assume that $H^1_{DR}(X,\mathbb{C})$ is pure. This means that any zigzag contributing to the De Rham cohomology $H^1_{DR}(X,\mathbb{C})$ is of length one, i.e. drawing only the odd zigzags and leaving out the squares and the even zigzags (which do not contribute to the De Rham cohomology), the lower part of the double complex looks like this:



Here, a $\bullet$ denotes a dot, i.e. a zigzag of length one and multiplicity one. The symmetry along the diagonal comes from the real structure of A given by complex conjugation. A priori, there may be other zigzags passing through $(1,0)$ and $(0,1)$. Schematically, these would all arise by choosing some connected subgraph with at least one arrow of the diagram

$$\begin{tikzcd}
\bullet & & \\
\bullet^{0,1} \arrow[u, dashed, "\bar\partial"] \arrow[r, dashed, "\partial"] & \bullet & \\
& \bullet^{1,0} \arrow[u, dashed, "\bar\partial"] \arrow[r, dashed, "\partial"] & \bullet
\end{tikzcd}$$

They could either be of even-length or of odd-length and not contributing to $H^1_{DR}(X,\mathbb{C})$ but contributing to $H^2(X,\mathbb{C})$. Note that the subdiagram

$$\begin{tikzcd}
\bullet^{1,0} \arrow[r, "\partial"] & \bullet \\
& \bullet^{0,1} \arrow[u, "\bar\partial"]
\end{tikzcd}$$

is not allowed since this would give rise to a non-pure class in $H^1_{DR}(X,\mathbb{C})$, i.e. a class which does not lie in $H^{1,0}_{DR}(X)+H^{0,1}_{DR}(X)$). But we have already ruled this out by purity.

However, since $h^{1,0}_{\bar\partial}+h^{0,1}_{\bar\partial}=b_4$, there can be no differentials in the Frölicher spectral sequence starting or ending in degree $(1,0)$ or $(0,1)$. In terms of zigzags, this means no even-length zigzag passes through these bidegrees. This rules out the zigzags

$$\begin{tikzcd}
\bullet^{0,1} \arrow[r, "\partial"] & \bullet & \\
& \bullet^{1,0} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \bullet
\end{tikzcd}
\qquad
\bullet \xrightarrow{\partial} \bullet^{1,0}
\qquad
\bullet^{1,0} \xrightarrow{\partial} \bullet$$

and their reflections along the diagonal (which have to occur with the same multiplicity since A is equipped with a real structure). So, the only options for zigzags passing through $(1,0)$ that are left are

$$\begin{tikzcd}
\bullet & & \\
\bullet^{0,1} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \bullet & \\
& \bullet^{1,0} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \bullet
\end{tikzcd}
\qquad \text{or} \qquad
\begin{tikzcd}
\bullet & \\
\bullet^{1,0} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \bullet
\end{tikzcd}$$

and one of these has to occur since otherwise $H_A^{1,0}$ would be of dimension 2, contradicting the assumptions. But the occurence of either one implies that $H^2_{DR}(X,\mathbb{C})$ is not pure. □

(III) Case of complex parallelisable nilmanifolds

We will now prove that all complex parallelisable nilmanifolds are page-1-$\partial\bar\partial$-manifolds. On the one hand, this generalises one implication in (i) of Proposition 3.3.25. On the other hand, it provides a large class of page-1-$\partial\bar\partial$-manifolds that are not $\partial\bar\partial$-manifolds. Indeed, it is known that a nilmanifold $\Gamma\backslash G$ is never $\partial\bar\partial$ (or even formal in the sense of [DGMS75]) unless it is Kähler (i.e. a complex torus, or equivalently, the Lie group G is abelian).

Recall that a compact complex parallelisable manifold X is a manifold whose holomorphic tangent bundle is trivial. By Wang's theorem [Wan54], X is a quotient $\Gamma\backslash G$ of a *complex* Lie group G by a co-compact, discrete subgroup Γ. When G is nilpotent, the manifold X is a complex parallelisable nilmanifold. The Iwasawa manifold is an example of this type. We first need an algebraic result.

Lemma 3.3.27. *Let $(A^\bullet, d_A)$ and $(B^\bullet, d_B)$ be two complexes of vector spaces and $C = A\otimes B$ their tensor product, considered as a double complex, i.e.:*

$$\begin{aligned}
C^{p,q} &:= A^p \otimes B^q\\
\partial_1(a\otimes b) &:= d_A a \otimes b\\
\partial_2(a\otimes b) &:= (-1)^{|a|} a\otimes d_B b
\end{aligned}$$

Then C satisfies the page-1-$\partial_1\partial_2$-property.

Proof. First, we compute the first and second pages of the column Frölicher spectral sequence. (We only treat the column case, the row case being analogous.) The first page is the column cohomology:

$$({}_1E_1^{\bullet,\bullet}, d_1) = (H^q(C^{p,\bullet}, \partial_2), \partial_1)$$

Since ∂_2 is, up to sign, $\mathrm{Id}_A \otimes d_B$, one has $H^q(C^{p,\bullet},\partial_2) = A^p \otimes H^q(B, d_B)$ and $d_1 = d_A \otimes \mathrm{Id}_{H(B)}$. Therefore, ${}_2E_2^{p,q} = H^p(A, d_A)\otimes H^q(B, d_B)$. Now, for every d_A-closed element $a \in A^p$ and every d_B-closed element $b\in B^q$, the element $a\otimes b \in C^{p+q}$ is $d = \partial_1 + \partial_2$ closed. Similarly, if one of the two is d_A or d_B exact, the form $a\otimes b$ will be d-exact. Hence we get a natural map $\bigoplus_{p+q=k} H^p(A, d_A) \otimes H^q(B, d_B) \to H^k_{dR}(C)$. Since we are working over a field, the Künneth formula tells us that this is an isomorphism. □

Given a complex parallelisable nilmanifold $\Gamma\backslash G$, let $\mathfrak{g}$ be the (real) Lie algebra of G, and denote by $J\colon \mathfrak{g} \to \mathfrak{g}$ the endomorphism induced by the complex structure of the Lie group G. Then $J^2 = -\mathrm{Id}$ and

$$[Jx, y] = J[x, y], \tag{3.30}$$

for all $x, y \in \mathfrak{g}$. Let $\mathfrak{g}^*_{\mathbb{C}}$ be the dual of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ and denote by $\mathfrak{g}^{1,0}$ (respectively $\mathfrak{g}^{0,1}$) the eigenspace of the eigenvalue i (resp. $-i$) of J considered as an endomorphism of $\mathfrak{g}^*_{\mathbb{C}}$. Condition (3.30) is equivalent to $[\mathfrak{g}^{0,1}, \mathfrak{g}^{1,0}] = 0$ which is equivalent to $d(\mathfrak{g}^{1,0}) \subset \bigwedge^2(\mathfrak{g}^{1,0})$, i.e. there is no component of bidegree $(1,1)$. Therefore, $\bar\partial$ is identically zero on $\bigwedge^p(\mathfrak{g}^{1,0})$, and ∂ is identically zero on $\bigwedge^q(\mathfrak{g}^{0,1})$, that is,

$$\partial_{|_{\bigwedge^p(\mathfrak{g}^{1,0})}} = d_{|_{\bigwedge^p(\mathfrak{g}^{1,0})}}, \qquad \bar\partial_{|_{\bigwedge^p(\mathfrak{g}^{1,0})}} = 0, \qquad \partial_{|_{\bigwedge^q(\mathfrak{g}^{0,1})}} = 0, \qquad \text{and} \qquad \bar\partial_{|_{\bigwedge^q(\mathfrak{g}^{0,1})}} = d_{|_{\bigwedge^q(\mathfrak{g}^{0,1})}}. \tag{3.31}$$

Theorem 3.3.28. *Complex parallelisable nilmanifolds are page-1-$\partial\bar\partial$-manifolds.*

Proof. Sakane [Sak76] showed that the inclusion of the double complex $(\bigwedge^{\bullet,\bullet}\mathfrak{g}^*_{\mathbb{C}}, \partial, \bar\partial)$ as left invariant forms into the complex of all forms on $\Gamma\backslash G$ induces an isomorphism of the respective first pages of the corresponding Frölicher spectral sequences (hence of all later pages). But the equations (3.31) mean that the double complex $(\bigwedge^{\bullet,\bullet}\mathfrak{g}^*_{\mathbb{C}}, \partial, \bar\partial)$ is the tensor product of the simple complexes $(\bigwedge^{\bullet}\mathfrak{g}^{1,0}, d)$ and $(\bigwedge^{\bullet}, \mathfrak{g}^{0,1}, d)$, so we can apply Lemma 3.3.27. □

(IV) Nilmanifolds with abelian complex structures

In this subsection, we construct two classes of page-1-$\partial\bar\partial$-manifolds which are not biholomorphic to complex parallelisable nilmanifolds (see Remarks 3.3.30 and 3.3.32). Indeed, they are nilmanifolds endowed with an invariant complex structure that is *abelian*, which means that, in contrast to (3.31), ∂ vanishes on left-invariant $(p,0)$-forms, i.e. $\partial_{|_{\bigwedge^p(\mathfrak{g}^{1,0})}} = 0$.

Theorem 3.3.29. *Let $n \geq 3$ and G be the nilpotent Lie group with abelian complex structure defined by the structure equations*

$$(Ab1_n)\quad d\omega^1 = 0,\ d\omega^2 = 0,\ d\omega^3 = \omega^2 \wedge \overline{\omega^1}, \ldots,\ d\omega^n = \omega^{n-1} \wedge \overline{\omega^1},$$

or

$(Ab2_n)$ $\quad d\omega^1 = 0, \ldots,\ d\omega^{n-1} = 0,\ d\omega^n = \omega^1 \wedge \overline{\omega^2} + \omega^3 \wedge \overline{\omega^4} + \cdots + \omega^{n-2} \wedge \overline{\omega^{n-1}}$ *(only for odd $n \geq 3$).*

Then, any nilmanifold $\Gamma\backslash G$ is a page-1-$\partial\bar\partial$-manifold.

Proof. Recall that for abelian complex structures, just as for complex parallelisable ones, Dolbeault, Aeppli and Bott-Chern cohomology groups can be computed via left-invariant forms (see e.g. [LU15] and the references therein).

We consider first a complex structure defined by $(Ab1_n)$. Denote by A the exterior algebra over the vector space $\langle \omega^1, \ldots, \omega^n, \overline{\omega^1}, \ldots, \overline{\omega^n}\rangle$, which is naturally identified with the space of left-invariant $\mathbb{C}$-valued forms on G. Let us write A_1 for $(A, \partial, \bar\partial)$, where the exterior algebra A is equipped with the differentials defined in the statement $(Ab1_n)$ of the theorem.

We can also equip A with a different differential d_{P_1}, which acts as follows in degree 1:

$$d_{P_1}(\omega^1) = 0,\ d_{P_1}(\omega^2) = 0,\ d_{P_1}(\omega^3) = \omega^2 \wedge \omega^1, \ldots,\ d_{P_1}(\omega^n) = \omega^{n-1} \wedge \omega^1.$$

One has $d_{P_1}^2 = 0$, $d_{P_1} = \partial_{P_1} + \bar\partial_{P_1}$, where ∂_{P_1} and $\bar\partial_{P_1}$ denote the components of bidegree $(1,0)$ and $(0,1)$, and $d_{P_1}(A^{1,0}) \subseteq A^{2,0}$. So, (A, d_{P_1}) can be considered as the space of left-invariant forms on a nilmanifold endowed with a complex parallelisable structure P_1. By Theorem 3.3.28, $A_{P_1} := (A, \partial_{P_1}, \bar\partial_{P_1})$ has the page-1-$\partial_{P_1}\bar\partial_{P_1}$-property. So, by Theorem 3.3.18, $h_{BC}(A_{P_1}) + h_A(A_{P_1}) = h_{\partial_{P_1}}(A_{P_1}) + h_{\bar\partial_{P_1}}(A_{P_1})$.

Define a $\mathbb{C}$-linear involution $C : A \to A$ in degree 1 by $C(\omega^1) = \overline{\omega^1}$ and $C(\omega^i) = \omega^i$, $C(\overline{\omega^i}) = \overline{\omega^i}$ for $i > 1$ and in degree k by $C(\alpha^1 \wedge \ldots \wedge \alpha^k) := C(\alpha^1) \wedge \ldots \wedge C(\alpha^k)$. This is compatible with the total degree, but not with the bigrading. One checks that

$$C \circ \partial = \bar\partial_{P_1} \circ C, \quad \text{and} \quad C \circ \bar\partial = \partial_{P_1} \circ C.$$

In fact, this holds in degree 1 and in higher degrees it follows from the Leibniz-rule. As a consequence, C induces isomorphisms

$$H_{BC}(A_1) \cong H_{BC}(A_{P_1}), \qquad H_A(A_1) \cong H_A(A_{P_1}), \qquad H_\partial(A_1) \cong H_{\bar\partial_{P_1}}(A_{P_1}), \qquad H_{\bar\partial}(A_1) \cong H_{\partial_{P_1}}(A_{P_1}).$$

Here we mean the total cohomologies, e.g. $H_{BC}(A_1) = \frac{\ker \partial \cap \ker \bar\partial}{\operatorname{im} \partial\bar\partial}(A_1) = \oplus_{p,q} H^{p,q}_{BC}(A_1)$ and the induced maps are not assumed to be compatible with the bigrading. The existence of such isomorphisms implies that we also have an equality $h_{BC}(A_1) + h_A(A_1) = h_\partial(A_1) + h_{\bar\partial}(A_1)$, i.e. the page-1-$\partial\bar\partial$-property holds for the space of left-invariant forms on G. Since G carries an abelian complex structure, this implies that $\Gamma\backslash G$ is a page-1-$\partial\bar\partial$-manifold.

We consider now a complex nilmanifold, of odd complex dimension $n \geq 3$, defined by $(Ab2_n)$. Let us write A_2 for $(A, \partial, \bar\partial)$, where the exterior algebra A is now equipped with the differentials defined in the statement $(Ab2_n)$ of the theorem. As before, we may also equip A with a different differential d_{P_2}, which acts as follows in degree 1:

$$d_{P_2}(\omega^1) = 0, \ldots,\ d_{P_2}(\omega^{n-1}) = 0,\ d_{P_2}(\omega^n) = \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4 + \cdots + \omega^{n-2} \wedge \omega^{n-1}.$$

One has $d_{P_2}(A^{1,0}) \subseteq A^{2,0}$, so (A, d_{P_2}) can be considered as the space of left-invariant forms on a nilmanifold endowed with a complex parallelisable structure P_2. Hence, $A_{P_2} := (A, \partial_{P_2}, \bar\partial_{P_2})$ has the page-1-$\partial_{P_2}\bar\partial_{P_2}$-property, by Theorem 3.3.28, and we have $h_{BC}(A_{P_2}) + h_A(A_{P_2}) = h_{\partial_{P_2}}(A_{P_2}) + h_{\bar\partial_{P_2}}(A_{P_2})$, by Theorem 3.3.18.

Let us define a $\mathbb{C}$-linear involution $C : A \to A$ in degree 1 by

$$C(\omega^{2i+1}) = \omega^{2i+1}, \quad C(\overline{\omega^{2i+1}}) = \overline{\omega^{2i+1}}, \quad C(\omega^{2i}) = \overline{\omega^{2i}}, \quad \text{for } 0 \leq i \leq \frac{n-1}{2},$$

together with $C(\omega^{2n}) = \omega^{2n}$ and $C(\overline{\omega^{2n}}) = \overline{\omega^{2n}}$. We extend C to degree k by $C(\alpha^1 \wedge \ldots \wedge \alpha^k) := C(\alpha^1) \wedge \ldots \wedge C(\alpha^k)$. One checks that $C \circ \partial = \bar\partial_{P_2} \circ C$ and $C \circ \bar\partial = \partial_{P_2} \circ C$ and may conclude as before. $\square$

Remark 3.3.30. *Note that $C \circ d = d_{P_1} \circ C$ (similarly for P_2), and C is compatible with the real structure. So it induces an isomorphism of the underlying real Lie groups. However, the corresponding complex nilmanifolds are not biholomorphic. Indeed, the Hodge number of bidegree $(1,0)$ is given by*

$$h^{1,0}_{\bar\partial} = 2 \ \text{ for } \ (Ab1_n), \qquad \text{and} \qquad h^{1,0}_{\bar\partial} = n-1 \ \text{ for } \ (Ab2_n),$$

whereas $h^{1,0}_{\bar\partial_P} = n$ for any complex parallelisable nilmanifold of complex dimension n.

Note that the abelian complex structures defined by $(Ab1_n)$ and $(Ab2_n)$ coincide precisely when $n = 3$. We will denote it by $\tilde J$, and we write $\widetilde X = (\Gamma\backslash G, \tilde J)$ for any nilmanifold endowed with such complex structure.

In the following proposition we prove that, in complex dimension 3, the only complex nilmanifolds which are page-$(r-1)$-$\partial\bar\partial$ for some $r \in \mathbb{N}^\star$ are, apart from a torus, the Iwasawa manifold $I^{(3)}$ and the nilmanifold $\widetilde{X}$.

Proposition 3.3.31. *Let $X = (\Gamma\backslash G, J)$ be a complex nilmanifold of complex dimension 3, different from a torus, endowed with an invariant complex structure J. If there exists $r \in \mathbb{N}^\star$ such that X is a page-$(r-1)$-$\partial\bar\partial$-manifold, then J is equivalent to the complex parallelisable structure of $I^{(3)}$ or to the abelian complex structure $\tilde{J}$ defined by $(Ab1_n)$ in Theorem 3.3.29 for $n = 3$. In both cases $r = 2$, i.e. both of these manifolds are page-1-$\partial\bar\partial$-manifolds.*

Proof. We already know by Theorems 3.3.28 and 3.3.29 that $I^{(3)}$ and $\widetilde{X}$ are page-1-$\partial\bar\partial$-manifolds. On the other hand, it is proved in [LU15] that for any other invariant complex structure J (i.e. not equivalent to $\tilde{J}$ or to the complex parallelisable structure of $I^{(3)}$), the nilmanifold $X = (\Gamma\backslash G, J)$ fails to be pure in degree 4 or 5, that is, the direct sum decomposition of Definition 1.2.17 is not satisfied for $k = 4$ or $k = 5$ (or both). So, such complex nilmanifolds $X = (\Gamma\backslash G, J)$ are not page-$(r-1)$-$\partial\bar\partial$-manifolds for any $r \in \mathbb{N}^\star$. □

Remark 3.3.32. *According to [Pop15] and [PU18], a compact complex manifold X is called an* **sGG manifold** *if every Gauduchon metric ω on X is sG, i.e. $\partial\omega^{n-1}$ is $\bar\partial$-exact. By the numerical characterisation proved in [PU18, Theorem 1.6], a compact complex manifold is sGG if and only if $b_1 = 2h^{0,1}_{\bar\partial}$. For instance, the Iwasawa manifold is sGG (see [PU18]), and more generally any complex parallelisable nilmanifold is sGG, due to (3.31).*

For the nilmanifolds endowed with the abelian complex structures defined in Theorem 3.3.29, we have the following Betti and Hodge numbers:

$$b_1 = 4 \neq 2n = 2\,h^{0,1}_{\bar\partial} \quad \textit{for} \quad (Ab1_n), \qquad \textit{and} \qquad b_1 = 2(n-1) \neq 2n = 2\,h^{0,1}_{\bar\partial} \quad \textit{for} \quad (Ab2_n).$$

Hence, such complex nilmanifolds are not sGG-manifolds.

On the other hand, all the sGG nilmanifolds of complex dimension $n = 3$ are identified in [PU18, Theorem 6.1]. In particular, there exist complex nilmanifolds different from the Iwasawa manifold and $\widetilde{X}$ which are sGG, so by Proposition 3.3.31, they are not page-$(r-1)$-$\partial\bar\partial$-manifolds for any $r \in \mathbb{N}^\star$.

Therefore, the page-1-$\partial\bar\partial$ and the sGG properties of compact complex manifolds are unrelated.

3.3.5 Construction methods for page-r-$\partial\bar\partial$-manifolds

Theorem 3.3.33. *Let X and Y be compact complex manifolds.*

1. *If X is a page-r-$\partial\bar\partial$-manifold and Y is a page-r'-$\partial\bar\partial$-manifold, the product $X \times Y$ is a page-$\tilde{r}$-$\partial\bar\partial$-manifold, where $\tilde{r} = \max\{r, r'\}$.*

   Conversely, if the product is a page-r-$\partial\bar\partial$-manifold, so are both factors.

2. *For any vector bundle $\mathcal{V}$ over X, the projectivised bundle $\mathbb{P}(\mathcal{V})$ is a page-r-$\partial\bar\partial$-manifold if and only if X is.*

3. *Suppose X is a page-r-$\partial\bar\partial$-manifold. Let $f : X \longrightarrow Y$ be a surjective holomorphic map and assume there exists a d-closed (l,l)-current Ω on X (with $l = \dim X - \dim Y$) such that $f_*\Omega \neq 0$. Then Y is also a page-r-$\partial\bar\partial$-manifold.*

   In particular, this implication always holds when $\dim X = \dim Y$, e.g. for contractions (take Ω to be a constant).

4. *Given a a submanifold $Z \subset X$, denote by $\widetilde{X}$ the blow-up of X along Z. If X is page-r-$\partial\bar{\partial}$ and Z is page-r'-$\partial\bar{\partial}$, then $\widetilde{X}$ is a page-$\tilde{r}$-$\partial\bar{\partial}$ manifold, where $\tilde{r} = \max\{r, r'\}$.*

   Conversely, if $\widetilde{X}$ is page r-$\partial\bar{\partial}$, so are X and Z.

5. *The page-r-$\partial\bar{\partial}$-property of compact complex manifolds is a bimeromorphic invariant if and only if it is stable under passage to submanifolds.*

Proof. The proofs are very similar to those in [Ste20, Cor. 28]. We will be using the characterisation of the page-r-$\partial\bar{\partial}$-property in terms of occuring zigzags (Theorem 3.3.9) and E_1-isomorphisms (Def. 3.3.11), in particular Lemma 3.3.13.

Writing A_X as shorthand for the double complex $(C^\infty_{\bullet,\bullet}(X, \mathbb{C}), \partial, \bar{\partial})$ and $A_X[i]$ for the shifted double complex with bigrading $(A_X[i])^{p,q} := A_X^{p-i,q-i}$, we have the following E_1**-isomorphisms**:[1]

$$A_{X\times Y} \simeq_1 A_X \otimes A_Y \tag{3.32}$$

$$A_{\mathbb{P}(\mathcal{V})} \simeq_1 \bigoplus_{i=0}^{\operatorname{rk}\mathcal{V}-1} A_X[i] \tag{3.33}$$

$$A_X \simeq_1 A_Y \oplus A_X/f^*A_Y \tag{3.34}$$

$$A_{\widetilde{X}} \simeq_1 A_X \oplus \bigoplus_{i=1}^{\operatorname{codim} Z-1} A_Z[i], \tag{3.35}$$

Since the occuring zigzags get only shifted, $A_X[i]$ satisfies the page-r-$\partial\bar{\partial}$-property if and only if A_X does. Furthermore, a direct sum of complexes satisfies the page-r-$\partial\bar{\partial}$-property if and only if each summand does. So, the second, third and fourth E_1-isomorphisms imply 2., 3. and 4.

For the first part of (1), we use the first isomorphism and the fact that one knows how irreducible subcomplexes behave under tensor product (see [Ste20, Prop. 16]). In particular, even-length zigzags do not get longer and the product of two length-one zigzags is again of length one. For the converse, note that A_X and A_Y are direct summands in their tensor product, so we can argue as before.

The 'if' statement in the last part of (5) is a direct consequence of (4) and the weak factorization theorem [AKMW02], which says that every bimeromorphic map can be factored as a sequence of blow-ups and blow-downs with smooth centres. The 'only if' part also follows from 4. (cf. also [Me19a]). Indeed, let X be page-r-$\partial\bar{\partial}$ and let $Z \subset X$ be a submanifold. If Z has codimension one, we replace X by $X \times \mathbb{P}^1_{\mathbb{C}}$ (which is still page-r-$\partial\bar{\partial}$ by 1.) and Z by $Z \times \{0\}$. By assumption, the blow-up is still page-r-$\partial\bar{\partial}$ and one can apply (4) to infer that the same holds for Z. $\square$

Since for surfaces and threefolds, the centre of a nontrivial blow-up is a point or a curve, we get

Corollary 3.3.34. *Fix any $r \in \mathbb{N}$. The page-r-$\partial\bar{\partial}$-property of compact complex surfaces and threefolds is a bimeromorphic invariant.*

Since it can be proved by the same methods as above, let us also record the following result, although it is not strictly related to page-r-$\partial\bar{\partial}$-manifolds. According to [Pop19], a compact complex

[1] Proved in [Ste20, Sect. 4] and [Ste19], apart from the case of dominations of unequal dimensions, i.e. the maps addressed in the third isomorphism, which is treated in [Meng19], together with some related work. Cf. also [RYY19], [YY18], [Meng20] and [ASTT19] for different approaches to the blow-up question in the setting of particular cohomologies.

manifold X is called an **E_r-sGG manifold** if every Gauduchon metric on X is E_r-sG. Let $T_r : H_A^{n-1,\,n-1}(X) \to E_r^{n,\,n-1}(X)$ be the natural linear map given by $[\alpha] \mapsto [\partial\alpha]$. Then, a Gauduchon metric ω on X is E_r-sG if and only if $[\omega^{n-1}]_A \in \ker T_r$. Since the Gauduchon cone is open and non-empty, the following is an easy generalisation of an observation in [Pop15]:

Lemma 3.3.35. *A compact complex manifold X is E_r-sGG if and only if $T_r = 0$.*

As a consequence of this, we get the **bimeromorphic invariance** of the **E_r-sGG** property.

Corollary 3.3.36. *Let X and $\widetilde{X}$ be* **bimeromorphically equivalent** *compact complex manifolds. Then, every Gauduchon metric on X is E_r-sG if and only if this is true on $\widetilde{X}$.*

Proof. By the weak factorisation theorem [AKMW02], it suffices again to check this for blow-ups $\widetilde{X} \to X$ with d-dimensional smooth centers Z of codimension ≥ 2. After picking any isomorphism realising formula (3.35), any class $c \in H_A^{n-1,\,n-1}(\widetilde{X})$ can be written as $c = c_X + c_Z$, with $c_X \in H_A^{n-1,\,n-1}(X)$ and $c_Z \in H_A^{d,\,d}(Z)$. So, $T_r c = T_r c_X + T_r c_Z = T_r c_X$ since $\partial\eta = 0$ for all (d,d)-forms on Z for dimension reasons. $\square$

Note that the above map T_r is given in all cases by applying ∂. Generally speaking, if $A = B \oplus C$, then $H_A(A) = H_A(B) \oplus H_A(C)$ and $E_r(A) = E_r(B) \oplus E_r(C)$ and $T_r^A = T_r^B + T_r^C$. We omitted the superscripts on T_r in the above proof for the sake of simplicity.

3.4 Higher-page Bott-Chern and Aeppli cohomologies

The main goal of this section, whose material is taken from [PSU20b], is to continue the construction of the higher-page Hodge Theory begun in §.3.3 by introducing *higher-page analogues of the Bott-Chern and Aeppli cohomologies* that provide a natural link between their classical counterparts and the Frölicher spectral sequence and, simultaneously, parallel the higher-page Frölicher cohomologies $E_r^{\bullet,\bullet}(X)$ when $r \geq 2$.

3.4.1 Basic definitions

Let X be an n-dimensional compact complex manifold. Fix an arbitrary positive integer r and a bidegree $(p,\,q)$. In §.3.2.2, we defined the notions of E_r-closedness and E_r-exactness for forms $\alpha \in C^\infty_{p,\,q}(X)$ as higher-page analogues of $\bar\partial$-closedness (that can now be called E_1-closedness) and $\bar\partial$-exactness (that can now be called E_1-exactness). We then gave these notions explicit descriptions in Proposition 3.2.4.

In the same vein, we say that α is *$\overline{E}_r$-closed* if $\bar\alpha$ is E_r-closed and we say that α is *$\overline{E}_r$-exact* if $\bar\alpha$ is E_r-exact. In particular, characterisations of $\overline{E}_r$-closedness and $\overline{E}_r$-exactness are obtained by permuting ∂ and $\bar\partial$ in the characterisations of E_r-closedness and E_r-exactness of Proposition 3.2.4.

Moreover, we can take our cue from Proposition 3.2.4 to define higher-page analogues of $\partial\bar\partial$-closedness and $\partial\bar\partial$-exactness in the following way.

Definition 3.4.1. *Suppose that $r \geq 2$.*

(i) We say that a form $\alpha \in C^\infty_{p,\,q}(X)$ is **$E_r\overline{E}_r$-closed** *if there exist smooth forms $\eta_1, \ldots, \eta_{r-1}$ and*

$\rho_1, \dots, \rho_{r-1}$ *such that the following two towers of* $r-1$ *equations are satisfied:*

$$\begin{aligned} \partial\alpha &= \bar\partial\eta_1 & \bar\partial\alpha &= \partial\rho_1 \\ \partial\eta_1 &= \bar\partial\eta_2 & \bar\partial\rho_1 &= \partial\rho_2 \\ &\vdots & & \\ \partial\eta_{r-2} &= \bar\partial\eta_{r-1}, & \bar\partial\rho_{r-2} &= \partial\rho_{r-1}. \end{aligned}$$

(ii) We refer to the properties of α *in the two towers of* $(r-1)$ *equations under (i) by saying that* $\partial\alpha$*, resp.* $\bar\partial\alpha$*,* **runs at least** $(r-1)$ **times***.*

(iii) We say that a form $\alpha \in C^\infty_{p,\,q}(X)$ *is* $E_r\overline{E}_r$**-exact** *if there exist smooth forms* ζ, ξ, η *such that*

$$\alpha = \partial\zeta + \partial\bar\partial\xi + \bar\partial\eta \tag{3.36}$$

and such that ζ *and* η *further satisfy the following conditions. There exist smooth forms* $v_{r-3}, \dots, v_0$ *and* $u_{r-3}, \dots, u_0$ *such that the following two towers of* $r-1$ *equations are satisfied:*

$$\begin{aligned} \bar\partial\zeta &= \partial v_{r-3} & \partial\eta &= \bar\partial u_{r-3} \\ \bar\partial v_{r-3} &= \partial v_{r-4} & \partial u_{r-3} &= \bar\partial u_{r-4} \\ &\vdots & & \\ \bar\partial v_0 &= 0, & \partial u_0 &= 0. \end{aligned}$$

(iv) We refer to the properties of ζ*, resp.* η*, in the two towers of* $(r-1)$ *equations under (iii) by saying that* $\bar\partial\zeta$*, resp.* $\partial\eta$*,* **reaches** 0 **in at most** $(r-1)$ **steps***.*

When $r-1=1$, the properties of $\bar\partial\zeta$, resp. $\partial\eta$, *reaching* 0 *in* $(r-1)$ *steps* translate to $\bar\partial\zeta = 0$, resp. $\partial\eta = 0$.

To unify the definitions, we will also say that a form $\alpha \in C^\infty_{p,\,q}(X)$ is $E_1\overline{E}_1$**-closed** (resp. $E_1\overline{E}_1$**-exact**) if α is $\partial\bar\partial$-closed (resp. $\partial\bar\partial$-exact).

As with E_r and $\overline{E}_r$, it follows at once from Definition 3.4.1 that the $E_r\overline{E}_r$-closedness condition becomes stronger and stronger as r increases, while the $E_r\overline{E}_r$-exactness condition becomes weaker and weaker as r increases. In other words, the following inclusions of vector spaces hold:

$$\begin{aligned} &\{\partial\bar\partial\text{-exact forms}\} \subset \dots \subset \{E_r\overline{E}_r\text{-exact forms}\} \subset \{E_{r+1}\overline{E}_{r+1}\text{-exact forms}\} \subset \dots \\ &\qquad \dots \subset \{E_{r+1}\overline{E}_{r+1}\text{-closed forms}\} \subset \{E_r\overline{E}_r\text{-closed forms}\} \subset \dots \subset \{\partial\bar\partial\text{-closed forms}\}. \end{aligned}$$

The following statement collects a few other immediate relations among these notions.

Lemma 3.4.2. *Fix an arbitrary* $r \in \mathbb{N}^\star$*.*

(i) A pure-type form α *is simultaneously* E_r*-closed and* $\overline{E}_r$*-closed if and only if* α *is simultaneously* ∂*-closed and* $\bar\partial$*-closed. This is further equivalent to* α *being* d*-closed.*

(ii) If α *is* $E_r\overline{E}_r$*-exact, then each of the classes* $\{\alpha\}_{E_r}$ *and* $\{\alpha\}_{\overline{E}_r}$ *contains a* $\partial\bar\partial$*-exact form and* α *is both* E_r*-exact and* $\overline{E}_r$*-exact.*

(iii) Fix any bidegree $(p,\,q)$ *and let* $\alpha \in C^\infty_{p,\,q}(X)$*. If* α *is* $E_r\overline{E}_r$*-exact for some* $r \in \mathbb{N}^\star$*, then* α *is* d*-exact.*

Proof. (i) is obvious. To see (ii), let $\alpha = \partial\zeta + \partial\bar\partial\xi + \bar\partial\eta$ be $E_r\overline{E}_r$-exact, with ζ and η satisfying the conditions under (ii) of Definition 3.4.1. Then

$$\{\alpha\}_{E_r} = \{\alpha - \partial\zeta - \bar\partial\eta\}_{E_r} = \{\partial\bar\partial\xi\}_{E_r} \quad \text{and} \quad \{\alpha\}_{\overline{E}_r} = \{\alpha - \partial\zeta - \bar\partial\eta\}_{\overline{E}_r} = \{\partial\bar\partial\xi\}_{\overline{E}_r},$$

while $\alpha = \partial\zeta + \bar\partial(-\partial\xi + \eta)$ is E_r-exact and $\alpha = \partial(\zeta + \bar\partial\xi) + \bar\partial\eta$ is $\overline{E}_r$-exact.

To prove (iii), let $\alpha = \partial\zeta + \partial\bar\partial\xi + \bar\partial\eta$, where ζ and η satisfy the conditions in the two towers under (ii) of Definition 3.4.1. Going down the first tower, we get

$$\partial\zeta = d\zeta - \bar\partial\zeta = d\zeta - \partial v_{r-3} = d(\zeta - v_{r-3}) + \partial v_{r-4} = \cdots = d(\zeta - v_{r-3} + \cdots + (-1)^r\, v_0).$$

In particular, $\partial\zeta$ is d-exact.

Similarly, going down the second tower, we get

$$\bar\partial\eta = d(\eta - u_{r-3} + \cdots + (-1)^r\, u_0).$$

In particular, $\bar\partial\eta$ is d-exact.

Since $\partial\bar\partial\xi$ is also d-exact, we infer that α is d-exact. Explicitly, we have

$$\alpha = d[(\zeta + \eta) + \bar\partial\xi - w_{r-3} + \cdots + (-1)^r\, w_0],$$

where $w_j := u_j + v_j$ for all j. $\square$

The main takeaway from Lemma 3.4.2 is that $E_r\overline{E}_r$-exactness implies E_r-exactness, $\overline{E}_r$-exactness and d-exactness. Let us now pause briefly to notice a property involving the spaces $\mathcal{C}_r^{p,\,q}$ of E_r-exact $(p,\,q)$-forms, resp. $\overline{\mathcal{C}}_r^{p,\,q}$ of $\overline{E}_r$-exact $(p,\,q)$-forms.

Lemma 3.4.3. *Fix an arbitrary $r \in \mathbb{N}^\star$. For any bidegree $(p,\,q)$, the following identity of vector subspaces of $C^\infty_{p,\,q}(X)$ holds:*

$$\mathcal{C}_r^{p,\,q} + \overline{\mathcal{C}}_r^{p,\,q} = \mathit{Im}\,\partial + \mathit{Im}\,\bar\partial.$$

Proof. For any bidegree $(p,\,q)$, consider the vector spaces (see (iv) of Definition 3.4.1 for the terminology):

$$\begin{aligned}\mathcal{E}^{p,\,q}_{\partial,\,r} &:= \{\alpha \in C^\infty_{p,\,q}(X) \mid \partial\alpha \text{ reaches 0 in at most r steps}\},\\ \mathcal{E}^{p,\,q}_{\bar\partial,\,r} &:= \{\beta \in C^\infty_{p,\,q}(X) \mid \bar\partial\beta \text{ reaches 0 in at most r steps}\}.\end{aligned}$$

From the definitions, we get: $\mathcal{C}_r^{p,\,q} = \partial(\mathcal{E}^{p,\,q}_{\bar\partial,\,r-1}) + \mathrm{Im}\,\bar\partial$ and $\overline{\mathcal{C}}_r^{p,\,q} = \mathrm{Im}\,\partial + \bar\partial(\mathcal{E}^{p,\,q}_{\partial,\,r-1})$. This trivially implies the contention. $\square$

We now come to the main definitions of this subsection.

Definition 3.4.4. *Let X be an n-dimensional compact complex manifold. Fix $r \in \mathbb{N}^\star$ and a bidegree $(p,\,q)$.*

*(i) The E_r-**Bott-Chern** cohomology group of bidegree $(p,\,q)$ of X is defined as the following quotient complex vector space:*

$$E^{p,\,q}_{r,\,BC}(X) := \frac{\{\alpha \in C^\infty_{p,\,q}(X) \mid d\alpha = 0\}}{\{\alpha \in C^\infty_{p,\,q}(X) \mid \alpha \text{ is } E_r\overline{E}_r\text{-exact}\}}.$$

*(ii) The E_r-**Aeppli** cohomology group of bidegree $(p,\,q)$ of X is defined as the following quotient complex vector space:*

$$E^{p,\,q}_{r,\,A}(X) := \frac{\{\alpha \in C^\infty_{p,\,q}(X) \mid \alpha \text{ is } E_r\overline{E}_r - \text{closed}\}}{\{\alpha \in C^\infty_{p,\,q}(X) \mid \alpha \in \mathit{Im}\,\partial + \mathit{Im}\,\bar\partial\}}.$$

When $r=1$, the above groups coincide with the standard Bott-Chern, respectively Aeppli, cohomology groups (see [BC65] and [Aep62]). Note that, by (i) of Lemma 3.4.2, the representatives of E_r-Bott-Chern classes can be alternatively described as the forms that are simultaneously E_r-closed and $\overline{E}_r$-closed, while by Lemma 3.4.3, the E_r-Aeppli-exact forms can be alternatively described as those forms lying in $\mathcal{C}_r^{p,\,q}+\overline{\mathcal{C}}_r^{p,\,q}$.

Also note that the inclusions of vector spaces spelt out just before Lemma 3.4.2 and their analogues for the E_r- and $\overline{E}_r$-cohomologies imply the following inequalities of dimensions:

$$\cdots\leq\dim E_{r,\,BC}^{p,\,q}(X)\leq\dim E_{r-1,\,BC}^{p,\,q}(X)\leq\cdots\leq\dim E_{1,\,BC}^{p,\,q}(X)=\dim H_{BC}^{p,\,q}(X)$$

and their analogues for the E_r-Aeppli cohomology spaces.

3.4.2 Serre-type duality between $E_{r,\,BC}$ and $E_{r,\,A}$ cohomologies

The first step towards extending to the higher pages of the Frölicher spectral sequence the standard Serre-type duality between the classical Bott-Chern and Aeppli cohomology groups of complementary bidegrees is the following

Proposition 3.4.5. *Let X be a compact complex manifold with $\dim_{\mathbb{C}}X=n$. For every $r\in\mathbb{N}^\star$ and all $p,q\in\{0,\dots,n\}$, the following bilinear pairing is* **well defined***:*

$$E_{r,\,BC}^{p,\,q}(X)\times E_{r,\,A}^{n-p,\,n-q}(X)\longrightarrow\mathbb{C},\qquad\bigg(\{\alpha\}_{E_{r,\,BC}},\,\{\beta\}_{E_{r,\,A}}\bigg)\mapsto\int\limits_X\alpha\wedge\beta,$$

in the sense that it is independent of the choice of representative of either of the classes $\{\alpha\}_{E_{r,\,BC}}$ and $\{\beta\}_{E_{r,\,A}}$.

Proof. The proof consists in a series of integrations by parts (mathematical ping-pong).

• To prove independence of the choice of representative of the E_r-Bott-Chern class, let us modify a representative α to some representative $\alpha+\partial\zeta+\partial\bar\partial\xi+\bar\partial\eta$ of the same E_r-Bott-Chern class. This means that $\partial\zeta+\partial\bar\partial\xi+\bar\partial\eta$ is $E_r\overline{E}_r$-exact, so ζ and η satisfy the towers of $r-1$ equations under (ii) of Definition 3.4.1. We have

$$\int\limits_X(\alpha+\partial\zeta+\partial\bar\partial\xi+\bar\partial\eta)\wedge\beta\;\;=\;\;\int\limits_X\alpha\wedge\beta\pm\int\limits_X\zeta\wedge\partial\beta\pm\int\limits_X\xi\wedge\partial\bar\partial\beta\pm\int\limits_X\eta\wedge\bar\partial\beta.$$

Since β is $E_r\overline{E}_r$-closed, it is also $\partial\bar\partial$-closed (see (i) of Lemma 3.4.2), so the last but one integral above vanishes.

Using the $r-1$ equations in the first tower under (i) of Definition 3.4.1 (with β in place of α) and the first tower under (ii) of the same definition, we get:

$$\begin{aligned}\int\limits_X\zeta\wedge\partial\beta\;\;&=\;\;\int\limits_X\zeta\wedge\bar\partial\eta_1=\pm\int\limits_X\bar\partial\zeta\wedge\eta_1=\pm\int\limits_X\partial v_{r-3}\wedge\eta_1=\pm\int\limits_X v_{r-3}\wedge\partial\eta_1\\&=\;\;\pm\int\limits_X v_{r-3}\wedge\bar\partial\eta_2=\pm\int\limits_X\bar\partial v_{r-3}\wedge\eta_2=\pm\int\limits_X\partial v_{r-4}\wedge\eta_2=\pm\int\limits_X v_{r-4}\wedge\partial\eta_2\\&\;\;\vdots\\&=\;\;\pm\int\limits_X v_0\wedge\bar\partial\eta_{r-1}=\pm\int\limits_X\bar\partial v_0\wedge\eta_{r-1}=0,\end{aligned}$$

where the last identity follows from $\bar\partial v_0 = 0$.

Playing the analogous mathematical ping-pong while using the second tower under both (i) and (ii) of Definition 3.4.1, we get:

$$\begin{aligned}\int\limits_X \eta\wedge\bar\partial\beta &= \int\limits_X \eta\wedge\partial\rho_1 = \pm\int\limits_X \partial\eta\wedge\rho_1 = \pm\int\limits_X \bar\partial u_{r-3}\wedge\rho_1 = \pm\int\limits_X u_{r-3}\wedge\bar\partial\rho_1\\ &= \pm\int\limits_X u_{r-3}\wedge\partial\rho_2 = \pm\int\limits_X \partial u_{r-3}\wedge\rho_2 = \pm\int\limits_X \bar\partial u_{r-4}\wedge\rho_2 = \pm\int\limits_X u_{r-4}\wedge\bar\partial\rho_2\\ &\;\;\vdots\\ &= \pm\int\limits_X u_0\wedge\partial\rho_{r-1} = \pm\int\limits_X \partial u_0\wedge\rho_{r-1} = 0,\end{aligned}$$

where the last identity follows from $\partial u_0 = 0$.

We conclude that $\int_X(\alpha+\partial\zeta+\partial\bar\partial\xi+\bar\partial\eta)\wedge\beta = \int_X\alpha\wedge\beta$.

• To prove independence of the choice of representative of the E_r-Aeppli class, let us modify a representative β to some representative $\beta+\partial\zeta+\bar\partial\xi$ of the same E_r-Aeppli class. So, ζ and ξ are arbitrary forms. We get:

$$\int\limits_X \alpha\wedge(\beta+\partial\zeta+\bar\partial\xi) = \int\limits_X \alpha\wedge\beta \pm \int\limits_X \partial\alpha\wedge\zeta \pm \int\limits_X \bar\partial\alpha\wedge\xi = 0,$$

where the last identity follows from $\partial\alpha = 0$ and $\bar\partial\alpha = 0$. □

We now take up the issue of the non-degeneracy of the above bilinear pairing. For the sake of expediency, we start by defining the dual notion to the $E_r\overline{E}_r$-closedness of Definition 3.4.1 after we have fixed a metric.

Definition 3.4.6. *Let $(X,\,\omega)$ be a compact complex Hermitian manifold. Fix an integer $r\geq 2$ and a bidegree $(p,\,q)$.*

We say that a form $\alpha\in C^\infty_{p,\,q}(X)$ is $E_r^\star\overline{E}_r^\star$**-closed** *with respect to the Hermitian metric ω if there exist smooth forms $a_1,\dots,a_{r-1}$ and $b_1,\dots,b_{r-1}$ such that the following two towers of $r-1$ equations are satisfied:*

$$\begin{array}{ll}\partial^\star\alpha = \bar\partial^\star a_1 & \qquad\bar\partial^\star\alpha = \partial^\star b_1\\ \partial^\star a_1 = \bar\partial^\star a_2 & \qquad\bar\partial^\star b_1 = \partial^\star b_2\\ \quad\vdots & \\ \partial^\star a_{r-2} = \bar\partial^\star a_{r-1}, & \qquad\bar\partial^\star b_{r-2} = \partial^\star b_{r-1}.\end{array}$$

That this notion is indeed dual to the $E_r\overline{E}_r$-closedness via the Hodge star operator $\star=\star_\omega$ associated with the metric ω is the content of the following analogue of Corollary 3.2.9.

Lemma 3.4.7. *In the setting of Definition 3.4.6, the following equivalence holds for every form $\alpha\in C^\infty_{p,\,q}(X)$:*

$$\alpha \text{ is } E_r\overline{E}_r\text{-closed} \iff \star\bar\alpha \text{ is } E_r^\star\overline{E}_r^\star\text{-closed}.$$

Proof. Thanks to conjugations, to the fact that $\star\star = \pm\,\mathrm{Id}$ (with the sign depending on the parity of the total degree of the forms involved) and to $\star$ being an isomorphism, the two towers of $r-1$ equations that express the $E_r\overline{E}_r$-closedness of α (cf. (i) of Definition 3.4.1) translate to

$$\begin{array}{cc} (-\star\bar\partial\star)(\star\bar\alpha) = (-\star\partial\star)(\star\bar\eta_1) & (-\star\partial\star)(\star\bar\alpha) = (-\star\bar\partial\star)(\star\bar\rho_1) \\ (-\star\bar\partial\star)(\star\bar\eta_1) = (-\star\partial\star)(\star\bar\eta_2) & (-\star\partial\star)(\star\bar\rho_1) = (-\star\bar\partial\star)(\star\bar\rho_2) \\ \vdots & \\ (-\star\bar\partial\star)(\star\bar\eta_{r-2}) = (-\star\partial\star)(\star\bar\eta_{r-1}), & (-\star\partial\star)(\star\bar\rho_{r-2}) = (-\star\bar\partial\star)(\star\bar\rho_{r-1}). \end{array}$$

Now, put $a_j := \star\bar\eta_j$ and $b_j := \star\bar\rho_j$ for all $j \in \{1, \dots, r-1\}$. Since $-\star\bar\partial\star = \partial^\star$ and $-\star\partial\star = \bar\partial^\star$, these two towers amount to $\star\bar\alpha$ being $E_r^\star\overline{E}_r^\star$-closed. (See Definition 3.4.6). □

We now come to two crucial lemmas from which Hodge isomorphisms for the E_r-Bott-Chern and the E_r-Aeppli cohomologies will follow. Based on the terminology introduced in (ii) of Definition 3.4.1, we define the vector spaces:

$$\begin{aligned} \mathcal{F}^{p,\,q}_{\partial,\,r} &:= \{\alpha \in C^\infty_{p,\,q}(X) \mid \partial\alpha \text{ runs at least r times}\}, \\ \mathcal{F}^{p,\,q}_{\bar\partial,\,r} &:= \{\beta \in C^\infty_{p,\,q}(X) \mid \bar\partial\beta \text{ runs at least r times}\} \end{aligned}$$

and their analogues $\mathcal{F}^{p,\,q}_{\partial^\star,\,r}$ and $\mathcal{F}^{p,\,q}_{\bar\partial^\star,\,r}$ when ∂ is replaced by $\partial^\star$ and $\bar\partial$ is replaced by $\bar\partial^\star$. Note that the space of $E_r^\star\overline{E}_r^\star$-closed $(p,\,q)$-forms defined in Definition 3.4.6 is precisely the intersection $\mathcal{F}^{p,\,q}_{\partial^\star,\,r-1} \cap \mathcal{F}^{p,\,q}_{\bar\partial^\star,\,r-1}$.

Lemma 3.4.8. *Let $(X,\,\omega)$ be a compact complex Hermitian manifold. Fix an integer $r \geq 2$, a bidegree $(p,\,q)$ and a form $\alpha \in C^\infty_{p,\,q}(X)$.*

The following two statements are **equivalent***.*

(i) α is $E_r^\star\overline{E}_r^\star$-closed (w.r.t. ω);

(ii) α is L^2_ω-orthogonal to the space of smooth $E_r\overline{E}_r$-exact $(p,\,q)$-forms.

Proof. "(i) $\Longrightarrow$ (ii)" Suppose that α is $E_r^\star\overline{E}_r^\star$-closed. This means that α satisfies the two towers of $(r-1)$ equations in Definition 3.4.6. Let $\beta = \partial\zeta + \partial\bar\partial\xi + \bar\partial\eta$ be an arbitrary $E_r\overline{E}_r$-exact $(p,\,q)$-form. So, ζ and η satisfy the respective towers of $r-1$ equations under (ii) of Definition 3.4.1. For the L^2_ω-inner product of α and β, we get:

$$\langle\langle\alpha,\,\beta\rangle\rangle = \langle\langle\partial^\star\alpha,\,\zeta\rangle\rangle + \langle\langle\bar\partial^\star\partial^\star\alpha,\,\xi\rangle\rangle + \langle\langle\bar\partial^\star\alpha,\,\eta\rangle\rangle. \tag{3.37}$$

Since $\bar\partial^\star\partial^\star\alpha = \bar\partial^\star\bar\partial^\star a_1 = 0$, the middle term on the r.h.s. of (3.37) vanishes.

For the first term on the r.h.s. of (3.37), we use the towers of equations satisfied by α and ζ to get:

$$\begin{aligned} \langle\langle\partial^\star\alpha,\,\zeta\rangle\rangle &= \langle\langle\bar\partial^\star a_1,\,\zeta\rangle\rangle = \langle\langle a_1,\,\bar\partial\zeta\rangle\rangle = \langle\langle a_1,\,\partial v_{r-3}\rangle\rangle = \langle\langle\partial^\star a_1,\,v_{r-3}\rangle\rangle = \langle\langle\bar\partial^\star a_2,\,v_{r-3}\rangle\rangle \\ &= \langle\langle a_2,\,\bar\partial v_{r-3}\rangle\rangle = \langle\langle a_2,\,\partial v_{r-4}\rangle\rangle \\ &\vdots \\ &= \langle\langle a_{r-1},\,\bar\partial v_0\rangle\rangle = 0, \end{aligned}$$

where the last identity followed from the property $\bar\partial v_0 = 0$.

For the last term on the r.h.s. of (3.37), we use the towers of equations satisfied by α and η to get:

$$\begin{aligned}\langle\langle\bar\partial^\star\alpha,\,\eta\rangle\rangle &= \langle\langle\partial^\star b_1,\,\eta\rangle\rangle = \langle\langle b_1,\,\partial\eta\rangle\rangle = \langle\langle b_1,\,\bar\partial u_{r-3}\rangle\rangle = \langle\langle\bar\partial^\star b_1,\,u_{r-3}\rangle\rangle = \langle\langle\partial^\star b_2,\,u_{r-3}\rangle\rangle \\ &= \langle\langle b_2,\,\partial u_{r-3}\rangle\rangle = \langle\langle b_2,\,\bar\partial u_{r-4}\rangle\rangle \\ &\vdots \\ &= \langle\langle b_{r-1},\,\partial u_0\rangle\rangle = 0,\end{aligned}$$

where the last identity followed from the property $\partial u_0 = 0$.

"(ii) $\implies$ (i)" Suppose that α is orthogonal to all the smooth $E_r\overline{E}_r$-exact $(p,\,q)$-forms β. These forms are of the shape $\beta = \partial\zeta + \partial\bar\partial\xi + \bar\partial\eta$, where ξ is subject to no condition, while $\zeta\in\mathcal{E}^{p-1,\,q}_{\bar\partial,\,r-1}$ and $\eta\in\mathcal{E}^{p,\,q-1}_{\partial,\,r-1}$. (See notation introduced in the proof of Lemma 3.4.3).

The orthogonality condition is equivalent to the following three identities:

$$(a)\,\langle\langle\bar\partial^\star\partial^\star\alpha,\,\xi\rangle\rangle = 0, \qquad (b)\,\langle\langle\partial^\star\alpha,\,\zeta\rangle\rangle = 0, \qquad (c)\,\langle\langle\bar\partial^\star\alpha,\,\eta\rangle\rangle = 0$$

holding for all the forms ζ,ξ,η satisfying the above conditions.

Since ξ is subject to no condition, (a) amounts to $\bar\partial^\star\partial^\star\alpha = 0$. This means that $\partial^\star\alpha\in\ker\bar\partial^\star$ and $\bar\partial^\star\alpha\ker\partial^\star$. Condition (b) requires $\partial^\star\alpha\perp\mathcal{E}^{p-1,\,q}_{\bar\partial,\,r-1}$, while (c) requires $\bar\partial^\star\alpha\perp\mathcal{E}^{p,\,q-1}_{\partial,\,r-1}$.

• *Unravelling condition (b).* The forms $\zeta\in\mathcal{E}^{p-1,\,q}_{\bar\partial,\,r-1}$ are characterised by the existence of forms $v_{r-3},\dots,v_0$ satisfying the first tower of $(r-1)$ equations in (iii) of Definition 3.4.1. That tower imposes the condition $v_{r-j}\in\mathcal{E}_{\bar\partial,\,r-j+1}\cap\mathcal{F}_{\partial,\,j-2}$ for every $j\in\{3,\dots,j\}$. (We have dropped the superscripts to lighten the notation.)

Now, every form $\zeta\in\ker\Delta''$ satisfies the condition $\bar\partial\zeta = 0$, hence $\zeta\in\mathcal{E}_{\bar\partial,\,1}\subset\mathcal{E}_{\bar\partial,\,r-1}$. From condition (b), we get $\partial^\star\alpha\perp\ker\Delta''$. Since $\ker\bar\partial^\star$ (to which $\partial^\star\alpha$ belongs by condition (a)) is the orthogonal direct sum between $\ker\Delta''$ and $\mathrm{Im}\,\bar\partial^\star$, we get $\partial^\star\alpha\in\mathrm{Im}\,\bar\partial^\star$, so

$$\partial^\star\alpha = \bar\partial^\star a_1 \tag{3.38}$$

for some form a_1. Condition (b) becomes:

$$0 = \langle\langle\partial^\star\alpha,\,\zeta\rangle\rangle = \langle\langle a_1,\,\bar\partial\zeta\rangle\rangle = \langle\langle a_1,\,\partial v_{r-3}\rangle\rangle = \langle\langle\partial^\star a_1,\,v_{r-3}\rangle\rangle \quad \text{for all} \quad v_{r-3}\in\mathcal{E}_{\bar\partial,\,r-2}\cap\mathcal{F}_{\partial,\,1}.$$

In other words, $\partial^\star a_1\perp(\mathcal{E}_{\bar\partial,\,r-2}\cap\mathcal{F}_{\partial,\,1})$.

We will now use the 3-space decomposition (3.28) of $C^\infty_{p,\,q}(X)$ for the case $r=2$. (See Proposition 3.2.12 in Appendix one.) It is immediate to check the inclusion $\mathcal{E}_{\bar\partial,\,r-2}\cap\mathcal{F}_{\partial,\,1}\supset\mathcal{H}_2\oplus(\mathrm{Im}\,\bar\partial + \partial(\mathcal{E}_{\bar\partial,\,1}))$. Therefore, condition (b) implies that $\partial^\star a_1\perp(\mathcal{H}_2\oplus(\mathrm{Im}\,\bar\partial+\partial(\mathcal{E}_{\bar\partial,\,1})))$. Since the orthogonal complement of $\mathcal{H}_2\oplus(\mathrm{Im}\,\bar\partial+\partial(\mathcal{E}_{\bar\partial,\,1}))$ is $\partial^\star(\mathcal{E}_{\bar\partial^\star,\,1})+\mathrm{Im}\,\bar\partial^\star$ by the 3-space decomposition (3.28) for $r=2$, we infer that $\partial^\star a_1\in\partial^\star(\mathcal{E}_{\bar\partial^\star,\,1})+\mathrm{Im}\,\bar\partial^\star$. Therefore, there exist forms $b_1\in\ker\bar\partial^\star$ and a_2 such that

$$\partial^\star a_1 = \partial^\star b_1 + \bar\partial^\star a_2. \tag{3.39}$$

Since $\bar\partial^\star b_1 = 0$, equations (3.38) and (3.39) yield:

$$\begin{aligned}\partial^\star\alpha &= \bar\partial^\star(a_1-b_1) \\ \partial^\star(a_1-b_1) &= \bar\partial^\star a_2.\end{aligned} \tag{3.40}$$

Thus, condition (b) becomes:

$$\begin{aligned}0 &= \langle\langle\partial^\star\alpha,\,\zeta\rangle\rangle = \langle\langle\bar\partial^\star(a_1-b_1),\,\zeta\rangle\rangle = \langle\langle a_1-b_1,\,\partial v_{r-3}\rangle\rangle = \langle\langle\partial^\star(a_1-b_1),\,v_{r-3}\rangle\rangle \\ &= \langle\langle\bar\partial^\star a_2,\,v_{r-3}\rangle\rangle = \langle\langle a_2,\,\partial v_{r-4}\rangle\rangle = \langle\langle\partial^\star a_2,\,v_{r-4}\rangle\rangle \quad \text{for all} \quad v_{r-4}\in\mathcal{E}_{\bar\partial,\,r-3}\cap\mathcal{F}_{\partial,\,2}.\end{aligned}$$

In other words, $\partial^\star a_2 \perp (\mathcal{E}_{\bar\partial,\,r-3}\cap\mathcal{F}_{\partial,\,2})$.

Now, it is immediate to check the inclusion $\mathcal{E}_{\bar\partial,\,r-3}\cap\mathcal{F}_{\partial,\,2}\supset\mathcal{H}_3\oplus(\mathrm{Im}\,\bar\partial+\partial(\mathcal{E}_{\bar\partial,\,2}))$. Since the orthogonal complement of $\mathcal{H}_3\oplus(\mathrm{Im}\,\bar\partial+\partial(\mathcal{E}_{\bar\partial,\,2}))$ is $\partial^\star(\mathcal{E}_{\bar\partial^\star,\,2})+\mathrm{Im}\,\bar\partial^\star$ by the 3-space decomposition (3.28) for $r=3$, we infer that $\partial^\star a_2\in\partial^\star(\mathcal{E}_{\bar\partial^\star,\,1})+\mathrm{Im}\,\bar\partial^\star$. Therefore, there exist forms $b_2\in\mathcal{E}_{\bar\partial^\star,\,2}$ and a_3 such that

$$\partial^\star a_2=\partial^\star b_2+\bar\partial^\star a_3. \tag{3.41}$$

Since the condition $b_2\in\mathcal{E}_{\bar\partial^\star,\,2}$ translates to the equations

$$\bar\partial^\star b_2=\partial^\star c_1 \quad\text{and}\quad \bar\partial^\star c_1=0, \tag{3.42}$$

for some form c_1, equations (3.40) and (3.41) yield:

$$\begin{aligned}&\partial^\star\alpha=\bar\partial^\star(a_1-b_1-c_1)\\&\partial^\star(a_1-b_1-c_1)=\bar\partial^\star(a_2-b_2)\\&\partial^\star(a_2-b_2)=\bar\partial^\star a_3.\end{aligned}$$

Continuing in this way, we inductively get the following tower of $(r-1)$ equations:

$$\begin{array}{rcl}\partial^\star\alpha&=&\bar\partial^\star(a_1-b_1-c_1-c_1^{(3)}-\cdots-c_1^{(r-2)})\\ \partial^\star(a_1-b_1-c_1-c_1^{(3)}-\cdots-c_1^{(r-2)})&=&\bar\partial^\star(a_2-b_2-c_2^{(3)}-\cdots-c_2^{(r-2)})\\ &\vdots&\\ \partial^\star(a_{r-2}-b_{r-2})&=&\bar\partial^\star a_{r-1},\end{array} \tag{3.43}$$

where $b_j\in\mathcal{E}_{\bar\partial^\star,\,j}$ for all $j\in\{1,\dots,r-2\}$, so b_j satisfies the following tower of j equations:

$$\begin{array}{rcl}\bar\partial^\star b_j&=&\partial^\star c_{j-1}^{(j)}\\ \bar\partial^\star c_{j-1}^{(j)}&=&\partial^\star c_{j-2}^{(j)}\\ &\vdots&\\ \bar\partial^\star c_2^{(j)}&=&\partial^\star c_1^{(j)}\\ \bar\partial^\star c_1^{(j)}&=&0,\end{array}$$

for some forms $c_l^{(j)}$.

Consequently, conditions (a) and (b) to which α is subject imply that $\alpha\in\mathcal{F}_{\partial^\star,\,r-1}$ (cf. tower (3.43)), which is the first of the two conditions required for α to be $E_r^\star\overline{E}_r^\star$-closed under Definition 3.4.6.

• *Unravelling condition (c).* Proceeding in a similar fashion, with $\partial^\star$ and $\bar\partial^\star$ permuted, we infer that conditions (a) and (c) to which α is subject imply that $\alpha\in\mathcal{F}_{\bar\partial^\star,\,r-1}$, which is the second of the two conditions required for α to be $E_r^\star\overline{E}_r^\star$-closed under Definition 3.4.6.

• We conclude that α is indeed $E_r^\star\overline{E}_r^\star$-closed. $\square$

The immediate consequence of Lemma 3.4.8 is the following *Hodge isomorphism* for the E_r-Bott-Chern cohomology.

Corollary and Definition 3.4.9. *Let $(X,\,\omega)$ be a compact complex Hermitian manifold. For every bidegree $(p,\,q)$ and every $r\in\mathbb{N}^\star$, every E_r-Bott-Chern cohomology class $\{\alpha\}_{E_{r,\,BC}}\in E_{r,\,BC}^{p,\,q}(X)$ can be represented by a unique form $\alpha\in C_{p,\,q}^\infty(X)$ satisfying the following three conditions:*

$$\alpha \text{ is } \partial\text{-closed, } \bar\partial\text{-closed and } E_r^\star\overline{E}_r^\star\text{-closed.}$$

*Any such form α is called E_r-**Bott-Chern harmonic** with respect to the metric ω.*
*There is a vector-space **isomorphism** depending on the metric ω:*

$$E_{r,\,BC}^{p,\,q}(X) \simeq \mathcal{H}_{r,\,BC}^{p,\,q}(X),$$

where $\mathcal{H}_{r,\,BC}^{p,\,q}(X) \subset C_{p,\,q}^\infty(X)$ is the space of E_r-Bott-Chern harmonic $(p,\,q)$-forms associated with ω.

Of course, the above isomorphism maps every class $\{\alpha\}_{E_{r,\,BC}} \in E_{r,\,BC}^{p,\,q}(X)$ to its unique E_r-Bott-Chern harmonic representative.

The analogous statement for the E_r-Aeppli cohomology follows at once from standard material. Indeed, it is classical that the L^2_ω-orthogonal complement of $\mathrm{Im}\,\partial$ (resp. $\mathrm{Im}\,\bar\partial$) in $C_{p,\,q}^\infty(X)$ is $\ker\partial^\star$ (resp. $\ker\bar\partial^\star$). The immediate consequence of this is the following *Hodge isomorphism* for the E_r-Aeppli cohomology.

Corollary and Definition 3.4.10. *Let $(X,\,\omega)$ be a compact complex Hermitian manifold. For every bidegree $(p,\,q)$, every E_r-Aeppli cohomology class $\{\alpha\}_{E_{2,\,A}} \in E_{r,\,A}^{p,\,q}(X)$ can be represented by a unique form $\alpha \in C_{p,\,q}^\infty(X)$ satisfying the following three conditions:*

$$\alpha \text{ is } E_r\overline{E}_r\text{-closed, } \partial^\star\text{-closed and } \bar\partial^\star\text{-closed.}$$

*Any such form α is called E_r-**Aeppli harmonic** with respect to the metric ω.*
*There is a vector-space **isomorphism** depending on the metric ω:*

$$E_{r,\,A}^{p,\,q}(X) \simeq \mathcal{H}_{r,\,A}^{p,\,q}(X),$$

where $\mathcal{H}_{r,\,A}^{p,\,q}(X) \subset C_{p,\,q}^\infty(X)$ is the space of E_r-Aeppli harmonic $(p,\,q)$-forms associated with ω.

Of course, the above isomorphism maps every class $\{\alpha\}_{E_{r,\,A}} \in E_{r,\,A}^{p,\,q}(X)$ to its unique E_r-Aeppli harmonic representative.

We can now conclude from the above results that there is a Serre-type **canonical duality** between the E_r-Bott-Chern cohomology and the E_r-Aeppli cohomology of complementary bidegrees.

Theorem 3.4.11. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. For all $p, q \in \{0,\dots,n\}$, the following bilinear pairing is **well defined** and **non-degenerate**:*

$$E_{r,\,BC}^{p,\,q}(X) \times E_{r,\,A}^{n-p,\,n-q}(X) \longrightarrow \mathbb{C}, \qquad \bigg(\{\alpha\}_{E_{r,\,BC}},\,\{\beta\}_{E_{r,\,A}}\bigg) \mapsto \int\limits_X \alpha\wedge\beta.$$

Proof. The well-definedness was proved in Proposition 3.4.5. The non-degeneracy is proved in the usual way on the back of the above preliminary results, as follows.

Let $\{\alpha\}_{E_{r,\,BC}} \in E_{r,\,BC}^{p,\,q}(X)$ be an arbitrary non-zero class. Fix an arbitrary Hermitian metric ω on X and let α be the unique *E_r-Bott-Chern harmonic* representative (w.r.t. ω) of the class $\{\alpha\}_{E_{r,\,BC}}$ (whose existence and uniqueness are guaranteed by Corollary and Definition 3.4.9). In particular, $\alpha \neq 0$.

Based on the characterisations of the E_r-Bott-Chern and E_r-Aeppli harmonicities given in Corollaries and Definitions 3.4.9 and 3.4.10, Lemma 3.4.7 and the standard equivalences ($\alpha \in \ker\partial \iff$

$\star\bar{\alpha} \in \ker \partial^\star$) and ($\alpha \in \ker \bar{\partial} \iff \star\bar{\alpha} \in \ker \bar{\partial}^\star$) ensure that $\star\bar{\alpha}$ is *E_r-Aeppli harmonic*. In particular, $\star\bar{\alpha}$ represents an E_r-Aeppli class $\{\star\bar{\alpha}\}_{E_r,A} \in E_{r,A}^{n-p,n-q}(X)$. Moreover, pairing $\{\alpha\}_{E_r,BC}$ with $\{\star\bar{\alpha}\}_{E_r,A}$ yields $\int_X \alpha \wedge \star\bar{\alpha} = ||\alpha||^2 \neq 0$, where $||\ ||$ stands for the L^2_ω-norm.

Similarly, starting off with a non-zero class $\{\beta\}_{E_r,A} \in E_{r,A}^{n-p,n-q}(X)$ and selecting its unique *E_r-Aeppli harmonic* representative β, we get that $\beta \neq 0$, $\star\bar{\beta}$ is *E_r-Bott-Chern harmonic* (hence it represents a class in $E_{r,BC}^{p,q}(X)$) and the classes $\{\star\bar{\beta}\}_{E_r,BC}$ and $\{\beta\}_{E_r,A}$ pair to $\pm||\beta||^2 \neq 0$. □

3.4.3 Characterisation in terms of exactness properties

We now state and prove the following higher-page analogue of Proposition 3.3.22.

Theorem 3.4.12. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Fix an arbitrary integer $r \geq 1$. The properties (A) and (B) below are equivalent.*

(A) X is a **page-$(r-1)$-$\partial\bar{\partial}$-manifold**.

(B) For all $p, q \in \{0, \dots, n\}$ and for every form $\alpha \in C^\infty_{p,q}(X)$ such that $d\alpha = 0$, the following equivalences hold:

$$\alpha \in Im\, d \iff \alpha \text{ is } E_r\text{-exact} \iff \alpha \text{ is } \overline{E}_r\text{-exact} \iff \alpha \text{ is } E_r\overline{E}_r\text{-exact}. \tag{3.44}$$

Except for one step, the proof is purely algebraic, so let us first do the algebra in the following

Lemma 3.4.13. *Let $(A, \partial_1, \partial_2)$ be a bounded double complex of vector spaces. Then, property (B) in Theorem 3.4.12 (with ∂_1 in place of ∂ and ∂_2 in place of $\bar{\partial}$) is equivalent to A being isomorphic to a direct sum of squares, even-length zigzags of length $< 2r$ and odd-length zigzags of length $2l+1 \leq 2r-1$ and type M.*

Proof. Since $(A, \partial_1, \partial_2)$ is always isomorphic to a sum of indecomposable complexes, it suffices to check each possible indecomposable summand separately. We refer to Theorem 3.3.8 for diagrams of all possibilities and will use the same notation as the one introduced there.

• Case of squares

Every d-closed pure-type form is a multiple of $\partial_2\partial_1 a = -\partial_1\partial_2 a$ and the latter form is easily seen to be exact in all four ways appearing in (B), so the properties are equivalent in this case.

• Case of even-length zigzags

In the type 1 case, every d-closed pure-type form is a multiple of some $\partial_1 a_i$, so we have to investigate these forms more closely. They have the following properties.

1. All the $\partial_1 a_i$'s are d-exact (indeed, $\partial_1 a_i = d(a_1 + \cdots + a_i)$) and ∂_1-(hence $\bar{E}_r$-)exact.

2. Using the tower of equations in the definition of E_r-exactness, one sees that $\partial_1 a_i$ is E_r-exact if and only if $i + 1 \leq r$.

3. Since for this double complex $\partial_1\partial_2 = 0$ and a nontrivial tower of equations for $\bar{E}_r$-exactness can never be found, $\partial_1 a_i$ is E_r-exact if and only if it is $E_r\bar{E}_r$-exact.

Hence, one sees that for even-length zigzags, all four properties are equivalent if and only if $l < r$. The type 2 case is handled analogously.

• **Case of odd-length zigzags**

For type M, the pure-type d-closed forms are again the $\partial_1 a_i$'s. Their exactness properties are as follows.

1. Each $\partial_1 a_i$ is d-, ∂_1- and ∂_2-exact, so in particular also E_r- and $\bar{E}_r$-exact.

2. Using the towers of equations in the definition of $E_r\overline{E}_r$-exactness, we see that $\partial_1 a_i$ is $E_r\overline{E}_r$-exact if and only if $i < r$ or $l - i + 1 < r$.

Indeed, if $\partial_1 a_i$ is viewed as ∂_1-exact with potential a_i, it is $E_r\overline{E}_r$-exact if and only if $\partial_2 a_i$ reaches 0 in at most $r-1$ steps. Since $\partial_2 a_i = -\partial_1 a_{i-1}, \partial_2 a_{i-1} = -\partial_1 a_{i-2}, \dots, \partial_2 a_2 = -\partial_1 a_1, \partial_2 a_1 = 0$, this is the case if and only if $i \leq r-1$.

Meanwhile, if $\partial_1 a_i = -\partial_2 a_{i+1}$ is viewed as ∂_2-exact with potential a_{i+1}, it is $E_r\overline{E}_r$-exact if and only if $\partial_1 a_{i+1}$ reaches 0 in at most $r-1$ steps. Since $\partial_1 a_{i+1} = -\partial_2 a_{i+2}, \partial_1 a_{i+2} = -\partial_2 a_{i+3}, \dots \partial_1 a_{l+1} = 0$, this is the case if and only if $l - i + 1 \leq r - 1$.

Hence, one sees that in this type of zigzags the exactness properties are equivalent for all the bidegrees if and only if $l + 1 < r$. (In particular, this is always true for $l = 0$).

It is left to check type L: the d-closed pure forms are the $\partial_1 a_i$. None of these is d-exact, but all except $\partial_1 a_l$ are ∂_2-exact, hence E_r-exact and all but $\partial_2 a_1$ are ∂_1-exact, so $\bar{E}_r$-exact. In particular, the exactness properties under (B) are never equivalent. □

Proof of Theorem 3.4.12. The symmetry of occuring zigzags along the antidiagonal $p+q=n$ stated in Lemma 3.3.15 exchanges, among the odd-length zigzags of length > 1, those of type M with those of type L. It also and sends dots to dots. Thus, by Lemma 3.4.13, in the case of $A = A_X = (C^\infty_{\bullet,\bullet}(X, \mathbb{C}), \partial, \bar\partial)$ for a compact connected complex manifold X, Property (B) in Theorem 3.4.12 is equivalent to the existence of a decomposition of A_X into squares, odd-length zigzags of length one (giving rise to pure De Rham classes) and even-length zigzags of length $< 2r$ (responsible for possible differentials in early pages). This, in turn, has already been seen to be equivalent to the page-$(r-1)$-$\partial\bar\partial$-property of X. □

3.4.4 Characterisation in terms of maps to and from higher-page Bott-Chern and Aeppli cohomology groups

Let X be an n-dimensional compact complex manifold. Fix $r \in \mathbb{N}^\star$ and a bidegree (p, q). Recall that $\mathcal{Z}_r^{p,q}$ and $\mathcal{C}_r^{p,q}$ stand for the space of E_r-closed, resp. E_r-exact, smooth (p, q)-forms on X. (See section 3.2.2.) Let $\mathcal{D}_r^{p,q}$ stand for the space of $E_r\overline{E}_r$-exact smooth (p, q)-forms on X.

Lemma 3.4.14. *(i) The following inclusions of vector subspaces of $C^\infty_{p+1,q}(X)$ hold:*

$$\begin{array}{ccccc} Im\,(\partial\bar\partial) \subset \partial(\mathcal{Z}_r^{p,q}) \subset & \mathcal{D}_r^{p+1,q} & \subset \mathcal{C}_r^{p+1,q} \cap \ker d \\ & \cap & \\ & Im\, d & \end{array} \tag{3.45}$$

(ii) Every E_r-class $\{\alpha\}_{E_r} \in E_r^{p,q}(X)$ can be represented by a d-closed form if and only if $\partial(\mathcal{Z}_r^{p,q}) \subset Im\,(\partial\bar\partial)$. In other words, this happens if and only if the first inclusion in (3.45) is an equality.

Proof. (i) To prove the first inclusion, it suffices to show that every $\bar\partial$-exact (p, q)-form is E_r-closed. Let $\alpha = \bar\partial\beta$ be a (p, q)-form. Then, $\bar\partial\alpha = 0$ and $\partial\alpha = \bar\partial(-\partial\beta)$. Putting $u_1 := -\partial\beta$, we have $\partial u_1 = 0$, so we can choose $u_2 = 0, \dots, u_{r-1} = 0$ to satisfy the tower of equations under (i) of Proposition 3.2.4. This shows that α is E_r-closed.

To prove the second inclusion, let $\alpha \in \mathcal{Z}_r^{p,\,q}$. By (i) of Proposition 3.2.4, this implies that $\bar\partial\alpha = 0$, so if we write $\partial\alpha = \partial\zeta + \partial\bar\partial\xi + \bar\partial\eta$ with $\zeta = \alpha$, $\xi = 0$ and $\eta = 0$, we satisfy the conditions under (ii) of Definition 3.4.1 with $v_j = 0$ and $u_j = 0$ for all $j \in \{0, \dots, r-3\}$. This proves that $\partial\alpha$ is $E_r\overline{E}_r$-exact.

The third inclusion on the first row is a consequence of (iii) and (iv) of Lemma 3.4.2, while the "vertical" inclusion is a translation of (iv) of the same lemma.

(ii) Let $\{\alpha\}_{E_r} \in E_r^{p,\,q}(X)$ be an arbitrary class and let α be an arbitrary representative of it. Then, $\{\alpha\}_{E_r} \in E_r^{p,\,q}(X)$ can be represented by a d-closed form if and only if there exists an E_r-exact form $\rho = \partial a + \bar\partial b$, with a satisfying the conditions $\bar\partial a = \partial c_{r-3}$, $\bar\partial c_{r-3} = \partial c_{r-4}, \dots \bar\partial c_0 = 0$ for some forms c_j, such that $\partial(\alpha - \rho) = 0$. This last identity is equivalent to

$$\partial\bar\partial b = \partial\alpha.$$

Thus, the class $\{\alpha\}_{E_r}$ contains a d-closed form if and only if the form $\partial\alpha$, which already lies in $\partial(\mathcal{Z}_r^{p,\,q})$, is $\partial\bar\partial$-exact.

This proves the contention. $\square$

Theorem 3.4.15. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Fix an arbitrary integer $r \geq 2$. The following properties are equivalent.*

(A) X is a **page-$(r-1)$-$\partial\bar\partial$-manifold***.*

(C) For all $p, q \in \{0, \dots, n\}$, the following identities of vector subspaces of $C^\infty_{p+1,\,q}(X)$ hold:

$$(i)\quad Im\,(\partial\bar\partial) = \partial(\mathcal{Z}_r^{p,\,q}) \quad and \quad (ii)\quad \mathcal{C}_r^{p,\,q} \cap \ker d = Im\,d. \tag{3.46}$$

Proof. By (ii) of Lemma 3.4.14, identity (i) in (C) is equivalent to every E_r-class of type $(p,\,q)$ being representable by a d-closed form. On the other hand, if this is the case, then identity (ii) in (C) is equivalent to the map $E_r^{p,\,q}(X) \ni \{\alpha\}_{E_r} \mapsto \{\alpha\}_{DR} \in H^{p+q}_{DR}(X,\,\mathbb{C})$ (with $\alpha \in \ker d$) being well defined and injective.

The contention follows from Theorem and Definition 3.3.3. $\square$

As an aside, note that identity (ii) in (C) of Theorem 3.4.15 is a reformulation of the first equivalence in (B) of Theorem 3.4.12.

We will now relate the page-$(r-1)$-$\partial\bar\partial$-property of compact complex manifolds to the E_r-Bott-Chern and E_r-Aeppli cohomologies introduced in §3.4.1. The study will reveal an analogy with the standard $\partial\bar\partial$-property in relation to the standard Bott-Chern and Aeppli cohomologies (corresponding to the case $r = 1$).

Lemma 3.4.16. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Fix any integer $r \geq 2$.*

(i) There are well-defined canonical linear maps induced by the identity:

$$\begin{array}{ccc} E_{r,\,BC}^{p,\,q}(X) \stackrel{T_r^{p,\,q}}{\longrightarrow} E_r^{p,\,q}(X) \stackrel{S_r^{p,\,q}}{\longrightarrow} E_{r,\,A}^{p,\,q}(X) & \quad and \quad & E_{r,\,BC}^{p,\,q}(X) \longrightarrow H_{DR}^{p,\,q}(X) \longrightarrow E_{r,\,A}^{p,\,q}(X) \\ \{\alpha\}_{E_{r,\,BC}} \mapsto \{\alpha\}_{E_r} \mapsto \{\alpha\}_{E_{r,\,A}} & & \{\alpha\}_{E_{r,\,BC}} \mapsto \{\alpha\}_{DR} \mapsto \{\alpha\}_{E_{r,\,A}}. \end{array}$$

(ii) The following equivalences hold:

$$T_r^{p,\,q} \text{ is injective } \iff \mathcal{D}_r^{p,\,q} \supset \mathcal{C}_r^{p,\,q} \cap \ker d \iff \mathcal{D}_r^{p,\,q} = \mathcal{C}_r^{p,\,q} \cap \ker d$$

$$T_r^{p,\,q} \text{ is surjective } \iff Im\,(\partial\bar\partial) \supset \partial(\mathcal{Z}_r^{p,\,q}) \iff Im\,(\partial\bar\partial) = \partial(\mathcal{Z}_r^{p,\,q})$$

$$S_r^{p,\,q} \text{ is injective } \iff \overline{\mathcal{C}}_r^{p,\,q} \cap \ker d \subset \mathcal{C}_r^{p,\,q}$$

$$S_r^{p,\,q} \text{ is surjective } \iff Im\,(\partial\bar\partial) \supset \bar\partial(\mathcal{Z}_{r\bar r}^{p,\,q}) \iff Im\,(\partial\bar\partial) = \bar\partial(\mathcal{Z}_{r\bar r}^{p,\,q}),$$

where $\mathcal{Z}_{r\bar r}^{p,\,q}$ stands for the space of smooth $E_r\overline{E}_r$-closed $(p,\,q)$-forms.

(iii) If the map $T_r^{p,\,q}$ is bijective, the identity induces a well-defined surjection

$$E_r^{p,\,q}(X) \longrightarrow H_{DR}^{p,\,q}(X),$$

in the sense that every class $\{\alpha\}_{E_r} \in E_r^{p,\,q}(X)$ contains a d-closed representative and the linear map $E_r^{p,\,q}(X) \ni \{\alpha\}_{E_r} \mapsto \{\alpha\}_{DR} \in H_{DR}^{p,\,q}(X)$ is independent of the choice of d-closed representative α of the class $\{\alpha\}_{E_r} \in E_r^{p,\,q}(X)$.

Proof. It consists of immediate verifications based on the definitions and is left to the reader. □

We now come to the main result of this subsection.

Theorem 3.4.17. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Fix an arbitrary integer $r \geq 1$. The following statements are* **equivalent***.*

(A) X is a **page-$(r-1)$-$\partial\bar\partial$-manifold***.*

(D) For all $p, q \in \{0, \dots, n\}$, the canonical linear maps $T_r^{p,\,q} : E_{r,\,BC}^{p,\,q}(X) \longrightarrow E_r^{p,\,q}(X)$ and $S_r^{p,\,q} : E_r^{p,\,q}(X) \longrightarrow E_{r,\,A}^{p,\,q}(X)$ are **isomorphisms***.*

(E) For all $p, q \in \{0, \dots, n\}$, the canonical linear map $S_r^{p,\,q} \circ T_r^{p,\,q} : E_{r,\,BC}^{p,\,q}(X) \longrightarrow E_{r,\,A}^{p,\,q}(X)$ is **injective***.*

Proof. "(A) $\implies$ (D)" Suppose that X is a page-$(r-1)$-$\partial\bar\partial$-manifold. Thanks to Theorem 3.4.15, the page-$(r-1)$-$\partial\bar\partial$-property of X is equivalent to the first inclusion on the left in (3.45) being an identity (which is further equivalent to $T_r^{p,\,q}$ being surjective) and to the last space on the right in (3.45) being equal to $\operatorname{Im} d$ (which, after conjugation of its occurence in bidegree $(q,\,p)$, implies that $S_r^{p,\,q}$ is injective).

On the other hand, the equivalence "α is E_r-exact $\iff$ α is $E_r\overline{E}_r$-exact", ensured for d-closed forms α by characterisation (B) of the page-$(r-1)$-$\partial\bar\partial$-property given in Theorem 3.4.12, is equivalent to the third inclusion on the first row in (3.45) being an identity. Thanks to (ii) of Lemma 3.4.16, this is further equivalent to $T_r^{p,\,q}$ being injective. Thus, $T_r^{p,\,q}$ is bijective.

It remains to show that $S_r^{p,\,q}$ is surjective. The duality results of Corollary 3.2.10 and Theorem 3.4.11 ensure that the dual map of $S_r^{p,\,q} : E_r^{p,\,q}(X) \to E_{r,\,A}^{p,\,q}(X)$ is the map $T_r^{n-p,\,n-q} : E_{r,\,BC}^{n-p,\,n-q}(X) \to E_r^{n-p,\,n-q}(X)$. Meanwhile, $T_r^{n-p,\,n-q}$ has been proved above to be *injective* in all bidegrees. This is equivalent to its dual map $S_r^{p,\,q}$ being *surjective*, as desired.

"(D) $\implies$ (E)" This implication is trivial.

"(E) $\implies$ (A)" Suppose that $S_r^{p,\,q} \circ T_r^{p,\,q}$ is injective, hence $T_r^{p,\,q}$ is injective, for every $(p,\,q)$. By (ii) of Lemma 3.4.16, the injectivity of $T_r^{p,\,q}$ translates to the following equivalence for all d-closed $(p,\,q)$-forms α:

$$\alpha \text{ is } E_r\text{-exact} \iff \alpha \text{ is } E_r\overline{E}_r\text{-exact}.$$

Since this holds in every bidegree $(p,\,q)$, taking conjugates we get the following equivalence for d-closed forms α of every bidegree:

$$\alpha \text{ is } \overline{E}_r\text{-exact} \iff \alpha \text{ is } E_r\overline{E}_r\text{-exact}.$$

Thanks to characterisation (B) of the page-$(r-1)$-$\partial\bar\partial$-property given in Theorem 3.4.12, it remains to prove the implication

$$\alpha \text{ is } d\text{-exact} \implies \alpha \text{ is } E_r\overline{E}_r\text{-exact}$$

for every d-closed form α of every bidegree $(p,\,q)$. (Recall that the implication "α is $E_r\overline{E}_r$-exact $\implies$ α is d-exact" always holds by (iii) of Lemma 3.4.2.) Therefore, let $\alpha = dv = \partial v^{p-1,\,q} + \bar\partial v^{p,\,q-1} \in C^\infty_{p,\,q}(X)$ be d-exact. Then, $\alpha \in \operatorname{Im}\partial + \operatorname{Im}\bar\partial$, hence $\{\alpha\}_{E_{r,\,A}} = 0 \in E^{p,\,q}_{r,\,A}(X)$. Since $S^{p,\,q}_r \circ T^{p,\,q}_r : E^{p,\,q}_{r,\,BC}(X) \longrightarrow E^{p,\,q}_{r,\,A}(X)$ is injective, we get $\{\alpha\}_{E_{r,\,BC}} = 0 \in E^{p,\,q}_{r,\,BC}(X)$, which translates to α being $E_r\overline{E}_r$-exact. $\square$

3.4.5 The role of the dimensions of $E^{\bullet,\,\bullet}_{r,\,BC}(X)$ and $E^{\bullet,\,\bullet}_{r,\,A}(X)$

Many of the results in sections §.3.3 and §.3.4 can be summed up as follows. As we have seen, requiring the existence of d-closed pure-type representatives for the E_r-classes in (1) of the next statement is a key condition.

Conclusion 3.4.18. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Fix an arbitrary $r \in \mathbb{N}^\star$. The following statements are equivalent.*

(1) *For every bidegree $(p,\,q)$, every class $\{\alpha^{p,\,q}\}_{E_r} \in E^{p,\,q}_r(X)$ can be represented by a* d**-closed** $(p,\,q)$**-form** *and for every k, the linear map*

$$\bigoplus_{p+q=k} E^{p,\,q}_r(X) \ni \sum_{p+q=k}\{\alpha^{p,\,q}\}_{E_r} \mapsto \left\{\sum_{p+q=k}\alpha^{p,\,q}\right\}_{DR} \in H^k_{DR}(X,\,\mathbb{C})$$

is **well-defined** *by means of d-closed pure-type representatives and* **bijective***.*

In this case, X is said to have the E_r**-Hodge Decomposition** *property.*

(2) *The Frölicher spectral sequence of X* **degenerates at** E_r *and the De Rham cohomology of X is* **pure***.*

(3) *For all $p,q \in \{0,\dots,n\}$ and for every form $\alpha \in C^\infty_{p,\,q}(X)$ such that $d\alpha = 0$, the following equivalences hold:*

$$\alpha \in Im\, d \iff \alpha \text{ is } E_r\text{-exact} \iff \alpha \text{ is } \overline{E}_r\text{-exact} \iff \alpha \text{ is } E_r\overline{E}_r\text{-exact}.$$

(4) *For all $p,q \in \{0,\dots,n\}$, the canonical linear maps*

$$E^{p,\,q}_{r,\,BC}(X) \longrightarrow E^{p,\,q}_r(X) \quad \text{and} \quad E^{p,\,q}_r(X) \longrightarrow E^{p,\,q}_{r,\,A}(X)$$

are **isomorphisms***, where $E^{p,\,q}_{r,\,BC}(X)$ and $E^{p,\,q}_{r,\,A}(X)$ are the* E_r**-Bott-Chern***, respectively the* E_r**-Aeppli***, cohomology groups of bidegree $(p,\,q)$ introduced in Definition 3.4.4.*

(5) *For all $p,q \in \{0,\dots,n\}$, the canonical linear map $E^{p,\,q}_{r,\,BC}(X) \longrightarrow E^{p,\,q}_{r,\,A}(X)$ is* **injective***.*

(6) *We have: $\dim E^k_{r,\,BC}(X) = \dim E^k_{r,\,A}(X)$ for all $k \in \{0, ..., 2n\}$.*

A compact complex manifold X that satisfies any of the equivalent conditions (1)–(6) *is said to be a* **page-$(r-1)$-$\partial\bar{\partial}$-manifold**.

Only the equivalence of (6) with the other conditions still needs a proof. We will use the language of abstract double complexes, including squares and zigzags, of §.3.3.2 in the proof that we now spell out and that reproves other equivalences of Conclusion 3.4.18.

(Alternative) proof of Conclusion 3.4.18

In order to compute $E_{r,\,BC}$ and $E_{r,\,A}$ on abstract double complexes, first recall that $E_{r,\,BC}$ is a quotient of H_{BC} and $E_{r,\,A}$ is a subspace of H_A. In particular, if H_{BC} or H_A are zero on some double complex, so are their lower dimensional counterparts. Further note that $\partial_1\partial_2 = 0$ on zigzags and we even have $\partial_1 = \partial_2 = 0$ on dots. These observations yield

Observation 3.4.19. *For a square S we have $E_{r,\,A}(S) = E_{r,\,BC}(S) = 0$, while for a dot $D = \langle a\rangle$ we have $E_{r,\,BC}(D) = E_{r,\,A}(D) = \langle a\rangle$ for all $r \geq 1$.*

For higher length zigzags Z, generated by $a_1, ..., a_l$, we get that $H_A(Z) = \langle a_1, ..., a_l\rangle$ keeps the lower antidiagonal, while $H_{BC}(Z) = \langle \partial_2 a_1, ..., \partial_2 a_l, \partial_1 a_l\rangle$ remembers the higher antidiagonal. To describe their higher-page analogues, it suffices to understand the kernel, resp. cokernel, of the projection $H_{BC}(Z) \twoheadrightarrow E_{r,\,BC}(Z)$, resp. the inclusion $E_{r,\,A}(Z) \hookrightarrow H_A(Z)$. These are described as follows.

Lemma 3.4.20. *Let Z be a zigzag of length at least two, generated by $a_1, ..., a_l$. For any $i \notin \{1, ..., l\}$, set $a_i := 0$. Then, for any $r \geq 2$:*

1. *Even length type I: If $\partial_2 a_1 = 0$ and $\partial_1 a_l \neq 0$, one has:*

$$\ker\left(H_{BC}(Z) \to E_{r,\,BC}(Z)\right) = \langle \partial_1 a_1, ..., \partial_1 a_{r-1}\rangle$$
$$\operatorname{coker}\left(E_{r,\,A}(Z) \to H_A(Z)\right) = \langle a_{l-r+2}, ..., a_l\rangle.$$

2. *Even length type II: If $\partial_2 a_1 \neq 0$ and $\partial_1 a_l = 0$, one has:*

$$\ker\left(H_{BC}(Z) \to E_{r,\,BC}(Z)\right) = \langle \partial_1 a_{l-r+2}, ..., \partial_1 a_l\rangle$$
$$\operatorname{coker}\left(E_{r,\,A}(Z) \to H_A(Z)\right) = \langle a_1, ..., a_{r-1}\rangle.$$

3. *Odd length type M: If $\partial_2 a_1 = 0 = \partial_2 a_l$, one has $E_{r,\,A}(Z) = H_A(Z)$ and*

$$\ker\left(H_{BC}(Z) \to E_{r,\,BC}(Z)\right) = \langle \partial_1 a_1, ..., \partial_1 a_{r-1}, \partial 2 a_{l-r+2}, ..., \partial_2 a_l\rangle.$$

4. *Odd length type L: If $\partial_2 a_1 \neq 0 \neq \partial_2 a_l$, one has $H_{BC}(Z) = E_{r,\,BC}(Z)$ and*

$$\operatorname{coker}\left(E_{r,\,A}(Z) \to H_A(Z)\right) = \langle a_1, ..., a_{r-1}, a_{l-r+2}, ..., a_l\rangle.$$

Note that for large r, some of the written generators could be zero or there could be some overlap in the last two cases.

Proof. Let us only do the computation for $E_{r,BC}$. The elements that get modded out here in addition to the $\partial_1\partial_2$-exact ones are the E_r-exact ones and the $\bar{E}_r$-exact ones. By the definition of E_r-exactness, this means that, whenever a zigzag has top left corner generated by a_1 with $\partial_2 a_1 = 0$, i.e.:

$$\begin{tikzcd}
\langle a_1\rangle \arrow[r] & \langle \partial_1 a_1\rangle & & \\
& \langle a_2\rangle \arrow[u] \arrow[r] & \langle \partial_1 a_2\rangle & \\
& & & \ddots
\end{tikzcd}$$

the classes of $\partial_1 a_1, ..., \partial_1 a_{r-1}$ are zero in $E_{r,BC}(Z)$. Along the same lines, if a zigzag has bottom right corner generated by a_1 with $\partial_2 a_1 = 0$, i.e.:

$$\begin{tikzcd}
\ddots & & \\
& & \\
& \langle a_2\rangle \arrow[u] \arrow[r] & \langle \partial_2 a_1\rangle \\
& & \langle a_1\rangle \arrow[u]
\end{tikzcd}$$

the classes of $\partial_2 a_1, ..., \partial_2 a_{r-1}$ are zero in $E_{r,BC}(Z)$. This yields the result for $E_{r,BC}$. The calculation for H_A is analogous. □

Let us also record what this yields for the dimensions of the new cohomology groups.

Corollary 3.4.21. *Let Z be an indecomposable bounded double complex and let $r \geq 2$.*

1. *If Z is a square, then $e_{r,BC}(Z) = 0 = e_{r,A}(Z)$.*
2. *If Z is a dot, then $e_{r,BC}(Z) = 1 = e_{r,A}(Z)$.*
3. *If Z is a zigzag of odd length $2l+1 \geq 3$ of type L, one has $e_{r,BC}(Z) = h_{BC}(Z) = l+1$ and $e_{r,A}(Z) = \max\{l - 2(r-1), 0\}$.*
4. *If Z is a zigzag of odd length $2l+1 \geq 3$ of type M, one has $e_{r,BC}(Z) = \max\{l-2(r-1), 0\}$ and $e_{r,A}(Z) = h_A(Z) = l+1$.*
5. *If Z is a zigzag of even length $2l$, one has $e_{r,BC}(Z) = e_{r,A}(Z) = \max\{l-r+1, 0\}$.*

Recall ([KQ20], [PSU20], [Ste20]) that any bounded double complex A can be written as a direct sum of indecomposable ones, all indecomposable ones are either squares or zigzags and A has the page-r-$\partial_1\partial_2$-property if and only if in any decomposition into indecomposables, there are no odd length zigzags other than dots (length one) and no even length zigzags of length greater than $2r$. Thus, we get

Corollary 3.4.22. *For any bounded double complex A such that all the numerical quantities involved are finite, there is an inequality:*

$$e_{r,A}(A) + e_{r,BC}(A) \geq e_r(A) + \bar{e}_r(A) \geq 2b(A).$$

Equality holds if A satisfies the page-$(r-1)$-$\partial_1\partial_2$-property

Proof. Since all the quantities involved are additive under direct sums, it suffices to show this for indecomposable double complexes Z. The middle and right hand side were computed in [Ste20]: $e_r(Z) + e_r(Z)$ equals 0 for a square and for a zigzag of length $2l \leq 2(r-1)$, while it equals 2 for all other zigzags. Also, $b(Z) = 1$ for odd length zigzags and $b(Z) = 0$ otherwise. In particular, for an arbitrary double complex, the middle quantity is just twice the number of all zigzags which have odd length or even length at least $2r$. By the previous Corollary 3.4.21, $e_{r,A}(Z) + e_{r,BC}(Z) = 0$ for squares and even length zigzags of length $2l \leq 2(r-1)$, while it equals 2 for dots and zigags of length $2r$ and is greater than or equal to 2 for all other zigzags. □

Remark 3.4.23. *Somewhat unexpectedly, the equality $e_{r,A}(A) + e_{r,BC}(A) = 2b(A)$ does **not** imply the page-$(r-1)$-$\partial_1\partial_2$-property for $r \geq 2$, contrary to the case $r = 1$ ([AT13]). For example, both sides are equal to 2 for $r \geq 2$ and A a zigzag of length 3. As one may see, for example from a Hopf surface, this behaviour really occurs in geometric situations. A different generalisation of the case $r = 1$ has been obtained in [PSU20].*

Corollary 3.4.24. *Let A be a bounded double complex such that $e^k_{r,BC}(A)$ and $e^k_{r,A}(A)$ are finite. The following properties are equivalent:*

*(A') A has the **page-$(r-1)$-$\partial_1\partial_2$-property**.*

(B') The map $E_{r,BC}(A) \to E_{r,A}(A)$ is an isomorphism.

(C') One has $e^k_{r,BC}(A) = e^k_{r,A}(A)$ for all $k \in \mathbb{Z}$.

Proof. Note that property *(B')* implies property *(C')*. Thus, it suffices to show that property *(B')* is satisfied for squares, dots and even length zigzags of length $\leq 2(r-1)$ and property *(C')* is violated for all other zigzags. This is a direct consequence of Lemma 3.4.20 and Corollary 3.4.21. □

3.5 Adiabatic limit for complex structures

This section, taken from [Pop17], should be compared with §.3.1 and §.3.2. Indeed, we now give an alternative analytic approach to the Frölicher Spectral Sequence (FSS) by means of a *rescaled Laplacian*, a differential operator that can be seen as complementary to the *pseudo-differential Laplacians* of §.3.1 and §.3.2.

The main result of this section (see Theorem 3.5.13) is a general formula for the dimensions of the vector spaces featuring in the Frölicher spectral sequence in terms of the asymptotics, as a positive constant h decreases to zero, of the small eigenvalues of the rescaled Laplacian Δ_h, introduced in [Pop17] in the present form, that we adapt to the context of a complex structure from the well-known construction of the adiabatic limit and from the analogous result for Riemannian foliations of Álvarez López and Kordyukov in [ALK00].

3.5.1 Rescaled Laplacians

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. We fix a Hermitian metric ω on X.

(I) Rescaling the metric

The first operation we will consider is a *partial rescaling of ω* in a way that depends solely on the *holomorphic* degree of forms.

Definition 3.5.1. *For all $p, q \in \{0, \dots, n\}$, all $(p,\, q)$-forms u, v and every constant $h > 0$, we define the following* **pointwise inner product**

$$\langle u,\, v\rangle_{\omega_h} := h^{2p}\,\langle u,\, v\rangle_\omega$$

where $\langle\,,\;\rangle_\omega$ stands for the pointwise inner product defined by the original Hermitian metric ω.

Note that, for every $h > 0$, we obtain in this way a Hermitian metric ω_h on every vector bundle $\Lambda^{p,\,q}T^\star X$ of $(p,\, q)$-forms on X. The maps

$$\theta_h : \Lambda^{p,\,q}T^\star X \longrightarrow \Lambda^{p,\,q}T^\star X, \qquad u \mapsto \theta_h u := h^p u,$$

induce an **isometry** of Hermitian vector bundles $\theta_h : (\Lambda T^\star X,\, \omega_h) \longrightarrow (\Lambda T^\star X,\, \omega)$ since

$$\langle u,\, v\rangle_{\omega_h} = \langle h^p u,\, h^p v\rangle_\omega = \langle \theta_h u,\, \theta_h v\rangle_\omega \quad \text{for all } u, v \in \Lambda^{p,\,q}T^\star X.$$

In particular, we have defined a Hermitian metric

$$\omega_h = \frac{1}{h^2}\,\omega, \qquad h > 0,$$

on the holomorphic tangent bundle $T^{1,\,0}X$ of vector fields of type $(1,\, 0)$, or equivalently, a rescaled C^∞ positive-definite $(1,\, 1)$-form $\omega_h = h^{-2}\,\omega$ on X. This induces a C^∞ positive volume form

$$dV_{\omega_h} := \frac{\omega_h^n}{n!} = \frac{1}{h^{2n}}\,\frac{\omega^n}{n!} = \frac{1}{h^{2n}}\,dV_\omega$$

on X, which in turn gives rise, in conjunction with the above pointwise inner product $\langle\;,\;\rangle_{\omega_h}$, to the following L^2 **inner product**

$$\langle\langle u,\, v\rangle\rangle_{\omega_h} := \int\limits_X \langle u,\, v\rangle_{\omega_h}\, dV_{\omega_h} = \frac{1}{h^{2n}}\int\limits_X \langle \theta_h u,\, \theta_h v\rangle_\omega\, dV_\omega = \frac{1}{h^{2n}}\,\langle\langle \theta_h u,\, \theta_h v\rangle\rangle_\omega$$

for all forms $u, v \in C^\infty_{p,\,q}(X,\, \mathbb{C})$ and all bidegrees $(p,\, q)$.

Formula 3.5.2. *For all $(p,\, q)$-forms u, v, we have*

$$\langle\langle u,\, v\rangle\rangle_{\omega_h} = \frac{1}{h^{2(n-p)}}\,\langle\langle u,\, v\rangle\rangle_\omega, \qquad \textit{hence} \qquad ||u||_{\omega_h} = h^{-(n-p)}\,||u||_\omega.$$

Proof. The formula follows at once from the last identity and from the fact that $\theta_h u = h^p u$ for all $(p,\, q)$-forms u. $\square$

Definition 3.5.3. *Let $(X,\, \omega)$ be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$. For every $k = 0, \dots, 2n$ and every constant $h > 0$, we consider the* d**-Laplacian w.r.t. the rescaled metric** ω_h *acting on C^∞ k-forms on X:*

$$\Delta_{\omega_h} : C^\infty_k(X,\, \mathbb{C}) \longrightarrow C^\infty_k(X,\, \mathbb{C}), \qquad \Delta_{\omega_h} := dd^\star_{\omega_h} + d^\star_{\omega_h} d,$$

where $d^\star_{\omega_h}$ is the formal adjoint of d w.r.t. $\langle\langle\;,\;\rangle\rangle_{\omega_h}$ and $\langle\langle\;,\;\rangle\rangle_{\omega_h}$ has been extended from the spaces $C^\infty_{p,\,q}(X,\, \mathbb{C})$ to $C^\infty_k(X,\, \mathbb{C}) = \oplus_{p+q=k} C^\infty_{p,\,q}(X,\, \mathbb{C})$ by sesquilinearity and by imposing that $\langle\langle u,\, v\rangle\rangle_{\omega_h} = 0$ whenever $u \in C^\infty_{p,\,q}(X,\, \mathbb{C})$ and $v \in C^\infty_{r,\,s}(X,\, \mathbb{C})$ with $(p,\, q) \neq (r,\, s)$.

(II) Rescaling the differential

The second operation we will consider is a *partial rescaling of* $d = \partial + \bar\partial$ that applies solely to its component of type (1, 0).

Definition 3.5.4. *Let X be a compact complex manifold, $dim_{\mathbb{C}}X = n$. For every constant $h > 0$, let*

$$d_h := h\partial + \bar\partial : C^\infty_k(X,\,\mathbb{C}) \longrightarrow C^\infty_{k+1}(X,\,\mathbb{C}), \qquad k \in \{0,\,\dots,2n\}.$$

Some basic properties of the rescaled differential d_h are summed up in the following

Lemma 3.5.5. (i) *The operators d and d_h are related by the identity*

$$d_h = \theta_h d \theta_h^{-1}.$$

(ii) *$d_h^2 = 0$ and the d- and d_h-cohomologies are related by the* **isomorphism**

$$H^k_d(X,\,\mathbb{C}) \stackrel{\simeq}{\longrightarrow} H^k_{d_h}(X,\,\mathbb{C}), \qquad \{u\}_d \mapsto \{\theta_h u\}_{d_h}$$

where $H^k_d(X,\,\mathbb{C}) = H^k_{DR}(X,\,\mathbb{C})$ are the usual De Rham cohomology groups, while $H^k_{d_h}(X,\,\mathbb{C}) :=$ $\ker(d_h : C^\infty_k(X,\,\mathbb{C}) \longrightarrow C^\infty_{k+1}(X,\,\mathbb{C}))/Im\,(d_h : C^\infty_{k-1}(X,\,\mathbb{C}) \longrightarrow C^\infty_k(X,\,\mathbb{C}))$ are the d_h-cohomology groups.

Proof. (i) If u is a $(p,\,q)$-form, we have

$$(\theta_h d\theta_h^{-1})(u) = \theta_h d(h^{-p}u) = h^{-p}\theta_h(\partial u) + h^{-p}\theta_h(\bar\partial u) = h^{-p}h^{p+1}\partial u + h^{-p}h^p\bar\partial u = h\partial u + \bar\partial u = d_h u.$$

Thus, $d_h = \theta_h d\theta_h^{-1}$ on pure-type forms, so this identity extends to arbitrary forms by linearity.

(ii) On the one hand, $d_h^2 = \theta_h d^2\theta_h^{-1} = 0$; on the other hand,

$$d_h(\theta_h u) = \theta_h du, \qquad \text{so we have the equivalence:} \qquad \theta_h u \in \ker(d_h) \iff u \in \ker d;$$

$$\theta_h u = d_h v \text{ iff } u = d(\theta_h^{-1}v), \qquad \text{so we have the equivalence:} \qquad \theta_h u \in \mathrm{Im}\,(d_h) \iff u \in \mathrm{Im}\,d.$$

These equivalences show that the linear map $H^k_d(X,\,\mathbb{C}) \ni \{u\}_d \mapsto \{\theta_h u\}_{d_h} \in H^k_{d_h}(X,\,\mathbb{C})$ is well defined and bijective. $\square$

In particular, the spectral sequences induced by the pairs of differentials $(\partial,\,\bar\partial)$ and $(h\partial,\,\bar\partial)$ are *isomorphic*, so degenerate at the same page. The first of them is the Frölicher spectral sequence of X.

Definition 3.5.6. *Let $(X,\,\omega)$ be a compact Hermitian manifold with $dim_{\mathbb{C}}X = n$. For every constant $h > 0$ and every degree $k \in \{0,\dots,2n\}$, we consider the d_h-***Laplacian w.r.t. the given metric** *ω acting on C^∞ k-forms on X:*

$$\Delta_h : C^\infty_k(X,\,\mathbb{C}) \longrightarrow C^\infty_k(X,\,\mathbb{C}), \qquad \Delta_h := d_h d_h^\star + d_h^\star d_h,$$

where $d_h^\star$ is the formal adjoint of d_h w.r.t. the L^2 inner product induced by ω.

(III) Comparison of the two rescaled Laplacians

We now bring together the above two operations by comparing the corresponding Laplace-type operators. Note that Δ_{ω_h} was defined by the rescaled differential d_h and the original metric ω, while Δ_h was induced by the rescaled metric ω_h and the original differential d.

Lemma 3.5.7. (i) *If $\theta_h^\star$ and $d_h^\star$ stand for the formal adjoints of θ_h, resp. d_h, w.r.t. the pointwise, resp. L^2, inner product induced by ω, we have*

$$\theta_h^\star = \theta_h \quad \text{and} \quad d_h^\star = \theta_h^{-1} d^\star \theta_h.$$

(ii) *The adjoints $\partial^\star_{\omega_h}$, $\bar\partial^\star_{\omega_h}$ w.r.t. to the metric ω_h, as well as the adjoints $\partial^\star_\omega = \partial^\star$, $\bar\partial^\star_\omega = \bar\partial^\star$ w.r.t. to the metric ω, of ∂, resp. $\bar\partial$ are related by the formulae:*

$$\partial^\star_{\omega_h} = h^2 \partial^\star \quad \text{and} \quad \bar\partial^\star_{\omega_h} = \bar\partial^\star.$$

Consequently, we get

$$\begin{aligned}\Delta_{\omega_h} &= h^2\Delta' + \Delta'' + [\partial,\,\bar\partial^\star] + h^2[\bar\partial,\,\partial^\star]\\ &= h^2\Delta' + \Delta'' - [\partial,\,\bar\tau^\star] - h^2[\bar\tau,\,\partial^\star] = h^2\Delta' + \Delta'' - [\tau,\,\bar\partial^\star] - h^2[\bar\partial,\,\tau^\star],\end{aligned}$$

and

$$\begin{aligned}\Delta_h &= h^2\Delta' + \Delta'' + h[\partial,\,\bar\partial^\star] + h[\bar\partial,\,\partial^\star]\\ &= h^2\Delta' + \Delta'' - h[\partial,\,\bar\tau^\star] - h[\bar\tau,\,\partial^\star] = h^2\Delta' + \Delta'' - h[\tau,\,\bar\partial^\star] - h[\bar\partial,\,\tau^\star],\end{aligned}$$

where the adjoints $\partial^\star, \bar\partial^\star, \tau^\star, \bar\tau^\star$ and the Laplacians Δ', Δ'' are computed w.r.t. the metric ω, while

$$\tau = \tau_\omega := [\Lambda_\omega,\,\partial\omega\wedge\cdot] : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p+1,\,q}(X,\,\mathbb{C})$$

is the **torsion operator** *(of type $(1,\,0)$ and order zero, acting on smooth forms of any bidegree $(p,\,q)$, where Λ_ω is the adjoint of the multiplication operator $\omega\wedge\cdot$) associated with the metric ω as defined in* [Dem84] *(see also* [Dem97, VII, §.1]*).*

In particular, the second-order Laplacians Δ_{ω_h} and Δ_h are **elliptic** *since the second-order Laplacians Δ' and Δ'' are and the deviation terms $-[\partial,\,\bar\tau^\star] - h^2[\bar\tau,\,\partial^\star]$ and $-h[\partial,\,\bar\tau^\star] - h[\bar\tau,\,\partial^\star]$ are only of order* 1.

Note that $\langle\langle[\partial,\,\bar\partial^\star]u,\,u\rangle\rangle = \langle\langle[\bar\partial,\,\partial^\star]u,\,u\rangle\rangle = 0$ whenever the form u is **of pure type** *and whatever metric is used to define $\langle\langle\ ,\ \rangle\rangle$ (because pure-type forms of different bidegrees are orthogonal w.r.t. any metric), so*

$$\langle\langle\Delta_{\omega_h}u,\,u\rangle\rangle = \langle\langle\Delta_h u,\,u\rangle\rangle = h^2\,\langle\langle\Delta' u,\,u\rangle\rangle + \langle\langle\Delta'' u,\,u\rangle\rangle \quad \textit{for every } \textbf{pure-type} \textit{ form } u. \tag{3.47}$$

(This fails, in general, if u is not of pure type, unless the metric ω is Kähler.)

(iii) *The rescaled Laplacians Δ_{ω_h} and Δ_h are related by the formula*

$$\Delta_h = \theta_h \Delta_{\omega_h} \theta_h^{-1}. \tag{3.48}$$

Proof. (i) For any k-forms $u = \sum\limits_{p+q=k} u^{p,\,q}$ and $v = \sum\limits_{p+q=k} v^{p,\,q}$, we have

$$\langle\theta_h u,\,v\rangle_\omega = \sum_{p+q=k}\langle h^p u^{p,\,q},\,v^{p,\,q}\rangle_\omega = \sum_{p+q=k}\langle u^{p,\,q},\,h^p v^{p,\,q}\rangle_\omega = \langle u,\,\theta_h v\rangle_\omega, \qquad \text{so } \theta_h^\star = \theta_h.$$

The second identity in (i) follows by taking conjugates in $d_h = \theta_h d\theta_h^{-1}$.

(ii) For any forms $\alpha \in C^\infty_{p-1,\,q}(X,\,\mathbb{C})$ and $\beta \in C^\infty_{p,\,q}(X,\,\mathbb{C})$, we have

$$\begin{aligned}\langle\langle\alpha,\,\partial^\star_\omega\beta\rangle\rangle_\omega &= \langle\langle\partial\alpha,\,\beta\rangle\rangle_\omega = \int\limits_X \langle\partial\alpha,\,\beta\rangle_\omega\,dV_\omega = \int\limits_X \frac{1}{h^{2p}}\,\langle\partial\alpha,\,\beta\rangle_{\omega_h}\,h^{2n}dV_{\omega_h} = h^{2(n-p)}\,\langle\langle\partial\alpha,\,\beta\rangle\rangle_{\omega_h}\\ &= h^{2(n-p)}\,\langle\langle\alpha,\,\partial^\star_{\omega_h}\beta\rangle\rangle_{\omega_h} = h^{2(n-p)}\int\limits_X h^{2(p-1)}\,\langle\alpha,\,\partial^\star_{\omega_h}\beta\rangle_\omega\,\frac{1}{h^{2n}}\,dV_\omega = \frac{1}{h^2}\,\langle\langle\alpha,\,\partial^\star_{\omega_h}\beta\rangle\rangle_\omega.\end{aligned}$$

We get $\partial^\star_\omega = h^{-2}\,\partial^\star_{\omega_h}$, which is the first identity under (ii).

The identity $\bar\partial^\star_{\omega_h} = \bar\partial^\star_\omega$ is proved in the same way by using the fact that $\bar\partial$ acts only on the anti-holomorphic degree of forms which is unaffected by the change of metric from ω to ω_h.

Using these formulae, we get

$$\begin{aligned}\Delta_{\omega_h} &= [\partial+\bar\partial,\,\partial^\star_{\omega_h}+\bar\partial^\star_{\omega_h}] = [\partial,\,h^2\partial^\star] + [\bar\partial,\,\bar\partial^\star] + [\partial,\,\bar\partial^\star] + [\bar\partial,\,h^2\partial^\star]\\ &= h^2\Delta' + \Delta'' + [\partial,\,\bar\partial^\star] + h^2[\bar\partial,\,\partial^\star]\end{aligned}$$

and

$$\begin{aligned}\Delta_h &= [h\partial+\bar\partial,\,h\partial^\star+\bar\partial^\star] = h^2[\partial,\,\partial^\star] + [\bar\partial,\,\bar\partial^\star] + h[\partial,\,\bar\partial^\star] + h[\bar\partial,\,\partial^\star]\\ &= h^2\Delta' + \Delta'' + h[\partial,\,\bar\partial^\star] + h[\bar\partial,\,\partial^\star].\end{aligned}$$

On the other hand, we know from [Dem84] (or [Dem97, VII, §.1]) that

$$[\partial,\,\bar\partial^\star] = -[\partial,\,\bar\tau^\star] = -[\tau,\,\bar\partial^\star] \quad \text{and, by conjugation, we get} \quad [\bar\partial,\,\partial^\star] = -[\bar\partial,\,\tau^\star] = -[\bar\tau,\,\partial^\star].$$

So, the terms measuring the deviations of Δ_{ω_h} and Δ_h from $h^2\Delta' + \Delta''$ are of order 1 and we get the alternative formulae for Δ_{ω_h} and Δ_h spelt out in the statement.

(iii) For any smooth $(p,\,q)$-form α, we have

$$\begin{aligned}(\theta_h\Delta_{\omega_h}\theta_h^{-1})\alpha &= \frac{1}{h^p}\theta_h\Delta_{\omega_h}\alpha = \frac{1}{h^p}\theta_h(h^2\Delta'\alpha) + \frac{1}{h^p}\theta_h(\Delta''\alpha) + \frac{1}{h^p}\theta_h([\partial,\,\bar\partial^\star]\alpha) + \frac{1}{h^p}\theta_h(h^2[\bar\partial,\,\partial^\star]\alpha)\\ &= \frac{h^2h^p}{h^p}\Delta'\alpha + \frac{h^p}{h^p}\Delta''\alpha + \frac{h^{p+1}}{h^p}[\partial,\,\bar\partial^\star]\alpha + \frac{h^2h^{p-1}}{h^p}[\bar\partial,\,\partial^\star]\alpha\\ &= h^2\Delta'\alpha + \Delta''\alpha + h[\partial,\,\bar\partial^\star]\alpha + h[\bar\partial,\,\partial^\star]\alpha = \Delta_h\alpha.\end{aligned}$$

Thus, $\theta_h\Delta_{\omega_h}\theta_h^{-1} = \Delta_h$ on pure-type forms and this identity extends to arbitrary forms by linearity. □

Corollary 3.5.8. *Let $(X,\,\omega)$ be a compact Hermitian manifold with $dim_{\mathbb{C}}X = n$. For every constant $h > 0$ and every degree $k \in \{0,\dots,2n\}$, the* **spectra** *of the rescaled Laplacians $\Delta_h, \Delta_{\omega_h} : C^\infty_k(X,\,\mathbb{C}) \longrightarrow C^\infty_k(X,\,\mathbb{C})$* **coincide***, i.e.*

$$Spec(\Delta_h) = Spec(\Delta_{\omega_h}), \tag{3.49}$$

and their respective **eigenspaces** *are obtained from each other via the rescaling isometry θ_h:*

$$\theta_h(E_{\Delta_{\omega_h}}(\lambda)) = E_{\Delta_h}(\lambda) \qquad \text{for every } \lambda \in Spec(\Delta_h) = Spec(\Delta_{\omega_h}), \tag{3.50}$$

where $E_{\Delta_{\omega_h}}(\lambda)$, resp. $E_{\Delta_h}(\lambda)$, stands for the eigenspace corresponding to the eigenvalue λ of the operator Δ_{ω_h}, resp. Δ_h.

Thus, Δ_h and Δ_{ω_h} have the **same eigenvalues** *with the* **same multiplicities***.*

Proof. Let $\lambda \in \mathrm{Spec}(\Delta_{\omega_h})$ and let $\alpha \in E_{\Delta_{\omega_h}}(\lambda) \subset C^\infty_k(X,\,\mathbb{C})$. So $\Delta_{\omega_h}\alpha = \lambda\alpha$, hence

$$\Delta_h(\theta_h\alpha) = (\theta_h\Delta_{\omega_h}\theta_h^{-1})(\theta_h\alpha) = \theta_h(\lambda\alpha) = \lambda(\theta_h\alpha).$$

Thus, $\lambda \in \mathrm{Spec}(\Delta_h)$ and $\theta_h\alpha \in E_{\Delta_h}(\lambda)$. These implications also hold in reverse order, so we get the equivalences:

$$\lambda \in \mathrm{Spec}(\Delta_h) \iff \lambda \in \mathrm{Spec}(\Delta_{\omega_h}) \quad \text{and} \quad \alpha \in E_{\Delta_{\omega_h}}(\lambda) \iff \theta_h\alpha \in E_{\Delta_h}(\lambda).$$

These equivalences amount to (3.49) and (3.50). □

Another consequence of the above discussion is a Hodge Theory for the d_h-cohomology and the resulting equidimensionality of the kernels of Δ and Δ_h in every degree.

Corollary 3.5.9. *Let $(X,\,\omega)$ be a compact Hermitian manifold with $\dim_{\mathbb{C}}X = n$. For every constant $h>0$ and every degree $k \in \{0,\dots,2n\}$, the operator $d_h : C^\infty_k(X,\,\mathbb{C}) \longrightarrow C^\infty_k(X,\,\mathbb{C})$ induces the following L^2_ω-orthogonal direct-sum decomposition:*

$$C^\infty_k(X,\,\mathbb{C}) = \mathcal{H}^k_{\Delta_h}(X,\,\mathbb{C}) \oplus Im\,d_h \oplus Im\,d_h^\star,$$

where $\mathcal{H}^k_{\Delta_h}(X,\,\mathbb{C})$ is the kernel of $\Delta_h : C^\infty_k(X,\,\mathbb{C}) \longrightarrow C^\infty_k(X,\,\mathbb{C})$ and $\ker d_h = \mathcal{H}^k_{\Delta_h}(X,\,\mathbb{C}) \oplus Im\,d_h$. *The vector space $\mathcal{H}^k_{\Delta_h}(X,\,\mathbb{C})$ is* **finite-dimensional**, *while $Im\,d_h$ and $Im\,d_h^\star$ are* **closed** *subspaces of $C^\infty_k(X,\,\mathbb{C})$.*

This, in turn, induces the **Hodge isomorphism**

$$\mathcal{H}^k_{\Delta_h}(X,\,\mathbb{C}) \simeq H^k_{d_h}(X,\,\mathbb{C}), \qquad \alpha \mapsto \{\alpha\}_{d_h}.$$

Since $H^k_d(X,\,\mathbb{C})$ and $H^k_{d_h}(X,\,\mathbb{C})$ are isomorphic (via θ_h, see Lemma 3.5.5) and $\mathcal{H}^k_\Delta(X,\,\mathbb{C}) \simeq H^k_d(X,\,\mathbb{C})$ (by standard Hodge theory), we infer that $\mathcal{H}^k_\Delta(X,\,\mathbb{C})$ and $\mathcal{H}^k_{\Delta_h}(X,\,\mathbb{C})$ are **isomorphic** *(although the isomorphism need not be defined by θ_h). In particular,*

$$dim\,\mathcal{H}^k_{\Delta_h}(X,\,\mathbb{C}) = dim\,\mathcal{H}^k_\Delta(X,\,\mathbb{C}) \quad \textit{for all } h>0.$$

Proof. Since X is compact and Δ_h is elliptic and self-adjoint, a standard consequence of Gårding's inequality (see e.g. [Dem97, VI]) yields the two-space orthogonal decomposition $C^\infty_k(X,\,\mathbb{C}) = \mathcal{H}^k_{\Delta_h}(X,\,\mathbb{C}) \oplus \mathrm{Im}\,\Delta_h$, while this, together with the integrability property $d_h^2 = 0$, further induces the orthogonal splitting $\mathrm{Im}\,\Delta_h = \mathrm{Im}\,d_h \oplus \mathrm{Im}\,d_h^\star$. The same consequence of Gårding's inequality ensures that $\ker\Delta_h$ is finite-dimensional and that the images in $C^\infty_k(X,\,\mathbb{C})$ of d_h and $d_h^\star$ are closed. □

3.5.2 The differentials in the Frölicher spectral sequence

Let X be a compact complex manifold with $\dim_{\mathbb{C}}X = n$. Recall the construction of the Frölicher spectral sequence and the notation in §.1.2. For the sake of simplicity, we will write $E_r^{p,\,q}$ instead of $E_r^{p,\,q}(X)$. On the 0-th page, we have the *Dolbeault complex*:

$$\dots \xrightarrow{d_0} E_0^{p,\,q-1} \xrightarrow{d_0} E_0^{p,\,q} \xrightarrow{d_0} E_0^{p,\,q+1} \xrightarrow{d_0} \dots.$$

Thus, in every bidegree $(p,\,q)$, the inclusions $\mathrm{Im}\,d_0^{p,\,q-1} \subset \ker d_0^{p,\,q} \subset E_0^{p,\,q}$ induce (infinitely many, non-canonical) isomorphisms:

$$C^\infty_{p,\,q}(X,\,\mathbb{C}) \simeq \operatorname{Im} d_0^{p,\,q-1} \oplus E_1^{p,\,q} \oplus (E_0^{p,\,q}/\ker d_0^{p,\,q}), \tag{3.51}$$

where $d_0 = d_0^{p,\,q} : E_0^{p,\,q} \longrightarrow E_0^{p,\,q+1}$ is the differential d_0 acting in bidegree $(p,\,q)$ and the $E_1^{p,\,q} := \ker d_0^{p,\,q}/\operatorname{Im} d_0^{p,\,q-1} = H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ are the *Dolbeault cohomology groups* of X.

On the first page E_1 of the FSS, we have the type-$(1,\,0)$ differentials d_1:

$$\dots \xrightarrow{d_1} E_1^{p-1,\,q} \xrightarrow{d_1} E_1^{p,\,q} \xrightarrow{d_1} E_1^{p+1,\,q} \xrightarrow{d_1} \dots.$$

induced in cohomology by ∂ (i.e. $d_1(\{\alpha\}_{\bar\partial}) := \{\partial\alpha\}_{\bar\partial}$). Thus, in every bidegree $(p,\,q)$, the inclusions $\operatorname{Im} d_1^{p-1,\,q} \subset \ker d_1^{p,\,q} \subset E_1^{p,\,q}$ induce (infinitely many, non-canonical) isomorphisms:

$$E_1^{p,\,q} \simeq \operatorname{Im} d_1^{p-1,\,q} \oplus E_2^{p,\,q} \oplus (E_1^{p,\,q}/\ker d_1^{p,\,q}), \tag{3.52}$$

where $d_1^{p,\,q}$ is d_1 acting in bidegree $(p,\,q)$, while the spaces $E_2^{p,\,q} := \ker d_1^{p,\,q}/\operatorname{Im} d_1^{p-1,\,q}$ form the cohomology of the page E_1.

The remaining pages are constructed inductively: the differentials $d_r = d_r^{p,\,q} : E_r^{p,\,q} \longrightarrow E_r^{p+r,\,q-r+1}$ are of type $(r,\,-r+1)$ for every r, while the spaces $E_r^{p,\,q} := \ker d_{r-1}^{p,\,q}/\operatorname{Im} d_{r-1}^{p-r+1,\,q+r-2}$ on the r^{th} page are defined as the cohomology of the previous page E_{r-1}. On every page E_r and in every bidegree $(p,\,q)$, the inclusions $\operatorname{Im} d_r^{p-r,\,q+r-1} \subset \ker d_r^{p,\,q} \subset E_r^{p,\,q}$ induce (infinitely many, non-canonical) isomorphisms:

$$E_r^{p,\,q} \simeq \operatorname{Im} d_r^{p-r,\,q+r-1} \oplus E_{r+1}^{p,\,q} \oplus (E_r^{p,\,q}/\ker d_r^{p,\,q}), \tag{3.53}$$

where $E_{r+1}^{p,\,q} := \ker d_r^{p,\,q}/\operatorname{Im} d_r^{p-r,\,q+r-1}$.

It is worth stressing that (3.51), (3.52) and (3.53) only assert that the vector spaces on either side of $\simeq$ are isomorphic, but no choice of preferred isomorphism is possible at this stage.

(I) Identification of the d_r's with restrictions of d

Summing up (3.51), (3.52), (3.53) over $r = 0, \dots, N-1$, we get (infinitely many, non-canonical) isomorphisms

$$C^\infty_{p,\,q}(X,\,\mathbb{C}) \simeq \bigoplus_{r=0}^{N-1} \operatorname{Im} d_r^{p-r,\,q+r-1} \oplus E_\infty^{p,\,q} \oplus \bigoplus_{r=0}^{N-1} (E_r^{p,\,q}/\ker d_r^{p,\,q})$$

for every bidegree $(p,\,q)$. Note that the isomorphisms (3.51), (3.52), (3.53) identify the spaces $\operatorname{Im} d_r^{p-r,\,q+r-1}$, $E_r^{p,\,q}$ (including for $r = \infty$) and $E_r^{p,\,q}/\ker d_r^{p,\,q}$ with certain subspaces of $C^\infty_{p,\,q}(X,\,\mathbb{C})$. However, these subspaces have not been specified yet since multiple choices (and no canonical choice) are possible for the isomorphisms (3.51), (3.52), (3.53). These choices can only be made unique once a Hermitian metric has been fixed on X. (See (II) below.)

Now, since $C^\infty_k(X,\,\mathbb{C}) = \oplus_{p+q=k} C^\infty_{p,\,q}(X,\,\mathbb{C})$ for all $k = 0, \dots, 2n$, we get

$$\begin{array}{ccc}
C^\infty_k(X,\,\mathbb{C}) \simeq & & \displaystyle\bigoplus_{\substack{0\leq r\leq N-1\\ p+q=k}} \operatorname{Im} d_r^{p-r,\,q+r-1} \oplus \bigoplus_{p+q=k} E_\infty^{p,\,q} \oplus \bigoplus_{\substack{0\leq r\leq N-1\\ p+q=k}} (E_r^{p,\,q}/\ker d_r^{p,\,q}) \\
\Big\downarrow d & & \\
C^\infty_{k+1}(X,\,\mathbb{C}) \simeq & & \displaystyle\bigoplus_{\substack{0\leq r\leq N-1\\ p+q=k}} \operatorname{Im} d_r^{p,\,q} \oplus \bigoplus_{p'+q'=k+1} E_\infty^{p',\,q'} \oplus \bigoplus_{\substack{0\leq r\leq N-1\\ p+q=k}} (E_r^{p+r,\,q-r+1}/\ker d_r^{p+r,\,q-r+1}).
\end{array}$$

Thus, under these isomorphisms, the operator $d = d^{(k)} : C^\infty_k(X,\,\mathbb{C}) \longrightarrow C^\infty_{k+1}(X,\,\mathbb{C})$ identifies as

$$d^{(k)} \simeq \bigoplus_{\substack{0\leq r\leq N-1\\ p+q=k}} d_r^{p,\,q}, \tag{3.54}$$

where the isomorphism $d_r^{p,\,q} : E_r^{p,\,q}/\ker d_r^{p,\,q} \longrightarrow \operatorname{Im} d_r^{p,\,q}$ is the restriction of $d_r = d_r^{p,\,q} : E_r^{p,\,q} \longrightarrow E_r^{p+r,\,q-r+1}$ to the third piece on the r.h.s. of (3.53). The fact that d_r is of type $(r,\,-r+1)$ will play a key role in the sequel.

On the other hand, summing up the splittings of $C^\infty_{p,\,q}(X,\,\mathbb{C})$ over $p\geq s$ for any given s, we get

$$\mathcal{A}^k_s := \bigoplus_{\substack{p\geq s\\ p+q=k}} C^\infty_{p,\,q}(X,\,\mathbb{C}) \simeq \bigoplus_{\substack{p\geq s\\ p+q=k}} \left[\bigoplus_{r=0}^{N-1} \operatorname{Im} d_r^{p-r,\,q+r-1} \oplus E_\infty^{p,\,q} \oplus \bigoplus_{r=0}^{N-1}(E_r^{p,\,q}/\ker d_r^{p,\,q})\right].$$

Lemma 3.5.10. (i) *For every r and every k, let $E_r^k := \bigoplus_{p+q=k} E_r^{p,\,q}$. Then*

$$dim\, E_r^k = \sum_{p+q=k} dim\, E_r^{p,\,q} = b_k + m_r^{k-1} + m_r^k, \qquad 0\leq r\leq N,\ \ 0\leq k\leq 2n, \tag{3.55}$$

where we set $m_r^k := \sum_{l\geq r}\sum_{p+q=k} dim\,(E_l^{p,\,q}/\ker d_l^{p,\,q})$.

(ii) *For every r and every k, let $L_r^{p,\,q} := \bigoplus_{l\geq r}(E_l^{p,\,q}/\ker d_l^{p,\,q})$ and $L_r^k := \bigoplus_{p+q=k} L_r^{p,\,q}$. Then, $dim\, L_r^k = m_r^k$ (obvious) and, under the identifications defined by the isomorphisms (3.51), (3.52), (3.53), the following inclusions hold:*

$$d(L_r^{p,\,q}) \subset \mathcal{A}^{p+q+1}_{p+r}, \qquad 0\leq r\leq N,\ \ 0\leq p,q\leq n, \tag{3.56}$$

where $d(L_r^{p,\,q}) := \oplus_{l\geq r} d_l^{p,\,q}(E_l^{p,\,q}/\ker d_l^{p,\,q})$ in keeping with identification (3.54).

Proof. (i) For every fixed r, summing up the splittings (3.53) with l in place of r over $l\geq r$ and then summing up over $p+q=k$ for every fixed k, we get

$$E_r^k \simeq \bigoplus_{p+q=k} E_\infty^{p,\,q} \oplus \bigoplus_{l\geq r}\bigoplus_{p+q=k} \operatorname{Im} d_l^{p-l,\,q+l-1} \oplus \bigoplus_{l\geq r}\bigoplus_{p+q=k}(E_l^{p,\,q}/\ker d_l^{p,\,q}).$$

Since $\operatorname{Im} d_l^{p-l,\,q+l-1} \simeq E_l^{p-l,\,q+l-1}/\ker d_l^{p-l,\,q+l-1}$ for all p,q,l, if we set $p' := p-l$ and $q' := q+l-1$, we have $p'+q' = k-1$ when $p+q=k$ and the above isomorphism translates to

$$E_r^k \simeq \bigoplus_{p+q=k} E_\infty^{p,\,q} \oplus \bigoplus_{l\geq r}\bigoplus_{p'+q'=k-1}(E_l^{p',\,q'}/\ker d_l^{p',\,q'}) \oplus \bigoplus_{l\geq r}\bigoplus_{p+q=k}(E_l^{p,\,q}/\ker d_l^{p,\,q})$$

for every k. Now, $\dim \oplus_{p+q=k} E_\infty^{p,\,q} = b_k$ (the k^{th} Betti number of X) thanks to (1.16), so taking dimensions in the above isomorphism, we get (3.55).

(ii) Since $d_l^{p,\,q} : E_l^{p,\,q}/\ker d_l^{p,\,q} \longrightarrow \operatorname{Im} d_l^{p,\,q}$ is an isomorphism of type $(l,\,-l+1)$ for all l,p,q, we get for all $l\geq r$:

$$d(L_r^{p,\,q}) = \bigoplus_{l\geq r} d_l^{p,\,q}(E_l^{p,\,q}/\ker d_l^{p,\,q}) \quad \text{and} \quad d_l^{p,\,q}(E_l^{p,\,q}/\ker d_l^{p,\,q}) \subset E_l^{p+l,\,q-l+1} \subset C^\infty_{p+l,\,q-l+1} \subset \mathcal{A}^{p+q+1}_{p+r}$$

under the identification of each space $E_l^{p+l,\,q-l+1}$ with a subspace of $C^\infty_{p+l,\,q-l+1}$ defined by the isomorphisms (3.51), (3.52), (3.53). This proves (3.56). □

(II) Explicit description of the above identifications

We take this opportunity to point out an explicit description of the differentials d_r in cohomology and of their unique realisations induced by a given Hermitian metric on X. Recall that d acts as d_r on representatives of E_r-classes (cf. (3.54)).

Every fixed Hermitian metric ω on X selects a unique realisation of each of the isomorphisms (3.51), (3.52) and (3.53) and, implicitly, identifies each space $E_r^{p,\,q}$ with a precise subspace $\mathcal{H}_r^{p,\,q}$ (depending on ω) of $C^\infty_{p,\,q}(X,\,\mathbb{C})$ via an isomorphism $E_r^{p,\,q}\simeq\mathcal{H}_r^{p,\,q}$ depending on ω. These *harmonic* subspaces $\mathcal{H}_r^{p,\,q}\subset C^\infty_{p,\,q}(X,\,\mathbb{C})$ are constructed by induction on $r\geq 1$ as follows.

Definition 3.5.11. *Let $\mathcal{H}_1^{p,\,q}\subset C^\infty_{p,\,q}(X,\,\mathbb{C})$ be the orthogonal complement for the L^2_ω-norm of $Im\, d_0^{p,\,q-1}$ in $\ker d_0^{p,\,q}$. Due to (3.51), $\mathcal{H}_1^{p,\,q}$ is isomorphic to $E_1^{p,\,q}$. In every bidegree $(p,\,q)$, the linear map $d_1^{p,\,q}:E_1^{p,\,q}\longrightarrow E_1^{p+1,\,q}$ induces a linear map (denoted by the same symbol) $d_1^{p,\,q}:\mathcal{H}_1^{p,\,q}\longrightarrow\mathcal{H}_1^{p+1,\,q}$ via the isomorphisms $\mathcal{H}_1^{p,\,q}\simeq E_1^{p,\,q}$ and $\mathcal{H}_1^{p+1,\,q}\simeq E_1^{p+1,\,q}$. Let $\mathcal{H}_2^{p,\,q}\subset\mathcal{H}_1^{p,\,q}\subset C^\infty_{p,\,q}(X,\,\mathbb{C})$ be the orthogonal complement for the L^2_ω-norm of $Im\, d_1^{p-1,\,q}$ in $\ker d_1^{p,\,q}$ (viewed as subspaces of $\mathcal{H}_1^{p,\,q}$). Due to (3.52), $\mathcal{H}_2^{p,\,q}$ is isomorphic to $E_2^{p,\,q}$. Continuing inductively, when the linear maps $d_r^{p,\,q}:E_r^{p,\,q}\longrightarrow E_r^{p+r,\,q-r+1}$ have induced counterparts (denoted by the same symbol) $d_r^{p,\,q}:\mathcal{H}_r^{p,\,q}\longrightarrow\mathcal{H}_r^{p+r,\,q-r+1}$ between the already constructed subspaces $\mathcal{H}_r^{p,\,q}\subset C^\infty_{p,\,q}(X,\,\mathbb{C})$ and $\mathcal{H}_r^{p+r,\,q-r+1}\subset C^\infty_{p+r,\,q-r+1}(X,\,\mathbb{C})$, we let $\mathcal{H}_{r+1}^{p,\,q}\subset\mathcal{H}_r^{p,\,q}\subset C^\infty_{p,\,q}(X,\,\mathbb{C})$ be the orthogonal complement for the L^2_ω-norm of $Im\, d_r^{p-r,\,q+r-1}$ in $\ker d_r^{p,\,q}$ (viewed as subspaces of $\mathcal{H}_r^{p,\,q}$). Due to (3.53), $\mathcal{H}_{r+1}^{p,\,q}$ is isomorphic to $E_{r+1}^{p,\,q}$.*

Note that we have

$$\begin{aligned}\mathcal{H}_1^{p,\,q} &= \ker(\Delta'':C^\infty_{p,\,q}(X,\,\mathbb{C})\longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}))=\{u\in C^\infty_{p,\,q}(X,\,\mathbb{C})\ \mid\ \bar\partial u=0 \text{ and } \bar\partial^\star u=0\},\\ \mathcal{H}_2^{p,\,q} &= \ker(\widetilde\Delta:C^\infty_{p,\,q}(X,\,\mathbb{C})\longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}))\\ &= \{u\in C^\infty_{p,\,q}(X,\,\mathbb{C})\ \mid\ \bar\partial u=0,\ \bar\partial^\star u=0,\ p''(\partial u)=0 \text{ and } p''\partial^\star u=0\},\end{aligned}\tag{3.57}$$

where $\widetilde\Delta=\partial p''\partial^\star+\partial^\star p''\partial+\Delta''$ is the pseudo-differential Laplacian of Definition 3.1.2.

Also note that standard Hodge theory (for the elliptic differential operator Δ'') is used to ensure that $\mathrm{Im}\, d_0^{p,\,q-1}$ is closed in $C^\infty_{p,\,q}(X,\,\mathbb{C})$ and that $\mathcal{H}_1^{p,\,q}$ is finite-dimensional. However, all the other images $\mathrm{Im}\, d_r^{p-r,\,q+r-1}$ are automatically closed since they are (necessarily finite-dimensional) vector subspaces of a finite-dimensional vector space.

When the vector space $C^\infty_{p,\,q}(X,\,\mathbb{C})$ is endowed with the L^2-norm induced by ω, every subspace $\mathcal{H}_r^{p,\,q}$ inherits the restricted norm. On the other hand, every space $E_r^{p,\,q}$ has a quotient norm induced by the L^2_ω-norm. The isomorphisms $E_r^{p,\,q}\simeq\mathcal{H}_r^{p,\,q}$ constructed above are isometries when $E_r^{p,\,q}$ and $\mathcal{H}_r^{p,\,q}$ are endowed with the quotient, resp. L^2 norms.

Conclusion 3.5.12. *Let X be a compact complex manifold and let ω be any Hermitian metric on X. Let $\cdots\subset\mathcal{H}_{r+1}^{p,\,q}\subset\mathcal{H}_r^{p,\,q}\subset\cdots\subset\mathcal{H}_1^{p,\,q}\subset C^\infty_{p,\,q}(X,\,\mathbb{C})$ be the subspaces of Definition 3.5.11 induced by ω.*

*For every r and every bidegree $(p,\,q)$, each class $\{\alpha\}_{E_r}\in E_r^{p,\,q}$ contains a unique representative $\alpha\in\mathcal{H}_r^{p,\,q}$ (necessarily satisfying condition (P_r)). For $l\in\{1,\dots,r-1\}$, let $u_l\in C^\infty_{p+l,\,q-l}(X,\,\mathbb{C})$ be the **unique** solutions with **minimal** L^2_ω-norms of the equations:*

$$\bar\partial\alpha=0,\ \partial\alpha=\bar\partial u_1,\ \partial u_1=\bar\partial u_2,\dots,\partial u_{r-2}=\bar\partial u_{r-1}$$

constructed inductively from one another. The well-known Neumann formula yields:

$$u_1=\Delta''^{-1}\bar\partial^\star(\partial\alpha)\quad\text{and}\quad u_l=\Delta''^{-1}\bar\partial^\star(\partial u_{l-1})\quad\text{for}\quad l\in\{2,\dots,r-1\}.$$

In particular, the maps $\alpha \mapsto u_1$ and $u_{l-1} \mapsto u_l$ are linear.

For all r, p, q, we define the linear operator

$$T_r = T_r^{p,\,q} : \mathcal{H}_r^{p,\,q} \longrightarrow C^\infty_{p+r,\,q-r+1}(X,\,\mathbb{C}), \qquad \alpha \mapsto T_r(\alpha) := \partial u_{r-1}.$$

Since $\mathcal{H}_r^{p,\,q}$ is finite-dimensional, T_r is bounded, so there exists a constant $C_r^{p,\,q} > 0$ such that

$$||T_r(\alpha)|| = ||\partial u_{r-1}|| \leq C_r^{p,\,q}\,||\alpha|| \quad \textit{for all } \alpha \in \mathcal{H}_r^{p,\,q}.$$

It is easy to see that $T_r(\alpha)$ need not belong to $\mathcal{H}_r^{p+r,\,q-r+1}$ when $\alpha \in \mathcal{H}_r^{p,\,q}$. If we let $P_r^{p,\,q} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow \mathcal{H}_r^{p,\,q}$ be the L_ω-orthogonal projection onto $\mathcal{H}_r^{p,\,q}$, we get

$$||(P_r^{p,\,q} \circ T_r)(\alpha)|| = ||P_r^{p,\,q}(\partial u_{r-1})|| \leq ||\partial u_{r-1}|| \leq C_r^{p,\,q}\,||\alpha|| \quad \textit{for all } \alpha \in \mathcal{H}_r^{p,\,q}.$$

3.5.3 Use of the rescaled Laplacians in the study of the Frölicher spectral sequence

In this subsection, we prove the main result of this section, namely

Theorem 3.5.13. *Let $(X,\,\omega)$ be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$. For every $r \in \mathbb{N}^\star$ and every $k = 0, \dots, 2n$, the following identity holds:*

$$\dim_{\mathbb{C}} E_r^k = \sharp\{i \mid \lambda_i^k(h) \in O(h^{2r}) \quad \textit{as } h \downarrow 0\}, \tag{3.58}$$

where $E_r^k := \oplus_{p+q=k} E_r^{p,\,q}$ is the direct sum of the spaces of total degree k on the r^{th} page of the Frölicher spectral sequence of X, while $0 \leq \lambda_1^k(h) \leq \lambda_2^k(h) \leq \dots \leq \lambda_i^k(h) \leq \dots$ are the eigenvalues, counted with multiplicities, of the rescaled Laplacian $\Delta_h : C_k^\infty(X,\,\mathbb{C}) \longrightarrow C_k^\infty(X,\,\mathbb{C})$ (= those of $\Delta_{\omega_h} : C_k^\infty(X,\,\mathbb{C}) \longrightarrow C_k^\infty(X,\,\mathbb{C})$) acting on k-forms. As usual, $\sharp$ stands for the cardinal of a set.

This result and its proof are strongly inspired by the analogous result for foliations proved by Álvarez López and Kordyukov in [ALK00]. However, to our knowledge, this particular form of the result in the context of the Frölicher spectral sequence did not appear anywhere before [Pop17] and is of independent interest.

As in [ES89], [GS91], [ALK00], we consider the *spectrum distribution function* associated with any of the rescaled Laplacians Δ_h, Δ_{ω_h} in our context. Its definition and its study are made far simpler in this setting than in those references by the manifold X being *compact* and by the Laplacians Δ', Δ'' being *elliptic*.

Notation 3.5.14. *Let $(X,\,\omega)$ be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$. For every $k \in \{0, \dots, n\}$ and every constant $\lambda \geq 0$, let $N_h^k(\lambda)$ stand for the number of eigenvalues (counted with multiplicities) of Δ_h that are $\leq \lambda$.*

Replacing Δ_h with Δ_{ω_h} does not change the spectrum distribution function $N_h^k : [0,\,+\infty) \longrightarrow \mathbb{N}$ since Δ_h and Δ_{ω_h} have the same eigenvalues with the same multiplicities (cf. Corollary 3.5.8). Theorem 3.5.13 can be reworded as ensuring the existence of a constant $C > 0$ independent of h such that, for all r and k, we have

$$\dim E_r^k = N_h^k(Ch^{2r}) \quad \text{when } 0 < h \ll 1. \tag{3.59}$$

(I) The Efremov-Shubin variational principle

The main technical ingredient we will need is the following variant of the *variational principle* proved in a more general context in [ES89] and used extensively thereafter (e.g. [GS91], [ALK00]) in settings different from ours. We adapt to our situation the result of [ES89].

Proposition 3.5.15. *(see e.g. Efremov-Shubin [ES89]) Let $(X,\,\omega)$ be a compact Hermitian manifold with $dim_{\mathbb{C}}X = n$. For every $k = 0, \dots, 2n$ and every $\lambda \geq 0$, the following identity holds*

$$N_h^k(\lambda) = F_h^{k-1}(\lambda) + b_k + F_h^k(\lambda), \tag{3.60}$$

where b_k is the k^{th} Betti number of X and the function $F_h^k : [0,\,+\infty) \longrightarrow \mathbb{N}$ is defined by

$$F_h^k(\lambda) = \sup_L dim\, L, \tag{3.61}$$

where L ranges over the **closed** *vector subspaces of the quotient space $C_k^\infty(X,\,\mathbb{C})/\ker\, d$ on which the operator $d : C_k^\infty(X,\,\mathbb{C})/\ker\, d \longrightarrow C_{k+1}^\infty(X,\,\mathbb{C})$ induced by $d : C_k^\infty(X,\,\mathbb{C}) \longrightarrow C_{k+1}^\infty(X,\,\mathbb{C})$ satisfies the following $L^2_{\omega_h}$-norm estimate:*

$$||d\zeta||_{\omega_h} \leq \sqrt{\lambda}\,||\zeta||_{\omega_h}, \quad \text{for every } \zeta \in L. \tag{3.62}$$

(The understanding is that $||d\zeta||_{\omega_h}$ stands for the usual L^2-norm induced by the metric ω_h, while $||\zeta||_{\omega_h}$ stands for the **quotient norm** *induced on $C_k^\infty(X,\,\mathbb{C})/\ker\, d$ by the $L^2_{\omega_h}$-norm.)*

We will present a detailed proof of this statement along the lines of [ES89] with a few minor simplifications afforded by our special setting where the manifold X is **compact** and the operator Δ_h is **elliptic**. While a more general version for unbounded operators on L^2 spaces was needed in [ALK00], we stress that, in this context, we can confine ourselves to the case of operators on spaces of C^∞ differential forms.

The main step is the following statement (a version of the classical Min-Max Principle) that was proved in a more general setting in [ES89].

Proposition 3.5.16. *Let $(X,\,\omega)$ be a* **compact** *Hermitian manifold with $dim_{\mathbb{C}}X = n$. For an arbitrary $k \in \{0, \dots, 2n\}$, let $P : C_k^\infty(X,\,\mathbb{C}) \longrightarrow C_k^\infty(X,\,\mathbb{C})$ be an* **elliptic**, *self-adjoint and non-negative differential operator of order ≥ 1.*

Then, for every $\lambda \geq 0$, the **spectrum distribution function** *N_k* **of** *P (i.e. $N_k(\lambda)$ is defined to be the number of eigenvalues of P, counted with multiplicities, that are $\leq \lambda$) is given by the following identities (in which the suprema are actually maxima):*

$$N_k(\lambda) = \sup_{L \in \mathcal{L}_\lambda^{(k)}} dim\, L = \sup_{E \in \mathcal{P}_\lambda^{(k)}} Tr\, E, \tag{3.63}$$

where $\mathcal{L}_\lambda^{(k)}$ stands for the set of **closed** *vector subspaces $L \subset C_k^\infty(X,\,\mathbb{C})$ such that*

$$\langle\langle Pu,\,u\rangle\rangle \leq \lambda||u||^2 \quad \text{for all } u \in L,$$

while $\mathcal{P}_\lambda^{(k)}$ stands for the set of all bounded linear operators $E : C_k^\infty(X,\,\mathbb{C}) \longrightarrow C_k^\infty(X,\,\mathbb{C})$ satisfying the conditions:

(i) $E^2 = E = E^\star$ *(i.e. E is an* **orthogonal projection** *w.r.t. the L^2_ω inner product);*

(ii) $\langle\langle Pu,\,u\rangle\rangle \leq \lambda||u||^2$ *for all $u \in Im\, E$.*

(In other words, E is the orthogonal projection onto one of the subspaces $L \in \mathcal{L}_\lambda^{(k)}$, so $L = Im\,E$ for some $L \in \mathcal{L}_\lambda^{(k)}$.)

Proof. The second identity in (3.63) follows at once from the fact that the dimension of any closed subspace $L \subset C_k^\infty(X,\,\mathbb{C})$ equals the trace of the orthogonal projection onto L. So, we only have to prove the first identity in (3.63).

Since X is compact and P is elliptic, self-adjoint and non-negative, the spectrum of P is discrete and consists of non-negative eigenvalues, while there exists a countable orthonormal (w.r.t. the L^2_ω-inner product) basis of $C_k^\infty(X,\,\mathbb{C})$ (and of the Hilbert space $L^2_k(X,\,\mathbb{C})$ of L^2 k-forms) consisting of eigenvectors of P. For every $\mu \geq 0$, let $E_P(\mu) \subset C_k^\infty(X,\,\mathbb{C})$ be the eigenspace of P corresponding to the eigenvalue μ (with the understanding that $E_P(\mu) = \{0\}$ if μ is not an actual eigenvalue). The spaces $E_P(\mu)$ are finite-dimensional and consist of C^∞ forms since P is assumed to be elliptic (hence also hypoelliptic) and X is compact.

For every $\lambda \geq 0$, let $L_\lambda := \bigoplus\limits_{0\leq\mu\leq\lambda} E_P(\mu) \subset C_k^\infty(X,\,\mathbb{C})$. Thus, L_λ is finite-dimensional and $\dim L_\lambda = N_k(\lambda)$, while $\langle\langle Pu,\,u\rangle\rangle \leq \lambda\,||u||^2$ for all $u \in L_\lambda$. Hence $L_\lambda \in \mathcal{L}_\lambda^{(k)}$, so $N_k(\lambda) \leq \sup\limits_{L\in\mathcal{L}_\lambda^{(k)}} \dim L$.

To prove the reverse inequality, let $\lambda \geq 0$ and let $L \in \mathcal{L}_\lambda^{(k)}$. The existence of an orthonormal basis of eigenvectors implies the orthogonal direct-sum decomposition

$$C_k^\infty(X,\,\mathbb{C}) = \bigoplus_{0\leq\mu\leq\lambda} E_P(\mu) \oplus \bigoplus_{\mu>\lambda} E_P(\mu).$$

In particular, $\oplus_{\mu>\lambda} E_P(\mu) = \ker E_\lambda$, where E_λ is the orthogonal projection onto $\oplus_{0\leq\mu\leq\lambda} E_P(\mu)$.

Now, $\langle\langle Pu,\,u\rangle\rangle > \lambda||u||^2$ for all $u \in \oplus_{\mu>\lambda} E_P(\mu) \setminus \{0\}$, while $\langle\langle Pu,\,u\rangle\rangle \leq \lambda||u||^2$ for all $u \in L$. So, $L \cap \ker E_\lambda = L \cap \oplus_{\mu>\lambda} E_P(\mu) = \{0\}$. This implies that the restriction

$$E_{\lambda|L} : L \longrightarrow \mathrm{Im}\,E_\lambda = \bigoplus_{0\leq\mu\leq\lambda} E_P(\mu)$$

is **injective**. In particular, $\dim L \leq \dim \oplus_{0\leq\mu\leq\lambda} E_P(\mu) = N_k(\lambda)$. Since L has been chosen arbitrarily in $\mathcal{L}_\lambda^{(k)}$, we conclude that $\sup\limits_{L\in\mathcal{L}_\lambda^{(k)}} \dim L \leq N_k(\lambda)$ and we are done. $\square$

The second step towards proving Proposition 3.5.15 is the standard 3-space decomposition used in Hodge theory. For every $k = 0, \dots, 2n$, the operator $\Delta_{\omega_h} : C_k^\infty(X,\,\mathbb{C}) \longrightarrow C_k^\infty(X,\,\mathbb{C})$ is elliptic and since the manifold X is compact and $d^2 = 0$, we have the $L^2_{\omega_h}$-orthogonal decomposition:

$$C_k^\infty(X,\,\mathbb{C}) = \mathcal{H}^k_{\Delta_{\omega_h}}(X,\,\mathbb{C}) \oplus E_k(X,\,\mathbb{C}) \oplus E_k^\star(X,\,\mathbb{C}), \quad \text{where} \quad \ker d = \mathcal{H}^k_{\Delta_{\omega_h}}(X,\,\mathbb{C}) \oplus E_k(X,\,\mathbb{C}), \tag{3.64}$$

and where $\mathcal{H}^k_{\Delta_{\omega_h}}(X,\,\mathbb{C})$ is the kernel of $\Delta_{\omega_h} : C_k^\infty(X,\,\mathbb{C}) \longrightarrow C_k^\infty(X,\,\mathbb{C})$, $E_k(X,\,\mathbb{C}) := \mathrm{Im}\,(d : C_{k-1}^\infty(X,\,\mathbb{C}) \longrightarrow C_k^\infty(X,\,\mathbb{C}))$ and $E_k^\star(X,\,\mathbb{C}) := \mathrm{Im}\,(d^\star_{\omega_h} : C_{k+1}^\infty(X,\,\mathbb{C}) \longrightarrow C_k^\infty(X,\,\mathbb{C}))$.

Moreover, each of the three subspaces into which $C_k^\infty(X,\,\mathbb{C})$ splits in (3.64) is **Δ_{ω_h}-invariant**, i.e.

$$\Delta_{\omega_h}(\mathcal{H}^k_{\Delta_{\omega_h}}(X,\,\mathbb{C})) \subset \mathcal{H}^k_{\Delta_{\omega_h}}(X,\,\mathbb{C}), \quad \Delta_{\omega_h}(E_k(X,\,\mathbb{C})) \subset E_k(X,\,\mathbb{C}), \quad \Delta_{\omega_h}(E_k^\star(X,\,\mathbb{C})) \subset E_k^\star(X,\,\mathbb{C})$$

because Δ_{ω_h} commutes with d and with $d^\star_{\omega_h}$. The invariance implies that an $L^2_{\omega_h}$-orthonormal basis $\{e_i^k(h)\}_{i\in\mathbb{N}^\star}$ of $C_k^\infty(X,\,\mathbb{C})$ consisting of eigenvectors for Δ_{ω_h} (whose existence follows from the standard elliptic theory) can be chosen such that each $e_i^k(h)$ belongs to one and only one of the subspaces $\mathcal{H}^k_{\Delta_{\omega_h}}(X,\,\mathbb{C})$, $E_k(X,\,\mathbb{C})$ and $E_k^\star(X,\,\mathbb{C})$. Let $0\leq\lambda_1^k(h)\leq\dots\leq\lambda_i^k(h)\leq\dots$ be the corresponding eigenvalues, counted with multiplicities, of the rescaled Laplacian $\Delta_h : C^\infty_k(X,\,\mathbb{C})\longrightarrow C^\infty_k(X,\,\mathbb{C})$ (= those of $\Delta_{\omega_h} : C^\infty_k(X,\,\mathbb{C})\longrightarrow C^\infty_k(X,\,\mathbb{C})$). Thus, $\Delta_{\omega_h}e_i^k(h)=\lambda_i^k(h)\,e_i^k(h)$ for all i.

Consequently, we can define functions $F_h^k : [0,\,+\infty)\longrightarrow\mathbb{N}$ and $G_h^k : [0,\,+\infty)\longrightarrow\mathbb{N}$ by

$$F_h^k(\lambda):=\sharp\{i\mid e_i^k(h)\in E_k^\star(X,\,\mathbb{C})\quad\text{and}\quad\lambda_i^k(h)\leq\lambda\}$$

and

$$G_h^k(\lambda):=\sharp\{i\mid e_i^k(h)\in E_k(X,\,\mathbb{C})\quad\text{and}\quad\lambda_i^k(h)\leq\lambda\}.$$

These definitions of F_h^k and $G_h^k(\lambda)$ are independent of the choice of orthonormal basis $\{e_i^k(h)\}_{i\in\mathbb{N}^\star}$ of $C^\infty_k(X,\,\mathbb{C})$ satisfying the above properties.

Lemma 3.5.17. *The functions F_h^k and G_h^k are the spectrum distribution functions of the restrictions $\Delta_{\omega_h|E_k^\star(X,\,\mathbb{C})} : E_k^\star(X,\,\mathbb{C})\longrightarrow E_k^\star(X,\,\mathbb{C})$, resp. $\Delta_{\omega_h|E_k(X,\,\mathbb{C})} : E_k(X,\,\mathbb{C})\longrightarrow E_k(X,\,\mathbb{C})$.*

In other words, they are described as follows:

$$\begin{aligned} F_h^k(\lambda) &= \sup_{L\in\mathcal{L}_\lambda^{''(k)}} dim\,L, \\ G_h^k(\lambda) &= \sup_{L\in\mathcal{L}_\lambda^{'(k)}} dim\,L \end{aligned}\tag{3.65}$$

where $\mathcal{L}_\lambda^{''(k)}$ stands for the set of **closed** *vector subspaces $L\subset E_k^\star(X,\,\mathbb{C})$ such that*

$$||du||^2_{\omega_h}\leq\lambda||u||^2_{\omega_h}\qquad\textit{for all } u\in L,\tag{3.66}$$

and $\mathcal{L}_\lambda^{'(k)}$ stands for the set of **closed** *vector subspaces $L\subset E_k(X,\,\mathbb{C})$ such that*

$$||d^\star_{\omega_h}u||^2_{\omega_h}\leq\lambda||u||^2_{\omega_h}\qquad\textit{for all } u\in L.\tag{3.67}$$

Proof. This is an immediate application of the *variational principle* of Proposition 3.5.16 to the restrictions $\Delta_{\omega_h|E_k^\star(X,\,\mathbb{C})} : E_k^\star(X,\,\mathbb{C}))\longrightarrow E_k^\star(X,\,\mathbb{C})$ and $\Delta_{\omega_h|E_k(X,\,\mathbb{C})} : E_k(X,\,\mathbb{C}))\longrightarrow E_k(X,\,\mathbb{C})$. Estimates (3.66) and (3.67) are consequences of the identity $\langle\langle\Delta_{\omega_h}u,\,u\rangle\rangle_{\omega_h}=||du||^2_{\omega_h}+||d^\star_{\omega_h}u||^2_{\omega_h}$ and of the fact that $d^\star_{\omega_h}u=0$ whenever $u\in E_k^\star(X,\,\mathbb{C})$ (since $\mathrm{Im}\,d^\star_{\omega_h}\subset\ker d^\star_{\omega_h}$) and that $du=0$ whenever $u\in E_k(X,\,\mathbb{C})$ (since $\mathrm{Im}\,d\subset\ker d$). $\square$

The last ingredient we need is the following very simple observation.

Lemma 3.5.18. *For every $\lambda\geq 0$ and every $k\in\{-1,0,\dots,2n\}$, we have*

$$F_h^k(\lambda)=G_h^{k+1}(\lambda)\qquad\textit{with the understanding that}\qquad F_h^{-1}(\lambda)=G_h^{2n+1}(\lambda)=0.$$

Proof. We know from the orthogonal decompositions (3.64) that the restriction of d to $E_k^\star(X,\,\mathbb{C})$ is injective, so

$$d_{|E_k^\star(X,\,\mathbb{C})} : E_k^\star(X,\,\mathbb{C})\longrightarrow E_{k+1}(X,\,\mathbb{C})$$

is an isomorphism. Moreover, $d\Delta_{\omega_h}=\Delta_{\omega_h}d$, so whenever $\Delta_{\omega_h}u_i=\lambda_i^k(h)\,u_i$, we get $\Delta_{\omega_h}(du_i)=\lambda_i^k(h)\,(du_i)$. Combined with the above isomorphism, with the invariance of $E_k^\star(X,\,\mathbb{C})$ under Δ_{ω_h} and with the definitions of $F_k^h(\lambda)$ and $G_{k+1}^h(\lambda)$, this implies the contention. $\square$.

Proof of Proposition 3.5.15. Putting together (3.64), the definitions of $F_h^k(\lambda)$ and $G_h^k(\lambda)$ and the fact that the Hodge isomorphism $\mathcal{H}^k_{\Delta_{\omega_h}} \simeq H^k_{DR}(X, \mathbb{C})$ (which follows at once from (3.64)) implies $b_k = \dim \mathcal{H}^k_{\Delta_{\omega_h}}$, we get

$$N_h^k(\lambda) = b_k + G_h^k(\lambda) + F_h^k(\lambda)$$

for all k and all $\lambda \geq 0$. Using Lemma 3.5.18, this is equivalent to (3.60).

On the other hand, the descriptions (3.65) and (3.66) of $F_h^k(\lambda)$ coincide with the descriptions (3.61) and (3.62) thanks to the isomorphism $E_k^\star(X, \mathbb{C}) \simeq C_k^\infty(X, \mathbb{C})/\ker d$, which is another consequence of the decompositions (3.64). □

(II) Metric independence of asymptotics

Although the following statement has no impact on either the statement of Theorem 3.5.13 or its proof, we pause briefly to show, exactly as in the foliated case of [ALK00], that the asymptotics of the eigenvalues $\lambda_i^k(h)$ and of the spectrum distribution function N_h^k as $h \downarrow 0$ depend only on the complex structure of X. The proof is an easy application of the Variational Principle of Proposition 3.5.15.

Proposition 3.5.19. *The asymptotics of the $\lambda_i^k(h)$'s and of N_h^k as $h \downarrow 0$ are independent of the choice of Hermitian metric ω.*

Proof. We adapt to our setting the proof of the corresponding result in [ALK00]. Let ω and ω' be two Hermitian metrics on X. They induce, respectively, rescaled metrics $(\omega_h)_{h>0}$ and $(\omega'_h)_{h>0}$. Let $N_h^{\prime k}(\lambda) = F_h^{\prime k-1}(\lambda) + b_k + F_h^{\prime k}(\lambda)$ be the spectrum distribution function associated with the rescaled Laplacian $\Delta_{\omega'_h} : C_k^\infty(X, \mathbb{C}) \longrightarrow C_k^\infty(X, \mathbb{C})$, written as in (3.60).

Since X is compact, there exists a constant $C > 0$ such that the respective L^2-norms satisfy the following inequalities in every bidegree (p, q):

$$\frac{1}{C}\,||\;||_\omega \leq ||\;||_{\omega'} \leq C\,||\;||_\omega, \quad \text{hence} \quad \frac{1}{C}\,||\;||_{\omega_h} \leq ||\;||_{\omega'_h} \leq C\,||\;||_{\omega_h} \quad \text{on } L^2_{p,q}(X, \mathbb{C}) \quad \text{for every } h > 0.$$

The constant C is independent of $h > 0$ thanks to Formula 3.5.2.

Hence, for every $\zeta \in C_k^\infty(X, \mathbb{C})/\ker d$ such that $||d\zeta||_{\omega_h} \leq \sqrt{\lambda}\,||\zeta||_{\omega_h}$, we get $||d\zeta||_{\omega'_h} \leq \sqrt{C^4\lambda}\,|\zeta||_{\omega'_h}$. Thanks to Proposition 3.5.15, this implies that

$$F_h^k(\lambda) \leq F_h^{\prime k}(C^4\lambda), \qquad \lambda \geq 0, \;\; h > 0.$$

By symmetry, we also get $F_h^{\prime k}(\lambda) \leq F_h^k(C^4\lambda)$, so putting the last two inequalities together, we get

$$F_h^{\prime k}(C^{-4}\lambda) \leq F_h^k(\lambda) \leq F_h^{\prime k}(C^4\lambda), \qquad \lambda \geq 0, \;\; h > 0.$$

The proof is complete. □

(III) Proof of the inequality "≤" in Theorem 3.5.13

We are now in a position to prove the following

Theorem 3.5.20. *Let (X, ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$. For every r and every $k = 0, \dots, 2n$, the following inequality holds:*

$$\dim E_r^k \leq \sharp\{i \mid \lambda_i^k(h) \in O(h^{2r}) \quad \text{as } h \downarrow 0\}. \tag{3.68}$$

Proof. We have to prove the existence of a uniform constant $C>0$ such that $\dim E_r^k \leq N_h^k(Ch^{2r})$ for all r,k and all $0<h\ll 1$. Recall the following facts:

(i) $\dim E_r^k = b_k + m_r^{k-1} + m_r^k$, where $m_r^k := \dim L_r^k$ and $L_r^k := \bigoplus_{p+q=k} L_r^{p,\,q} = \bigoplus_{p+q=k}\bigoplus_{l\geq r}(E_l^{p,\,q}/\ker d_l^{p,\,q})$

(proved in (3.55) of Lemma 3.5.10);

(ii) $N_h^k(\lambda) = b_k + F_h^{k-1}(\lambda) + F_h^k(\lambda)$ for all $\lambda \geq 0$

(cf. (3.60) of Proposition 3.5.15).

Thus, it suffices to prove that

$$m_r^k \leq F_h^k(Ch^{2r}) \quad \text{for all } 0<h\ll 1, \tag{3.69}$$

for a uniform constant $C>0$ and for all r and k.

Now, thanks to the definition (3.61) of F_h^k, to prove (3.69) it suffices to prove that L_r^k is one of the subspaces of $C_k^\infty(X,\,\mathbb{C})/\ker d$ contributing to the definition of $F_h^k(Ch^{2r})$ for some uniform constant $C>0$. In other words, it suffices to prove that there exists $C>0$ such that

$$||d\zeta||_{\omega_h} \leq \sqrt{C}\, h^r\, ||\zeta||_{\omega_h}, \quad \text{for all } \zeta\in L_r^k \text{ and all } 0<h\ll 1. \tag{3.70}$$

Meanwhile, every $\zeta \in L_r^k = \bigoplus_{p+q=k} L_r^{p,\,q}$ splits uniquely as $\zeta = \sum_{p+q=k}\zeta^{p,\,q}$ with $\zeta^{p,\,q}\in L_r^{p,\,q}$ for all p,q. Thus, it suffices to prove that, for a uniform constant $C>0$, we have

$$||d\zeta^{p,\,q}||_{\omega_h} \leq \sqrt{C}\, h^r\, ||\zeta^{p,\,q}||_{\omega_h}, \quad \text{for all } p,q, \text{ all } \zeta^{p,\,q}\in L_r^{p,\,q} \text{ and all } 0<h\ll 1. \tag{3.71}$$

This holds mainly because d_r is of type $(r,\,-r+1)$, so d_r increases the holomorphic degree by r and thus the norm $|\ \ |_{\omega_h}$ brings out an extra factor h^r. Specifically, for every $\zeta^{p,\,q}\in L_r^{p,\,q}$, (3.56) of Lemma 3.5.10 yields $d\zeta^{p,\,q}\in d(L_r^{p,\,q})\subset \mathcal{A}_{p+r}^{p+q-1}$. Therefore, the holomorphic degree of $d\zeta^{p,\,q}$ is $\geq p+r$, so from Formula 3.5.2 we get

$$||d\zeta^{p,\,q}||_{\omega_h} \leq \frac{h^{p+r}}{h^n}\, ||d\zeta^{p,\,q}||_{\omega} \quad \text{for all } p,q, \text{ all } \zeta^{p,\,q}\in L_r^{p,\,q} \text{ and all } 0<h<1.$$

Now, $L_r^{p,\,q}$ is a *finite-dimensional* vector subspace of $C_k^\infty(X,\,\mathbb{C})/\ker d$, so there exists a constant $C_r>0$ (depending on r,p,q, but independent of h) such that $||d\zeta^{p,\,q}||_\omega \leq C_r\, ||\zeta^{p,\,q}||_\omega$ for all $\zeta^{p,\,q}\in L_r^{p,\,q}$. Meanwhile, Formula 3.5.2 tells us again that $||\zeta^{p,\,q}||_\omega = (h^n/h^p)\, ||\zeta^{p,\,q}||_{\omega_h}$, so putting the last three relations together, we get

$$||d\zeta^{p,\,q}||_{\omega_h} \leq C_r\, h^r\, ||\zeta^{p,\,q}||_{\omega_h} \quad \text{for all } p,q, \text{ all } \zeta^{p,\,q}\in L_r^{p,\,q} \text{ and all } 0<h<1.$$

This proves (3.71) after setting $C := \max_{\substack{0\leq r\leq N\\ 0\leq p,q\leq n}} C_r^2 > 0$.

The proof is complete. $\square$

Note that L_r^k is a vector space of classes of cohomology classes, rather than of differential forms, so what is meant by L_r^k in the above proof is its image in $C_k^\infty(X,\,\mathbb{C})/\ker d$ under the isometries explained in (II) of §3.5.2. We can use these isometries, the identification of d acting on $\mathcal{H}_r^{p,\,q}$ with d_r and Conclusion 3.5.12 in the following way to make the above proof even more explicit. If we choose $\zeta^{p,\,q}$ to be the ω_h-*harmonic* representative of its class (also denoted by $\zeta^{p,\,q}$) and to play the role of α of Conclusion 3.5.12, we can re-write the above inequalities in a more detailed form as follows:

$$\begin{aligned}||d\zeta^{p,\,q}||_{\omega_h} & = ||(P(\partial u_{r-1})||_{\omega_h} \leq \frac{h^{p+r}}{h^n}\,||(P\circ T)(\zeta^{p,\,q})||_\omega \\ & \leq \frac{h^{p+r}}{h^n}\,C_r\,||\zeta^{p,\,q}||_\omega = C_r\,h^r||\alpha||_{\omega_h},\end{aligned}$$

where P and T are the linear maps $P_r^{p,\,q}$ and $T_r^{p,\,q}$ (with indices removed) of Conclusion 3.5.12 that was used above, while $||\;||_{\omega_h}$ stands for the $L^2_{\omega_h}$-norm when applied to a form and for the induced quotient norm when applied to a class.

(IV) Preliminaries to the proof of the inequality "$\geq$" in Theorem 3.5.13

We will need a few simple observations.

Lemma 3.5.21. *Let $(X,\,\omega)$ be a compact Hermitian manifold with $\dim_{\mathbb{C}}X=n$. For every bidegree $(p,\,q)$ and every $(p,\,q)$-form u on X, the following identities hold:*

$$\langle\langle\Delta_h u,\,u\rangle\rangle_\omega = h^{2(n-p)}\,\langle\langle\Delta_{\omega_h}u,\,u\rangle\rangle_{\omega_h} = h^{2(n-p)}\,(||du||^2_{\omega_h} + ||d^\star_{\omega_h}u||^2_{\omega_h}). \tag{3.72}$$

Proof. The latter identity is obvious, so we will only prove the former one. Since u is of pure type, (3.47) yields the first identity below, while the second identity follows from Formula 3.5.2:

$$\begin{aligned}\langle\langle\Delta_h u,\,u\rangle\rangle_\omega & = h^2\,\langle\langle\Delta' u,\,u\rangle\rangle_\omega + \langle\langle\Delta'' u,\,u\rangle\rangle_\omega = h^2\,h^{2(n-p)}\,\langle\langle\Delta' u,\,u\rangle\rangle_{\omega_h} + h^{2(n-p)}\,\langle\langle\Delta'' u,\,u\rangle\rangle_{\omega_h} \\ & = h^{2(n-p)}\,\langle\langle\Delta_{\omega_h}u,\,u\rangle\rangle_{\omega_h}.\end{aligned}$$

The last identity followed again from (3.47). ☐

Lemma 3.5.22. *Let $u\in C^\infty_{p,\,q}(X,\,\mathbb{C})$ be an arbitrary form. Considering the splitting $d=d^{(k)}=\bigoplus_{\substack{0\leq r\leq N-1\\ p+q=k}} d_r^{p,\,q}: C^\infty_k(X,\,\mathbb{C})\longrightarrow C^\infty_{k+1}(X,\,\mathbb{C})$ of the operator d (see (3.54)) and the splitting*

$$u=\sum_{r=0}^{N-1}u_r + \ker d, \qquad \textit{implying} \qquad du=\sum_{r=0}^{N-1}d_r u_r,$$

with $u_r\in E_r^{p,\,q}/\ker d_r^{p,\,q}$ (see §.3.5.2 and recall that $d_r: E_r^{p,\,q}/\ker d_r^{p,\,q}\longrightarrow Im\, d_r^{p,\,q}\subset C^\infty_{p+r,\,q-r+1}(X,\,\mathbb{C})$ is an isomorphism), the following identity holds:

$$h^{2(n-p)}\,||du||^2_{\omega_h} = \sum_{r=0}^{N-1}h^{2r}\,||d_r u_r||^2_\omega \quad \textit{for all } h>0. \tag{3.73}$$

Proof. Since d_r is of type $(r,\,-r+1)$, $d_r u_r$ is of type $(p+r,\,q-r+1)$, so the $d_r u_r$'s are mutually orthogonal (w.r.t. any metric) when r varies. We get

$$||du||^2_{\omega_h} = \sum_{r=0}^{N-1}||d_r u_r||^2_{\omega_h} = \sum_{r=0}^{N-1}\frac{h^{2(p+r)}}{h^{2n}}\,||d_r u_r||^2_\omega,$$

where for the last identity we used Formula 3.5.2. ☐

Lemma 3.5.23. *For every r and every bidegree $(p,\,q)$, the formal adjoints of d_r w.r.t. the metrics ω_h and ω compare as follows:*

$$(d_r)^\star_{\omega_h} = h^{2r}\,(d_r)^\star_\omega. \tag{3.74}$$

Consequently, for every form $u \in C^\infty_{p,\,q}(X,\,\mathbb{C})$, the following counterpart of Lemma 3.5.22 for the adjoints holds. Considering the splitting $(d^{(k)})^\star_{\omega_h} = \bigoplus\limits_{\substack{0\leq r\leq N-1\\ p+q=k}} (d_r^{p,\,q})^\star_{\omega_h} : C^\infty_{k+1}(X,\,\mathbb{C}) \longrightarrow C^\infty_k(X,\,\mathbb{C})$ of the operator $d^\star$ and the splitting

$$u = \sum_{r=0}^{N-1} v_r + \ker\, d^\star_{\omega_h}, \qquad \textit{implying} \qquad d^\star_{\omega_h} u = \sum_{r=0}^{N-1} (d_r)^\star_{\omega_h} v_r,$$

with $v_r \in Im\, d_r^{p-r,\,q+r-1}$ (see (I) of §.3.5.2), the following identity holds:

$$h^{2(n-p)}\,||d^\star_{\omega_h} u||^2_{\omega_h} = \sum_{r=0}^{N-1} h^{2r}\,||(d_r)^\star_\omega v_r||^2_\omega \quad \textit{for all } h>0. \tag{3.75}$$

Proof. For every $(p,\,q)$-form v and every $(p-r,\,q+r-1)$-form u, we have

$$\frac{h^{2(p-r)}}{h^{2n}}\,\langle\langle (d_r)^\star_{\omega_h} v,\, u\rangle\rangle_\omega = \langle\langle (d_r)^\star_{\omega_h} v,\, u\rangle\rangle_{\omega_h} = \langle\langle v,\, d_r u\rangle\rangle_{\omega_h} = \frac{h^{2p}}{h^{2n}}\,\langle\langle v,\, d_r u\rangle\rangle_\omega = \frac{h^{2p}}{h^{2n}}\,\langle\langle (d_r)^\star_\omega v,\, u\rangle\rangle_\omega.$$

This proves (3.74). Using the mutual orthogonality of the $(d_r)^\star_{\omega_h} v_r$'s (due to bidegree reasons) and Formula 3.5.2, we get

$$||d^\star_{\omega_h} u||^2_{\omega_h} = \sum_{r=0}^{N-1} ||(d_r)^\star_{\omega_h} v_r||^2_{\omega_h} = \sum_{r=0}^{N-1} \frac{h^{2(p-r)}}{h^{2n}}\,||(d_r)^\star_{\omega_h} v_r||^2_\omega = \sum_{r=0}^{N-1} \frac{h^{2(p-r)}}{h^{2n}}\,h^{4r}\,||(d_r)^\star_\omega v_r||^2_\omega.$$

This proves (3.75). □

Putting together (3.72), (3.73) and (3.75), we get

Corollary 3.5.24. *Let $(X,\,\omega)$ be a compact Hermitian manifold with $dim_{\mathbb{C}} X = n$. For every bidegree $(p,\,q)$ and every $(p,\,q)$-form u on X, the following identity holds:*

$$\langle\langle \Delta_h u,\, u\rangle\rangle_\omega = \sum_{r'=0}^{N-1} h^{2r'}\,||d_{r'} u_{r'}||^2_\omega + \sum_{r'=0}^{N-1} h^{2r'}\,||(d_{r'})^\star_\omega v_{r'}||^2_\omega,$$

where u splits uniquely (cf. (I) of §.3.5.2) as

$$u = \sum_{r'=0}^{N-1} u_{r'} + \ker\, d = \sum_{r'=0}^{N-1} v_{r'} + \ker\, d^\star = \sum_{r'=0}^{N-1} u_{r'} + \sum_{r'=0}^{N-1} v_{r'} + w$$

with $u_{r'} \in E_{r'}^{p,\,q}/\ker d_{r'}^{p,\,q}$, $v_{r'} \in Im\, d_{r'}^{p-r',\,q+r'-1}$ and $w \in E_\infty^{p,\,q}$.

(V) Proof of the inequality "≥" in Theorem 3.5.13

Following again the analogy with the foliated case of [ALK00], we will actually prove a stronger statement from which the following result will follow as a corollary.

Theorem 3.5.25. *Let $(X,\,\omega)$ be a compact Hermitian manifold with $dim_{\mathbb{C}}X=n$. For every r and every $k=0,\dots,2n$, the following inequality holds:*

$$dim\,E_r^k\geq\sharp\{i\mid\lambda_i^k(h)\in O(h^{2r})\quad as\ h\downarrow 0\}. \tag{3.76}$$

The first main ingredient we will use is the *pseudo-differential Laplacian*

$$\widetilde{\Delta}=\partial p''\partial^\star+\partial^\star p''\partial+\Delta'':C^\infty_{p,\,q}(X,\,\mathbb{C})\longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C})$$

defined in arbitrary bidegree $(p,\,q)$ and introduced in Definition 3.1.2, where $p'':C^\infty_{p,\,q}(X,\,\mathbb{C})\longrightarrow\ker\Delta''$ is the orthogonal projection (w.r.t. the L^2_ω-norm) onto the Δ''-harmonic subspace of $C^\infty_{p,\,q}(X,\,\mathbb{C})$. The pseudo-differential Laplacian $\widetilde{\Delta}$ gives a Hodge theory for the second page of the Frölicher spectral sequence in the sense that there is a *Hodge isomorphism*

$$E_2^{p,\,q}\xrightarrow{\simeq}\mathcal{H}^{p,\,q}_{\widetilde{\Delta}}(X,\,\mathbb{C}):=\ker(\widetilde{\Delta}:C^\infty_{p,\,q}(X,\,\mathbb{C})\longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C}))\quad\text{for all } p,q=0,\dots,n. \tag{3.77}$$

Note that $(p'')^2=p''=(p'')^\star$, so $\partial p''\partial^\star=(p''\partial^\star)^\star(p''\partial^\star)$ and $\partial^\star p''\partial=(p''\partial)^\star(p''\partial)$. Thus, $\widetilde{\Delta}$ is a sum of non-negative operators, so its kernel is the intersection of the respective kernels. Since $\ker(A^\star A)=\ker A$ for any operator A, we get

$$\ker\widetilde{\Delta}=\ker(p''\partial)\cap\ker(p''\partial^\star)\cap\ker\bar\partial\cap\ker\bar\partial^\star.$$

The second main ingredient we will use is the following lower estimate of the rescaled Laplacian Δ_h. It is the analogue in our context of a result in [ALK00].

Lemma 3.5.26. *Let $(X,\,\omega)$ be a compact Hermitian manifold with $dim_{\mathbb{C}}X=n$. There exists a constant $C>0$ such that the following inequality of linear operators holds on differential forms of any degree $k=0,\dots,2n$:*

$$\Delta_h\geq\frac{3}{4}\,\Delta''+h^2\,\Delta'-Ch^2\qquad\textit{for all } h>0,$$

where $\Delta''=\bar\partial\bar\partial^\star+\bar\partial^\star\bar\partial$ and $\Delta'=\partial\partial^\star+\partial^\star\partial$ are the usual $\bar\partial$- and ∂-Laplacians.

The coefficients $3/4$ and 1 are not optimal, but they suffice for our purposes and the proof provided below shows that they can be made optimal if this is desired.

Proof of Lemma 3.5.26. We know from (ii) of Lemma 3.5.7 that

$$\Delta_h=\Delta''+h^2\Delta'-h([\tau,\,\bar\partial^\star]+[\tau^\star,\,\bar\partial]), \tag{3.78}$$

where $\tau=\tau_\omega:=[\Lambda,\,\partial\omega\wedge\cdot]$ is the zero-th order *torsion operator* of type $(1,\,0)$ associated with ω.

For any form u, the first-order terms on the r.h.s. of (3.78) are easily estimated using the Cauchy-Schwarz inequality as follows:

$$\begin{aligned}h\,|\langle\langle[\tau,\,\bar\partial^\star]u+[\tau^\star,\,\bar\partial]u,\,u\rangle\rangle| &= h\,|\langle\langle\bar\partial^\star u,\,\tau^\star u\rangle\rangle+\langle\langle\tau u,\,\bar\partial u\rangle\rangle+\langle\langle\bar\partial u,\,\tau u\rangle\rangle+\langle\langle\tau^\star u,\,\bar\partial^\star u\rangle\rangle|\\ &\leq 2h||\tau u||\,||\bar\partial u||+2h||\tau^\star u||\,||\bar\partial^\star u||\\ &\leq \frac{1}{4}\,(||\bar\partial u||^2+||\bar\partial^\star u||^2)+4h^2\,(||\tau u||^2+||\tau^\star u||^2)\\ &\leq \frac{1}{4}\,\langle\langle\Delta''u,\,u\rangle\rangle+Ch^2\,||u||^2,\end{aligned}$$

where the constant $C>0$ exists because the linear operators τ and $\tau^\star$ are of order zero, hence bounded. In particular, we get the operator inequality $-h([\tau,\,\bar\partial^\star]+[\tau^\star,\,\bar\partial])\geq -\frac{1}{4}\,\Delta''-Ch^2$ which, alongside (3.78), proves the contention. $\square$

We are now ready to state and prove a general result that will imply Theorem 3.5.25.

Theorem 3.5.27. *Let $(X,\,\omega)$ be a compact Hermitian manifold with $\dim_{\mathbb{C}}X=n$. Let $k\in\{0,\dots,2n\}$ and $r\geq 1$ be fixed integers. Suppose there exist a sequence $(h_i)_{i\in\mathbb{N}}$ of constants $h_i>0$ such that $h_i\downarrow 0$ and a sequence $(u_i)_{i\in\mathbb{N}}$ of k-forms $u_i\in C^\infty_k(X,\,\mathbb{C})$ such that $||u_i||_\omega=1$ for every i and*

$$\langle\langle\Delta_{h_i}u_i,\,u_i\rangle\rangle_\omega\in o(h_i^{2(r-1)})\qquad \text{as } i\to+\infty. \tag{3.79}$$

Then, there exists a subsequence $(u_{i_l})_{l\in\mathbb{N}}$ of $(u_i)_{i\in\mathbb{N}}$ such that $(u_{i_l})_{l\in\mathbb{N}}$ converges in the L^2_ω-topology to some k-form $u\in\mathcal{H}^k_r:=\oplus_{p+q=k}\mathcal{H}^{p,\,q}_r\simeq E^k_r$, where the $\mathcal{H}^{p,\,q}_r\subset C^\infty_{p,\,q}(X,\,\mathbb{C})$ are the "harmonic" vector subspaces of Definition 3.5.11 induced by the metric ω.

Proof. • **Case $r=1$.** In this case, Hypothesis (3.79) means that $\langle\langle\Delta_{h_i}u_i,\,u_i\rangle\rangle_\omega\longrightarrow 0$ as $i\to+\infty$. Then also $\langle\langle\Delta_{h_i}u_i,\,u_i\rangle\rangle_\omega+Ch_i^2\longrightarrow 0$ as $i\to+\infty$. Since, by Lemma 3.5.26, we have

$$\langle\langle\Delta_{h_i}u_i,\,u_i\rangle\rangle_\omega+Ch_i^2\geq\frac{3}{4}\,\langle\langle\Delta''u_i,\,u_i\rangle\rangle_\omega+h_i^2\,\langle\langle\Delta'u_i,\,u_i\rangle\rangle_\omega\geq 0\qquad\text{for all } i\in\mathbb{N},$$

we get

$$(i)\;\langle\langle\Delta''u_i,\,u_i\rangle\rangle_\omega\longrightarrow 0\quad\text{as } i\to+\infty\quad\text{and}\quad(ii)\;h_i^2\,\langle\langle\Delta'u_i,\,u_i\rangle\rangle_\omega\longrightarrow 0\quad\text{as } i\to+\infty. \tag{3.80}$$

Meanwhile, the $\bar\partial$-Laplacian Δ'' is *elliptic* and the manifold X is *compact*, so the Gårding inequality yields constants $\delta_1,\delta_2>0$ such that the first inequality below holds:

$$\delta_2\,||u_i||_{W^1}\leq\langle\langle\Delta''u_i,\,u_i\rangle\rangle_\omega+\delta_1\,||u_i||_\omega\leq C_1,\qquad\text{for all } i\in\mathbb{N},$$

where $||\;\;||_{W^1}$ stands for the Sobolev norm W^1 induced by the metric ω. The second inequality above holds for some constant $C_1>0$ since the quantity $\langle\langle\Delta''u_i,\,u_i\rangle\rangle_\omega$ converges to zero (cf. (3.80)), hence is bounded, and $||u_i||_\omega=1$ by the hypothesis of Theorem 3.5.27.

Consequently, the sequence $(u_i)_{i\in\mathbb{N}}$ is bounded in the Sobolev space W^1 (a Hilbert space), so by the Banach-Alaoglu Theorem there exists a subsequence $(u_{i_l})_{l\in\mathbb{N}}$ that converges in the weak topology of W^1 to some k-form $u\in W^1$. In particular, the following convergences hold in the weak topology of distributions:

$$\bar\partial u_{i_l}\longrightarrow\bar\partial u\quad\text{and}\quad\bar\partial^\star u_{i_l}\longrightarrow\bar\partial^\star u\quad\text{as}\quad l\to+\infty.$$

On the other hand, $||\bar\partial u_i||^2+||\bar\partial^\star u_i||^2=\langle\langle\Delta''u_i,\,u_i\rangle\rangle_\omega\longrightarrow 0$ as $i\to+\infty$, so $\bar\partial u_i\longrightarrow 0$ and $\bar\partial^\star u_i\longrightarrow 0$ in the L^2-topology as $i\to+\infty$. Comparing this with the above convergences in the weak topology of distributions, we get

$$\bar\partial u=0\quad\text{and}\quad\bar\partial^\star u=0,$$

which, by (3.57), is equivalent to $u\in\ker(\Delta'':C^\infty_k(X,\,\mathbb{C})\longrightarrow C^\infty_k(X,\,\mathbb{C}))=\mathcal{H}^k_1\simeq E^k_1$.

Note that by the Rellich Lemma (asserting the compactness of the inclusion $W^1\hookrightarrow L^2$), the convergence of $(u_{i_l})_{l\in\mathbb{N}}$ to u in the weak topology of W^1 implies that $(u_{i_l})_{l\in\mathbb{N}}$ also converges in the L^2-topology to u. Moreover, the ellipticity of Δ'' and the relation $u\in\ker\Delta''$ imply that u is C^∞.

• **Case** $r = 2$. In this case, Hypothesis (3.79) means that $\langle\langle\Delta_{h_i}u_i,\, u_i\rangle\rangle_\omega \in o(h_i^2)$ as $i \to +\infty$. Since $\langle\langle\Delta_{h_i}u_i,\, u_i\rangle\rangle_\omega = ||d_{h_i}u_i||^2 + ||d_{h_i}^\star u_i||^2 = ||h_i\partial u_i + \bar\partial u_i||^2 + ||h_i\partial^\star u_i + \bar\partial^\star u_i||^2$, this implies that

$$\partial u_i + \frac{1}{h_i}\,\bar\partial u_i \longrightarrow 0 \quad \text{and} \quad \partial^\star u_i + \frac{1}{h_i}\,\bar\partial^\star u_i \longrightarrow 0 \quad \text{in the } L^2\text{-topology, as } i \to +\infty. \tag{3.81}$$

Since the orthogonal projection p'' onto $\ker\Delta''$ is continuous w.r.t. the L^2-topology and since $p''\bar\partial = 0$ and $p''\bar\partial^\star = 0$ (because $\operatorname{Im}\bar\partial \perp \ker\Delta''$ and $\operatorname{Im}\bar\partial^\star \perp \ker\Delta''$), an application of p'' to (3.81) yields

$$p''\partial u_i \longrightarrow 0 \quad \text{and} \quad p''\partial^\star u_i \longrightarrow 0 \quad \text{in the } L^2\text{-topology, as } i \to +\infty. \tag{3.82}$$

On the other hand, we know from the discussion of the case $r = 1$ (whose weaker assumption is still valid in the case $r = 2$) that there exists a subsequence $(u_{i_l})_{l\in\mathbb{N}}$ that converges in the weak topology of W^1 to some k-form $u \in W^1$. Thus, $\partial u_{i_l} \longrightarrow \partial u \in L^2$ in the weak topology of L^2 as $l \to +\infty$. This means that

$$\langle\langle\partial u_{i_l},\, v\rangle\rangle_\omega \longrightarrow \langle\langle\partial u,\, v\rangle\rangle_\omega \quad \text{for all } v \in L^2, \quad \text{hence } \langle\langle\partial u_{i_l},\, p''v\rangle\rangle_\omega \longrightarrow \langle\langle\partial u,\, p''v\rangle\rangle_\omega \quad \text{for all } v \in L^2,$$

as $l \to +\infty$. (The second convergence follows from the first since $||p''v|| \leq ||v||$ for all $v \in L^2$, so $p''(L^2) \subset L^2$.) Now, p'' is self-adjoint, so the last convergence translates to

$$\langle\langle p''\partial u_{i_l},\, v\rangle\rangle_\omega \longrightarrow \langle\langle p''\partial u,\, v\rangle\rangle_\omega \quad \text{as } l \to +\infty, \text{ for all } v \in L^2.$$

This means that $p''\partial u_{i_l}$ converges to $p''\partial u$ in the weak topology of L^2 as $l \to +\infty$. However, we know from (3.82) that $p''\partial u_{i_l}$ converges to 0 in the L^2-topology. Hence $p''\partial u = 0$. The same argument run with $\partial^\star$ in place of ∂ yields that $p''\partial^\star u = 0$. On the other hand, we know from the discussion of the case $r = 1$ that $u \in \ker\bar\partial \cap \ker\bar\partial^\star = \ker\Delta''$, so we get

$$u \in \ker(p''\partial) \cap \ker(p''\partial^\star) \cap \ker\bar\partial \cap \ker\bar\partial^\star = \mathcal{H}_2^k \simeq E_2^k$$

after remembering the description (3.57) of the spaces $\mathcal{H}_2^{p,\,q}$ and that $\mathcal{H}_2^k = \oplus_{p+q=k}\mathcal{H}_2^{p,\,q}$.

• **Case** $r \geq 3$. Using the information from the first two cases and from subsection §.**??**, this last case can easily be dealt with as follows.

For each of the k-forms u_i given by the hypotheses of Theorem 3.5.27, we consider the splitting

$$u_i = \sum_{r'=0}^{N-1} u_{r'}^{(i)} + \sum_{r'=0}^{N-1} v_{r'}^{(i)} + w_i,$$

with $u_{r'}^{(i)} \in E_{r'}^{p,\,q}/\ker d_{r'}^{p,\,q}$, $v_{r'}^{(i)} \in \operatorname{Im} d_{r'}^{p-r',\,q+r'-1}$ and $w_i \in E_\infty^{p,\,q}$, and the corresponding splitting

$$\langle\langle\Delta_{h_i}u_i,\, u_i\rangle\rangle_\omega = \sum_{r'=0}^{N-1} h_i^{2r'}\,||d_{r'}u_{r'}^{(i)}||_\omega^2 + \sum_{r'=0}^{N-1} h_i^{2r'}\,||(d_{r'})_\omega^\star v_{r'}^{(i)}||_\omega^2$$

obtained in Corollary 3.5.24

On the other hand, (3.79) ensures that $\langle\langle\Delta_{h_i}u_i,\, u_i\rangle\rangle_\omega \in o(h_i^{2(r-1)})$ as $i \to +\infty$. Together with the above identity, this implies the following convergences in the L_ω^2-norm as $i \to +\infty$:

$$d_{r'}u_{r'}^{(i)} \longrightarrow 0 \quad \text{and} \quad (d_{r'})_\omega^\star v_{r'}^{(i)} \longrightarrow 0 \quad \text{for every } \; r' \in \{0, \dots, r-1\}.$$

We even get

$$\frac{1}{h_i^{r-r'-1}}\,d_{r'}u_{r'}^{(i)} \longrightarrow 0 \quad \text{and} \quad \frac{1}{h_i^{r-r'-1}}\,(d_{r'})^\star_\omega v_{r'}^{(i)} \longrightarrow 0 \quad \text{for every} \;\; r' \in \{0, \dots, r-1\}.$$

Defining in an ad hoc way a "formal" Laplacian by $\Delta_{r'}^{formal} := d_{r'}(d_{r'})^\star_\omega + (d_{r'})^\star_\omega d_{r'}$, we get that the limit u of a subsequence of $(u_i)_{i\in\mathbb{N}}$ lies in

$$\ker\left(\Delta_{r-1}^{formal} : \bigoplus_{p+q=k} E_{r-1}^{p,\,q} \longrightarrow \bigoplus_{p+q=k} E_{r-1}^{p,\,q}\right) \simeq \mathcal{H}_r^k \simeq E_r^k$$

and we are done. □

Proof of Theorem 3.5.25. It is an immediate consequence of Theorem 3.5.27. Indeed, fix any $r \in \mathbb{N}^\star$ and $k \in \{0, \dots, 2n\}$ and suppose that inequality (3.76) does not hold. Then, the reverse strict inequality holds, so there exists a sequence $(h_i)_{i\in\mathbb{N}}$ of positive constants such that $h_i \downarrow 0$ when $i \to +\infty$ and a sequence $(u_i)_{i\in\mathbb{N}}$ of eigenvectors for the Laplacians Δ_{h_i} acting on k-forms such that $||u_i||_\omega = 1$, $u_i \perp \mathcal{H}_r^k$ for all i and $\langle\langle\Delta_{h_i}u_i,\, u_i\rangle\rangle \in o(h_i^{2(r-1)})$ as $i \to +\infty$.

Thanks to Theorem 3.5.27, there exists a subsequence $(u_{i_l})_{l\in\mathbb{N}}$ of $(u_i)_{i\in\mathbb{N}}$ such that $(u_{i_l})_{l\in\mathbb{N}}$ converges in the L^2_ω-topology to some k-form $u \in \mathcal{H}_r^k \simeq E_r^k$. However, the form u is orthogonal to $\mathcal{H}_r^k$ since $u_i \perp \mathcal{H}_r^k$ for all i and the orthogonality property is preserved in the limit. Since $||u||_\omega = 1$ (because $||u_i||_\omega = 1$ for all i), $u \neq 0$, so u cannot be at once orthogonal to and a member of $\mathcal{H}_r^k$. This is a contradiction. □

Chapter 4

Special Hermitian Metrics on Compact Complex Manifolds

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. As already stated in §.1.1.1, a *Hermitian metric* on X identifies with a C^∞ positive definite $(1,1)$-form ω on X. Hermitian metrics exist on every X. However, if an extra (closedness or exactness) condition is imposed on ω, metrics of the resulting nature may not exist. When they do, they impose a certain kind of geometry on the underlying manifold X. The purpose of this chapter is to study the geometry of compact complex manifolds X that admit one of a series of special Hermitian metrics that will be specified.

Three different kinds of metrics ω satisfying a condition in bidegree $(1,1)$, resp. in bidegree $(n-1, n-1)$, are given on the first, resp. second, line in the following picture. The implications among these conditions are also indicated. These metrics will be analysed one by one in separate sections of this chapter.

$$
\begin{array}{lclcl}
d\omega = 0 & \Longrightarrow & \exists\ \rho^{0,2} \in C^\infty_{0,2}(X, \mathbb{C}) \text{ s.t.} & \Longrightarrow & \partial\bar\partial\omega = 0 \\
 & & d(\overline{\rho^{0,2}} + \omega + \rho^{0,2}) = 0 & & \\
(\omega \text{ is Kähler}) & & (\omega \text{ is Hermitian-symplectic}) & & (\omega \text{ is SKT}) \\
\Downarrow & & & & \\
d\omega^{n-1} = 0 & \Longrightarrow & \exists\ \Omega^{n-2,n} \in C^\infty_{n-2,n}(X, \mathbb{C}) \text{ s.t.} & \Longrightarrow & \partial\bar\partial\omega^{n-1} = 0 \\
 & & d(\overline{\Omega^{n-2,n}} + \omega^{n-1} + \Omega^{n-2,n}) = 0 & & \\
(\omega \text{ is balanced}) & & (\omega \text{ is strongly Gauduchon (sG)}) & & (\omega \text{ is Gauduchon}).
\end{array}
\tag{P}
$$

The manifold X is called *Kähler*, *Hermitian-symplectic* (H-S), *SKT*, *balanced*, *strongly Gauduchon* (sG) if it carries a Hermitian metric ω of the corresponding type. Meanwhile, Gauduchon metrics always exist on any X by [Gau77a] (see Theorems 4.1.2 and 4.1.7).

The classical *Serre duality* between the Dolbeault cohomology groups $H^{1,1}_{\bar\partial}(X, C)$ and $H^{n-1,n-1}_{\bar\partial}(X, C)$, as well as its analogues between:

(i) $H^{1,1}_{BC}(X, C)$ and $H^{n-1,n-1}_{A}(X, C)$ (see Theorem 1.1.15);

(ii) $E^{1,1}_r(X)$ and $E^{n-1,n-1}_r(X)$ for every $r \geq 2$ (see Theorems 3.2.1 and 3.2.3);

(iii) $E^{1,1}_{r,BC}(X)$ and $E^{n-1,n-1}_{r,A}(X)$ (see Theorem 3.4.11);

suggest an interplay between the bidegrees $(1,1)$ and $(n-1, n-1)$ at the level of the special

Hermitian metrics featuring in picture (P). This duality interplay will be investigated in the course of this chapter.

The relationship between the bidegrees (1, 1) and $(n-1,\, n-1)$ is also expressed by the following simple observation of Michelsohn's in linear algebra asserting that every *positive definite* $(n-1,\, n-1)$-form admits a unique *positive definite* $(n-1)$*-st root.* This is a purely pointwise statement, so any closedness or exactness properties the $(n-1,\, n-1)$-form may have are not inherited by its root. The proof will show that the positive definiteness assumption cannot be relaxed to semi-positivity.

Lemma 4.0.1. *([Mic82, p.279-280]) Let X be a complex n-dimensional manifold. For every positive definite form $\Omega \in C^\infty_{n-1,\,n-1}(X,\, \mathbb{C})$ there exists a unique positive definite form $\omega \in C^\infty_{1,\,1}(X,\, \mathbb{C})$ such that*

$$\omega^{n-1} = \Omega.$$

Proof. The result being pointwise, we fix an arbitrary point $x_0 \in X$ and choose local holomorphic coordinates $z_1, \dots, z_n$ about x_0 such that

$$\Omega = \sum_{j=1}^n \Omega_j \, i\widehat{dz_j \wedge d\bar{z}_j} \quad \text{at } x_0,$$

where $i\widehat{dz_j \wedge d\bar{z}_j}$ stands for the product of all the (1, 1)-forms $idz_k \wedge d\bar{z}_k$ except the one corresponding to $k = j$. The positive definiteness of Ω is equivalent to $\Omega_j > 0$ for every $j = 1, \dots, n$.

We wish to find a positive definite (1, 1)-form

$$\rho = \sum_{j=1}^n \lambda_j \, idz_j \wedge d\bar{z}_j \quad \text{at } x_0,$$

such that $(1/(n-1)!)\, \rho^{n-1} = \Omega$ at x_0. The positive definiteness of ρ amounts to $\lambda_j > 0$ for every $j = 1, \dots, n$. Since

$$\frac{\rho^{n-1}}{(n-1)!} = \sum_{j=1}^n \frac{\lambda_1 \dots \lambda_n}{\lambda_j} \, i\widehat{dz_j \wedge d\bar{z}_j} \quad \text{at } x_0,$$

the condition $(1/(n-1)!)\, \rho^{n-1} = \Omega$ at x_0 is equivalent to the system of equations:

$$\frac{\lambda_1 \dots \lambda_n}{\lambda_j} = \Omega_j, \quad j \in \{1, \dots, n\},$$

whose unique solution is

$$\lambda_j = \frac{(\Omega_1 \dots \Omega_n)^{1/(n-1)}}{\Omega_j}, \quad j \in \{1, \dots, n\}.$$

□

Thanks to Lemma 4.0.1, we will sometimes identify an $(n-1,\, n-1)$-form $\Omega > 0$ with the $(1,\, 1)$-form $\omega > 0$ that is its $(n-1)$-st root.

4.1 Gauduchon metrics

These metrics were introduced in [Gau77a] under the name of *metrics with vanishing excentricity.*

Definition 4.1.1. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. A C^∞ positive definite $(1,\,1)$-form ω on X is said to be a* **Gauduchon metric** *if $\partial\bar\partial\omega^{n-1} = 0$.*

We will present two points of view on the existence of these metrics. The weaker statement of §.4.1.1 suffices in many applications, but Gauduchon's stronger theorem of §.4.1.2 is of vital importance in understanding the pervasiveness of Gauduchon metrics.

4.1.1 Existence of Gauduchon metrics on manifolds

The fundamental fact of life about Gauduchon metrics is the following statement that is a special case of Gauduchon's main result in [Gau77a] presented as Theorem 4.1.7 below.

Theorem 4.1.2. *Every compact complex manifold carries Gauduchon metrics.*

This result follows at once from the following

Lemma 4.1.3. *Let X be a compact complex manifold. Then, X carries a Gauduchon metric if and only if there is no non-zero $\partial\bar\partial$-***exact** *positive current T of bidegree $(1,\,1)$ on X.*

Proof of Theorem 4.1.2 granted that Lemma 4.1.3 has been proved. We will show that a current as in Lemma 4.1.3 never exists. Suppose there exists a non-zero positive $(1,\,1)$-current $T = i\partial\bar\partial\varphi \geq 0$ on X, where φ is a distribution. Since X is *compact*, the maximum principle implies that φ is a constant function on X, hence $T = 0$, a contradiction. □

Proof of Lemma 4.1.3. Let $n := \dim_{\mathbb{C}} X$.

• We will first prove that a Gauduchon metric ω and a non-zero $\partial\bar\partial$-exact positive $(1,\,1)$-current $T = i\partial\bar\partial\varphi \geq 0$ cannot simultaneously exist on X, where φ is a distribution. This will prove one of the implications in the equivalence stated in Lemma 4.1.3. Indeed, if both ω and T existed, then

$$\int\limits_X \omega^{n-1}\wedge T = \int\limits_X i\varphi\,\partial\bar\partial\omega^{n-1} = 0, \tag{4.1}$$

since $\partial\bar\partial\omega^{n-1} = 0$ by the Gauduchon property of ω. On the other hand, $\omega > 0$ and $T \geq 0$, so $\omega^{n-1}\wedge T$ is a positive (i.e. non-negative) $(n,\,n)$-current on X. Therefore, property (4.1) forces it to be the zero current, hence $T = 0$, a contradiction.

• To prove the reverse implication in the equivalence stated in Lemma 4.1.3, suppose there exists no non-zero *$\partial\bar\partial$-exact positive* $(1,\,1)$-current T on X. We will prove the existence of a Gauduchon metric under this assumption.

The idea goes back to [Sul76] and it has been used in various contexts and with different details by different authors, such as [HL83], [Mic83], [Pop09a]. It relies on a classical result in Functional Analysis: the *Hahn-Banach Separation Theorem*. Consider the locally convex space $\mathcal{D}'^{(1,\,1)}_{\mathbb{R}}(X)$ of real bidegree $(1,\,1)$ currents on X. On the one hand, the real $\partial\bar\partial$-**exact** currents of bidegree $(1,\,1)$ form a **closed** vector subspace

$$\mathcal{A} \subset \mathcal{D}'^{(1,\,1)}_{\mathbb{R}}(X).$$

Meanwhile, if we fix a smooth, strictly positive $(n-1,n-1)$-form $\Theta > 0$ on X, positive non-zero $(1,1)$-currents T on X can be normalised such that $\int\limits_X T\wedge\Theta = 1$. These normalised **positive** $(1,1)$-currents form a **compact** (in the locally convex topology of weak convergence of currents) convex subset

$$\mathcal{B} \subset \mathcal{D}'^{(1,\,1)}_{\mathbb{R}}(X).$$

Now, the Hahn-Banach separation theorem for locally convex spaces guarantees the existence of a linear functional $F : \mathcal{D}'^{(1,1)}_{\mathbb{R}}(X) \to \mathbb{R}$ vanishing identically on a given closed subset (hence $F \equiv 0$ on $\mathcal{A}$) and assuming only positive values on a given compact subset (hence $F > 0$ on $\mathcal{B}$) if the two subsets are convex and do not intersect. In our case, $\mathcal{A} \cap \mathcal{B} = \emptyset$, by assumption. Moreover, $F > 0$ on $\mathcal{B}$ if and only if $F(T) > 0$ for every (not necessarily normalised) positive non-zero $(1,1)$-current T on X.

Meanwhile, the duality between (strictly positive), smooth $(n-1, n-1)$-forms and non-zero (positive, i.e. non-negative) $(1,1)$-currents on X entails that the linear functional $F : \mathcal{D}'^{(1,1)}_{\mathbb{R}}(X) \to \mathbb{R}$ is defined by integration over X of the wedge product against a *strictly positive* form $\Omega \in C^\infty_{n-1,\, n-1}(X, \mathbb{R})$, namely $F(T) = \int_X T \wedge \Omega$ for every $T \in \mathcal{D}'^{(1,1)}_{\mathbb{R}}(X)$. Moreover, by Lemma 4.0.1, there is a unique *positive definite* smooth form ω of type $(1,1)$ on X such that $\Omega = \omega^{n-1}$. Hence, the property $F = 0$ on $\mathcal{A}$ translates to:

$$\int\limits_X \omega^{n-1} \wedge i\partial\bar\partial\varphi = 0$$

for every distribution φ on X. This further translates to $\partial\bar\partial\omega^{n-1} = 0$. Therefore, ω is a Gauduchon metric on X. $\square$

4.1.2 Existence of Gauduchon metrics in conformal classes

The starting point of this discussion is the following simple notion.

Definition 4.1.4. *Let ω be a Hermitian metric on a complex manifold X. The* **conformal class** *of ω is the set of all Hermitian metrics of the form $\omega' = \varphi\omega$, where $\varphi : X \longrightarrow (0, +\infty)$ is a positive-valued C^∞ function on X.*

Two Hermitian metrics lying in the same conformal class are said to be **conformally equivalent**.

We will need two versions of the *maximum principle* that are two facets of a same result. The version for open subsets in $\mathbb{R}^m$ reads as follows.

Lemma 4.1.5. *(see e.g. [LT95,7.2.8.]) Let $U \subset \mathbb{R}^m$ be an open connected subset, let $a_{ij}, b_i : U \longrightarrow \mathbb{R}$ be C^∞ functions for $i, j = 1, \dots, m$ such that the matrix $(a_{ij}(x))_{i,j}$ is positive definite and symmetric at every point $x \in U$.*

If a C^∞ function $f : U \longrightarrow \mathbb{R}$ satisfies the condition:

$$\sum_{i,j=1}^m a_{ij}\,\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^m b_i\,\frac{\partial f}{\partial x_i} \geq 0 \quad \textit{on } U$$

and if f has a relative **maximum** *at some point $x_0 \in U$, then f is* **constant** *in a neighbourhood of x_0. In particular, if f has an absolute* **maximum** *at some point $x_0 \in U$, then f is* **constant** *on U.*

The version for compact manifolds reads as follows. As usual, $C^\infty(M)$ stands for the vector space of smooth *functions* on a given manifold M.

Lemma 4.1.6. *(see e.g. [Gau77a, II]) Let M be a* **compact** *connected manifold and let $L : C^\infty(M) \longrightarrow C^\infty(M)$ be a real* **elliptic** *differential operator of order two with C^∞ coefficients and with* **no zero-th order terms** *(i.e. $L(1) = 0$).*

If, for a function $f \in C^\infty(M)$, $L(f) \geq 0$ at every point of M or $L(f) \leq 0$ at every point of M, then $L(f) \equiv 0$ and f is **constant** *on M.*

The main result of this subsection is the following fundamental theorem of Gauduchon's. It strengthens Theorem 4.1.2 by showing that not only does a Gauduchon metric exist on the ambient compact complex manifold, but that every Hermitian metric is conformally equivalent to a Gauduchon metric.

Theorem 4.1.7. *([Gau77a, Théorème 1]) Let X be a compact complex manifold. Every* **conformal class** *of Hermitian metrics on X contains a* **unique** *(up to multiplications by positive constants)* **Gauduchon metric**.

Proof. Let $n = \dim_{\mathbb{C}} X$ and fix an arbitrary Hermitian metric ω on X. We wish to prove the *existence* of a *unique* (up to multiplications by positive constants) C^∞ function $\psi : X \longrightarrow (0, +\infty)$ such that the Hermitian metric $\psi\omega$ is *Gauduchon* (i.e. $\partial\bar\partial(\psi^{n-1}\omega^{n-1}) = 0$).

Consider the Laplace-type operator:

$$P_\omega := i\Lambda_\omega\bar\partial\partial : C^\infty(X,\, \mathbb{C}) \longrightarrow C^\infty(X,\, \mathbb{C}).$$

Its adjoint is the operator $P_\omega^\star : C^\infty(X,\, \mathbb{C}) \longrightarrow C^\infty(X,\, \mathbb{C})$ given by

$$P_\omega^\star(f) = i\star_\omega\bar\partial\partial\bigg(f\,\frac{\omega^{n-1}}{(n-1)!}\bigg),$$

where $\star = \star_\omega$ is the Hodge star operator induced by ω. This follows at once from the formulae: $\partial^\star = -\star\bar\partial\star$, $\bar\partial^\star = -\star\partial\star$ and $\star\omega = \omega^{n-1}/(n-1)!$.

Our goal is to prove the *existence* of a *unique* (up to multiplications by positive constants) C^∞ function $\varphi : X \longrightarrow (0, +\infty)$ such that $\varphi \in \ker P_\omega^\star$. The Hermitian metric $\varphi^{\frac{1}{n-1}}\omega$ will then be *Gauduchon*.

The operators P_ω and $P_\omega^\star$ are elliptic, ≥ 0, and of vanishing index (as the principal symbols are self-adjoint). Moreover, $\ker P_\omega = \mathbb{C}$ (i.e. the constant functions) by the obvious inclusion $\mathbb{C} \subset \ker P_\omega$ and the *maximum principle* of Lemma 4.1.6. Hence, by ellipticity and vanishing index, $\dim\ker P_\omega^\star = 1$.

We will now give a sequence of lemmas describing the kernel of $P_\omega^\star$.

Lemma 4.1.8. *([Gau77a, (5)]) Let $f_0 : X \longrightarrow \mathbb{R}$ be a C^∞ function. The following implication holds:*

$$f_0 \in \ker P_\omega^\star \quad \textit{and} \quad f_0 \not\equiv 0 \implies \langle\langle f_0,\, 1\rangle\rangle_\omega \neq 0,$$

where $\langle\langle\ ,\ \rangle\rangle_\omega$ is the L^2_ω inner product on functions.

Proof. If $\langle\langle f_0,\, 1\rangle\rangle_\omega = 0$, then the constant function 1 is orthogonal to $\ker P_\omega^\star$ because $\dim\ker P_\omega^\star = 1$, $f_0 \in \ker P_\omega^\star$ and f_0, being non-identically zero, must be a generator of $\ker P_\omega^\star$. Therefore, the L^2_ω-orthogonal two-space decomposition:

$$C^\infty(X,\, \mathbb{C}) = \ker P_\omega^\star \oplus \operatorname{Im} P_\omega, \tag{4.2}$$

which follows from the ellipticity of P_ω via the standard elliptic *a priori estimate* (cf. e.g. (2) of Theorem 1.1.6), implies that $1 \in \operatorname{Im} P_\omega$.

Thus, there exists a C^∞ function $g : X \longrightarrow \mathbb{C}$ such that $P_\omega(g) = 1 > 0$ on X. The *maximum principle* of Lemma 4.1.6 implies that g must be *constant*, hence $P_\omega(g) = 0 \neq 1$, a contradiction. □

Lemma 4.1.9. *([Gau77a, (8)]) For all C^∞ functions $f : X \longrightarrow \mathbb{R}$ and $\varphi : X \longrightarrow (0, +\infty)$, the following identity and equivalence hold:*

(1) $\langle\langle\varphi^{1-n}f,\, 1\rangle\rangle_{\varphi\omega} = \langle\langle\varphi f,\, 1\rangle\rangle_\omega$;

(2) $f \in \ker P^\star_\omega \iff \varphi^{1-n}f \in \ker P^\star_{\varphi\omega}$.

Proof. (1) follows from the identities:

$$\langle\langle\varphi^{1-n}f,\, 1\rangle\rangle_{\varphi\omega} = \int\limits_X \varphi^{1-n}f\,\frac{\varphi^n\omega^n}{n!} = \int\limits_X \varphi f\,\frac{\omega^n}{n!} = \langle\langle\varphi f,\, 1\rangle\rangle_\omega.$$

(2) follows from the equivalences:

$$\varphi^{1-n}f \in \ker(P^\star_{\varphi\omega}) \iff \bar\partial\partial\bigg(\varphi^{1-n}f\,\frac{\varphi^{n-1}\omega^{n-1}}{(n-1)!}\bigg) = 0 \iff \bar\partial\partial\bigg(f\,\frac{\omega^{n-1}}{(n-1)!}\bigg) = 0 \iff f \in \ker(P^\star_\omega).$$

□

Lemma 4.1.10. *([Gau77a, Lemme 1]) Let $f_0 : X \longrightarrow \mathbb{R}$ be a C^∞ function. If f_0 is a* **generator** *of* $\ker P^\star_\omega$*, then*

$$f_0 \geq 0 \;\; on \;\; X \qquad or \qquad f_0 \leq 0 \;\; on \;\; X.$$

Proof. Since f_0 generates $\ker P^\star_\omega$, we have $f_0 \not\equiv 0$. Thus, Lemma 4.1.8 tells us that $\langle\langle f_0,\, 1\rangle\rangle_\omega \neq 0$. To make a choice, let us suppose that $\langle\langle f_0,\, 1\rangle\rangle_\omega > 0$. We will show that $f_0 \geq 0$ on X.

Suppose there exists a point $x_0 \in X$ such that $f_0(x_0) < 0$. By continuity of f, there exists an open neighbourhood U of x_0 in X such that $f_{|U} < 0$. Consider the *open* subset

$$V := \bigg\{x \in X \;\mid\; f_0(x) < 0\bigg\} \subset X.$$

Obviously, $U \subset V$. It is easy to construct a C^∞ function $\varphi : X \longrightarrow (0, +\infty)$ such that

$$\int\limits_V \varphi f_0\, dV_\omega + \int\limits_{X\setminus V} \varphi f_0\, dV_\omega = \int\limits_X \varphi f_0\, dV_\omega = 0,$$

by adjusting it so that the negative values of f_0 on V get compensated for on average by the positive values of f_0 on $X \setminus V$.

Thus, $\langle\langle\varphi f_0,\, 1\rangle\rangle_\omega = 0$. Thanks to (1) of Lemma 4.1.9, this amounts to $\langle\langle\varphi^{1-n}f_0,\, 1\rangle\rangle_{\varphi\omega} = 0$. On the other hand, (2) of Lemma 4.1.9 tells us that $\varphi^{1-n}f_0 \in \ker P^\star_{\varphi\omega}$ because we already have $f_0 \in \ker P^\star_\omega$ by hypothesis. Moreover, $\varphi^{1-n}f_0 \not\equiv 0$ because $f_0 \not\equiv 0$ and $\varphi > 0$. Summing up, we have:

$$\varphi^{1-n}f_0 \in \ker P^\star_{\varphi\omega}, \quad \varphi^{1-n}f_0 \not\equiv 0 \quad \text{and} \quad \langle\langle\varphi^{1-n}f_0,\, 1\rangle\rangle_{\varphi\omega} = 0.$$

This contradicts Lemma 4.1.8. □

The following lemma is a general result that is a useful complement to the *maximum principle* of Lemma 4.1.5. It applies to second-order elliptic operators that may have zero-th order terms.

Lemma 4.1.11. *([Gau77a, Lemme 2]) Let X be a (not necessarily compact) connected differentiable manifold and let $Q : C^\infty(X, \mathbb{R}) \longrightarrow C^\infty(X, \mathbb{R})$ be an* **elliptic** *differential operator of* **order two** *with real C^∞ coefficients.*

For every C^∞ function $f_0 : X \longrightarrow \mathbb{R}$ such that $Q(f_0) = 0$ and $f_0 \geq 0$ on X, we have:

$$f_0 > 0 \text{ on } X \quad \text{or} \quad f_0 \equiv 0.$$

Proof. We will reproduce part of the proof of Theorem 1.2.4 in [LT95]. Suppose there exists a point $x_0 \in X$ such that $f_0(x_0) = 0$. In an open neighbourhood U of x_0, the operator Q has the shape:

$$Q(f) = \sum_{i,\,j} a_{ij}\,\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_i b_i\,\frac{\partial f}{\partial x_i} + cf,$$

where a_{ij}, b_i, c are C^∞ functions of the real coordinates $x_1, \dots, x_m$ on $U \subset X$ and, to make a choice, the matrix $(a_{ij}(x))_{i,\,j}$ is *positive definite* and symmetric at every point $x \in U$.

After possibly shrinking U about x_0, we may assume that a_{ij}, b_i, c are *bounded* on U and

$$a_{11} \geq \varepsilon > 0 \quad \text{on } U$$

for some constant $\varepsilon > 0$.

Meanwhile, for a constant $a > 0$ that will be specified later, we consider the C^∞ function $g_0 : U \longrightarrow \mathbb{R}$ defined as

$$g_0(x) := e^{-ax_1}\,f_0(x).$$

The hypothesis $Q(f_0) = 0$ translates as follows:

$$\begin{aligned} Q(f_0) = 0 \quad & \iff \quad Q(g_0) + 2a \sum_i a_{1i}\frac{\partial g_0}{\partial x_i} + a^2 a_{11}\,g_0 + ab_1\,g_0 = 0 \\ & \iff \quad \sum_{i,\,j} a_{ij}\,\frac{\partial^2 g_0}{\partial x_i \partial x_j} + \sum_i (b_i + 2a\,a_{1i})\,\frac{\partial g_0}{\partial x_i} = -(a^2 a_{11} + ab_1 + c)\,g_0. \end{aligned} \tag{4.3}$$

Now, if we choose $a > 0$ large enough, we have $a^2 a_{11} + ab_1 + c > 0$ at every point of U. Since $g_0 \geq 0$ on U (because $f_0 \geq 0$ on X), we get $-(a^2 a_{11} + ab_1 + c)\,g_0 \leq 0$ on U. Meanwhile, $g_0(x) \geq g_0(x_0) = 0$ for every $x \in U$, so x_0 is a *minimum* for g in U.

From these pieces of information, including (4.3), and from the *maximum principle* of Lemma 4.1.5 we conclude that $g_0(x) = 0$ for all $x \in U$. Equivalently, $f_0(x) = 0$ for all $x \in U$.

The proof so far implies that the zero locus of f_0 is *open* in X. On the other hand, it is also *closed* by continuity of f_0. Since X is *connected*, we get $f_0^{-1}(0) = X$ since $x_0 \in f_0^{-1}(0)$. This means that $f_0 \equiv 0$ on X. □

As a consequence of Lemmas 4.1.10 and 4.1.11, we get the following

Corollary 4.1.12. *Let $f_0 : X \longrightarrow \mathbb{R}$ be a C^∞ function such that $f_0 \in \ker P_\omega^\star$. Then*

$$f_0 > 0 \text{ on } X \qquad \text{or} \qquad f_0 < 0 \text{ on } X \qquad \text{or} \qquad f_0 \equiv 0.$$

Proof. Since $\dim \ker P_\omega^\star = 1$, f_0 generates $\ker P_\omega^\star$ if $f_0 \not\equiv 0$. In this case, Lemma 4.1.10 tells us that either $f_0 \geq 0$ on X or $f_0 \leq 0$ on X. To make a choice, let us suppose that $f_0 \geq 0$ on X. Choosing $Q = P_\omega^\star$, Lemma 4.1.11 tells us that $f_0 > 0$ on X since we are in the case where $f_0 \not\equiv 0$. □

End of proof of Theorem 4.1.7. Recall that proving Theorem 4.1.7 amounts to proving the existence of a unique (up to multiplications by positive constants) C^∞ function $\varphi : X \longrightarrow (0, +\infty)$ such that $\varphi \in \ker P^\star_\omega$.

The fact that $\dim \ker P^\star_\omega = 1$ proves the uniqueness of φ up to positive multiplicative constants.

Let $\varphi : X \longrightarrow \mathbb{R}$ be a generator of $\ker P^\star_\omega$. Then $\varphi \not\equiv 0$, so by Corollary 4.1.12 we either have $\varphi > 0$ on X or $\varphi < 0$ on X. To make a choice, we may assume that $\varphi > 0$ on X and we are done. $\Box$

The proof of Gauduchon's Theorem 4.1.7 shows that any Gauduchon metric on some fibre X_0 of a holomorphic family deforms in a C^∞ way to Gauduchon metrics on the nearby fibres X_t.

Proposition 4.1.13. *Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds, where $B \subset \mathbb{C}^m$ is an open ball about 0 for some $m \in \mathbb{N}^\star$. Put $X_t := \pi^{-1}(t)$ for $t \in B$.*

Let ω_0 be a **Gauduchon metric** *on X_0. After possibly shrinking B about 0, there exists a C^∞ family $(\omega_t)_{t\in B}$ of 2-forms on the fibres $(X_t)_{t\in B}$ whose member for $t = 0$ is ω_0 and such that ω_t is a* **Gauduchon metric** *on X_t for every $t \in B$.*

Proof. Let $(\gamma_t)_{t\in B}$ be any family of Hermitian metrics varying in a C^∞ way with t on $(X_t)_{t\in\Delta}$ such that $\gamma_0 = \omega_0$. We will prove a slightly stronger statement in the sense that we need not assume γ_0 to be Gauduchon from the start.

The family $(P^\star_{\gamma_t})_{t\in B}$ is a C^∞ family of *elliptic* differential operators on the fibres X_t with kernels of *equal dimensions* ($= 1$). By the Kodaira-Spencer Theorem C reproduced in §.2.5.1, the kernels define a C^∞ vector bundle $B \ni t \mapsto \ker(P^\star_{\omega'_t})$. Meanwhile, by Corollary 4.1.12, there exists $f_0 \in \ker(P^\star_{\gamma_0|C^\infty(X,\mathbb{R})})$ such that $f_0 > 0$. Then, it suffices to extend f_0 to a local C^∞ section $B \ni t \mapsto f_t$ of the C^∞ real vector bundle $B \ni t \mapsto \ker(P^\star_{\gamma_t|C^\infty(X,\mathbb{R})})$ which is a trivial bundle if B has been shrunk sufficiently about 0. By continuity, $f_t > 0$ on X_t for all t sufficiently close to $0 \in B$. Indeed, by Corollary 4.1.12, $f_t > 0$ on X_t or $f_t < 0$ on X_t or $f_t \equiv 0$, but the last two possibilities are ruled out by the continuous dependence of f_t on t and by $f_0 > 0$.

Thus, we get a family $\omega_t := f_t^{\frac{1}{n-1}}\,\gamma_t$, $t \in B$, of *Gauduchon metrics* varying in a C^∞ way with t on the fibres X_t. If $\gamma_0 = \omega_0$ is already Gauduchon, we may choose $f_0 \equiv 1$. $\Box$

4.1.3 The Gauduchon cone

Recall a classical notion: the *Kähler cone* $\mathcal{K}_X$ of a compact complex manifold X is the set of all Bott-Chern cohomology classes of Kähler metrics on X. As such, it is an *open convex cone* in $H^{1,1}_{BC}(X, \mathbb{R})$. Moreover, $\mathcal{K}_X = \emptyset$ if X is not Kähler.

We start by introducing the following analogue of the Kähler cone in bidegree $(n-1, n-1)$. (For this analogy, see the new approach to Mirror Symmetry described in chapter 6.) It is never empty, it generalises the Kähler cone and it has come to play a key role in various aspects of both Kähler and non-Kähler geometry, some of which will be described further down.

Definition 4.1.14. *([Pop15a, Definition 5.1]) Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The* **Gauduchon cone** *of X is the set*

$$\mathcal{G}_X := \left\{[\omega^{n-1}]_A \in H^{n-1,\,n-1}_A(X, \mathbb{R}) \mid \omega \text{ is a Gauduchon metric on } X\right\} \subset H^{n-1,\,n-1}_A(X, \mathbb{R}).$$

Any element $[\omega^{n-1}]_A$ of the Gauduchon cone $\mathcal{G}_X$ is called an **Aeppli-Gauduchon class***.*

In chapter 6, the Gauduchon cone will be seen to also offer an analogue of the various cones of cohomology classes of curves when the given manifold supports no curves.

Since $\partial\bar\partial\omega^{n-1}=0$ for every Gauduchon metric ω, the $(n-1,\,n-1)$-form $\partial\bar\partial\omega^{n-1}$ does define a real Aeppli cohomology class $[\omega^{n-1}]_A$ of bidegree $(n-1,\,n-1)$. Moreover, we have

Proposition 4.1.15. *The Gauduchon cone $\mathcal{G}_X$ is an* **open convex cone** *in $H^{n-1,\,n-1}_A(X,\,\mathbb{R})$.*

Proof. • If $[\omega^{n-1}]_A\in\mathcal{G}_X$ and $\lambda>0$ is a constant, then $\lambda\,[\omega^{n-1}]_A\in\mathcal{G}_X$, so $\mathcal{G}_X$ is indeed a *cone*.

• To show *convexity*, let $[\omega_1^{n-1}]_A,[\omega_2^{n-1}]_A\in\mathcal{G}_X$ and let $\lambda,\mu>0$ be constants such that $\lambda+\mu=1$. Then, $\lambda\,[\omega_1^{n-1}]_A+\mu\,[\omega_2^{n-1}]_A=[\omega^{n-1}]_A\in\mathcal{G}_X$ where $\omega>0$ is the $(n-1)^{st}$ root of $\lambda\,\omega_1^{n-1}+\mu\,\omega_2^{n-1}>0$. (See Lemma 4.0.1 for the existence of the root.)

• To show *openness*, let us equip the finite-dimensional vector space $H^{n-1,\,n-1}_A(X,\,\mathbb{R})$ with an arbitrary norm $||\ ||$ (e.g. the Euclidian norm after we have fixed a basis; at any rate, all the norms are equivalent). Let $[\omega^{n-1}]_A\in\mathcal{G}_X$ be an arbitrary element, where $\omega>0$ is some Gauduchon metric on X. Let $\alpha\in H^{n-1,\,n-1}_A(X,\,\mathbb{R})$ be a class such that $||\alpha-[\omega^{n-1}]_A||<\varepsilon$ for some small $\varepsilon>0$. Fix any Hermitian metric ω_0 on X and consider the Aeppli Laplacian Δ_A defined by ω_0 inducing the *Hodge isomorphism* $H^{n-1,\,n-1}_A(X,\,\mathbb{R})\simeq\mathcal{H}^{n-1,\,n-1}_{\Delta_A}(X,\,\mathbb{R})$. (See (2) of Corollary 1.1.13.) Let $\Omega_\alpha\in\mathcal{H}^{n-1,\,n-1}_{\Delta_A}(X,\,\mathbb{R})$ be the Δ_A-harmonic representative of the class α. Since $\omega^{n-1}\in\ker(\partial\bar\partial)$, (2) of Corollary 1.1.13 gives a unique decomposition:

$$\omega^{n-1}=\Omega+(\partial u+\bar\partial v)\quad\text{with}\ \ \Delta_A\Omega=0.$$

If we set $\Gamma:=\Omega_\alpha+(\partial u+\bar\partial v)$ (with the same forms u,v as for ω^{n-1}), then $\partial\bar\partial\Gamma=0$, Γ represents the Aeppli class α and we have

$$||\Gamma-\omega^{n-1}||_{C^0}=||\Omega_\alpha-\Omega||_{C^0}\leq C\,||\alpha-[\omega^{n-1}]_A||<C\varepsilon,$$

for some constant $C>0$ induced by the Hodge isomorphism. (We have chosen the C^0 norm on $\mathcal{H}^{n-1,\,n-1}_{\Delta_A}(X,\,\mathbb{R})$ only for the sake of convenience.) Thus, if $\varepsilon>0$ is chosen sufficiently small, the $(n-1,\,n-1)$-form Γ must be positive definite since ω^{n-1} is, so there exists a unique positive definite $(1,\,1)$-form γ such that $\gamma^{n-1}=\Gamma$. Thus γ is a Gauduchon metric and γ^{n-1} represents the original Aeppli class α, so $\alpha\in\mathcal{G}_X$. □

So, Definition 4.1.14 is meaningful. Moreover, thanks to Theorem 4.1.2, $\mathcal{G}_X\neq\emptyset$ for every compact complex manifold X. Intuitively, Gauduchon's Theorem 4.1.7 shows that the Gauduchon cone is fairly large. We will see later that the smaller $\mathcal{G}_X$, the better the properties of X.

We will now discuss the link between the Gauduchon cone and the cone of Bott-Chern cohomology classes of d-closed semi-positive $(1,\,1)$-currents introduced by Demailly. By a *semi-positive* current we mean what is called a *positive* current in the French terminology, where the notion originated, and in the terminology employed by many authors. (See [Dem97, chapter III].) This property of currents is denoted by $\geq$. We may also refer to such a current as *positive*, so, when applied to currents, the terms *semi-positive* and *positive* will be interchangeable in this book.

Definition 4.1.16. *([Dem92, Definition 1.3]) Let X be a compact complex manifold. The* **pseudo-effective cone** *of X is the set*

$$\mathcal{E}_X:=\bigg\{[T]_{BC}\in H^{1,\,1}_{BC}(X,\,\mathbb{R})\,/\,T\geq 0\ \textit{d-closed}\ (1,\,1)\textit{-current on}\ X\bigg\}.$$

Any element $[T]_{BC}$ of the pseudo-effective cone $\mathcal{E}_X$ is called a **pseudo-effective class***.*

Like the De Rham, Dolbeault and Aeppli cohomologies, Bott-Cern cohomology can be computed using either smooth differential forms or currents. The latter point of view on the cohomology defined in §.1.1 is taken in the above definition.

Proposition 4.1.17. *([Dem92, Proposition 6.1]) The pseudo-effective cone $\mathcal{E}_X$ is a* **closed convex cone** *in $H^{1,1}_{BC}(X,\,\mathbb{R})$.*

Proof. If T_1 and T_2 are d-closed semi-positive $(1,\,1)$-currents on X, so is any linear combination $\lambda T_1 + \mu T_2$ with λ, μ non-negative reals. Therefore, $\mathcal{E}_X$ is a convex cone.

To show that $\mathcal{E}_X$ is *closed* in $H^{1,1}_{BC}(X,\,\mathbb{R})$, fix an arbitrary Gauduchon metric ω on X (which exists thanks to Theorem 4.1.2) and let $(T_k)_{k\in\mathbb{N}}$ be a sequence of d-closed semi-positive $(1,\,1)$-currents on X such that their classes $[T_k]_{BC}$ converge to a limit $\mathfrak{c}^{1,1}_{BC} \in H^{1,1}_{BC}(X,\,\mathbb{R})$. We will show that $\mathfrak{c}^{1,1}_{BC} \in \mathcal{E}_X$.

By continuity of the evaluation map

$$H^{1,1}_{BC}(X,\,\mathbb{R}) \ni \mathfrak{b}^{1,1}_{BC} \mapsto \mathfrak{b}^{1,1}_{BC} \cdot [\omega^{n-1}]_A \in \mathbb{R},$$

$[T_k]_{BC} \cdot [\omega^{n-1}]_A = \int_X T_k \wedge \omega^{n-1}$ converges to $\mathfrak{c}^{1,1}_{BC} \cdot [\omega^{n-1}]_A$ as $k \to +\infty$. In particular, the sequence $(\int_X T_k \wedge \omega^{n-1})_{k\in\mathbb{N}}$ is *bounded* in $\mathbb{R}$. This means that the sequence of currents $(T_k)_{k\in\mathbb{N}}$ is *bounded in mass*. Since the T_k's are *semi-positive* currents, this implies that the sequence $(T_k)_{k\in\mathbb{N}}$ is *compact* in the weak topology of currents. Therefore, there exists a weakly convergent subsequence $T_{k_\nu} \longrightarrow T$.

The limit T must be a d-closed semi-positive $(1,\,1)$-current on X and, since taking the cohomology class $[\ ]_{BC}$ is a continuous operation w.r.t. the weak topology of currents, we get $[T]_{BC} = \mathfrak{c}^{1,1}_{BC}$. Therefore, $\mathfrak{c}^{1,1}_{BC} \in \mathcal{E}_X$. □

The main link between the cones $\mathcal{G}_X$ and $\mathcal{E}_X$ is provided by the following reformulation of a result of Lamari's from [Lam99].

Theorem 4.1.18. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The pseudo-effective cone $\mathcal{E}_X \subset H^{1,1}_{BC}(X,\,\mathbb{R})$ and the closure of the Gauduchon cone $\overline{\mathcal{G}}_X \subset H^{n-1,\,n-1}_A(X,\,\mathbb{R})$ are* **dual** *to each other under the duality:*

$$H^{1,1}_{BC}(X,\,\mathbb{C}) \times H^{n-1,\,n-1}_A(X,\,\mathbb{C}) \to \mathbb{C}, \qquad ([\alpha]_{BC},\,[\beta]_A) \mapsto \int\limits_X \alpha \wedge \beta,$$

of Theorem 1.1.15.

The meaning of the above duality of cones is that the following two statements hold.

(1) Given any class $\mathfrak{c}^{1,1}_{BC} \in H^{1,1}_{BC}(X,\,\mathbb{R})$, the following equivalence holds:

$$\mathfrak{c}^{1,1}_{BC} \in \mathcal{E}_X \iff \mathfrak{c}^{1,1}_{BC}.\mathfrak{c}^{n-1,\,n-1}_A \geq 0 \quad \text{for every class } \mathfrak{c}^{n-1,\,n-1}_A \in \mathcal{G}_X.$$

(2) Given any class $\mathfrak{c}^{n-1,\,n-1}_A \in H^{n-1,\,n-1}_A(X,\,\mathbb{R})$, the following equivalence holds:

$$\mathfrak{c}^{n-1,\,n-1}_A \in \overline{\mathcal{G}}_X \iff \mathfrak{c}^{1,1}_{BC}.\mathfrak{c}^{n-1,\,n-1}_A \geq 0 \quad \text{for every class } \mathfrak{c}^{1,1}_{BC} \in \mathcal{E}_X.$$

In a similar vein, the various dualities between smooth forms and currents on an n-dimensional complex manifold X run along the following general principles (cf. e.g. [Dem97, chapter III]) that we will be using throughout this book:

(a) *smooth* objects (i.e. C^∞ differential forms) of any bidegree $(p,\,q)$ are dual to *singular* objects (i.e. currents) of the complementary bidegree $(n-p,\,n-q)$;

(b) *semi-positive* objects are dual to *strictly positive* objects of the complementary bidegree;

(c) *strongly positive* objects are dual to *weakly positive* objects of the complementary bidegree.

For example, combining (b) and (c) for a given current $T\in\mathcal{D}'^{p,p}(X)$ of bidegree $(p,\,p)$, we get the equivalence:

$$T\geq 0 \text{ weakly} \iff \int\limits_X T\wedge\alpha\geq 0 \quad \text{for every strongly strictly positive form } \alpha\in C^\infty_{n-p,\,n-p}(X,\,\mathbb{C}).$$

(d) *closed* objects (w.r.t. any of the operators d, ∂, $\bar\partial$, $\partial\bar\partial$) are dual to *exact* objects of the complementary bidegree (w.r.t. the same operator) when X is *compact*.

For example, for a given form $\alpha\in C^\infty_{p,q}(X,\,\mathbb{C})$, we have the equivalence:

$$\bar\partial\alpha=0 \iff \int\limits_X \alpha\wedge\bar\partial S=0 \quad \text{for every current } S\in\mathcal{D}'^{n-p,\,n-q-1}(X,\,\mathbb{C}).$$

Theorem 4.1.18 follows by applying the following result of Lamari's to *d-closed* forms α.

Lemma 4.1.19. *(Lamari's duality lemma, [Lam99, Lemme 3.3]) Let X be a compact complex manifold with $\dim_{\mathbb{C}}X=n$ and let α be a C^∞ real $(1,\,1)$-form on X. The following two statements are equivalent.*

(i) There exists a distribution ψ on X such that $\alpha+i\partial\bar\partial\psi\geq 0$ in the sense of $(1,\,1)$-currents on X.

(ii) $\int_X\alpha\wedge\omega^{n-1}\geq 0$ for any Gauduchon metric ω on X.

Proof. (i) $\implies$ (ii). This implication is obvious. Indeed, $\int_X\alpha\wedge\omega^{n-1}=\int_X(\alpha+i\partial\bar\partial\psi)\wedge\omega^{n-1}$ for every Gauduchon metric ω, by Stokes and $\partial\bar\partial\omega^{n-1}=0$, while the latter integral is non-negative if $\alpha+i\partial\bar\partial\psi\geq 0$.

(ii) $\implies$ (i). Let us consider the following:

- *vector subspace* $E:=C^\infty_{n-1,\,n-1}(X,\,\mathbb{R})\cap\ker(\partial\bar\partial)\subset C^\infty_{n-1,\,n-1}(X,\,\mathbb{R})$;

- *open convex* subset $U:=\left\{\Omega\in C^\infty_{n-1,\,n-1}(X,\,\mathbb{R})\;\mid\;\Omega>0\right\}\subset C^\infty_{n-1,\,n-1}(X,\,\mathbb{R})$;

- *open convex* subset $V:=U\cap E\subset E$. So, thanks to the existence and uniqueness of the $(n-1)$-*st root* (see Lemma 4.0.1), V identifies with the set of *Gauduchon metrics* on X.

Meanwhile, the given form $\alpha\in C^\infty_{1,1}(X,\,\mathbb{R})$ defines an $\mathbb{R}$-linear *evaluation map*:

$$\alpha:E\longrightarrow\mathbb{R},\qquad \Omega\longmapsto\alpha(\Omega):=\int\limits_X\alpha\wedge\Omega,$$

that we denote by the same symbol. Property (ii) in the statement, serving as our current hypothesis, translates to

$$\alpha_{|V}\geq 0. \tag{4.4}$$

Given hypothesis (4.4), there are two cases.

Case 1. Suppose there exists a *Gauduchon metric* ω_0 (i.e. $\omega_0^{n-1} \in V$) such that $\int_X \alpha \wedge \omega_0^{n-1} = 0$. We will prove that $\alpha_{|E} \equiv 0$ in this case.

Let $\Omega \in E$. For every $t \in \mathbb{R}$, let

$$\Omega_t := (1-t)\,\omega_0^{n-1} + t\,\Omega.$$

Meanwhile, consider the *affine function*

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \qquad f(t) := \int\limits_X \alpha \wedge \Omega_t = \alpha(\Omega_t).$$

Since $\Omega_t \in E$ for every $t \in \mathbb{R}$, Ω_t depends continuously on t, $\Omega_0 = \omega_0^{n-1} \in V$ and V is *open* in E, there exists $\varepsilon > 0$ such that $\Omega_t \in V$ for all $t \in [-\varepsilon,\, \varepsilon]$. Since $\alpha_{|V} \geq 0$, we get $f(t) \geq 0$ for all $t \in [-\varepsilon,\, \varepsilon]$. Moreover, $f(0) = 0$ by the assumption of *Case* 1 and f is *affine*. We infer that f is the *identically zero* function on $\mathbb{R}$. In particular, $f(1) = \int_X \alpha \wedge \Omega = 0$, namely $\alpha(\Omega) = 0$.

Case 2. Suppose that $\alpha_{|V} > 0$. Since V is non-empty (thanks to the existence of Gauduchon metrics on X), this implies that

$$F := \ker(\alpha : E \to \mathbb{R})$$

is a *closed hyperplane* of E and $U \cap F = \emptyset$.

Therefore, by the *Hahn-Banach Separation Theorem*, we can separate U and F by an element T of the dual space of $C^\infty_{n-1,\,n-1}(X,\, \mathbb{R})$ (equivalently, by a real $(1,\,1)$-current T on X) such that

$$(i)\ T_{|U} > 0 \quad \text{and} \quad (ii)\ T_{|F} = 0. \tag{4.5}$$

By the duality principles (b) and (c), property (i) of (4.5) translates to $T \geq 0$ and $T \neq 0$ as a $(1,\,1)$-current on X.

Now, fix an arbitrary Gauduchon metric ω_1 on X. So, $\omega_1^{n-1} \in V$. Since $\alpha_{|V} > 0$ and $T_{|U} > 0$ while $U \supset V$, we get:

$$\int\limits_X \alpha \wedge \omega_1^{n-1} > 0 \quad \text{and} \quad \int\limits_X T \wedge \omega_1^{n-1} > 0.$$

Therefore, there exists a real $\lambda > 0$ such that $\int_X \alpha \wedge \omega_1^{n-1} = \lambda \int_X T \wedge \omega_1^{n-1}$. This means that

$$\int\limits_X (\alpha - \lambda T) \wedge \omega_1^{n-1} = 0,$$

or equivalently that $(\alpha - \lambda T)_{|\mathbb{R}\omega_1^{n-1}} \equiv 0$.

On the other hand, $(\alpha - \lambda T)_{|F} \equiv 0$ because $T_{|F} \equiv 0$ by (ii) of (4.5) and $\alpha_{|F} \equiv 0$ since $F = \ker \alpha$.

Since the codimension of F in E is 1 and the vector line $\mathbb{R}\omega_1^{n-1}$ of E meets F only at 0, we infer from the above properties that $(\alpha - \lambda T)_{|E} \equiv 0$. Since $E = \ker(\partial\bar\partial)$, the duality principle (d) ensures the existence of a distribution ψ on X such that $\alpha - \lambda T = -i\partial\bar\partial\psi$. In other words, the $(1,\,1)$-current

$$\alpha + i\partial\bar\partial\psi = \lambda T \geq 0$$

is *semi-positive* on X and lies in the Bott-Chern cohomology class of α. $\square$

4.1.4 The nef and big cones

Besides the Kähler, pseudo-effective and Gauduchon cones discussed in §.4.1.3, we now present two more positivity cones that have played major parts in algebraic and transcendental aspects of compact complex manifolds. Gauduchon metrics will again prove critical to certain proofs.

(A) The nef cone

The starting point of this discussion is the following definition of Demailly's.

Definition 4.1.20. *([Dem92, Definition 1.3]) Let X be a compact complex manifold. A cohomology class $[\alpha]_{BC}\in H^{1,1}_{BC}(X,\,\mathbb{R})$ is said to be* **nef** *if for some fixed Hermitian metric ω on X and for every constant $\varepsilon>0$ there exists a C^∞ form $\alpha_\varepsilon\in[\alpha]_{BC}$ such that $\alpha_\varepsilon\geq -\varepsilon\omega$.*

The **nef cone** *of X is the set*

$$\mathcal{N}_X:=\bigg\{[\alpha]_{BC}\in H^{1,1}_{BC}(X,\,\mathbb{R})\,/\,[\alpha]_{BC}\ \textit{is nef}\bigg\}.$$

Since X is compact, any two Hermitian metrics ω_1 and ω_2 on X are comparable, namely there exists a constant $c>0$ such that $(1/c)\,\omega_1\leq\omega_2\leq c\,\omega_1$ on X. Thus, the choice of background Hermitian metric ω on X is irrelevant in the Definition 4.1.20 which requires a nef class to be representable by *smooth* forms of *arbitrarily small negative part* (if any). Of course, any class $[\alpha]_{BC}\in H^{1,1}_{BC}(X,\,\mathbb{R})$ that can be represented by a *semi-positive* C^∞ $(1,\,1)$-form $\alpha\geq 0$ is nef, but there are examples of nef classes $[\alpha]_{BC}$ that do not contain semi-positive representatives.

Moreover, the above transcendental definition of *nefness* generalises the analogous classical definition for *integral* cohomology classes (i.e. first Chern classes $c_1(L)\in H^{1,1}_{BC}(X,\,\mathbb{R})$ of holomorphic line bundles $L\longrightarrow X$) on *projective* manifolds. Specifically, if X is projective and $[\alpha]\in H^{1,\,1}(X,\,\mathbb{R})\cap H^2_{DR}(X,\,\mathbb{Z})$, then it is standard (see e.g. [Dem90, Proposition 4.2]) that $[\alpha]$ is *nef* if and only if $[\alpha].C:=\int_C\alpha\geq 0$ for every curve $C\subset X$. Since any projective manifold is a $\partial\bar\partial$-manifold, the Bott-Chern, Dolbeault and Aeppli cohomologies are canonically isomorphic (cf. §.1.3), accounting for the dropping of the subscript BC from $H^{1,\,1}(X,\,\mathbb{R})$ above.

Proposition 4.1.21. *([Dem92, Proposition 6.1]) The nef cone $\mathcal{N}_X$ of any compact complex manifold X is a* **closed convex cone** *in $H^{1,1}_{BC}(X,\,\mathbb{R})$. Moreover,*

$$\mathcal{K}_X\subset\mathcal{N}_X\subset\mathcal{E}_X,$$

where $\mathcal{K}_X$ is the Kähler cone of X.

If X is **Kähler***, the nef cone is the closure of the Kähler cone. If X is* **non-Kähler***, its Kähler cone is empty.*

Proof. • If $[\alpha_1]_{BC},[\alpha_2]_{BC}\in\mathcal{N}_X$ and λ,μ are non-negative reals, then $\lambda\,[\alpha_1]_{BC}+\mu\,[\alpha_2]_{BC}\in\mathcal{N}_X$, proving that $\mathcal{N}_X$ is a *convex cone*. Indeed, fix any Hermitian metric ω on X and any constant $\varepsilon>0$. Then, there exist C^∞ forms $\alpha_1(\varepsilon)\in[\alpha_1]_{BC}$ and $\alpha_2(\varepsilon)\in[\alpha_2]_{BC}$ such that $\alpha_1(\varepsilon)\geq-(\varepsilon/2\lambda)\,\omega$ and $\alpha_2(\varepsilon)\geq-(\varepsilon/2\mu)\,\omega$. Hence, $\lambda\,\alpha_1(\varepsilon)+\mu\,\alpha_2(\varepsilon)\in C^\infty_{1,1}(X,\,\mathbb{R})$ represents the class $\lambda\,[\alpha_1]_{BC}+\mu\,[\alpha_2]_{BC}$ and $\lambda\alpha_1(\varepsilon)+\mu\alpha_2(\varepsilon)\geq-\varepsilon\,\omega$.

To show that $\mathcal{N}_X$ is *closed* in $H^{1,1}_{BC}(X,\,\mathbb{R})$, let $([\alpha_k]_{BC})_{k\in\mathbb{N}}$ be a sequence of classes converging to $[\alpha]_{BC}$ in $H^{1,1}_{BC}(X,\,\mathbb{R})$ as $k\to+\infty$. Then, the class $[\alpha-\alpha_k]_{BC}$ converges to the zero class in $H^{1,1}_{BC}(X,\,\mathbb{R})$. From the definition of the quotient topology on $H^{1,1}_{BC}(X,\,\mathbb{R})$ we infer the existence of C^∞ representatives $\beta_k\in[\alpha-\alpha_k]_{BC}$ converging to 0 in the C^∞ topology.

Now, fix $\varepsilon > 0$. If the classes $[\alpha_k]_{BC}$ are *nef*, then for each k, there exists a C^∞ representative $\alpha_k(\varepsilon) \in [\alpha_k]_{BC}$ such that $\alpha_k(\varepsilon) \geq (-\varepsilon/2)\,\omega$. Meanwhile, since β_k converges to 0 in the C^0 topology, $\beta_k \geq (-\varepsilon/2)\,\omega$ whenever k is large enough. Therefore, $\alpha_k(\varepsilon) + \beta_k \geq -\varepsilon\,\omega$ for all $k \gg 1$ and $\alpha_k(\varepsilon) + \beta_k \in [\alpha]_{BC}$. Hence, $[\alpha]_{BC}$ is *nef*, proving that $\mathcal{N}_X$ is closed in $H^{1,1}_{BC}(X,\,\mathbb{R})$.

• The inclusion $\mathcal{K}_X \subset \mathcal{N}_X$ is obvious. The inclusion $\mathcal{N}_X \subset \mathcal{E}_X$ is a consequence of the existence of Gauduchon metrics. (See Theorem 4.1.2.) Indeed, fix an arbitrary Gauduchon metric ω on X. Let $[\alpha]_{BC} \in \mathcal{N}_X$. We wish to prove the existence of a semi-positive current T in the class $[\alpha]_{BC}$.

For every $\varepsilon > 0$, we know that there exists a C^∞ form $\alpha_\varepsilon \in [\alpha]_{BC}$ such that $\alpha_\varepsilon \geq -\varepsilon\omega$. Thus, $\alpha_\varepsilon + \varepsilon\omega \geq 0$ and the ω-mass of this semi-positive form satisfies:

$$\int\limits_X (\alpha_\varepsilon + \varepsilon\omega)\wedge\omega^{n-1} = \int\limits_X \alpha\wedge\omega^{n-1} + \varepsilon\int\limits_X \omega^n, \qquad \varepsilon > 0,$$

thanks to $\alpha_\varepsilon = \alpha + i\partial\bar\partial\varphi_\varepsilon$, to Stokes's Theorem and to the Gauduchon property $\partial\bar\partial\omega^{n-1} = 0$.

We infer that the family $(\alpha_\varepsilon + \varepsilon\omega)_{\varepsilon>0}$ of semi-positive forms is *uniformly bounded in mass*, hence relatively *weakly compact*. Thus, there exists a subsequence $(\alpha_{\varepsilon_k} + \varepsilon_k\omega)_{k\in\mathbb{N}}$, with $\varepsilon_k \downarrow 0$, converging in the weak topology of currents to some current T. This limit current T is necessarily d-closed (since all the α_{ε_k}'s are), semi-positive (since all the $(\alpha_{\varepsilon_k} + \varepsilon_k\omega)$'s are) and lies in the cohomology class $[\alpha]_{BC}$ (since $\alpha_{\varepsilon_k} \in [\alpha]_{BC}$ for every k and taking the class is a continuous operation w.r.t. the weak topology of currents).

We conclude that $[\alpha]_{BC} \in \mathcal{E}_X$.

• If X is non-Kähler, there are no Kähler metrics on X, so $\mathcal{K}_X = \emptyset$.

Suppose that X is Kähler and fix an arbitrary Kähler metric ω on X. Since $\mathcal{K}_X \subset \mathcal{N}_X$ and $\mathcal{N}_X$ is closed, we have $\overline{\mathcal{K}}_X \subset \mathcal{N}_X$. Conversely, let $[\alpha]_{BC} \in \mathcal{N}_X$. Then, there exist C^∞ forms $\alpha_\varepsilon \in [\alpha]_{BC}$ such that $\alpha_\varepsilon \geq -\varepsilon\omega$. Hence, $\alpha_\varepsilon + 2\varepsilon\omega \geq \varepsilon\omega$ is a Kähler metric and

$$[\alpha_\varepsilon + 2\varepsilon\omega]_{BC} = [\alpha]_{BC} + 2\varepsilon\,[\omega]_{BC} \in \mathcal{K}_X$$

converges to $[\alpha]_{BC}$ when $\varepsilon \downarrow 0$. Therefore, $[\alpha]_{BC} \in \overline{\mathcal{K}}_X$. □

(B) The big cone

The starting point of this discussion is the following definition of Demailly's.

Definition 4.1.22. *([Dem90]) Let X be a compact complex manifold. A* **Kähler current** *on X is a d-closed current T of bidegree $(1,\,1)$ such that $T \geq \varepsilon\omega$ on X for some constant $\varepsilon > 0$ and some Hermitian metric ω on X.*

When $L \longrightarrow X$ is a holomorphic line bundle on a *projective* n-dimensional manifold X, L is *big* (in the sense that $\dim_{\mathbb{C}} H^0(X,\,L^k)$ has the maximal growth order of $\mathcal{O}(k^n)$ when $k \to +\infty$) if and only if its first Chern class $c_1(L)$ can be represented by a *Kähler current*. (See e.g. [Dem90, Proposition 4.2].)

The main appeal of Kähler currents stems from the following result of Demailly and Paun.

Theorem 4.1.23. *([DP04, Theorem 3.4]) A compact complex manifold X carries a* **Kähler current** *if and only if X is a* **class $\mathcal{C}$ manifold**.

Proof. • Suppose that X is a **class $\mathcal{C}$ manifold**. Then, there exists a modification $\mu : \widetilde{X} \longrightarrow X$ from a compact Kähler manifold $\widetilde{X}$. Let $\widetilde{\omega}$ be a Kähler metric on $\widetilde{X}$.

Fix an arbitrary Hermitian metric ω on X. Since $\widetilde{X}$ is compact, $\widetilde{\omega} > 0$ and $\widetilde{\omega}$ and $\mu^\star\omega$ are C^∞ (hence continuous) forms on $\widetilde{X}$, there exists a constant $C > 0$ such that

$$\widetilde{\omega} \geq C\mu^\star\omega \quad \text{on } \widetilde{X}.$$

Taking pushforwards under μ, we get

$$\mu_\star\widetilde{\omega} \geq C\omega \quad \text{on } X.$$

Meanwhile, $d(\mu_\star\widetilde{\omega}) = \mu_\star(d\widetilde{\omega}) = 0$, so we conclude that $\mu_\star\widetilde{\omega}$ is a Kähler current on X. (Recall that the pushforward of a smooth form is, in general, only a current.)

• Conversely, suppose that there exists a **Kähler current** $T \geq \varepsilon\omega$ on X, where ω is an arbitrary Hermitian metric and $\varepsilon > 0$ is a constant. Demailly's Regularisation-of-Currents Theorem 1.1. in [Dem92] implies that T is the weak limit of a sequence $(T_m)_{m\in\mathbb{N}}$ of Kähler currents with *analytic singularities* lying in the Bott-Chern cohomology class of T. In particular, for every m, the analytic singularities of T_m can be resolved in the sense that there exists a modification $\mu_m : \widetilde{X}_m \longrightarrow X$ from some compact complex manifold $\widetilde{X}_m$ such that

$$\mu_m^\star T_m = \lambda_m[\widetilde{D}_m] + \widetilde{\alpha}_m \quad \text{on } \widetilde{X}_m, \tag{4.6}$$

where $\widetilde{D}_m$ is a normal crossing divisor and $\widetilde{\alpha}_m$ is a C^∞ d-closed (1, 1)-form on $\widetilde{X}_m$, while $\lambda_m > 0$ is a constant.

A few explanations are in order.

(i) What is meant by the singularities of T_m being *analytic* is the fact that, once we have selected a C^∞ representative α of $[T]_{BC}$ and written $T = \alpha + i\partial\bar\partial\psi$ on X for some quasi-psh function $\psi : X \longrightarrow \mathbb{R}\cup\{-\infty\}$, we can choose the approximating currents T_m such that $T_m = \alpha + i\partial\bar\partial\psi_m$ on X for quasi-psh functions $\psi_m : X \longrightarrow \mathbb{R}\cup\{-\infty\}$ whose only singularities are *logarithmic poles*. This means that each ψ_m is C^∞ on $X \setminus Z_m$ for some *analytic subset* $Z_m \subset X$ and that in a neighbourhhod of every point of Z_m the function ψ_m is of the shape

$$\psi_m(z) = c_m \log\sum_l |g_{m,l}|^2 + C^\infty,$$

where the $g_{m,l}$'s are locally defined *holomorphic* functions and the c_m's are positive constants. The singularities (= the logarithmic poles) of ψ_m are the points of Z_m; they arise locally as the common zeros of the $g_{m,l}$'s when l varies.

(ii) The regularisation process introduces a loss of positivity which can be made arbitrarily small provided that m is chosen big enough. Thus, T_m is slightly less positive than T in the sense that $T_m \geq (\varepsilon - \varepsilon_m)\,\omega$ on X for some sequence of constants $\varepsilon_m \downarrow 0$ as $m \to +\infty$. However, we still have $\varepsilon - \varepsilon_m > 0$, so T_m is still a Kähler current, if $m \gg 1$.

(iii) By [Meo96], the pullback of a d-closed semipositive (1, 1)-current T (in particular of T_m) under a holomorphic map $\mu$is always well-defined. Indeed, we can write locally $T = i\partial\bar\partial\varphi$ for a locally defined psh function φ and we put $\mu^\star T = i\partial\bar\partial(\varphi\circ\mu)$ locally. It can then be shown that these local pieces glue together into a globally defined d-closed semipositive (1, 1)-current $\mu^\star T$ and that the operation $\mu^\star$ is continuous w.r.t. the weak topology of currents.

Going back to (4.6), we see that $\widetilde{\alpha}_m \geq \varepsilon\mu_m^\star\omega \geq 0$. So, in particular, $\widetilde{\alpha}_m$ is a smooth d-closed semi-positive $(1,\,1)$-form on $\widetilde{X}_m$. It remains to correct $\widetilde{\alpha}_m$ to a *strictly positive* d-closed form (i.e. to a Kähler metric) on $\widetilde{X}_m$.

To this end, fix an m so large that T_m is a Kähler current on X. Let $\widetilde{X} := \widetilde{X}_m$, $\mu := \mu_m$, $\widetilde{\alpha} := \widetilde{\alpha}_m$ and $T' := T_m$. We may suppose that $\mu : \widetilde{X} \longrightarrow X$ is obtained as a composition of finitely many blow-ups:

$$\widetilde{X} = X_N \stackrel{\mu_{N-1}}{\longrightarrow} X_{N-1} \stackrel{\mu_{N-2}}{\longrightarrow} \cdots \longrightarrow X_1 \stackrel{\mu_0}{\longrightarrow} X_0 = X,$$

where each $\mu_j : X_{j+1} \longrightarrow X_j$ is the blow-up of X_j along a (smooth) submanifold $Y_j \subset X_j$. We denote the exceptional divisor of each μ_j by $E_{j+1} \subset X_{j+1}$.

It is standard (see e.g. [Dem97, VII, §.12.4.]) that, for every j, $\mathcal{O}(-E_{j+1})_{|E_{j+1}} \simeq \mathcal{O}_{P(N_j)}(1)$, where $P(N_j)$ is the projectivised normal bundle of Y_j in X_j. Pick an arbitrary C^∞ Hermitian metric on each N_j, consider the induced Fubini-Study metric on $\mathcal{O}_{P(N_j)}(1)$ and then extend the latter as a C^∞ Hermitian fibre metric on the line bundle $\mathcal{O}(-E_{j+1}) \longrightarrow X_{j+1}$. The curvature of the last fibre metric is *positive* on those tangent vectors to X_{j+1} that are also tangent to the fibres of the line bundle $E_{j+1} \longrightarrow X_{j+1}$ (i.e. transversal to the divisor $E_{j+1} \subset X_{j+1}$).

Starting from the Kähler current T' with analytic singularities on X, we can construct by induction on j, by means of a resolution of the singularities of the Kähler current involved that produces at every inductive step a decomposition of the shape (4.6), a Kähler current T_{j+1} on X_{j+1} with the following property. Once the Kähler current T_j has been constructed on X_j, we notice that there exists a constant $\varepsilon_{j+1} > 0$ and a C^∞ $(1,\,1)$-form u_{j+1} on X_{j+1} lying in the Bott-Chern cohomology class of the current of integration $[E_{j+1}]$ on the divisor E_{j+1} such that

$$T_{j+1} = \mu_j^\star T_j - \varepsilon_{j+1}u_{j+1}$$

is a Kähler current on X_{j+1}. This is possible thanks to the positivity property of $\mathcal{O}(-E_{j+1})_{|E_{j+1}} \longrightarrow X_{j+1}$.

So, essentially, we correct the semi-positive form $\widetilde{\alpha}$ by exploiting the positivity of the line bundles $\mathcal{O}(-E_{j+1}) \longrightarrow X_{j+1}$ in the directions transversal to the exceptional divisors E_{j+1}. Specifically, if for every j we let $\tilde{u}_j$ be the pullback of u_j to the final blown-up manifold $\widetilde{X}$, we get that the d-closed C^∞ $(1,\,1)$-form

$$\widetilde{\omega} := \widetilde{\alpha} - \sum_j \varepsilon_j\tilde{u}_j$$

is positive definite, hence a Kähler metric, on $\widetilde{X}$. □

The natural generalisation of the algebraic situation described just after Definition 4.1.22 to possibly non-Kähler compact complex manifolds and possibly non-integral cohomology classes is the following

Definition 4.1.24. *([Bou02]) Let X be a compact complex manifold. A cohomology class $\mathfrak{c} \in H^{1,1}_{BC}(X,\,\mathbb{R})$ is said to be* **big** *if it can be represented by a Kähler current.*

In a slightly non-standard way, we propose the following

Definition 4.1.25. *Let X be a compact complex manifold. The* **big cone** *of X is the set*

$$\mathcal{BIG}_X := \bigg\{[T]_{BC} \in H^{1,1}_{BC}(X,\,\mathbb{R})\,/\,T \;\; \textit{is a Kähler current on} \;\; X\bigg\}.$$

In other words, $\mathcal{BIG}_X$ is the set of all big classes on X. Identity (4.7) below accounts for the fact that many authors call the interior $\mathring{\mathcal{E}}_X$ of the pseudo-effective cone $\mathcal{E}_X$ the big cone of X when X is Kähler.

Proposition 4.1.26. *The big cone $\mathcal{BIG}_X$ of any compact complex manifold X is an* **open convex cone** *in $H^{1,1}_{BC}(X,\,\mathbb{R})$. Moreover,*

$$\mathcal{K}_X \subset \mathcal{BIG}_X \subset \mathcal{E}_X.$$

If X is a **class C manifold***, the big cone of X is the interior of its pseudo-effective cone:*

$$\mathcal{BIG}_X = \mathring{\mathcal{E}}_X. \tag{4.7}$$

Proof. Any convex combination of Kähler currents is a Kähler current, so $\mathcal{BIG}_X$ is a convex cone.

To show that $\mathcal{BIG}_X$ is open in $H^{1,1}_{BC}(X,\,\mathbb{R})$, fix a class $[T]_{BC} \in \mathcal{BIG}_X$ and represent it by a Kähler current $T \geq \varepsilon\omega$, where ω is an arbitrary Hermitian metric on X and $\varepsilon > 0$ is a constant. Then, for any d-closed form $\alpha \in C^\infty_{1,1}(X,\,\mathbb{R})$ and for any constant $0 < \varepsilon_0 < \varepsilon$, we get

$$T + \delta\alpha \geq \varepsilon\omega + \delta\alpha \geq \varepsilon_0\omega$$

whenever $\delta > 0$ is small enough. Thus, $T + \delta\alpha$ is a Kähler current, hence $[T]_{BC} + \delta[\alpha]_{BC} \in \mathcal{BIG}_X$, for all $0 < \delta \ll 1$. We conclude that $\mathcal{BIG}_X$ is open in $H^{1,1}_{BC}(X,\,\mathbb{R})$.

The inclusion $\mathcal{K}_X \subset \mathcal{BIG}_X$ holds because any Kähler metric is a Kähler current, while the inclusion $\mathcal{BIG}_X \subset \mathcal{E}_X$ holds because any Kähler current is a semi-positive current. Moreover, the latter inclusion and the openness of $\mathcal{BIG}_X$ imply that $\mathcal{BIG}_X \subset \mathring{\mathcal{E}}_X$.

Now, suppose that X is of *class $\mathcal{C}$*. By Theorem 4.1.23, a Kähler current $R \geq \delta\omega$ exists on X, where ω is a Hermitian metric and $\delta > 0$ is a constant. To prove the inclusion $\mathring{\mathcal{E}}_X \subset \mathcal{BIG}_X$, fix a class $[T]_{BC} \in \mathring{\mathcal{E}}_X$ and represent it by a semi-positive current $T \geq 0$. Since $[T]_{BC}$ is an interior point of $\mathcal{E}_X$ and $[R]_{BC} \in H^{1,1}_{BC}(X,\,\mathbb{R})$, $[T]_{BC} - \varepsilon[R]_{BC} \in \mathcal{E}_X$ for every $\varepsilon > 0$ small enough. This means that $T - \varepsilon R$ is Bott-Chern cohomologous to a semi-positive current $S \geq 0$. Equivalently, there exists a distribution ψ on X such that

$$T + i\partial\bar\partial\psi = S + \varepsilon R \geq \varepsilon\delta\omega.$$

Thus, $T + i\partial\bar\partial\psi$ is a Kähler current lying in the Bott-Chern class of T. Hence, $[T]_{BC} \in \mathcal{BIG}_X$.

This proves identity (4.7) when X is of *class $\mathcal{C}$*. $\square$

An immediate consequence of Theorem 4.1.23 is the following

Corollary 4.1.27. *A compact complex manifold X is a* **class $\mathcal{C}$ manifold** *if and only if its big cone $\mathcal{BIG}_X$ is* **non-empty***.*

We will see later that there exist (necessarily *non-class $\mathcal{C}$*) compact complex manifolds X such that $\mathcal{BIG}_X = \emptyset$ but $\mathring{\mathcal{E}}_X \neq \emptyset$. This is why we defined the big cone as in Definition 4.1.25 for arbitrary compact complex manifolds X, rather than as $\mathring{\mathcal{E}}_X$ as many authors do in the Kähler case (a case in which the two definitions coincide thanks to (4.7)).

On the other hand, let us also notice that the big cone $\mathcal{BIG}_X$ of a compact complex manifold X never contains the zero class. Since $\mathcal{BIG}_X$ is an open convex cone in $H^{1,1}_{BC}(X,\,\mathbb{R})$, this is equivalent to the inclusion $\mathcal{BIG}_X \subset H^{1,1}_{BC}(X,\,\mathbb{R})$ being always strict. We will see in §4.5.6 that, in stark contrast with the big cone, the Gauduchon cone $\mathcal{G}_X$ may equal the whole vector space $H^{n-1,\,n-1}_A(X,\,\mathbb{R})$ in which it is contained.

Corollary 4.1.28. *Let X be a compact complex manifold X. No Kähler current T (if any) on X can be either d-exact or $\partial\bar\partial$-exact.*

In particular, if the second Betti number b_2 of X vanishes, X is not a **class $\mathcal{C}$ manifold**.

Proof. If $T = i\partial\bar\partial\psi \geq 0$ on X for some distribution ψ, the compactness of X and the maximum principle imply that ψ must be a constant function on each connected component of X, hence $T = 0$ on X. This would contradict the strict positivity of the Kähler current T.

If $T = dS \geq 0$ on X for some 1-current S, then Proposition 4.2.5 in the next section and the fact that $T \neq 0$ (which follows from T being a Kähler current) imply that X is *not* a *strongly Gauduchon manifold*. (See terminology in the next section.)

On the other hand, the existence of the Kähler current T on X implies that X is a *class $\mathcal{C}$ manifold*. (See Theorem 4.1.23.) Finally, Corollary 4.2.12 in the next section implies that X is then also a *strongly Gauduchon manifold*, contradicting the previous conclusion.

As for the last statement, if X is a *class $\mathcal{C}$* manifold, there exists a Kähler current T on X by the easy implication in Theorem 4.1.23. However, if $b_2(X) = 0$, T must be d-exact, contradicting the first statement. $\square$

4.1.5 Qualitative part of Demailly's Transcendental Morse Inequalities Conjecture for a difference of two nef classes

We now present an application of the duality between the pseudo-effective cone and the closure of the Gauduchon cone established in Theorem 4.1.18.

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. We make the following assumption:

(H) *there exists a Hermitian metric ω on X such that*

$$\partial\bar\partial\omega^k = 0 \text{ for all } k = 1, 2, \dots, n-1.$$

It is clear that (H) holds if X is a Kähler manifold. It is also standard (see e.g. [GL10]) and easy to check that condition (H) is equivalent to either of the following two equivalent conditions:

$$\partial\bar\partial\omega = 0 \text{ and } \partial\bar\partial\omega^2 = 0 \iff \partial\bar\partial\omega = 0 \text{ and } \partial\omega \wedge \bar\partial\omega = 0.$$

Following the method of Chiose [Chi13] and Xiao [Xia15], itself inspired by earlier authors, we prove the following statement.

Theorem 4.1.29. *([Pop15b, Theorem 1.1]) Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ satisfying the assumption (H). Then, for any* **nef** *Bott-Chern cohomology classes $\{\alpha\}$, $\{\beta\} \in H^{1,1}_{BC}(X, \mathbb{R})$, the following implication holds:*

$$\{\alpha\}^n - n\,\{\alpha\}^{n-1}.\,\{\beta\} > 0 \Longrightarrow \text{the class } \{\alpha - \beta\} \text{ contains a } \textbf{Kähler current}.$$

In other words, the difference of two *nef* cohomology classes for which a certain intersection number is *positive* is a *big* class. (See Definition 4.1.24.)

Terminology and preliminary comments on Theorem 4.1.29.

(a) For the sake of convenience, Bott-Chern cohomology classes in the statement and the proof of Theorem 4.1.29 are denoted by $\{\ \}$ rather than $[\ ]_{BC}$. The *intersection numbers* featuring in the statement are defined as

$$\{\alpha\}^n = \int\limits_X \alpha^n \quad \text{and} \quad \{\alpha\}^{n-1}.\{\beta\} = \int\limits_X \alpha^{n-1}\wedge\beta.$$

They are independent of the choices of representatives α and β of the classes $\{\alpha\}$, $\{\beta\}\in H^{1,1}_{BC}(X,\,\mathbb{R})$.

(b) Theorem 4.1.29 answers affirmatively the *qualitative* part of a special version (i.e. the one for a difference of two nef classes) of Demailly's *transcendental Morse inequalities conjecture* (see [BDPP13, Conjecture 10.1, (*ii*)]) and will be crucial to the eventual extension of the duality theorem proved in [BDPP13, Theorem 2.2] to transcendental classes in the fairly general context of compact Kähler (not necessarily projective) manifolds. Although the method we propose here also produces a lower bound for the volume of the difference class $\{\alpha-\beta\}$, this bound (that we will not present here) is weaker than the lower bound $\{\alpha\}^n - n\,\{\alpha\}^{n-1}.\{\beta\}$ predicted in the *quantitative* part of Conjecture 10.1, (*ii*) in [BDPP13].

Xiao proves in [Xia15] the existence of a Kähler current in the class $\{\alpha-\beta\}$ under the stronger assumption $\{\alpha\}^n - 4n\,\{\alpha\}^{n-1}.\{\beta\}>0$ and the same assumption (H) on X. The two ingredients he uses are Lamari's duality lemma 4.1.19 and the following

Theorem 4.1.30. *(the Tosatti-Weinkove resolution of Hermitian Monge-Ampère equations, [TW10]) Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ and let ω be a Hermitian metric on X.*

Then, for any C^∞ function $F : X\to\mathbb{R}$, there exist a unique constant $C>0$ and a unique C^∞ function $\varphi : X\to\mathbb{R}$ such that

$$(\omega+i\partial\bar\partial\varphi)^n = Ce^F\omega^n, \quad \omega+i\partial\bar\partial\varphi>0 \quad \textit{and} \quad \sup_X\varphi=0.$$

As a matter of fact, Yau's classical theorem that solved the Calabi Conjecture, of which Theorem 4.1.30 is a generalisation to the possibly non-Kähler context, suffices for the proof of Theorem 4.1.29 whose assumptions imply that X must be Kähler (as already pointed out by Xiao in his situation based on [Chi13, Theorem 0.2]) although this is not used either here or in Xiao's work.

Proof of Theorem 4.1.29

We first reproduce Xiao's arguments (themselves inspired by earlier authors such as Chiose) up to the point where we will branch off in a different direction to handle certain estimates.

(I) Xiao's approach in [Xia15]

Let us fix a Hermitian metric ω on X such that $\partial\bar\partial\omega^k=0$ for all k. We also fix nef Bott-Chern $(1,1)$-classes $\{\alpha\},\{\beta\}$. By the nef assumption, for every $\varepsilon>0$, there exist C^∞ functions $\varphi_\varepsilon,\psi_\varepsilon : X\to\mathbb{R}$ such that

$$\alpha_\varepsilon := \alpha+\varepsilon\,\omega+i\partial\bar\partial\varphi_\varepsilon>0 \quad \text{and} \quad \beta_\varepsilon := \beta+\varepsilon\,\omega+i\partial\bar\partial\psi_\varepsilon>0 \quad \text{on} \quad X. \tag{4.8}$$

Note that α_ε and β_ε need not be d-closed, but the property $\partial\bar\partial\omega^k=0$ yields

$$\partial\bar\partial\alpha_\varepsilon^k = \partial\bar\partial\beta_\varepsilon^k = 0 \quad \text{and} \quad \partial\bar\partial(\alpha+\varepsilon\,\omega)^k = \partial\bar\partial(\beta+\varepsilon\,\omega)^k=0 \tag{4.9}$$

for all $k=1,2,\dots,n-1$. We normalise $\sup_X\varphi_\varepsilon=\sup_X\psi_\varepsilon=0$ for every $\varepsilon>0$.

Let us fix $\varepsilon>0$. The existence of a Kähler current in the class $\{\alpha-\beta\}=\{\alpha_\varepsilon-\beta_\varepsilon\}$ is equivalent to

$$\exists\,\delta>0,\ \exists \text{ a distribution}\,\theta_\delta \text{ on } X \quad \text{such that} \quad \alpha_\varepsilon-\beta_\varepsilon+i\partial\bar\partial\theta_\delta\geq\delta\,\alpha_\varepsilon,$$

which, in view of Lamari's duality lemma 4.1.19, is equivalent to

$$\exists\delta>0 \quad \text{such that} \quad \int\limits_X(\alpha_\varepsilon-\beta_\varepsilon)\wedge\gamma^{n-1}\geq\delta\int\limits_X\alpha_\varepsilon\wedge\gamma^{n-1}$$

for every Gauduchon metric γ on X. This is, of course, equivalent to

$$\exists\delta>0 \quad \text{such that} \quad (1-\delta)\int\limits_X\alpha_\varepsilon\wedge\gamma^{n-1}\geq\int\limits_X\beta_\varepsilon\wedge\gamma^{n-1}$$

for every Gauduchon metric γ on X.

Xiao's approach is to prove the existence of a Kähler current in the class $\{\alpha-\beta\}=\{\alpha_\varepsilon-\beta_\varepsilon\}$ by contradiction. Suppose that no such current exists. Then, for every $\varepsilon>0$ and every sequence of positive reals $\delta_m\downarrow 0$, there exist Gauduchon metrics $\gamma_{m,\,\varepsilon}$ on X such that

$$(1-\delta_m)\int\limits_X\alpha_\varepsilon\wedge\gamma_{m,\,\varepsilon}^{n-1}<\int\limits_X\beta_\varepsilon\wedge\gamma_{m,\,\varepsilon}^{n-1}=1 \qquad \text{for all } m\in\mathbb{N}^\star,\ \varepsilon>0. \tag{4.10}$$

The last identity is a normalisation of the Gauduchon metrics $\gamma_{m,\,\varepsilon}$ which is clearly always possible by rescaling $\gamma_{m,\,\varepsilon}$ by a positive factor. This normalisation implies that for every $\varepsilon>0$, the positive definite $(n-1,\,n-1)$-forms $(\gamma_{m,\,\varepsilon}^{n-1})_m$ are uniformly bounded in mass, hence after possibly extracting a subsequence we can assume the convergence $\gamma_{m,\,\varepsilon}^{n-1}\to\Gamma_{\infty,\,\varepsilon}$ in the weak topology of currents as $m\to+\infty$, where $\Gamma_{\infty,\,\varepsilon}\geq 0$ is an $(n-1,\,n-1)$-current on X. Taking limits as $m\to+\infty$ in (4.10), we get

$$\int\limits_X\alpha_\varepsilon\wedge\Gamma_{\infty,\,\varepsilon}\leq 1 \qquad \text{for all } \varepsilon>0. \tag{4.11}$$

Note that the l.h.s. of (4.10) does not change if α_ε is replaced with any $\alpha_\varepsilon+i\partial\bar\partial u$ (thanks to $\gamma_{m,\,\varepsilon}$ being Gauduchon), while $\alpha_\varepsilon\wedge\gamma_{m,\,\varepsilon}^{n-1}$ is (after division by $\gamma_{m,\,\varepsilon}^n$) the trace of α_ε w.r.t. $\gamma_{m,\,\varepsilon}$ divided by n (i.e. the arithmetic mean of the eigenvalues). To find a lower bound for the trace that would contradict (4.10), it is natural to prescribe the volume form (i.e. the product of the eigenvalues) of some $\alpha_\varepsilon+i\partial\bar\partial u_{m,\,\varepsilon}$ by imposing that it be, up to a constant factor, the strictly positive $(n,\,n)$-form featuring in the r.h.s. of (4.10). More precisely, the Tosatti-Weinkove theorem 4.1.30 allows us to solve the Monge-Ampère equation

$$(\star)_{m,\,\varepsilon} \qquad\qquad (\alpha_\varepsilon+i\partial\bar\partial u_{m,\,\varepsilon})^n=c_\varepsilon\,\beta_\varepsilon\wedge\gamma_{m,\,\varepsilon}^{n-1}$$

for any $\varepsilon>0$ and any $m\in\mathbb{N}^\star$ by ensuring the existence of a unique constant $c_\varepsilon>0$ and of a unique C^∞ function $u_{m,\,\varepsilon}\,:\,X\to\mathbb{R}$ satisfying $(\star)_{m,\,\varepsilon}$ so that

$$\widetilde\alpha_{m,\,\varepsilon}:=\alpha_\varepsilon+i\partial\bar\partial u_{m,\,\varepsilon}>0, \quad \sup_X(\varphi_\varepsilon+u_{m,\,\varepsilon})=0.$$

Note that c_ε is independent of m since we must have

$$c_\varepsilon = \int\limits_X \widetilde{\alpha}^n_{m,\,\varepsilon} = \int\limits_X (\alpha + \varepsilon\omega)^n \downarrow \int\limits_X \alpha^n := c_0 > 0, \tag{4.12}$$

where the non-increasing convergence is relative to $\varepsilon \downarrow 0$. Indeed, the second identity in (4.12) follows from $\partial\bar\partial(\alpha + \varepsilon\omega)^k = 0$ for all $k = 1, 2, \dots, n-1$ (cf. (4.9)). Thus, it is significant that c_ε does not change if we add any $i\partial\bar\partial u$ to α, i.e. c_ε depends only on the Bott-Chern class $\{\alpha\}$, on ω and on ε. Analogously, one defines

$$M_\varepsilon := \int\limits_X \widetilde{\alpha}^{n-1}_{m,\,\varepsilon} \wedge \beta_\varepsilon = \int\limits_X (\alpha + \varepsilon\omega)^{n-1} \wedge (\beta + \varepsilon\omega) \downarrow \int\limits_X \alpha^{n-1} \wedge \beta := M_0 \geq 0, \tag{4.13}$$

where the non-increasing convergence is relative to $\varepsilon \downarrow 0$. Clearly, M_ε is independent of m and depends only on the Bott-Chern classes $\{\alpha\}$, $\{\beta\}$, on ω and on ε. Note that the second integral in (4.13) equals $\int_X (\alpha + \varepsilon\omega + i\partial\bar\partial\varphi_\varepsilon)^{n-1} \wedge (\beta + \varepsilon\omega + i\partial\bar\partial\psi_\varepsilon)$ which is positive since $\alpha_\varepsilon, \beta_\varepsilon > 0$ by (4.8). Since $M_0 \geq 0$, the hypothesis $c_0 - nM_0 > 0$ made in Theorem 4.1.29 implies $c_0 > 0$. This justifies the final claim in (4.12).

(II) The arguments introduced in [Pop15b]

The handling of the estimates in the Monge-Ampère equation in [Pop15b], that we now proceed to present, was different from Xiao's in [Xia15]. The starting point is the following simple, elementary observation.

Lemma 4.1.31. *([Pop15b, Lemma 3.1]) For any Hermitian metrics α, β, γ on a complex manifold, the following inequality holds at every point:*

$$(\Lambda_\alpha\beta) \cdot (\Lambda_\beta\gamma) \geq \Lambda_\alpha\gamma. \tag{4.14}$$

Proof. Since (4.14) is a pointwise inequality, we fix an arbitrary point x and choose local coordinates about x such that

$$\beta(x) = \sum_j idz_j \wedge d\bar z_j, \quad \alpha(x) = \sum_j \alpha_j\, idz_j \wedge d\bar z_j \quad \text{and} \quad \gamma(x) = \sum_{j,\,k} \gamma_{j\bar k}\, idz_j \wedge d\bar z_k.$$

Then $\alpha_j > 0$ and $\gamma_{j\bar j} > 0$ for every j. If we denote by the same symbol any (1, 1)-form and its coefficient matrix in the chosen coordinates, we have

$$\alpha^{-1}\gamma = (\tfrac{1}{\alpha_j}\gamma_{j\bar k})_{j,\,k}, \text{ hence } \mathrm{Tr}(\alpha^{-1}\gamma) = \sum_j \tfrac{1}{\alpha_j}\gamma_{j\bar j}.$$

Thus (4.14) translates to $(\sum\limits_j \frac{1}{\alpha_j}) \sum\limits_k \gamma_{k\bar k} \geq \sum\limits_j \frac{1}{\alpha_j}\gamma_{j\bar j}$ which clearly holds since $\sum\limits_{j\neq k} \frac{1}{\alpha_j}\gamma_{k\bar k} > 0$ because all the α_j and all the $\gamma_{k\bar k}$ are positive. □

The main observation in [Pop15b] was the following statement.

Lemma 4.1.32. *([Pop15b, Lemma 3.2]) For every $m \in \mathbb{N}^\star$ and every $\varepsilon > 0$, we have:*

$$\left(\int\limits_X \widetilde{\alpha}_{m,\,\varepsilon} \wedge \gamma^{n-1}_{m,\,\varepsilon}\right) \cdot \left(\int\limits_X \widetilde{\alpha}^{n-1}_{m,\,\varepsilon} \wedge \beta_\varepsilon\right) \geq \frac{1}{n} \int\limits_X \widetilde{\alpha}^n_{m,\,\varepsilon} = \frac{c_\varepsilon}{n}. \tag{4.15}$$

Proof. Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, resp. $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, be the eigenvalues of $\widetilde{\alpha}_{m,\varepsilon}$, resp. β_ε, w.r.t. $\gamma_{m,\varepsilon}$. We have:

$$\widetilde{\alpha}^n_{m,\varepsilon} = \lambda_1 \dots \lambda_n \, \gamma^n_{m,\varepsilon} \;\text{ and }\; \widetilde{\alpha}_{m,\varepsilon} \wedge \gamma^{n-1}_{m,\varepsilon} = \frac{1}{n}\,(\Lambda_{\gamma_{m,\varepsilon}}\widetilde{\alpha}_{m,\varepsilon})\,\gamma^n_{m,\varepsilon} = \frac{\lambda_1 + \cdots + \lambda_n}{n}\,\gamma^n_{m,\varepsilon}.$$

Similarly, $\beta_\varepsilon \wedge \gamma^{n-1}_{m,\varepsilon} = \frac{1}{n}\,(\Lambda_{\gamma_{m,\varepsilon}}\beta_\varepsilon)\,\gamma^n_{m,\varepsilon} = \frac{\mu_1 + \cdots + \mu_n}{n}\,\gamma^n_{m,\varepsilon}$.

Thus, the Monge-Ampère equation $(\star)_{m,\varepsilon}$ translates to

$$\lambda_1 \dots \lambda_n = c_\varepsilon \, \frac{\mu_1 + \cdots + \mu_n}{n}. \tag{4.16}$$

In particular, the normalisation $\int_X \beta_\varepsilon \wedge \gamma^{n-1}_{m,\varepsilon} = 1$ reads

$$\frac{1}{c_\varepsilon} \int\limits_X \lambda_1 \dots \lambda_n \, \gamma^n_{m,\varepsilon} = \int\limits_X \frac{\mu_1 + \cdots + \mu_n}{n}\,\gamma^n_{m,\varepsilon} = 1. \tag{4.17}$$

Note that we also have

$$\widetilde{\alpha}^{n-1}_{m,\varepsilon} \wedge \beta_\varepsilon = \frac{1}{n}\,(\Lambda_{\widetilde{\alpha}_{m,\varepsilon}}\beta_\varepsilon)\,\widetilde{\alpha}^n_{m,\varepsilon} = \frac{1}{n}\,(\Lambda_{\widetilde{\alpha}_{m,\varepsilon}}\beta_\varepsilon)\,\lambda_1 \dots \lambda_n \, \gamma^n_{m,\varepsilon}. \tag{4.18}$$

Putting all of the above together, we get:

$$\begin{aligned}
& \left(\int\limits_X \widetilde{\alpha}_{m,\varepsilon} \wedge \gamma^{n-1}_{m,\varepsilon}\right) \cdot \left(\int\limits_X \widetilde{\alpha}^{n-1}_{m,\varepsilon} \wedge \beta_\varepsilon\right) \\
= \; & \left(\int\limits_X \frac{1}{n}\,(\Lambda_{\gamma_{m,\varepsilon}}\widetilde{\alpha}_{m,\varepsilon})\,\gamma^n_{m,\varepsilon}\right) \cdot \left(\int\limits_X \frac{1}{n}\,(\Lambda_{\widetilde{\alpha}_{m,\varepsilon}}\beta_\varepsilon)\,\lambda_1 \dots \lambda_n \, \gamma^n_{m,\varepsilon}\right) \\
\overset{(a)}{\geq} \; & \frac{1}{n^2}\left(\int\limits_X \left[(\Lambda_{\gamma_{m,\varepsilon}}\widetilde{\alpha}_{m,\varepsilon})\,(\Lambda_{\widetilde{\alpha}_{m,\varepsilon}}\beta_\varepsilon)\right]^{\frac{1}{2}} (\lambda_1 \dots \lambda_n)^{\frac{1}{2}}\,\gamma^n_{m,\varepsilon}\right)^2 \\
\overset{(b)}{\geq} \; & \frac{1}{n^2}\left(\int\limits_X (\Lambda_{\gamma_{m,\varepsilon}}\beta_\varepsilon)^{\frac{1}{2}}\,(\lambda_1 \dots \lambda_n)^{\frac{1}{2}}\,\gamma^n_{m,\varepsilon}\right)^2 \overset{(c)}{=} \frac{1}{n^2}\left(\int\limits_X \frac{\sqrt{n}}{\sqrt{c_\varepsilon}}\,\lambda_1 \dots \lambda_n \, \gamma^n_{m,\varepsilon}\right)^2 \\
= \; & \frac{1}{n\,c_\varepsilon}\left(\int\limits_X \widetilde{\alpha}^n_{m,\varepsilon}\right)^2 \overset{(d)}{=} \frac{1}{n\,c_\varepsilon}\left(\int\limits_X c_\varepsilon\,\beta_\varepsilon \wedge \gamma^{n-1}_{m,\varepsilon}\right)^2 \overset{(e)}{=} \frac{c_\varepsilon}{n}.
\end{aligned}$$

This proves (4.15). Inequality (a) is an application of the Cauchy-Schwarz inequality, inequality (b) has followed from (4.14), identity (c) has followed from (4.16), identity (d) has followed from $\widetilde{\alpha}^n_{m,\varepsilon} = c_\varepsilon\,\beta_\varepsilon \wedge \gamma^{n-1}_{m,\varepsilon}$ (which is nothing but the Monge-Ampère equation $(\star)_{m,\varepsilon}$), while identity (e) has followed from the normalisation $\int_X \beta_\varepsilon \wedge \gamma^{n-1}_{m,\varepsilon} = 1$ (cf. (4.10)). The proof of Lemma 4.1.32 is complete. $\square$

End of proof of Theorem 4.1.29. Now, $\widetilde{\alpha}_{m,\varepsilon} = \alpha_\varepsilon + i\partial\bar\partial u_{m,\varepsilon}$ and $\partial\bar\partial\gamma^{n-1}_{m,\varepsilon} = 0$, so

$$\int\limits_X \widetilde{\alpha}_{m,\varepsilon} \wedge \gamma^{n-1}_{m,\varepsilon} = \int\limits_X \alpha_\varepsilon \wedge \gamma^{n-1}_{m,\varepsilon} \longrightarrow \int\limits_X \alpha_\varepsilon \wedge \Gamma_{\infty,\varepsilon} \leq 1 \quad \text{for all } \varepsilon > 0, \tag{4.19}$$

where the above arrow stands for convergence as $m \to +\infty$ and the last inequality is nothing but (4.11) (which, recall, is a consequence of the assumption that no Kähler current exists in $\{\alpha - \beta\}$ — an assumption that we are going to contradict). On the other hand, the second factor on the l.h.s. of (4.15) is precisely M_ε defined in (4.13), so in particular it is independent of m. Fixing any $\varepsilon > 0$, taking limits as $m \to +\infty$ in (4.15) and using (4.19), we get

$$M_\varepsilon \geq \frac{c_\varepsilon}{n} \quad \text{for every } \varepsilon > 0. \tag{4.20}$$

Taking now limits as $\varepsilon \downarrow 0$ and using (4.13) and (4.12), we get

$$M_0 \geq \frac{c_0}{n}, \quad \text{i.e.} \quad \{\alpha\}^{n-1}.\{\beta\} \geq \frac{\{\alpha\}^n}{n}.$$

The last identity means that $\{\alpha\}^n - n\,\{\alpha\}^{n-1}.\{\beta\} \leq 0$ which is impossible if we suppose that $\{\alpha\}^n - n\,\{\alpha\}^{n-1}.\{\beta\} > 0$. This is the desired contradiction proving the existence of a Kähler current in the class $\{\alpha - \beta\}$ under the assumption $\{\alpha\}^n - n\,\{\alpha\}^{n-1}.\{\beta\} > 0$. □

4.1.6 The Demailly-Paun numerical characterisation of the Kähler cone of a compact Kähler manifold

As an application of Theorem 4.1.29, we now present a proof of an important result of Demailly and Paun characterising *Kähler classes* on a compact Kähler manifold. Since the proof of Theorem 4.1.29 based on a use of the Gauduchon cone and spelt out above in §.4.1.5 is far simpler than the proof of a special case of it that featured as Theorem 2.12 in [DP04] and constituted the main ingredient in the proof of the main result of [DP04], this approach will implicitly underscore the key role played by the Gauduchon cone even in the Kähler context. The historical order of events is, of course, different from the one presented here since Theorem 4.1.29 and the Gauduchon cone were unavailable at the time of the Demailly-Paun work [DP04].

As in §.4.1.5, Bott-Chern cohomology classes will be denoted by $\{\ \}$ rather than $[\ ]_{BC}$, while the subscript BC will be removed from $H^{1,1}(X,\mathbb{R})$ since X is assumed Kähler. As usual, by a *Kähler class* we mean a cohomology class in $H^{1,1}(X,\mathbb{R})$ that can be represented by a *Kähler metric*.

Theorem 4.1.33. *([DP04, Main Theorem 0.1 and Theorem 4.2]) Let X be a compact* **Kähler** *manifold with $\dim_{\mathbb{C}} X = n$.*

(a) The Kähler cone $\mathcal{K}_X$ of X is one of the **connected components** *of the set:*

$$\mathcal{P}_X := \left\{\{\alpha\} \in H^{1,1}(X,\mathbb{R}) \mid \int\limits_Y \alpha^p > 0 \;\; \textit{for all } p = 0,\dots,n \textit{ and all } Y \subset X \textit{ with } \dim_{\mathbb{C}} Y = p\right\},$$

where Y runs through the **irreducible analytic subsets** *of X.*

(b) Let $\{\omega\} \in H^{1,1}(X,\mathbb{R})$ be an arbitrary **Kähler class** *on X. For every $\{\alpha\} \in H^{1,1}(X,\mathbb{R})$ the following properties are equivalent:*

(i) the class $\{\alpha\}$ is **Kähler***;*

(ii) $\int\limits_Y (\alpha + t\omega)^p > 0$ for all $p \in \{0,\dots,n\}$, all irreducible analytic subsets $Y \subset X$ with $\dim_{\mathbb{C}} Y = p$ and all $t \geq 0$;

(iii) $\int\limits_Y \alpha^k \wedge \omega^{p-k} > 0$ for all $p \in \{0,\dots,n\}$, all $k \in \{1,\dots,p\}$ and all irreducible analytic subsets $Y \subset X$ with $\dim_{\mathbb{C}} Y = p$.

As already mentioned, Theorem 4.1.33 will follow easily from the next statement that is obtained by taking $\{\beta\} = 0$ in Theorem 4.1.29.

Theorem 4.1.34. *([DP04, Theorem 2.12]) Let (X, ω) be a compact* **Kähler** *manifold with $\dim_{\mathbb{C}} X = n$. Suppose that $\{\alpha\} \in H^{1,1}_{BC}(X, \mathbb{R})$ is a* **nef** *Bott-Chern cohomology class such that $\{\alpha\}^n > 0$. Then, $\{\alpha\}$ contains a* **Kähler current**.

Another ingredient needed in the proof of Theorem 4.1.33 will be the following result of Paun's. Recall that for a current T on a complex space X and for an arbitrary real $c > 0$, the Lelong upper-level set $E_c(T)$ consists of the points $x \in X$ at which the Lelong number $\nu(T, x)$ of T is $\geq c$. By an important theorem of Siu's from [Siu74], $E_c(T)$ is an *analytic* subset of X.

Proposition 4.1.35. *([Pau98]) Let X be a compact complex space and let $\{\alpha\} \in H^{1,1}_{BC}(X, \mathbb{R})$.*
If the class $\{\alpha\}$ contains a **Kähler current** *T and its restriction $\{\alpha\}_{|Y}$ is a* **Kähler** *class on Y for every constant $c > 0$ and every irreducible component Y of $E_c(T)$, then $\{\alpha\}$ is a* **Kähler** *class on X.*

Proof. See [Pau98]. □

Proof of Theorem 4.1.33. (a) It is obvious that $\mathcal{K}_X$ is an *open convex cone* and that $\mathcal{K}_X \subset \mathcal{P}_X$. It remains to prove that $\mathcal{K}_X$ is *closed* in $\mathcal{P}_X$, or equivalently that $\overline{\mathcal{K}}_X \cap \mathcal{P}_X = \mathcal{K}_X$.

The inclusion $\mathcal{K}_X \subset \overline{\mathcal{K}}_X \cap \mathcal{P}_X$ being obvious, it remains to prove the reverse inclusion. Let $\{\alpha\} \in \overline{\mathcal{K}}_X \cap \mathcal{P}_X$. This means that $\{\alpha\}$ is *nef* (see Proposition 4.1.21) and $\int_Y \alpha^p > 0$ for all $p \in \{0, \dots, n\}$ and all irreducible analytic subsets $Y \subset X$ with $\dim_{\mathbb{C}} Y = p$.

We now prove that, for every irreducible analytic subset $Y \subset X$, the restriction $\{\alpha\}_{|Y}$ is a Kähler class on Y. This is done by induction on $\dim_{\mathbb{C}} Y$. We will take $Y = X$ in the end to conclude that $\{\alpha\}$ is a Kähler class on X.

Fix an arbitrary such $Y \subset X$ with $\dim_{\mathbb{C}} Y = p$ and let $\mu : \widetilde{Y} \longrightarrow Y$ be a *desingularisation* of Y obtained as a finite sequence of blow-ups with smooth centres in Y. Then, $\{\alpha\}_{|Y}$ is a nef class on Y, hence $\mu^\star(\{\alpha\}_{|Y})$ is a nef class on $\widetilde{Y}$. Moreover, we have:

$$\Big(\mu^\star(\{\alpha\}_{|Y})\Big)^p = \int\limits_{\widetilde{Y}} (\mu^\star\alpha)^p = \int\limits_{Y} \alpha^p > 0,$$

the last inequality holding by assumption.

From Theorem 4.1.34 we get the existence of a Kähler current $\widetilde{T} \in \mu^\star(\{\alpha\}_{|Y}$ on $\widetilde{Y}$. Then, $T := \mu_\star\widetilde{T}$ is a Kähler current on Y and $T \in \{\alpha\}_{|Y} = \mu_\star\mu^\star\{\alpha\}_{|Y}$.

On the other hand, the induction hypothesis ensures that the class $\{\alpha\}_{|Z}$ is *Kähler* for every constant $c > 0$ and every irreducible component $Z \subset E_c(T)$ of the Lelong upper-level subset $E_c(T) \subset Y$ of T. Moreover, we necessarily have $\dim E_c(T) \leq p - 1$ for all $c > 0$, so the induction hypothesis applies.

Using Paun's Proposition 4.1.35, we infer that $\{\alpha\}_{|Y}$ is a Kähler class on Y. This completes the induction argument and the proof of (a).

(b) The implications $(i) \implies (iii) \implies (ii)$ are trivial. To prove the implication $(ii) \implies (i)$, start by noticing that, if (ii) holds, the half-line

$$[0, +\infty) \ni t \longmapsto \{\alpha + t\omega\} \in H^{1,1}(X, \mathbb{R})$$

is contained in $\mathcal{P}_X$, hence in one of the connected components of $\mathcal{P}_X$.

Meanwhile, $\alpha + t\omega > 0$, hence $\{\alpha + t\omega\}$ is a Kähler class, for all t large enough (since X is compact, so smooth forms on X are bounded). Since $\mathcal{K}_X$ is one of the connected components of $\mathcal{P}_X$ by (a), we infer that the half line $[0, +\infty) \ni t \longmapsto \{\alpha + t\omega\} \in \mathcal{P}_X$ must be contained in the connected component of $\mathcal{P}_X$ that is $\mathcal{K}_X$. □

4.1.7 Variation of the Kähler cone under deformations of compact Kähler manifolds

The material in this subsection is taken from [DP04, §.5] where Demailly and Paun apply their main result Theorem 4.1.33 to obtain the following

Theorem 4.1.36. *([DP04, Theorem 0.9]) Let $\pi : \mathcal{X} \longrightarrow B$ be a proper analytic map between reduced complex spaces such that B is irreducible. Suppose that π is a locally C^∞ trivial fibration whose fibres $X_t := \pi^{-1}(t)$ are (smooth)* **Kähler** *manifolds for all $t \in B$.*

Then, there exists a countable union $B' = \cup_{\nu\in\mathbb{Z}}\Sigma_\nu$ of **analytic** *subsets $\Sigma_\nu \subsetneq B$ such that the Kähler cones $\mathcal{K}_t := \mathcal{K}_{X_t} \subset H^{1,1}(X_t, \mathbb{R})$ are* **invariant** *over $B \setminus B'$ under parallel transport with respect to the $(1, 1)$-projection $\nabla^{1,1}$ of the Gauss-Manin connection ∇.*

The base B of the family in Theorem 4.1.36 is not assumed to be smooth. Since π is locally C^∞ trivial, for every k the C^∞ vector bundle $B \ni t \longmapsto H^k(X_t, \mathbb{C})$ is locally constant and carries a natural flat connection ∇ known as the *Gauss-Manin connection*.

On the other hand, for every p, let $\mathcal{C}^p(\mathcal{X}/B) \subset \mathcal{C}^p(\mathcal{X})$ be the *relative Barlet space* of p-dimensional cycles contained in the fibres X_t. It is a subspace of the *Barlet cycle space* $\mathcal{C}^p(\mathcal{X})$ of all p-dimensional cycles in $\mathcal{X}$ and is equipped with a canonical holomorphic projection

$$\sigma_p : \mathcal{C}^p(\mathcal{X}/B) \longrightarrow B$$

mapping every relative cycle $Z \subset X_t$ to the point $t \in B$ above which the fibre that contains it lives.

Now, for every bidegree (p, q), the Dolbeault cohomology groups $H^{p,q}(X_t, \mathbb{C})$ of the fibres form a *real analytic* vector subbundle $B \ni t \longmapsto H^{p,q}(X_t, \mathbb{C})$ of the locally constant vector bundle $B \ni t \longmapsto H^{p+q}(X_t, \mathbb{C})$. Since X_t is Kähler, there is a canonical Hodge decomposition:

$$H^2(X_t, \mathbb{C}) \simeq H^{2,0}(X_t, \mathbb{C}) \oplus H^{1,1}(X_t, \mathbb{C}) \oplus H^{0,2}(X_t, \mathbb{C}), \qquad t \in B,$$

which, together with the Gauss-Manin connection ∇ of the vector bundle $B \ni t \longmapsto H^2(X_t, \mathbb{C})$, induces a real analytic, not necessarily flat, connection $\nabla^{p,q}$ on the vector subbundle $B \ni t \longmapsto H^{p,q}(X_t, \mathbb{C})$. It is this connection $\nabla^{1,1}$ obtained when $(p, q) = (1, 1)$ that is involved in Theorem 4.1.36.

Proof of Theorem 4.1.36. The statement being local over B, we can shrink B about an arbitrary pregiven point to ensure that the locally constant vector bundles $B \ni t \longmapsto H^k(X_t, \mathbb{C})$ are constant.

On the other hand, the fibres X_t being Kähler, it is standard that, for every p, the restrictions of the canonical holomorphic projection $\sigma_p : \mathcal{C}^p(\mathcal{X}/B) \longrightarrow B$ to the connected components of $\mathcal{C}^p(\mathcal{X}/B)$ are *proper* maps. We define the subsets $\Sigma_\nu \subsetneq B$ as the images in B under the maps σ_p of those connected components of the relative cycle spaces $\mathcal{C}^p(\mathcal{X}/B)$ that do not surject onto B. Since the σ_p's are proper on each such connected component, each Σ_ν is an *analytic* subset of B. Moreover, from the definition of the Σ_ν's we infer that the cohomology classes $\{[Z]\}$ of the (currents of integration over the) analytic cycles $Z \subset X_t$ remain *constant* as t varies in $B \setminus B'$ and Z varies in a given connected component of the relative cycle space $\mathcal{C}^p(\mathcal{X}/B)$.

Moreover, $B\setminus B'$ is arcwise connected by piecewise smooth arcs since B is irreducible and B' is a countable union of proper analytic subsets of B. We fix such a smooth arc

$$\gamma : [0,\,1] \longrightarrow B\setminus B'$$

and we let $\alpha(u)\in H^{1,\,1}(X_{\gamma(u)},\,\mathbb{R})$, with $u\in[0,\,1]$, be a family of cohomology classes that is *constant* by parallel transport under $\nabla^{1,\,1}$. This constancy property amounts to

$$\nabla(\alpha(u))\in H^{2,\,0}(X_{\gamma(u)},\,\mathbb{C})\oplus H^{0,\,2}(X_{\gamma(u)},\,\mathbb{C}), \qquad u\in[0,\,1]. \tag{4.21}$$

• We first prove that the cones $\mathcal{P}_t := \mathcal{P}_{X_t}\subset H^{1,\,1}(X_t,\,\mathbb{R})$ are **invariant** over $B\setminus B'$ under parallel transport with respect to the $(1,\,1)$-projection $\nabla^{1,\,1}$ of the Gauss-Manin connection ∇.

To this end, let us suppose that $\alpha(0)\in\mathcal{P}_0 := \mathcal{P}_{X_0}$. This means that

$$\alpha(0)^p.\{[Z]\} := \int\limits_Z \alpha(0)^p > 0 \tag{4.22}$$

for all p and all p-dimensional analytic cycles Z in $X_{\gamma(0)}$. Let us fix such a cycle $Z\subset X_{\gamma(0)}$ and let us transport its cohomology class $\{[Z]\}$ along the arc γ such that it remains constant with respect to the Gauss-Manin connection ∇. We get a family $(\zeta_Z(\gamma(u)))_{u\in[0,\,1]}$ of cohomology classes $\zeta_Z(\gamma(u))\in H^{2q}(X_{\gamma(u)},\,\mathbb{Z})$, where $q=\dim_{\mathbb{C}}X_t-p$, whose member for $\gamma(0)$ is $\{[Z]\}$ and which satisfies the ∇-parallelism condition:

$$\nabla\zeta_Z(\gamma(u))=0, \qquad u\in[0,\,1]. \tag{4.23}$$

By the choices of γ and B', $\zeta_Z(\gamma(u))$ is of type $(q,\,q)$ for every $u\in[0,\,1]$ and the cohomology class of (the current of integration on) every analytic cycle in any fibre $X_{\gamma(u)}$ with $u\in[0,\,1]$ can be realised as a $\zeta_Z(\gamma(u))$ for some originally given $Z\subset X_{\gamma(0)}$ and some p.

Using (4.21) and (4.23), we get:

$$\frac{d}{du}\bigg(\alpha(u)^p.\zeta_Z(\gamma(u))\bigg) = p\,\alpha(u)^{p-1}\cdot\nabla\alpha(u)\cdot\zeta_Z(\gamma(u)) = 0,$$

where the last identity is the consequence of the bidegrees of the classes involved. Hence, the map

$$[0,\,1]\ni u\longmapsto \alpha(u)^p.\zeta_Z(\gamma(u))\in\mathbb{R}$$

is constant. Since its value at $u=0$ is positive (see (4.22)), we infer that $\alpha(u)^p.\zeta_Z(\gamma(u))>0$ for all $u\in[0,\,1]$ (and, of course, all p and all classes $\zeta_Z(\gamma(u))$ of analytic p-cycles in $X_{\gamma(u)}$).

By the definition of $\mathcal{P}_{\gamma(u)} := \mathcal{P}_{X_{\gamma(u)}}$, the last fact amounts to $\alpha(u)\in\mathcal{P}_{\gamma(u)}$ for all $u\in[0,\,1]$.

• To conclude the proof of Theorem 4.1.36, we apply part (b) of the Kodaira-Spencer Theorem 2.6.6. It ensures that, for every $t_0\in B$, every Kähler class on X_{t_0} can be deformed in a C^∞ way to Kähler classes on the nearby fibres X_t. Together with what was proved above about the variation of the cone $\mathcal{P}_t$, this implies that the connected component of $\mathcal{P}_t$ which constitutes the Kähler cone $\mathcal{K}_t$ (cf. (a) of Theorem 4.1.33) must remain constant as specified in the statement of Theorem 4.1.36. □

4.2 Strongly Gauduchon metrics and manifolds

The material in this section is taken from [Pop09a] and [Pop14]. The starting point is the following notion that was introduced for the purpose of controlling masses of metrics or currents and volumes of divisors in families of compact complex manifolds.

Definition 4.2.1. *([Pop09a, Definition 4.1]) Let X be a compact complex manifold, $\dim_{\mathbb{C}}X = n$.*

(i) A C^∞ positive definite $(1,1)$-form ω on X is said to be a **strongly Gauduchon (sG) metric** *if the $(n,n-1)$-form $\partial\omega^{n-1}$ is $\bar\partial$-exact on X.*

(ii) If X carries such a metric, X is said to be a **strongly Gauduchon (sG) manifold***.*

Notice that the Gauduchon condition only requires $\partial\omega^{n-1}$ to be $\bar\partial$-closed on X. Hence, every *strongly Gauduchon* metric is a Gauduchon metric. Conversely, we have

Proposition 4.2.2. *Let X be a compact* $\partial\bar\partial$**-manifold***. Every Gauduchon metric on X is strongly Gauduchon.*

Proof. Let $n := \dim_{\mathbb{C}}X$ and let ω be a Gauduchon metric on X. Then $\partial\omega^{n-1}$ is $\bar\partial$-closed and ∂-closed, hence also d-closed. Since $\partial\omega^{n-1}$ is both d-closed and ∂-exact, the $\partial\bar\partial$ hypothesis on X implies that $\partial\omega^{n-1}$ is also $\bar\partial$-exact. Hence, ω is a strongly Gauduchon metric. □

Thus, the Gauduchon and strongly Gauduchon notions coincide on $\partial\bar\partial$-manifolds. However, we will see that the strongly Gauduchon condition is strictly stronger than the Gauduchon condition in general. Furthermore, unlike Gauduchon metrics which exist on any compact complex manifold, sG metrics need not exist in general, as hinted in (ii) of Definition 4.2.1.

4.2.1 Intrinsic characterisations of strongly Gauduchon manifolds

We will give two necessary and sufficient conditions for the manifold X to carry an sG metric.

(I) The first intrinsic characterisation of sG manifolds that we give is the following

Lemma 4.2.3. *([Pop09a, Lemma 4.2]) Let X be a compact complex manifold of complex dimension n. Then, X carries an sG metric if and only if there exists a C^∞ $(2n-2)$-form Ω on X satisfying the following three conditions:*

(a) $\Omega = \overline{\Omega}$ (i.e. Ω is real);
(b) $d\Omega = 0$;
(c) $\Omega^{n-1,n-1} > 0$ on X (i.e. the component of type $(n-1,\, n-1)$ of Ω w.r.t. the complex structure of X is positive definite).

Note that conditions (a) and (b) are independent of the complex structure of X, while a change of complex structure changes the $(n-1,\, n-1)$-component of a given $(2n-2)$-form Ω. Thus condition (c) is the only one to be dependent on the complex structure of X.

Proof of Lemma 4.2.3. The vanishing of the $(2n-1)$-form $d\Omega$ (cf. (b)) amounts to the simultaneous vanishing of its components $\partial\Omega^{n-1,\,n-1} + \bar\partial\Omega^{n,\,n-2}$ (of type $(n,\, n-1)$) and $\partial\Omega^{n-2,\,n} + \bar\partial\Omega^{n-1,\,n-1}$ (of type $(n-1,\, n)$). These two components are conjugate to each other if Ω satisfies (a). Thus, if (a) holds, (b) is equivalent to $\partial\Omega^{n-1,\,n-1} + \bar\partial\Omega^{n,\,n-2} = 0$.

Suppose there exists an sG metric ω on X. Then, the $(n-1,\, n-1)$-form $\Omega^{n-1,\,n-1} := \omega^{n-1}$ is positive definite on X and there exists a C^∞ $(n,\, n-2)$-form $\Omega^{n,\,n-2}$ on X satisfying $\partial\Omega^{n-1,\,n-1} = -\bar\partial\Omega^{n,\,n-2}$. Considering the $(n-2,\, n)$-form $\Omega^{n-2,\,n} := \overline{\Omega^{n,\,n-2}}$, we see that the C^∞ $(2n-2)$-form

$$\Omega := \Omega^{n,\,n-2} + \Omega^{n-1,\,n-1} + \Omega^{n-2,\,n}$$

satisfies conditions (a), (b), (c).

Conversely, suppose there exists a C^∞ $(2n-2)$-form Ω on X satisfying conditions (a), (b), (c). By Lemma 4.0.1, the form $\Omega^{n-1,\,n-1} > 0$ has a unique $(n-1)^{st}$ root, namely there exists a unique C^∞ positive definite $(1,\,1)$-form $\omega > 0$ on X such that

$$\omega^{n-1} = \Omega^{n-1,\,n-1}.$$

By condition (b) satisfied by Ω, we see that $\partial\omega^{n-1}$ is $\bar\partial$-exact, which means that the Hermitian metric ω of X is *strongly Gauduchon*. □

An immediate consequence of Lemma 4.2.3 is that the **sG property** of compact complex manifolds is **open** under holomorphic deformations of the complex structure. (See Definition 2.6.1 for this last piece of terminology.)

Theorem 4.2.4. *([Pop10a, Conclusion 2.4.]) Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds. Fix a point $0 \in B$ and suppose that the fibre $X_0 := \pi^{-1}(0)$ is a* **strongly Gauduchon manifold**. *Then, $X_t := \pi^{-1}(t)$ is a* **strongly Gauduchon manifold** *for every $t \in B$ sufficiently close to 0.*

Proof. Let $n := \dim_{\mathbb{C}} X_t$ for $t \in B$. As usual, we denote by X the C^∞ manifold underlying the fibres X_t and by J_t the complex structure of X_t for all $t \in B$. If we have a C^∞ $(2n-2)$-form Ω on X, its components $\Omega_t^{n-1,\,n-1}$ of type $(n-1,\,n-1)$ w.r.t. the complex structures J_t vary in a C^∞ way with $t \in B$. Consequently, if $\Omega_0^{n-1,\,n-1} > 0$ then $\Omega_t^{n-1,\,n-1} > 0$ for $t \in B$ sufficiently close to $0 \in B$. Thus condition (c) of Lemma 4.2.3 is preserved under small deformations by mere continuity. Since conditions (a) and (b) of Lemma 4.2.3 are independent of the complex structure of X, it follows that any C^∞ $(2n-2)$-form Ω on X satisfying conditions (a), (b) and (c) of Lemma 4.2.3 w.r.t. J_0 also satisfies these conditions w.r.t. J_t for all t sufficiently near 0. The proof of Theorem 4.2.4 is complete. □

(II) The second intrinsic characterisation of sG manifolds that we give is the following

Proposition 4.2.5. *([Pop09a, Proposition 4.3.]) Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Then, X carries a* **strongly Gauduchon** *metric ω if and only if there is no non-zero current T of bidegree $(1,1)$ on X such that $T \geq 0$ and T is d-exact on X.*

Proof. We shall use Sullivan's technique of [Sul76] based on the elementary Hahn-Banach theorem, as we did to prove Theorem 4.1.2 and Lemma 4.1.3 and as had been done earlier in e.g. [HL83] and [Mic83].

We start by determining when a $(2n-2)$-form as in Lemma 4.2.3 exists. Let Ω be any real C^∞ form of degree $2n-2$ on X.

The condition $d\Omega = 0$ is equivalent, by the duality between d-closed smooth real $(2n-2)$-forms and real exact 2-currents $T = dS$ on X, to the property:

$$\int\limits_X \Omega \wedge dS = 0 \quad \text{for every real 1-current } S \text{ on } X. \tag{4.24}$$

On the other hand, the duality between strictly positive, smooth $(n-1,n-1)$-forms and non-zero positive $(1,1)$-currents on X shows that the condition $\Omega^{n-1,n-1} > 0$ is equivalent to the property:

$$\int_X \Omega^{n-1,n-1}\wedge T > 0 \quad \text{for every non-zero } (1,1)\text{-current } T\geq 0 \text{ on } X. \tag{4.25}$$

Now, if T is of type $(1,1)$, we clearly have $\int_X \Omega^{n-1,n-1}\wedge T = \int_X \Omega\wedge T$. Furthermore, real d-exact 2-currents $T = dS$ form a closed vector subspace $\mathcal{A}$ of the locally convex space $\mathcal{D}'_{\mathbb{R}}(X)$ of real 2-currents on X. Meanwhile, if we fix a smooth, strictly positive $(n-1,n-1)$-form Θ on X, positive non-zero $(1,1)$-currents T on X can be normalised such that $\int_X T\wedge\Theta = 1$ and it suffices to guarantee property (4.25) for normalised currents. Clearly, these normalised positive $(1,1)$-currents form a compact (in the locally convex topology of weak convergence of currents) convex subset $\mathcal{B}$ of the locally convex space $\mathcal{D}'_{\mathbb{R}}(X)$ of real 2-currents on X.

The *Hahn-Banach Separation Theorem* for locally convex spaces guarantees the existence of a linear functional vanishing identically on a given closed subset and assuming only positive values on a given compact subset if the two subsets are convex and do not intersect. Hence, in our case, there exists a real smooth $(2n-2)$-form Ω on X satisfying both conditions (4.24) and (4.25) if and only if $\mathcal{A}\cap\mathcal{B}=\emptyset$. This amounts to there existing no non-trivial exact $(1,1)$-current $T = dS$ such that $T\geq 0$ on X. $\square$

4.2.2 Examples of non-sG compact complex manifolds

In this subsection we provide two groups of such examples according to whether the complex dimension is 2 or ≥ 3.

(I) Surface examples

We show that no *non-Kähler* compact complex surface is strongly Gauduchon.

Theorem 4.2.6. *Let X be a compact complex surface. The following equivalence holds:*

$$X \text{ is Kähler} \iff X \text{ is strongly Gauduchon.}$$

Proof. The implication " $\implies$ " is obvious and holds in every dimension.

It is well-known that a compact complex surface is Kähler if and only if its first Betti number b_1 is even. (See Kodaira's classification of surfaces, Miyaoka's result [Miy74] asserting that an elliptic surface is Kähler if and only if its first Betti number is even and Siu's result [Siu83] asserting that every K3 surface is Kähler. Alternatively, see [Buc99] and [Lam99] for independent direct proofs.).

Now, it can be easily shown by the same duality method of Sullivan's (see, e.g. [Lam99, Théorème 6.1]) as the one employed in the proof of Proposition 4.2.5 that a non-zero d-exact semipositive (1, 1)-current always exists on any compact complex surface with b_1 odd. Threfore, by Proposition 4.2.5, no compact complex surface with b_1 odd can be strongly Gauduchon. $\square$

(II) Examples of dimension ≥ 3

The sG property of compact complex manifolds X with $\dim_{\mathbb{C}}X\geq 3$ is tremendously weaker than the Kähler property, in stark contrast with the case of complex surfaces. To exemplify this fact, we prove the following

Theorem 4.2.7. *([Pop14, Theorem 1.9.]) The* **Calabi-Eckmann manifolds** *[CE53], the* **Hopf manifolds** *[Hop48] and* **Tsuji's manifolds** *[Tsu84] are not strongly Gauduchon.*

This will be proved by analysing the three well-known classes of compact complex manifolds mentioned in the statement. The underlying C^∞ manifold of all these complex manifolds is a product $X := S^{2p+1} \times S^{2q+1}$ of two real odd-dimensional spheres, so they all share the property $H^2_{DR}(X, \mathbb{R}) = 0$ for the second De Rham cohomology group. This implies that any d-closed positive current T of bidegree $(1, 1)$ on X, should it exist, must be d-exact since the associated De Rham cohomology 2-class $\{T\} \in H^2_{DR}(X, \mathbb{R})$ must vanish. However, we know from Proposition 4.2.5 that the existence of a non-trivial $(1, 1)$-current T on X that is both positive and d-exact amounts to X being non-sG.

We shall briefly review the three classes of compact complex manifolds mentioned above and notice that every such manifold X possesses complex hypersurfaces $Y \subset X$. Thus, since $H^2_{DR}(X, \mathbb{R}) = 0$, the current of integration on any of these complex hypersurfaces Y is a current as in Proposition 4.2.5, ruling out the possibility that any manifold X in one of these classes be sG.

(a) **Calabi-Eckmann manifolds.** For all $p, q \in \mathbb{N}$, Calabi and Eckmann [CE53] constructed a complex structure on the Cartesian product $S^{2p+1} \times S^{2q+1}$ of odd-dimensional spheres. The case $p = q = 0$ being equivalent to a closed Riemann surface of genus 1 and periods $1, \tau$, they assume $p > 0$. In the case $q = 0$, the Calabi-Eckmann complex structure on $S^{2p+1} \times S^1$, although constructed by a different method, coincides with the complex structure constructed earlier by Hopf in [Hop48] starting from the universal covering space of $S^{2p+1} \times S^1$ equipped with the complex structure of $\mathbb{C}^{p+1} \setminus \{0\}$. The simply connected manifolds $S^{2p+1} \times S^{2q+1}$ $(p, q > 0)$ are given in [CE53] complex structures making them into compact, simply connected, non-Kähler complex manifolds $M^{p,q}$ of complex dimension $p + q + 1$ enjoying, among other things, the following properties (for all p, q, including $q = 0$):

(i) there exists a complex analytic fibring $\sigma : M^{p,q} \to \mathbb{P}^p \times \mathbb{P}^q$ over the product of complex projective spaces $\mathbb{P}^p$ and $\mathbb{P}^q$ whose fibres are tori of real dimension 2 (or algebraic curves of genus 1) (cf. [CE56, Theorem II]);

(ii) every compact complex subvariety of $M^{p,q}$ is the set of all points that are mapped by σ onto an algebraic subvariety of $\mathbb{P}^p \times \mathbb{P}^q$; it is therefore also fibred by tori (cf. [CE56, Theorem IV]).

It is clear that the inverse image under σ of any complex hypersurface of $\mathbb{P}^p \times \mathbb{P}^q$ defines a complex hypersurface of the Calabi-Eckmann manifold $M^{p,q}$. Thus no Calabi-Eckmann manifold $M^{p,q}$ $(p > 0)$ can be an sG manifold.[1]

(b) **Hopf manifolds.** As mentioned above (and proved in §.3 of [CE56]), the Hopf manifolds $S^{2p+1} \times S^1$ $(p > 0)$ endowed with the complex structure constructed in [Hop48] can be seen in retrospect as special cases for $q = 0$ of Calabi-Eckmann manifolds. Thus they contain complex hypersurfaces and are not sG manifolds by the above arguments.

(c) **Tsuji's manifolds.** Generalising the Calabi-Eckmann complex structures, Tsuji constructed in [Tsu84] complex structures on $S^3 \times S^3$ in the following way. Starting from an arbitrary $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$ satisfying

$$0 < |\alpha_1| \leq |\alpha_2| < 1 \quad \text{and} \quad 0 < |\alpha_3| < 1,$$

the author of [Tsu84] considers the primary Hopf manifold

[1] This same argument was invoked in [Mic82, p.263] to show that Calabi-Eckmann manifolds are not balanced.

$$H(\alpha) := \mathbb{C}^3 \setminus \{0\}/\langle h\rangle$$

of complex dimension 3, where the automorphism $h : \mathbb{C}^3 \to \mathbb{C}^3$ is defined by $h(z_1, z_2, z_3) := (\alpha_1 z_1, \alpha_2 z_2, \alpha_3 z_3)$ for all $(z_1, z_2, z_3) \in \mathbb{C}^3$ and $\langle h\rangle \subset \mathrm{Aut}(\mathbb{C}^3)$ denotes the automorphism group generated by h. He then goes on to consider

$$C := \{[z_1, z_2, z_3] \in H(\alpha) \ ; \ z_1 = z_2 = 0\} \subset H(\alpha),$$

an elliptic curve contained in $H(\alpha)$ and

$$S_0 := \{[z_1, z_2, z_3] \in H(\alpha) \ ; \ z_3 = 0\} \subset H(\alpha),$$

a primary Hopf surface which is a complex hypersurface of $H(\alpha)$. For every

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad \text{and} \quad m = (m_1, m_2) \in \mathbb{Z}^2, m_1, m_2 \gg 1,$$

he shows the existence of $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{C}^3$ defining biholomorphisms

$$L^\star(\beta) \overset{\Phi^\pm}{\simeq} L^\star(\alpha),$$

where $L^\star(\alpha)$ and $L^\star(\beta)$ are obtained from $L(\alpha)$ and $L(\beta)$ by removing the respective zero section, while $L(\alpha)$ and $L(\beta)$ are holomorphic line bundles over the respective primary Hopf surfaces

$$S_{\alpha_1,\alpha_2,0} := \mathbb{C}^2 \setminus \{0\}/\langle g_\alpha\rangle \quad \text{and} \quad S_{\beta_1,\beta_2,0} := \mathbb{C}^2 \setminus \{0\}/\langle g_\beta\rangle$$

associated with automorphisms of $\mathbb{C}^2$

$$g_\alpha(z_1, z_2) := (\alpha_1 z_1, \alpha_2 z_2) \quad \text{and} \quad g_\beta(z_1, z_2) := (\beta_1 z_1, \beta_2 z_2)$$

defined by

$$L(\alpha) := \mathbb{C}^2 \setminus \{0\} \times \mathbb{C}/\langle h_\alpha\rangle \quad \text{and} \quad L(\beta) := \mathbb{C}^2 \setminus \{0\} \times \mathbb{C}/\langle h_\beta\rangle,$$

where the automorphisms h_α and h_β of $\mathbb{C}^3$ are defined by

$$h_\alpha(z_1, z_2, z_3) := (\alpha_1 z_1, \alpha_2 z_2, \alpha_3 z_3) \quad \text{and} \quad h_\beta(z_1, z_2, z_3) := (\beta_1 z_1, \beta_2 z_2, \beta_3 z_3).$$

Considering a compactification of $L(\beta)$ as a $\mathbb{P}^1$-bundle $\mathbb{P}(\beta) \to S_{\beta_1,\beta_2,0}$, the infinity section of $\mathbb{P}(\beta)$ is denoted S_∞, while $U(S_\infty)$ denotes a tubular neighbourhood of S_∞ in $\mathbb{P}(\beta)$. The author defines compact complex manifolds

$$M^\pm(\alpha, A, m)$$

by identifying

$$L^\star(\beta) \subset \mathbb{P}(\beta) \setminus (\text{zero section})$$

with

$$L^\star(\alpha) \simeq H(\alpha) \setminus (S_0 \cup C) \subset H(\alpha)$$

using $\Phi^\pm$. These compact complex manifolds are seen to arise as

$$M^{\pm}(\alpha, A, m) = (H(\alpha) \setminus C) \cup U(S_\infty), \tag{4.26}$$

or equivalently, $M^{\pm}(\alpha, A, m)$ are obtained from $H(\alpha)$ by a surgery which replaces C with $U(S_\infty)$.

Theorem 4.2.8. *([Tsu84, Theorem 1.13]) $M^{\pm}(\alpha, A, m)$ is diffeomorphic to $S^3 \times S^3$ if and only if A is of the form $A = \begin{pmatrix} a & b \\ \pm 1 & d \end{pmatrix}$.*

Consequently, if A has the above shape, $M^{\pm}(\alpha, A, m)$ is diffeomorphic to an S^3-bundle over a lens space, hence $M^{\pm}(\alpha, A, m)$ has a complex structure.

With this outline of Tsuji's construction understood, we see that the complex hypersurface $S_0 \subset H(\alpha)$ satisfies $S_0 \cap C = \emptyset$. Thus, in view of the description (4.26) of $M^{\pm}(\alpha, A, m)$, we get a complex hypersurface

$$S_0 \subset M^{\pm}(\alpha, A, m)$$

whose existence, along with the property $H^2_{DR}(M^{\pm}(\alpha, A, m), \mathbb{R}) = 0$, shows that Tsuji's compact complex manifolds $M^{\pm}(\alpha, A, m)$ are not sG for any $\alpha \in \mathbb{C}^3, A \in SL(2, \mathbb{Z}), m = (m_1, m_2) \in \mathbb{Z}^2$ as above.

4.2.3 Stability of strongly Gauduchon manifolds under modifications

In this subsection we show that the *strongly Gauduchon* property of compact complex manifolds is stable under modifications (i.e. proper, holomorphic, bimeromorphic maps). This provides a sharp contrast to the Kähler property of these manifolds which is only preserved under blowing up (smooth) submanifolds ([Bla58]).

Theorem 4.2.9. *([Pop10b, Theorem 1.3.]) Let $\mu : \widetilde{X} \to X$ be a modification of compact complex manifolds. Then, $\widetilde{X}$ is a* **strongly Gauduchon manifold** *if and only if X is a* **strongly Gauduchon manifold***.*

This result parallels the main result of Alessandrini and Bassanelli in [AB95] (see also [AB91b] and [AB93]) which asserts that *balanced manifolds* (to be defined and studied in the next section) enjoy the same stability property under modifications as above. The proof of Theorem 4.2.9 will draw on some of the results in [AB91b], [AB93] and [AB95], but with certain arguments handled slightly differently while others are considerably simplified by the fact that d-closed positive $(1, 1)$-currents always admit unambiguously defined inverse images constructed from their local potentials, unlike the much more delicate-to-handle $\partial\bar\partial$-closed positive $(1, 1)$-currents that were relevant to the case of *balanced manifolds*. Inverse images for this latter class of currents were painstakingly constructed in [AB93] and a unique choice was shown to enjoy the necessary cohomological properties, rendering the case treated in [AB93] and [AB95] conspicuously more involved than ours.

Proof of Theorem 4.2.9

Let $\mu : \widetilde{X} \to X$ be a modification of compact complex manifolds and let $n = \dim_{\mathbb{C}}\widetilde{X} = \dim_{\mathbb{C}}X$. Let E be the exceptional divisor of μ on $\widetilde{X}$ and let $\Sigma \subset X$ be the analytic subset of codimension ≥ 2 such that the restriction $\mu_{|\widetilde{X}\setminus E} : \widetilde{X} \setminus E \longrightarrow X \setminus \Sigma$ is a biholomorphism. Theorem 4.2.9 comprises two parts.

(I) One implication of the equivalence in Theorem 4.2.9 is dealt with in the following

Theorem 4.2.10. *([Pop10b, Theorem 2.1.]) If $\mu : \widetilde{X} \to X$ is a modification of compact complex manifolds and X is* **strongly Gauduchon**, *then $\widetilde{X}$ is again* **strongly Gauduchon**.

Proof. We proceed by contradiction. Suppose that $\widetilde{X}$ is not *strongly Gauduchon*. Then, by Proposition 4.2.5, there exists a current $T \neq 0$ of type (1, 1) on $\widetilde{X}$ such that

$$T \geq 0 \quad \text{and} \quad T \in \operatorname{Im} d \quad \text{on} \quad \widetilde{X}.$$

By compactness of $\widetilde{X}$, the map μ is proper and therefore the direct image under μ of any current on $\widetilde{X}$ is well-defined. Thus $\mu_\star T$ is a well-defined current of type (1, 1) on X. It is clear that

$$\mu_\star T \geq 0 \quad \text{and} \quad \mu_\star T \in \operatorname{Im} d \quad \text{on} \quad X.$$

Indeed, for every C^∞ (1, 1)-form $\omega > 0$ on X, we have

$$\int\limits_X \mu_\star T \wedge \omega^{n-1} = \int\limits_{\widetilde{X}} T \wedge (\mu^\star \omega)^{n-1} \geq 0,$$

a fact that proves the positivity of $\mu_\star T$, while the d-exactness follows from $\mu_\star$ commuting with d. Now we have the following dichotomy.

If $\mu_\star T$ is non-zero, we get a contradiction to the *strongly Gauduchon* assumption on X thanks to Proposition 4.2.5.

If $\mu_\star T = 0$ on X, we show that $T = 0$ on $\widetilde{X}$, contradicting the choice of T. Indeed, if $\mu_\star T = 0$, the support of T must be contained in the support of E. Since T is a closed positive current of bidegree (1, 1) and the irreducible components E_j of E are all of codimension 1 in $\widetilde{X}$, a classical theorem of support (see e.g. [Dem97, Chapter III, Corollary 2.14]) forces T to have the shape

$$T = \sum_{j \in J} \lambda_j \, [E_j], \quad \text{with coefficients} \quad \lambda_j \geq 0 \quad \text{and some index set } J.$$

Since all the irreducible components of S are of codimension ≥ 2 in X, $\operatorname{codim}_X \mu(E_j) \geq 2$ for every $j \in J$. All we have to do is repeat the argument of [AB91b, p. 5] that we now recall for the reader's convenience. By [GR70, p. 286], for every $i \geq 0$, there exists a vector subspace $H_i^\star(E) \subset H_i(E)$ and a commutative diagram whose rows are short exact sequences featuring the homology groups H_i of the various spaces involved:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_i^\star(E) & \hookrightarrow & H_i(E) & \xrightarrow{\beta_i} & H_i(S) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_i^\star(E) & \longrightarrow & H_i(\widetilde{X}) & \xrightarrow{\alpha_i} & H_i(X) & \to & 0, \end{array}$$

where $\hookrightarrow$ stands for inclusion. If we denote by $\{\ \}_E$ (respectively $\{\ \}_{\widetilde{X}}$) the homology class of a subvariety of real dimension $2(n-1)$ in the ambient space Supp E (respectively $\widetilde{X}$), we see that

$$\beta_{2(n-1)}\{T\}_E = \sum_j \lambda_j \, \beta_{2(n-1)}\{E_j\}_E = 0$$

since $\mu(E_j) \subset S$ for all j and $\dim_{\mathbb{C}} S \leq n-2$. Thus, from the top exact sequence, we get that $\{T\}_E$ belongs to $H^\star_{2(n-1)}(E)$. The diagram being commutative, the image of $\{T\}_E \in H^\star_{2(n-1)}(E)$ in $H_{2(n-1)}(\widetilde{X})$ under the injective arrow of the bottom exact sequence is $\{T\}_{\widetilde{X}}$. Meanwhile $\{T\}_{\widetilde{X}} = 0$ since T is d-exact on $\widetilde{X}$. We get that $\{T\}_E = 0$, namely $\sum\limits_{j=1}^{s} \lambda_j \, \{E_j\}_E = 0$. This implies that

$\lambda_j = 0$ for every j. (See e.g. [BH61, Theorem 3.2] for the existence and uniqueness of the integral fundamental class of a complex analytic space inducing the natural orientation at every simple point.) Hence $T = 0$ as a current on $\widetilde{X}$, a contradiction.

The proof is complete. □

(II) The other implication of the equivalence in Theorem 4.2.9 is dealt with in the following

Theorem 4.2.11. *([Pop10b, Theorem 2.2.]) If $\mu : \widetilde{X} \to X$ is a modification of compact complex manifolds and $\widetilde{X}$ is* **strongly Gauduchon**, *then X is again* **strongly Gauduchon**.

Proof. We proceed once more by contradiction. Suppose that X is not *strongly Gauduchon*. Then, in view of Proposition 4.2.5, there exists a current $T \neq 0$ of type $(1, 1)$ on X such that

$$T \geq 0 \quad \text{and} \quad T = dS \quad \text{for some real } 1-\text{current } S \text{ on } \quad X.$$

We shall show that the inverse image current $\mu^\star T$ is a well-defined $(1, 1)$-current on $\widetilde{X}$ enjoying the same properties as T on X, thus contradicting the *strongly Gauduchon* assumption on $\widetilde{X}$ in view of Proposition 4.2.5.

Although the inverse image of an arbitrary current is not defined in general, the inverse image of a d-closed positive $(1, 1)$-current is well-defined under μ by the inverse images of its local $\partial\bar\partial$-potentials (see e.g. [Meo96]). Indeed, following [Meo96], for every open subset $U \subset X$ such that $T_{|U} = i\partial\bar\partial\varphi$ for a psh function φ on U, one defines $(\mu^\star T)_{|\mu^{-1}(U)} := i\partial\bar\partial(\varphi \circ \mu)$. The psh function $\varphi \circ \mu$ is $\not\equiv -\infty$ on every connected component of $\mu^{-1}(U)$ since μ has generically maximal rank and the local pieces $(\mu^\star T)_{|\mu^{-1}(U)}$ glue together into a globally defined d-closed positive $(1, 1)$-current $\mu^\star T$ on $\widetilde{X}$ that is independent of the choice of open subsets $U \subset X$ and local potentials φ.

It is clear that $\mu^\star T$ is not the zero current on $\widetilde{X}$. Indeed, if we had $\mu^\star T = 0$, the support of T would be contained in S. If all the irreducible components of S were of codimension ≥ 2 in X, a classical theorem of support (see e.g. [Dem97, Chapter III, Corollary 2.11]) would guarantee that the closed positive $(1, 1)$-current T must be the zero current on X, a contradiction. If S had certain global irreducible components S_j of codimension 1 in X, another theorem of support (cf. [Dem97, Chapter III, Corollary 2.14]) would ensure that T has the shape $T = \sum \lambda_j [S_j]$ for some constants $\lambda_j \geq 0$. Then $\mu^\star[S_j]$ would be the current of integration on the inverse-image divisor $\mu^{-1}(S_j) \subset \widetilde{X}$ and $\mu^\star T$ cannot be the zero current unless $\lambda_j = 0$ for all j. However, in this event $T = 0$ on X, a contradiction.

The only thing that has yet to be checked before reaching the desired contradiction is that the non-trivial d-closed positive $(1, 1)$-current $\mu^\star T$ is d-exact on $\widetilde{X}$. Since the 1-current S cannot be pulled back to $\widetilde{X}$ (the local potential technique is no longer available), we shall use Demailly's regularisation-of-currents theorem [Dem92, Main Theorem 1.1] to get a sequence $(v_j)_{j\in\mathbb{N}}$ of C^∞ $(1, 1)$-forms on X such that every v_j lies in the same Bott-Chern (hence also De Rham) cohomology class as T with convergence

$$v_j \longrightarrow T \quad \text{weakly as} \quad j \to +\infty, \quad \text{while} \quad v_j \geq -C\omega, \quad j \in \mathbb{N},$$

where ω is any Hermitian metric on X fixed beforehand and $C > 0$ is a constant independent of $j \in \mathbb{N}$.

Since T is d-exact and cohomologous to each v_j, every form v_j is d-exact. Thus, for all $j \in \mathbb{N}$, $v_j = du_j$ for some C^∞ 1-form u_j on X. Unlike S, the C^∞ forms u_j have inverse images under μ and we get

$$\mu^\star v_j = d(\mu^\star u_j) \longrightarrow \mu^\star T \quad \text{weakly as} \quad j \to +\infty, \tag{4.27}$$

after possibly extracting a subsequence. Indeed, it was shown in [Meo96, Proposition 1] that for every sequence of d-closed positive $(1,\,1)$-currents T_j converging weakly to T, the sequence of inverse-image currents $\mu^\star T_j$ converges weakly to $\mu^\star T$. In our case, the $(1,\,1)$-forms v_j are not necesarily *positive* but only *almost positive* (the negative part being uniformly bounded by $C\omega$). We now spell out the reason why $\mu^\star v_j$ converges weakly to the current $\mu^\star T$ in this slightly more general context. The argument is virtually the same as that of [Meo96].

Pick any C^∞ $(1,\,1)$-form α in the Bott-Chern class of the forms v_j (= the class of T). Then, for every $j \in \mathbb{N}$, we have

$$v_j = \alpha + i\partial\bar\partial\psi_j \geq -C\omega \quad \text{on} \quad X,$$

with C^∞ functions $\psi_j : X \to \mathbb{R}$ that we normalise such that $\int\limits_X \psi_j\,\omega^n = 0$ for every j. This normalisation makes ψ_j unique. Applying the trace w.r.t. ω and using the corresponding Lapacian $\Delta_\omega(\cdot) = \text{Trace}_\omega(i\partial\bar\partial(\cdot))$, we get

$$\Delta_\omega\psi_j = \text{Trace}_\omega(v_j - \alpha), \quad j \in \mathbb{N}.$$

Applying now the Green operator G of Δ_ω and using the normalisation of ψ_j, we get

$$\psi_j = G\,\text{Trace}_\omega(v_j - \alpha), \quad j \in \mathbb{N}.$$

Since G is a compact operator from the Banach space of bounded Borel measures on X to $L^1(X)$ and since the forms v_j converge weakly to T, we infer that some subsequence $(\psi_{j_k})_k$ converges to a limit $\psi \in L^1(X)$ in $L^1(X)$-topology. Thus the weak continuity of $\partial\bar\partial$ gives

$$T = \lim_k(\alpha + i\partial\bar\partial\psi_{j_k}) = \alpha + i\partial\bar\partial\psi \quad \text{on} \quad X.$$

Now the sequence $(\psi_j)_j$ is uniformly bounded above on X by some constant $C_1 > 0$ thanks to the normalisation imposed on ψ_j and the Green-Riesz representation formula for ψ_j, Δ_ω and G. Hence the sequence $(\psi_j \circ \mu)_j$ is uniformly bounded above on $\widetilde{X}$ by $C_1 > 0$. On the other hand, $\psi_{j_k} \circ \mu$ converges almost everywhere to $\psi \circ \mu$ on $\widetilde{X}$. Since the forms $i\partial\bar\partial(\psi_{j_k} \circ \mu)$ are uniformly bounded below on $\widetilde{X}$ by $-(\mu^\star\alpha + C\mu^\star\omega)$, the almost psh functions $\psi_{j_k} \circ \mu$ can be simultaneously made psh on small open subsets of $\widetilde{X}$ by the addition of a same locally defined smooth psh function. We can thus apply the classical result stating that a sequence of psh functions that are locally uniformly bounded above either converges locally uniformly to $-\infty$ (a case that is ruled out in our present situation), or has a subsequence that converges in L^1_{loc} topology (see e.g. [Hor94, Theorem 3.2.12., p. 149]). We infer that the almost psh functions $\psi_{j_k} \circ \mu$ actually converge in $L^1(\widetilde{X})$-topology (hence also in the weak topology of distributions) and implicitly the forms

$$\mu^\star v_{j_k} = \mu^\star\alpha + i\partial\bar\partial(\psi_{j_k} \circ \mu)$$

converge weakly to the current $\mu^\star T = \mu^\star\alpha + i\partial\bar\partial(\psi \circ \mu)$. Thus the convergence statement (4.27) is proved.

Since the De Rham class is continuous w.r.t. the weak topology of currents and since each form $\mu^\star v_j = d(\mu^\star u_j)$ has vanishing De Rham class, the limit current $\mu^\star T$ must have vanishing De Rham class. Equivalently, $\mu^\star T$ is d-exact, providing a contradiction to the *strongly Gauduchon* assumption on $\widetilde{X}$ in view of Proposition 4.2.5. The proof is complete. $\square$

An immediate consequence of Theorem 4.2.9 is the following

Corollary 4.2.12. *Every* **class $\mathcal{C}$ manifold** *is a* **strongly Gauduchon manifold**.

Proof. If X is of *class* $\mathcal{C}$, there exists a modification $\mu : \widetilde{X} \longrightarrow X$ such that $\widetilde{X}$ is compact Kähler. Then $\widetilde{X}$ is also sG, hence X is sG by Theorem 4.2.9. □

4.3 The class of sGG manifolds

We saw in Proposition 4.2.2 that on compact $\partial\bar\partial$-manifolds, every Gauduchon metric on X is strongly Gauduchon. However, we will see in this section, whose material is mostly taken from [PU18], that the class of compact complex manifolds on which the notions of **Gauduchon** and **strongly Gauduchon** metrics coincide is strictly larger than the class of $\partial\bar\partial$-manifolds. The manifolds in this larger class will be called sGG.

4.3.1 Original motivation for the introduction of sGG manifolds

It has long been conjectured that the deformation limit of any holomorphic family of *class* $\mathcal{C}$ manifolds ought to be a *class* $\mathcal{C}$ manifold (cf. Conjecture 7.0.5).

A two-step strategy for tackling this conjecture was briefly outlined in [Pop15a]:

Step 1: *prove that a compact complex manifold X belongs to the class $\mathcal{C}$ if and only if there are "many" closed positive $(1,\,1)$-currents on X.*

Step 2: *prove that there can only be "more" closed positive $(1,\,1)$-currents on X_0 than on the generic fibre X_t.*

A key tool in this section will be the following

Definition 4.3.1. *([Pop15a, §.5) Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The* **sG cone** *of X is the set*

$$\mathcal{SG}_X = \mathcal{G}_X \cap \ker T \subset \mathcal{G}_X \subset H^{n-1,\,n-1}_A(X,\,\mathbb{R}),$$

namely the intersection of the Gauduchon cone with the kernel of the canonical linear map:

$$T\,:\,H^{n-1,\,n-1}_A(X,\,\mathbb{C}) \longrightarrow H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C}), \qquad T([\Omega]_A) := [\partial\Omega]_{\bar\partial}. \tag{4.28}$$

Note that the map T is well defined (i.e. independent of the choice of representative Ω of the Aeppli class $[\Omega]_A$) and it shows that the sG property is cohomological: either all the Gauduchon metrics ω for which ω^{n-1} belongs to a given Aeppli-Gauduchon class $[\omega^{n-1}]_A \in \mathcal{G}_X$ are strongly Gauduchon (in which case we say that $[\omega^{n-1}]_A$ is an *sG class*), or none of them is. In other words, the sG cone $\mathcal{SG}_X$ is the set of all sG classes on X. It is empty if X does not support any sG metric.

Here is the meaning of the two-step approach to the conjecture mentioned above.

Step 1 would be the transcendental analogue of the following well-known fact: a compact complex manifold X is Moishezon (= bimeromorphically equivalent to a projective manifold) if and only if there are "many" divisors on X (in the sense that the algebraic dimension of X is maximal, i.e. equal to the dimension of X as a complex manifold).

The meaning of "many" in connection with closed positive (1, 1)-currents has yet to be probed, but we suspect that it will mean that the pseudo-effective cone $\mathcal{E}_X$ of X is "maximal" at least in the following sense:

$$(i)\ \mathring{\mathcal{E}}_X \neq \emptyset \quad \text{and} \quad (ii)\ \mathcal{SG}_X = \mathcal{G}_X, \qquad\qquad (\star)$$

where $^\circ$ stands for "interior", while $\mathcal{SG}_X$ and $\mathcal{G}_X$ are respectively the sG cone and the Gauduchon cone of X. Property (i) uses the non-emptiness of the interior as a way of requiring $\mathcal{E}_X$ to be fairly large, while property (ii) requires $\mathcal{G}_X$ to be fairly small, hence by duality $\mathcal{E}_X$ to be again fairly large by a different criterion.

Each of the two properties in $(\star)$ is necessary for X to be of class $\mathcal{C}$, but none of them is sufficient on its own (see Proposition 4.5.66 for examples of manifolds not in the class $\mathcal{C}$ whose pseudo-effective cone has non-empty interior). However, together they may become sufficient, or should condition $(\star)$ turn out to be insufficient for X to be in the *class* $\mathcal{C}$, it will have to be reinforced.

Step 2 means that the pseudo-effective cone $\mathcal{E}_{X_t}$ can only increase in the limit as $t \to 0$ (i.e. it behaves upper-semicontinuously under deformations of the complex structure of X_t), while its dual, the (closure of the) Gauduchon cone $\mathcal{G}_{X_t}$, can only decrease in the limit (i.e. it behaves lower-semicontinuously).

In this section, we begin the implementation of this two-step strategy by studying the manifolds defined by property (ii) in $(\star)$ and by giving a complete affirmative answer to the problem raised at *Step* 2 of this line of argument for deformations of such manifolds.

4.3.2 Definition and first properties of sGG manifolds

In line with the goals in the first step of the approach to the conjecture mentioned in §.4.3.1, we will investigate the following class of manifolds.

Definition 4.3.2. *([Pop15a] and [PU18, Definition 1.2.]) Let X be a compact complex manifold. We say that X is an* **sGG manifold** *if the sG cone of X coincides with the Gauduchon cone of X, i.e. if $\mathcal{SG}_X = \mathcal{G}_X$.*

We have the following equivalent descriptions of the sGG property. (See [Pop15a, section §.5] for $(i) - (iii)$).

Lemma 4.3.3. *The following statements are equivalent:*

(i) *X is an sGG manifold;*

(ii) *the map T defined in (4.28) vanishes identically;*

(iii) *the following special case of the $\partial\bar\partial$-property holds: for every d-closed $(n,\, n-1)$-form Γ on X, if Γ is ∂-exact, then Γ is also $\bar\partial$-exact;*

(iv) *every Gauduchon metric ω on X is strongly Gauduchon.*

Proof. Since the kernel of the linear map T is a vector subspace of $H^{n-1,\,n-1}_A(X,\,\mathbb{C})$, its intersection with the open convex Gauduchon cone leaves the latter unchanged if and only if $\ker T = H^{n-1,\,n-1}_A(X,\,\mathbb{C})$, i.e. if and only if T vanishes identically. This proves the equivalence of (i) and (ii).

The equivalence of (ii) and (iii) is an immediate consequence of the definition (4.28) of T.

If T vanishes identically, then for every Gauduchon metric ω, $T([\omega^{n-1}]_A) = [\partial\omega^{n-1}]_{\bar\partial} = 0$, so $\partial\omega^{n-1}$ is $\bar\partial$-exact, which means that ω is strongly Gauduchon. This proves the implication "$(ii) \implies (iv)$ ".

Now, suppose that (iv) holds. If $[\omega^{n-1}]_A \in \mathcal{G}_X$ with ω a Gauduchon metric, then ω is sG by (iv), hence $[\omega^{n-1}]_A \in \mathcal{SG}_X$. Thus, $\mathcal{G}_X = \mathcal{SG}_X$. This proves the implication "$(iv) \implies (i)$ ". □

An obvious consequence of Lemma 4.3.3 is that the first of the following two implications holds for every compact complex manifold X:

$$X \text{ is a } \partial\bar\partial\text{-manifold} \implies X \text{ is an sGG manifold} \implies X \text{ is an sG manifold}, \tag{4.29}$$

while the second implication follows from the existence of Gauduchon metrics (i.e. $\mathcal{G}_X \neq \emptyset$). We shall see later that both converses fail. In other words, the sGG class of compact complex manifolds strictly contains the $\partial\bar\partial$ class and is strictly contained in the sG class.

4.3.3 First numerical characterisation of sGG manifolds

We will first give a numerical characterisation in terms of the Bott-Chern number $h^{0,\,1}_{BC} := \dim_{\mathbb{C}} H^{0,\,1}_{BC}(X,\,\mathbb{C})$ and the Hodge number $h^{0,\,1}_{\bar\partial} := \dim_{\mathbb{C}} H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})$.

Theorem 4.3.4. *([PU18, Theorem 1.4]) On any compact complex manifold X we have $h^{0,\,1}_{BC} \leq h^{0,\,1}_{\bar\partial}$. Moreover, X is an sGG manifold if and only if $h^{0,\,1}_{BC} = h^{0,\,1}_{\bar\partial}$.*

An immediate consequence is the following

Corollary 4.3.5. *([PU18, Corollary 1.5]) The Iwasawa manifold and all its small deformations in its Kuranishi family are sGG manifolds (but, of course, not $\partial\bar\partial$-manifolds).*

Thanks to Corollary 4.3.5, the Iwasawa manifold is our main example of sGG manifold that is not $\partial\bar\partial$. Its Kuranishi family was explicitly computed by Nakamura in [Nak75].

We will actually prove the following more precise version of Theorem 4.3.4.

Theorem 4.3.6. *([PU18, Theorem 2.1]) Let X be any compact complex manifold, $\dim_{\mathbb{C}} X = n$.*

(i) There is a well-defined canonical $\mathbb{C}$-linear map

$$S\,:\, H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C}) \longrightarrow H^{n,\,n-1}_{A}(X,\,\mathbb{C}), \quad S([\Gamma]_{\bar\partial}) := [\Gamma]_A.$$

Moreover, the map S is **surjective**, *and we have an exact sequence*

$$H^{n-1,\,n-1}_{A}(X,\,\mathbb{C}) \xrightarrow{T} H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C}) \xrightarrow{S} H^{n,\,n-1}_{A}(X,\,\mathbb{C}) \longrightarrow 0,$$

i.e. $Im\,T = \ker S$, where T is the map defined in (4.28). In particular, X is an sGG manifold if and only if S is injective (i.e. if and only if S is bijective).

(ii) There are well-defined canonical $\mathbb{C}$-linear maps and an exact sequence

$$0 \longrightarrow H^{0,\,1}_{BC}(X,\,\mathbb{C}) \xrightarrow{S^\star} H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C}) \xrightarrow{T^\star} H^{1,\,1}_{BC}(X,\,\mathbb{C})$$

defined by $S^\star([u]_{BC}) := [u]_{\bar\partial}$ for any d-closed $(0,\,1)$-form u and $T^\star([v]_{\bar\partial}) := [\partial v]_{BC}$ for any $\bar\partial$-closed $(0,\,1)$-form v. Thus $Im\,S^\star = \ker T^\star$.

Moreover, the maps $S^\star$ and $T^\star$ are dual to S and respectively T. Thus $S^\star$ is **injective**, *hence* $h^{0,1}_{BC} \leq h^{0,1}_{\bar\partial}$.

(iii) It follows that X is an sGG manifold if and only if $S^\star$ is surjective (i.e. if and only if $S^\star$ is bijective) if and only if $h^{0,1}_{BC} = h^{0,1}_{\bar\partial}$.

Proof. By S being well defined, we mean that $S([\Gamma]_{\bar\partial})$ (i.e. $[\Gamma]_A$) is meaningful and does not depend on the choice of representative Γ of the class $[\Gamma]_{\bar\partial}$. The immediate verification of this fact is left to the reader.

It is clear that $\operatorname{Im} T \subset \ker S$ since $\operatorname{Im}\partial \subset \operatorname{Im}\partial + \operatorname{Im}\bar\partial$. To show the reverse inclusion, let $[\Gamma]_{\bar\partial} \in H^{n,n-1}_{\bar\partial}(X, \mathbb{C})$ such that $S([\Gamma]_{\bar\partial}) = 0$. Then there are forms Ω, Λ of respective bidegrees $(n-1, n-1)$ and $(n, n-2)$ such that $\Gamma = \partial\Omega + \bar\partial\Lambda$, i.e. $\Gamma - \bar\partial\Lambda = \partial\Omega$. Hence $\partial\bar\partial\Omega = 0$, $[\Gamma]_{\bar\partial} = [\Gamma - \bar\partial\Lambda]_{\bar\partial}$ and $T([\Omega]_A) = [\partial\Omega]_{\bar\partial} = [\Gamma]_{\bar\partial}$. Thus $[\Gamma]_{\bar\partial} \in \operatorname{Im} T$. This proves the identity $\operatorname{Im} T = \ker S$.

The surjectivity of S will follow from the injectivity of its dual map $S^\star$ that will be proved below.

The well-definedness of $S^\star$ and $T^\star$ are proved in a similar way. The identity $\operatorname{Im} S^\star = \ker T^\star$ follows by duality from $\operatorname{Im} T = \ker S$ or directly in the following way. Let $[u]_{BC} \in H^{0,1}_{BC}(X, \mathbb{C})$, i.e. u is a d-closed $(0, 1)$-form. Then $\partial u = 0$ and $\bar\partial u = 0$, hence $T^\star(S^\star[u]_{BC}) = T^\star([u]_{\bar\partial}) = [\partial u]_{BC} = 0$. Thus $\operatorname{Im} S^\star \subset \ker T^\star$. To show the reverse inclusion, let $[v]_{\bar\partial} \in \ker T^\star$, i.e. v is a $\bar\partial$-closed $(0, 1)$-form such that $\partial v = \partial\bar\partial f$ for some function f. Then $\partial(v - \bar\partial f) = 0$, hence $d(v - \bar\partial f) = 0$ and $[v]_{\bar\partial} = [v - \bar\partial f]_{\bar\partial} = S^\star([v - \bar\partial f]_{BC})$, so $[v]_{\bar\partial} \in \operatorname{Im} S^\star$.

Let us now show that $S^\star$ is injective. Let $[u]_{BC} \in \ker S^\star$, i.e. u is a d-closed $(0, 1)$-form such that $u = \bar\partial f$ for some function f. Since $du = 0$, we also have $\partial u = 0$, hence $\partial\bar\partial f = 0$ on X. Because X is compact, the function f must be constant, hence $u = \bar\partial f = 0$. In particular, $[u]_{BC} = 0$.

Let us now check that the maps T and $T^\star$ are dual to each other under the duality (**??**) and under the Serre duality

$$H^{0,1}_{\bar\partial}(X, \mathbb{C}) \times H^{n,n-1}_{\bar\partial}(X, \mathbb{C}) \longrightarrow \mathbb{C}, \quad ([v]_{\bar\partial}, [\Gamma]_{\bar\partial}) \mapsto \int\limits_X v \wedge \Gamma,$$

the latter being defined for every $\bar\partial$-closed forms v and Γ of respective bidegrees $(0, 1)$ and $(n, n-1)$. We have to check that for every $[v]_{\bar\partial} \in H^{0,1}_{\bar\partial}(X, \mathbb{C}) \simeq (H^{n,n-1}_{\bar\partial}(X, \mathbb{C}))^\star$, if we denote by

$$\sigma_v \,:\, H^{n,n-1}_{\bar\partial}(X, \mathbb{C}) \longrightarrow \mathbb{C}$$

the linear map induced by $[v]_{\bar\partial}$ under duality, then the linear map

$$\tau_{\partial v} \,:\, H^{n-1,n-1}_{A}(X, \mathbb{C}) \longrightarrow \mathbb{C}$$

induced by $T^\star([v]_{\bar\partial}) = [\partial v]_{BC} \in H^{1,1}_{BC}(X, \mathbb{C}) \simeq (H^{n-1,n-1}_{A}(X, \mathbb{C}))^\star$ under duality is $\sigma_v \circ T$. This is indeed the case since, for every $[\Omega]_A \in H^{n-1,n-1}_A(X, \mathbb{C})$, we have

$$(\sigma_v \circ T)([\Omega]_A) = \sigma_v([\partial\Omega]_{\bar\partial}) = \int\limits_X v \wedge \partial\Omega = \int\limits_X \partial v \wedge \Omega = \tau_{\partial v}([\Omega]_A),$$

having used the Stokes formula $\int_X \partial(v \wedge \Omega) = 0$ and $\partial(v \wedge \Omega) = \partial v \wedge \Omega - v \wedge \partial\Omega$.

We can now check the equivalence:

$$T \text{ vanishes identically} \iff T^\star \text{ vanishes identically.}$$

Indeed, T vanishes identically if and only if for every $[\Omega]_A \in H^{n-1,n-1}_A(X, \mathbb{C})$ and every $[v]_{\bar\partial} \in H^{0,1}_{\bar\partial}(X, \mathbb{C})$ we have $\int_X \partial\Omega \wedge v = 0$. Since $\int_X \partial\Omega \wedge v = \int_X \Omega \wedge \partial v$ by the Stokes formula, this is equivalent to the map $\tau_{\partial v} \,:\, H^{n-1,n-1}_A(X, \mathbb{C}) \to \mathbb{C}$ vanishing identically, i.e. to $[\partial v]_{BC} = 0$, for every $[v]_{\bar\partial} \in H^{0,1}_{\bar\partial}(X, \mathbb{C})$. Since $[\partial v]_{BC} = T^\star([v]_{\bar\partial})$, this is still equivalent to the map $T^\star$ vanishing identically.

Thus, if we put the various bits together, we get the equivalences:

$$\begin{aligned}X \text{ is an sGG manifold } &\iff \operatorname{Im} T = 0 \iff \ker S = 0 \iff S \text{ is injective}\\ &\iff \ker T^\star = H^{0,1}_{\bar\partial}(X,\mathbb{C}) \iff \operatorname{Im} S^\star = H^{0,1}_{\bar\partial}(X,\mathbb{C})\\ &\iff S^\star \text{ is surjective.}\end{aligned}$$

It can be checked that the maps S and $S^\star$ are dual to each other in the same way as the duality between T and $T^\star$ has been checked. □

Proof of Corollary 4.3.5. Reading the dimension tables for the Hodge and Bott-Chern numbers given in [Nak75, p.96] and resp. [Ang11, Theorem 5.1], we gather that

$$h^{0,1}_{BC} = h^{0,1}_{\bar\partial} = 2$$

for the Iwasawa manifold and all its small deformations. Thus, the conclusion follows from Theorem 4.3.4. □

Remark 4.3.7. In the context of solvmanifolds some examples of sGG manifolds can be obtained. For instance, for the completely-solvable Nakamura manifold, studied first by Nakamura in [Nak75], it is shown by Angella and Kasuya that the corresponding Lie group G admits lattices Γ (see cases (i)–(iii) in [AK12, Example 2.17]) for which the Bott-Chern cohomology of the compact solvmanifolds G/Γ can be determined. For the lattices Γ in cases (ii) and (iii) the solvmanifolds satisfy $h^{0,1}_{BC}(G/\Gamma) = 1 = h^{0,1}_{\bar\partial}(G/\Gamma)$ (see [AK12, Table 6]), so by Theorem 4.3.4 they are sGG. Note that in [AK12, Remark 2.19] it is proved that G/Γ is not a $\partial\bar\partial$-manifold only for Γ in case (ii).

4.3.4 Second numerical characterisation of sGG manifolds

We will now give a numerical characterisation in terms of the first Betti number $b_1 := \dim_{\mathbb{C}} H^1_{DR}(X,\mathbb{C})$ and the Hodge number $h^{0,1}_{\bar\partial}$.

Theorem 4.3.8. *([PU18, Theorem 1.6]) On any compact complex manifold X we have $b_1 \leq 2h^{0,1}_{\bar\partial}$. Moreover, X is an sGG manifold if and only if $b_1 = 2h^{0,1}_{\bar\partial}$.*

This makes sGG manifolds reminiscent of compact Kähler surfaces. (See proof of Theorem 4.2.6.)

We will actually prove the following more precise version of Theorem 4.3.8. For any form α, we denote by $\alpha^{p,q}$ its component of bidegree (p,q).

Theorem 4.3.9. *([PU18, Theorem 3.1]) Let X be any compact complex manifold, $\dim_{\mathbb{C}} X = n$.*

(i) There is a well-defined canonical $\mathbb{C}$-linear map

$$\begin{aligned}F : H^1_{DR}(X,\mathbb{C}) &\longrightarrow H^{0,1}_{\bar\partial}(X,\mathbb{C}) \oplus \overline{H^{0,1}_{\bar\partial}(X,\mathbb{C})},\\ F(\{\alpha\}_{DR}) &:= ([\alpha^{0,1}]_{\bar\partial}, \overline{[\overline{\alpha^{1,0}}]_{\bar\partial}}).\end{aligned} \tag{4.30}$$

Moreover, the map F is **injective**. *Consequently, the following inequality holds on any compact complex manifold:*

$$b_1 \leq 2h^{0,1}_{\bar\partial}.$$

(ii) There is a well-defined canonical $\mathbb{C}$-linear map:

$$\begin{aligned} F^\star : H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C}) \oplus \overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})} & \longrightarrow H^{2n-1}_{DR}(X,\,\mathbb{C}), \\ F^\star([\beta]_{\bar\partial},\,\overline{[\gamma]_{\bar\partial}}) & := \{\beta+\bar\gamma\}_{DR}. \end{aligned} \tag{4.31}$$

Moreover, the map $F^\star$ is dual to the map F. Hence $F^\star$ is **surjective**.

(iii) The following equivalence holds:

$$X \text{ is an } sGG \text{ manifold} \iff F^\star \text{ is injective.}$$

Since F is always injective by (i), this means that X is an sGG manifold if and only if the linear map F is bijective. In other words, the following equivalence holds:

$$X \text{ is an } sGG \text{ manifold} \iff b_1 = 2\,h^{0,\,1}_{\bar\partial}.$$

Proof. (i) For any 1-form α, the condition $d\alpha=0$ is equivalent to

$$\bar\partial\alpha^{0,\,1}=0, \quad \partial\alpha^{1,\,0}=0 \ (\Leftrightarrow \bar\partial\overline{\alpha^{1,\,0}}=0), \quad \partial\alpha^{0,\,1}+\bar\partial\alpha^{1,\,0}=0.$$

Thus, if $d\alpha=0$, $\alpha^{0,\,1}$ and $\overline{\alpha^{1,\,0}}$ define Dolbeault cohomology classes of type (0, 1). To show that the map F is independent of the choice of representative in a given De Rham class $\{\alpha\}_{DR}$, let α be any d-exact 1-form on X. Then, there exists a function f on X such that $\alpha = df = \partial f + \bar\partial f$. Hence $\alpha^{0,\,1} = \bar\partial f$ and $\overline{\alpha^{1,\,0}} = \bar\partial\bar f$, so $[\alpha^{0,\,1}]_{\bar\partial} = [\overline{\alpha^{1,\,0}}]_{\bar\partial} = 0$. This proves the well-definedness of the map F.

To prove that F is injective, let α be a d-closed 1-form such that $F(\{\alpha\}_{DR})=0$, i.e. $\alpha^{0,\,1}=\bar\partial f$ and $\overline{\alpha^{1,\,0}} = \bar\partial g$ (i.e. $\alpha^{1,\,0}=\partial\bar g$) for some functions f, g on X. Then

$$0 = \partial\alpha^{0,\,1}+\bar\partial\alpha^{1,\,0} = \partial\bar\partial(f-\bar g) \quad \text{on } X,$$

where the first identity follows from $\partial\alpha^{0,\,1}+\bar\partial\alpha^{1,\,0}$ being the component of bidegree (1, 1) of $d\alpha=0$. Since X is compact, $f-\bar g$ must be constant on X, hence $\partial\bar g = \partial f$, so we get

$$\alpha = \alpha^{1,\,0}+\alpha^{0,\,1} = \partial f + \bar\partial f = df.$$

Thus $\{\alpha\}_{DR}=0$. Consequently, F is injective.

(ii) For any $\bar\partial$-closed $(n,\,n-1)$-forms β,γ, we have $\partial\beta=\partial\gamma=0$ for bidegree reasons, hence $d\beta=d\gamma=0$, so $\beta+\bar\gamma$ is d-closed and therefore it defines a De Rham class. To show that $F^\star$ is independent of the choice of representatives of the classes $[\beta]_{\bar\partial}, [\gamma]_{\bar\partial}$, suppose that $\beta=\bar\partial u$ and $\gamma=\bar\partial v$ for some $(n,\,n-2)$-forms u, v. Since $\partial u=\partial v=0$ for bidegree reasons, we see that $\beta=du$ and $\gamma=dv$, hence $\beta+\bar\gamma = d(u+\bar v)$, so $\{\beta+\bar\gamma\}_{DR}=0$. We conclude that $F^\star$ is well defined.

We now prove that the maps F and $F^\star$ are dual to each other. Under the Serre duality $H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C}) \simeq (H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C}))^\star$, every pair $([\beta]_{\bar\partial},\,\overline{[\gamma]_{\bar\partial}}) \in H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C}) \oplus \overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})}$ can be identified with the pair $(u,\,\bar v)$ in which $u, v : H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C}) \to \mathbb{C}$ are the $\mathbb{C}$-linear maps acting as

$$u([\alpha^{0,\,1}]_{\bar\partial}) = \int\limits_X \beta\wedge\alpha^{0,\,1} \quad \text{and} \quad v([\alpha^{0,\,1}]_{\bar\partial}) = \int\limits_X \gamma\wedge\alpha^{0,\,1}$$

for every class $[\alpha^{0,\,1}]_{\bar\partial} \in H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})$ and $\bar v : \overline{H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})} \to \mathbb{C}$ is the $\mathbb{C}$-linear map defined by $\bar v(\overline{[\alpha^{0,\,1}]_{\bar\partial}}) := \overline{v([\alpha^{0,\,1}]_{\bar\partial})}$. Proving the duality between F and $F^\star$ amounts to proving that

$$\sigma_{\beta+\bar{\gamma}} = (u+\bar{v})\circ F \tag{4.32}$$

for any $[\beta]_{\bar{\partial}}, [\gamma]_{\bar{\partial}} \in H^{n,\,n-1}_{\bar{\partial}}(X,\,\mathbb{C})$, where $\sigma_{\beta+\bar{\gamma}} : H^1_{DR}(X,\,\mathbb{C}) \to \mathbb{C}$ is the $\mathbb{C}$-linear map representing the De Rham class

$$F^\star([\beta]_{\bar{\partial}},\,\overline{[\gamma]_{\bar{\partial}}}) = \{\beta+\bar{\gamma}\}_{DR} \in H^{2n-1}_{DR}(X,\,\mathbb{C}) \simeq (H^1_{DR}(X,\,\mathbb{C}))^\star$$

under Poincaré duality. By definition, this means that for every $\{\alpha\}_{DR} \in H^1_{DR}(X,\,\mathbb{C})$ we have

$$\sigma_{\beta+\bar{\gamma}}(\{\alpha\}_{DR}) = \int\limits_X (\beta+\bar{\gamma})\wedge\alpha = \int\limits_X \beta\wedge\alpha^{0,\,1} + \int\limits_X \bar{\gamma}\wedge\alpha^{1,\,0} = \bigg((u+\bar{v})\circ F\bigg)(\{\alpha\}_{DR}).$$

This proves (4.32). We conclude that F and $F^\star$ are dual to each other.

(iii) Let us first prove the implication " $\implies$ ". Suppose that X is an sGG manifold. Let $[\beta]_{\bar{\partial}}, [\gamma]_{\bar{\partial}} \in H^{n,\,n-1}_{\bar{\partial}}(X,\,\mathbb{C})$ such that $\beta+\bar{\gamma} = d(\Omega^{n,\,n-2}+\Omega^{n-1,\,n-1}+\Omega^{n-2,\,n})$ for some forms $\Omega^{p,\,q}$ of the specified bidegrees. This amounts to having

$$\beta = \bar{\partial}\Omega^{n,\,n-2} + \partial\Omega^{n-1,\,n-1} \quad \text{and} \quad \bar{\gamma} = \bar{\partial}\Omega^{n-1,\,n-1} + \partial\Omega^{n-2,\,n},$$

therefore to having

$$\beta - \bar{\partial}\Omega^{n,\,n-2} = \partial\Omega^{n-1,\,n-1} \quad \text{and} \quad \gamma - \bar{\partial}\overline{\Omega^{n-2,\,n}} = \partial\overline{\Omega^{n-1,\,n-1}}.$$

Now, $\beta - \bar{\partial}\Omega^{n,\,n-2} \in [\beta]_{\bar{\partial}}$ and $\gamma - \bar{\partial}\overline{\Omega^{n-2,\,n}} \in [\gamma]_{\bar{\partial}}$. On the other hand, $\partial\Omega^{n-1,\,n-1}$ and $\partial\overline{\Omega^{n-1,\,n-1}}$ are d-closed and ∂-exact $(n,\,n-1)$-forms on X, so the sGG assumption on X implies (thanks to (iii) of Lemma 4.3.3) that they are both $\bar{\partial}$-exact, i.e. $[\beta]_{\bar{\partial}} = [\gamma]_{\bar{\partial}} = 0$ in $H^{n,\,n-1}_{\bar{\partial}}(X,\,\mathbb{C})$. We conclude that $F^\star$ is injective if X is sGG.

We now prove the reverse implication "$\Longleftarrow$". Suppose that $F^\star$ is injective. We will show that every Gauduchon metric on X is actually strongly Gauduchon. This will imply that X is an sGG manifold thanks to (iv) of Lemma 4.3.3.

Let ω be any Gauduchon metric on X. Then $\partial\omega^{n-1} \in \ker\bar{\partial}$, so we have a Dolbeault class $[\partial\omega^{n-1}]_{\bar{\partial}} \in H^{n,\,n-1}_{\bar{\partial}}(X,\,\mathbb{C})$. Now, $\partial\omega^{n-1} + \overline{\partial\omega^{n-1}} = d\omega^{n-1}$ and, moreover,

$$F^\star([\partial\omega^{n-1}]_{\bar{\partial}},\,\overline{[\partial\omega^{n-1}]_{\bar{\partial}}}) = \{\partial\omega^{n-1} + \overline{\partial\omega^{n-1}}\}_{DR} = \{d\omega^{n-1}\}_{DR} = 0.$$

Since $F^\star$ is supposed injective, we infer that $[\partial\omega^{n-1}]_{\bar{\partial}} = 0$, i.e. ω is strongly Gauduchon. $\square$

As a consequence of Theorem 4.3.8, we get the following

Corollary 4.3.10. *([PU18, Corollary 1.7]) Let $(X_t)_{t\in\Delta}$ be any holomorphic family of compact complex manifolds. Fix an arbitrary $t_0 \in \Delta$. If X_{t_0} is an sGG manifold, then:*

(i) X_t is an sGG manifold for all $t\in\Delta$ close enough to t_0;

(ii) $h^{0,\,1}_{\bar{\partial}}(t) = h^{0,\,1}_{\bar{\partial}}(t_0)$ and $h^{0,\,1}_{BC}(t) = h^{0,\,1}_{BC}(t_0)$ for all $t\in\Delta$ close enough to t_0.

Proof. We will use the (local) invariance of the Betti numbers of the fibres in a C^∞ family of compact complex manifolds and the upper-semicontinuity of the Hodge numbers $h^{p,\,q}(t)$ as t varies in Δ (cf. [KS60, Theorem 4]). Indeed, if X_{t_0} is an sGG manifold, we have:

$$b_1 = 2h^{0,1}_{\bar\partial}(t_0) \geq 2h^{0,1}_{\bar\partial}(t) \geq b_1 \quad \text{for all } t \text{ sufficiently close to } t_0.$$

Thus, we must have equalities $b_1 = 2h^{0,1}_{\bar\partial}(t_0) = 2h^{0,1}_{\bar\partial}(t)$ for all t close to t_0. In particular, by Theorem 4.3.8, X_t must be an sGG manifold for all t close to t_0. Then Theorem 4.3.4 implies $h^{0,1}_{BC}(t_0) = h^{0,1}_{BC}(t)$ for t close to t_0. □

Another consequence of Theorem 4.3.8 is the following

Corollary 4.3.11. *([PU18, Corollary 1.8]) Let $\mu : \widetilde{X} \to X$ be a holomorphic bimeromorphic map between compact complex manifolds $\widetilde{X}$ and X. The following equivalence holds:*

$$\widetilde{X} \text{ is an sGG manifold} \quad \iff \quad X \text{ is an sGG manifold.}$$

Proof. Thanks to the characterisation of sGG manifolds given in Theorem 4.3.8, it suffices to ensure the invariance of b_1 and of $h^{0,1}_{\bar\partial}$ under modifications, both of which are standard. Indeed, the fundamental group is known to be a bimeromorphic invariant of complex manifolds, hence so is its abelianisation H_1, so also b_1.

We recall for the reader's convenience the well-known argument showing the modification invariance of every $h^{0,k}_{\bar\partial}$ for any compact complex manifold (not necessarily satisfying the Hodge symmetry)[2]. This invariance follows from the combination of two things. The first thing is the following standard fact (cf. e.g. [Har77]) giving the vanishing of the higher direct image sheaves of the structural sheaf under modifications:

Let $f : X \to Y$ be a **bimeromorphic** *morphism between (smooth) compact complex manifolds. Then:*

(i) $f_\star\mathcal{O}_X = \mathcal{O}_Y$;

(ii) $R^i f_\star\mathcal{O}_X = 0$ *for all* $i > 0$.

The second thing is the Leray spectral sequence associated with f and $\mathcal{O}_X$. Recall that this is the spectral sequence starting at $E_2^{p,q} := H^p(Y, R^q f_\star\mathcal{O}_X)$ and converging to $H^{p+q}(X, \mathcal{O}_X)$. The shape of the direct image sheaves of $\mathcal{O}_X$ under f implies at once that

$$E_2^{p,0} = H^p(Y, \mathcal{O}_Y) \simeq H^{0,p}(Y, \mathbb{C}) \quad \text{and} \quad E_2^{p,q} = 0, \ q \geq 1.$$

It follows that the Leray spectral sequence degenerates at E_2 and we have

$$H^{0,k}(X, \mathbb{C}) = H^k(X, \mathcal{O}_X) \simeq \bigoplus_{p+q=k} E_2^{p,q} = E_2^{k,0} \simeq H^{0,k}(Y, \mathbb{C}) \quad \text{for all } k.$$

□

4.3.5 The cones $\mathcal{G}_X$ and $\mathcal{E}_X$ under deformations of sGG manifolds

In connection with the conjecture mentioned at the beginning of §.4.3.1 and the second step of the approach to it outlined there, we prove the following semi-continuity properties of the pseudo-effective and Gauduchon cones in families of sGG manifolds.

[2]This simple argument was pointed out to the author by A. Fujiki and F. Campana.

Theorem 4.3.12. *([PU18, Theorem 1.9]) Let $(X_t)_{t\in\Delta}$ be any holomorphic family of* **sGG** *compact complex manifolds. Then $\mathcal{G}_{X_t}$ behaves* **lower-semicontinuously**, *while $\mathcal{E}_{X_t}$ behaves* **upper-semicontinuously** *w.r.t. the usual topology of Δ as $t\in\Delta$ varies.*

More precise statements will be given in Theorems 4.3.19 and 4.3.26.

As usual, for any differential form Ω of any degree k and for any $(p,\,q)$ such that $p+q=k$, we denote by $\Omega^{p,\,q}$ the component of Ω of bidegree $(p,\,q)$. Thus $\Omega=\sum_{p+q=k}\Omega^{p,\,q}$.

It will be seen in (II) below that the discussion of the variation of the cones $\mathcal{G}_X$ and $\mathcal{E}_X$ under deformations of the complex structure of a compact sGG manifold X would be greatly simplified if the Bott-Chern number $h^{1,1}_{BC}(X)$ were locally deformation constant. Unfortunately, this is not the case as Proposition 4.3.32 will show, rendering indispensable the introduction of some technical work in (II).

(I) Fake Hodge-Aeppli decomposition of $H^{2n-2}_{DR}(X,\,\mathbb{R})$ when X is sGG

If our n-dimensional compact complex manifold X were supposed to be a $\partial\bar\partial$-manifold, there would exist a canonical isomorphism $H^{2n-2}_{DR}(X,\,\mathbb{C})\simeq H^{n,\,n-2}_A(X,\,\mathbb{C})\oplus H^{n-1,\,n-1}_A(X,\,\mathbb{C})\oplus H^{n-2,\,n}_A(X,\,\mathbb{C})$ (cf. [Pop13b] where this splitting was called a Hodge-Aeppli decomposition), hence in particular a canonical surjection $H^{2n-2}_{DR}(X,\,\mathbb{C})\twoheadrightarrow H^{n-1,\,n-1}_A(X,\,\mathbb{C})$ and a canonical injection $H^{n-1,\,n-1}_A(X,\,\mathbb{C})\hookrightarrow H^{2n-2}_{DR}(X,\,\mathbb{C})$ which is a section of the surjection. However, under the weaker sGG assumption on X, a complete Hodge-Aeppli decomposition in degree $2n-2$ need not exist, but we will show that a weaker substitute thereof (that will prove sufficient for our purposes later on) exists: a canonical surjection and a non-canonical but naturally-associated-with-any-given-metric injection as above exist if we restrict attention to the real cohomologies.

We start by noticing the existence of the canonical surjection.

Proposition 4.3.13. *([PU18, Proposition 5.1]) Let X be an arbitrary compact complex manifold, $dim_{\mathbb{C}}X=n$. The following canonical linear map*

$$P\,:\,H^{2n-2}_{DR}(X,\,\mathbb{R})\to H^{n-1,\,n-1}_A(X,\,\mathbb{R}),\quad \{\Omega\}_{DR}\mapsto[\Omega^{n-1,\,n-1}]_A, \tag{4.33}$$

is well defined. Furthermore, if X is an sGG manifold, P is surjective.

Proof. Let $\Omega=\Omega^{n,\,n-2}+\Omega^{n-1,\,n-1}+\Omega^{n-2,\,n}$ be any d-closed (not necessarily real) C^∞ form of degree $2n-2$. We have:

$$\begin{aligned} d\Omega=0 \quad &\iff \quad \partial\Omega^{n-1,\,n-1}+\bar\partial\Omega^{n,\,n-2}=0 \text{ and } \partial\Omega^{n-2,\,n}+\bar\partial\Omega^{n-1,\,n-1}=0\\ &\implies \quad \partial\bar\partial\Omega^{n-1,\,n-1}=0. \end{aligned} \tag{4.34}$$

The last identity shows that $\Omega^{n-1,\,n-1}$ defines indeed an Aeppli cohomology class of bidegree $(n-1,\,n-1)$. To show well-definedness for P, we still have to show that the definition is independent of the choice of representative of the De Rham class $\{\Omega\}_{DR}$. Let Ω_1,Ω_2 represent the same De Rham class, i.e. $\Omega:=\Omega_1-\Omega_2$ is d-exact. Then there exists a $(2n-3)$-form Γ such that $\Omega=d(\Gamma^{n,\,n-3}+\Gamma^{n-1,\,n-2}+\Gamma^{n-2,\,n-1}+\Gamma^{n-3,\,n})$. Thus, the d-exactness of a $(2n-2)$-form Ω is equivalent to the existence of $\Gamma\in C^\infty_{2n-3}(X,\,\mathbb{C})$ such that

$$\begin{aligned} \Omega^{n,\,n-2} &\overset{(i)}{=} \partial\Gamma^{n-1,\,n-2}+\bar\partial\Gamma^{n,\,n-3}\\ \Omega^{n-1,\,n-1} &\overset{(ii)}{=} \partial\Gamma^{n-2,\,n-1}+\bar\partial\Gamma^{n-1,\,n-2}\\ \Omega^{n-2,\,n} &\overset{(iii)}{=} \partial\Gamma^{n-3,\,n}+\bar\partial\Gamma^{n-2,\,n-1}. \end{aligned} \tag{4.35}$$

Identity (ii) above means that $[\Omega^{n-1,\,n-1}]_A = 0$, i.e. $[\Omega_1^{n-1,\,n-1}]_A = [\Omega_2^{n-1,\,n-1}]_A$.

Let us now suppose that X is an sGG manifold. Pick any class $[\Omega^{n-1,\,n-1}]_A \in H_A^{n-1,\,n-1}(X,\,\mathbb{R})$ with $\Omega^{n-1,\,n-1}$ *real*. Then $\partial\bar\partial\Omega^{n-1,\,n-1} = 0$, hence $d(\partial\Omega^{n-1,\,n-1}) = 0$. Since $\partial\Omega^{n-1,\,n-1}$ is a ∂-exact d-closed $(n,\,n-1)$-form, the sGG assumption on X implies that $\partial\Omega^{n-1,\,n-1}$ is $\bar\partial$-exact (see Lemma 4.3.3). Thus, there exists an $(n,\,n-2)$-form $\Omega^{n,\,n-2}$ such that

$$\partial\Omega^{n-1,\,n-1} = -\bar\partial\Omega^{n,\,n-2}.$$

Hence, since $\Omega^{n-1,\,n-1}$ is real, $\bar\partial\Omega^{n-1,\,n-1} = -\partial\overline{\Omega^{n,\,n-2}}$. Therefore, the $(2n-2)$-form $\Omega := \Omega^{n,\,n-2} + \Omega^{n-1,\,n-1} + \overline{\Omega^{n,\,n-2}}$ is real and $d\Omega = 0$ (cf. (4.34)). It is clear that $P(\{\Omega\}_{DR}) = [\Omega^{n-1,\,n-1}]_A$. This proves that P is surjective. $\square$

Corollary 4.3.14. *([PU18, Corollary 5.2]) If X is an sGG compact complex manifold with $\dim_{\mathbb{C}} X = n$, the dual map of P:*

$$P^\star\,:\,H_{BC}^{1,\,1}(X,\,\mathbb{R}) \to H_{DR}^2(X,\,\mathbb{R}),\quad [\alpha]_{BC} \mapsto \{\alpha\}_{DR}, \tag{4.36}$$

is injective. (Of course, $P^\star$ is canonically well defined for any X but it need not be injective if X is not sGG.)

That $P^\star$ is indeed the dual map of P follows immediately from the identity $\int_X \alpha\wedge\Omega = \int_X \alpha\wedge\Omega^{n-1,\,n-1}$ which holds for bidegree reasons for any class $[\alpha]_{BC} \in H_{BC}^{1,\,1}(X,\,\mathbb{R}))$ and any class $\{\Omega\}_{DR} \in H_{DR}^{2n-2}(X,\,\mathbb{R})$.

No canonical right inverse of P need exist when X is only an sGG manifold, but for any given Hermitian metric on X we will now construct a right inverse of P depending on the chosen metric.

Definition 4.3.15. *([PU18, Definition 5.3]) Let X be an sGG compact complex manifold with $\dim_{\mathbb{C}} X = n$ and let ω be an arbitrary Hermitian metric on X. With ω we associate the following injective linear map*

$$Q_\omega\,:\,H_A^{n-1,\,n-1}(X,\,\mathbb{R}) \to H_{DR}^{2n-2}(X,\,\mathbb{R}),\quad [\Omega^{n-1,\,n-1}]_A \mapsto \{\Omega\}_{DR}, \tag{4.37}$$

where the real d-closed $(2n-2)$-form Ω on X is determined by a given real $\partial\bar\partial$-closed $(n-1,\,n-1)$-form $\Omega^{n-1,\,n-1}$ and by the metric ω in the following way.

(i) If Δ_A denotes the Aeppli Laplacian associated with ω, we have an orthogonal (w.r.t. the L^2 inner product defined by ω) splitting

$$\ker(\partial\bar\partial) = \ker\Delta_A \oplus (Im\,\partial + Im\,\bar\partial) \qquad \textit{(see e.g. [Pop13, §.2])},$$

which induces a splitting of $\Omega^{n-1,\,n-1} \in \ker(\partial\bar\partial)$ as

$$\Omega^{n-1,\,n-1} = \Omega_A^{n-1,\,n-1} + \partial\Gamma^{n-2,\,n-1} + \bar\partial\Gamma^{n-1,\,n-2}, \tag{4.38}$$

where $\Delta_A\Omega_A^{n-1,\,n-1} = 0$. The forms $\Gamma^{n-2,\,n-1}$, $\Gamma^{n-1,\,n-2}$ are of the shown bidegrees and are not uniquely determined, but we will see that the definition of Q_ω does not depend on their choices. (Since $\Omega^{n-1,\,n-1}$ is real, we can always choose $\Gamma^{n-2,\,n-1} = \overline{\Gamma^{n-1,\,n-2}}$.)

(ii) Since $\bar\partial(\partial\Omega_A^{n-1,\,n-1}) = 0$, we also have $d(\partial\Omega_A^{n-1,\,n-1}) = 0$. Thus the sGG assumption on X and Lemma 4.3.3 ensure that $\partial\Omega_A^{n-1,\,n-1}$ is $\bar\partial$-exact, i.e. there exists a smooth $(n,\,n-2)$-form $\Omega_A^{n,\,n-2}$ such that

$$\partial\Omega_A^{n-1,\,n-1} = \bar\partial(-\Omega_A^{n,\,n-2}). \tag{4.39}$$

We choose $\Omega_A^{n,\,n-2}$ *to be the solution of equation (4.39) of minimal* L^2*-norm (defined by* ω*). Thus,* $\Omega_A^{n,\,n-2}$ *is uniquely determined by the formula*

$$\Omega_A^{n,\,n-2} = -\bar\partial^\star\Delta''^{-1}(\partial\Omega_A^{n-1,\,n-1}), \tag{4.40}$$

where $\Delta'' = \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial$ *is the* $\bar\partial$*-Laplacian associated with the given metric* ω *and* Δ''^{-1} *is its Green operator (i.e. the inverse of its restriction to the orthogonal complement of its kernel).*

(iii) Taking ∂ *in (4.38) and using (4.39), we get:*

$$\partial\Omega^{n-1,\,n-1} = \partial\Omega_A^{n-1,\,n-1} + \partial\bar\partial\Gamma^{n-1,\,n-2} = \bar\partial(-\Omega_A^{n,\,n-2} - \partial\Gamma^{n-1,\,n-2}).$$

We set $\Omega^{n,\,n-2} := \Omega_A^{n,\,n-2} + \partial\Gamma^{n-1,\,n-2}$*. Thus we get:*

$$\partial\Omega^{n-1,\,n-1} = \bar\partial(-\Omega^{n,\,n-2}), \quad \textit{hende also} \quad \bar\partial\Omega^{n-1,\,n-1} = \partial(-\overline{\Omega^{n,\,n-2}}), \tag{4.41}$$

where the latter identity follows from the former by taking conjugates and using the fact that $\Omega^{n-1,\,n-1}$ *is real.*

(iv) We set $\Omega := \Omega^{n,\,n-2} + \Omega^{n-1,\,n-1} + \overline{\Omega^{n,\,n-2}}$*. It is clear that* Ω *is a real* $(2n-2)$*-form on* X *and* $d\Omega = 0$ *(compare (4.41) with (4.34)).*

We now make the trivial observation that Definition 4.3.15 is correct.

Lemma 4.3.16. *([PU18, Lemma 5.4]) The map* Q_ω *is well defined and injective. Moreover, the composed linear map* $P\circ Q_\omega : H_A^{n-1,\,n-1}(X,\,\mathbb{R}) \to H_A^{n-1,\,n-1}(X,\,\mathbb{R})$ *is the identity map of* $H_A^{n-1,\,n-1}(X,\,\mathbb{R})$ *(so* Q_ω *is a section of* P*).*

Proof. For well-definedness, we need to show that $Q_\omega([\Omega^{n-1,\,n-1}]_A)$ does not depend either on the choice of representative of the Aeppli class $[\Omega^{n-1,\,n-1}]_A$ or on the choice of the forms $\Gamma^{n-2,\,n-1}, \Gamma^{n-1,\,n-2}$ in (4.38). Let us consider two *real* representatives of a same *real* Aeppli class:

$$[\Omega_1^{n-1,\,n-1}]_A = [\Omega_2^{n-1,\,n-1}]_A.$$

Let $\Omega_j = \Omega_j^{n,\,n-2} + \Omega_j^{n-1,\,n-1} + \overline{\Omega_j^{n,\,n-2}}$ $(j=1,2)$ be the real d-closed $(2n-2)$-forms on X determined by $\Omega_j^{n-1,\,n-1}$ and ω as described in Definition 4.3.15.

Since the Δ_A-harmonic representative of a given Aeppli class is unique, we infer that $\Omega_{1,\,A}^{n-1,\,n-1} = \Omega_{2,\,A}^{n-1,\,n-1}$ (i.e. $\Omega_1^{n-1,\,n-1}$ and $\Omega_2^{n-1,\,n-1}$ have the same Δ_A-harmonic projection). This implies that $\Omega_{1,\,A}^{n,\,n-2} = \Omega_{2,\,A}^{n,\,n-2}$ since the solution of minimal L^2-norm of a $\bar\partial$-equation (equation (4.39) here) is unique. This further implies that

$$\Omega_1^{n,\,n-2} - \Omega_2^{n,\,n-2} = \partial(\Gamma_1^{n-1,\,n-2} - \Gamma_2^{n-1,\,n-2}). \tag{4.42}$$

On the other hand, (4.38) spells

$$\Omega_j^{n-1,\,n-1} = \Omega_{j,\,A}^{n-1,\,n-1} + \partial\Gamma_j^{n-2,\,n-1} + \bar\partial\Gamma_j^{n-1,\,n-2}, \qquad j=1,2,$$

which gives, since $\Omega_{1,\,A}^{n-1,\,n-1} = \Omega_{2,\,A}^{n-1,\,n-1}$, the identity

$$\Omega_1^{n-1,\,n-1} - \Omega_2^{n-1,\,n-1} = \partial(\Gamma_1^{n-2,\,n-1} - \Gamma_2^{n-2,\,n-1}) + \bar\partial(\Gamma_1^{n-1,\,n-2} - \Gamma_2^{n-1,\,n-2}). \tag{4.43}$$

We see that (4.42) and (4.43) amount to the d-exactness condition (4.35) for the real $(2n-2)$-form $\Omega := \Omega_1 - \Omega_2$ (where we choose $\Gamma^{n,\,n-3} = 0$ and $\Gamma^{n-3,\,n} = 0$). Thus $\Omega_1 - \Omega_2$ is d-exact, i.e. $\{\Omega_1\}_{DR} = \{\Omega_2\}_{DR}$, so $Q_\omega([\Omega_1^{n-1,\,n-1}]_A) = Q_\omega([\Omega_2^{n-1,\,n-1}]_A)$.

To show that Q_ω is injective, let $Q_\omega([\Omega^{n-1,\,n-1}]_A)=0$, i.e. $\{\Omega\}_{DR}=0$. This means that Ω is d-exact, which in turn means that the identities (4.35) hold. It is clear that (ii) of (4.35) expresses the fact that $[\Omega^{n-1,\,n-1}]_A=0$.

The fact that $P\circ Q_\omega([\Omega^{n-1,\,n-1}]_A)=[\Omega^{n-1,\,n-1}]_A$ for any class $[\Omega^{n-1,\,n-1}]_A\in H^{n-1,\,n-1}_A(X,\,\mathbb{R})$ follows immediately from the definitions of P and Q_ω: the original $(n-1,\,n-1)$-form $\Omega^{n-1,\,n-1}$ is indeed the $(n-1,\,n-1)$-component of the $(2n-2)$-form constructed from $\Omega^{n-1,\,n-1}$ in Definition 4.3.15. $\square$

Putting these pieces of information together, we immediately get the

Corollary 4.3.17. *([PU18, Corollary 5.5]) Let X be an sGG compact complex manifold, $dim_{\mathbb{C}}X=n$. For any Hermitian metric ω on X, the dual map of Q_ω:*

$$Q_\omega^\star\,:\,H^2_{DR}(X,\,\mathbb{R})\rightarrow H^{1,\,1}_{BC}(X,\,\mathbb{R}) \tag{4.44}$$

is surjective. Moreover, the composition $Q_\omega^\star\circ P^\star\,:\,H^{1,\,1}_{BC}(X,\,\mathbb{R})\rightarrow H^{1,\,1}_{BC}(X,\,\mathbb{R})$ is the identity map.

Note that the dual map $Q_\omega^\star$ has the following explicit form

$$\int\limits_X Q_\omega^\star(\{\alpha\}_{DR})\wedge[\Omega^{n-1,\,n-1}]_A=\int\limits_X\{\alpha\}_{DR}\wedge Q_\omega([\Omega^{n-1,\,n-1}]_A) \tag{4.45}$$

for any classes $\{\alpha\}_{DR}\in H^2_{DR}(X,\,\mathbb{R})$ and $[\Omega^{n-1,\,n-1}]_A\in H^{n-1,\,n-1}_A(X,\,\mathbb{R})$. (The meaning of cohomology classes in (4.45) is that the integrals do not depend on the choice of representatives in those classes.)

We can well call the pair of maps $(P,\,Q_\omega)$ a *fake Hodge-Aeppli decomposition* of $H^{2n-2}_{DR}(X,\,\mathbb{R})$ and the dual pair of maps $(P^\star,\,Q_\omega^\star)$ the *dual fake Hodge-Bott-Chern decomposition* of $H^2_{DR}(X,\,\mathbb{R})$.

(II) Deformation semicontinuity of $\mathcal{G}_X$ and $\mathcal{E}_X$ when X is sGG

We now use the fake Hodge decomposition of the previous subsection in the context of small deformations of an sGG complex structure.

Let $\pi\,:\,\mathcal{X}\rightarrow\Delta$ be a holomorphic family of compact complex manifolds. Without loss of generality, we may suppose that $\Delta\subset\mathbb{C}$ is an open disc about the origin. The fibres $X_t:=\pi^{-1}(t)\subset\mathcal{X}$ $(t\in\Delta)$ are thus compact complex manifolds of equal dimension n and the family is C^∞ locally trivial, hence the De Rham cohomology groups of the fibres can be identified with a fixed space $H^k(X,\,\mathbb{C})$ for every $k=0,1,\dots,2n$. As the complex structure of X_t varies with $t\in\Delta$, the Bott-Chern, Dolbeault and Aeppli cohomologies of the fibres depend on t.

Suppose moreover that X_0 is an sGG manifold. Then X_t is an sGG manifold for all $t\in\Delta$ sufficiently close to 0 by our Corollary 4.3.10. After shrinking Δ about 0, we can assume that X_t is an sGG manifold for all $t\in\Delta$. We fix any C^∞ family $(\omega_t)_{t\in\Delta}$ of Hermitian metrics on the respective fibres $(X_t)_{t\in\Delta}$. Let $t_0\in\Delta$ be an arbitrary point (e.g. we take $t_0=0$).

(1) *Variation of the Gauduchon cone*

The *fake Hodge-Aeppli decomposition* constructed in the previous subsection on each fibre X_t gives us maps as follows:

$$\begin{array}{ccccc} H_A^{n-1,\,n-1}(X_0,\,\mathbb{R}) & \overset{Q_{\omega_0}}{\hookrightarrow} & H_{DR}^{2n-2}(X,\,\mathbb{R}) & \overset{P_t}{\twoheadrightarrow} & H_A^{n-1,\,n-1}(X_t,\,\mathbb{R}), \quad t\in\Delta. \\ \bigcup & & & & \bigcup \\ \mathcal{G}_{X_0} & & & & \mathcal{G}_{X_t} \end{array}$$

Thus the image of the Gauduchon cone $\mathcal{G}_{X_0}$ of X_0 under the composition $P_t\circ Q_{\omega_0}\,:\,H_A^{n-1,\,n-1}(X_0,\,\mathbb{R})\to H_A^{n-1,\,n-1}(X_t,\,\mathbb{R})$ can be compared to $\mathcal{G}_{X_t}$ as subsets of $H_A^{n-1,\,n-1}(X_t,\,\mathbb{R})$. Note that $P_0\circ Q_{\omega_0} = \mathrm{Id}_{H_A^{n-1,\,n-1}(X_0,\,\mathbb{R})}$ and it follows from [KS60, Theorem 5] that the surjections $(P_t)_{t\in\Delta}$ vary in a C^∞ way with t (hence the maps $P_t\circ Q_{\omega_0}$ are isomorphisms) if the dimension of $H_A^{n-1,\,n-1}(X_t,\,\mathbb{R})$ $(= h_{BC}^{1,1}(t)$ by duality) is independent of t. However, if $h_{BC}^{1,1}(0) > h_{BC}^{1,1}(t)$ for $t\neq 0$ close to 0, the Gauduchon cone $\mathcal{G}_{X_0}$ of X_0 has more dimensions than its counterparts $\mathcal{G}_{X_t}$ on the nearby fibres, but the projections P_t for $t\neq 0$ eliminate the extra dimensions of $(P_t\circ Q_{\omega_0})(\mathcal{G}_{X_0})$. It seems sensible to introduce the following definition.

Definition 4.3.18. *([PU18, DEfinition 5.6]) If $(X_t)_{t\in\Delta}$ is a holomorphic family of* **sGG** *compact complex n-dimensional manifolds, the* **limit as $t\to t_0$ of the Gauduchon cones** *$\mathcal{G}_{X_t}$ of the fibres X_t for $t\neq t_0$ is defined as the following subset of $H_A^{n-1,\,n-1}(X_{t_0},\,\mathbb{R})$:*

$$\lim_{t\to t_0}\mathcal{G}_{X_t} := \left\{[\Omega^{n-1,\,n-1}]_A\in H_A^{n-1,\,n-1}(X_{t_0},\,\mathbb{R})\ \mid\ (P_t\circ Q_{\omega_{t_0}})([\Omega^{n-1,\,n-1}]_A)\in\mathcal{G}_{X_t}\ \ \forall t\sim t_0\right\},$$

where "$\forall t\sim t_0$" means "for all t sufficiently close to t_0".

Note that $\lim_{t\to t_0}\mathcal{G}_{X_t}$ depends on the metric ω_{t_0} (since $Q_{\omega_{t_0}}$ depends thereon) but does not depend on the way in which ω_{t_0} has been extended in a C^∞ fashion to metrics ω_t on the nearby fibres (since the maps P_t are canonical).

We can now prove that the Gauduchon cone $\mathcal{G}_{X_t}$ of the sGG fibre X_t behaves lower-semicontinuously w.r.t. $t\in\Delta$ in the usual topology of Δ much as it did in the special case of families of $\partial\bar\partial$-manifolds treated in [Pop13b].

Theorem 4.3.19. *([PU18, Theorem 5.7]) Let $(X_t)_{t\in\Delta}$ be any holomorphic family of* **sGG** *compact complex manifolds endowed with any C^∞ family $(\omega_t)_{t\in\Delta}$ of Hermitian metrics.*

Then, for all $t_0\in\Delta$, the following inclusion holds:

$$\mathcal{G}_{X_{t_0}}\subset\lim_{t\to t_0}\mathcal{G}_{X_t}.$$

Proof. We may assume that $t_0=0$. Denote by n the complex dimension of the fibres and let $[\gamma_0^{n-1}]_A\in\mathcal{G}_{X_0}$ for some Gauduchon metric $\gamma_0>0$ on X_0. Let Ω be the C^∞ real d-closed $(2n-2)$-form determined by γ_0^{n-1} and by the Hermitian metric ω_0 as in Definition 4.3.15. For every $t\in\Delta$, the splitting of Ω into pure-type forms reads:

$$\Omega = \Omega_t^{n,\,n-2} + \Omega_t^{n-1,\,n-1} + \Omega_t^{n-2,\,n},\quad t\in\Delta,$$

where $\Omega_0^{n-1,\,n-1}=\gamma_0^{n-1}$. Then $(P_t\circ Q_{\omega_0})([\gamma_0^{n-1}]_A) = P_t(\{\Omega\}_{DR}) = [\Omega_t^{n-1,\,n-1}]_A\in H_A^{n-1,\,n-1}(X_t,\,\mathbb{R})$ for all $t\in\Delta$. Since $\Omega_0^{n-1,\,n-1}>0$ and the $\Omega_t^{n-1,\,n-1}$ vary in a C^∞ way with $t\in\Delta$ (as components of the fixed form Ω), we get

$$\Omega_t^{n-1,\,n-1}>0\quad\text{for all } t \text{ sufficiently close to } 0,$$

hence there exists a unique Gauduchon metric γ_t on X_t such that $\gamma_t^{n-1} = \Omega_t^{n-1,\,n-1}$, so $[\Omega_t^{n-1,\,n-1}]_A \in \mathcal{G}_{X_t}$ for all t close to 0. Thus $[\gamma_0^{n-1}]_A \in \lim\limits_{t\to t_0} \mathcal{G}_{X_t}$.

□

(2) *Dual situation: variation of the pseudo-effective cone*

The dual of the *fake Hodge-Aeppli decomposition* constructed in the previous subsection on each fibre X_t gives us maps as follows:

$$\begin{array}{ccccc} H^{1,\,1}_{BC}(X_0,\,\mathbb{R}) & \overset{P_0^\star}{\hookrightarrow} & H^2_{DR}(X,\,\mathbb{R}) & \overset{Q_{\omega_t}^\star}{\twoheadrightarrow} & H^{1,\,1}_{BC}(X_t,\,\mathbb{R}), \quad t\in\Delta. \\ \bigcup & & & & \bigcup \\ \mathcal{E}_{X_0} & & & & \mathcal{E}_{X_t} \end{array}$$

Clearly, $Q_{\omega_0}^\star \circ P_0^\star = \mathrm{Id}_{H^{1,\,1}_{BC}(X_0,\,\mathbb{R})}$ and it follows from [KS60, Theorem 5] that the surjections $(Q_{\omega_t}^\star)_{t\in\Delta}$ vary in a C^∞ way with t (hence the maps $Q_{\omega_t}^\star \circ P_0^\star$ are isomorphisms) if the dimension of $H^{1,\,1}_{BC}(X_t,\,\mathbb{R})$ $(= h^{1,\,1}_{BC}(t))$ is independent of t. However, if $h^{1,\,1}_{BC}(0) > h^{1,\,1}_{BC}(t)$ for $t \neq 0$ close to 0, the pseudo-effective cone $\mathcal{E}_{X_0}$ of X_0 lies in a space which has a higher dimension than the ambient spaces of its counterparts $\mathcal{E}_{X_t}$ on the nearby fibres.

• We shall now define a complex vector subspace

$$H^{'1,\,1}_{BC}(X_0,\,\mathbb{C}) \subset H^{1,\,1}_{BC}(X_0,\,\mathbb{C}) \tag{4.46}$$

depending on the chosen family of Hermitian metrics $(\omega_t)_{t\in\Delta}$ such that:

· $\dim H^{'1,\,1}_{BC}(X_0,\,\mathbb{C}) = \dim H^{1,\,1}_{BC}(X_t,\,\mathbb{C})$ for all $t\in\Delta$ close to 0;

· $H^{'1,\,1}_{BC}(X_0,\,\mathbb{C}) = H^{1,\,1}_{BC}(X_0,\,\mathbb{C})$ when $h^{1,\,1}_{BC}(0) = h^{1,\,1}_{BC}(t)$ for t close to 0.

• We will need a general fact, which is essentially known but may not be written as such in the literature as far as we are aware, that will prove useful in other similar situations. We now proceed to explain it.

The starting point is the following classical result of Grauert's.

Theorem 4.3.20. *([Gra58])*[3] *Let $E \longrightarrow X$ be a* **real-analytic** *vector bundle over a* **real-analytic** *manifold X. Then, the space of* **real-analytic** *sections of E is* **dense** *in the space of C^∞ sections of E.*

Grauert proved this using the technique of Stein tubular neighbourhoods in the complexified manifold $\widetilde{X}$. As a consequence, we get the following

Corollary 4.3.21. *Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds $X_t := \pi^{-1}(t)$, with $t\in B$, over an open ball $B\subset\mathbb{C}^N$ about the origin for some $N\in\mathbb{N}^\star$.*

(1) There exists a **real-analytic** *family $(\omega_t)_{t\in B}$ of Hermitian metrics on the respective fibres X_t.*

(2) Taking adjoints and Laplacians w.r.t. to the ω_t's, the familiar differential operators:

$$\partial_t^\star,\ \ \bar\partial_t^\star,\ \ ,\Delta_t,\ \ ,\Delta'_t,\ \ ,\Delta''_t,\ \ ,\Delta_{BC,\,t},\ \ ,\Delta_{A,\,t},$$

vary in a **real-analytic** *way with $t\in B$.*

[3]The author is grateful to J.-P. Demailly for pointing out to him this result of Grauert's and for confirming that Corollary 4.3.21 holds as a consequence thereof.

(3) For any bidegree (p, q), any **real-analytic** *family $(P_t)_{t\in B}$ of* **elliptic** *differential operators $P_t : C^\infty_{p,q}(X_t, \mathbb{C}) \longrightarrow C^\infty_{p,q}(X_t, \mathbb{C})$ and for any* **Jordan curve** *$C \subset \mathbb{C}$ that contains $0 \in \mathbb{C}$ in its interior and does not meet the spectrum of P_0, there exists a small neighbourhood $\Delta \subset B$ of 0 in $\mathbb{C}$ such that C does not meet the spectrum of P_t for any $t \in \Delta$ and the vector bundle given by the Kodaira-Spencer theory presented in §.2.5.2:*

$$\Delta \ni t \mapsto \bigoplus_{\lambda(t)\in int(C)} E_{\lambda(t)}(P_t)$$

is **real-analytic***, where $E_{\lambda(t)}(P_t)$ is the $\lambda(t)$-eigenspace of P_t.*

Sketch of proof. Part (1) follows at once from Grauert's Theorem 4.3.20 and immediately implies part (2). Part (3) follows from parts (1) and (2) and from the Cauchy integral formula given in the Kodaira-Spencer Lemma 2.5.18, by integrating w.r.t. λ the Green operators $(P_t - \lambda\,\mathrm{Id})^{-1}$ on the Jordan curve. $\square$

The problem we will now address is the following.

Question 4.3.22. *Let $V \longrightarrow D \subset \mathbb{C}$ be a C^∞ vector bundle over an open disc about 0 in the complex plane. Suppose V is equipped with a C^∞ fibre metric and that $H : V \longrightarrow V$ is a C^∞ self-adjoint endomorphism of V (i.e. a C^∞ family of self-adjoint operators $H_t : V_t \longrightarrow V_t$) such that $H_t \geq 0$ for all $t \in D$.*

Suppose that $H_0 = 0$ (so, $\ker H_0 = V_0$) and that dim $\ker H_t <$ dim V_t for all $t \in D \setminus \{0\}$.

Does $\ker H_t$ have a limiting position when $D \setminus \{0\} \ni t \longrightarrow 0$?

By $\ker H_t$ having a limiting position as $t \in D \setminus \{0\}$ converges to 0 we mean that there exists a C^∞ vector bundle over an open subset Ω of D containing 0 whose fibre at every $t \in \Omega \setminus \{0\}$ is $\ker H_t$.

The answer to Question 4.3.22 is **negative** in general.

Counter-example 4.3.23. [4] *Let $V = \mathbb{C} \longrightarrow D \subset \mathbb{C}$ be the constant real vector bundle of rank 2 and let*

$$H_t : \mathbb{C} \longrightarrow \mathbb{C}, \quad z \longmapsto Re\,(\bar{t}z)\,z, \qquad t \in D.$$

Then,

$$\ker H_t = \{z = x + iy \in \mathbb{C} \mid ux + vy = 0\}, \qquad t = u + iv \in D \setminus \{0\},$$

with $x, y, u, v \in \mathbb{R}$.

Consequently, if we fix $\lambda \in \mathbb{R}$ and choose $t = u + iv = (\lambda + i)\,v \in D \setminus \{0\}$, with $v \in \mathbb{R} \setminus \{0\}$, then $\ker H_t$ is a real line d_λ depending only on λ for all $v \neq 0$, namely

$$\ker H_t = \{z = x + iy \in \mathbb{C} \mid \lambda x + y = 0\} := d_\lambda, \qquad t = (\lambda + i)\,\mathbb{R} \cap (D \setminus \{0\}).$$

Hence, when $t \in D \setminus \{0\}$ approaches 0 along the real line $(\lambda + i)\,\mathbb{R}$, $\ker H_t$ remains constant, equal to d_λ, so the limiting position of $\ker H_t$ is the line d_λ. However, d_λ changes into the different line d_μ when $\mu \neq \lambda$, so the limiting position of $\ker H_t$ is the different line d_μ when $t \in D \setminus \{0\}$ approaches 0 along the real line $(\mu + i)\,\mathbb{R}$.

Consequently, $\ker H_t$ has no limiting position when $t \in D \setminus \{0\}$ approaches 0 from all possible real directions.

[4] The author is very grateful to P. Deligne for pointing out to him this counter-example and the proof of the next Proposition 4.3.24.

What we can now show using Corollary 4.3.21 is that the answer to Question 4.3.22 becomes **affirmative** if the regularity of the vector bundle $V \longrightarrow D \subset \mathbb{C}$, of the fibre metric thereon and of the self-adjoint endomorphism $H : V \longrightarrow V$ is **real-analytic** and if the disc D is replaced by any **path** $\Gamma \subset D$.

Proposition 4.3.24. *Let $V \longrightarrow D \subset \mathbb{C}$ be a* **real-analytic** *$\mathbb{C}$-vector bundle over an open disc about 0 in the complex plane. Suppose V is equipped with a* **real-analytic** *fibre metric and that $H : V \longrightarrow V$ is a* **real-analytic** *Hermitian endomorphism of V (i.e. a* **real-analytic** *family of self-adjoint operators $H_t : V_t \longrightarrow V_t$) such that $H_t \geq 0$ for all $t \in D$.*

Suppose that $H_0 = 0$ (so, $\ker H_0 = V_0$) and that $\dim \ker H_t < \dim V_t$ for all $t \in D \setminus \{0\}$.

Then, for any **real curve** *$\Gamma \subset D$ ending at 0, $\ker H_t$ has a limiting position when $\Gamma \setminus \{0\} \ni t \to 0$.*

Proof. Restrict V to Γ and complexify to get a *holomorphic* vector bundle $\widetilde{V} \longrightarrow U \cap D$, where U is a neighbourhood of Γ in $\mathbb{C}$. Similarly, restrict H to Γ and complexify to get a *holomorphic* endomorphism $\widetilde{H} : \widetilde{V} \longrightarrow \widetilde{V}$. In particular,

$$\widetilde{V}_{|\Gamma} = V_{|\Gamma} \quad \text{and} \quad \widetilde{H}_{|\Gamma} = H_{|\Gamma}.$$

Then, $\ker \widetilde{H}$ is a *coherent* subsheaf of the locally free sheaf $\mathcal{O}(\widetilde{V})$ (because the kernel of a morphism of coherent sheaves is coherent). Hence, $\ker \widetilde{H}$ is also *torsion-free* (because any coherent subsheaf of a torsion-free sheaf is torsion-free). Similarly, $\widetilde{V}/\ker \widetilde{H}$ is a *torsion-free coherent sheaf* on $U \cap D$.

Now, every torsion-free coherent sheaf is locally free outside an analytic subset of codimension ≥ 2. (See e.g. [Kob87, V].) Since the complex dimension of $U \cap D$ is 1, we get that $\ker \widetilde{H}$ and $\widetilde{V}/\ker \widetilde{H}$ are *locally free* on $U \cap D$ and $\widetilde{V}/\ker \widetilde{H}$ is *locally a direct factor*.

Moreover, the fibre at t of the holomorphic vector bundle $\ker \widetilde{H}$ is $\ker H_t$ for all $t \in U \cap D$ except, possibly, on a discrete subset. We conclude that the fibre at $t = 0$ of $\ker \widetilde{H}$ is a limiting position for $\ker H_t$ when $t \in \Gamma$ approaches 0. □

• Using Proposition 4.3.24, we define a complex vector subspace $H'^{1,1}_{BC}(X_0, \mathbb{C}) \subset H^{1,1}_{BC}(X_0, \mathbb{C})$ with the above-mentioned properties by taking V to be the real-analytic vector bundle

$$\Delta \ni t \mapsto \bigoplus_{0 \leq \lambda < \varepsilon} E^{1,1}_{\Delta_{BC,t}}(\lambda)$$

and $H_t := \Delta_{BC,t}$ (the Bott-Chern Laplacians induced by a real-analytic family of Hermitian metrics on the respective fibres X_t), where $\Delta \subset \mathbb{C}$ is a sufficiently small open disc about 0 and $\varepsilon > 0$ is so small that:

· 0 is the only eigenvalue of $\Delta_{BC,0}$ in the interval $[0, \varepsilon)$;

· the boundary of the open disc $\Delta_\varepsilon \subset \mathbb{C}$ of radius ε about 0 does not meet the spectrum of any $\Delta_{BC,t}$ with $t \in \Delta$;

· the number of eigenvalues (counted with multiplicities) of $\Delta_{BC,t}$ lying inside the disc Δ_ε is independent of $t \in \Delta$.

We then pick an arbitrary path $\Gamma \subset \Delta$ ending at 0 and we use Proposition 4.3.24 to define $H'^{1,1}_{BC}(X_0, \mathbb{C})$ as the limiting position of $\ker H_t = \ker \Delta_{BC,t}$ as $t \in \Gamma$ approaches 0.

To lighten the exposition, we will henceforth assume, including in Definition 4.3.25 and in Theorem 4.3.26, that a limiting position of $\ker \Delta_{BC,t}$ exists as $t \in \Delta$ approaches 0. (Otherwise, replace Δ by a path $\Gamma \subset \Delta$.)

• Continuing the work in our situation, for every $t \in \Delta$, let $\Delta_{A,t} : C^\infty_{n-1,n-1}(X_t, \mathbb{C}) \to C^\infty_{n-1,n-1}(X_t, \mathbb{C})$ be the Aeppli Laplacian in bidegree $(n-1, n-1)$ defined by the Hermitian metric ω_t on X_t. Since $\Delta_{A,t}$ is a non-negative self-adjoint elliptic operator (of order 4), it has a discrete spectrum $0 \le \lambda_1(t) \le \lambda_2(t) \le \dots$ with $+\infty$ as sole accumulation point and the space $C^\infty_{n-1,n-1}(X_t, \mathbb{C})$ has an orthonormal basis $(e_j(t))_{j\ge 1}$ consisting of eigenvectors such that

$$\Delta_{A,t} e_j(t) = \lambda_j(t)\, e_j(t), \quad j \ge 1, t \in \Delta.$$

Let $N := h^{1,1}_{BC}(0)$ and $p := h^{1,1}_{BC}(t)$ for t close to 0. Thus $N \ge p$ by the Kodaira-Spencer upper-semicontinuity property [KS60, Theorem 4]. Let

$$0 < \varepsilon < \min\Bigg(\operatorname{Spec}\Delta_{A,0} \cap (0, +\infty)\Bigg) \quad \text{such that} \quad \varepsilon \notin \operatorname{Spec}\Delta_{A,t} \ \forall t \sim 0.$$

Then, thanks to fundamental Kodaira-Spencer theorems on smooth families of elliptic operators [KS60, Theorems 1-5], we have the following picture:

$0 = \lambda_1(0) = \dots = \lambda_N(0) < \varepsilon < \lambda_{N+1}(0)$, while for all $t \sim 0, t \ne 0$, we have:
$0 = \lambda_1(t) = \dots = \lambda_p(t) < \lambda_{p+1}(t) \le \dots \le \lambda_N(t) < \varepsilon < \lambda_{N+1}(t)$,

i.e. the number of eigenvalues (counted with multiplicities) of $\Delta_{A,t}$ lying in the open interval $(-1, \varepsilon)$ is independent of t if $t \in \Delta$ is sufficiently close to 0. Moreover, if $E_{\Delta_{A,t}}(\lambda)$ denotes the eigenspace of $\Delta_{A,t}$ corresponding to the eigenvalue λ, the Kodaira-Spencer theorems further ensure that

$$\Delta \ni t \mapsto \bigoplus_{0\le\lambda<\varepsilon} E_{\Delta_{A,t}}(\lambda) := \mathcal{E}^{n-1,n-1}_{A,\varepsilon}(t)$$

is a C^∞ vector bundle of finite rank (equal to N here) after possibly shrinking Δ about 0 and that the orthogonal projections

$$C^\infty_{n-1,n-1}(X_t, \mathbb{C}) \xrightarrow{\sigma_t} \mathcal{E}^{n-1,n-1}_{A,\varepsilon}(t) \tag{4.47}$$

vary in a C^∞ way with $t \in \Delta$.
Thus $\{e_1(t), \dots, e_N(t)\}$ is a local frame for the vector bundle $\mathcal{E}^{n-1,n-1}_{A,\varepsilon}$ and we have:

$$\begin{aligned} \ker\Delta_{A,0} &= \langle e_1(0), \dots, e_p(0), \dots, e_N(0)\rangle = \mathcal{E}^{n-1,n-1}_{A,\varepsilon}(0), \\ \ker\Delta_{A,t} &= \langle e_1(t), \dots, e_p(t)\rangle \subset \langle e_1(t), \dots, e_N(t)\rangle = \mathcal{E}^{n-1,n-1}_{A,\varepsilon}(t), \quad t \ne 0. \end{aligned}$$

Thus we have an orthogonal splitting

$$\ker\Delta_{A,0} = \langle e_1(0), \dots, e_p(0)\rangle \oplus \langle e_{p+1}(0), \dots, e_N(0)\rangle$$

which induces under the Hodge isomorphism $\ker\Delta_{A,0} \simeq H^{n-1,n-1}_A(X_0, \mathbb{C})$ a splitting

$$H^{n-1,n-1}_A(X_0, \mathbb{C}) = H^{'n-1,n-1}_A(X_0, \mathbb{C}) \oplus H^{''n-1,n-1}_A(X_0, \mathbb{C}) \tag{4.48}$$

where $H^{'n-1,n-1}_A(X_0, \mathbb{C}) \simeq \langle e_1(0), \dots, e_p(0)\rangle$ and
$H^{''n-1,n-1}_A(X_0, \mathbb{C}) \simeq \langle e_{p+1}(0), \dots, e_N(0)\rangle$. Now, $H^{1,1}_{BC}(X_0, \mathbb{C})$ and $H^{n-1,n-1}_A(X_0, \mathbb{C})$ are dual to each other, so identifying $H^{1,1}_{BC}(X_0, \mathbb{C})$ with $H^{n-1,n-1}_A(X_0, \mathbb{C})^\star$ we define

$$H^{'1,1}_{BC}(X_0, \mathbb{C}) := \Bigg\{[\alpha]_{BC} \in H^{1,1}_{BC}(X_0, \mathbb{C}) \ \mid\ [\alpha]_{BC|H^{''n-1,n-1}_A(X_0,\mathbb{C})} = 0\Bigg\}. \tag{4.49}$$

Thus, $H^{'1,1}_{BC}(X_0,\,\mathbb{C})$ consists of the linear maps $[\alpha]_{BC}\,:\,H^{n-1,\,n-1}_A(X_0,\,\mathbb{C})\to\mathbb{C}$ vanishing on $H^{''n-1,\,n-1}_A(X_0,\,\mathbb{C})$, i.e. identifies with the dual of $H^{'n-1,\,n-1}_A(X_0,\,\mathbb{C})$.

It is clear that $H^{'1,1}_{BC}(X_0,\,\mathbb{C})$ coincides with $H^{1,1}_{BC}(X_0,\,\mathbb{C})$ if $h^{1,1}_{BC}(0)=h^{1,1}_{BC}(t)$ for $t\sim 0$, but it depends on the choice of the C^∞ family of metrics $(\omega_t)_{t\in\Delta}$, so it is not canonical, if $h^{1,1}_{BC}(0)>h^{1,1}_{BC}(t)$. The same construction can, of course, be run for any $t_0\in\Delta$ in place of 0.

Definition 4.3.25. *([PU18, Definition 5.8] Let $(X_t)_{t\in\Delta}$ be a holomorphic family of* **sGG** *compact complex manifolds equipped with a C^∞ family of Hermitian metrics $(\omega_t)_{t\in\Delta}$.*

For any $t_0\in\Delta$, the **limit as $t\to t_0$ of the pseudo-effective cones $\mathcal{E}_{X_t}$** *of the fibres X_t for $t\neq t_0$ is defined as the following subset of $H^{1,1}_{BC}(X_{t_0},\,\mathbb{R})$:*

$$\lim_{t\to t_0}\mathcal{E}_{X_t}:=\left\{[\alpha]_{BC}\in H^{1,1}_{BC}(X_{t_0},\,\mathbb{R})\cap H^{'1,1}_{BC}(X_{t_0},\,\mathbb{C})\mid (Q^\star_{\omega_t}\circ P^\star_{t_0})([\alpha]_{BC})\in\mathcal{E}_{X_t}\;\;\forall t\sim t_0\right\},$$

where "$\forall t\sim t_0$" means "for all t sufficiently close to t_0".

Note that we restrict from the start to classes in the subspace $H^{'1,1}_{BC}(X_{t_0},\,\mathbb{C})\subset H^{1,1}_{BC}(X_{t_0},\,\mathbb{C})$ to trim off the extra dimensions that the limit may acquire if the dimension of $H^{1,1}_{BC}(X_t,\,\mathbb{C})$ increases in the limit. Note also that, unlike its Gauduchon-cone counterpart, $\lim\limits_{t\to t_0}\mathcal{E}_{X_t}$ depends not only on the metric ω_{t_0} but on the whole family of metrics $(\omega_t)_{t\in\Delta}$ for $t\sim 0$.

We can now prove that the pseudo-effective cone $\mathcal{E}_{X_t}$ behaves upper-semicontinuously in families of sGG manifolds.

Theorem 4.3.26. *([PU18, Theorem 5.9]) Let $(X_t)_{t\in\Delta}$ be any holomorphic family of* **sGG** *compact complex manifolds endowed with any C^∞ family $(\omega_t)_{t\in\Delta}$ of Hermitian metrics.*

Then, for all $t_0\in\Delta$, the following inclusion holds:

$$\mathcal{E}_{X_{t_0}}\supset\lim_{t\to t_0}\mathcal{E}_{X_t}.$$

Proof. Without loss of generality, we may suppose that $t_0=0$. Let $[T]_{BC}\in\lim\limits_{t\to 0}\mathcal{E}_{X_t}$, where T is a d-closed real $(1,\,1)$-current on X_0. (We implicitly use the fact that the Bott-Chern cohomology can be computed using either smooth forms or currents.) Since $\mathcal{E}_{X_t}$ is the dual of $\overline{\mathcal{G}_{X_t}}$ by Lamari's duality lemma, for all $t\neq 0$ with $t\sim 0$ and for any Gauduchon metric γ_t on X_t, we have

$$\begin{aligned}\int\limits_{X_t}(Q^\star_{\omega_t}\circ P^\star_0)([T]_{BC})\wedge[\gamma_t^{n-1}]_A &= \int\limits_{X_t}\{T\}_{DR}\wedge Q_{\omega_t}([\gamma_t^{n-1}]_A)\\ &= \int\limits_{X_t}T\wedge(\Omega_t^{n,\,n-2}+\gamma_t^{n-1}+\overline{\Omega_t^{n,\,n-2}})\geq 0.\end{aligned}\tag{4.50}$$

Indeed, the first identity in (4.50) is (4.45), while the second identity holds for the $(n,\,n-2)$-form $\Omega_t^{n,\,n-2}$ on X_t determined as described in Definition 4.3.15 by γ_t^{n-1} and the Hermitian metric ω_t of X_t.

We will show that $[T]_{BC}\in\mathcal{E}_{X_0}$. Since $\mathcal{E}_{X_0}$ is the dual of $\overline{\mathcal{G}_{X_0}}$ by Theorem 4.1.18, this amounts to showing that

$$\int\limits_{X_0}T\wedge\gamma_0^{n-1}\geq 0\tag{4.51}$$

for any Gauduchon metric γ_0 on X_0.

Let us fix an arbitrary Gauduchon metric γ_0 on X_0. Pick any C^∞ deformation of γ_0 to Gauduchon metrics $(\gamma_t)_{t\in\Delta}$ on the fibres $(X_t)_{t\in\Delta}$. (This is always possible as the proof of Gauduchon's theorem shows – see e.g. [Pop13a, §.3]). Since $(\gamma_t^{n-1})_{t\in\Delta}$ is a C^∞ family of $(n-1,\, n-1)$-forms and since $(\sigma_t)_{t\in\Delta}$ is a C^∞ family of orthogonal projections (defined in (4.47)), $(\sigma_t\gamma_t^{n-1})_{t\in\Delta}$ is a C^∞ family of $(n-1,\, n-1)$-forms.

We use the notation of Definition 4.3.15 with $\Omega^{n-1,\,n-1}$ replaced with γ_t^{n-1} on each fibre X_t. Thus $\Omega_{A,\,t}^{n-1,\,n-1}$ stands for the $\Delta_{A,\,t}$-harmonic component of γ_t^{n-1}, so for every $t\in\Delta$ we have:

$$\gamma_t^{n-1} = \Omega_{A,\,t}^{n-1,\,n-1} + \partial_t\Gamma_t^{n-2,\,n-1} + \bar\partial_t\overline{\Gamma_t^{n-2,\,n-1}},$$

$$\Omega_{A,\,t}^{n,\,n-2} := -\bar\partial_t^\star\Delta_t^{''-1}(\partial_t\Omega_{A,\,t}^{n-1,\,n-1}) \quad \text{and} \quad \Omega_t^{n,\,n-2} := \Omega_{A,\,t}^{n,\,n-2} + \partial_t\overline{\Gamma_t^{n-2,\,n-1}}$$

On the other hand, we have:

$$\sigma_0\gamma_0^{n-1} = \Omega_{A,\,0}^{n-1,\,n-1} = \sum_{j=1}^{p} c_j(0)\, e_j(0) + \sum_{j=p+1}^{N} c_j(0)\, e_j(0) := \Omega'_{A,\,0} + \Omega''_{A,\,0},$$

$$\sigma_t\gamma_t^{n-1} = \sum_{j=1}^{p} c_j(t)\, e_j(t) + \sum_{j=p+1}^{N} c_j(t)\, e_j(t) = \Omega_{A,\,t}^{n-1,\,n-1} + \sum_{j=p+1}^{N} c_j(t)\, e_j(t),$$

for $t\sim 0, t\neq 0$. Thus $\Omega'_{A,\,0}, \Omega''_{A,\,0}\in\ker\Delta_{A,\,0}$ and $[\Omega'_{A,\,0}]_A\in H_A^{'n-1,\,n-1}(X_0,\,\mathbb{C})$ while $[\Omega''_{A,\,0}]_A\in H_A^{''n-1,\,n-1}(X_0,\,\mathbb{C})$. The coefficients $c_j(t)\in\mathbb{C}$ vary continuously with $t\in\Delta$, so $c_j(t)\to c_j(0)$ as $t\to 0$ for every j. We get:

$$\Omega_{A,\,t}^{n-1,\,n-1} = \sum_{j=1}^{p} c_j(t)\, e_j(t) \longrightarrow \sum_{j=1}^{p} c_j(0)\, e_j(0) = \Omega'_{A,\,0}\in\ker\Delta_{A,\,0} \quad \text{as } t\to 0, \tag{4.52}$$

hence, from ∂_t varying in a C^∞ way with t up to $t=0$, we infer

$$\partial_t\Omega_{A,\,t}^{n-1,\,n-1} \longrightarrow \partial_0\Omega'_{A,\,0} \quad \text{as } t\to 0.$$

Now comes a crucial argument. The forms $\partial_t\Omega_{A,\,t}^{n-1,\,n-1}$ and $\partial_0\Omega'_{A,\,0}$ are of bidegree $(n,\, n-1)$ for their respective complex structures. On the other hand, by Serre duality we have $h_{\bar\partial}^{n,\,n-1}(t) = h_{\bar\partial}^{0,\,1}(t)$, hence part (ii) of our Corollary 4.3.10 and the sGG assumption ensure that

$$h_{\bar\partial}^{n,\,n-1}(0) = h_{\bar\partial}^{n,\,n-1}(t) \quad \text{for all } t\sim 0.$$

Therefore, the Green operators $(\Delta_t^{''-1})_{t\in\Delta}$ vary in a C^∞ way with t (up to $t=0$) by the Kodaira-Spencer theorem [KS60, Theorem 5] which applies when the relevant Hodge numbers ($h_{\bar\partial}^{n,\,n-1}(t)$ here) do not jump. Thus,

$$\Delta_t^{''-1}(\partial_t\Omega_{A,\,t}^{n-1,\,n-1}) \longrightarrow \Delta_0^{''-1}(\partial_0\Omega'_{A,\,0}) \quad \text{as } t\to 0$$

and since $\bar\partial_t^\star$ varies in a C^∞ way with t up to $t=0$, we infer

$$\Omega_{A,\,t}^{n,\,n-2} = -\bar\partial_t^\star\Delta_t^{''-1}(\partial_t\Omega_{A,\,t}^{n-1,\,n-1}) \longrightarrow -\bar\partial_0^\star\Delta_0^{''-1}(\partial_0\Omega'_{A,\,0}) := \Omega_{A,\,0}^{'n,\,n-2} \tag{4.53}$$

as $t\to 0$. It is clear that the form $\Omega_{A,\,0}^{'n,\,n-2}$ is of bidegree $(n,\, n-2)$ on X_0.

We can now finish the proof of the theorem. Recall that we have to prove inequality (4.51). With the above preparations, we have:

$$\begin{aligned}\int\limits_{X_0} T\wedge\gamma_0^{n-1} &= \int\limits_{X_0} T\wedge(\Omega_{A,0}^{n-1,\,n-1}+\partial_0\Gamma_0^{n-2,\,n-1}+\bar\partial_0\overline{\Gamma_0^{n-2,\,n-1}}) \stackrel{(a)}{=} \int\limits_{X_0} T\wedge\Omega_{A,0}^{n-1,\,n-1} \\ &= \int\limits_{X_0} T\wedge\Omega'_{A,0}+\int\limits_{X_0} T\wedge\Omega''_{A,0} \stackrel{(b)}{=} \int\limits_{X_0} T\wedge\Omega'_{A,0},\end{aligned} \tag{4.54}$$

where identity (a) follows by Stokes' theorem from $\partial_0 T = 0$ and $\bar\partial_0 T = 0$ (due to $dT = 0$), while identity (b) follows from the definition of $H_{BC}^{'1,1}(X_0,\,\mathbb{C})$, from $[T]_{BC}\in H_{BC}^{'1,1}(X_0,\,\mathbb{C})$ and from $[\Omega''_{A,0}]_A\in H_A^{''n-1,\,n-1}(X_0,\,\mathbb{C})$.

On the other hand, the last integral in (4.50), which is non-negative for all $t\sim 0$ and $t\neq 0$, transforms as follows:

$$\begin{aligned}\int\limits_{X_t} T\wedge(\Omega_t^{n,\,n-2}+\gamma_t^{n-1}+\overline{\Omega_t^{n,\,n-2}}) &= \int\limits_{X_t} T\wedge\gamma_t^{n-1} \\ &+ \int\limits_{X_t} T\wedge\Omega_{A,t}^{n,\,n-2}+\int\limits_{X_t} T\wedge\partial_t\Gamma_t^{n-1,\,n-2} \\ &+ \int\limits_{X_t} T\wedge\overline{\Omega_{A,t}^{n,\,n-2}}+\int\limits_{X_t} T\wedge\bar\partial_t\Gamma_t^{n-2,\,n-1}.\end{aligned} \tag{4.55}$$

Now, $\int_{X_t} T\wedge\gamma_t^{n-1}$ converges to $\int_{X_0} T\wedge\gamma_0^{n-1}$, while $\int_{X_t} T\wedge\Omega_{A,t}^{n,\,n-2}$ converges to $\int_{X_0} T\wedge\Omega_{A,0}^{'n,\,n-2}=0$ by the crucial convergence (4.53). The last identity follows from T being of bidegree $(1,\,1)$ and $\Omega_{A,0}^{'n,\,n-2}$ being of bidegree $(n,\,n-2)$, hence $T\wedge\Omega_{A,0}^{'n,\,n-2}=0$ as an $(n+1,\,n-1)$-current. By conjugation, we infer that $\int_{X_t} T\wedge\overline{\Omega_{A,t}^{n,\,n-2}}$ converges to $\int_{X_0} T\wedge\overline{\Omega_{A,0}^{'n,\,n-2}}=0$. Furthermore, we have:

$$\int\limits_{X_t} T\wedge\partial_t\overline{\Gamma_t^{n-2,\,n-1}} = \int\limits_{X_t} T\wedge d\overline{\Gamma_t^{n-2,\,n-1}} - \int\limits_{X_t} T\wedge\bar\partial_t\overline{\Gamma_t^{n-2,\,n-1}} = -\int\limits_{X_t} T\wedge\bar\partial_t\overline{\Gamma_t^{n-2,\,n-1}},$$

the last identity following from Stokes' theorem and $dT=0$. We also have the conjugate identity: $\int_{X_t} T\wedge\bar\partial_t\Gamma_t^{n-2,\,n-1} = -\int_{X_t} T\wedge\partial_t\Gamma_t^{n-2,\,n-1}$, hence:

$$\begin{aligned}\int\limits_{X_t} T\wedge\partial_t\overline{\Gamma_t^{n-2,\,n-1}}+\int\limits_{X_t} T\wedge\bar\partial_t\Gamma_t^{n-2,\,n-1} &= -\int\limits_{X_t} T\wedge(\bar\partial_t\overline{\Gamma_t^{n-2,\,n-1}}+\partial_t\Gamma_t^{n-2,\,n-1}) \\ &= -\int\limits_{X_t} T\wedge(\gamma_t^{n-1}-\Omega_{A,t}^{n-1,\,n-1}).\end{aligned} \tag{4.56}$$

Now, $\int_{X_t} T\wedge\gamma_t^{n-1}$ converges to $\int_{X_0} T\wedge\gamma_0^{n-1}$ and, by (4.52), $\int_{X_t} T\wedge\Omega_{A,t}^{n-1,\,n-1}$ converges to $\int_{X_0} T\wedge\Omega'_{A,0}$ as $t\to 0$. Putting together (4.55), (4.56) and all the pieces of convergence information just mentioned, we get the convergence:

$$\int\limits_{X_t} T\wedge(\Omega_t^{n,\,n-2}+\gamma_t^{n-1}+\overline{\Omega_t^{n,\,n-2}}) \longrightarrow \int\limits_{X_0} T\wedge\Omega'_{A,0} = \int\limits_{X_0} T\wedge\gamma_0^{n-1} \quad \text{as } t\to 0, \tag{4.57}$$

where the last identity is nothing but (4.54).

Recall that $\int_{X_t} T \wedge (\Omega_t^{n,n-2} + \gamma_t^{n-1} + \overline{\Omega_t^{n,n-2}}) \geq 0$ for all $t \sim 0$ with $t \neq 0$ by (4.50). Hence (4.57) implies $\int\limits_{X_0} T \wedge \gamma_0^{n-1} \geq 0$ and we are done. □

4.3.6 Relations between the sGG class and other classes of compact complex manifolds

In this section we show that sGG manifolds are unrelated to balanced manifolds and to those whose Frölicher spectral sequence degenerates at E_1. Examples of compact complex manifolds X with $\mathcal{SG}_X \neq \mathcal{G}_X$ but admitting strongly Gauduchon metrics are also given. To construct appropriate examples we will consider the class of nilmanifolds endowed with an invariant complex structure.

(I) Generalities on nilmanifolds

Recall that a nilmanifold $N = G/\Gamma$ is a compact quotient of a connected and simply-connected nilpotent real Lie group G by a lattice Γ of maximal rank in G. Let $\mathfrak{g}$ be the Lie algebra of the group G. We will say that *"N has underlying Lie algebra $\mathfrak{g}$"* or that *"$\mathfrak{g}$ is the Lie algebra underlying N"*. We will denote 6-dimensional real Lie algebras in the usual abbreviated form; for instance, $(0^4, 12, 34)$ denotes the Lie algebra $\mathfrak{g}$ with generators $\{e_i\}_{i=1}^6$ satisfying the bracket relations $[e_1, e_2] = -e_5$, $[e_3, e_4] = -e_6$, or equivalently there exists a basis $\{\alpha^i\}_{i=1}^6$ of the dual $\mathfrak{g}^*$ such that $d\alpha^1 = d\alpha^2 = d\alpha^3 = d\alpha^4 = 0$, $d\alpha^5 = \alpha^1 \wedge \alpha^2$, $d\alpha^6 = \alpha^3 \wedge \alpha^4$.

Notice that by Nomizu's theorem [Nom54], the integer k appearing in 0^k in the notation above is precisely the first Betti number of N, i.e. $b_1(N) = k$.

The complex structures that we will consider on N are invariant in the sense that they stem naturally from "complex" structures J on the Lie algebra $\mathfrak{g}$ of G. For any such J, the i-eigenspace $\mathfrak{g}_{1,0}$ of J in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex subalgebra. When $\mathfrak{g}_{1,0}$ is abelian we will refer to J as an *abelian* complex structure.

The following result identifies the compact complex nilmanifolds of complex dimension 3 that are sGG.

Theorem 4.3.27. *([PU18, Theorem 6.1]) Let N be a nilmanifold of (real) dimension six not isomorphic to a torus and let J be an invariant complex structure on N. Then, the compact complex manifold $X = (N, J)$ is sGG if and only if the Lie algebra underlying N is isomorphic to*

$$(0^4, 12, 34), \quad (0^4, 12, 14 + 23), \quad (0^4, 13 + 42, 14 + 23) \quad \text{or} \quad (0^4, 12, 13)$$

and the complex structure J is not abelian.

Proof. By Theorem 4.3.8, if N admits an invariant complex structure J such that $X = (N, J)$ is sGG then the first Betti number is even. From the classification of nilpotent Lie algebras admitting a complex structure [Sal01], this condition implies that the Lie algebra underlying N belongs to the following list: $(0^4, 12, 34)$, $(0^4, 12, 14 + 23)$, $(0^4, 13 + 42, 14 + 23)$, $(0^4, 12, 13)$, $(0^4, 12, 14 + 25)$, $(0^2, 12, 13, 23, 14 + 25)$.

We first rule out the last two cases. It was proved in [UV14, Proposition 2.4] that for any invariant complex structure J on a nilmanifold N with underlying Lie algebra $(0^2, 12, 13, 23, 14+25)$ there is a global basis $\{\eta^1, \eta^2, \eta^3\}$ of forms of bidegree (1,0), with respect to J, satisfying complex equations of the shape:

$$d\eta^1 = 0, \quad d\eta^2 = \eta^{13} + \eta^{1\bar{3}}, \quad d\eta^3 = i\eta^{1\bar{1}} \pm i(\eta^{1\bar{2}} - \eta^{2\bar{1}}),$$

(where we use the standard notation: $\eta^{jk} := \eta^j \wedge \eta^k$, $\eta^{j\bar{k}} := \eta^j \wedge \overline{\eta^k}$.)
That is to say, up to equivalence there exist exactly two invariant complex structures on N depending on the choice of sign in the third equation. Hence, $H^{0,1}_{\bar\partial}(N, J) = \langle [\eta^{\bar 1}]_{\bar\partial}, [\eta^{\bar 3}]_{\bar\partial}\rangle$ by a result in [Rol09, Section 4.2], and we get $b_1(N) = 2 < 4 = 2h^{0,1}_{\bar\partial}(N, J)$. It follows from Theorem 4.3.8 that there is no invariant complex structure on N satisfying the sGG property.

By [COUV11], for any invariant complex structure J on a nilmanifold N with underlying Lie algebra $(0^4, 12, 14 + 25)$ there is a global basis $\{\eta^1, \eta^2, \eta^3\}$ of (1,0)-forms satisfying

$$d\eta^1 = 0, \quad d\eta^2 = \eta^{1\bar 1}, \quad d\eta^3 = \eta^{1\bar 2} + \eta^{2\bar 1},$$

which implies that $h^{0,1}_{\bar\partial}(N, J) = 3$. Therefore, $b_1(N) = 4 < 6 = 2h^{0,1}_{\bar\partial}(N, J)$, so there is no invariant complex structure on N satisfying the sGG property.

It is well known that on 6-dimensional nilmanifolds different from the complex tori there exists (up to equivalence) only one complex-parallelisable complex structure given by the complex equations

$$d\eta^1 = d\eta^2 = 0, \quad d\eta^3 = \eta^{12}.$$

This corresponds to the Iwasawa manifold (which is an sGG manifold by e.g. Corollary 4.3.5) whose underlying Lie algebra is precisely $(0^4, 13 + 42, 14 + 23)$. Now, for any other complex structure J on a nilmanifold N with underlying Lie algebra $(0^4, 12, 34)$, $(0^4, 12, 14 + 23)$, $(0^4, 13 + 42, 14 + 23)$ or $(0^4, 12, 13)$, it is proved in [COUV11] that there is a (1,0)-basis satisfying

$$d\eta^1 = d\eta^2 = 0, \quad d\eta^3 = \rho\, \eta^{12} + \eta^{1\bar 1} + \lambda\, \eta^{1\bar 2} + D\, \eta^{2\bar 2}, \tag{4.58}$$

where $\rho \in \{0, 1\}$, $\lambda \in \mathbb{R}^{\geq 0}$ and $D = x + iy \in \mathbb{C}$ with $y \geq 0$. Notice that J is abelian if and only if $\rho = 0$.

To complete the proof we must show that for any complex structure J given by (4.58) the compact complex manifold (N, J) satisfies $b_1(N) = 4 = 2h^{0,1}_{\bar\partial}(N, J)$ if and only if $\rho = 1$. But this is clear because $H^{0,1}_{\bar\partial}(N, J) = \langle [\eta^{\bar 1}]_{\bar\partial}, [\eta^{\bar 2}]_{\bar\partial}, [\eta^{\bar 3}]_{\bar\partial}\rangle$ when $\rho = 0$, and $H^{0,1}_{\bar\partial}(N, J) = \langle [\eta^{\bar 1}]_{\bar\partial}, [\eta^{\bar 2}]_{\bar\partial}\rangle$ for $\rho = 1$. □

(II) sG vs. sGG manifolds

For any compact complex manifold X, it is immediate that if X is sGG then X has an sG metric. The following example shows that the converse does not hold in general even if the sG hypothesis is reinforced to the balanced hypothesis and even with an extra property.

Proposition 4.3.28. *([PU18, Proposition 6.2]) There exists a compact complex manifold X having a balanced metric, with Frölicher spectral sequence degenerating at the first step and with first Betti number $b_1(X)$ odd. Thus, X is not sGG.*

Proof. Let N be a nilmanifold with underlying Lie algebra isomorphic to $(0^5, 12 + 34)$. Then, the first Betti number of N is 5. We consider on N the complex structure J defined by the complex equations

$$d\eta^1 = d\eta^2 = 0, \quad d\eta^3 = \eta^{1\bar 1} - \eta^{2\bar 2}.$$

By Theorem 4.3.8 we know that $X = (N, J)$ is not sGG because $b_1(N) = 5$. It is proved in [COUV11] that $E_1(X) \cong E_\infty(X)$. Moreover, X is balanced; for instance, $\omega = \frac{i}{2}(\eta^{1\bar 1} + \eta^{2\bar 2} + \eta^{3\bar 3})$ satisfies $d\omega^2 = 0$, that is, ω is a balanced metric on X. □

Proposition 4.3.29. *([PU18, Proposition 6.3]) The balanced property and the sGG property are unrelated. Moreover, the Frölicher spectral sequence degenerating at E_1 and the sGG property are also unrelated.*

Proof. In Proposition 4.3.28 we proved that "balanced" does not imply "sGG", and that $E_1(X) \cong E_\infty(X)$ does not imply X to be sGG. We now show that there exists an sGG compact complex manifold X that is not balanced and whose Frölicher spectral sequence does not degenerate at E_1.

Let N be a nilmanifold with underlying Lie algebra isomorphic to $(0^4, 13+42, 14+23)$, that is, N is the (real) manifold underlying the Iwasawa manifold. We consider on N the complex structure J defined by the complex equations

$$d\eta^1 = d\eta^2 = 0, \quad d\eta^3 = \eta^{12} + \eta^{1\bar{1}}. \tag{4.59}$$

By Theorem 4.3.27 the compact complex manifold $X = (N, J)$ is sGG because the complex structure J is not abelian. However, from the general study in [COUV11] one has that $E_1(X) \not\cong E_2(X) \cong E_\infty(X)$ and X does not admit any balanced metric. □

(III) Superstrong Gauduchon vs. sGG manifolds

We now briefly discuss a class of manifolds resembling sGG manifolds. Then term in the following definition was coined by M. Verbitsky in a private communication with the author who was simultaneously contemplating the same notion.

Definition 4.3.30. *Let X be a complex manifold with $\dim_{\mathbb{C}} X = n$. A Hermitian metric ω on X is said to be* **superstrong Gauduchon (super sG)** *if $\partial\omega^{n-1}$ is $\partial\bar{\partial}$-exact.*

The manifold X is said to be a **superstrong Gauduchon manifold (super sG manifold)** *if it supports such a metric.*

Any superstrong Gauduchon metric is trivially strongly Gauduchon and the two notions are equivalent if X is a $\partial\bar{\partial}$-manifold.

Proposition 4.3.31. *([PU18, Proposition 6.4]) The superstrong Gauduchon property and the sGG property are unrelated.*

Proof. Let us show first that there exists an sGG compact complex manifold X that does not admit any superstrong Gauduchon metric. Consider $X = (N, J)$ a (real) $2n$-dimensional nilmanifold N endowed with an invariant complex structure J. By the usual symmetrisation process, if ω is a superstrong Gauduchon metric on X, then there also exists an invariant superstrong Gauduchon metric $\hat{\omega}$ on X. Indeed, if $\Omega = \omega^{n-1}$ satisfies $\partial\Omega = \partial\bar{\partial}\alpha$ for some $(n-1, n-2)$-form α, then by symmetrisation we get that the positive definite invariant $(n-1, n-1)$-form $\widetilde{\Omega}$ (obtained from Ω) satisfies $\partial\widetilde{\Omega} = \partial\bar{\partial}\tilde{\alpha}$ for an invariant $(n-1, n-2)$-form $\tilde{\alpha}$. Now, since $\widetilde{\Omega} > 0$, it is well known that there exists an invariant Hermitian metric $\hat{\omega}$ such that $\widetilde{\Omega} = \hat{\omega}^{n-1}$. Thus $\hat{\omega}$ is necessarily an invariant superstrong Gauduchon metric on X.

Now, let us consider $X = (N, J)$ defined by (4.59), which by Theorem 4.3.27 is sGG. A direct calculation shows that $\partial\bar{\partial}\Lambda^{2,1}(\mathfrak{g}^*) \equiv 0$. Therefore, if a superstrong Gauduchon metric existed on X, it would have to be an invariant balanced metric. However, we pointed out in the proof of Proposition 4.3.29 that X is not balanced. Thus, X is sGG but does not admit any superstrong Gauduchon metric.

Conversely, we notice that the superstrong Gauduchon property does not imply the sGG property because, thanks to Proposition 4.3.28, there exists a balanced manifold which is not sGG. □

4.3.7 Examples of deformation limits of sGG manifolds

The following result shows that the sGG hypothesis on X does not ensure the Bott-Chern number $h^{1,1}_{BC}(X)$ to be locally deformation constant.

Proposition 4.3.32. *([PU18, Proposition 7.1]) There exists a holomorphic family of compact complex* **sGG** *manifolds $(X_t)_{t\in\Delta}$ such that $h^{1,1}_{BC}(0) > h^{1,1}_{BC}(t)$ for all $t \in \Delta \setminus C$, where $\Delta \subset \mathbb{C}$ is a small open disc about 0 and C is a real curve through 0.*

Proof. Let $X_0 = (N, J_0)$ be a complex nilmanifold of real dimension 6 defined by the equations

$$d\eta^1 = d\eta^2 = 0, \quad d\eta^3 = \eta^{12} + \eta^{1\bar{1}} + \eta^{1\bar{2}} - 2\,\eta^{2\bar{2}}. \tag{4.60}$$

By [COUV11, Table 1] the Lie algebra $\mathfrak{g}$ underlying N is isomorphic to $(0^4, 12, 14+23)$. Since the complex structure J_0 is not abelian, Theorem 4.3.27 implies that X_0 is sGG.

By [Ang11, Theorem 2.7], the Bott-Chern cohomology groups of X_0 can be calculated at the level of the Lie algebra underlying N, in particular, $H^{1,1}_{BC}(X_0) \cong H^{1,1}_{BC}(\mathfrak{g}, J_0) = \ker\{d\colon \Lambda^{1,1}(\mathfrak{g}^*) \longrightarrow \Lambda^3(\mathfrak{g}^*_{\mathbb{C}})\}$. From the equations (4.60) we get

$$H^{1,1}_{BC}(X_0) \cong \langle [\eta^{1\bar{1}}]_{BC}, [\eta^{1\bar{2}}]_{BC}, [\eta^{2\bar{1}}]_{BC}, [\eta^{2\bar{2}}]_{BC}, [\eta^{1\bar{3}}+2\,\eta^{2\bar{3}}+\eta^{3\bar{1}}+2\,\eta^{3\bar{2}}]_{BC}\rangle,$$

therefore $h^{1,1}_{BC}(X_0) = 5$.

Now we consider a small deformation J_t given by

$$t\frac{\partial}{\partial z_2} \otimes d\bar{z}_2 \in H^{0,1}(X_0,\, T^{1,0}X_0),$$

where z_2 is a complex coordinate such that $\eta^2 = dz_2$. By Corollary 4.3.10 we know that the compact complex manifold $X_t = (N, J_t)$ is sGG for all $t \in \mathbb{C}$ close enough to 0. In fact, for $t \in \mathbb{C}$ with $|t| < 1$, if we consider the basis $\{\nu^1_t = \eta^1, \nu^2_t = \frac{1-\bar{t}}{1-|t|^2}(\eta^2 + t\,\eta^{\bar{2}}), \nu^3_t = \eta^3\}$ of complex forms of type (1,0) with respect to J_t, then the complex structure equations along the deformation are:

$$d\nu^1_t = d\nu^2_t = 0, \quad d\nu^3_t = \nu^{12}_t + \nu^{1\bar{1}}_t + \nu^{1\bar{2}}_t - 2\,\frac{1-|t|^2}{|1-t|^2}\,\nu^{2\bar{2}}_t. \tag{4.61}$$

Next we compute the dimension of the Bott-Chern cohomology group $H^{1,1}_{BC}(X_t)$ of X_t. Since the complex structure J_t is invariant, we can use again [Ang11, Theorem 2.7] to reduce the calculation to the invariant forms. By (4.61) it is clear that $\nu^{1\bar{1}}_t$, $\nu^{1\bar{2}}_t$, $\nu^{2\bar{1}}_t$ and $\nu^{2\bar{2}}_t$ define Bott-Chern classes in $H^{1,1}_{BC}(X_t)$. To see if there are some other classes, we need to compute the differentials of the remaining basic (1,1)-forms $\nu^{j\bar{k}}_t$. From the equations (4.61) we get:

$$d\nu^{1\bar{3}}_t = \nu^{12\bar{1}}_t - 2\,\tfrac{1-|t|^2}{|1-t|^2}\nu^{12\bar{2}}_t - \nu^{1\bar{1}\bar{2}}_t,$$

$$d\nu^{2\bar{3}}_t = -\nu^{12\bar{1}}_t - \nu^{2\bar{1}\bar{2}}_t,$$

$$d\nu^{3\bar{1}}_t = \nu^{12\bar{1}}_t - \nu^{1\bar{1}\bar{2}}_t + 2\,\tfrac{1-|t|^2}{|1-t|^2}\nu^{2\bar{1}\bar{2}}_t,$$

$$d\nu^{3\bar{2}}_t = \nu^{12\bar{2}}_t + \nu^{1\bar{1}\bar{2}}_t,$$

$$d\nu^{3\bar{3}}_t = \nu^{12\bar{3}}_t - \nu^{13\bar{1}}_t - \nu^{23\bar{1}}_t + 2\,\tfrac{1-|t|^2}{|1-t|^2}\nu^{23\bar{2}}_t + \nu^{1\bar{1}\bar{3}}_t + \nu^{1\bar{2}\bar{3}}_t - 2\,\tfrac{1-|t|^2}{|1-t|^2}\nu^{2\bar{2}\bar{3}}_t - \nu^{3\bar{1}\bar{2}}_t.$$

From these expressions, it is easy to check that there exists at most one more closed (1,1)-form, and that such a form exists if and only if $1-|t|^2=|1-t|^2$.

Let $C=\{t\in\mathbb{C}\mid |t|^2+|1-t|^2=1\}$. Note that C is a circle centered at $t=1/2$ passing through $t=0$. Our discussion above shows that $h^{1,1}_{BC}(X_t)=4$ for all $t\in\Delta^\star\setminus C$, where $\Delta=\{t\in\mathbb{C}\mid |t|<1\}\subset\mathbb{C}$, that is, the Bott-Chern number $h^{1,1}_{BC}$ is not locally deformation constant. $\square$

In the following result we show by means of three examples that the sGG property of compact complex manifolds is not closed under holomorphic deformations. The behaviour of the holomorphic families in the three examples is different and illustrate several possibilities for the limiting fibre.

Proposition 4.3.33. *([PU18, Proposition 7.2]) There exist holomorphic families of compact complex manifolds $(X_t)_{t\in\Delta}$ over an open disc $\Delta\subset\mathbb{C}$ about 0 such that X_t is sGG for all $t\in\Delta\setminus\{0\}$, but X_0 is not sGG.*

Proof. We will describe three examples in succession.
First example. Let us consider the compact complex manifold $X_0=(N,J_0)$, where N is the nilmanifold with underlying Lie algebra $(0^4,13+42,14+23)$ and J_0 is the abelian structure defined by the complex structure equations

$$d\eta^1=d\eta^2=0,\quad d\eta^3=\eta^{1\bar{1}}+\eta^{1\bar{2}}.$$

Since J_0 is abelian, the manifold X_0 is not sGG (by Theorem 4.3.27), and $H^{0,1}_{\bar\partial}(X_0,\mathbb{C})=\langle[\eta^{\bar1}]_{\bar\partial},[\eta^{\bar2}]_{\bar\partial},[\eta^{\bar3}]_{\bar\partial}\rangle$. Consider a small deformation J_t given by

$$t\frac{\partial}{\partial z_2}\otimes d\bar{z}_2\in H^{0,1}(X_0,\,T^{1,0}X_0),$$

where z_2 is a complex coordinate such that $\eta^2=dz_2$. Let us consider the basis $\{\tau^1_t=\eta^1,\tau^2_t=\eta^2+t\,\eta^{\bar2},\tau^3_t=\eta^3\}$ of complex forms of type (1,0) with respect to J_t. Then, for $t\in\mathbb{C}$ with $|t|<1$, the complex structure equations of the deformation are:

$$d\tau^1_t=d\tau^2_t=0,\quad d\tau^3_t=-\frac{\bar t}{1-|t|^2}\,\tau^{12}_t+\tau^{1\bar1}_t+\frac{1}{1-|t|^2}\,\tau^{1\bar2}_t.$$

For any $t\neq0$, the complex structure is not abelian because the differential of the (1,0)-form τ^3_t has a non-zero component of bidegree (2,0), so the compact complex manifold $X_t=(N,J_t)$ is sGG for any $t\neq0$ by Theorem 4.3.27.

Note that X_0 admits a balanced metric by [COUV11, Proposition 7.7], hence also an sG metric, so $\mathcal{SG}_{X_0}\neq\emptyset$.

Second example. In [COUV11, Theorem 7.9] it is constructed a holomorphic family of compact complex (nil)manifolds $(X_t)_{t\in\Delta}$ over an open disc $\Delta\subset\mathbb{C}$ about 0, where X_t is balanced for any $t\neq0$ and such that the central limit X_0 is a complex nilmanifold with underlying Lie algebra $(0^4,12,14+23)$ endowed with an abelian complex structure J_0. The complex structure on X_t is invariant and non-abelian for any $t\neq0$, so by Theorem 4.3.27 the compact complex manifold X_t is sGG, but the central limit X_0 is not sGG. Moreover, it is proved in [COUV11, Proposition 7.7] that X_0 is not sG, so $\mathcal{SG}_{X_0}=\emptyset$.

Third example. Angella and Kasuya obtain in [AK14, Proposition 4.1 (i)] a holomorphic family of compact complex manifolds X_t over an open disc in $\mathbb{C}$ about 0, satisfying the $\partial\bar\partial$-lemma for any $t\neq0$ and such that the central limit X_0 is the complex-parallelisable Nakamura manifold [Nak75].

By Theorem 4.3.8 we conclude that X_0 is not sGG because $b_1(X_0) = 2 < 6 = 2h^{0,1}_{\bar\partial}(X_0)$ (see [AK14, Table 10]). Note however that the central limit X_0 is balanced. $\square$

In [Pop09, Proposition 4.1] it is proved that given a holomorphic family of compact complex manifolds $(X_t)_{t\in\Delta}$ over an open disc $\Delta \subset \mathbb{C}$ about 0, if X_t satisfies the $\partial\bar\partial$-lemma for all $t \in \Delta \setminus \{0\}$ then X_0 is sG. However, the central limit X_0 may be neither sGG (see *Third example* in the proof of Proposition 4.3.33) nor balanced (see [FOU14, Theorem 5.2]). Furthermore, in the following proposition we show that in general X_0 does not admit superstrong Gauduchon metrics.

Proposition 4.3.34. *([PU18, Proposition 7.3]) There exists a holomorphic family of compact complex manifolds $(X_t)_{t\in\Delta}$ over an open disc $\Delta \subset \mathbb{C}$ about 0 such that X_t satisfies the $\partial\bar\partial$-lemma for all $t \in \Delta \setminus \{0\}$, but X_0 is not superstrong Gauduchon.*

Proof. We consider the holomorphic family of compact complex manifolds $(X_t)_{t\in\Delta}$ constructed in [FOU14, Theorem 5.2], which satisfies the $\partial\bar\partial$-lemma for all $t \in \Delta \setminus \{0\}$. The central limit of that family is $X_0 = (G/\Gamma, J_0)$, where G/Γ is a solvmanifold (i.e. a compact quotient of a connected and simply-connected *solvable* real Lie group G by a lattice Γ of maximal rank in G) and J_0 is the invariant complex structure defined by the complex structure equations

$$d\eta^1 = 2i\,\eta^{13} + \eta^{3\bar 3}, \quad d\eta^2 = -2i\,\eta^{23}, \quad d\eta^3 = 0,$$

where $\{\eta^1, \eta^2, \eta^3\}$ is a (1,0)-basis. Thus, we can apply the symmetrisation process and proceed as in the proof of Proposition 4.3.31. So it suffices to show that there do not exist superstrong Gauduchon metrics on the underlying solvable Lie algebra $\mathfrak{g}$. From the complex structure equations above, it is easy to check that $\partial\bar\partial\Lambda^{2,1}(\mathfrak{g}^*) \equiv 0$, which implies that any invariant superstrong Gauduchon metric must be balanced. But this is not possible by [FOU14, Theorem 5.2], so we conclude that X_0 is not superstrong Gauduchon. (However, X_0 is sG as pointed out in [FOU14].) $\square$

4.4 E_r-sG metrics and manifolds

In this section, we generalise the notion of *strongly Gauduchon (sG)* metric described in §.4.2. The starting point is the observation that, for any Gauduchon metric ω on a compact complex n-dimensional manifold X, the $(n, n-1)$-form $\partial\omega^{n-1}$ is E_r-closed for every $r \in \mathbb{N}^\star$. Indeed, in (i) of Definition 1.2.9 we can choose $u_1 = \cdots = u_{r-1} = 0$.

Definition 4.4.1. *([Pop19, Definition 3.2.]) Let ω be a Gauduchon metric on a compact complex manifold X with $\dim_{\mathbb{C}} X = n$. Fix an arbitrary integer $r \geq 1$.*

*(i) We say that ω is an E_r-**sG metric** if $\partial\omega^{n-1}$ is E_r-exact.*

*(ii) A compact complex manifold X is said to be an E_r-**sG manifold** if an E_r-sG metric exists on X.*

*(iii) A compact complex manifold X is said to be an E_r-**sGG manifold** if every Gauduchon metric on X is an E_r-sG metric.*

The term chosen in the last definition is a nod to the notion of sGG manifold introduced in [PU14] as any compact complex manifold on which every Gauduchon metric is strongly Gauduchon. (See §.4.3.) It follows from the above definitions that the E_1-sG property is equivalent to the sG property and that the following implications hold for any Hermitian metric γ and every $r \in \mathbb{N}^\star$:

$$\gamma \text{ is } E_1\text{-sG} \implies \gamma \text{ is } E_2\text{-sG} \implies \cdots \implies \gamma \text{ is } E_r\text{-sG} \implies \gamma \text{ is } E_{r+1}\text{-sG} \implies \ldots.$$

Actually, for bidegree reasons, if a Hermitian metric γ is E_r-sG for some integer $r \geq 1$, then $r \leq 3$. Indeed, if $(p,\,q) = (n,\,n-1)$, the tower of relations (1.29) reduces to its first two lines since ζ_{r-2} is of bidegree $(n-1,\,n-1)$, hence $v_{r-3}^{(r-2)}$ is of bidegree $(n-2,\,n)$, hence $\bar\partial v_{r-3}^{(r-2)} = 0$ for bidegree reasons, so $v_{r-4}^{(r-2)}, \dots, v_0^{(r-2)}$ can all be chosen to be zero.

We now notice that the E_r-sG property is open under deformations of the complex structure.

Lemma 4.4.2. *([Pop19, Lemma 3.3]) Let $\pi : \mathcal{X} \longrightarrow B$ be a C^∞ family of compact complex n-dimensional manifolds over an open ball $B \subset \mathbb{C}^N$ about the origin. Fix an integer $r \geq 1$.*

If γ_0 is an E_r-sG metric on $X_0 := \pi^{-1}(0)$, after possibly shrinking B about 0 there exists a C^∞ family $(\gamma_t)_{t\in B}$ of E_r-sG metrics on the respective fibres $X_t := \pi^{-1}(t)$ whose element for $t = 0$ is the original γ_0.

Moreover, this family can be chosen such that $\partial_t\gamma_t^{n-1} = \bar\partial_t\Gamma_t^{n,\,n-2} + \partial_t\zeta_{r-2,\,t}$ for all t, with J_t-type $(n,\,n-2)$-forms $\Gamma_t^{n,\,n-2}$ and J_t-type $(n-1,\,n-1)$-forms $\zeta_{r-2,\,t}$ depending in a C^∞ way on t.

The forms $\Gamma_t^{n,\,n-2}$, $\zeta_{r-2,\,t}$ and the induced $v_{k,\,t}^{(r-2)}$ (with $0 \leq k \leq r-3$) satisfying the tower of relations (1.29) that are (non-uniquely) associated with an E_r-sG metric γ_t will be called **potentials** of γ_t. So, the above lemma says that not only can any E_r-sG metric γ_0 on X_0 be deformed in a smooth way to E_r-sG metrics γ_t on the nearby fibres X_t, but so can its potentials.

Proof of Lemma 4.4.2. By (ii) of Proposition 3.2.4, the E_r-sG assumption on γ_0 implies the existence of a J_0-type $(n,\,n-2)$-form $\Gamma_0^{n,\,n-2}$ and of a J_0-type $(n-1,\,n-1)$-form $\zeta_{r-2,\,0}$ such that $\partial_0\gamma_0^{n-1} = \bar\partial_0\Gamma_0^{n,\,n-2} + \partial_0\zeta_{r-2,\,0}$ and such that

$$\bar\partial_0\zeta_{r-2,\,0} = \partial_0 v_{r-3,\,0}^{(r-2)}, \qquad \text{and} \qquad \bar\partial_0 v_{r-3,\,0}^{(r-2)} = 0, \tag{4.62}$$

for some J_0-type $(n-2,\,n)$-form $v_{r-3,\,0}^{(r-2)}$. (As already pointed out, for bidegree reasons, the general tower (1.29) reduces to (4.62) in this case.)

We get $\partial_0(\gamma_0^{n-1} - \zeta_{r-2,\,0} - \overline{\zeta_{r-2,\,0}}) = \bar\partial_0(\Gamma_0^{n,\,n-2} - \overline{v_{r-3,\,0}^{(r-2)}})$, so the $(2n-2)$-form

$$\Omega := -(\Gamma_0^{n,\,n-2} - \overline{v_{r-3,\,0}^{(r-2)}}) + (\gamma_0^{n-1} - \zeta_{r-2,\,0} - \overline{\zeta_{r-2,\,0}}) - \overline{(\Gamma_0^{n,\,n-2} - \overline{v_{r-3,\,0}^{(r-2)}})}$$

is real and d-closed and its J_0-pure-type components $\Omega_0^{n,\,n-2}, \Omega_0^{n-1,\,n-1}, \Omega_0^{n-2,\,n}$ are given by the respective paratheses, with their respective signs, on the right of the above identity defining Ω.

If $\Omega_t^{n,\,n-2}, \Omega_t^{n-1,\,n-1}, \Omega_t^{n-2,\,n}$ stand for the J_t-pure-type components of Ω for any $t \in B$, they all depend in a C^∞ way on t. On the other hand, deforming identities (4.62) in a C^∞ way when the complex structure J_0 deforms to J_t, we find (non-unique) C^∞ families of J_t-type $(n-1,\,n-1)$-forms $(\zeta_{r-2,\,t})_{t\in B}$ and J_t-type $(n-2,\,n)$-forms $(v_{r-3,\,t}^{(r-2)})_{t\in B}$, whose elements for $t = 0$ are $\zeta_{r-2,\,0}$, respectively $v_{r-3,\,0}^{(r-2)}$, such that $\bar\partial_t\zeta_{r-2,\,t} = \partial_t v_{r-3,\,t}^{(r-2)}$ and $\bar\partial_t v_{r-3,\,t}^{(r-2)} = 0$ for $t \in B$. Then, the J_t-type $(n-1,\,n-1)$-form $\Omega_t^{n-1,\,n-1} + \zeta_{r-2,\,t} + \overline{\zeta_{r-2,\,t}}$ depends in a C^∞ way on $t \in B$. When $t = 0$, it equals γ_0^{n-1}, so it is positive definite. By continuity, it remains positive definite for all $t \in B$ sufficiently close to $0 \in B$, so it has a unique $(n-1)$-st root and the root is positive definite. In other words, there exists a unique C^∞ positive definite J_t-type $(1,\,1)$-form γ_t such that

$$\gamma_t^{n-1} = \Omega_t^{n-1,\,n-1} + \zeta_{r-2,\,t} + \overline{\zeta_{r-2,\,t}} > 0, \qquad t \in B,$$

after possibly shrinking B about 0. By construction, γ_t depends in a C^∞ way on t.

If we set $\Gamma_t^{n,\,n-2} := -\Omega_t^{n,\,n-2} + \overline{v_{r-3,\,t}^{(r-2)}}$ for all $t \in B$ close to 0, we get $\partial_t\gamma_t^{n-1} = \bar\partial_t\Gamma_t^{n,\,n-2} + \partial_t\zeta_{r-2,\,t}$. Since $\bar\partial_t\zeta_{r-2,\,t} = \partial_t v_{r-3,\,t}^{(r-2)}$ and $\bar\partial_t v_{r-3,\,t}^{(r-2)} = 0$, we conclude that γ_t is an E_r-sG metric for the complex structure J_t for all $t \in B$ close to 0. □

The link between the page-$(r-1)$-$\partial\bar\partial$ and the E_r-sG properties is spelt out in the following

Proposition 4.4.3. *([PSU20b, Proposition 5.2.]) Let $r \in \mathbb{N}^\star$ and let X be a* **page-$(r-1)$-$\partial\bar\partial$-manifold***. Then, every Gauduchon metric on X is* **E_r-sG***. In particular, X is an* **E_r-sG manifold**.

Proof. Let ω be a Gauduchon metric on X. Then, $\partial\omega^{n-1}$ is $\bar\partial$-closed and ∂-closed, hence d-closed. It is also ∂-exact, or equivalently, $\overline{E}_1$-exact, hence also $\overline{E}_r$-exact.

Now, thanks to Theorem 3.4.12, the page-$(r-1)$-$\partial\bar\partial$-property of X implies the equivalence between $\overline{E}_r$-exactness and E_r-exactness for d-closed pure-type forms. Consequently, $\partial\omega^{n-1}$ must be E_r-exact, so ω is an E_r-sG metric. □

Let $X_{u,v}$ be a Calabi-Eckmann manifold, i.e. any of the complex manifolds C^∞-diffeomorphic to $S^{2u+1} \times S^{2v+1}$ constructed by Calabi and Eckmann in [CE53] (see §.4.2.2). Recall that the $X_{0,v}$'s and the $X_{u,0}$'s are Hopf manifolds. By Theorem 4.2.7, $X_{u,v}$ does not admit any sG metric. However, we now prove the existence of E_2-sG metrics when $uv > 0$.

Proposition 4.4.4. *([PSU20b, Proposition 5.3.]) Let $X_{u,v}$ be a* **Calabi-Eckmann manifold** *of complex dimension ≥ 2. Let $u \leq v$.*

(i) *If $u > 0$, then $X_{u,v}$ does* **not** *admit sG* **metrics***, but it is an E_2-***sG manifold**.

(ii) *If $u = 0$, $X_{u,v}$ does* **not** *admit* **E_r-sG metrics** *for any r.*

Proof. By Borel's result in [Hir78, Appendix Two by A. Borel], we have

$$H_{\bar\partial}^{\bullet,\bullet}(X_{u,v}) \cong \frac{\mathbb{C}[x_{1,1}]}{(x_{1,1}^{u+1})} \otimes \bigwedge(x_{v+1,v}, x_{0,1}).$$

In other words, a model for the Dolbeault cohomology of the Calabi-Eckmann manifold $X_{u,v}$ is provided by the CDGA (see [NT78])

$$(V\langle x_{0,1},\, x_{1,1},\, y_{u+1,u},\, x_{v+1,v}\rangle,\ \bar\partial),$$

with differential

$$\bar\partial x_{0,1} = 0, \quad \bar\partial x_{1,1} = 0, \quad \bar\partial y_{u+1,u} = x_{1,1}^{u+1}, \quad \bar\partial x_{v+1,v} = 0.$$

Thus, if $u > 0$, we have a minimal model. (For $u = 0$ we have only a cofibrant model in the sense of [NT78].)

Moreover, ∂ acts on generators as follows [NT78]:

$$\partial x_{0,1} = x_{1,1} \ \ (\text{hence } \partial x_{1,1} = 0), \quad \partial y_{u+1,u} = 0, \quad \partial x_{v+1,v} = 0.$$

Next we determine the spaces $E_r^{n,\,n-1}$, for any $r \geq 1$, where $n = u + v + 1$.

Let us first focus on the case (i), i.e. $u > 0$. We need to consider the Dolbeault cohomology groups $H_{\bar\partial}^{n-1,\,n-1}(X_{u,v})$ and $H_{\bar\partial}^{n,\,n-1}(X_{u,v})$. They are given by

$$H_{\bar\partial}^{u+v,\,u+v}(X_{u,v}) = \langle x_{0,1} \cdot x_{1,1}^{u-1} \cdot x_{v+1,v}\rangle, \quad H_{\bar\partial}^{u+v+1,\,u+v}(X_{u,v}) = \langle x_{1,1}^{u} \cdot x_{v+1,v}\rangle.$$

Now we consider
$$H_{\bar\partial}^{u+v,\,u+v}(X_{u,v}) \xrightarrow{\partial} H_{\bar\partial}^{u+v+1,\,u+v}(X_{u,v}) \longrightarrow 0.$$
Since $\partial(x_{0,1}\cdot x_{1,1}^{u-1}\cdot x_{v+1,v}) = \partial(x_{0,1})\cdot x_{1,1}^{u-1}\cdot x_{v+1,v} = x_{1,1}^{u}\cdot x_{v+1,v}$, the first map is surjective. Therefore, $E_2^{n,n-1}(X_{u,v}) = 0$. Thus, any Gauduchon metric on $X_{u,v}$ is an E_2-sG metric.

Next we focus on the case (ii), i.e. $u = 0$, so $v \geq 1$. In this case $H_{\bar\partial}^{v+1,v}(X_{0,v}) = \langle x_{v+1,v}\rangle$. Notice that the Dolbeault cohomology groups $H_{\bar\partial}^{v-r+1,v+r-1}(X_{0,v})$ are all zero for every $r \geq 1$. Therefore, $E_r^{v-r+1,v+r-1}(X_{0,v}) = \{0\}$ for every $r \geq 1$. Meanwhile, from
$$\{0\} = E_r^{v-r+1,v+r-1}(X_{0,v}) \xrightarrow{d_r} E_r^{v+1,v}(X_{0,v}) \longrightarrow 0,$$
we get $E_r^{v+1,v}(X_{0,v}) = H_{\bar\partial}^{v+1,v}(X_{0,v})$ for every $r \geq 2$. So, the existence of an E_r-sG metric on $X_{0,v}$ would imply the existence of an sG metric, which would contradict Theorem 4.2.7.

□

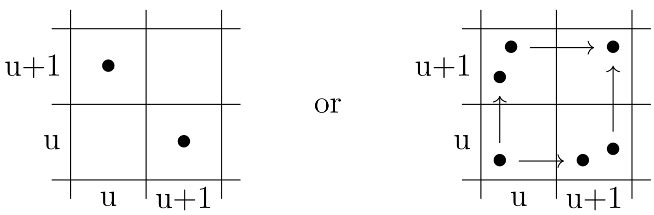
As a by-product of Borel's description of the Dolbeault cohomology of the Calabi-Eckmann manifolds $X_{u,v}$ used in the above proof, one gets that the Frölicher spectral sequence of $X_{u,v}$ satisfies $E_1 \neq E_2 = E_\infty$ when $u > 0$, whereas it degenerates at E_1 when $u = 0$. This latter fact implies that no Hopf manifold $X_{0,v}$ can have a pure De Rham cohomology, hence cannot be a page-r-$\partial\bar\partial$-manifold for any r. Indeed, if the De Rham cohomology were pure, then $X_{0,v}$ would be a $\partial\bar\partial$-manifold, a fact that is trivial to contradict.

By the previous result, all the Calabi-Eckmann manifolds that are not Hopf manifolds are E_2-sG manifolds. However, the next observation shows that they are not page-r-$\partial\bar\partial$-manifolds for any $r \in \mathbb{N}$.

Lemma 4.4.5. *([PSU20b, Lemma 5.4.]) Let $u, v \geq 0$ and let $X = S^{2u+1} \times S^{2v+1}$ be equipped with any of the Calabi-Eckmann complex structures. Assume that either $u \neq v$ or $u = v = 1$.*

Then, the De Rham cohomology of X is not pure.

Proof. When $u \neq v$, this was proved in [Ste18, p.29ff] as a consequence of the computation of the Hodge numbers of X by Borel in [Hir78, Appendix Two by A. Borel]. As explained in [Ste18], when $u = v$, the only consequence that one can draw from the numerical information given by Borel is that the only possible zigzags passing through the middle degree $2u + 1$ are either two dots or two length-three zigzags situated in the following bidegrees:



In the first case, the De Rham cohomology is pure, while in the second one it is not. They cannot be distinguished by the Hodge numbers. However, they may be distinguished by the Bott-Chern numbers. Specifically, $h_{BC}^{u+1,u+1} = 0$ in the former case and $h_{BC}^{u+1,u+1} = 1$ in the latter. (Recall that $h_{BC}^{p,q}$ counts 'top right' corners of zigzags in bidegree (p, q), i.e. those which have no outgoing edges). A calculation in [TT17] shows that the latter case occurs when $u = v = 1$. □

Remark 4.4.6. *It appears to be very likely that this Lemma also holds for arbitrary $u = v > 1$. To settle this issue, it suffices to determine whether $h_{BC}^{v+1,u+1} = 1$ for the higher-dimensional Calabi-Eckmann manifolds with $u = v$.*

4.5 Balanced metrics and manifolds

The notion that will be discussed in this section was introduced by Gauduchon in [Gau77b] under the name of *semi-Kähler* metric. These metrics were renamed *balanced* by Michelsohn in [Mic83] and this latter terminology is now widely used in the literature.

Definition 4.5.1. *([Gau77b, Définition 2], [Mic83]) Let X be a complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$.*

(i) A C^∞ positive definite $(1,1)$-form ω on X is said to be a **balanced metric** *if $d\omega^{n-1} = 0$.*

(ii) If X carries such a metric, X is said to be a **balanced manifold**.

Obviously, when $n = 2$, balanced metrics coincide with Kähler metrics, but we will see that in dimension $n \geq 3$, there exist many non-Kähler balanced manifolds.

4.5.1 Basic properties of balanced metrics

Since the form ω^{n-1} is *real* whenever ω is a positive definite $(1,1)$-form, the balanced condition $d\omega^{n-1} = 0$ is trivially equivalent to either of the following two equivalent conditions:

$$\partial\omega^{n-1} = 0 \iff \bar\partial\omega^{n-1} = 0.$$

Part (ii) of the next result shows that the balanced condition is a kind of dual to the Kähler condition. This accounts for the balanced metrics being sometimes called **co-Kähler**.

Lemma 4.5.2. *Let ω be a Hermitian metric on a complex manifold X with $\dim_{\mathbb{C}} X = n \geq 2$.*

(i) If ω is Kähler, then ω is balanced.

(ii) The metric ω is **balanced** *if and only if it is* **co-closed**.

Specifically, the following equivalences hold:

$$\omega \text{ is balanced} \iff d^\star\omega = 0 \iff \partial^\star\omega = 0 \iff \bar\partial^\star\omega = 0,$$

where $d^\star = d^\star_\omega$, $\partial^\star = \partial^\star_\omega$ and $\bar\partial^\star = \bar\partial^\star_\omega$ are the formal adjoints of $d : C^\infty_1(X, \mathbb{C}) \longrightarrow C^\infty_2(X, \mathbb{C})$, $\partial : C^\infty_{0,1}(X, \mathbb{C}) \longrightarrow C^\infty_{1,1}(X, \mathbb{C})$, resp. $\bar\partial : C^\infty_{1,0}(X, \mathbb{C}) \longrightarrow C^\infty_{1,1}(X, \mathbb{C})$ w.r.t. the L^2 inner product $\langle\langle\ ,\ \rangle\rangle = \langle\langle\ ,\ \rangle\rangle_\omega$ induced by ω.

(iii) The metric ω is **balanced** *if and only if $d\omega$ (or $\partial\omega$ or $\bar\partial\omega$) is ω-***primitive**.

Specifically, the following equivalences hold:

$$\begin{aligned}\omega \text{ is balanced} &\iff \Lambda_\omega(d\omega) = 0 \iff \Lambda_\omega(\partial\omega) = 0 \iff \Lambda_\omega(\bar\partial\omega) = 0\\ &\iff \omega^{n-2}\wedge d\omega = 0 \iff \omega^{n-2}\wedge\partial\omega = 0 \iff \omega^{n-2}\wedge\bar\partial\omega = 0,\end{aligned}$$

where Λ_ω is the adjoint of the Lefschetz operator $L_\omega := \omega\wedge\cdot$ w.r.t. the pointwise inner product $\langle\ ,\ \rangle = \langle\ ,\ \rangle_\omega$ induced by ω.

Proof. (i) The Leibniz rule gives $d\omega^{n-1} = (n-1)\,\omega^{n-2}\wedge d\omega$, so $d\omega^{n-1} = 0$ if $d\omega = 0$.

(ii) The equivalences follow at once from the Hodge star operator $\star = \star_\omega : \Lambda^{p,q}T^\star X \longrightarrow \Lambda^{n-q,n-p}T^\star X$ induced by ω being an *isomorphism* and from the standard formulae:

$$d^\star = -\star d\star, \quad \partial^\star = -\star\bar\partial\star, \quad \bar\partial^\star = -\star\partial\star, \quad \star\omega = \frac{\omega^{n-1}}{(n-1)!}.$$

(iii) The equivalences follow at once from the obvious identities:

$$d\omega^{n-1} = (n-1)\,\omega^{n-2}\wedge d\omega, \qquad \partial\omega^{n-1} = (n-1)\,\omega^{n-2}\wedge\partial\omega, \qquad \bar\partial\omega^{n-1} = (n-1)\,\omega^{n-2}\wedge\bar\partial\omega$$

and from the following standard pointwise characterisation of ω-*primitive* k-forms u on an n-dimensional complex manifold with $n\geq k$:

$$u \text{ is } \omega\text{-primitive} \stackrel{def}{\iff} \omega^{n-k+1}\wedge u = 0 \iff \Lambda_\omega u = 0. \tag{4.63}$$

In our case, $u\in\{d\omega,\,\partial\omega,\,\bar\partial\omega\}$, so $k=3$. □

(I) Balanced vs. locally conformally Kähler (lck) metrics

Picking up on (iii) of Lemma 4.5.2, we now point out the fact that balanced metrics are, in a certain sense, quite the opposite of another heavily studied class of metrics that will only be mentioned in passing in this book.

• Preliminaries on primitivity and the Hodge $\star$ operator

We start by recalling the following standard result for a complete proof of which the reader may consult e.g. [Dem96] or [Voi02].

Theorem 4.5.3. *(***Lefschetz decomposition of differential forms***) Let ω be a Hermitian metric on a complex manifold X with $\dim_{\mathbb{C}}X = n\geq 2$. For every $k\in\{0,\dots,2n\}$ and for every form $\alpha\in\Lambda^kT^\star X$, there exists a* **unique** *decomposition of α as*

$$\alpha = \alpha_0 + \sum_{1\leq r\leq N_{n,\,k}}\omega^r\wedge\alpha_r, \tag{4.64}$$

*where every form $\alpha_r\in\Lambda^{k-2r}T^\star X$ is ω-***primitive** *and $N_{n,\,k}$ is the maximum value of $r\geq 0$ such that $2r\leq k$.*

Moreover, the Lefschetz decompostion (4.64) of α is pointwise and its terms are mutually **orthogonal** *w.r.t. the pointwise inner product induced by ω.*

Sketch of proof. The statement is a consequence of the fact that, for any $k\leq n$ and any Hermitian metric ω on X, the multiplication map

$$L^l_\omega = \omega^l\wedge\cdot : \Lambda^kT^\star X\longrightarrow\Lambda^{k+2l}T^\star X$$

defined at every point of X is an **isomorphism** if $l=n-k$, is **injective** (but in general not surjective) for every $l<n-k$ and is **surjective** (but in general not injective) for every $l>n-k$. A k-form is said to be ω-**primitive** if it lies in the kernel of the multiplication map L^{n-k+1}_ω. Equivalently, the ω-primitive k-forms are precisely those that lie in the kernel of $\Lambda_\omega:\Lambda^kT^\star X\longrightarrow\Lambda^{k-2}T^\star X$. Thus, for every $k\leq n$ and every form $\alpha\in\Lambda^kT^\star X$, we have:

$$\alpha \text{ is } \omega\text{-primitive} \stackrel{def}{\iff} \omega^{n-k+1}\wedge\alpha = 0 \iff \Lambda_\omega\alpha = 0. \tag{4.65}$$

These facts follow from the formula:

$$[L^r_\omega,\,\Lambda_\omega] = r(k-n+r-1)\,L^{r-1}_\omega \quad \text{on } k\text{-forms}, \qquad r\geq 1, \tag{4.66}$$

which, in turn, can be proved by induction on $r \geq 1$ starting from its well-known version for $r = 1$:

$$[L_\omega,\, \Lambda_\omega] = (k-n)\,\mathrm{Id} \quad \text{on } k\text{-forms}, \tag{4.67}$$

where Id is the identity operator on forms. $\square$

We now recall the following standard formula (cf. e.g. [Voi02, Proposition 6.29, p. 150]) for images of *primitive* forms v of arbitrary bidegree $(p,\, q)$ under the Hodge $\star$ operator $\star = \star_\omega$ associated with an arbitrary Hermitian metric ω:

$$\star\, v = (-1)^{k(k+1)/2}\, i^{p-q}\, \frac{\omega^{n-p-q}\wedge v}{(n-p-q)!}, \quad \text{where } k := p+q, \tag{4.68}$$

We now use formula (4.68) to derive the following result that will come in handy later on.

Lemma 4.5.4. *Let $(X,\, \omega)$ be a complex Hermitian manifold with $\dim_{\mathbb{C}} X = n$ and let $\star = \star_\omega$ be the Hodge $\star$ operator associated with the metric ω. For any $(2,\, 1)$-form α, we have:*

$$\star\bigg(\alpha\wedge\frac{\omega^{n-2}}{(n-2)!}\bigg) = -i\,\Lambda_\omega\alpha. \tag{4.69}$$

Proof. Let α be a $(2,\, 1)$-form and let $\alpha = \alpha_{prim} + \omega\wedge u$ be its Lefschetz decomposition, where α_{prim} is a *primitive* (w.r.t. ω) $(2,\, 1)$-form and u is a $(1,\, 0)$-form. We can compute u by applying Λ_ω to the above identity: since $\Lambda_\omega\alpha_{prim} = 0$, we get $\Lambda_\omega\alpha = \Lambda_\omega(\omega\wedge u) = [\Lambda_\omega,\, L_\omega]\, u = (n-1)\, u$ (since $\Lambda_\omega u = 0$ and $[\Lambda_\omega,\, L_\omega] = -(p+q-n)\,\mathrm{Id}$ on $(p,\, q)$-forms – see e.g. [Dem97, Chapter VI, §.5.2]). Thus,

$$u = \frac{1}{n-1}\,\Lambda_\omega\alpha, \quad \text{hence} \quad \alpha = \alpha_{prim} + \frac{1}{n-1}\,\Lambda_\omega\alpha\wedge\omega.$$

Now, multiplying the last identity by $\omega^{n-2}/(n-2)!$ and because $\alpha_{prim}\wedge\omega^{n-2} = 0$ (due to α_{prim} being a primitive 3-form), we get

$$\alpha\wedge\frac{\omega^{n-2}}{(n-2)!} = \Lambda_\omega\alpha\wedge\frac{\omega^{n-1}}{(n-1)!}, \quad \text{hence} \quad \star\bigg(\alpha\wedge\frac{\omega^{n-2}}{(n-2)!}\bigg) = \star\bigg(\Lambda_\omega\alpha\wedge\frac{\omega^{n-1}}{(n-1)!}\bigg) = -i\,\Lambda_\omega\alpha.$$

This proves (4.69). To see the last identity, recall that $\Lambda_\omega\alpha$ is a $(1,\, 0)$-form, hence primitive (for bidegree reasons), hence the general formula (4.68) applied with $v = \Lambda_\omega\alpha$ reads

$$\star\Lambda_\omega\alpha = -i\,\Lambda_\omega\alpha\wedge\frac{\omega^{n-1}}{(n-1)!}.$$

It remains to apply $\star$ on either side of the last identity and to use the fact that $\star\star\,\Lambda_\omega\alpha = -\Lambda_\omega\alpha$. $\square$

• Locally conformally Kähler (lck) metrics

On the other hand, let us recall the following standard

Definition 4.5.5. *A C^∞ positive definite $(1,\, 1)$-form (i.e. a Hermitian metric) ω on a complex manifold X is said to be* **locally conformally Kähler (lck)** *if*

$$d\omega = \omega\wedge\theta \quad \textit{for some } C^\infty \text{ 1-}\textit{form } \theta \textit{ satisfying } d\theta = 0. \tag{4.70}$$

The 1-form θ is uniquely determined, real and called the **Lee form** of ω, often denoted by θ_ω.

Lemma 4.5.6. *An lck metric ω on a complex manifold X with $\dim_{\mathbb{C}} X = n \geq 2$ is Kähler if and only if its Lee form θ_ω vanishes.*

Proof. Since $1 \leq n-1$, we saw above that the Lefschetz linear map $L_\omega : C^\infty_1(X,\,\mathbb{C}) \longrightarrow C^\infty_3(X,\,\mathbb{C})$ of multiplication by ω is *injective*. Hence, for any 1-form θ on X, $\omega\wedge\theta = 0$ if and only if $\theta = 0$. $\square$

The obstruction to a given Hermitian metric ω being lck depends on whether $n=2$ or $n\geq 3$.

Lemma 4.5.7. *Let X be a complex manifold with $dim_{\mathbb{C}}X = n$.*

(i) If $n=2$, for any Hermitian metric ω there exists a unique, possibly non-closed, C^∞ 1-form $\theta=\theta_\omega$, that we still call the **Lee form** *of ω, such that $d\omega = \omega\wedge\theta$. Therefore,*

$$\omega \quad \textit{is lck} \iff d\theta_\omega = 0. \tag{4.71}$$

*Moreover, for any Hermitian metric ω, the 2-form $d\theta_\omega$ is ω-***primitive***, i.e. $\Lambda_\omega(d\theta_\omega)=0$, or equivalently, $\omega\wedge d\theta_\omega = 0$, while the Lee form is real and is explicitly given by the formula*

$$\theta_\omega = \Lambda_\omega(d\omega). \tag{4.72}$$

Alternatively, if $\theta_\omega = \theta^{1,\,0}_\omega + \theta^{0,\,1}_\omega$ is the splitting of θ_ω into components of pure types, we have

$$\theta^{1,\,0}_\omega = \Lambda_\omega(\partial\omega) = -i\bar\partial^\star\omega \tag{4.73}$$

and the analogous formulae for $\theta^{0,\,1}_\omega = \overline{\theta^{1,\,0}_\omega}$ obtained by taking conjugates.

(ii) If $n\geq 3$, for any Hermitian metric ω there exists a unique ω-primitive C^∞ 3-form $(d\omega)_{prim}$ and a unique C^∞ 1-form $\theta=\theta_\omega$, that we still call the **Lee form** *of ω, such that*

$$d\omega = (d\omega)_{prim} + \omega\wedge\theta.$$

The Lee form is real and is explicitly given by the formula

$$\theta_\omega = \frac{1}{n-1}\,\Lambda_\omega(d\omega). \tag{4.74}$$

Moreover,

$$\omega \quad \textit{is lck} \iff (d\omega)_{prim} = 0. \tag{4.75}$$

If ω is lck, the component of type $(1,\,0)$ of its Lee form is given by

$$\theta^{1,\,0}_\omega = \frac{1}{n-1}\,\Lambda_\omega(\partial\omega) = -\frac{i}{n-1}\,\bar\partial^\star\omega \tag{4.76}$$

and the analogous formulae obtained by taking conjugates hold for the $(0,\,1)$-component $\theta^{0,\,1}_\omega = \overline{\theta^{1,\,0}_\omega}$ of θ_ω.

Proof. Recall the torsion operator $\tau=\tau_\omega := [\Lambda_\omega,\,\partial\omega\wedge\cdot] : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p+1,\,q}(X,\,\mathbb{C})$ of order 0 and bidegree $(1,\,0)$ associated with ω in every bidegree $(p,\,q)$. (See [Dem84], [Dem97, VII, §.1] and Proposition 4.5.11 below for further details.) This definition of τ_ω yields:

$$\bar\tau^\star_\omega\omega = [(\bar\partial\omega\wedge\cdot)^\star, L_\omega](\omega) = (\bar\partial\omega\wedge\cdot)^\star(\omega^2).$$

On the other hand, if $\alpha^{1,\,0}$ is any $(1,\,0)$-form on X, let $\bar\xi_\alpha$ be the $(0,\,1)$-vector field defined by the requirement $\bar\xi_\alpha\lrcorner\omega = \alpha^{1,\,0}$. It is easily checked in local coordinates chosen about a given point x such that the metric ω is defined by the identity matrix at x, that the adjoint w.r.t. $\langle\ ,\ \rangle_\omega$ of the contraction operator by $\bar\xi_\alpha$ is given by the formula

$$(\bar{\xi}_\alpha\lrcorner\cdot)^\star = -i\alpha^{0,\,1}\wedge\cdot, \qquad \text{or equivalently} \qquad -i\bar{\xi}_\alpha\lrcorner\cdot = (\alpha^{0,\,1}\wedge\cdot)^\star,$$

where $\alpha^{0,\,1} = \overline{\alpha^{1,\,0}}$. Explicitly, if $\alpha^{0,\,1} = \sum\limits_k \bar{a}_k d\bar{z}_k$ on a neighbourhood of x, then $-i\bar{\xi}_\alpha\lrcorner\cdot = (\alpha^{0,\,1}\wedge\cdot)^\star = \sum\limits_k a_k\,\frac{\partial}{\partial\bar{z}_k}\lrcorner\cdot$ at x. Hence, $-i\bar{\xi}_\alpha\lrcorner\alpha^{0,\,1} = \sum\limits_k |a_k|^2 = |\alpha^{0,\,1}|^2_\omega$ at x. We have just got the pointwise formula

$$-i\bar{\xi}_\alpha\lrcorner\alpha^{0,\,1} = |\alpha^{0,\,1}|^2_\omega = |\alpha^{1,\,0}|^2_\omega \tag{4.77}$$

at every point of X.

Now, suppose that $d\omega = \omega\wedge\theta_\omega$ for some (necessarily real) 1-form θ_ω. Then, $\bar\partial\omega = \omega\wedge\theta_\omega^{0,\,1}$, so $(\bar\partial\omega\wedge\cdot)^\star = -i\Lambda_\omega(\bar{\xi}_\theta\lrcorner\cdot)$, where $\bar{\xi}_\theta := \bar{\xi}_\alpha$ with $\alpha^{1,\,0} = \theta_\omega^{1,\,0}$. The above formula for $\bar\tau^\star_\omega\omega$ translates to

$$\bar\tau^\star_\omega\omega = -i\Lambda_\omega(\bar{\xi}_\theta\lrcorner\omega^2) = -2i\Lambda_\omega(\omega\wedge(\bar{\xi}_\theta\lrcorner\omega)) = -2i[\Lambda_\omega,\,L_\omega](\bar{\xi}_\theta\lrcorner\omega) = -2i(n-1)\theta_\omega^{1,\,0}$$

The conclusion of this discussion is that, when $d\omega = \omega\wedge\theta_\omega$, formula (4.74) translates to

$$\theta_\omega^{1,\,0} = \frac{1}{n-1}\,\Lambda_\omega(\partial\omega) = \frac{1}{n-1}\,[\Lambda_\omega,\,\partial](\omega) = \frac{1}{n-1}\,i\bar\partial^\star_\omega\omega + \frac{1}{n-1}\,i\bar\tau^\star_\omega\omega = \frac{1}{n-1}\,i\bar\partial^\star_\omega\omega + 2\theta_\omega^{1,\,0},$$

which amounts to $\theta_\omega^{1,\,0} = -\frac{1}{n-1}\,i\bar\partial^\star_\omega\omega$. This proves (4.76) for an arbitrary n, hence also (4.73) when $n=2$, if the other statements in Lemma 4.5.7 have been proved. We now proceed to prove them.

(i) When $n=2$, the map $\omega\wedge\cdot : \Lambda^1T^\star X \longrightarrow \Lambda^3T^\star X$ is an isomorphism at every point of X. In particular, the 3-form $d\omega$ is the image of a unique 1-form θ under this map.

To see that $d\theta$ is primitive, we apply d to the identity $d\omega = \omega\wedge\theta$ to get

$$0 = d^2\omega = d\omega\wedge\theta + \omega\wedge d\theta.$$

Meanwhile, multiplying the same identity by θ, we get $d\omega\wedge\theta = \omega\wedge\theta\wedge\theta = 0$ since $\theta\wedge\theta = 0$ due to the degree of θ being 1. Therefore, $\omega\wedge d\theta = 0$, which means that the 2-form $d\theta$ is ω-primitive.

To prove formula (4.72), we apply Λ_ω to the identity $d\omega = \omega\wedge\theta$ to get

$$\Lambda_\omega(d\omega) = [\Lambda_\omega,\,L_\omega](\theta) = -[L_\omega,\,\Lambda_\omega](\theta) = -(1-2)\,\theta = \theta,$$

where we used the identities $\Lambda_\omega(\theta) = 0$ (for bidegree reasons) and $[L_\omega,\,\Lambda_\omega] = (k-n)\,\mathrm{Id}$ on k-forms (while here $k=1$ and $n=2$).

(ii) The splitting $d\omega = (d\omega)_{prim} + \omega\wedge\theta$ is the Lefschetz decomposition of $d\omega$ w.r.t. the metric ω. Applying Λ_ω, we get $\Lambda_\omega(d\omega) = [\Lambda_\omega,\,L_\omega](\theta) = -[L_\omega,\,\Lambda_\omega](\theta) = -(1-n)\,\theta = (n-1)\,\theta$, which proves (4.74).

The implication "ω lck $\implies$ $(d\omega)_{prim} = 0$" follows at once from the definitions. To prove the reverse implication, suppose that $(d\omega)_{prim} = 0$. We have to show that θ is d-closed. The assumption means that $d\omega = \omega\wedge\theta$, so $d\omega\wedge\theta = \omega\wedge\theta\wedge\theta = 0$ and $0 = d^2\omega = d\omega\wedge\theta + \omega\wedge d\theta$. Consequently, $\omega\wedge d\theta = 0$. Now, the multiplication of k-forms by ω^l is injective whenever $l\leq n-k$. When $n\geq 3$, if we choose $l=1$ and $k=2$ we get that the multiplication of 2-forms by ω is injective. Hence, the identity $\omega\wedge d\theta = 0$ implies $d\theta = 0$, so ω is lck. □

Note that, among other things, Lemma 4.5.7 proposes a generalisation of the Lee form to arbitrary (i.e. not necessarily lck) Hermitian metrics in a departure from the standard terminology.

Another standard observation is that the Lefschetz decomposition transforms nicely, hence the lck property is preserved, under conformal rescaling.

Lemma 4.5.8. *Let ω be an arbitrary Hermitian metric and let f be any smooth real-valued function on a compact complex n-dimensional manifold X. If $d\omega = (d\omega)_{prim} + \omega \wedge \theta_\omega$ is the Lefschetz decomposition of $d\omega$ w.r.t. the metric ω (with the understanding that $(d\omega)_{prim} = 0$ when $n = 2$), then*

$$d(e^f\omega) = e^f(d\omega)_{prim} + e^f\omega \wedge (\theta_\omega + df) \tag{4.78}$$

is the Lefschetz decomposition of $d(e^f\omega)$ w.r.t. the metric $\tilde{\omega} := e^f\omega$.

Consequently, ω is lck if and only if any conformal rescaling $e^f\omega$ of ω is lck, while the Lee form transforms as $\theta_{e^f\omega} = \theta_\omega + df$. In particular, when the lck metric ω varies in a fixed conformal class, the Lee form θ_ω varies in a fixed De Rham 1-class $\{\theta_\omega\}_{DR} \in H^1(X, \mathbb{R})$ called the **Lee De Rham class** *associated with the given conformal class. Moreover, the map $\omega \mapsto \theta_\omega$ defines a bijection from the set of lck metrics in a given conformal class to the set of elements of the corresponding Lee De Rham 1-class.*

Proof. Differentiating, we get $d(e^f\omega) = e^f d\omega + e^f\omega \wedge df = e^f(d\omega)_{prim} + e^f\omega \wedge (\theta_\omega + df)$. Meanwhile, it can immediately be checked that

$$\Lambda_{e^f\omega} = e^{-f}\Lambda_\omega,$$

so $\ker \Lambda_{e^f\omega} = \ker \Lambda_\omega$. Thus, the ω-primitive forms coincide with the $\tilde{\omega}$-primitive forms. Since $\Lambda_{\tilde{\omega}}$ commutes with the multiplication by any real-valued function, $e^f(d\omega)_{prim}$ is $\tilde{\omega}$-primitive, so (4.78) is the Lefschetz decompostion of $d\tilde{\omega}$ w.r.t. $\tilde{\omega}$. □

The point we will make is that the *balanced* and the *lck* conditions are opposite to each other and can only coexist in the Kähler case.

Lemma 4.5.9. *Let ω be a Hermitian metric on a complex manifold X with $\dim_{\mathbb{C}} X = n \geq 2$.*

Then, ω is both **balanced** *and* **lck** *if and only if ω is* **Kähler***.*

Proof. Let $d\omega = (d\omega)_{prim} + \omega \wedge \theta$ be the Lefschetz decomposition of the 3-form $d\omega$, where $(d\omega)_{prim}$ is an *ω-primitive* 3-form on X (the primitive part of $d\omega$) and θ is a smooth 1-form on X.

We know from (iii) of Lemma 4.5.2 that ω is balanced if and only if $d\omega = (d\omega)_{prim}$ (i.e. $d\omega$ is reduced to its primitive part), while Definition 4.5.5 tells us that ω is lck if and only if $d\omega = \omega \wedge \theta$ (i.e. $d\omega$ is reduced to its anti-primitive part).

We infer that ω is both balanced and lck if and only if $(d\omega)_{prim} = \omega\wedge\theta = 0$. Since $(d\omega)_{prim} \perp \omega\wedge\theta$, this is equivalent to $d\omega = 0$, namely to ω being Kähler. □

Based on the above statement at the level of metrics and on other reasons, we conjecture that the analogous statement at the level of compact manifolds ought to be true. (Cf. the similar Conjecture 4.5.10 in the balanced/skt case.)

Conjecture 4.5.10. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$.*

Then, X carries both a **balanced** *metric and a (possibly different)* **lck** *metric if and only if X carries a* **Kähler** *metric.*

(II) Michelsohn's torsion $(1, 0)$-form

We start by briefly recalling some standard commutation formulae that will be used further down.

Proposition 4.5.11. *([Dem84], see also [Dem97, VII, §.1]) Let* $(X,\,\omega)$ *be a complex Hermitian manifold with* $\dim_{\mathbb{C}} X = n$*. Consider the* **torsion operator** *(of order zero and type* $(1,\,0)$*) associated with the metric* ω*:*

$$\tau = \tau_\omega := [\Lambda,\, \partial\omega \wedge \cdot] : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p+1,\,q}(X,\,\mathbb{C}), \qquad p, q \in \{0, \dots, n\}.$$

The following **Hermitian commutation relations** *hold on differential forms on* X*:*

$$\begin{aligned}&(i)\ (\partial + \tau)^\star = i\,[\Lambda,\, \bar\partial]; \quad &(ii)\ (\bar\partial + \bar\tau)^\star = -i\,[\Lambda,\, \partial];\\ &(iii)\ \partial + \tau = -i\,[\bar\partial^\star,\, L]; \quad &(iv)\ \bar\partial + \bar\tau = i\,[\partial^\star,\, L],\end{aligned} \tag{4.79}$$

where the upper symbol $\star$ *stands for the formal adjoint w.r.t. the* L^2 *inner product induced by* ω*,* $L = L_\omega := \omega \wedge \cdot$ *is the Lefschetz operator of multiplication by* ω *and* $\Lambda = \Lambda_\omega := L^\star$*.*

Again following [Dem97, VII, §.1], recall that the commutation relations (4.79) immediately induce, via the Jacobi identity, the Bochner-Kodaira-Nakano-type identity:

$$\Delta'' = \Delta' + [\partial,\, \tau^\star] - [\bar\partial,\, \bar\tau^\star] \tag{4.80}$$

relating the $\bar\partial$-Laplacian $\Delta'' = [\bar\partial,\, \bar\partial^\star] = \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial$ and the ∂-Laplacian $\Delta' = [\partial,\, \partial^\star] = \partial\partial^\star + \partial^\star\partial$. This, in turn, induces the following **Bochner-Kodaira-Nakano-type identity** (cf. [Dem84]) in which the first-order terms have been absorbed in the twisted Laplace-type operator $\Delta'_\tau := [\partial + \tau,\, (\partial + \tau)^\star]$:

$$\Delta'' = \Delta'_\tau + T_\omega, \tag{4.81}$$

where $T_\omega := \left[\Lambda,\, [\Lambda,\, \frac{i}{2}\,\partial\bar\partial\omega]\right] - [\partial\omega \wedge \cdot,\, (\partial\omega \wedge \cdot)^\star]$ is a zeroth order operator of type $(0,\,0)$ associated with the torsion of ω. Formula (4.81) is obtained from (4.80) via the following identities (cf. [Dem84] or [Dem97, VII, §.1]) which have an interest of their own:

$$\begin{aligned}&(i)\ [L,\, \tau] = 3\,\partial\omega \wedge \cdot, \quad &&(ii)\ [\Lambda,\, \tau] = -2i\,\bar\tau^\star,\\ &(iii)\ [\partial,\, \bar\tau^\star] = -[\partial,\, \bar\partial^\star] = [\tau,\, \bar\partial^\star], \quad &&(iv)\ -[\bar\partial,\, \bar\tau^\star] = [\tau,\, (\partial + \tau)^\star] + T_\omega.\end{aligned} \tag{4.82}$$

Note that (iii) yields, in particular, that ∂ and $\bar\partial^\star + \bar\tau^\star$ anti-commute, hence by conjugation, $\bar\partial$ and $\partial^\star + \tau^\star$ anti-commute, i.e.

$$[\partial,\, \bar\partial^\star + \bar\tau^\star] = 0 \quad \text{and} \quad [\bar\partial,\, \partial^\star + \tau^\star] = 0. \tag{4.83}$$

With these commutation relations understood, we can prove the following addition to Lemma 4.5.2.

Lemma 4.5.12. *Let* $(X,\,\omega)$ *be a compact complex Hermitian manifold with* $\dim_{\mathbb{C}} X = n$*. The following identities hold:*

$$-\frac{1}{2}\,\bar\tau^\star_\omega\omega \overset{(i)}{=} \bar\partial^\star_\omega\omega \overset{(ii)}{=} i\,\Lambda_\omega(\partial\omega). \tag{4.84}$$

In particular, ω *is* **balanced** *if and only if* $\bar\tau^\star_\omega\omega = 0$*.*

Proof. • To prove identity (i) in (4.84), we will show that the multiplication operators by the $(1,\,0)$-forms $\bar\tau^\star\omega$ and $-2\bar\partial^\star\omega$ acting on functions, namely

$$\bar\tau^\star\omega\wedge\cdot,\;\; -2\bar\partial^\star\omega\wedge\cdot : C^\infty_{0,\,0}(X,\,\mathbb{C})\longrightarrow C^\infty_{1,\,0}(X,\,\mathbb{C}),$$

coincide by showing that their adjoints

$$(\bar\tau^\star\omega\wedge\cdot)^\star,\;\; (-2\bar\partial^\star\omega\wedge\cdot)^\star : C^\infty_{1,\,0}(X,\,\mathbb{C})\longrightarrow C^\infty_{0,\,0}(X,\,\mathbb{C})$$

coincide.

Let $\alpha\in C^\infty_{1,\,0}(X,\,\mathbb{C})$ and $g\in C^\infty_{0,\,0}(X,\,\mathbb{C})$ be arbitrary. We have:

$$\langle\langle(\bar\partial^\star\omega\wedge\cdot)^\star\alpha,\,g\rangle\rangle = \langle\langle\bar g\alpha,\,\bar\partial^\star\omega\rangle\rangle = \langle\langle\bar\partial(\bar g\alpha),\,\omega\rangle\rangle = \int\limits_X\bar\partial(\bar g\alpha)\wedge\star\omega = \int\limits_X\bar g\alpha\wedge\bar\partial\omega_{n-1}, \tag{4.85}$$

where we put $\omega_{n-1}:=\omega^{n-1}/(n-1)!$ and we used the standard identity $\star\omega=\omega_{n-1}$.

Meanwhile, we have:

$$\begin{aligned}\langle\langle(\bar\tau^\star\omega\wedge\cdot)^\star\alpha,\,g\rangle\rangle &= \langle\langle\bar g\alpha,\,\bar\tau^\star\omega\rangle\rangle = \langle\langle\bar g\,\bar\tau(\alpha),\,\omega\rangle\rangle = \langle\langle\bar g\,\Lambda(\bar\partial\omega\wedge\alpha),\,\omega\rangle\rangle\\ &= \langle\langle\bar\partial\omega\wedge\alpha,\,g\,\omega^2\rangle\rangle = \int\limits_X\bar\partial\omega\wedge\alpha\wedge\star(\bar g\omega^2) = -2\int\limits_X\bar g\alpha\wedge\bar\partial\omega\wedge\omega_{n-2}\\ &= -2\int\limits_X\bar g\alpha\wedge\bar\partial\omega_{n-1},\end{aligned} \tag{4.86}$$

where for the third identity on the first line we used the definition $\bar\tau=[\Lambda,\,\bar\partial\omega\wedge\cdot]$ of $\bar\tau$ and the fact that $\Lambda(\alpha)=0$ for bidegree reasons, while for the third identity on the second line we used the standard identity $\star\omega_2=\omega_{n-2}$, where $\omega_2:=\omega^2/2!$.

Comparing (4.85) and (4.86), we get $\langle\langle(\bar\tau^\star\omega\wedge\cdot)^\star\alpha,\,g\rangle\rangle=-2\,\langle\langle(\bar\partial^\star\omega\wedge\cdot)^\star\alpha,\,g\rangle\rangle$ for all α and g. Hence $(\bar\tau^\star\omega\wedge\cdot)^\star=-2\,(\bar\partial^\star\omega\wedge\cdot)^\star$, which proves (i) of (4.84).

• To prove identity (ii) in (4.84), we start from the Hermitian commutation relation (ii) in (4.79):

$$[\Lambda,\,\partial]=i\,(\bar\partial^\star+\bar\tau^\star)$$

that we apply to ω. We get the equivalent identities:

$$[\Lambda,\,\partial]\,\omega=i\bar\partial^\star\omega+i\bar\tau^\star\omega \iff \Lambda(\partial\omega)-\partial(\Lambda\omega)=-i\bar\partial^\star\omega \iff \Lambda(\partial\omega)=-i\bar\partial^\star\omega,$$

the last of which is (ii) of (4.84), where for the first equivalence we used the identity $\bar\tau^\star\omega=-2\,\bar\partial^\star\omega$ proved above as (i) in (4.84), while for the second equivalence we used the fact that $\Lambda\omega=n$, hence $\partial(\Lambda\omega)=0$. □

• **Alternative proof of identity (ii) in (4.84).** From $\bar\partial^\star=-\star\,\partial\star$ and $\star\omega=\omega^{n-1}/(n-1)!$ we get the first identity below:

$$\bar\partial^\star\omega=-\star\,\partial\frac{\omega^{n-1}}{(n-1)!}=-\star\left(\partial\omega\wedge\frac{\omega^{n-2}}{(n-2)!}\right)=i\,\Lambda(\partial\omega),$$

where the last identity follows from the general formula (4.69) applied to $\alpha=\partial\omega$.

Definition 4.5.13. *Let X be a compact complex manifold. For any Hermitian metric ω on X,* **Michelsohn's torsion $(1,\,0)$-form** *of ω is the $(1,\,0)$-form*

$$\rho = \rho_\omega := \Lambda_\omega(\partial\omega) = -i\bar\partial^\star_\omega\omega.$$

The two definitions of ρ_ω as either of the forms $\Lambda(\partial\omega)$ and $-i\bar\partial^\star\omega$ coincide thanks to (ii) of (4.84). Moreover, Lemma 4.5.7 shows that the $(1,\,0)$-component $\theta^{1,\,0}_\omega$ of the Lee form θ_ω is proportional to Michelsohn's torsion $(1,\,0)$-form ρ_ω for every Hermitian metric ω on X:

$$\theta^{1,\,0}_\omega = \frac{1}{n-1}\,\rho_\omega. \tag{4.87}$$

We conclude that much of Lemma 4.5.2 can be reworded as

Corollary 4.5.14. *Let X be a compact complex manifold. A Hermitian metric ω on X is* **balanced** *if and only if its* **Michelsohn's torsion $(1,\,0)$-form ρ_ω vanishes**.

This is still equivalent to the **vanishing** *of the* **Lee form θ_ω** *of ω.*

The order of events in Michelsohn's paper was different. The torsion $(1,\,0)$-form was defined in a different way (using the torsion tensor) in [Mic83, Definition 1.3] and one of the two alternative definitions of what we call Michelsohn's torsion $(1,\,0)$-form in the above Definition 4.5.13 was proved as a formula in [Mic83, Proposition 1.5].

(III) Link between the torsion tensor and Michelsohn's torsion $(1,\,0)$-form

We start by recalling some well-known definitions. Let $(X,\,\omega)$ be a compact complex Hermitian manifold with $\dim_{\mathbb{C}}X = n$. Let

$$D : C^\infty(X,\,TX) \longrightarrow C^\infty(X,\,T^\star X\otimes TX)$$

be the *Chern connection* of the Hermitian holomorphic vector bundle $(T^{1,\,0}X,\,\omega)$.

The *torsion tensor* $T_D \in C^\infty(X,\,\Lambda^2T^\star X\otimes TX)$ of D is defined as

$$T_D(V,\,W) := D_VW - D_WV - [V,\,W] \in C^\infty(X,\,TX), \qquad V, W \in C^\infty(X,\,TX). \tag{4.88}$$

Meanwhile, the splitting

$$\mathbb{C}\otimes TX = T^{1,\,0}X \oplus T^{0,\,1}X$$

of the complexified tangent bundle of X induces, for every vector field $V \in C^\infty(X,\,\mathbb{C}\otimes TX)$, a splitting $V = V^{1,\,0} + V^{0,\,1}$, where

$$V^{1,\,0} := \frac{1}{2}\,(V - iJV) \quad \text{and} \quad V^{0,\,1} := \frac{1}{2}\,(V + iJV).$$

The fact that a connection D is the *Chern connection* of $(T^{1,\,0}X,\,\omega)$ is equivalent to the following three properties holding simultaneously:

$$\begin{aligned}
&(a) \quad D(\omega) = 0 \quad \left(\overset{def}{\Longleftrightarrow} \; U\cdot\omega(V,\,W) = \omega(D_UV,\,W) + \omega(V,\,D_UW) \quad \forall\; U, V, W \in C^\infty(X,\,TX)\right);\\
&(b) \quad D(J) = 0 \quad \left(\overset{def}{\Longleftrightarrow} \; D_V(J\,W) = J(D_VW) \quad \forall\; V, W \in C^\infty(X,\,TX)\right);\\
&(c) \quad T_D(J\,V,\,J\,W) = T_D(V,\,W) \quad \forall\; V, W \in C^\infty(X,\,TX),
\end{aligned} \tag{4.89}$$

where J is the complex structure of X. Property (c) means that the $(1,\,1)$-part of the torsion vanishes in the sense that (c) is equivalent to any of the following equivalent properties holding on every coordinate open subset $U\subset X$:

$$D_{V^{0,\,1}}\varphi=0 \qquad \forall V^{0,\,1}\in C^\infty(U,\,T^{1,\,0}U),\;\; \forall\varphi=\sum_{j=1}^n\varphi_j\,\frac{\partial}{\partial z_j}\in H^0(U,\,T^{1,\,0}U)$$

$$\iff \quad D_{JV}\varphi=i\,D_V\varphi \qquad \forall\varphi\in H^0(U,\,T^{1,\,0}U),\;\; \forall V\in C^\infty(U,\,{}^{\mathbb{R}}TU)\ \text{such that}\ \overline{V}=V,$$

where ${}^{\mathbb{R}}TU$ is the *real* tangent bundle restricted to U.

• Computation of the torsion in local coordinates

Let $\omega=i\sum\limits_{i,\,j=1}^n\omega_{i\bar{j}}\,dz_i\wedge d\bar{z}_j$ be the expression of the Hermitian metric ω in local holomorphic coordinates $z_1,\dots,z_n$ on some open subset $U\subset X$. So, $\omega_{i\bar{j}}=\omega(\partial/\partial z_i,\,\partial/\partial z_j)$ on U for all i,j. If we denote by $a=(a_{jk})_{1\leq j,k\leq n}$ the matrix of $(1,\,0)$-forms on U that expresses the connection D in the local coordinates $(z_1,\dots,z_n)$, we have

$$D\bigg(\frac{\partial}{\partial z_j}\bigg)=\sum_{k=1}^n a_{jk}\,\frac{\partial}{\partial z_k}, \qquad j\in\{1,\dots,n\}. \tag{4.90}$$

For all i,j, we get:

$$\begin{aligned} d\omega_{i\bar{j}} &= d\bigg(\omega\bigg(\frac{\partial}{\partial z_i},\,\frac{\partial}{\partial z_j}\bigg)\bigg)=\omega\bigg(D\,\frac{\partial}{\partial z_i},\,\frac{\partial}{\partial z_j}\bigg)+\omega\bigg(\frac{\partial}{\partial z_i},\,D\,\frac{\partial}{\partial z_j}\bigg)\\ &= \sum_{k=1}^n\omega_{k\bar{j}}\,a_{ik}+\sum_{k=1}^n\omega_{i\bar{k}}\,\bar{a}_{jk}, \end{aligned}$$

where the second identity on the first line follows from (a) of (4.89) and the last identity follows from (4.90). In particular, we get $\partial\omega_{i\bar{j}}=\sum_{k=1}^n a_{ik}\,\omega_{k\bar{j}}$ for all i,j. In matrix form, this reads:

$$a=(\partial\omega)\,\omega^{-1}, \tag{4.91}$$

where we have put $\omega:=(\omega_{i\bar{j}})_{1\leq i,\,j\leq n}$ (a matrix of functions) and $\partial\omega:=(\partial\omega_{i\bar{j}})_{1\leq i,\,j\leq n}$ (a matrix of $(1,\,0)$-forms).

Recall that the torsion T_D is a *real* TX-valued 2-form with no $(1,\,1)$-component (by (c) of (4.89)), so if $T_D^{p,\,q}$ stands for the component of type $(p,\,q)$ of T_D, we have:

$$T_D=T_D^{2,\,0}+T_D^{0,\,2} \quad \text{and} \quad T_D^{2,\,0}=\overline{T_D^{0,\,2}}.$$

Proposition 4.5.15. *The expression in local coordinates $(z_1,\dots,z_n)$ of the TX-valued $(2,\,0)$-form $T_D^{2,\,0}$ (the $(2,\,0)$-component of the torsion form $T_D\in C^\infty(X,\,\Lambda^2T^\star X\otimes TX)$) is*

$$T_D^{2,\,0}=\sum_{\substack{1\leq j<k\leq n\\ 1\leq l\leq n}}T_{jk}^l\,dz_j\wedge dz_k\otimes\frac{\partial}{\partial z_l},$$

where

$$T_{jk}^l=\sum_\alpha\bigg(\frac{\partial\omega_{k\bar{\alpha}}}{\partial z_j}\,\omega^{\bar{\alpha}\,l}-\frac{\partial\omega_{j\bar{\alpha}}}{\partial z_k}\,\omega^{\bar{\alpha}\,l}\bigg)=\Gamma_{jk}^l-\Gamma_{kj}^l, \qquad 1\leq j,\,k,\,l\leq n, \tag{4.92}$$

and the matrix $(\omega^{\bar{j}\,k})_{j,\,k}$ is the inverse of the matrix $(\omega_{j\,\bar{k}})_{j,\,k}$, while the Γ_{jk}^l's are the Christoffel coefficients.

Proof. By the definition of T_D, we have

$$T_D^{2,\,0}\bigg(\frac{\partial}{\partial z_j},\,\frac{\partial}{\partial z_k}\bigg) = D_{\frac{\partial}{\partial z_j}}\frac{\partial}{\partial z_k} - D_{\frac{\partial}{\partial z_k}}\frac{\partial}{\partial z_j}, \qquad 1\leq j,\,k\leq n.$$

On the other hand, (4.90) yields: $D_{\frac{\partial}{\partial z_j}}\frac{\partial}{\partial z_k} = \sum\limits_{l=1}^n a_{kl}(\frac{\partial}{\partial z_j})\,\frac{\partial}{\partial z_l} = \sum\limits_{l=1}^n \Gamma_{jk}^l\,\frac{\partial}{\partial z_l}$ for all $j,\,k$. Meanwhile, (4.91) yields: $a_{kl} = \sum\limits_\alpha \partial\omega_{k\bar\alpha}\cdot\omega^{\bar\alpha\,l}$, hence $\Gamma_{jk}^l = a_{kl}(\frac{\partial}{\partial z_j}) = \sum\limits_\alpha \frac{\partial\omega_{k\bar\alpha}}{\partial z_j}\,\omega^{\bar\alpha\,l}$.

Putting together the last identities, we get:

$$D_{\frac{\partial}{\partial z_j}}\frac{\partial}{\partial z_k} = \sum_{\alpha,\,l=1}^n\frac{\partial\omega_{k\bar\alpha}}{\partial z_j}\,\omega^{\bar\alpha\,l}\,\frac{\partial}{\partial z_l} \quad\text{and, permuting } j \text{ and } k,\quad D_{\frac{\partial}{\partial z_k}}\frac{\partial}{\partial z_j} = \sum_{\alpha,\,l=1}^n\frac{\partial\omega_{j\bar\alpha}}{\partial z_k}\,\omega^{\bar\alpha\,l}\,\frac{\partial}{\partial z_l}.$$

Subtracting the last identity from the previous one, we get:

$$T_D^{2,\,0}\bigg(\frac{\partial}{\partial z_j},\,\frac{\partial}{\partial z_k}\bigg) = \sum_{l=1}^n T_{jk}^l\,\frac{\partial}{\partial z_l}, \qquad 1\leq j,\,k\leq n,$$

with the T_{jk}^l's given by formula (4.92). □

As an immediate consequence of Proposition 4.5.15, we get the first explicit link between the torsion and the metric ω at the level of differential forms.

Corollary 4.5.16. *The scalar-valued* $(2,\,1)$*-form* $\mathcal{T} = \mathcal{T}_\omega := T_D^{2,\,0}\lrcorner\omega \in C^\infty_{2,\,1}(X,\,\mathbb{C})$ *(obtained by contracting* ω *with the* $T^{1,\,0}X$*-components of* $T_D^{2,\,0}$ *and then multiplying the result by the scalar-valued* $(2,\,0)$*-form components of* $T_D^{2,\,0}$*) is given by the explicit formula:*

$$\mathcal{T}_\omega = \partial\omega.$$

Proof. In local coordinates, we have:

$$\mathcal{T} = \sum_{\substack{j<k\\ l}} T_{jk\bar l}\,dz_j\wedge dz_k\wedge d\bar z_l, \qquad\text{where}\quad T_{jk\bar l} = i\sum_\alpha T_{jk}^\alpha\,\omega_{\alpha\bar l}\quad\text{for all } j,\,k,\,l.$$

From this and from (4.92), for all $j,\,k,\,l$ we get:

$$\begin{aligned} T_{jk\bar l} &= i\sum_{\alpha,\,\beta}\bigg(\frac{\partial\omega_{k\bar\beta}}{\partial z_j}\,\omega^{\bar\beta\alpha} - \frac{\partial\omega_{j\bar\beta}}{\partial z_k}\,\omega^{\bar\beta\alpha}\bigg)\,\omega_{\alpha\bar l} = i\sum_\beta\bigg(\frac{\partial\omega_{k\bar\beta}}{\partial z_j}\,\delta_{l\beta} - \frac{\partial\omega_{j\bar\beta}}{\partial z_k}\,\delta_{l\beta}\bigg) \\ &= i\,\bigg(\frac{\partial\omega_{k\bar l}}{\partial z_j} - \frac{\partial\omega_{j\bar l}}{\partial z_k}\bigg). \end{aligned} \tag{4.93}$$

On the other hand, we have

$$\begin{aligned} \partial\omega &= i\sum_{j,\,k,\,l}\frac{\partial\omega_{k\bar l}}{\partial z_j}\,dz_j\wedge dz_k\wedge d\bar z_l = i\sum_{\substack{j<k\\ l}}\bigg(\frac{\partial\omega_{k\bar l}}{\partial z_j} - \frac{\partial\omega_{j\bar l}}{\partial z_k}\bigg)\,dz_j\wedge dz_k\wedge d\bar z_l \\ &= \sum_{\substack{j<k\\ l}} T_{jk\bar l}\,dz_j\wedge dz_k\wedge d\bar z_l = \mathcal{T}, \end{aligned}$$

where the last but one identity follows from (4.93). □

As a result of Lemma 4.5.12 and of Corollary 4.5.16, we get that Michelsohn's torsion $(1,0)$-form ρ_ω introduced in Definition 4.5.13 for any Hermitian metric ω arises as the *trace* of the *torsion* $(2,\,1)$-*form* $\mathcal{T}_\omega$.

Corollary 4.5.17. *Let X be a compact complex manifold. For any Hermitian metric ω on X, we have:*

$$\rho_\omega = \Lambda_\omega(\mathcal{T}_\omega) = \Lambda_\omega(T_D^{2,\,0} \lrcorner \omega). \tag{4.94}$$

In particular, we see that a Hermitian metric ω is *Kähler* if and only if its *torsion vanishes* (i.e. $\mathcal{T}_\omega = 0$), while ω is *balanced* if and only if the *trace of its torsion vanishes* (i.e. $\Lambda_\omega(\mathcal{T}_\omega) = 0$).

- **Definition of Michelsohn's torsion $(1,\,0)$-form in terms of $T_D^{2,\,0}$ in local coordinates**

The following definition is Michelsohn's original definition of the torsion $(1,\,0)$-form.

Definition 4.5.18. *([Mic83, Definition 1.3]) Let X be a compact complex manifold. For any Hermitian metric ω on X,* **Michelsohn's torsion $(1,\,0)$-form** *of ω is the global scalar-valued $(1,\,0)$-form $\rho = \rho_\omega \in C^\infty_{1,\,0}(X,\,\mathbb{C})$ defined in local coordinates as $\rho = \sum\limits_{k=1}^n \rho_k\, dz_k$, where*

$$\rho_k := \sum_{j=1}^n T_{kj}^j, \qquad k \in \{1,\dots,n\},$$

and where the T_{jk}^l's are defined in (4.92).

The following result shows that ρ_ω is indeed globally defined on X and that it coincides with the differential form introduced in Definition 4.5.13.

Lemma 4.5.19. *([Mic83, Proposition 1.5]) Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. For any Hermitian metric ω on X, the following identity holds:*

$$i\rho_\omega = \bar\partial^\star\omega, \tag{4.95}$$

where ρ_ω is Michelsohn's torsion $(1,\,0)$-form of ω from Definition 4.5.18.

Proof. We have $i\rho_\omega = i\sum_k \rho_k\, dz_k$, where

$$\rho_k = \sum_{j,\,\alpha}\left(\frac{\partial\omega_{j\bar\alpha}}{\partial z_k}\,\omega^{\bar\alpha j} - \frac{\partial\omega_{k\bar\alpha}}{\partial z_j}\,\omega^{\bar\alpha j}\right), \qquad k \in \{1,\dots,n\}, \tag{4.96}$$

as we see by permuting j and k and taking $l = j$ in formula (4.92) for the T_{jk}^l's.

On the other hand, we know from (ii) of (4.84) that $\bar\partial^\star\omega = i\Lambda_\omega(\partial\omega)$. It follows from this and from (4.96) that proving (4.95) is equivalent to proving the following formula in local coordinates:

$$\Lambda_\omega(\partial\omega) = \sum_k \rho_k\, dz_k. \tag{4.97}$$

Since this is a pointwise formula, proving it amounts to proving that

$$\langle\Lambda_\omega(\partial\omega),\, u\rangle_\omega = \langle\sum_k \rho_k\, dz_k,\, u\rangle_\omega \tag{4.98}$$

at every point for every $(1,\,0)$-form $u = \sum_r u_r\, dz_r$. Thus, we can fix an arbitrary point x_0 and choose the coordinates $z_1, \dots, z_n$ about it such that ω is given by the identity matrix at x_0 (i.e. $\omega_{i\bar{j}}(x_0) = \delta_{ij}$ for all i, j). It suffices to prove (4.98) at x_0 for every $(1,\,0)$-form $u = \sum_r u_r\, dz_r$.

Meanwhile, in local coordinates, we get:

$$\partial\omega = i\sum_{i,\,j,\,k}\frac{\partial\omega_{i\bar{j}}}{\partial z_k}\, dz_k\wedge dz_i\wedge d\bar{z}_j = i\sum_{\substack{i<k\\ j}}\left(\frac{\partial\omega_{k\bar{j}}}{\partial z_i} - \frac{\partial\omega_{i\bar{j}}}{\partial z_k}\right) dz_i\wedge dz_k\wedge d\bar{z}_j := \sum_{\substack{i<k\\ j}}\Gamma_{ik\bar{j}}\, dz_i\wedge dz_k\wedge d\bar{z}_j,$$

where in the last identity we denoted by $\Gamma_{ik\bar{j}}$ the coefficients of $\partial\omega$. Hence, for every $u = \sum_r u_r\, dz_r$, the l.h.s. term of (4.98) reads:

$$\begin{aligned}
\langle\Lambda_\omega(\partial\omega),\, u\rangle_\omega &= \langle\partial\omega,\, \omega\wedge u\rangle_\omega = -i\sum_{\substack{i<k\\ j}}\Gamma_{ik\bar{j}}\,\overline{(\omega_{k\bar{j}}\, u_i - \omega_{i\bar{j}}\, u_k)}\\
&= -i\sum_{i<k}\Gamma_{ik\bar{k}}\,\bar{u}_i + i\sum_{i<k}\Gamma_{ik\bar{i}}\,\bar{u}_k\\
&= \sum_{i<k}\left(\frac{\partial\omega_{k\bar{k}}}{\partial z_i} - \frac{\partial\omega_{i\bar{k}}}{\partial z_k}\right)\bar{u}_i - \sum_{i<k}\left(\frac{\partial\omega_{k\bar{i}}}{\partial z_i} - \frac{\partial\omega_{i\bar{i}}}{\partial z_k}\right)\bar{u}_k\\
&= \sum_{i,\,k}\frac{\partial\omega_{k\bar{k}}}{\partial z_i}\,\bar{u}_i - \sum_{i,\,k}\frac{\partial\omega_{i\bar{k}}}{\partial z_k}\,\bar{u}_i,
\end{aligned}\tag{4.99}$$

where the last three identities hold only at x_0 and the last but one follows from the explicit expression of the $\Gamma_{ik\bar{j}}$'s.

On the other hand, for every $u = \sum_r u_r\, dz_r$, the r.h.s. term of (4.98) at x_0 reads:

$$\begin{aligned}
\langle\sum_k \rho_k\, dz_k,\, u\rangle_\omega &= \sum_r \rho_r\bar{u}_r = \sum_{r,\,j,\,\alpha}\left(\frac{\partial\omega_{j\bar{\alpha}}}{\partial z_r}\,\omega^{\bar{\alpha}\,j} - \frac{\partial\omega_{r\bar{\alpha}}}{\partial z_j}\,\omega^{\bar{\alpha}\,j}\right)\bar{u}_r\\
&= \sum_{r,\,j}\frac{\partial\omega_{j\bar{j}}}{\partial z_r}\,\bar{u}_r - \sum_{r,\,j}\frac{\partial\omega_{r\bar{j}}}{\partial z_j}\,\bar{u}_r.
\end{aligned}\tag{4.100}$$

A comparison of (4.99) and (4.100) proves (4.98) at x_0. $\square$

(IV) Equality of the Laplacians Δ'_ω and Δ''_ω on functions when ω is balanced

We will prove the following general formula.

Lemma 4.5.20. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. For any Hermitian metric ω on X, the $\bar\partial$-Laplacian $\Delta'' = \Delta''_\omega$ and the ∂-Laplacian $\Delta' = \Delta'_\omega$ are connected in the following way on smooth* **functions** *$f : X \to \mathbb{C}$:*

$$(\Delta'' - \Delta')\, f = \star\Big(i\partial f\wedge\bar\partial\omega_{n-1} + i\bar\partial f\wedge\partial\omega_{n-1}\Big), \tag{4.101}$$

where $\omega_{n-1} := \omega^{n-1}/(n-1)!$.

Proof. When applied to functions $f \in C^\infty_{0,0}(X, \mathbb{C})$, the Hermitian Bochner-Kodaira-Nakano-type formula (4.80) reads:

$$(\Delta'' - \Delta')\, f = \tau^\star(\partial f) - \bar\tau^\star(\bar\partial f). \tag{4.102}$$

Now, the definition of τ implies that it is given by the multiplication by the $(1, 0)$-form $\Lambda(\partial\omega) = -i\bar\partial^\star\omega$ (see (4.84) for this equality of forms) on functions:

$$C^\infty_{0,0}(X, \mathbb{C}) \ni f \overset{\tau}{\longmapsto} (-i\bar\partial^\star\omega)\, f \in C^\infty_{1,0}(X, \mathbb{C}).$$

Hence, its adjoint is $\tau^\star = (-i\bar\partial^\star\omega \wedge \cdot)^\star : C^\infty_{1,0}(X, \mathbb{C}) \longrightarrow C^\infty_{0,0}(X, \mathbb{C})$. Therefore, (4.102) transforms to

$$(\Delta'' - \Delta')\, f = i\,(\bar\partial^\star\omega \wedge \cdot)^\star(\partial f) + i\,(\partial^\star\omega \wedge \cdot)^\star(\bar\partial f). \tag{4.103}$$

On the other hand, to compute $(\bar\partial^\star\omega \wedge \cdot)^\star(\partial f)$, notice that, when $\alpha = \partial f$ for some C^∞ function $f : X \to \mathbb{C}$, the equality of the first and last terms in (4.85) reads:

$$\int\limits_X \bar g\,(\bar\partial^\star\omega \wedge \cdot)^\star(\partial f)\, dV_\omega = \int\limits_X \bar g\,\partial f \wedge \bar\partial\omega_{n-1}$$

for every function $g : X \to \mathbb{C}$. Hence,

$$(\bar\partial^\star\omega \wedge \cdot)^\star(\partial f) = \star(\partial f \wedge \bar\partial\omega_{n-1}) \tag{4.104}$$

for every function f.

We see that (4.101) follows by putting (4.103) and (4.104) together. $\square$

Corollary 4.5.21. *([Gau77b, Proposition 1.]) Suppose there exists a* **balanced** *metric ω on a compact complex manifold X. Then*

$$\Delta''_\omega\, f = \Delta'_\omega\, f \tag{4.105}$$

for any C^∞ function $f : X \to \mathbb{C}$.

Recall that $\Delta''_\omega = \Delta'_\omega$ in every bidegree when ω is **Kähler**.

4.5.2 Basic properties of balanced manifolds

On a given complex manifold X, we will deal with semi-positive bidegree $(1, 1)$-currents T that are the $(1, 1)$-components of d-exact currents of degree 2. Specifically, they are of the shape $T = (dS)^{1,1}$, where S is a current of degree 1. Since T is real, S can be chosen to be real as well and will henceforth be supposed to be real. If $S = S^{1,0} + S^{0,1}$ is the decomposition of S into pure-type currents, then $S^{0,1} = \overline{S^{1,0}}$ and $T = \partial S^{0,1} + \bar\partial S^{1,0}$. For bidegree reasons, we also have: $\partial S^{1,0} = 0$ (the vanishing $(2, 0)$-component of T) and $\bar\partial S^{0,1} = 0$ (the vanishing $(0, 2)$-component of T). In particular, T need not be d-closed, but $\partial\bar\partial T = 0$ and $T \in \operatorname{Im}\partial + \operatorname{Im}\bar\partial$. In other words, T is Aeppli cohomologous to 0, namely $\{T\}_A = 0 \in H^{1,1}_A(X, \mathbb{R})$.

Definition 4.5.22. *A $\partial\bar\partial$-closed form α or current T that represents the zero Aeppli cohomology class is called* **Aeppli-exact***.*

If T is a *real* Aeppli-exact bidegree $(1,\,1)$-current, there exist $(1,\,0)$, resp. $(0,\,1)$-currents $S^{1,\,0}$ and $S^{0,\,1}$ such that $S^{0,\,1}=\overline{S^{1,\,0}}$ and $T=\partial S^{0,\,1}+\bar\partial S^{1,\,0}$. (Indeed, if $T=\partial S_1+\bar\partial S_2$ with S_1 of bidegree $(1,\,0)$ and S_2 of bidegree $(0,\,1)$, we can take $S^{1,\,0}:=(S_1+\overline{S}_2)/2$ and $S^{0,\,1}:=(S_2+\overline{S}_1)/2$.) Moreover, T is the $(1,\,1)$-component of the d-exact current dS of degree 2, where $S:=S^{1,\,0}+S^{0,\,1}$.

We have thus noticed the following fact.

Lemma 4.5.23. *Let T be a real current or form of bidegree $(1,\,1)$ on a complex manifold X. Then, T is* **Aeppli-exact** *if and only if T is the $(1,\,1)$-component of a d-exact current dS of degree 2.*

In this case, the current S of degree 1 can be chosen to be real.

Michelsohn's following intrinsic characterisation of balanced manifolds says that a compact complex manifold X is *balanced* if and only if any *semi-positive Aeppli-exact* current T of bidegree $(1,\,1)$ on X is *zero*.

Proposition 4.5.24. *([Mic83, Theorem 4.7.]) Let X be a compact complex manifold. Then, X carries a* **balanced** *metric ω if and only if there is no non-zero current T of bidegree $(1,1)$ on X such that $T\geq 0$ and T is the $(1,\,1)$-component of some d-exact current of degree 2 on X.*

Proof. Let $\dim_{\mathbb{C}}X=n$.

• To prove the trivial implication, suppose the existence of a balanced metric ω on X. If a non-zero $(1,\,1)$-current $T=(dS)^{1,\,1}\geq 0$ existed on X for some current S of degree 1 (where the superscript $(1,\,1)$ stands for the component of bidegree $(1,\,1)$ of the 2-current dS), then on the one hand we would have:

$$\int\limits_X T\wedge\omega^{n-1}>0$$

since $T\wedge\omega^{n-1}\geq 0$ as a non-zero $(n,\,n)$-current on X, while on the other hand we would have:

$$\int\limits_X T\wedge\omega^{n-1}=\int\limits_X dS\wedge\omega^{n-1}=\int\limits_X S\wedge d\omega^{n-1}=0$$

by Stokes and the balanced property $d\omega^{n-1}=0$ of ω. This would be a contradiction.

• To prove the reverse implication, we use Sullivan's technique of [Sul76] based on the elementary Hahn-Banach theorem. The proof is very similar to that of Proposition 4.2.5, but needs the following extra ingredient that uses the following pieces of notation on a given n-dimensional compact complex manifold X:

· $\mathcal{E}'_p(X)$ stands for the space of *currents of dimension p (= of degree $2n-p$)* on X. By definition, this is the topological dual of the space $C^\infty_p(X,\,\mathbb{C})$ of C^∞ $\mathbb{C}$-valued p-forms on X equipped with the C^∞ topology. Moreover, $\mathcal{E}'_p(X)_{\mathbb{R}}$ is the subspace of $\mathcal{E}'_p(X)$ consisting of *real* currents.

· $\mathcal{E}'_{r,\,s}(X)$ stands for the space of *currents of bidimension $(r,\,s)$ (= of degree $(n-r,\,n-s)$)* on X. By definition, this is the topological dual of the space $C^\infty_{r,\,s}(X,\,\mathbb{C})$ of C^∞ $\mathbb{C}$-valued $(r,\,s)$-forms on X equipped with the C^∞ topology. Moreover, $\mathcal{E}'_{r,\,r}(X)_{\mathbb{R}}$ is the subspace of $\mathcal{E}'_{r,\,r}(X)$ consisting of *real* currents.

· $\pi_{r,\,r}:\bigoplus\limits_{r',\,s'}\mathcal{E}'_{r',\,s'}(X)\longrightarrow\mathcal{E}'_{r,\,r}(X)$ is the *projection* and $d_{r,\,r}:=\pi_{r,\,r}\circ d$.

· a **superscript** $(r,\,s)$ stands for the component of **bidegree** $(r,\,s)$ of the form or current to which it is attached.

Lemma 4.5.25. *([Mic83, Lemma 4.8]) Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The vector space*

$$d_{n-1,\,n-1}\mathcal{E}'_{2n-1}(X)_{\mathbb{R}} := \bigg\{(dS)^{1,\,1} \;\mid\; S \text{ is a real current of degree } 1 \text{ on } X\bigg\}$$

is **closed** *in $\mathcal{E}'_{n-1,\,n-1}(X)_{\mathbb{R}}$ with respect to the weak topology of currents.*

Proof. The linear map $d_{n-1,\,n-1} : \mathcal{E}'_{2n-1}(X)_{\mathbb{R}} \longrightarrow \mathcal{E}'_{n-1,\,n-1}(X)_{\mathbb{R}}$ is the dual of the linear map $d : C^\infty_{n-1,\,n-1}(X,\,\mathbb{R}) \longrightarrow C^\infty_{2n-1}(X,\,\mathbb{R})$. By a standard fact in functional analysis, any of these two maps has *closed* image if and only if the other does. We will prove that the latter map has *closed* image.

Another standard fact in functional analysis stipulates that if a continuous linear operator T has a *finite codimensional* image $\operatorname{Im} T$, then this image is *closed*.

Now, fix an arbitrary Hermitian metric γ on X. We know from standard Hodge theory (ellipticity of the d-Laplacian Δ induced by γ, compactness of X and the standard elliptic theory) that there is an L^2_γ-orthogonal 3-space decomposition:

$$C^\infty_p(X,\,\mathbb{R}) = \ker\Delta \oplus \operatorname{Im} d \oplus \operatorname{Im} d^\star$$

in which $\ker d = \ker\Delta \oplus \operatorname{Im} d$, $\mathcal{H}^p_\Delta(X,\,\mathbb{C}) := \ker\Delta$ is finite-dimensional and the subspaces $\operatorname{Im} d$ and $\operatorname{Im} d^\star$ are closed. In particular, the subspace:

$$B := d(C^\infty_{2n-2}(X,\,\mathbb{R})) \subset C^\infty_{2n-1}(X,\,\mathbb{R})$$

is *closed* in $C^\infty_{2n-1}(X,\,\mathbb{R})$ because it is closed in $C^\infty_{2n-1}(X,\,\mathbb{R}) \cap \ker d$ (thanks to $\dim_{\mathbb{R}}\mathcal{H}^p_\Delta(X,\,\mathbb{R}) < +\infty$) and $\ker d$ is closed in $C^\infty_{2n-1}(X,\,\mathbb{R})$.

Thus, it suffices to prove that

$$\operatorname{codim}_B B_0 < +\infty, \tag{4.106}$$

where $B_0 := d(C^\infty_{n-1,\,n-1}(X,\,\mathbb{R})) \subset B = d(C^\infty_{2n-2}(X,\,\mathbb{R}))$. Indeed, (4.106) will imply that B_0 is closed in B, hence B_0 will be closed in $C^\infty_{2n-1}(X,\,\mathbb{R})$ since we already know that B is closed in $C^\infty_{2n-1}(X,\,\mathbb{R})$.

To prove (4.106), for every form $\alpha \in C^\infty_{2n-2}(X,\,\mathbb{R})$ consider its decomposition $\alpha = \alpha^{n,\,n-2} + \alpha^{n-1,\,n-1} + \overline{\alpha^{n,\,n-2}}$ into pure-type forms, where $\alpha^{n-1,\,n-1}$ is real. Now, for bidegree reasons we have $\bar\partial\overline{\alpha^{n,\,n-2}} = 0$ and since $\ker\bar\partial = \mathcal{H}^{n-2,\,n}_{\Delta''}(X,\,\mathbb{C}) \oplus \operatorname{Im}\bar\partial$, we get a unique splitting:

$$\overline{\alpha^{n,\,n-2}} = \alpha_h^{n-2,\,n} + \bar\partial a^{n-2,\,n-1}$$

for some Δ''_γ-harmonic $(n-2,\,n)$-form $\alpha_h^{n-2,\,n}$ and some $(n-2,\,n-1)$-form $a^{n-2,\,n-1}$.

Thus, for every form $\alpha \in C^\infty_{2n-2}(X,\,\mathbb{R})$, we get:

$$\begin{aligned}\alpha &= \left(\overline{\alpha_h^{n-2,\,n}} + \partial\overline{a^{n-2,\,n-1}}\right) + \alpha^{n-1,\,n-1} + \left(\alpha_h^{n-2,\,n} + \bar\partial a^{n-2,\,n-1}\right)\\ &= \overline{\alpha_h^{n-2,\,n}} + d\overline{a^{n-2,\,n-1}} + (\alpha^{n-1,\,n-1} - \partial a^{n-2,\,n-1} - \bar\partial\overline{a^{n-2,\,n-1}}) + \alpha_h^{n-2,\,n} + da^{n-2,\,n-1}.\end{aligned}$$

Taking d, we further get:

$$d\alpha = d\beta^{n-1,\,n-1} + d(\alpha_h^{n-2,\,n} + \overline{\alpha_h^{n-2,\,n}}),$$

where $\beta^{n-1,\,n-1} := \alpha^{n-1,\,n-1} - \partial a^{n-2,\,n-1} - \bar\partial\overline{a^{n-2,\,n-1}} \in C^\infty_{n-1,\,n-1}(X,\,\mathbb{R})$.

We conclude that $B = B_0 + dV$, where $V := \{\alpha_h^{n-2,\,n} + \overline{\alpha_h^{n-2,\,n}} \;\mid\; \alpha_h^{n-2,\,n} \in \mathcal{H}^{n-2,\,n}_{\Delta''}(X,\,\mathbb{C})\}$. Hence

$$\dim_{\mathbb{R}}(B/B_0) \leq \dim_{\mathbb{R}} V < +\infty,$$

where the last inequality follows from $\dim\mathcal{H}^{n-2,\,n}_{\Delta''}(X,\,\mathbb{C})<+\infty$. This proves (4.106) and we are done. $\square$

End of proof of Proposition 4.5.24. By Lemma 4.5.25, the vector subspace $\mathcal{A}:=d_{n-1,\,n-1}\mathcal{E}'_{2n-1}(X)_{\mathbb{R}}$ of the locally convex space $\mathcal{E}'_{n-1,\,n-1}(X)_{\mathbb{R}}$ is **closed**.

Meanwhile, if we fix an arbitrary Hermitian metric γ on X, the subset

$$\mathcal{B}:=\left\{T\in\mathcal{E}'_{n-1,\,n-1}(X)_{\mathbb{R}}\ \mid\ T\geq 0\ \text{and}\ \int\limits_X T\wedge\gamma^{n-1}=1\right\}\subset\mathcal{E}'_{n-1,\,n-1}(X)_{\mathbb{R}}$$

is **compact** and **convex**.

If we assume that there exists no non-zero current T of bidegree $(1,\,1)$ (equivalently, of bidimension $(n-1,\,n-1)$, i.e. lying in $\mathcal{E}'_{n-1,\,n-1}(X)$) on X such that $T\geq 0$ and T is the $(1,\,1)$-component of some d-exact current of degree 2 on X, we have:

$$\mathcal{A}\cap\mathcal{B}=\emptyset.$$

By the *Hahn-Banach Separation Theorem* for locally convex spaces, there exists a real C^∞ $(n-1,\,n-1)$ form Ω on X such that:

(a) $\Omega_{|\mathcal{A}}\equiv 0$, which is equivalent to $\int\limits_X dS\wedge\Omega=0$ for every real 1-current S on X. This is further equivalent to $d\Omega=0$.

(b) $\Omega_{|\mathcal{B}}>0$, which is equivalent to $\int\limits_X T\wedge\Omega>0$ for every non-zero $(1,\,1)$-current $T\geq 0$ on X. This is further equivalent to $\Omega>0$ on X.

Using Lemma 4.0.1, we conclude from (a) and (b) that there is a unique C^∞ $(1,\,1)$-form $\omega>0$ on X such that $\omega^{n-1}=\Omega>0$ and $d\omega^{n-1}=0$ on X. This means that ω is a balanced metric on X. $\square$

Next, we give two functorial properties of balanced manifolds.

Proposition 4.5.26. *([Mic83, Proposition 1.9.]) Let X and Y be complex manifolds.*

(i) If X and Y are **balanced***, the product manifold $X\times Y$ is* **balanced***.*

(ii) If X is **balanced** *and if there exists a surjective proper holomorphic submersion $\pi:X\longrightarrow Y$, then Y is* **balanced***.*

Proof. (i) Let $n=\dim_{\mathbb{C}}X$ and $m=\dim_{\mathbb{C}}Y$. Let ω_X, resp. ω_Y, be a balanced metric on X, resp. Y. Finally, let π_X and π_Y be the projections of $X\times Y$ onto X, resp. Y. Since d commutes with the inverse image maps, we get: $d\pi_X^\star\omega_X^{n-1}=0$ and $d\pi_Y^\star\omega_Y^{m-1}=0$.

The induced product metric on $X\times Y$ is $\omega=\pi_X^\star\omega_X+\pi_Y^\star\omega_Y$. We have:

$$\omega^{n+m-1}=\binom{n+m-1}{n-1}\pi_X^\star\omega_X^{n-1}\wedge\pi_Y^\star\omega_Y^m+\binom{n+m-1}{n}\pi_X^\star\omega_X^n\wedge\pi_Y^\star\omega_Y^{m-1}.$$

Since each of the forms $\omega_X^{n-1},\omega_X^n,\omega_Y^{m-1},\omega_Y^m$ is d-closed on its respective manifold (either thanks to the balanced hypothesis or for bidegree reasons), we infer that $d\omega^{n+m-1}=0$ on $X\times Y$. Thus, ω is a balanced metric on $X\times Y$.

(ii) Let $n := \dim_{\mathbb{C}} X \geq m := \dim_{\mathbb{C}} Y$ and let ω_X be a balanced metric on X. Let

$$\Omega_Y := \pi_\star(\omega_X^{n-1})$$

be the direct image (i.e. the push-forward) of ω_X^{n-1} to Y under π. Since π is proper and holomorphic, Ω_Y is a well-defined current on Y of the same bidimension $(= (1,\, 1))$ as ω_X^{n-1}. Equivalently, Ω_Y is a current of bidegree $(m-1,\, m-1)$ on Y. Moreover, since π is a submersion, Ω_Y is actually a C^∞ form on Y obtained by integrating ω_X^{n-1} on the fibres $(\pi^{-1}(y))_{y\in Y}$ of π. (See [Dem97, I, §.2.C.1] for the basics of direct images of forms and currents.)

On the one hand, d commutes with the map $\pi_\star$. Hence, $d\Omega_Y = \pi_\star(d\omega_X^{n-1}) = 0$ on Y.

On the other hand, direct images of positive forms are positive. (See e.g. [Dem97, III, §.1].) In our case where π is a submersion, this is equivalent to saying that, in local coordinates, the matrix formed by the coefficients of Ω_Y, which are obtained by integrating over the fibres of π the entries of the positive definite matrix formed by the coefficients of ω_X^{n-1}, is positive definite. Thus, $\Omega_Y > 0$ on Y. By Lemma 4.0.1, there exists a unique C^∞ positive definite $(1,\, 1)$-form $\omega_Y > 0$ on Y such that $\omega_Y^{m-1} = \Omega_Y$.

Summing up, $\omega_Y \in C^\infty_{1,\,1}(Y,\, \mathbb{R})$, $\omega_Y > 0$ and $d\omega_Y^{m-1} = 0$ on Y, so ω_Y is a balanced metric on Y. □

4.5.3 The Iwasawa manifold revisited and deformations of balanced manifolds

In this subsection, we show that the Iwasawa manifold $I^{(3)}$ discussed in §.1.3.3 is *balanced*. Actually, all the manifolds in a class containing the Iwasawa manifold will be seen to be balanced. Thus, we get a large class of examples of balanced non-Kähler manifolds.

We will also include a general discussion of some related issues. The point of view presented here is mostly that of [AB91a].

(I) Complex parallelisable manifolds

We start with the simple observation that any holomorphic $(n-1)$-form on an n-dimensional compact complex manifold is d-closed.

Observation 4.5.27. *([Nak75, Lemma 1.2.] or [AB91a, Remark 3.1.]) Let X be any compact complex manifold with $\dim_{\mathbb{C}} X = n$. Then, for every form $u \in C^\infty(X,\, \Lambda^{n-1,\,0}T^\star X)$ such that $\bar\partial u = 0$, we have $du = 0$.*

Proof. Let $u \in C^\infty(X,\, \Lambda^{n-1,\,0}T^\star X)$ such that $\bar\partial u = 0$. Then $du = \partial u$ is of type $(n,\, 0)$ and $d\bar u = \overline{du} = \overline{\partial u}$ is of type $(0,\, n)$.

On the one hand, $i^{n^2} du\wedge d\bar u \geq 0$ as an $(n,\, n)$-form on X. Indeed, in local holomorphic coordinates $z_1, \dots, z_n$ we can write:

$$du = f\, dz_1 \wedge \dots dz_n, \quad \text{hence} \quad i^{n^2} du \wedge d\bar u = |f|^2\, idz_1 \wedge d\bar z_1 \dots idz_n \wedge d\bar z_n \geq 0,$$

for some smooth function f. Thus, $i^{n^2} du \wedge d\bar u = 0$ if and only if $f \equiv 0$ if and only if $du = 0$.

Meanwhile, Stokes gives:

$$\int\limits_X i^{n^2} du \wedge d\bar u = i^{n^2} \int\limits_X d(u \wedge d\bar u) = 0.$$

Hence, $i^{n^2} du \wedge d\bar u = 0$ everywhere on X, hence $du = 0$ everywhere on X. □

Corollary 4.5.28. *Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. Suppose there is a form $u \in C^\infty(X, \Lambda^{n-1,0}T^\star X)$ such that $\bar\partial u = 0$. Then, the $(n-1, n-1)$-form $i^{(n-1)^2} u \wedge \bar u$ satisfies:*

$$i^{(n-1)^2} u \wedge \bar u \geq 0 \quad \text{and} \quad d\left(i^{(n-1)^2} u \wedge \bar u\right) = 0 \quad \text{on} \quad X.$$

Proof. The pointwise inequality can be checked for any $(n-1, 0)$-form u by a trivial calculation. If $\bar\partial u = 0$, then $du = 0$ by Observation 4.5.27. Then $d\bar u = 0$ and the second part follows. □

Now recall the following standard notion.

Definition 4.5.29. *([Wan54]) A compact complex manifold X is said to be* **complex parallelisable** *if its holomorphic tangent bundle $T^{1,0}X$ is* **trivial**.

This condition is, of course, equivalent to the sheaf of germs of holomorphic 1-forms Ω^1_X being trivial. If $n = \dim_{\mathbb{C}} X$, the *complex parallelisable* condition is equivalent to the existence of n holomorphic vector fields $\theta_1, \dots, \theta_n \in H^0(X, T^{1,0}X)$ that are linearly independent at every point of X. It is again equivalent to the existence of n holomorphic 1-forms $\varphi_1, \dots, \varphi_n \in H^0(X, \Omega^1_X)$ that are linearly independent at every point of X.

Theorem 4.5.30. *([Wan54]) A compact complex manifold X is* **complex parallelisable** *if and only if X is the compact quotient $X = G/\Gamma$ of a simply connected, connected* **complex** *Lie group G by a discrete subgroup $\Gamma \subset G$.*

Proof. See [Wan54]. □

In particular, for any compact *complex parallelisable* manifold X, $H^0(X, T^{1,0}X) \simeq \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of G.

Definition 4.5.31. *A* **nilmanifold** *(resp.* **solvmanifold***) X is a compact complex manifold $X = G/\Gamma$ that can be realised as a compact quotient of a simply connected, connected* **nilpotent** *(resp.* **solvable***) real Lie group G by a lattice Γ of maximal rank in G.*

Note that although a nilmanifold or a solvmanifold X carries a complex structure, the Lie group G need not be a *complex* Lie group unless X is complex parallelisable. There exist many non-complex parallelisable nilmanifolds and solvmanifolds. The Heisenberg group defining the Iwasawa manifold being a *nilpotent complex Lie group*, we have

Corollary 4.5.32. *The Iwasawa manifold is a* **complex parallelisable nilmanifold**.

The main result of this subsection is

Corollary 4.5.33. *([AB91a, Remark 3.1.]) Every compact* **complex parallelisable** *manifold is* **balanced**. *In particular, the Iwasawa manifold is balanced.*

Proof. Let X be an arbitrary compact *complex parallelisable* manifold, $\dim_{\mathbb{C}} X = n$. Let $\varphi_1, \dots, \varphi_n \in H^0(X, \Omega^1_X)$ be n holomorphic 1-forms that are linearly independent at every point of X. Consider the $(n-1, n-1)$-form on X:

$$\Omega := i^{(n-1)^2} \sum_{i=1}^n \varphi_1 \wedge \dots \wedge \widehat{\varphi_i} \wedge \dots \wedge \varphi_n \wedge \bar\varphi_1 \wedge \dots \wedge \widehat{\bar\varphi_i} \wedge \dots \wedge \bar\varphi_n = \sum_{i=1}^n i^{(n-1)^2} u_i \wedge \bar u_i,$$

where $u_i := \varphi_1 \wedge \cdots \wedge \widehat{\varphi}_i \wedge \cdots \wedge \varphi_n \in C^\infty(X,\, \Lambda^{n-1,\,0}T^\star X)$ and $\widehat{\ }$ indicates a missing factor. Since $\bar\partial\varphi_k = 0$ for all $k = 1, \dots, n$, we see that $\bar\partial u_i = 0$ for all $i = 1, \dots, n$. Then, Observation 4.5.27 gives $du_i = 0$ for all $i = 1, \dots, n$, while Corollary 4.5.28 gives:

$$\Omega \geq 0 \quad \text{and} \quad d\Omega = 0 \quad \text{on } X.$$

Furthermore, since $\varphi_1, \dots, \varphi_n$ are linearly independent at every point of X, we must even have $\Omega > 0$. Thus Ω is a C^∞ $(n-1,\, n-1)$-form on X satisfying:

$$\Omega > 0 \quad \text{and} \quad d\Omega = 0 \quad \text{on } X.$$

Applying Lemma 4.0.1, there exists a unique C^∞ positive-definite $(1,\, 1)$-form $\omega > 0$ on X such that $\omega^{n-1} = \Omega$. From $d(\omega^{n-1}) = d\Omega = 0$, we infer that ω is a *balanced* metric on X. $\square$

Note, however, that very few compact *complex parallelisable* manifolds are Kähler thanks to a result of Wang.

Remark 4.5.34. *([Wan54]) Let $X = G/\Gamma$ be a compact complex parallelisable manifold. Then:*

$$X \text{ is Kähler} \quad \Longleftrightarrow \quad G \text{ is abelian} \quad \Longleftrightarrow \quad X \text{ is a complex torus.}$$

Proof. See Corollary 2, p. 776 in [Wan54]. $\square$

Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. Since there are no non-zero $\bar\partial$-exact $(1,\, 0)$-forms on X (for obvious bidegree reasons), we have

$$H^{1,\,0}(X,\, \mathbb{C}) = \{u \in C^\infty(X,\, \Lambda^{1,\,0}T^\star X)\ ;\ \bar\partial u = 0\},$$

i.e. $H^{1,\,0}(X,\, \mathbb{C})$ consists of holomorphic 1-forms on X. Denoting $h^{1,\,0}(X)$ its dimension, we have

Observation 4.5.35. *If X is complex parallelisable, then $h^{1,\,0}(X) = n$.*

Proof. By the *complex parallelisable* hypothesis on X, the rank-n analytic sheaf Ω^1_X is trivial, hence it is generated by n holomorphic 1-forms $\varphi_1, \dots, \varphi_n \in H^{1,\,0}(X,\, \mathbb{C})$ that are linearly independent at every point of X. In particular, $\{\varphi_1, \dots, \varphi_n\}$ is a basis of $H^{1,\,0}(X,\, \mathbb{C}) \simeq H^0(X,\, \Omega^1_X)$. $\square$

Suppose now that X is compact *complex parallelisable*. Let $\theta_1, \dots, \theta_n \in H^0(X,\, T^{1,\,0}X)$ be n holomorphic vector fields that are linearly independent at every point of X, chosen to be dual to the holomorphic $(1,\, 0)$-forms $\varphi_1, \dots, \varphi_n \in H^{1,\,0}(X,\, \mathbb{C})$ considered in the above proof. For every smooth function $g : X \to \mathbb{C}$, we have

$$\partial g = \sum_{\lambda=1}^n (\theta_\lambda g)\, \varphi_\lambda, \qquad \bar\partial g = \sum_{\lambda=1}^n (\bar\theta_\lambda g)\, \overline{\varphi}_\lambda, \tag{4.107}$$

i.e. the familiar formalism induced by local holomorphic coordinates finds a global analogue on a compact *complex parallelisable* manifold in a formalism where θ_λ replaces $\partial/\partial z_\lambda$ and φ_λ replaces dz_λ. Thus any $(0,\, 1)$-form φ on X has a unique decomposition

$$\varphi = \sum_{\lambda=1}^n f_\lambda \overline{\varphi}_\lambda$$

with $f_1, \dots, f_n : X \to \mathbb{C}$ functions on X. Thus there is an implicit L^2 inner product on $C^\infty(X,\, \Lambda^{0,\,1}T^\star X)$ defined as follows (no Hermitian metric is needed on X): for any $\varphi = \sum\limits_{\lambda=1}^{n} f_\lambda \overline{\varphi}_\lambda, \psi = \sum\limits_{\lambda=1}^{n} g_\lambda \overline{\varphi}_\lambda \in C^\infty(X,\, \Lambda^{0,\,1}T^\star X)$, set

$$\langle\langle \varphi,\, \psi \rangle\rangle := \int\limits_X \left(\sum_{\lambda=1}^{n} f_\lambda\, \bar{g}_\lambda\right) i^{n^2} \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \overline{\varphi}_1 \wedge \cdots \wedge \overline{\varphi}_n. \tag{4.108}$$

It is clear that $dV := i^{n^2} \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \overline{\varphi}_1 \wedge \cdots \wedge \overline{\varphi}_n > 0$ is a global volume form on X and that the above L^2 inner product is independent of the choices made. We can define the formal adjoint $\bar\partial^\star$ of $\bar\partial$ w.r.t. this L^2 inner product in the usual way: for any smooth $(0,\, 1)$-form φ, define $\bar\partial^\star\varphi$ to be the unique smooth function on X satisfying

$$\langle\langle \bar\partial^\star\varphi,\, g \rangle\rangle = \langle\langle \varphi,\, \bar\partial g \rangle\rangle$$

for any smooth function g on X. A trivial calculation using Stokes's theorem gives

$$\bar\partial^\star\varphi = -\sum_{\lambda=1}^{n} \theta_\lambda f_\lambda \tag{4.109}$$

for any smooth $(0,\, 1)$-form $\varphi = \sum\limits_{\lambda=1}^{n} f_\lambda \overline{\varphi}_\lambda$ on X. Thus we see that

$$\bar\partial^\star\overline{\varphi}_\nu = 0, \qquad \nu = 1, \dots, n, \tag{4.110}$$

because $\overline{\varphi}_\nu = \sum\limits_{\lambda=1}^{n} \delta_{\nu\lambda}\overline{\varphi}_\lambda$ and $\theta_\lambda\delta_{\nu\lambda} = 0$ (since the $\delta_{\nu\lambda}$ are constants).

Now denote by $r \in \{0, 1, \dots, n\}$ the number of d-closed forms among $\varphi_1, \dots, \varphi_n$. After a possible reordering, we can suppose that $\varphi_1, \dots, \varphi_r$ are d-closed and $\varphi_{r+1}, \dots, \varphi_n$ are not d-closed. Then we have

$$\partial\varphi_1 = \cdots = \partial\varphi_r = 0 \quad \text{or equivalently} \quad \bar\partial\overline{\varphi}_1 = \cdots = \bar\partial\overline{\varphi}_r = 0. \tag{4.111}$$

Thus the $\bar\partial$-closed $(0,\, 1)$-forms $\overline{\varphi}_1, \dots, \overline{\varphi}_r$ define Dolbeault $(0,\, 1)$-cohomology classes in $H^{0,\,1}(X,\, \mathbb{C})$.

We can define the $\bar\partial$-Laplacian on forms of X in the usual way:

$$\Delta'' := \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial.$$

The corresponding harmonic space of $(0,\, 1)$-forms $\mathcal{H}^{0,\,1}_{\Delta''}(X,\, \mathbb{C}) := \ker\Delta'' = \ker\bar\partial \cap \ker\bar\partial^\star$ satisfies the Hodge isomorphism $\mathcal{H}^{0,\,1}_{\Delta''}(X,\, \mathbb{C}) \simeq H^{0,\,1}(X,\, \mathbb{C})$. Notice that (4.110) and (4.111) give

$$\Delta''\overline{\varphi}_\nu = 0, \qquad \nu = 1, \dots, r, \tag{4.112}$$

i.e. the forms $\overline{\varphi}_1, \dots, \overline{\varphi}_r$ are Δ''-harmonic. On the other hand, $\overline{\varphi}_{r+1}, \dots, \overline{\varphi}_n$ are not Δ''-harmonic. Thus the number r of linearly independent d-closed holomorphic 1-forms of X (independent of the choice of $\varphi_1, \dots, \varphi_n$) satisfies:

$$r \leq h^{0,\,1}(X). \tag{4.113}$$

Suppose now that the compact *complex parallelisable* X is *nilpotent*.

Fact 4.5.36. *(see e.g. [Nak75] or [CFGU00, p.5405-5406]) If X is a compact* **complex parallelisable nilmanifold***, the holomorphic* 1*-forms* $\varphi_1, \dots, \varphi_n$ *that form a basis of* $H^{1,0}(X, \mathbb{C})$ *can be chosen such that*

$$d\varphi_\mu = \sum_{1\leq\lambda<\nu\leq n} c_{\mu\lambda\nu}\, \varphi_\lambda \wedge \varphi_\nu, \qquad 1 \leq \mu \leq n, \tag{4.114}$$

with constant coefficients $c_{\mu\lambda\nu} \in \mathbb{C}$ *satisfying*

$$c_{\mu\lambda\nu} = 0 \quad \textit{whenever} \quad \mu \leq \lambda \ \textit{or} \ \mu \leq \nu.$$

Taking this standard fact (which in [Nak75] follows from the existence of a *Chevalley decomposition* of the nilpotent Lie algebra $\mathfrak{g}$) for granted, we now spell out the details of the proof of the following result of Kodaira along the lines of [Nak75, Theorem 3, p. 100].

Theorem 4.5.37. *(Kodaira) If X is a compact* **complex parallelisable nilmanifold***, then* $h^{0,1}(X) = r$.

Moreover, the Δ''*-harmonic* $(0, 1)$*-forms* $\overline{\varphi}_1, \dots, \overline{\varphi}_r$ *form a basis of the harmonic space* $\mathcal{H}^{0,1}_{\Delta''}(X, \mathbb{C})$*. Equivalently, the Dolbeault* $(0, 1)$*-cohomology classes* $\{\overline{\varphi}_1\}, \dots, \{\overline{\varphi}_r\}$ *form a basis of* $H^{0,1}(X, \mathbb{C})$*.*

Proof. The only thing that needs proving is that the linearly independent forms $\overline{\varphi}_1, \dots, \overline{\varphi}_r \in \mathcal{H}^{0,1}_{\Delta''}(X, \mathbb{C})$ generate $\mathcal{H}^{0,1}_{\Delta''}(X, \mathbb{C})$. Pick an arbitrary C^∞ $(0, 1)$-form φ on X and write

$$\varphi = \sum_{\lambda=1}^{n} f_\lambda\, \overline{\varphi}_\lambda$$

with C^∞ functions $f_1, \dots, f_n$ on X. Using formula (4.107) for $\bar\partial$ and the obvious identities $\bar\partial\,\overline{\varphi}_\lambda = d\,\overline{\varphi}_\lambda$, $\lambda = 1, \dots, n$, due to φ_λ being holomorphic, we get:

$$\begin{aligned}
\bar\partial\varphi &= \sum_{\lambda,\,\nu=1}^{n} (\bar\theta_\nu f_\lambda)\, \overline{\varphi}_\nu \wedge \overline{\varphi}_\lambda + \sum_{\mu=1}^{n} f_\mu\, d\,\overline{\varphi}_\mu \\
&= \sum_{\lambda,\,\nu=1}^{n} (\bar\theta_\nu f_\lambda)\, \overline{\varphi}_\nu \wedge \overline{\varphi}_\lambda + \sum_{\mu=1}^{n} f_\mu \sum_{1\leq\nu<\lambda\leq n} \overline{c_{\mu\nu\lambda}}\, \overline{\varphi}_\nu \wedge \overline{\varphi}_\lambda \\
&= \sum_{1\leq\nu<\lambda\leq n} \left(\bar\theta_\nu f_\lambda - \bar\theta_\lambda f_\nu + \sum_{\mu=1}^{n} \overline{c_{\mu\nu\lambda}}\, f_\mu\right) \overline{\varphi}_\nu \wedge \overline{\varphi}_\lambda,
\end{aligned} \tag{4.115}$$

where the second line above follows from the conjugate of (4.114).

Now φ is Δ''-harmonic if and only if

(i) $\bar\partial\varphi = 0 \iff \bar\theta_\nu f_\lambda - \bar\theta_\lambda f_\nu + \sum_{\mu=1}^{n} \overline{c_{\mu\nu\lambda}}\, f_\mu = 0$ for $1 \leq \nu < \lambda \leq n$ (cf. (4.115));

and

(ii) $\bar\partial^\star\varphi = 0 \iff \sum_{\lambda=1}^{n} \theta_\lambda\, f_\lambda = 0$ (cf. (4.109)).

Suppose that φ is Δ''-harmonic. Then the above (i) reads:

$$\bar{\theta}_\lambda f_\nu = \sum_{\mu=1}^n \overline{c_{\mu\nu\lambda}}\, f_\mu + \bar{\theta}_\nu f_\lambda, \quad 1 \leq \nu < \lambda \leq n.$$

Summing over $\lambda = 1, \dots, n$ and using formula (4.107) for $\bar{\partial}$, we get:

$$\bar{\partial} f_\nu = \sum_{\lambda=1}^n (\bar{\theta}_\lambda f_\nu)\, \overline{\varphi}_\lambda = \sum_{\lambda,\, \mu=1}^n \overline{c_{\mu\nu\lambda}}\, f_\mu \overline{\varphi}_\lambda + \sum_{\lambda=1}^n (\bar{\theta}_\nu f_\lambda)\, \overline{\varphi}_\lambda, \quad \nu = 1, \dots, n,$$

with the understanding that $c_{\mu\nu\lambda} = 0$ if $\nu \geq \lambda$. Now $\Delta'' f_\nu = \bar{\partial}^\star \bar{\partial} f_\nu$ since f_ν is a function. Taking $\bar{\partial}^\star$ on either side above and using formula (4.109) for $\bar{\partial}^\star$, we get:

$$\begin{aligned} \Delta'' f_\nu \;\; &= \;\; - \sum_{\lambda,\, \mu=1}^n \theta_\lambda\, (\overline{c_{\mu\nu\lambda}}\, f_\mu) - \sum_{\lambda=1}^n \theta_\lambda\, (\bar{\theta}_\nu f_\lambda) \\ &= \;\; - \sum_{\lambda,\, \mu=1}^n \overline{c_{\mu\nu\lambda}}\, \theta_\lambda f_\mu, \quad \text{for all } \nu = 1, \dots, n, \end{aligned} \tag{4.116}$$

because $\theta_\lambda\, (\overline{c_{\mu\nu\lambda}} f_\mu) = \overline{c_{\mu\nu\lambda}}\, \theta_\lambda f_\mu$ due to $\overline{c_{\mu\nu\lambda}}$ being constant, while $\sum_{\lambda=1}^n \theta_\lambda f_\lambda = 0$ due to φ being Δ''-harmonic (cf. (ii) or (4.109)).

Taking now $\nu = n$ in (4.116), we get $\Delta'' f_n = 0$ since $c_{\mu n\lambda} = 0$ for all μ, λ by Fact 4.5.36 and the obvious inequality $\mu \leq \nu = n$. Thus the compactness of X and the ellipticity of Δ'' yield:

$$f_n \quad \text{is constant on } X \text{ if } \Delta''\varphi = 0. \tag{4.117}$$

Taking now $\nu = n - 1$ in (4.116) and using the fact that $\theta_\lambda f_n = 0$ for all λ (since f_n is constant by (4.117)), we get

$$\Delta'' f_{n-1} = - \sum_{\lambda=1}^n \left(\sum_{\mu=1}^{n-1} \overline{c_{\mu n-1\lambda}}\, \theta_\lambda f_\mu \right) = 0 \quad \text{on } X,$$

since $c_{\mu n-1\lambda} = 0$ for all $\mu = 1, \dots, n-1$ and $\lambda = 1, \dots, n$ by Fact 4.5.36 and the obvious inequality $\mu \leq \nu = n - 1$. Hence we get:

$$f_{n-1} \quad \text{is constant on } X \text{ if } \Delta''\varphi = 0. \tag{4.118}$$

We can now continue by decreasing induction on ν. Taking $\nu = n - 2$ in (4.116) and using the fact that $\theta_\lambda f_n = \theta_\lambda f_{n-1} = 0$ for all λ (since f_n is constant by (4.117) and f_{n-1} is constant by (4.118)), we get:

$$\Delta'' f_{n-2} = - \sum_{\lambda=1}^n \left(\sum_{\mu=1}^{n-2} \overline{c_{\mu n-2\lambda}}\, \theta_\lambda f_\mu \right) = 0 \quad \text{on } X,$$

since $c_{\mu n-2\lambda} = 0$ for all $\mu = 1, \dots, n-2$ and $\lambda = 1, \dots, n$ by Fact 4.5.36 and the obvious inequality $\mu \leq \nu = n - 2$. Hence we get:

$$f_{n-2} \quad \text{is constant on } X \text{ if } \Delta''\varphi = 0. \tag{4.119}$$

Running a decreasing induction on ν, we get:

$$f_\nu := C_\nu \quad \text{is constant on } X \text{ for all } \nu = 1, \dots, n \text{ if } \Delta''\varphi = 0. \tag{4.120}$$

We conclude that whenever $\Delta''\varphi = 0$ we have:

$$\varphi = \sum_{\nu=1}^{n} C_\nu\,\overline{\varphi}_\nu \quad \text{with } C_\nu \text{ constant for all } \nu = 1, \dots, n.$$

On the other hand, since $\Delta''\varphi = 0$, we must have $\bar\partial\varphi = 0$ which amounts to $\sum\limits_{\nu=1}^{n} C_\nu\,\bar\partial\overline{\varphi}_\nu = 0$. However, we know that $\bar\partial\overline{\varphi}_\nu = 0$ for all $\nu \in \{1, \dots, r\}$ (cf. (4.111)), hence $\sum\limits_{\nu=r+1}^{n} C_\nu\,\bar\partial\overline{\varphi}_\nu = 0$. Now the forms

$$\bar\partial\overline{\varphi}_\nu = d\overline{\varphi}_\nu = \sum_{\lambda,\mu} \overline{c_{\nu\lambda\mu}}\,\overline{\varphi}_\lambda \wedge \overline{\varphi}_\mu, \quad \nu = 1, \dots, n,$$

are linearly independent because $\overline{\varphi}_1, \dots, \overline{\varphi}_n$ are linearly independent at every point of X. Hence $C_\nu = 0$ for all $\nu = r+1, \dots, n$. We get:

$$\varphi = \sum_{\nu=1}^{r} C_\nu\,\overline{\varphi}_\nu \quad \text{with } C_\nu \text{ constant for all } \nu = 1, \dots, r.$$

Since φ has been chosen arbitrary in $\mathcal{H}^{0,1}_{\Delta''}(X, \mathbb{C})$, we have proved that the linearly independent forms $\overline{\varphi}_1, \dots, \overline{\varphi}_r \in \mathcal{H}^{0,1}_{\Delta''}(X, \mathbb{C})$ generate $\mathcal{H}^{0,1}_{\Delta''}(X, \mathbb{C})$. The proof of Kodaira's theorem 4.5.37 is complete. $\square$

When applying Observation 4.5.35 and Kodaira's Theorem 4.5.37 to the Iwasawa manifold (a compact *complex parallelisable nilmanifold* of dimension $n = 3$ having $r = 2$), we get the following classical fact.

Observation 4.5.38. *For the Iwasawa manifold, we have:*

$$h^{1,0} = 3 \quad \textit{and} \quad h^{0,1} = 2.$$

Since, on the other hand, the first Betti number is $b_1 = 4$ (see e.g. (1.56) in §.1.3.3), we see that $b_1 < h^{1,0} + h^{0,1}$. Thus, we infer again the non-degeneration at E_1 of the Frölicher spectral sequence of the Iwasawa manifold (cf. Proposition 1.3.22).

(II) The Kuranishi family of the Iwasawa manifold (after Nakamura [Nak75])

Let X be the *Iwasawa manifold*. In particular, X is a compact *complex parallelisable nilmanifold* of complex dimension 3. Let $\varphi_1 = dz_1$, $\varphi_2 = dz_2$, $\varphi_3 = dz_3 - z_1 dz_2$ be the holomrophic 1-forms on X defined in (1.54); they are linearly independent at every point of X. Since φ_1 and φ_2 are d-closed while φ_3 is not d-closed, $r = 2$ for the Iwasawa manifold. By Kodaira's theorem 4.5.37, the $\mathbb{C}$-vector space $H^{0,1}(X, \mathbb{C})$ has complex dimension 2 and is spanned by the Dolbeault cohomology classes $\{\overline{\varphi}_1\}$ and $\{\overline{\varphi}_2\}$. Let $\theta_1, \theta_2, \theta_3 \in H^0(X, \Omega^1_X)$ be the holomorphic vector fields dual to $\varphi_1, \varphi_2, \varphi_3$. They are given by

$$\theta_1 = \frac{\partial}{\partial z_1}, \;\; \theta_2 = \frac{\partial}{\partial z_2} + z_1\frac{\partial}{\partial z_3}, \;\; \theta_3 = \frac{\partial}{\partial z_3} \tag{4.121}$$

and satisfy the relations

$$[\theta_1, \theta_2] = -[\theta_2, \theta_1] = \theta_3, \qquad [\theta_1, \theta_3] = [\theta_2, \theta_3] = 0, \tag{4.122}$$

i.e. $[\theta_i,\,\theta_j]=0$ whenever $\{i,\,j\}\neq\{1,\,2\}$. In particular, we get:

$$[\theta_i\overline{\varphi}_\lambda,\,\theta_k\overline{\varphi}_\nu]=[\theta_i,\,\theta_k]\,\overline{\varphi}_\lambda\wedge\overline{\varphi}_\nu,\qquad i,k,\lambda,\nu=1,2,3, \tag{4.123}$$

Since the holomorphic tangent bundle $T^{1,\,0}X$ is trivial and spanned by $\theta_1,\theta_2,\theta_3$, the cohomology group $H^{0,\,1}(X,\,T^{1,\,0}X)$ of $T^{1,\,0}X$-valued $(0,\,1)$-forms on X is a $\mathbb{C}$-vector space of dimension 6 spanned by the classes of $\theta_i\,\overline{\varphi}_\lambda$:

$$H^{0,\,1}(X,\,T^{1,\,0}X)=\bigoplus_{1\leq i\leq 3,\,1\leq\lambda\leq 2}\mathbb{C}\{\theta_i\,\overline{\varphi}_\lambda\},\qquad \dim_{\mathbb{C}}H^{0,\,1}(X,\,T^{1,\,0}X)=6. \tag{4.124}$$

Consequently the Kuranishi family of X can be described by 6 parameters $t=(t_{i\lambda})_{1\leq i\leq 3,\,1\leq\lambda\leq 2}$.

By (4.112), the $T^{1,\,0}X$-valued $(0,\,1)$-forms $\theta_i\,\overline{\varphi_\lambda}$ are Δ''-harmonic when $1\leq\lambda\leq 2$. In order to construct the vector $(0,\,1)$-forms $\psi(t)\in C^\infty(X,\,\Lambda^{0,\,1}T^\star X\otimes T^{1,\,0}X)$ that describe the Kuranishi family of $X=\mathbb{C}^3/\Gamma$, the general theory presented in §.2.3.1 prescribes to start off by setting

$$\psi_1(t):=\sum_{i=1}^3\sum_{\lambda=1}^2 t_{i\lambda}\theta_i\overline{\varphi}_\lambda,\quad t=(t_{i\lambda})_{1\leq i\leq 3,\,1\leq\lambda\leq 2}, \tag{4.125}$$

for which we see that

$$\frac{1}{2}[\psi_1(t),\,\psi_1(t)]=\frac{1}{2}\sum_{i,j=1,2,3}\sum_{\lambda,\mu=1,2}t_{i\lambda}t_{j\mu}[\theta_i,\,\theta_j]\,\overline{\varphi}_\lambda\wedge\overline{\varphi}_\mu.$$

By (4.122), this translates to

$$\begin{aligned}\frac{1}{2}[\psi_1(t),\,\psi_1(t)]=\frac{1}{2}(t_{11}t_{22}\theta_3\,\overline{\varphi}_1\wedge\overline{\varphi}_2 \quad &+\quad t_{12}t_{21}\theta_3\,\overline{\varphi}_2\wedge\overline{\varphi}_1\\ &-\quad t_{21}t_{12}\theta_3\,\overline{\varphi}_1\wedge\overline{\varphi}_2-t_{22}t_{11}\theta_3\,\overline{\varphi}_2\wedge\overline{\varphi}_1).\end{aligned}$$

Since $\overline{\varphi}_1\wedge\overline{\varphi}_2=-\overline{\varphi}_2\wedge\overline{\varphi}_1$, we get

$$\frac{1}{2}[\psi_1(t),\,\psi_1(t)]=(t_{11}t_{22}-t_{12}t_{21})\,\theta_3\,\overline{\varphi}_1\wedge\overline{\varphi}_2. \tag{4.126}$$

On the other hand, for the choice (4.125) we see that

$$\bar\partial\psi_1(t)=d\psi_1(t)=\sum_{i=1}^3\sum_{\lambda=1}^2 t_{i\lambda}\,\theta_i\,d\overline{\varphi}_\lambda=0 \tag{4.127}$$

since $d\overline{\varphi}_1=d\overline{\varphi}_2=0$. Now setting

$$\psi_2(t):=-(t_{11}t_{22}-t_{12}t_{21})\,\theta_3\overline{\varphi}_3 \tag{4.128}$$

and using (1.55) and (4.126), we find:

$$\begin{aligned}\bar\partial\psi_2(t)\;&=\;d\psi_2(t)=(t_{11}t_{22}-t_{12}t_{21})\,\theta_3\,(-d\,\overline{\varphi}_3)\\ &=\;(t_{11}t_{22}-t_{12}t_{21})\,\theta_3\,\overline{\varphi}_1\wedge\overline{\varphi}_2=\frac{1}{2}\,[\psi_1(t),\,\psi_1(t)].\end{aligned} \tag{4.129}$$

In particular, $[\psi_1(t),\,\psi_1(t)]$ is seen to be $\bar\partial$-exact here (although it need not be so in the case of an arbitrary manifold). This readily yields the desired $\psi(t)$ by setting:

$$\psi(t) := \psi_1(t) + \psi_2(t) = \sum_{i=1}^{3}\sum_{\lambda=1}^{2} t_{i\lambda}\,\theta_i\,\overline{\varphi}_\lambda - (t_{11}t_{22} - t_{12}t_{21})\,\theta_3\,\overline{\varphi}_3, \tag{4.130}$$

for which we find

$$\frac{1}{2}[\psi(t),\,\psi(t)] = \sum_{j,,k=1}^{2}\frac{1}{2}[\psi_j(t),\,\psi_k(t)] = \frac{1}{2}[\psi_1(t),\,\psi_1(t)]. \tag{4.131}$$

Indeed, $[\psi_j(t),\,\psi_k(t)] = 0$ for all $(i,\,j)\neq(1,\,1)$ since these terms involve only brackets of the shape $[\theta_3,\,\theta_i] = 0$ and $[\theta_i,\,\theta_3] = 0$ which vanish by (4.122).

On the other hand, combining (4.127) and (4.129), we get:

$$\bar\partial\psi(t) = \bar\partial\psi_1(t) + \bar\partial\psi_2(t) = \bar\partial\psi_2(t) = \frac{1}{2}[\psi_1(t),\,\psi_1(t)]. \tag{4.132}$$

Then (4.131) and (4.132) yield:

$$\bar\partial\psi(t) = \frac{1}{2}[\psi(t),\,\psi(t)], \tag{4.133}$$

showing that $\psi(t)$ defined in (4.130) satisfies the integrability condition (2.16) of §.2.2.2.

Note that the above argument shows that such a $\psi(t)$ can be constructed from any $\psi_1(t)$ as in (4.125), namely for any coefficients $t = (t_{i\lambda})_{1\leq i\leq 3,\,1\leq\lambda\leq 2}\in\mathbb{C}^6$ with $|t|$ small enough.

By the general theory presented in §.2.3.1, we obtain the following description of the Kuranishi family of the Iwasawa manifold.

Theorem 4.5.39. *The* **Kuranishi family** *of the Iwasawa manifold is* **unobstructed***.*

Specifically, the $T^{1,0}X$-valued $(0,\,1)$-forms $\psi(t)$ of (4.130) define a locally complete complex analytic family (the Kuranishi family) of deformations X_t of the Iwasawa manifold depending on 6 effective parameters $t = (t_{i\lambda})_{1\leq i\leq 3, 1\leq\lambda\leq 2}$ such that the complex structure of each fibre X_t is defined by $\bar\partial_t := \bar\partial - \psi(t)$ and $X_0 = X = \mathbb{C}^3/\Gamma$ is the Iwasawa manifold.

It is noteworthy that in the special case of the Iwasawa manifold, the power series (2.21) of §.2.3.1 can be built with only two terms ($\psi_1(t)$ and $\psi_2(t)$) and the above simple calculations show $\psi(t) = \psi_1(t)+\psi_2(t)$ to satisfy the integrability condition (2.16) for all $t = (t_{i\lambda})_{1\leq i\leq 3,\,1\leq\lambda\leq 2}\in B_\varepsilon\subset\mathbb{C}^6$ if $\varepsilon > 0$ is small, where B_ε is the ball of radius ε centred at $0\in\mathbb{C}^6$.

Nakamura goes on to calculate holomorphic coordinates $\zeta_1 = \zeta_1(t), \zeta_2 = \zeta_2(t), \zeta_3 = \zeta_3(t)$ on X_t such that $\zeta_\nu(0) = z_\nu$ for $\nu = 1,2,3$ starting from arbitrary holomorphic coordinates z_1, z_2, z_3 given beforehand on the Iwasawa manifold $X_0 = X = \mathbb{C}^3/\Gamma$. Here is the way he proceeds.

We are looking for C^∞ functions $\zeta_\nu(t)$, $\nu = 1,2,3$, on X satisfying the holomorphicity condition:

$$\bar\partial_t\zeta_\nu(t) = 0 \iff \bar\partial\zeta_\nu(t) - \psi(t)\zeta_\nu(t) = 0, \quad \nu = 1,2,3. \tag{4.134}$$

Given the definition (4.130) of $\psi(t)$ and the formulae (4.121) for $\theta_1,\theta_2,\theta_3$, condition (4.134) reads for $\nu = 1,2,3$:

$$\begin{aligned}\bar\partial\zeta_\nu \quad & - \quad \sum_{\lambda=1}^{2} t_{1\lambda}\frac{\partial\zeta_\nu}{\partial z_1}\,d\bar z_\lambda - \sum_{\lambda=1}^{2} t_{2\lambda}\left(\frac{\partial\zeta_\nu}{\partial z_2} + z_1\frac{\partial\zeta_\nu}{\partial z_3}\right)d\bar z_\lambda \\ & - \quad \sum_{\lambda=1}^{2} t_{3\lambda}\frac{\partial\zeta_\nu}{\partial z_3}\,d\bar z_\lambda + (t_{11}t_{22} - t_{12}t_{21})\frac{\partial\zeta_\nu}{\partial z_3}(d\bar z_3 - \bar z_1 d\bar z_2) = 0. \end{aligned} \tag{4.135}$$

For $\nu=1$, we arrange to have $\frac{\partial\zeta_1}{\partial z_1}=1$ (in order to get $\zeta_1(t)=z_1+$(terms depending only on $\bar{z}_\lambda$)) and $\frac{\partial\zeta_1}{\partial z_2}=\frac{\partial\zeta_1}{\partial z_3}=0$. With these choices, condition (4.135) for $\nu=1$ becomes:

$$\bar\partial\zeta_1-\sum_{\lambda=1}^{2}t_{1\lambda}\bar\partial\bar{z}_\lambda=0 \quad\Longleftrightarrow\quad \bar\partial\zeta_1(t)=\bar\partial\bigg(\sum_{\lambda=1}^{2}t_{1\lambda}\bar{z}_\lambda\bigg).$$

Thus we can take:

$$\zeta_1(t)=z_1+\sum_{\lambda=1}^{2}t_{1\lambda}\bar{z}_\lambda. \tag{4.136}$$

For $\nu=2$, we similarly require $\frac{\partial\zeta_2}{\partial z_2}=1$ and $\frac{\partial\zeta_2}{\partial z_1}=\frac{\partial\zeta_2}{\partial z_3}=0$ and condition (4.135) for $\nu=2$ similarly yields:

$$\zeta_2(t)=z_2+\sum_{\lambda=1}^{2}t_{2\lambda}\bar{z}_\lambda. \tag{4.137}$$

For $\nu=3$, we require $\frac{\partial\zeta_3}{\partial z_3}=1$, $\frac{\partial\zeta_3}{\partial z_2}=0$ and $\frac{\partial\zeta_3}{\partial z_1}=\sum\limits_{\lambda=1}^{2}t_{2\lambda}\bar{z}_\lambda$. With these choices, (4.135) for $\nu=3$ reads:

$$\begin{aligned}\bar\partial\zeta_3 \quad &- \quad \bigg(\sum_{\lambda=1}^{2}t_{1\lambda}d\bar{z}_\lambda\bigg)\bigg(\sum_{\lambda=1}^{2}t_{2\lambda}\bar{z}_\lambda\bigg)-z_1\sum_{\lambda=1}^{2}t_{2\lambda}d\bar{z}_\lambda\\ &- \quad \sum_{\lambda=1}^{2}t_{3\lambda}d\bar{z}_\lambda+(t_{11}t_{22}-t_{12}t_{21})(d\bar{z}_3-\bar{z}_1d\bar{z}_2)=0.\end{aligned}$$

We thus get:

$$\zeta_3(t)=z_3+\sum_{\lambda=1}^{2}(t_{3\lambda}+t_{2\lambda}z_1)\bar{z}_\lambda+A(t,\,\bar{z})-D(t)\bar{z}_3, \tag{4.138}$$

where we have denoted $A(t,\,\bar{z}):=\frac{1}{2}(t_{11}t_{21}\bar{z}_1^2+2t_{11}t_{22}\bar{z}_1\bar{z}_2+t_{12}t_{22}\bar{z}_2^2)$ and $D(t):=(t_{11}t_{22}-t_{12}t_{21})$. We clearly have

$$d\zeta_1(t)\wedge d\zeta_2(t)\wedge d\zeta_3(t)=C(t)\,dz_1\wedge dz_2\wedge dz_3$$

for a constant $C(t)$ depending in a C^∞ way on t such that $C(0)=1$. Hence $\zeta_1(t),\zeta_2(t),\zeta_3(t)$ define holomorphic coordinates on X_t for all $t=(t_{i\lambda})_{1\leq i\leq 3,\,1\leq\lambda\leq 2}$ such that $\sum\limits_{i=1,2,3}\sum\limits_{\lambda=1,2}|t_{i\lambda}|<\varepsilon$ if $\varepsilon>0$ is small enough.

The conclusions of these computations are summed up in the following further description of the Kuranishi family of the Iwasawa manifold $I^{(3)}$ constructed in Theorem 4.5.39.

Theorem and Definition 4.5.40. *([Nak75, p. 96]) The 6-parameter space $(X_t)_{t=(t_{i\lambda})_{1\leq i\leq 3,\,1\leq\lambda\leq 2}}$ consisting of the small deformations of the Iwasawa manifold $I^{(3)}$ is divided in the following three classes that will be called* **Nakamura classes***:*

(i) $t_{11}=t_{12}=t_{21}=t_{22}=0$. All the manifolds X_t in this class are **complex parallelisable***.*

(ii) $(t_{11},\,t_{12},\,t_{21},\,t_{22})\neq(0,\,0,\,0,\,0)$ and $D(t)=0$. **No** *manifold X_t in this class is* **complex parallelisable***.*

(iii) $D(t) \neq 0$. **No** *manifold* X_t *in this class is* **complex parallelisable**.

The Iwasawa manifold $X_0 = I^{(3)}$ *belongs to the Nakamura class (i).*

Finally, we reproduce Nakamura's table ([Nak75, p. 96]) summing up the Betti and Hodge numbers of the Iwasawa manifold $X_0 = I^{(3)}$ and of its small deformations X_t according to their respective Nakamura class. They are easily obtained from the computations performed above.

	$h^{1,0}$	$h^{0,1}$	b_1	$h^{2,0}$	$h^{1,1}$	$h^{0,2}$	b_2	$h^{3,0}$	$h^{2,1}$	$h^{1,2}$	$h^{0,3}$	b_3
(i)	3	2	4	3	6	2	8	1	6	6	1	10
(ii)	2	2	4	2	5	2	8	1	5	5	1	10
(iii)	2	2	4	1	5	2	8	1	4	4	1	10

By Poincaré duality we have: $b_5 = b_1$, $b_4 = b_2$, while by Serre duality we have: $h^{p,\,q} = h^{3-p,\,3-q}$ for all $p,\, q \in \{0, \dots, 3\}$. Using Corollary 1.2.6 of §.1.2.1, an immediate consequence of the above table is the following

Corollary 4.5.41. *Let* X_t *be a small deformation of the Iwasawa manifold* $X_0 = I^{(3)}$.

(i) If X_t *lies in one of the Nakamura classes (i) and (ii), the Frölicher spectral sequence of* X_t *does not degenerate at* E_1.

(ii) If X_t *lies in the Nakamura class (iii), the Frölicher spectral sequence of* X_t *degenerates at* E_1.

Proof. If X_t is in class (i), $b_1 = 4 < 5 = h^{1,0} + h^{0,1}$. Hence, $E_1(X_t) \neq E_\infty(X_t)$.

If X_t is in class (ii), $b_2 = 8 < 9 = h^{2,0} + h^{1,1} + h^{0,2}$. Hence, $E_1(X_t) \neq E_\infty(X_t)$.

If X_t is in class (iii), $b_1 = 4 = h^{1,0} + h^{0,1}$, $b_2 = 8 = h^{2,0} + h^{1,1} + h^{0,2}$, $b_3 = 10 = h^{3,0} + h^{2,1} + h^{1,2} + h^{0,3}$. Hence, $E_1(X_t) = E_\infty(X_t)$. □

We will see later that the Frölicher spectral sequence of any small deformation X_t of the Iwasawa manifold $X_0 = I^{(3)}$ degenerates at E_2 at the latest.

(III) Non-deformation openness of the balanced property

We now point out a fundamental difference between balanced and sG manifolds that we shall later exploit to get examples of non-balanced sG manifolds. Unlike sG manifolds, balanced manifolds are not stable under small deformations. This result was first observed by Alessandrini and Bassanelli [AB90] and should be compared to Theorem 4.2.4.

Theorem 4.5.42. *([AB90]) The* **balanced** *property of compact complex manifolds is* **not open** *under holomorphic deformations.*

Proof. In the 6-parameter Kuranishi family $(X_t)_{t\in B}$, $t = (t_{i\lambda})_{1\leq i\leq 3,\, 1\leq\lambda\leq 2}$, of the Iwasawa manifold $X_0 = X = \mathbb{C}^3/\Gamma$, Alessandrini and Bassanelli [AB90] single out the direction corresponding to parameters t such that

$$t_{12} \neq 0, \quad t_{ij} = 0 \quad \text{for all } (i,\, j) \neq (1,\, 2). \tag{4.139}$$

With this choice of t, they have

$$A(t,\,\bar{z}) = 0 \quad \text{and} \quad D(t) = 0.$$

Thus, denoting $t := t_{12}$, the holomorphic coordinates of X_t computed in (4.136), (4.137) and (4.138) reduce to

$$\zeta_1(t) = z_1 + t\bar{z}_2, \quad \zeta_2(t) = z_2, \quad \zeta_3(t) = z_3. \tag{4.140}$$

Implicitly $z_1 = \zeta_1(t) - t\overline{\zeta}_2(t)$, which yields:

$$\begin{aligned} \varphi_3(t) : &= dz_3 - z_1 dz_2 = d\zeta_3(t) + (t\overline{\zeta}_2(t) - \zeta_1(t))\, d\zeta_2(t), \\ \varphi_2(t) : &= dz_2 = d\zeta_2(t), \qquad \widetilde{\varphi}_1(t) := dz_1 = d\zeta_1(t) - t d\overline{\zeta}_2(t). \end{aligned} \tag{4.141}$$

Set

$$\varphi_1(t) := d\zeta_1(t). \tag{4.142}$$

The above 1-forms $\varphi_1(t), \varphi_2(t), \varphi_3(t)$ are all of J_t-type $(1,\,0)$ since $\zeta_1(t), \zeta_2(t),\, \zeta_3(t)$ are holomorphic coordinates for the complex structure J_t of X_t.

The following result proves Theorem 4.5.42 by means of a counter-example which has an interest of its own.

Proposition 4.5.43. *(Alessandrini-Bassanelli [AB90, p. 1062]) Let $(X_t)_t$ be the Kuranishi family of the Iwasawa manifold $X = X_0$, $t = (t_{i\lambda})_{1\leq i\leq 3,\, 1\leq\lambda\leq 2}$.*

Then, for parameters such that $t_{i\lambda} = 0$ for all $(i,\,\lambda) \neq (1,\,2)$, X_t is **not balanced** *for any $t := t_{12} \neq 0$ satisfying $|t_{12}| < \varepsilon$ if $\varepsilon > 0$ is small enough.*

Proof. For the forms defined in (4.141) and (4.142), an immediate calculation shows

$$d\varphi_3(t) = (t\, d\bar{\zeta}_2(t) - d\zeta_1(t)) \wedge d\zeta_2(t) = -t\, \varphi_2(t) \wedge \overline{\varphi}_2(t) - \varphi_1(t) \wedge \varphi_2(t). \tag{4.143}$$

Thus the 2-form $d\varphi_3(t)$ has two components: $-t\, \varphi_2(t)\wedge\overline{\varphi}_2(t)$ is of J_t-type $(1,\,1)$, while $-\varphi_1(t)\wedge\varphi_2(t)$ is of J_t-type $(2,\,0)$.

Recall that $\dim_{\mathbb{C}} X_t = 3$ for all t. Suppose that X_t were balanced for some $t = t_{12} \neq 0$ satisfying $|t_{12}| < \varepsilon$ with $\varepsilon > 0$ small. Then, there would exist a balanced metric $\omega_t > 0$ on X_t. Thus, $\Omega_t := \omega_t^2$ would be a C^∞ $(2,\,2)$-form on X_t satisfying:

$$\Omega_t > 0 \quad \text{and} \quad d\Omega_t = 0.$$

In this case we would have:

$$0 = \int\limits_{X_t} d\Omega_t \wedge i\bar{t}\varphi_3(t) = -\int\limits_{X_t} \Omega_t \wedge i\bar{t}\, d\varphi_3(t) = |t|^2 \int\limits_{X_t} \Omega_t \wedge i\varphi_2(t) \wedge \overline{\varphi_2(t)}. \tag{4.144}$$

Indeed, the first identity above follows from $d\Omega_t = 0$, the second identity follows from Stokes's theorem, while the third identity follows from formula (4.143) for $d\varphi_3(t)$ and the fact that the $(2,\,0)$-component $-\varphi_1(t) \wedge \varphi_2(t)$ is annihilated when wedged with the $(2,\,2)$-form Ω_t.

Now $\Omega_t > 0$ and $i\varphi_2(t) \wedge \overline{\varphi_2(t)} \geq 0$, hence $\Omega_t \wedge i\varphi_2(t) \wedge \overline{\varphi_2(t)} \geq 0$ at every point of X_t. It follows that the right-hand term in (4.144) is non-negative. However, since it must vanish by the first identity in (4.144), the $(3,\,3)$-form $\Omega_t \wedge i\varphi_2(t) \wedge \overline{\varphi_2(t)}$ must vanish identically on X_t. This is equivalent to the vanishing of the trace of the semi-positive $(1,\,1)$-form $i\varphi_2(t)\wedge\overline{\varphi_2(t)}$ w.r.t. the root of the positive definite Ω_t. Hence, the $(1,\,1)$-form $i\varphi_2(t) \wedge \overline{\varphi_2(t)}$ too must vanish identically on X_t. This can only happen if $\varphi_2(t)$ vanishes identically on X_t, which is impossible since $\varphi_2(t) = d\zeta_2(t)$ and $\zeta_2(t)$ is a holomorphic coordinate on X_t if ε is small enough. This provides the desired contradiction.

Therefore X_t cannot be balanced for any $t = t_{12} \neq 0$ if $t_{i\lambda} = 0$ for all $(i,\,\lambda) \neq (1,\,2)$ and $\varepsilon > 0$ is small. The proofs of Proposition 4.5.43 and of Theorem 4.5.42 are complete. $\square$

We now make the following

Observation 4.5.44. *(implicit in [Nak75]) In the Kuranishi family of the Iwasawa manifold, the Frölicher spectral sequence does not degenerate at E_1 on any fibre X_t corresponding to parameters such that $t_{i\lambda} = 0$ for all $(i,\,\lambda) \neq (1,\,2)$ and $t := t_{12}$ satisfies $|t_{12}| < \varepsilon$ with $\varepsilon > 0$ small enough.*

Proof. These fibres lie in the Nakamura class (ii), so it suffices to apply Corollary 4.5.41. $\square$

(IV) Examples of non-balanced sG manifolds

We now show that, for compact complex manifolds X, the implication:

$$X \text{ is } \textbf{balanced} \implies X \text{ is } \textbf{strongly Gauduchon}$$

is strict. The counter-examples to the reverse implication are obtained by putting together Theorems 4.2.4 and 4.5.42 and Observation 4.5.44.

Theorem 4.5.45. *Let $(X_t)_t$ be the Kuranishi family of the Iwasawa manifold $X = X_0$, where $t = (t_{i\lambda})_{1\leq i\leq 3,\, 1\leq\lambda\leq 2}$.*

Then, for parameters t such that $t_{i\lambda} = 0$ for all $(i,\,\lambda) \neq (1,\,2)$, X_t is a **strongly Gauduchon** *manifold that is* **not balanced** *and whose* **Frölicher spectral sequence does not degenerate at** *E_1 for any $t = t_{12} \neq 0$ satisfying $|t_{12}| < \varepsilon$ if $\varepsilon > 0$ is small enough.*

Proof. Since the Iwasawa manifold is balanced (cf. Corollary 4.5.33), it is also an sG manifold. Since the sG property is open under holomorphic deformations (cf. Theorem 4.2.4), all sufficiently nearby fibres X_t in the Kuranishi family of the Iwasawa manifold X_0 are again sG manifolds.

However, by Proposition 4.5.43, the fibres X_t corresponding to parameters t for which $t_{i\lambda} = 0$ for all $(i,\,\lambda) \neq (1,\,2)$ are not balanced if $t_{12} \neq 0$ and t is sufficiently close to 0. By Observation 4.5.44, the Frölicher spectral sequence does not degenerate at E_1 on any of these fibres. $\square$

The analogue in bidegree $(1,\,1)$ of the existence of non-balanced sG manifolds proved in Theorem 4.5.45 is still an open problem. See the Streets-Tian Question 4.6.9.

(V) Deformation openness of the combined $\partial\bar\partial$ and balanced properties

We saw in Theorem 2.6.4 that the $\partial\bar\partial$-property of compact complex manifolds is **open** under holomorphic deformations of complex structures. This was Wu's first main result in [Wu06]. By contrast, the balanced property is **not open** under holomorphic deformations, as seen in the Alessandrini-Bassanelli Theorem 4.5.42 from [AB90].

We will now see that the simultaneous occurrence of these two properties is **open**. This was Wu's second main result in [Wu06].

Theorem 4.5.46. *([Wu06]) Let $(X_t)_{t\in B}$ be a holomorphic family of compact complex manifolds over an open ball B containing the origin in some $\mathbb{C}^N$.*

If the fibre X_0 is a **balanced $\partial\bar\partial$-manifold***, the fibre X_t is again a* **balanced $\partial\bar\partial$-manifold** *for every $t \in B$ sufficiently close to 0.*

Moreover, if X_0 is a **$\partial\bar\partial$-manifold***, any* **balanced** *metric ω_0 on X_0 deforms to a family of* **balanced** *metrics ω_t on X_t varying in a C^∞ way with t for t in a small enough neighbourhood of 0.*

Proof. We reproduce Wu's arguments in a slightly different notation. Let $(\gamma_t)_{t\in B}$ be an arbitrary C^∞ family of Hermitian metrics on the fibres $(X_t)_{t\in B}$. If $\Delta_{BC}(t)$ denotes the Bott-Chern Laplacian (cf. [KS60, §.6], see Definition 1.1.8) induced by the metric γ_t, we recall the following 3-space orthogonal decomposition of Corollary 1.1.10 in every bidegree $(p,\, q)$:

$$C^\infty_{p,\,q}(X_t,\,\mathbb{C}) = \ker\Delta_{BC}(t)\oplus \mathrm{Im}\,(\partial_t\bar\partial_t)\oplus(\mathrm{Im}\,\partial_t^\star + \mathrm{Im}\,\bar\partial_t^\star), \qquad t\in B,$$

where

$$\ker\partial_t\cap\ker\bar\partial_t = \ker\Delta_{BC}(t)\oplus\mathrm{Im}\,(\partial_t\bar\partial_t) \quad \text{and} \quad \mathrm{Im}\,\Delta_{BC}(t) = \mathrm{Im}\,(\partial_t\bar\partial_t)\oplus(\mathrm{Im}\,\partial_t^\star + \mathrm{Im}\,\bar\partial_t^\star), \qquad t\in B.$$

All the adjoints are w.r.t. γ_t.

Let F_t stand for the orthogonal projection onto $\ker\Delta_{BC}(t)$ w.r.t. the $L^2_{\gamma_t}$-inner product and let $\Delta_{BC}^{-1}(t)$ stand for the Green operator of the elliptic operator $\Delta_{BC}(t)$. By the above decompositions, every form $\alpha_t\in C^\infty_{p,\,q}(X_t,\,\mathbb{C})$ splits uniquely as $\alpha_t = F_t\alpha_t + \Delta_{BC}(t)\Delta_{BC}^{-1}(t)\alpha_t$. Moreover, if $\alpha_t\in\ker\partial_t\cap\ker\bar\partial_t$, this splitting reduces to

$$\alpha_t = F_t\alpha_t + \partial_t\bar\partial_t(\partial_t\bar\partial_t)^\star\Delta_{BC}^{-1}(t)\,\alpha_t$$

since $\alpha_t - F_t\alpha_t\in\mathrm{Im}\,(\partial_t\bar\partial_t)$ and the minimal $L^2_{\gamma_t}$-norm solution of the equation

$$\partial_t\bar\partial_t u_t = \alpha_t - F_t\alpha_t$$

is $u_t = (\partial_t\bar\partial_t)^\star\Delta_{BC}^{-1}(t)\,(\alpha_t - F_t\alpha_t) = (\partial_t\bar\partial_t)^\star\Delta_{BC}^{-1}(t)\,\alpha_t$. (See Wu's original argument or Theorem 4.5.47 below.)

Let ω_0 be a balanced metric on X_0 and n the complex dimension of X_t. Then $\omega_0^{n-1}\in\ker\partial_0\cap\ker\bar\partial_0$, so

$$\omega_0^{n-1} = F_0\,\omega_0^{n-1} + \partial_0\bar\partial_0(\partial_0\bar\partial_0)^\star\Delta_{BC}^{-1}(0)\,\omega_0^{n-1}.$$

Extend ω_0 in an arbitrary way to Hermitian metrics $\widetilde\omega_t$ varying in a C^∞ way with t on the nearby fibres X_t such that $\widetilde\omega_0 = \omega_0$. Put

$$\Omega_t := \mathrm{Re}\,(F_t\,\widetilde\omega_t^{n-1} + \partial_t\bar\partial_t(\partial_t\bar\partial_t)^\star\Delta_{BC}^{-1}(t)\,\widetilde\omega_t^{n-1}), \qquad t\in B.$$

By construction, every Ω_t is a C^∞, real, J_t-type $(n-1,\, n-1)$-form on X_t such that $d\Omega_t = 0$ for every t. Moreover, $\Omega_0 = \omega_0^{n-1}$.

Now, since X_0 is a $\partial\bar\partial$-manifold, Theorem 2.6.4 ensures that the fibres X_t are again $\partial\bar\partial$-manifolds for every t close to 0 and the dimensions $h^{p,\,q}_{BC}(t)$ of the Bott-Chern cohomology spaces $H^{p,\,q}_{BC}(t)$ are independent of t close to 0. Thanks to the Hodge isomorphism $H^{p,\,q}_{BC}(t)\simeq\ker\Delta_{BC}(t)$ (see Corollary 1.1.10) and to the Kodaira-Spencer theory for smooth families of elliptic operators (see Theorems C and D in §.2.5.1), this implies that the operators F_t and $\Delta_{BC}(t)^{-1}$ vary in a C^∞ way with t. Therefore, the real differential forms Ω_t vary in a C^∞ way with t. Since $\Omega_0 = \omega_0^{n-1} > 0$, we get by continuity that $\Omega_t > 0$ for every t sufficiently close to 0.

Taking the (unique) $(n-1)^{st}$ root $\omega_t > 0$ of $\Omega_t > 0$ for t close to 0 (see Michelsohn's Lemma 4.0.1), we get a C^∞ family of *balanced* metrics ω_t on the fibres X_t whose member corresponding to $t = 0$ coincides with the original ω_0. □

It remains to prove the following general fact, part of which was used in the above proof.

Theorem 4.5.47. *([Pop15a, Theorem 4.1]) Fix a compact Hermitian manifold $(X,\,\omega)$. For any C^∞ $(p,\,q)$-form $v\in Im\,(\partial\bar\partial)$, the (unique) minimal L^2-norm solution of the equation*

$$\partial\bar\partial u = v \tag{4.145}$$

is given by the Neumann-type formula:

$$u = (\partial\bar\partial)^\star\Delta_{BC}^{-1}v, \tag{4.146}$$

while its L^2-norm satisfies the estimate:

$$||u||^2 \leq \frac{1}{\lambda}\,||v||^2, \tag{4.147}$$

where Δ_{BC}^{-1} denotes the Green operator of Δ_{BC} and $\lambda>0$ is the smallest positive eigenvalue of Δ_{BC}. Furthermore, we have

$$\partial\Delta_{BC}^{-1}v = 0 \quad \textit{and} \quad \bar\partial\Delta_{BC}^{-1}v = 0. \tag{4.148}$$

Proof. Let $w:=\Delta_{BC}^{-1}v$, i.e. w is the unique $(p,\,q)$-form characterised by the following two properties:

$$\Delta_{BC}w = v \quad \text{and} \quad w\perp\ker\Delta_{BC}. \tag{4.149}$$

By the definition (1.2) of Δ_{BC}, the identity $\Delta_{BC}w = v = \partial\bar\partial u$ is equivalent to

$$A_1 + (A_2+A_3) = 0, \text{ where:}$$

$$\begin{aligned} A_1 &:= \partial\bar\partial\bigg((\partial\bar\partial)^\star w - u\bigg) \in \operatorname{Im}\partial\cap\operatorname{Im}\bar\partial, \\ A_2 &:= \partial^\star\partial w + (\partial^\star\bar\partial)(\partial^\star\bar\partial)^\star w \in \operatorname{Im}\partial^\star, \\ A_3 &:= \bar\partial^\star\bar\partial w + (\partial\bar\partial)^\star(\partial\bar\partial)w + (\partial^\star\bar\partial)^\star(\partial^\star\bar\partial) \in \operatorname{Im}\bar\partial^\star. \end{aligned}$$

Since $\operatorname{Im}\partial\perp\operatorname{Im}\partial^\star$ and $\operatorname{Im}\bar\partial\perp\operatorname{Im}\bar\partial^\star$, we infer that $A_1\perp A_2$ and $A_1\perp A_3$, hence $A_1\perp(A_2+A_3)$. It follows that the identity $\Delta_{BC}w = v = \partial\bar\partial u$ is equivalent to $A_1=0$ and $A_2+A_3=0$. Note that $A_1=0$ amounts to

$$(\partial\bar\partial)^\star w - u \in \ker(\partial\bar\partial). \tag{4.150}$$

Meanwhile, the solutions of equation (4.145) are unique up to $\ker(\partial\bar\partial)$, so if u is the minimal L^2-norm solution, then $u\in\ker(\partial\bar\partial)^\perp = \operatorname{Im}(\partial\bar\partial)^\star$. Thus

$$(\partial\bar\partial)^\star w - u \in \operatorname{Im}(\partial\bar\partial)^\star. \tag{4.151}$$

Now, $\ker(\partial\bar\partial)$ and $\operatorname{Im}(\partial\bar\partial)^\star$ are mutually orthogonal, so thanks to (4.150) and (4.151), the identity $A_1=0$ is equivalent to $(\partial\bar\partial)^\star w - u = 0$. This proves formula (4.146). On the other hand, the identity $A_2+A_3=0$ implies $\langle\langle A_2+A_3,\,w\rangle\rangle = 0$ which translates to

$$||\partial w||^2 + ||\bar\partial^\star\partial w||^2 + ||\bar\partial w||^2 + ||\partial\bar\partial w||^2 + ||\partial^\star\bar\partial w||^2 = 0.$$

This amounts to $\partial w = 0$ and $\bar\partial w = 0$, proving (4.148).

Let us now estimate the L^2 norm of $u = (\partial\bar\partial)^\star\Delta_{BC}^{-1}v$. We have:

$$\begin{aligned} ||u||^2 &= \langle\langle(\partial\bar\partial)(\partial\bar\partial)^\star\Delta_{BC}^{-1}v,\,\Delta_{BC}^{-1}v\rangle\rangle \stackrel{(a)}{=} \langle\langle\Delta_{BC}\Delta_{BC}^{-1}v,\,\Delta_{BC}^{-1}v\rangle\rangle \\ &= \langle\langle v,\,\Delta_{BC}^{-1}v\rangle\rangle \stackrel{(b)}{\leq} \frac{1}{\lambda}\,||v||^2, \end{aligned}$$

where identity (a) follows from (1.2) and from the identities:

$$\begin{aligned}\partial^\star\partial\Delta_{BC}^{-1}v &= 0, \quad (\partial^\star\bar\partial)(\partial^\star\bar\partial)^\star\Delta_{BC}^{-1}v = 0,\\ \bar\partial^\star\bar\partial\Delta_{BC}^{-1}v &= 0, \quad (\partial\bar\partial)^\star(\partial\bar\partial)\Delta_{BC}^{-1}v = 0, \quad (\partial^\star\bar\partial)^\star(\partial^\star\bar\partial)\Delta_{BC}^{-1}v = 0,\end{aligned}$$

all of which are consequences of $\partial\Delta_{BC}^{-1}v = 0$ and of $\bar\partial\Delta_{BC}^{-1}v = 0$ already proved as (4.148). Inequality (b) follows from $v \perp \ker\Delta_{BC}$ since $v \in \mathrm{Im}\,(\partial\bar\partial) \subset \mathrm{Im}\,\Delta_{BC}$ (see (1.10)). Estimate (4.147) is proved. □

4.5.4 Stability of balanced manifolds under modifications

In this subsection, we present the work of Alessandrini and Bassanelli in [AB91b], [AB93] and [AB95] proving that the *balanced* property of compact complex manifolds is stable under modifications (i.e. proper, holomorphic, bimeromorphic maps). We saw in §.4.2.3 that the analogous property holds for strongly Gauduchon manifolds.

Theorem 4.5.48. *([AB95]) Let $\mu : \widetilde{X} \to X$ be a modification of compact complex manifolds. Then, $\widetilde{X}$ is a* **balanced manifold** *if and only if X is a* **balanced manifold***.*

An immediate consequence is the following

Corollary 4.5.49. *Every* **class $\mathcal{C}$** *manifold is* **balanced***.*

Proof. If X is a *class $\mathcal{C}$* manifold, there exists a modification $\mu : \widetilde{X} \to X$ with $\widetilde{X}$ compact Kähler. Then $\widetilde{X}$ is also balanced, hence X is balanced by Theorem 4.5.48. □

Proof of Theorem 4.5.48

Let $\mu : \widetilde{X} \to X$ be a modification of compact complex manifolds and let $n = \dim_{\mathbb{C}}\widetilde{X} = \dim_{\mathbb{C}}X$. Let E be the exceptional divisor of μ on $\widetilde{X}$ and let $\Sigma \subset X$ be the analytic subset of codimension ≥ 2 such that the restriction $\mu_{|\widetilde{X}\setminus E} : \widetilde{X}\setminus E \longrightarrow X\setminus\Sigma$ is a biholomorphism.

Due to Proposition 4.5.24, we will have to deal with semi-positive Aeppli-exact bidegree $(1, 1)$-currents T. Alessandrini and Bassanelli obtained the following key result on $\partial\bar\partial$-closed semi-positive bidegree $(1, 1)$-currents.

Theorem 4.5.50. *(Theorem 5 in [AB91b]) Let M be an n-dimensional complex manifold and let $E \subset M$ be a compact analytic subset of pure complex dimension $n-1$ whose irreducible components are denoted by $(E_j)_{j=1,\dots,s}$. If $T \geq 0$ is a semi-positive bidegree $(1, 1)$-current on M such that*

$$\partial\bar\partial T = 0 \quad \text{and} \quad \mathrm{Supp}\,T \subset E,$$

there exist constants $\lambda_j \geq 0$ such that $T = \sum_{j=1}^{s}\lambda_j\,[E_j]$. In particular, T is d-closed.

Theorem 4.5.48 comprises two parts.

(I) One implication of the equivalence in Theorem 4.5.48 is dealt with in the following

Theorem 4.5.51. *([AB91b]) If $\mu : \widetilde{X} \to X$ is a modification of compact complex manifolds and X is* **balanced***, then $\widetilde{X}$ is again* **balanced***.*

Proof. The approach is very similar to the one in the proof of Theorem 4.2.10. The only difference is that we will have to deal with a $\partial\bar\partial$-closed current rather than a d-closed current at some point.

We proceed by contradiction. Suppose that $\widetilde{X}$ is not *balanced*. Then, by Proposition 4.5.24, there is a non-zero current $T = (dS)^{1,\,1} \neq 0$ of bidegree $(1,\,1)$ on $\widetilde{X}$ such that $T \geq 0$ and T is the $(1,\,1)$-component of some d-exact current dS of degree 2 on $\widetilde{X}$, with S a real current of degree 1.

For the direct image current $\mu_\star T$, a well-defined current of bidegree $(1,\,1)$ on X, we have:

$$\mu_\star T \geq 0 \quad\text{ and }\quad \mu_\star T = (d\mu_\star S)^{1,\,1} \quad\text{on}\quad X.$$

Now, we have the following dichotomy.

If $\mu_\star T$ is non-zero, we get a contradiction with the *balanced* assumption on X thanks to Proposition 4.5.24.

If $\mu_\star T = 0$ on X, we show that $T = 0$ on $\widetilde{X}$, contradicting the choice of T. Since $\mu_\star T = 0$, the support of T must be contained in the support of E. By Theorem 4.5.50, T is of the shape:

$$T = \sum_{j=1}^{s}\lambda_j\,[E_j],$$

with constant coefficients $\lambda_j \geq 0$, where the $(E_j)_{1\leq j\leq s}$ are the irreducible components of E.

In particular, T is d-closed. We will now see that T is even d-exact.

Lemma 4.5.52. *([AB91b, Lemma 8]) Let $\mu : \widetilde{X} \to X$ be a modification between n-dimensional compact complex manifolds. Let T be a real d-closed current of bidegree $(1,\,1)$ on $\widetilde{X}$ such that $T = (dS)^{1,\,1}$ for some real current S of degree 1 on $\widetilde{X}$ and $\mu_\star T = 0$ on X.*

Then, there exists a current Q such that $T = dQ$ on $\widetilde{X}$.

Proof of Lemma 4.5.52. We have $S^{0,\,1} = \overline{S^{1,\,0}}$, $T = \partial S^{0,\,1} + \bar\partial S^{1,\,0}$ and $0 = \partial T = \partial\bar\partial S^{1,\,0}$ on $\widetilde{X}$. Hence $\partial\bar\partial\mu_\star S^{1,\,0} = 0$ on X, so the $(1,\,0)$-current $\mu_\star S^{1,\,0}$ represents an Aeppli cohomology class on X. Since the Aeppli cohomology can be computed using either currents or smooth forms, we can find a smooth representative $\varphi^{1,\,0} \in \ker(\partial\bar\partial)$ of the Aeppli cohomology class of $\mu_\star S^{1,\,0}$. In other words, there exists a $\partial\bar\partial$-closed form $\varphi^{1,\,0} \in C^\infty_{1,\,0}(X,\,\mathbb{C})$ such that:

$$\mu_\star S^{1,\,0} = \varphi^{1,\,0} + \partial\theta \quad\text{ on }\; X,$$

for some distribution θ on X. (Note that the only $\bar\partial$-exact $(1,\,0)$-current is 0, so there is no Im $\bar\partial$-term on the r.h.s. of the above identity.) Let $\theta := a + ib$ with *real* distributions a and b on X.

Since $\mu_\star T = 0$, we get

$$0 = \mu_\star T = \partial\mu_\star S^{0,\,1} + \bar\partial\mu_\star S^{1,\,0} = \partial\varphi^{0,\,1} + \bar\partial\varphi^{1,\,0} - 2i\partial\bar\partial b \quad\text{ on }\; X,$$

where $\varphi^{0,\,1} := \overline{\varphi^{1,\,0}}$. In particular, $2i\partial\bar\partial b = \partial\varphi^{0,\,1} + \bar\partial\varphi^{1,\,0}$ is C^∞, hence b is a C^∞ real-valued function on X. Since C^∞ forms always have well-defined pull-backs under C^∞ maps, so does the C^∞ form $\varphi^{1,\,0} + i\partial b$ under the holomorphic map μ. Therefore, we get a well-defined $(1,\,0)$-current on $\widetilde{X}$ by putting:

$$S'^{1,\,0} = S^{1,\,0} - \mu^\star(\varphi^{1,\,0} + i\partial b), \quad\text{ on }\; \widetilde{X}.$$

Next, we put $S'^{0,\,1} = \overline{S'^{1,\,0}}$ and we get:

$$\begin{aligned}\partial S'^{0,\,1} + \bar\partial S'^{1,\,0} &= \partial S^{0,\,1} + \bar\partial S^{1,\,0} - \mu^\star(\partial\varphi^{0,\,1} - i\partial\bar\partial b) - \mu^\star(\bar\partial\varphi^{1,\,0} + i\bar\partial\partial b)\\ &= \partial S^{0,\,1} + \bar\partial S^{1,\,0} - \mu^\star(\partial\varphi^{0,\,1} + \bar\partial\varphi^{1,\,0} - 2i\partial\bar\partial b)\\ &= \partial S^{0,\,1} + \bar\partial S^{1,\,0} - \mu^\star(\mu_\star T) = \partial S^{0,\,1} + \bar\partial S^{1,\,0} = T.\end{aligned}$$

We now put $Q := S'^{1,0} + S'^{0,1}$ on $\widetilde{X}$. We will have $T = dQ$ once we have shown that $\partial S'^{1,0} = 0$ on $\widetilde{X}$. Indeed, we will then also have $\bar\partial S'^{0\,1} = 0$ by conjugation.

We have

$$\partial S'^{1,0} = \partial S^{1,0} - \mu^\star(\partial\varphi^{1,0}) = \partial S^{1,0} - \mu^\star\mu_\star(\partial S^{1,0}) \quad \text{on } \widetilde{X}.$$

Now, $\bar\partial(\partial S'^{1,0}) = \bar\partial\partial S^{1,0} - \mu^\star(\bar\partial\partial\varphi^{1,0}) = 0$ (recall that $\bar\partial\partial S^{1,0} = -\partial T = 0$), so $\partial S'^{1,0}$ is a $\bar\partial$-closed $(2,\,0)$-form on $\widetilde{X}$. Since the Laplacian Δ'' is hypoelliptic in bidegree $(p,\,0)$, $\partial S'^{1,0}$ is therefore a holomorphic 2-form on $\widetilde{X}$. Moreover, the restriction of μ to the complement of the support of the exceptional divisor E is a biholomorphism onto its image, so $\mu^\star\mu_\star(\partial S^{1,0}) = \partial S^{1,0}$ on $\widetilde{X}\setminus \operatorname{Supp} E$.

We conclude that $\partial S'^{1,0}$ is a holomorphic 2-form on $\widetilde{X}$ that vanishes identically on $\widetilde{X}\setminus \operatorname{Supp} E$. Therefore, $\partial S'^{1,0} = 0$ on $\widetilde{X}$. □

End of proof of Theorem 4.5.51. Since $\operatorname{codim}_X \mu(E_j) \geq 2$ for every $j = 1, \dots, s$ and T is d-exact on $\widetilde{X}$ by Lemma 4.5.52, the situation is now identical to the one at the end of the proof of Theorem 4.2.10. Repeating those arguments, we conclude that $T = 0$ as a current on $\widetilde{X}$, a contradiction.

The proof is complete. □

(II) The other implication of the equivalence in Theorem 4.5.48 is dealt with in the following

Theorem 4.5.53. *([AB95]) If $\mu : \widetilde{X} \to X$ is a modification of compact complex manifolds and $\widetilde{X}$ is* **balanced***, then X is again* **balanced***.*

The main ingredient in the proof of Theorem 4.5.53 is the following

Theorem 4.5.54. *([AB95, Theorem 3]) Let $\mu : \widetilde{X} \to X$ be a proper modification between complex manifolds and let $T \geq 0$ be a semi-positive $(1,\,1)$-current on X such that $\partial\bar\partial T = 0$. Then, there exists a* **unique** *semi-positive $(1,\,1)$-current $\widetilde{T} \geq 0$ on $\widetilde{X}$ such that*

$$\mu_\star\widetilde{T} = T \quad \textit{and} \quad \widetilde{T} \in \mu^\star\{T\}_A,$$

where $\{T\}_A \in H^{1,\,1}_A(X,\,\mathbb{R})$ is the Aeppli cohomology class represented by T on X.

In other words, T can be pulled back in a unique way to the pullback to $\widetilde{X}$ of the Aeppli cohomology class of T. Note that cohomology classes can always be unambiguously pulled back under holomorphic maps by pulling back their smooth representatives. Smooth forms have unique pullbacks as smooth forms of the same bidegree under holomorphic maps μ and the closedness and exactness conditions are preserved since μ commutes with ∂, $\bar\partial$ and d.

Proof of Theorem 4.5.53 assuming Theorem 4.5.54 has been proved.

We proceed by contradiction. Suppose that X is not balanced. Then, by Proposition 4.5.24 and Lemma 4.5.23 there exists a *non-zero* semi-positive Aeppli-exact bidegree $(1,\,1)$-current $T \geq 0$ on X. Then, by Theorem 4.5.54, there exists a unique semi-positive Aeppli-exact bidegree $(1,\,1)$-current $\widetilde{T} \geq 0$ on $\widetilde{X}$ such that $\mu_\star\widetilde{T} = T$.

However, $\widetilde{X}$ being balanced, we must have $\widetilde{T} = 0$ on $\widetilde{X}$ by Proposition 4.5.24 and Lemma 4.5.23. Then, $0 = \mu_\star\widetilde{T} = T$ on X, contradicting the assumption $T \neq 0$. □

Sketch of proof of Theorem 4.5.54.

We will only sketch the existence part since it is the only one needed for the proof of Theorem 4.5.54. Recall that $\mu_{|\widetilde{X}\setminus E} : \widetilde{X}\setminus E \longrightarrow X\setminus\Sigma$ is a biholomorphism, so we can define the semi-positive bidegree (1, 1)-current

$$R := \bigg((\mu_{|\widetilde{X}\setminus E})^{-1}\bigg)_\star (T_{|X\setminus\Sigma}) \geq 0$$

on $\widetilde{X}\setminus E$ as the direct image under the inverse of the restriction of μ to $\widetilde{X}\setminus E$ of the restriction of T to $X\setminus\Sigma$. Any current $\widetilde{T}$ as in the statement must be an extension of R to $\widetilde{X}$.

Now, suppose that μ is a blow-up with a smooth centre and fix an arbitrray point $x\in\Sigma$. By smoothing T through convolutions with regularising kernels ρ_ε in a neighbourhood $U\subset X$ of x, we get C^∞ semi-positive (1, 1)-forms T_ε in U such that $\partial\bar\partial T_\varepsilon = 0$ for all $\varepsilon>0$ and $T_\varepsilon \longrightarrow T$ in the weak topology of currents as $\varepsilon\downarrow 0$. Moreover, since $\partial\bar\partial T = 0$, there exists a possibly smaller open neighbourhood $W\subset U$ of x such that T is Aeppli-exact in W. Therefore, the smooth forms T_ε are also Aeppli-exact in W. Since the T_ε's are C^∞, they have well-defined pullbacks $\mu^\star T_\varepsilon\geq 0$ (which are again C^∞ semi-positive (1, 1)-forms) to $\mu^{-1}(U)\subset\widetilde{X}$ under μ.

After possibly shrinking W further about x, Alessandrini and Bassanelli showed in [AB93] that the masses of the C^∞ semi-positive (1, 1)-forms $(\mu^\star T_\varepsilon)_{|\mu^{-1}(W)}$ are uniformly bounded. This implies the existence of a weakly convergent subsequence $\mu^\star T_{\varepsilon_\nu} \longrightarrow \widetilde{T}\geq 0$ in $\mu^{-1}(W)$ as $\nu\to+\infty$. The authors of [AB93] go on to prove that the limiting current $\widetilde{T}$, a priori defined only in $\mu^{-1}(W)$, is independent of the subsequence $(\mu^\star T_{\varepsilon_\nu})_\nu$. This yields a globally defined $\partial\bar\partial$-closed semi-positive (1, 1)-current $\widetilde{T}\geq 0$ on $\widetilde{X}$ such that $\mu_\star\widetilde{T} = T$ even in the general case where μ is an arbitrary modification.

It remains to prove that $\widetilde{T}\in\mu^\star\{T\}_A$. Recall that $\mu:\widetilde{X}\to X$ is a proper modification. Hence, it is locally dominated by a blow-up in the following sense. For every point $x\in X$, there exists an open neighbourhood V of x, a complex manifold Z and holomorphic maps $g: Z\to\mu^{-1}(V)$, $q: Z\to V$ such that g is a blow-up and

$$q = \mu\circ g = q_1\circ\cdots\circ q_s$$

is the composition of finitely many blow-ups $q_j: V_j\to V_{j-1}$ with smooth centres, where $V_0 = V$ and $V_s = Z$.

For a fixed $x\in X$, we may assume that V is contained in a coordinate chart and, as seen above, that $T_{|V}$ is Aeppli-exact and a weak limit in V of a sequence of C^∞ semi-positive Aeppli-exact (1, 1)-forms T_ε. We put

$$T^j := \lim_{\varepsilon\to 0}(q_j^\star\circ\cdots\circ q_1^\star)(T_\varepsilon) \quad \text{in } V_j, \quad j\in\{1,\dots,s\}.$$

In particular, $T^s = \lim\limits_{\varepsilon\to 0} q^\star T_\varepsilon$ and $\widetilde{T}_{|\mu^{-1}(V)} = g_\star T^s$.

Now, T^1 is a $\partial\bar\partial$-closed semi-positive (1, 1)-current on V_1 and has the property that every $y\in V = V_0$ has an open neighbourhood W such that $T^1_{|q_1^{-1}(W)}$ is a weak limit of Aeppli-exact currents. Alessandrini and Bassanelli prove that, in this case, for every $y\in V = V_0$, W can be chosen such that $T^1_{|q_1^{-1}(W)}$ is Aeppli-exact. Moreover, $(q_1)_\star T^1 = T_{|V}$ is Aeppli-exact because V has trivial Aeppli cohomology. Another general result of Alessandrini-Bassanelli shows that T^1 is Aeppli-exact globally on V_1 because $(q_1)_\star T^1 = T_{|V}$ is Aeppli-exact globally on V and every $y\in V = V_0$ has an open neighbourhood W such that $T^1_{|q_1^{-1}(W)}$ is Aeppli-exact.

Since $T^1 = (q_2)_\star T^2$, we can continue inductively to conclude that T^s is Aeppli-exact globally on $V_s = Z$. This implies that $\widetilde{T}_{|\mu^{-1}(V)} = g_\star T^s$ is Aeppli-exact globally on $\mu^{-1}(V)$.

Finally, let α be a C^∞ representative of the Aeppli class $\{T\}_A \in H^{1,\,1}_A(X,\,\mathbb{R})$. Then, $\alpha_{|V}$ is Aeppli exact (because V has trivial Aeppli cohomology), hence $(\mu^\star\alpha)_{|\mu^{-1}(V)}$ is Aeppli exact. Applying the last general result of Alessandrini-Bassanelli used above to the current $\widetilde{T} - \mu^\star\alpha$ on $\widetilde{X}$, we get

$$\{\widetilde{T}\}_A = \{\mu^\star\alpha\}_A = \mu^\star\{T\}_A \in H^{1,\,1}_A(\widetilde{X},\,\mathbb{R})$$

because $\widetilde{T} - \mu^\star\alpha$ has the properties:

(i) $\mu_\star(\widetilde{T} - \mu^\star\alpha) = T - \alpha$ is Aeppli-exact on X;

(ii) every $x \in X$ has an open neighbourhood V such that $(\widetilde{T} - \mu^\star\alpha)_{|\mu^{-1}(V)}$ is Aeppli-exact;

hence $\widetilde{T} - \mu^\star\alpha$ is Aeppli-exact on $\widetilde{X}$. □

4.5.5 Balanced and $\partial\bar\partial$-manifolds

Corollary 4.5.49 was the last piece of information leading to the following

Conclusion 4.5.55. *The relations among various properties of a compact complex manifold X are summed up in the following diagram (skew arrows indicate implications):*

$(\star)$

$$\begin{tikzcd}
 & X \text{ is Kähler} \arrow[dr, Rightarrow] & & X \text{ is balanced} \arrow[dr, Rightarrow] & \\
X \text{ is projective} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & X \text{ is class } \mathcal{C} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & X \text{ is sG} \\
 & X \text{ is Moishezon} \arrow[ur, Rightarrow] & & X \text{ is } \partial\bar\partial \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\
 & & & & E_1(X) = E_\infty(X)
\end{tikzcd}$$

The above diagram simplifies dramatically, with many implications becoming equivalences, when $dim_{\mathbb{C}}X = 2$:

X projective $\Longleftrightarrow$ X Moishezon $\Longrightarrow$ X Kähler $\Longleftrightarrow$ X class $\mathcal{C}$ $\Longleftrightarrow$ X balanced $\Longleftrightarrow$ X satisfies the $\partial\bar\partial$-lemma $\Longleftrightarrow$ X sG

and we always have $E_1(X) = E_\infty(X)$ for surfaces.

Whether there is any relation between compact $\partial\bar\partial$-manifolds and balanced manifolds is still an open problem, but the following fact was conjectured in [Pop15c, §.6].

Conjecture 4.5.56. *Every compact $\partial\bar\partial$-manifold is balanced.*

This is part of a wider problem that we now briefly explain (and refer the reader to [Pop15c, §.6] for further details). Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. It is standard that the canonical linear map induced in cohomology by the identity:

$$I_{n-1} : H^{n-1,\,n-1}_{BC}(X,\,\mathbb{C}) \to H^{n-1,\,n-1}_{A}(X,\,\mathbb{C}), \qquad [\Omega]_{BC} \mapsto [\Omega]_A, \tag{4.152}$$

is well defined on every X, but it is neither injective, nor surjective in general. Moreover, the balanced cone $\mathcal{B}_X$ of X, consisting of the Bott-Chern cohomology classes of bidegree $(n-1,\,n-1)$ representable by balanced metrics ω^{n-1}, namely:

$$\mathcal{B}_X = \{[\omega^{n-1}]_{BC} \,/\, \omega > 0,\ C^\infty\ (1,\,1)\text{-form such that } d\omega^{n-1} = 0 \text{ on } X\} \subset H^{n-1,\,n-1}_{BC}(X,\,\mathbb{R}),$$

maps under I_{n-1} to a subset of the Gauduchon cone $\mathcal{G}_X$ of X.

The inclusion $I_{n-1}(\mathcal{B}_X) \subset \mathcal{G}_X$ is strict in general. So is the inclusion $I_{n-1}(\overline{\mathcal{B}_X}) \subset \overline{\mathcal{G}}_X$ involving the closures of these two open convex cones. When X is a $\partial\bar\partial$-manifold, I_{n-1} is an isomorphism of the vector spaces $H^{n-1,\,n-1}_{BC}(X,\,\mathbb{C})$ and $H^{n-1,\,n-1}_{A}(X,\,\mathbb{C})$.

Since the Gauduchon cone $\mathcal{G}_X$ is never empty, Conjecture 4.5.56 is a special case of the following

Conjecture 4.5.57. *([Pop15c, Conjecture 6.1]) If X is a compact $\partial\bar\partial$-manifold of dimension n, then $I_{n-1}(\overline{\mathcal{B}_X}) = \overline{\mathcal{G}}_X$.*

The reason for conjecturing this goes back to the *quantitative part* of Demailly's *Transcendental Morse Inequalities Conjecture* for differences of two nef classes:

Conjecture 4.5.58. *([BDPP13, Conjecture 10.1, (ii)]) Let X be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$ and let $\{\alpha\}, \{\beta\} \in H^{1,1}_{BC}(X,\,\mathbb{R})$ be* **nef** *cohomology classes such that*

$$\{\alpha\}^n - n\,\{\alpha\}^{n-1}.\,\{\beta\} > 0.$$

Then, the volume of the difference class $\{\alpha - \beta\} \in H^{1,1}_{BC}(X,\,\mathbb{R})$ satisfies the lower bound:

$$Vol\,(\{\alpha - \beta\}) \geq \{\alpha\}^n - n\,\{\alpha\}^{n-1}.\,\{\beta\}. \tag{4.153}$$

This is stated for arbitrary (i.e. possibly non-Kähler) compact complex manifolds in [BDPP13], but the volume is currently only known to be meaningful when X is of *class* $\mathcal{C}$, a case reducible to the Kähler case by modifications. Thus, we may assume without loss of generality that X is Kähler.

Recall that the volume is a way of gauging the "amount" of positivity of a class $\{\gamma\} \in H^{1,1}_{BC}(X,\,\mathbb{R})$ when X is Kähler (or merely of *class* $\mathcal{C}$) and was introduced in [Bou02, Definition 1.3] as

$$\text{Vol}\,(\{\gamma\}) := \sup_{T\in\{\gamma\},\,T\geq 0} \int\limits_X T^n_{ac}$$

if $\{\gamma\}$ is *pseudo-effective (psef)*, i.e. if $\{\gamma\}$ contains a *positive* $(1,\,1)$-current $T \geq 0$, where T_{ac} denotes the absolutely continuous part of T in the Lebesgue decomposition of its coefficients (which are complex measures when $T \geq 0$). If the class $\{\gamma\}$ is not psef, then its volume is set to be zero. It was proved in [Bou02, Theorem 1.2] that this volume (which is always a *finite* non-negative quantity thanks to the Kähler, or more generally *class* $\mathcal{C}$, assumption on X) coincides with the standard volume of a holomorphic line bundle L if the class $\{\gamma\}$ is integral (i.e. the first Chern class of some L). Moreover, the class $\{\gamma\}$ is big (i.e. contains a Kähler current) if and only if its volume is *positive*, by [Bou02, Theorem 4.7].

Theorem 4.1.29 proves the *qualitative part* of Conjecture 4.5.58. Its conclusion is equivalent to the fact that Vol $(\{\alpha-\beta\})>0$. The method of proof for Theorem 4.1.29 presented in §.4.1.5 also produces various lower bounds for Vol $(\{\alpha-\beta\})$, but they are weaker than the conjectured lower bound (4.153). (See [Pop15c] for quantitative results.)

One piece of evidence supporting Conjecture 4.5.57 is that it holds on every *class* $\mathcal{C}$ manifold X if Conjecture 4.5.58 is confirmed when X is Kähler. This is the gist of the observations made in [Tom10] and in [CRS14]. Indeed, if X is of *class* $\mathcal{C}$, we may assume without loss of generality that X is actually compact Kähler. As proved in [BDPP13], a complete positive answer to Conjecture 4.5.58 would imply that the pseudo-effective cone $\mathcal{E}_X\subset H^{1,1}(X,\mathbb{R})$ of classes of d-closed positive $(1,1)$-currents T is the dual of the cone $\mathcal{M}_X\subset H^{n-1,n-1}(X,\mathbb{R})$ of *movable classes* (i.e. the closure of the cone generated by classes of currents of the shape $\mu_\star(\widetilde{\omega}_1\wedge\cdots\wedge\widetilde{\omega}_{n-1})$, where $\mu:\widetilde{X}\to X$ is any modification of compact Kähler manifolds and the $\widetilde{\omega}_j$ are any Kähler metrics on $\widetilde{X}$ – see [BDPP13, Definition 1.3]). Since on $\partial\bar\partial$-manifolds (hence, in particular, on compact Kähler ones) the Bott-Chern, Dolbeault and Aeppli cohomologies are canonically equivalent, it is irrelevant in which of these cohomologies the groups $H^{1,1}(X,\mathbb{R})$ and $H^{n-1,n-1}(X,\mathbb{R})$ are considered.

The closure $\overline{\mathcal{G}}_X\subset H^{n-1,n-1}(X,\mathbb{R})$ of the Gauduchon cone is dual to the pseudo-effective cone $\mathcal{E}_X\subset H^{1,1}(X,\mathbb{R})$ by Theorem 4.1.18, while the same kind of argument (i.e. duality and Hahn-Banach) going back to Sullivan shows that the closure $\overline{\mathcal{B}}_X\subset H^{n-1,n-1}(X,\mathbb{R})$ of the balanced cone is dual to the cone

$$\mathcal{S}_X=\{[T]_A\,/\,T\geq 0,\ T\ \text{is a}\,(1,\,1)-\text{current such that}\ \partial\bar\partial T=0\ \text{on}\ X\}\subset H^{1,\,1}_A(X,\,\mathbb{R}).$$

Note that $\mathcal{S}_X$ is closed if X admits a balanced metric ω^{n-1} (against which the masses of positive $\partial\bar\partial$-closed $(1,\,1)$-currents T can be considered), hence so is it when X is Kähler. Thus, by duality, the identity $I_{n-1}(\overline{\mathcal{B}_X})=\overline{\mathcal{G}}_X$ is equivalent to $I_1(\mathcal{E}_X)=\mathcal{S}_X$, where I_1 is the canonical linear map induced in cohomology by the identity:

$$I_1:H^{1,\,1}_{BC}(X,\,\mathbb{C})\to H^{1,\,1}_A(X,\,\mathbb{C}),\qquad [\gamma]_{BC}\mapsto[\gamma]_A.$$

In general, I_1 is neither injective, nor surjective, but it is an isomorphism when X is a $\partial\bar\partial$-manifold.

With these facts understood, the identity $I_1(\mathcal{E}_X)=\mathcal{S}_X$ can be proved when X is Kähler (provided that Conjecture 4.5.58 can be solved in the affirmative) as explained in [CRS14, Proposition 2.5] by an argument generalising to transcendental classes an earlier argument from [Tom10] that we now recall for the reader's convenience.

The inclusion $I_1(\mathcal{E}_X)\subset\mathcal{S}_X$ is obvious. To prove the reverse inclusion, let $[T]_A\in\mathcal{S}_X$, i.e. $T\geq 0$ is a $(1,\,1)$-current such that $\partial\bar\partial T=0$. Since I_1 is an isomorphism, there exists a unique class $[\gamma]_{BC}\in H^{1,\,1}_{BC}(X,\,\mathbb{R})$ such that $I_1([\gamma]_{BC})=[T]_A$. This means that $[\gamma]_A=[T]_A$. We will show that $[\gamma]_{BC}\in\mathcal{E}_X$. If the [BDPP13] conjecture (predicated on Conjecture 4.5.58) predicting that $\mathcal{E}_X$ is dual to $\mathcal{M}_X$ is confirmed, showing that $[\gamma]_{BC}\in\mathcal{E}_X$ amounts to showing that

$$[\gamma]_{BC}\,.\,[\mu_\star(\widetilde{\omega}_1\wedge\cdots\wedge\widetilde{\omega}_{n-1})]_A\geq 0 \tag{4.154}$$

for all modifications $\mu:\widetilde{X}\to X$ and all Kähler metrics $\widetilde{\omega}_j$ on $\widetilde{X}$.

On the other hand, using the key Alessandrini-Bassanelli Theorem 4.5.54, we get:

$$\begin{aligned}[\gamma]_{BC}\,.\,[\mu_\star(\widetilde{\omega}_1\wedge\cdots\wedge\widetilde{\omega}_{n-1})]_A &= \int\limits_X \gamma\wedge\mu_\star(\widetilde{\omega}_1\wedge\cdots\wedge\widetilde{\omega}_{n-1}) = \int\limits_{\widetilde{X}}(\mu^\star\gamma)\wedge(\widetilde{\omega}_1\wedge\cdots\wedge\widetilde{\omega}_{n-1})\\ &= [\mu^\star\gamma]_A\,.\,[\widetilde{\omega}_1\wedge\cdots\wedge\widetilde{\omega}_{n-1}]_{BC} = \int\limits_{\widetilde{X}}(\mu^\star T)\wedge(\widetilde{\omega}_1\wedge\cdots\wedge\widetilde{\omega}_{n-1})\geq 0,\end{aligned}$$

which proves (4.154). Note that γ and $\mu^\star\gamma$ have no sign, so the key point has been the replacement in the integral over $\widetilde{X}$ of $\mu^\star\gamma$ by $\mu^\star T\geq 0$ which was made possible by $\widetilde{\omega}_1\wedge\cdots\wedge\widetilde{\omega}_{n-1}$ being d-closed (so we could switch the roles of the Bott-Chern and the Aeppli cohomologies) and by the identity $[\mu^\star\gamma]_A=[\mu^\star T]_A$ following from $[\gamma]_A=[T]_A$ (see above) and from $[\mu^\star T]_A=\mu^\star([T]_A)$.

The techniques employed in this section do not seem to be using the full force of the Kähler assumption on X and many of the arguments are valid in a more general context. This is part of the justification for proposing Conjecture 4.5.57.

4.5.6 Degenerate balanced structures

It is well known that no Kähler metric ω on a *compact* complex manifold X can be d-exact. Indeed, if $n=\dim_{\mathbb{C}}X$, then on the one hand we have $\int_X\omega^n>0$ (since $\omega>0$ implies $\omega^n>0$ everywhere on X), while if $\omega=d\alpha$ for some 1-form α on X, then $\omega^n=d(\alpha\wedge(d\alpha)^{n-1})$, hence $\int_X\omega^n=0$ by Stokes.

We will now see that, unlike Kähler metrics, balanced metrics ω may have the property that ω^{n-1} is d-exact.

Definition 4.5.59. *Let X be a compact complex manifold with $\dim_{\mathbb{C}}X=n$. A* **degenerate balanced metric** *on X is a C^∞ positive definite $(1,1)$-form ω on X such that ω^{n-1} is* **d-exact***.*

We say that X is a **degenerate balanced manifold** *if it carries a degenerate balanced metric.*

On the other hand, recall that the Gauduchon cone $\mathcal{G}_X\subset H_A^{n-1,\,n-1}(X,\,\mathbb{R})$ of an n-dimensional compact complex manifold X is never empty (since Gauduchon metrics always exist on X). However, it may be the whole ambient space $H_A^{n-1,\,n-1}(X,\,\mathbb{R})$, as we will see.

Definition 4.5.60. *Let X be a compact complex manifold with $\dim_{\mathbb{C}}X=n$. We say that the Gauduchon cone $\mathcal{G}_X$ of X* **degenerates** *if $\mathcal{G}_X=H_A^{n-1,\,n-1}(X,\,\mathbb{R})$.*

Before pointing out examples of (fairly exotic) manifolds X where the above two phenomena occur, we notice that they are equivalent. We will often denote De Rham cohomology groups $H^k_{DR}(X,\,\mathbb{C})$ simply by $H^k(X,\,\mathbb{C})$.

Proposition 4.5.61. *([Pop15a, Proposition 5.4]) Let X be a compact complex manifold.*

(a) The following three statements are equivalent.

(i) X is a **degenerate balanced manifold***.*

(ii) There exists no nonzero d-closed bidegree $(1,1)$-current $T\geq 0$ on X.

(iii) The Gauduchon cone of X **degenerates***.*

Furthermore, if any of the above three equivalent properties holds, X cannot be a class $\mathcal{C}$ manifold.

(b) If $H^2(X,\,\mathbb{C})=0$, the following equivalence holds:

$$X \text{ is an sG manifold} \iff X \text{ is a balanced manifold}$$

and each of these two equivalent properties implies that X is a degenerate balanced manifold.

Proof. Let $\dim_{\mathbb{C}} X = n$.

(a) The equivalence $(i) \Leftrightarrow (ii)$ follows by the standard duality and Hahn-Banach argument introduced in [Sul76] and already used in various situations in this book. Let Ω be a real C^∞ form of bidegree $(n-1,\,n-1)$ on X. Then Ω is d-exact if and only if

$$\int\limits_X \Omega\wedge T = 0 \quad \text{for every real } d\text{-closed } (1,\,1)\text{-current } T \text{ on } X,$$

while Ω is positive definite if and only if

$$\int\limits_X \Omega\wedge T > 0 \quad \text{for every nonzero } (1,\,1)\text{-current } T\geq 0 \text{ on } X.$$

It is thus clear that a form Ω as in (i) and a current T as in (ii) cannot simultaneously exist. Thus $(i) \Rightarrow (ii)$. Conversely, if there is no T as in (ii), the set $\mathcal{E}$ of real d-closed $(1,\,1)$-currents T on X is disjoint from the set $\mathcal{C}$ of $(1,\,1)$-currents $T\geq 0$ on X such that $\int_X T\wedge\gamma^{n-1}=1$ (where we have fixed an arbitrary smooth $(1,\,1)$-form $\gamma>0$ on X). Since $\mathcal{E}$ is a closed, convex subset of the locally convex space $\mathcal{D}'_{\mathbb{R}}$ of real $(1,\,1)$-currents on X, while $\mathcal{C}$ is a compact, convex subset of $\mathcal{D}'_{\mathbb{R}}$, by the Hahn-Banach separation theorem for locally convex spaces there must exist a linear functional on $\mathcal{D}'_{\mathbb{R}}$ that vanishes identically on $\mathcal{E}$ and is positive on $\mathcal{C}$ if $\mathcal{E}\cap\mathcal{C}=\emptyset$. This amounts to the existence of Ω as in (i). The implication $(ii) \Rightarrow (i)$ is proved.

We will now prove the equivalence "not $(ii) \Leftrightarrow$ not (iii)".

Suppose there exists a non-trivial closed positive $(1,\,1)$-current T on X. If $\mathcal{G}_X$ degenerates, it contains the zero Aeppli $(n-1,\,n-1)$-class, so there exists a C^∞ $(1,\,1)$-form $\omega>0$ on X such that $\omega^{n-1}=\partial u+\bar\partial v$ for some forms u, v of types $(n-2,\,n-1)$, resp. $(n-1,\,n-2)$. Thus, on the one hand, $\int_X T\wedge\omega^{n-1}>0$, while on the other hand Stokes's theorem would imply

$$\int\limits_X T\wedge\omega^{n-1} = \int\limits_X T\wedge(\partial u+\bar\partial v) = -\int\limits_X \partial T\wedge u - \int\limits_X \bar\partial T\wedge v = 0$$

since $\partial T=0$ and $\bar\partial T=0$ by the closedness assumption on T. This is a contradiction, so $\mathcal{G}_X$ cannot degenerate. We have thus proved the implication "not $(ii) \Rightarrow$ not (iii)".

Conversely, suppose that $\mathcal{G}_X \subsetneq H^{n-1,\,n-1}_A(X,\,\mathbb{R})$. If no non-trivial closed positive $(1,\,1)$-current existed on X, then by the implication $(ii) \Rightarrow (i)$ proved above, there would exist a d-exact C^∞ $(n-1,\,n-1)$-form $\Omega>0$ on X. Taking the $(n-1)^{st}$ root, there would exist a C^∞ $(1,\,1)$-form $\omega>0$ on X such that $\omega^{n-1}=\Omega$. Then $\omega^{n-1}\in \operatorname{Im} d \subset \operatorname{Im}\partial + \operatorname{Im}\bar\partial$, hence $[\omega^{n-1}]_A=0$. However, ω is a Gauduchon (even a balanced) metric, so $[\omega^{n-1}]_A\in\mathcal{G}_X$. We would thus have $0\in\mathcal{G}_X$, hence $\mathcal{G}_X = H^{n-1,\,n-1}_A(X,\,\mathbb{R})$, contradicting the assumption. This completes the proof of the implication "not $(iii) \Rightarrow$ not (ii)".

The last statement in (a) can be proved by contradiction. If X were of class $\mathcal{C}$, then by the easy implication in Theorem 4.1.23 of Demailly and Paun there would exist a Kähler current T on X. However, any Kähler current is, in particular, a nonzero d-closed positive $(1,\,1)$-current whose existence would violate (ii).

To prove (b), let us suppose that $H^2(X,\,\mathbb{C})=0$. Then $H^{2n-2}(X,\,\mathbb{C})=0$ by Poincaré duality, so for every balanced metric (if any) ω on X, ω^{n-1} must be d-exact, hence it must define a degenerate balanced structure on X. Thus, thanks to part (a), X is balanced if and only if there exists no nonzero d-closed $(1,\,1)$-current $T\geq 0$ on X. However, the assumption $H^2(X,\,\mathbb{C})=0$ ensures that any d-closed current of degree 2 is d-exact, so in this case the balanced condition on X is characterised by the same property as the one given in Proposition 4.2.5 to characterise the sG property of an arbitrary X. This proves the equivalence in (b).

The implication in (b) follows from the above discussion: the assumption $H^2(X,\,\mathbb{C})=0$ ensures that any balanced structure on X is degenerate, while the existence of a degenerate balanced structure implies that the Gauduchon cone contains the zero Aeppli class, hence it must be the whole space $H^{n-1,\,n-1}_A(X,\,\mathbb{R})$. $\square$

We notice that the Gauduchon cone $\mathcal{G}_X$ and the sG cone $\mathcal{SG}_X$ (see Definition 4.3.1 for the latter) cannot be simultaneously trivial, i.e. the following implication holds:

$$\mathcal{G}_X=H^{n-1,\,n-1}_A(X,\,\mathbb{R})\implies\mathcal{SG}_X\neq\emptyset.$$

Indeed, if $\mathcal{G}_X=H^{n-1,\,n-1}_A(X,\,\mathbb{R})$, then $\mathcal{SG}_X=\ker T\cap H^{n-1,\,n-1}_A(X,\,\mathbb{R})$ is an $\mathbb{R}$-vector subspace of $H^{n-1,\,n-1}_A(X,\,\mathbb{R})$, hence it contains at least the origin.

An immediate consequence of this and of Proposition 4.5.61 is the following.

Corollary 4.5.62. *If the Gauduchon cone $\mathcal{G}_X$ of a compact complex manifold X degenerates, then X is a strongly Gauduchon manifold but is not of class $\mathcal{C}$.*

Examples of degenerate balanced manifolds

We are aware of only two classes of such manifolds.

• **First class of examples**

We refer to [Fri91] and [Rei86] for the details of what follows.

A $(-1,\,-1)$-curve C in a smooth compact complex manifold Y with $\dim_{\mathbb{C}}Y=3$ is a rational curve $C\simeq\mathbb{P}^1$ such that $N_{Y|C}\simeq\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. By a contractibility criterion of Grauert and Artin, C can be contracted to a node by a morphism $f:Y\longrightarrow X'$ in the category of complex analytic varieties. Now, if Y contains a number of disjoint $(-1,\,-1)$-curves $C_1,\dots,C_m$, we can contract them by some $f:Y\longrightarrow X'$. By [Fri91], X' has smooth small deformations X. So, the nodes of X' disappear in X.

If, moreover, we can choose the $(-1,\,-1)$-curves $C_1,\dots,C_m$ in Y such that they span $H_2(Y,\,\mathbb{Z})$, then $H_2(X,\,\mathbb{C})=\{0\}$. In particular, X is not a *class $\mathcal{C}$* manifold. (See e.g. Corollary 4.1.28.) This construction of the 3-dimensional compact complex manifolds X by contractions of $(-1,\,-1)$-curves followed by small deformations are called *conifold transitions*.

Moreover, by surgery results of C. T. C. Wall, a 2-connected 6-manifold is a connected sum $\sharp_k(S^3\times S^3)$ of finitely many copies of $S^3\times S^3$.

Example 4.5.63. *Let $k\geq 2$ be an integer and let $X_k:=\sharp_k(S^3\times S^3)$ be the connected sum of k copies of $S^3\times S^3$ endowed with a complex structure J_k induced by* **conifold transitions**.

Then, $(X_k,\,J_k)$ is a **degenerate balanced manifold**.

Proof. It was shown in [FLY12, Corollary 1.3] that the complex structure constructed on $X:=X_k$ in [Fri91] and [LT96] by *conifold transitions* admits a balanced metric ω. Since $\dim_{\mathbb{C}}X=3$, ω^2 defines a De Rham cohomology class in $H^4(X,\,\mathbb{C})$. However, $H^4(X,\,\mathbb{C})=0$ for this particular X, so ω^2 must be d-exact. In particular, $\omega^2\in\operatorname{Im}\partial+\operatorname{Im}\bar\partial$, hence $[\omega^2]_A=0$. Since ω is necessarily a Gauduchon metric on X, it follows that $\mathcal{G}_X$ contains the origin. Hence, due to it being open, $\mathcal{G}_X$ must contain a neighbourhood of 0 in $H^{2,\,2}_A(X,\,\mathbb{R})$. Then $\mathcal{G}_X = H^{2,\,2}_A(X,\,\mathbb{R})$ by the convex cone property of $\mathcal{G}_X$. By Proposition 4.5.61, this is equivalent to $(X_k,\,J_k)$ being a degenerate balanced manifold. $\square$

It would be interesting to know whether the identity $\mathcal{G}_X = H^{n-1,\,n-1}_A(X,\,\mathbb{R})$ (which is equivalent to $0\in\mathcal{G}_X$ by the above arguments) can hold when $H^2(X,\,\mathbb{C})\neq 0$ or $H^{n-1,\,n-1}_A(X,\,\mathbb{R})\neq 0$.

• **Second class of examples**

We refer to [Yac98] for the details of what follows. Let G be a simply connected connected *complex* Lie group and let $\Gamma\subset G$ be a discrete co-compact subgroup. We saw in Theorem 4.5.30 that the quotient $X=G/\Gamma$ is a *complex parallelisable* compact complex manifold. Hence, by Corollary 4.5.33, $X=G/\Gamma$ is *balanced*. Actually, by [Yac98, Proposition 17], every left-invariant Hermitian metric $\widetilde\omega$ on G projects onto a balanced metric ω on $X=G/\Gamma$.

Example 4.5.64. *([Yac98, Proposition 18]) Let G be a* **semi-simple** *complex Lie group and let $\Gamma\subset G$ be a discrete co-compact subgroup.*

Then, every left-invariant Hermitian metric $\widetilde\omega$ on G projects onto a **degenerate balanced metric** *ω on $X=G/\Gamma$.*

Sketch of proof. We already know that ω is a balanced metric on $X=G/\Gamma$. It remains to prove that ω^{n-1} is d-exact on X, where $n:=\dim_{\mathbb{C}}X$.

Let $\mathfrak{g}$ be the (real) Lie algebra of G and denote by $J\colon\mathfrak{g}\to\mathfrak{g}$ the endomorphism induced by the complex structure of the Lie group G. Then $J^2=-\mathrm{Id}$ and $[Jx,y]=[x,Jy]=J[x,y]$ for all $x,y\in\mathfrak{g}$. Thus, J makes $\mathfrak{g}$ into a *complex* Lie algebra.

Since $\mathfrak{g}$ is a *semi-simple* complex Lie algebra with $\dim_{\mathbb{C}}g=n$, $H^{n-1}(\mathfrak{g},\,\mathbb{C})=0$. (Indeed, for any complex Lie algebra $\mathfrak{g}$, the first cohomology group $H^1(\mathfrak{g},\,\mathbb{C})$ is isomorphic to the quotient $\mathfrak{g}/[\mathfrak{g},\,\mathfrak{g}]$. Since $\mathfrak{g}$ is semi-simple, we have $[\mathfrak{g},\,\mathfrak{g}]=\mathfrak{g}$, hence $H^1(\mathfrak{g},\,\mathbb{C})=0$. By duality, we get $H^{n-1}(\mathfrak{g},\,\mathbb{C})=0$, where n is the complex dimension of $\mathfrak{g}$.) This further implies that

$$H^{n-1}(H^0(X,\,\Omega^\bullet),\,\partial)=0 \tag{4.155}$$

thanks to the isomorphism of complexes $(C^{(\bullet,\,0)}(\mathfrak{g},\,\mathbb{C}),\,\delta')\simeq(H^0(X,\,\Omega^\bullet),\,\partial)$, where $\delta: C^n(\mathfrak{g},\,\mathbb{C})\longrightarrow C^{n+1}(\mathfrak{g},\,\mathbb{C})$ is the differential in the Koszul complex and $\delta=\delta'+\delta''$ is its splitting into a $(1,\,0)$ part $\delta' : C^{p,\,q}(\mathfrak{g},\,\mathbb{C})\longrightarrow C^{p+1,\,q}(\mathfrak{g},\,\mathbb{C})$, resp. a $(0,\,1)$ part $\delta'' : C^{p,\,q}(\mathfrak{g},\,\mathbb{C})\longrightarrow C^{p,\,q+1}(\mathfrak{g},\,\mathbb{C})$. As usual, $C^{p,\,q}(\mathfrak{g},\,\mathbb{C}):=(\Lambda^p\mathfrak{g}^\star)\wedge(\Lambda^q\overline{\mathfrak{g}^\star})\subset C^{p+q}(\mathfrak{g},\,\mathbb{C}):=\Lambda^{p+q}(\mathfrak{g}\oplus\overline{\mathfrak{g}})$ for all p,q.

Now, (4.155) means that every form $\Gamma\in C^\infty_{n-1,\,0}(X,\,\mathbb{C})$ with the property $\bar\partial\Gamma=0$ is ∂-exact and d-exact. Indeed, by Observation 4.5.27, we have $\partial\Gamma=0$ for any such Γ. Therefore, (4.155) implies that $\Gamma=\partial\alpha$ for some $\alpha\in H^0(X,\,\Omega^{n-2})$. This means that $\alpha\in C^\infty_{n-2,\,0}(X,\,\mathbb{C})$ and $\bar\partial\alpha=0$. Hence, $d\alpha=\partial\alpha=\Gamma$.

Now, let $\{\Gamma_1,\dots,\Gamma_n\}$ be a $\mathbb{C}$-basis for $H^0(X,\,\Omega^{n-1})\simeq H^{n-1,\,0}(X,\,\mathbb{C})$. Since ω^{n-1} is the projection onto X of a left-invariant form on G, we have

$$\omega^{n-1}=i^{(n-1)^2}\sum_{1\leq k,\,l\leq n}a_{k\bar l}\,\Gamma_k\wedge\overline{\Gamma_l},$$

where the coefficients $a_{k\bar l}\in\mathbb{C}$ are *constant* (and the matrix $(a_{k\bar l})_{1\leq k,\,l\leq n}$ is positive definite because $\omega^{n-1}>0$, but this last point is irrelevant to us). Thus, we get the d-exactness of ω^{n-1} from the d-exactness of the Γ_k's. $\square$

The interior of the pseudo-effective cone

Starting from a handful of trivial observations, we now exhibit a few examples of compact complex manifolds which are not in the class $\mathcal{C}$ but whose pseudo-effective cone has non-empty interior. They motivate, in part, the two-step strategy for tackling the conjecture mentioned in §4.3.1, the introduction of sGG manifolds and the distinction we made between the big cone introduced in Definition 4.1.25 and the interior of the pseudo-effective cone on arbitrary, possibly *non-class* $\mathcal{C}$, manifolds.

Proposition 4.5.65. (I) *Let X be a compact complex* **surface**. *Then:*

(i) *there exists a non-zero d-closed $(1,\,1)$-current $T \geq 0$ on X;*

(ii) $h^{1,1}_{BC}(X,\,\mathbb{C}) \geq 1$;

(iii) *if $h^{1,1}_{BC}(X,\,\mathbb{C}) = 1$, then $\mathring{\mathcal{E}}_X \neq \emptyset$.*

(II) *Let X be a compact complex manifold of any dimension. If $h^{1,1}_{BC}(X,\,\mathbb{C}) = 1$ and if there exists a non-zero d-closed $(1,\,1)$-current $T \geq 0$ on X, then $\mathring{\mathcal{E}}_X \neq \emptyset$.*

Proof. $(I)(i)$ Suppose that such a current did not exist. Then, by (a) of Proposition 4.5.61, there would exist a degenerate balanced structure ω on X. Since $n-1=1$ on a surface, ω would be a Kähler metric, contradicting the supposed non-existence of a non-zero d-closed positive $(1,\,1)$-current.

(ii) Let $T \geq 0$ be a non-zero d-closed $(1,\,1)$-current on X (which exists by (i)). Then $[T]_{BC} \in H^{1,1}_{BC}(X,\,\mathbb{C})$ cannot be the zero Bott-Chern class since, otherwise, $T = i\partial\bar\partial\varphi \geq 0$ on X for some L^1_{loc} function φ, so φ would be a global psh function on the compact manifold X. Hence, φ would be constant and $T = i\partial\bar\partial\varphi = 0$, a contradiction.

(iii) For any non-zero d-closed $(1,\,1)$-current $T \geq 0$ on X, we have $0 \neq [T]_{BC} \in \mathcal{E}_X \subset H^{1,1}_{BC}(X,\,\mathbb{R})$. Since $\mathcal{E}_X$ is a convex cone, it must contain the whole ray $\mathbb{R}^+\cdot[T]_{BC}$, so it has non-empty interior in the ambient 1-dimensional real vector space.

(II) The proof of this statement is identical to that of $(I)(iii)$. It has been necessary to suppose the existence of a non-zero d-closed positive $(1,\,1)$-current since, unlike compact complex surfaces, arbitrary compact complex manifolds of dimension ≥ 3 need not possess such a current (see e.g. Example 4.5.63). □

We now notice a few examples showing that the property $\mathring{\mathcal{E}}_X \neq \emptyset$ does not imply that X is a class $\mathcal{C}$ manifold.

Proposition 4.5.66. *Let X be either a* **Hopf surface**, *or an* **Inoue S_M surface**, *or* **an Inoue $S_\pm$ surface**, *or a* **secondary Kodaira surface**.

Then, X is not in the class $\mathcal{C}$ but $\mathring{\mathcal{E}}_X \neq \emptyset$.

Proof. All the surfaces of the above types are non-Kähler, hence not in the class $\mathcal{C}$ (since the Kähler class coincides with the class $\mathcal{C}$ in the case of surfaces). Now, thanks to [ADT14, Theorem 2.2, Tables 1 and 2, p.8-9][5], $h^{1,1}_{BC}(X,\,\mathbb{C}) = 1$ for each of these surfaces. We get $\mathring{\mathcal{E}}_X \neq \emptyset$ from part $(I)(iii)$ of our Proposition 4.5.65. □

[5]The author is grateful to F. Campana for pointing out this reference to him.

4.5.7 The Laplacian $\Delta_\omega = \Lambda_\omega(i\partial\bar\partial)$ on functions when ω is Gauduchon or balanced

Let $(X,\,\omega)$ be a compact complex Hermitian manifold with $\dim_{\mathbb{C}}X = n \geq 2$. We denote by $\Delta_\omega\varphi = \Lambda_\omega(i\partial\bar\partial\varphi)$ the Laplace-type operator associated with ω acting on smooth complex-valued functions φ on X. Clearly, $\Delta_\omega = -P_\omega$, where

$$P_\omega := i\Lambda_\omega\bar\partial\partial : C^\infty(X,\,\mathbb{C}) \longrightarrow C^\infty(X,\,\mathbb{C})$$

is the operator considered in §.4.1.2. Let $\Delta^\star_\omega$ be the formal adjoint of Δ_ω.

By expanding the formula given for $P^\star_\omega$ in the proof of Theorem 4.1.7, one easily gets the following

Proposition 4.5.67. *(a) If ω is* **balanced**, *Δ_ω is* **self-adjoint**, *namely $\Delta^\star_\omega = \Delta_\omega$.*

(b) If ω is **Gauduchon**, *$\Delta^\star_\omega - \Delta_\omega$ is a* **first-order** *operator with* **no zero-th order terms**. *In particular, $\Delta^\star_\omega$ is* **elliptic** *of order two with* **no zero-th order terms**.

By a differential operator having no zero-th order terms we mean that it vanishes on constants. An immediate consequence of (b) of Proposition 4.5.67 is a key property of Gauduchon metrics: when ω is Gauduchon, $\Delta^\star_\omega$ satisfies the **maximum principle** which implies that $\ker\Delta^\star_\omega = \mathbb{C}$.

Proposition 4.5.67 will follow from the next result that subsumes it.

Lemma 4.5.68. *Let $(X,\,\omega)$ be a compact Hermitian manifold, $\dim_{\mathbb{C}}X = n \geq 2$.*

(i) For any smooth function $\varphi : X \longrightarrow \mathbb{C}$, we have:

$$\langle\langle\Delta_\omega\varphi,\,\varphi\rangle\rangle = -||\bar\partial\varphi||^2 - i\left\langle\left\langle\bar\partial\varphi,\,\varphi\star\left(\frac{\bar\partial\omega^{n-1}}{(n-1)!}\right)\right\rangle\right\rangle. \tag{4.156}$$

In particular, $\langle\langle\Delta_\omega\varphi,\,\varphi\rangle\rangle = -||\bar\partial\varphi||^2$ if the metric ω is **balanced**.

(ii) If the metric ω is **Gauduchon** *and the function φ is* **real-valued**, *then*

$$\langle\langle\Delta_\omega\varphi,\,\varphi\rangle\rangle = -||\bar\partial\varphi||^2. \tag{4.157}$$

(iii) Suppose that ω is a **Gauduchon** *metric. Then $\Delta^\star_\omega = \Delta_\omega + L_\omega$, where L_ω is the first-order operator with no zero-th order terms defined by*

$$L_\omega\varphi = \star\left(i\partial\varphi\wedge\frac{\bar\partial\omega^{n-1}}{(n-1)!} - i\bar\partial\varphi\wedge\frac{\partial\omega^{n-1}}{(n-1)!}\right) \tag{4.158}$$

for all smooth complex-valued functions φ on X.

In particular, $\Delta^\star_\omega$ is **elliptic** *of order two with* **no zero-th order terms**. *If ω is* **balanced**, *$L_\omega = 0$ and $\Delta^\star_\omega$ is* **self-adjoint**.

Moreover, for any **real-valued** *function φ on X, we have:*

$$\langle\langle L_\omega\varphi,\,\varphi\rangle\rangle = 0. \tag{4.159}$$

Proof. (i) Since $\partial^\star = -\star\bar\partial\star$, we get:

$$\begin{aligned}\langle\langle\Delta_\omega\varphi,\,\varphi\rangle\rangle &= \langle\langle\Lambda_\omega(i\partial\bar\partial\varphi),\,\varphi\rangle\rangle = \langle\langle i\bar\partial\varphi,\,\partial^\star(\varphi\omega)\rangle\rangle = -i\left\langle\left\langle\bar\partial\varphi,\,\star\bar\partial\left(\varphi\frac{\omega^{n-1}}{(n-1)!}\right)\right\rangle\right\rangle \\ &= -i\left\langle\left\langle\bar\partial\varphi,\,\varphi\star\left(\frac{\bar\partial\omega^{n-1}}{(n-1)!}\right)\right\rangle\right\rangle - i\left\langle\left\langle\bar\partial\varphi,\,\star\left(\bar\partial\varphi\wedge\frac{\omega^{n-1}}{(n-1)!}\right)\right\rangle\right\rangle. \end{aligned}\tag{4.160}$$

Since the form $\bar\partial\varphi$ is of bidegree (0, 1), it is *primitive*, so the standard formula (4.68) yields:

$$\star(i\bar\partial\varphi) = -\bar\partial\varphi\wedge\frac{\omega^{n-1}}{(n-1)!}, \quad \text{or equivalently} \quad \star\left(\bar\partial\varphi\wedge\frac{\omega^{n-1}}{(n-1)!}\right) = i\bar\partial\varphi,$$

since $\star\star = -\mathrm{Id}$ on forms of odd degree. Combined with the last identity in (4.160), this proves (4.156).

(ii) We now transform the first term on the r.h.s. of the last identity in (4.160). Since $\bar\partial^\star = -\star\partial\star$, we get:

$$\begin{aligned}\langle\langle\bar\partial\varphi,\,\varphi\star\bar\partial\omega^{n-1}\rangle\rangle &= \langle\langle\varphi,\,-\star\partial\star(\varphi\star(\bar\partial\omega^{n-1}))\rangle\rangle \stackrel{(a)}{=} \langle\langle\varphi,\,\star\partial(\varphi\,\bar\partial\omega^{n-1})\rangle\rangle \\ &= (n-1)\,\langle\langle\varphi,\,\star(\partial\varphi\wedge\bar\partial\omega\wedge\omega^{n-2})\rangle\rangle + \langle\langle\varphi,\,\star(\varphi\,\partial\bar\partial\omega^{n-1})\rangle\rangle,\end{aligned} \tag{4.161}$$

where (a) follows from the commutation of $\star$ with the function φ and from $\star\star = -\mathrm{Id}$ on forms of odd degree. If ω is Gauduchon, then $\partial\bar\partial\omega^{n-1} = 0$, so the last term above vanishes. To transform the previous term, we can use the Lefschetz decomposition of the (2, 2)-form $\partial\varphi\wedge\bar\partial\omega$ to write uniquely:

$$\partial\varphi\wedge\bar\partial\omega = (\partial\varphi\wedge\bar\partial\omega)_{prim} + \alpha\wedge\omega + f\,\omega^2,$$

where $(\partial\varphi\wedge\bar\partial\omega)_{prim}$ is a primitive (2, 2)-form, α is a primitive (1, 1)-form and f is a function. Multiplying by ω^{n-2}, we get:

$$\partial\varphi\wedge\bar\partial\omega\wedge\omega^{n-2} = (\partial\varphi\wedge\bar\partial\omega)_{prim}\wedge\omega^{n-3}\wedge\omega + \alpha\wedge\omega^{n-1} + f\,\omega^n = f\,\omega^n, \tag{4.162}$$

where the last identity follows from $(\partial\varphi\wedge\bar\partial\omega)_{prim}\wedge\omega^{n-3} = 0$ and $\alpha\wedge\omega^{n-1} = 0$ which reflect the definition of primitiveness for $(\partial\varphi\wedge\bar\partial\omega)_{prim}$ and α. Taking $\star$, we get $\star(\partial\varphi\wedge\bar\partial\omega\wedge\omega^{n-2}) = n!\,f$ since $\star(\omega^n/n!) = 1$.

Thus, when ω is Gauduchon, (4.156) and (4.161) read:

$$\langle\langle\Delta_\omega\varphi,\,\varphi\rangle\rangle = -||\bar\partial\varphi||^2 - (n-1)n\,i\,\langle\langle\varphi,\,f\rangle\rangle = -||\bar\partial\varphi||^2 - \frac{i}{(n-2)!}\int\limits_X\varphi\,\bar f\,\omega^n.$$

Now, suppose that φ is *real-valued*. Taking conjugates in (4.162), we get

$$\bar f\,\omega^n = \bar\partial\varphi\wedge\partial\omega\wedge\omega^{n-2} = \tfrac{1}{n-1}\,\bar\partial\varphi\wedge\partial\omega^{n-1},$$

which translates the previous identity to

$$\begin{aligned}\langle\langle\Delta_\omega\varphi,\,\varphi\rangle\rangle &= -||\bar\partial\varphi||^2 - \frac{i}{(n-1)(n-2)!}\int\limits_X\varphi\,\bar\partial\varphi\wedge\partial\omega^{n-1} \\ &= -||\bar\partial\varphi||^2 - \frac{i}{2(n-1)(n-2)!}\int\limits_X\bar\partial(\varphi^2)\wedge\partial\omega^{n-1} \\ &\stackrel{(a)}{=} -||\bar\partial\varphi||^2 - \frac{i}{2(n-1)(n-2)!}\int\limits_X\bar\partial(\varphi^2\wedge\partial\omega^{n-1}) \\ &\stackrel{(b)}{=} -||\bar\partial\varphi||^2,\end{aligned}$$

where identity (a) follows from $\partial\bar\partial\omega^{n-1} = 0$ (the Gauduchon assumption on ω) and (b) follows from Stokes's theorem. This proves (4.157).

(iii) From the standard formulae $\partial^\star = -\star\bar\partial\star$ and $\bar\partial^\star = -\star\partial\star$, we get $\Delta^\star_\omega = (-i)\,(-\star\partial\star)\,(-\star\bar\partial\star)(\omega\wedge\cdot)$. Since $\star$ commutes with multiplication by functions, $\star\omega = \omega^{n-1}/(n-1)!$ and $\star\star = -\mathrm{Id}$ on forms of odd degree, we get the first of the following identities:

$$\begin{aligned}\Delta^\star_\omega\varphi &= \star i\partial\bar\partial\bigg(\varphi\,\frac{\omega^{n-1}}{(n-1)!}\bigg) = \star\bigg(i\partial\bar\partial\varphi\wedge\frac{\omega^{n-1}}{(n-1)!}\bigg) + L_\omega + \star\bigg(\varphi\,\frac{i\partial\bar\partial\omega^{n-1}}{(n-1)!}\bigg)\\ &= \star\bigg(\Lambda_\omega(i\partial\bar\partial\varphi)\,\frac{\omega^n}{n!}\bigg) + L_\omega + \star\bigg(\varphi\,\frac{i\partial\bar\partial\omega^{n-1}}{(n-1)!}\bigg)\\ &= \Delta_\omega\varphi + L_\omega + \star\bigg(\varphi\,\frac{i\partial\bar\partial\omega^{n-1}}{(n-1)!}\bigg).\end{aligned}$$

The last term vanishes when ω is a Gauduchon metric, so we get the first statement in (iii).

To prove (4.159), we start by using the fact that the adjoint of $\star$ is $\star$ when acting on forms of even degree. This implies the first identity below:

$$\begin{aligned}\langle\langle L_\omega\varphi,\,\varphi\rangle\rangle &= \bigg\langle\bigg\langle i\partial\varphi\wedge\frac{\bar\partial\omega^{n-1}}{(n-1)!} - i\bar\partial\varphi\wedge\frac{\partial\omega^{n-1}}{(n-1)!},\,\varphi\,\frac{\omega^n}{n!}\bigg\rangle\bigg\rangle\\ &\overset{(a)}{=} i\,\bigg\langle\bigg\langle\varphi\,\partial\varphi\wedge\frac{\bar\partial\omega^{n-1}}{(n-1)!} - \varphi\,\bar\partial\varphi\wedge\frac{\partial\omega^{n-1}}{(n-1)!},\,\frac{\omega^n}{n!}\bigg\rangle\bigg\rangle\\ &= \frac{i}{2}\,\bigg\langle\bigg\langle\partial(\varphi^2)\wedge\frac{\bar\partial\omega^{n-1}}{(n-1)!} - \bar\partial(\varphi^2)\wedge\frac{\partial\omega^{n-1}}{(n-1)!},\,\frac{\omega^n}{n!}\bigg\rangle\bigg\rangle\\ &\overset{(b)}{=} \frac{i}{2}\int\limits_X\bigg\langle\partial\bigg(\varphi^2\wedge\frac{\bar\partial\omega^{n-1}}{(n-1)!}\bigg) - \bar\partial\bigg(\varphi^2\wedge\frac{\partial\omega^{n-1}}{(n-1)!}\bigg),\,\frac{\omega^n}{n!}\bigg\rangle\,\frac{\omega^n}{n!}\\ &\overset{(c)}{=} \frac{i}{2}\int\limits_X\partial\bigg(\varphi^2\wedge\frac{\bar\partial\omega^{n-1}}{(n-1)!}\bigg) - \frac{i}{2}\int\limits_X\bar\partial\bigg(\varphi^2\wedge\frac{\partial\omega^{n-1}}{(n-1)!}\bigg)\\ &\overset{(d)}{=} 0-0=0.\end{aligned}$$

Identity (a) above follows from φ being *real-valued* (otherwise, the r.h.s. φ would have become $\bar\varphi$ on the l.h.s.), while identity (b) is a consequence of the Gauduchon assumption on ω. To see (c), pick any point $x\in X$ and local coordinates $z_1,\dots,z_n$ about x such that

$$\omega(x) = \sum_{j=1}^n i\,dz_j\wedge d\bar z_j, \qquad \text{hence } \langle\omega_n(x),\,\omega_n(x)\rangle = 1, \text{ where } \omega_n := \omega^n/n!.$$

Now, for every $(n,\,n)$-form u, we have $u = f\,\omega_n$ for some function f, hence $\langle u,\,\omega_n\rangle\,\omega_n = f\,\omega_n = u$ at (any) x. This gives (c). Identity (d) follows from Stokes's theorem. The proof (4.159) is complete. □

4.6 SKT and Hermitian-symplectic metrics and manifolds

The analogue in bidegree (1, 1) of Gauduchon metrics is the following notion.

Definition 4.6.1. *Let X be a complex manifold.*

(i) A C^∞ positive definite $(1,1)$-form ω on X is said to be an **SKT (strong Kähler with torsion) metric**[6] *if $\partial\bar\partial\omega = 0$.*

[6] SKT metrics are also called **pluriclosed metrics**, a term used e.g. in [Egi01] and in [ST10].

(ii) If X carries such a metric, X is said to be an **SKT manifold**.

Meanwhile, thanks to Lemma 4.2.3, the analogue in bidegree $(1,\,1)$ of strongly Gauduchon metrics is the following notion.

Definition 4.6.2. *([ST10, Definition 1.5]) Let X be a complex manifold.*

(i) A C^∞ positive definite $(1,1)$-form ω on X is said to be a **Hermitian-symplectic (H-S) metric** *if ω is the component of bidegree $(1,1)$ of a real C^∞ d-closed 2-form $\widetilde{\omega}$ on X.*

(ii) If X carries such a metric, X is said to be a **Hermitian-symplectic manifold**.

The Hermitian-symplectic condition can be expressed as follows.

Lemma 4.6.3. *Let ω be a Hermitian metric on a compact complex manifold X.*

(I) The following statements are **equivalent**.

(a) ω is **Hermitian-symplectic**.

(b) There exists a form $\rho^{2,\,0}\in C^\infty_{2,\,0}(X,\,\mathbb{C})$ satisfying the equations:

$$(i)\ \partial\rho^{2,\,0}=0 \quad \textit{and} \quad (ii)\ \bar\partial\rho^{2,\,0}+\partial\omega=0. \tag{4.163}$$

(c) There exists a form $\rho^{0,\,2}\in C^\infty_{0,\,2}(X,\,\mathbb{C})$ satisfying the equations:

$$(iii)\ \bar\partial\rho^{0,\,2}=0 \quad \textit{and} \quad (iv)\ \partial\rho^{0,\,2}+\bar\partial\omega=0. \tag{4.164}$$

(II) If $dim_{\mathbb{C}}X=3$, the equivalences under (I) simplify to:

$$\omega \textit{ is } \textbf{Hermitian-symplectic} \iff \partial\omega\in Im\,\bar\partial \iff \bar\partial\omega\in Im\,\partial.$$

Proof. (I) (b) and (c) are equivalent by conjugation.

To prove the equivalence of (a) and (b), note that ω is H-S if and only if there exists a form $\rho^{2,\,0}\in C^\infty_{2,\,0}(X,\,\mathbb{C})$ such that $\widetilde\omega=\rho^{2,\,0}+\omega+\rho^{0,\,2}$ is d-closed, where $\rho^{0,\,2}=\overline{\rho^{2,\,0}}$. Meanwhile, the condition $d\widetilde\omega=0$ is equivalent to the four identities (i)-(iv) in (4.163) and (4.164). Since the pair of identities (4.163) is equivalent to the pair of identities (4.164), by conjugation, it follows that the condition $d\widetilde\omega=0$ is equivalent to either of these pairs.

(II) Let $\dim_{\mathbb{C}}X=3$. Thanks to (I), it suffices to prove that ω is Hermitian-symplectic whenever $\partial\omega\in\operatorname{Im}\bar\partial$. If $\partial\omega$ is supposed $\bar\partial$-exact, there exists $\rho^{2,\,0}\in C^\infty_{2,\,0}(X,\,\mathbb{C})$ such that $\bar\partial\rho^{2,\,0}+\partial\omega=0$. Thanks to the equivalence between (a) and (b) proved under (I), it suffices to show that $\partial\rho^{2,\,0}=0$.

Let $\rho^{0,\,2}:=\overline{\rho^{2,\,0}}$ and consider the $(3,\,3)$-form $i\partial\rho^{2,\,0}\wedge\bar\partial\rho^{0,\,2}$. Since $\dim_{\mathbb{C}}X=3$, the general formula (4.68) applied to the $(0,\,3)$-form $\bar\partial\rho^{0,\,2}$ (which is necessarily primitive, for bidegree reasons) yields $\star\bar\partial\rho^{0,\,2}=i\bar\partial\rho^{0,\,2}$. Hence

$$i\partial\rho^{2,\,0}\wedge\bar\partial\rho^{0,\,2}=\partial\rho^{2,\,0}\wedge\star\overline{\partial\rho^{2,\,0}}=|\partial\rho^{2,\,0}|^2_\omega\,dV_\omega\geq 0 \tag{4.165}$$

at every point of X. Meanwhile, it follows from $\bar\partial\rho^{2,\,0}+\partial\omega=0$ that $\bar\partial\partial\rho^{2,\,0}=0$. This implies the first identity below:

$$\int\limits_X i\partial\rho^{2,\,0}\wedge\bar\partial\rho^{0,\,2}=-\int\limits_X\bar\partial(i\partial\rho^{2,\,0}\wedge\rho^{0,\,2})=0, \tag{4.166}$$

where the last identity follows from Stokes. (We used again the fact that $\dim_{\mathbb{C}}X=3$ in order to be able to integrate $(3,\,3)$-forms on X.)

From (4.165) and (4.166), we infer that $|\partial\rho^{2,\,0}|^2_\omega=0$, hence $\partial\rho^{2,\,0}=0$, at every point of X. $\square$

An immediate consequence is the following.

Corollary 4.6.4. *Let ω be a Hermitian metric on a compact complex manifold X.*

(i) If ω is Hermitian-symplectic, then ω is also SKT.

(ii) If X is a $\partial\bar\partial$-manifold and ω is SKT, then ω is also Hermitian-symplectic.

Proof. (i) If ω is H-S, by taking $\bar\partial$ in (ii) of (4.163) we get $\bar\partial\partial\omega = 0$, so ω is SKT.

(ii) Suppose that X is a $\partial\bar\partial$-manifold. If $\partial\bar\partial\omega = 0$, the $(2,\,1)$-form $\partial\omega$ is $\bar\partial$-closed. Thus, $\partial\omega$ is ∂-exact and d-closed. By the $\partial\bar\partial$-property of X, $\partial\omega$ must also be $\bar\partial$-exact. Therefore, there exists a form $\rho^{2,\,0} \in C^\infty_{2,\,0}(X,\,\mathbb{C})$ such that $\partial\omega = -\bar\partial\rho^{2,\,0}$. Thus, $\rho^{2,\,0}$ satisfies condition (ii) in (4.163).

It remains to show that $\rho^{2,\,0}$ also satisfies condition (i) in (4.163). The $(3,\,0)$-form $\partial\rho^{2,\,0}$ is $\bar\partial$-closed (as we see by taking ∂ in $\partial\omega = -\bar\partial\rho^{2,\,0}$) and ∂-exact. Hence, it is d-closed and ∂-exact, so it is also $\bar\partial$-exact by the $\partial\bar\partial$-property of X. This means that $\partial\rho^{2,\,0} = \bar\partial\alpha^{3,\,-1}$ for some $(3,\,-1)$-form $\alpha^{3,\,-1}$. For bidegree reasons, we must have $\alpha^{3,\,-1} = 0$. We conclude that $\partial\rho^{2,\,0} = 0$.

Thus, ω is H-S by (b) of Lemma 4.6.3. $\square$

4.6.1 Basic properties of Hermitian-symplectic metrics and manifolds

(I) We first prove the implication:

$$X \text{ is a Hermitian-symplectic manifold} \implies X \text{ is a strongly Gauduchon manifold.}$$

Note that the analogous implication at the level of metrics ω does not hold.

Proposition 4.6.5. *([YZZ19, Lemma 1], [DP20, Proposition 2.1]) Every compact complex manifold X that admits a* **Hermitian-symplectic** *metric also admits a* **strongly Gauduchon (sG)** *metric.*

Proof. Let $n = \dim_{\mathbb{C}}X$. By Lemma 4.2.3, a strongly Gauduchon (sG) structure on X can be regarded as a real C^∞ d-closed $(2n-2)$-form Ω on X such that its $(n-1,\,n-1)$-component $\Omega^{n-1,\,n-1}$ is positive definite.

Now, suppose that an H-S structure $\widetilde\omega$ exists on X. This means that $\widetilde\omega = \rho^{2,\,0} + \omega + \rho^{0,\,2}$ is a real C^∞ d-closed 2-form on X such that its $(1,\,1)$-component ω is positive definite. Thus, $d\widetilde\omega^{n-1} = 0$ and

$$\widetilde\omega^{n-1} = [\omega + (\rho^{2,\,0} + \rho^{0,\,2})]^{n-1} = \sum_{k=0}^{n-1}\sum_{l=0}^{k}\binom{n-1}{k}\binom{k}{l}\,(\rho^{2,\,0})^l \wedge (\rho^{0,\,2})^{k-l} \wedge \omega^{n-k-1}.$$

In particular, the $(n-1,\,n-1)$-component of $\widetilde\omega^{n-1}$ is the sum of the terms for which $l = k-l$ in the above expression, i.e.

$$\Omega^{n-1,\,n-1} = \omega^{n-1} + \sum_{l=1}^{[\frac{n-1}{2}]}\binom{n-1}{2l}\binom{2l}{l}\,(\rho^{2,\,0})^l \wedge (\rho^{0,\,2})^l \wedge \omega^{n-2l-1}.$$

Thus, to prove the existence of an sG structure on X, it suffices to prove that the $(n-1,\,n-1)$-form $\Omega^{n-1,\,n-1}$ is positive definite. Its $(n-1)$-st root will then be an sG metric on X, by construction.

To show that $\Omega^{n-1,\,n-1} > 0$, it suffices to check that the real form $(\rho^{2,\,0})^l \wedge (\rho^{0,\,2})^l \wedge \omega^{n-2l-1}$ is weakly (semi)-positive at every point of X. (Recall that $\rho^{0,\,2}$ is the conjugate of $\rho^{2,\,0}$.) To this end, note that the $(2l, 2l)$-form $(\rho^{2,\,0})^l \wedge (\rho^{0,\,2})^l$ is weakly semi-positive as the wedge product of a $(2l, 0)$-form and its conjugate (see [Dem97, Chapter III, Example 1.2]). Therefore, the $(n-1,\,n-1)$-form $(\rho^{2,\,0})^l \wedge (\rho^{0,\,2})^l \wedge \omega^{n-2l-1}$ is (semi)-positive since the product of a weakly (semi)-positive form and a strongly (semi)-positive form is weakly (semi)-positive and ω is strongly positive (see [Dem97, Chapter III, Proposition 1.11]). (Recall that in bidegrees (1, 1) and $(n-1,\,n-1)$, the notions of weak and strong positivity coincide.) □

In the case $n = \dim_{\mathbb{C}} X = 2$, the notions of H-S and sG metrics coincide, as follows at once from Definition 4.6.2 and Lemma 4.2.3. Thus, from Theorem 4.2.6, we get the following fact that has been known for a while (cf. e.g. [LZ09] or [ST10, Proposition 1.6]).

Proposition 4.6.6. *Let X be a compact complex* **surface**. *The following equivalence holds:*

$$X \text{ is Kähler} \iff X \text{ is Hermitian-symplectic.}$$

(II) We now notice that the existence of Hermitian-symplectic metrics on a compact complex threefold implies a property that is well known to hold on compact complex manifolds whose Frölicher spectral sequence degenerates at E_1. (See Proposition 1.2.14 in the case $p = 1$.)

Proposition 4.6.7. *Let X be a compact complex* **Hermitian-symplectic** *manifold with $\dim_{\mathbb{C}} X = 3$. Then, every holomorphic 1-form (i.e. every smooth $\bar\partial$-closed $(1,\,0)$-form) on X is d-closed.*

Proof. Let ω be an H-S metric on X. Then, $\partial\omega \in \operatorname{Im}\bar\partial$ and $\bar\partial\omega \in \operatorname{Im}\partial$ (see e.g. Lemma 4.6.3). Choose any form $\rho^{2,\,0} \in C^\infty_{2,\,0}(X,\,\mathbb{C})$ such that $\partial\omega = -\bar\partial\rho^{2,\,0}$. Hence, $\bar\partial\omega = -\partial\rho^{0,\,2}$, where $\rho^{0,\,2} := \overline{\rho^{2,\,0}}$.

Now, let $\xi \in C^\infty_{1,\,0}(X,\,\mathbb{C})$ such that $\bar\partial\xi = 0$. We want to show that $\partial\xi = 0$.

On the one hand, if $\star = \star_\omega$ is the Hodge star operator induced by ω, the general formula (4.68) applied to the (necessarily primitive) $(0,\,2)$-form $\bar\partial\bar\xi$ yields: $\star(\bar\partial\bar\xi) = \bar\partial\bar\xi \wedge \omega$. Hence,

$$\partial\xi \wedge \bar\partial\bar\xi \wedge \omega = |\partial\xi|^2_\omega \, dV_\omega \geq 0 \tag{4.167}$$

at every point of X.

Meanwhile, an immediate calculation and the use of the identities $\bar\partial\xi = 0$ and $\partial\bar\xi = 0$ show that

$$\begin{aligned}
\partial\xi \wedge \bar\partial\bar\xi \wedge \omega &= -\partial\bar\partial(\xi \wedge \bar\xi \wedge \omega) + \xi \wedge \bar\partial\bar\xi \wedge \partial\omega + \partial\xi \wedge \bar\xi \wedge \bar\partial\omega + \xi \wedge \bar\xi \wedge \partial\bar\partial\omega \\
&= -\partial\bar\partial(\xi \wedge \bar\xi \wedge \omega) - \xi \wedge \bar\partial\bar\xi \wedge \bar\partial\rho^{2,0} - \partial\xi \wedge \bar\xi \wedge \partial\rho^{0,2} \\
&= -\partial\bar\partial(\xi \wedge \bar\xi \wedge \omega) + \bar\partial(\xi \wedge \bar\partial\bar\xi \wedge \rho^{2,0}) + \partial(\partial\xi \wedge \bar\xi \wedge \rho^{0,2}) \in \operatorname{Im}\partial + \operatorname{Im}\bar\partial,
\end{aligned}$$

where for the second identity we also used the property $\partial\bar\partial\omega = 0$ of the H-S metric ω. Using Stokes's theorem, we infer:

$$\int\limits_X \partial\xi \wedge \bar\partial\bar\xi \wedge \omega = 0. \tag{4.168}$$

Putting together (4.167) and (4.168), we get $\partial\xi = 0$ on X and we are done. □

(III) Hermitian-symplectic manifolds, which constitute a natural generalisation of compact Kähler manifolds, were given the following intrinsic characterisation by Sullivan.

Proposition 4.6.8. *([Sul76, Theorem III.2 and Remark III.11]) A compact complex manifold X is Hermitian-symplectic if and only if X carries no non-zero current T of bidegree $(n-1,\,n-1)$ such that $T \geq 0$ and T is d-exact.*

This can be proved using Sullivan's duality technique and the Hahn-Banach Theorem in a way similar to the proofs of Propositions 4.2.5 and 4.5.24. The details are left to the reader.

Nevertheless, Hermitian-symplectic manifolds remain poorly understood. As they lie at the interface between symplectic and complex Hermitian geometries, they seem to warrant further probing. It is an important open problem to find out whether or not the situation in complex dimension 2 described in Proposition 4.6.6 remains the same in higher dimensions. Streets and Tian asked the following

Question 4.6.9. *([ST10, Question 1.7]) Do there exist non-Kähler Hermitian-symplectic complex manifolds X with $\dim_{\mathbb{C}}X \geq 3$?*

Note that, by Theorem 4.5.45, the bidegree $(n-1, n-1)$ analogue of this question has an affirmative answer: there exist **non-balanced strongly Gauduchon** manifolds X with $\dim_{\mathbb{C}}X = n \geq 3$.

While the general case of Question 4.6.9 remains open, it has been answered negatively for a handful of special classes of manifolds, including all nilmanifolds endowed with an invariant complex structure by Enrietti, Fino and Vezzoni in [EFV12] and all twistor spaces by Verbitsky in [Ver14].

The Streets-Tian question is complementary to Donaldson's earlier

Question 4.6.10. *([Don06, Question 2]) If J is an almost-complex structure on a compact 4-manifold which is tamed by a symplectic form, is there a symplectic form compatible with J?*

Indeed, when the almost-complex structure J is integrable, a symplectic form $\widetilde{\omega}$ is a taming form for J if and only if the $(1, 1)$-component ω of $\widetilde{\omega}$ is a Hermitian-symplectic metric (i.e. positive definite). While J is assumed integrable in Question 4.6.9, the dimension of the underlying manifold is allowed to be arbitrary. Meanwhile, Question 4.6.10, that has come to be known in the literature as *Donaldson's tamed-to-compatible conjecture*, is peculiar to four real dimensions but J need not be integrable. Thus, the only known case so far lies at the intersection of Questions 4.6.10 and 4.6.9.

4.6.2 Basic properties of SKT metrics and manifolds

Note that the SKT condition coincides with the Gauduchon condition in complex dimension $n = 2$. Thus, thanks to Theorem 4.1.2, every compact complex surface is an **SKT manifold**. Higher-dimensional examples of SKT manifolds will be given later on. (See e.g. §.4.6.4.)

(I) The following fact was first noticed in [IP13] and in some of the references therein as a consequence of more general results. A quick proof appeared in [Pop15, Proposition 1.1].

Proposition 4.6.11. *If a Hermitian metric ω on a compact complex manifold X is both* **SKT** *and* **balanced**, *then ω is* **Kähler**.

Proof. The *SKT* assumption on ω translates to any of the following equivalent properties:

$$\partial\bar\partial\omega = 0 \iff \partial\omega \in \ker\bar\partial \iff \star(\partial\omega) \in \ker\partial^\star, \tag{4.169}$$

where the last equivalence follows from the standard formula $\partial^\star = -\star\bar\partial\star$ involving the Hodge-star isomorphism $\star = \star_\omega : \Lambda^{p,q}T^\star X \to \Lambda^{n-q,n-p}T^\star X$ defined by ω for arbitrary $p, q = 0, \dots, n$.

Meanwhile, the *balanced* assumption on ω translates to any of the following equivalent properties:

$$d\omega^{n-1} = 0 \iff \partial\omega^{n-1} = 0 \iff \omega^{n-2}\wedge\partial\omega = 0 \iff \partial\omega \text{ is primitive.}$$

Moreover, since $\partial\omega$ is primitive when ω is balanced, the general formula (4.68) yields:

$$\star(\partial\omega) = i\,\frac{\omega^{n-3}}{(n-3)!}\wedge\partial\omega = \frac{i}{(n-2)!}\,\partial\omega^{n-2}\in\operatorname{Im}\partial. \tag{4.170}$$

Thus, if ω is both SKT and balanced, we get from (4.169) and (4.170) that

$$\star(\partial\omega)\in\ker\partial^\star\cap\operatorname{Im}\partial = \{0\},$$

where the last identity follows from the subspaces $\ker\partial^\star$ and $\operatorname{Im}\partial$ of $C^\infty_{n-1,\,n-2}(X,\,\mathbb{C})$ being L^2_ω-orthogonal. We infer that $\partial\omega = 0$, i.e. ω is Kähler. $\square$

Based on the above statement at the level of metrics and on other reasons, we conjecture the analogous statement at the level of compact complex manifolds (cf. the similar Conjecture 4.5.10 in the balanced/lck case).

Conjecture 4.6.12. *Let X be a compact complex manifold with $\dim_{\mathbb{C}}X = n\geq 2$.*
Then, X carries both a **balanced** *metric and a (possibly different)* **SKT** *metric if and only if X carries a* **Kähler** *metric.*

Moreover, if Conjecture 4.5.56 turns out to be true, the following conjecture is weaker than the above Conjecture 4.6.12. It is also weaker than the Streets-Tian Question 4.6.9 thanks to Corollary 4.6.4.

Conjecture 4.6.13. *Let X be a compact* $\partial\bar\partial$**-manifold** *with $\dim_{\mathbb{C}}X = n\geq 2$.*
If X carries an **SKT** *metric, then X also carries a* **Kähler** *metric.*

(II) Recall that, for any Hermitian metric ω, $\star\omega = \omega^{n-1}/(n-1)!$, where $\star = \star_\omega$ is the Hodge star operator induced by ω. Consequently, if ω is Kähler, ω is harmonic for each of the Laplacians Δ, Δ', Δ'', Δ_{BC} and Δ_A it induces.

We now observe a kind of converse, namely that for an SKT metric, the balanced condition is equivalent to the Aeppli harmonicity.

Lemma 4.6.14. *Let $\omega > 0$ be a C^∞ positive definite $(1,\,1)$-form on X such that $\partial\bar\partial\omega = 0$. The following equivalence holds:*

$$\Delta_A\omega = 0 \iff d\omega^{n-1} = 0.$$

Proof. Since $\star\omega = \omega^{n-1}/(n-1)!$ and $d^\star = -\star d\star$, the balanced condition $d\omega^{n-1} = 0$ is equivalent to $d^\star\omega = 0$, hence to $\partial^\star\omega = 0$ and $\bar\partial^\star\omega = 0$. The contention is thus seen to follow from the vector space identity $\mathcal{H}^{1,\,1}_{\Delta_A} = \ker(\partial\bar\partial)\cap\ker\partial^\star\cap\ker\bar\partial^\star$ (see part (3) of Corollary 1.1.13). $\square$

Thus, Proposition 4.6.11 can be reworded in the following way.

Corollary 4.6.15. *Let $\omega > 0$ be a Hermitian metric on X. Then*

$$\omega \quad \textit{is Kähler} \quad \iff \Delta_A\omega = 0.$$

(III) One has the following intrinsic characterisation of SKT manifolds.

Proposition 4.6.16. *([Egi01, 3. of Theorem 3.3.]) An n-dimensional compact complex manifold X is SKT if and only if X carries no non-zero current T of bidegree $(n-1,\,n-1)$ such that $T\geq 0$ and T is $(\partial\bar\partial)$-exact.*

Proof. " $\implies$ " Suppose ω is an SKT metric X. If a current $T = \partial\bar\partial S$ (for some current S of bidegree $(n-2,\, n-2)$) as in the statement existed, the $(n,\, n)$-current $\omega\wedge T$ would be ≥ 0 and non-zero, implying the first inequality below:

$$0 < \int\limits_X \omega\wedge T = \int\limits_X \partial\bar\partial\omega\wedge S = 0,$$

a contradiction.

"$\Longleftarrow$" This can be proved using Sullivan's duality technique and the Hahn-Banach Theorem in a way similar to the proofs of Propositions 4.2.5, 4.5.24 and 4.6.8. The only extra thing one has to ensure is that the operator

$$\partial\bar\partial : \mathcal{D}'^{(n-2,\, n-2)}(X,\, \mathbb{R}) \longrightarrow \mathcal{D}'^{(n-1,\, n-1)}(X,\, \mathbb{R})$$

has a **closed image**, where $\mathcal{D}'^{(p,\, p)}(X,\, \mathbb{R}) = \mathcal{E}'_{(n-p,\, n-p)}(X)_{\mathbb{R}}$ is the space of real currents of bidegree $(p,\, p)$ (equivalently, of bidimension $(n-p,\, n-p)$) on X.

As explained in the proof of Lemma 4.5.25, this is equivalent to the dual map

$$\partial\bar\partial : C^\infty_{1,\,1}(X,\, \mathbb{R}) \longrightarrow C^\infty_{2,\,2}(X,\, \mathbb{R})$$

having a **closed image**. This follows from the Bott-Chern Laplacian Δ_{BC} being **elliptic** and from X being **compact**, leading to $\ker\Delta_{BC}$ being **finite-dimensional**, to the image of $\partial\bar\partial$ being **closed** and to the L^2_ω-orthogonal three-space decomposition

$$C^\infty_{p,\,q}(X, \mathbb{C}) = \ker\Delta_{BC}\oplus\operatorname{Im}\partial\bar\partial\oplus(\operatorname{Im}\partial^\star + \operatorname{Im}\bar\partial^\star).$$

(See (1) of Corollary 1.1.10.) $\square$

4.6.3 Bismut's holomorphic structure on $\Lambda^{1,\,0}T^\star X\oplus T^{1,\,0}X$ induced by an SKT metric

Most of the material in this subsection is taken from [Bis89]. Let X be a compact complex manifold with $\dim_{\mathbb{C}}X = n$ and let E be the holomorphic vector bundle

$$E := \Lambda^{1,\,0}T^\star X\oplus T^{1,\,0}X.$$

We also consider the exact sequence of holomorphic vector bundles on X:

$$0\longrightarrow\Lambda^{1,\,0}T^\star X\longrightarrow\Lambda^{1,\,0}T^\star X\oplus T^{1,\,0}X := E\longrightarrow T^{1,\,0}X\longrightarrow 0. \tag{4.171}$$

Now, let ω be a Hermitian metric on X. It defines a Hermitian fibre metric on the holomorphic tangent bundle $T^{1,\,0}X$ of X and induces Hermitian fibre metrics on the holomorphic cotangent bundle $\Lambda^{1,0}T^\star X$ and on E. On the other hand, the metric ω induces a vector-bundle-valued form $\alpha\in C^\infty_{0,\,1}(X,\, \operatorname{Hom}(T^{1,\,0}X,\, \Lambda^{1,\,0}T^\star X))$ in the following way:

$$(\alpha(\xi)\eta)(\nu) = (i\partial\omega)(\xi,\, \eta,\, \nu), \qquad \xi\in C^\infty(X,\, T^{0,\,1}X),\;\; \eta, \nu\in C^\infty(X,\, T^{1,\,0}X).$$

In particular, $\alpha(\xi)\in C^\infty(X,\, \operatorname{Hom}(T^{1,\,0}X,\, \Lambda^{1,\,0}T^\star X))$ for every $\xi\in C^\infty(X,\, T^{0,\,1}X)$.

Let $\bar\partial_E$ be the $\bar\partial$ operator that defines the natural holomorphic structure of E. The following result shows that, if it is SKT, the metric ω induces a different holomorphic structure on E by modifying $\bar\partial_E$.

Theorem 4.6.17. *([Bis89, Theorem 2.7]) Suppose that ω is* **SKT**. *Then, $\bar\partial_E+\gamma$ defines a* **holomorphic structure** *on E, namely $(\bar\partial_E+\gamma)^2=0$, where $\gamma\in C^\infty_{0,1}(X,\,End(E))$ is defined by*

$$\gamma(\xi):=\begin{pmatrix}0 & 0\\ \alpha(\xi) & 0\end{pmatrix}, \qquad \xi\in C^\infty(X,\,T^{0,1}X).$$

Proof. Due to its particular form in terms of α, γ only acts non-trivially on $T^{1,0}X$ and assumes its values in $\Lambda^{1,0}T^\star X$. So, we can identify γ with α and view it as an element of $C^\infty_{0,1}(X,\,\mathrm{Hom}\,(T^{1,0}X,\,\Lambda^{1,0}T^\star X))$, in which case $\bar\partial_{\mathrm{End}(E)}\gamma\in C^\infty_{0,2}(X,\,\mathrm{Hom}\,(T^{1,0}X,\,\Lambda^{1,0}T^\star X))$.

Since $\bar\partial_E^2=0$ and $\gamma^2=0$, we have

$$(\bar\partial_E+\gamma)^2=\bar\partial_E(\gamma\wedge\cdot)+\gamma\wedge\bar\partial_E=(\bar\partial_{\mathrm{End}(E)}\gamma)\wedge\cdot$$

So, we are left to prove that $\bar\partial_{\mathrm{End}(E)}\gamma=0$ in $C^\infty_{0,2}(X,\,\mathrm{Hom}\,(T^{1,0}X,\,\Lambda^{1,0}T^\star X))$.

Computing, we get:

$$(\bar\partial_{\mathrm{End}(E)}\gamma)(\xi_1,\,\xi_2)(\eta)(\nu)=(i\bar\partial\partial\omega)(\xi_1,\,\xi_2,\,\eta,\,\nu)=0,$$

for all $\xi_1,\xi_2\in C^\infty(X,\,T^{0,1}X)$ and all $\eta,\nu\in C^\infty(X,\,T^{1,0}X)$. The last identity follows from the SKT assumption $\bar\partial\partial\omega=0$ on ω. $\square$

The phenomenon described above, in which a metric induces a holomorphic structure on a vector bundle, lies at the heart of *Mirror Symmetry* that will be discussed in a later chapter of this book.

Another consequence of Theorem 4.6.17 is that, when ω is SKT, the form α is the *second fundamental form* (see terminology in [Dem97, V. §.14]) of the exact sequence

$$0\longrightarrow\Lambda^{1,0}T^\star X\longrightarrow(E,\,\bar\partial_E+\gamma)\longrightarrow T^{1,0}X\longrightarrow 0 \tag{4.172}$$

obtained from (4.171) by changing only the holomorphic structure on E from $\bar\partial_E$ to $\bar\partial_E+\gamma$. This means that the Chern connection $D_{E,\gamma}$ of E equipped with the holomorphic structure $\bar\partial_E+\gamma$ can be expressed in terms of the Chern connections D_S of $S:=\Lambda^{1,0}T^\star X$ and D_Q of $Q:=T^{1,0}X$ (all of them corresponding to the fibre metrics induced by ω) as

$$D_{E,\gamma}=\begin{pmatrix}D_S & \alpha^\star\\ -\alpha & D_Q\end{pmatrix}.$$

The restriction to $S:=\Lambda^{1,0}T^\star X$ of the new holomorphic structure $\bar\partial_E+\gamma$ of E coincides with the restriction of the original holomorphic structure $\bar\partial_E$, hence with the natural holomorphic structure of S, since γ vanishes on S, by definition.

Moreover, we know from the general theory (see [Dem97, V. §.14]) that the $\bar\partial$-cohomology class $\{\alpha\}\in H^{0,1}(X,\,\mathrm{Hom}\,(T^{1,0}X,\,\Lambda^{1,0}T^\star X))$ characterises the isomorphism class of $(E,\,\bar\partial_E+\gamma)$ among all the extensions of $S:=\Lambda^{1,0}T^\star X$ by $Q:=T^{1,0}X$. In particular, the exact sequence (4.172) splits holomorphically if and only if $\{\alpha\}=0\in H^{0,1}(X,\,\mathrm{Hom}\,(T^{1,0}X,\,\Lambda^{1,0}T^\star X))$, which is equivalent to $\partial\omega$ being $\bar\partial$-exact. When $\dim_{\mathbb{C}}X=3$, we know from (II) of Lemma 4.6.3 that $\partial\omega$ is $\bar\partial$-exact if and only if ω is Hermitian-symplectic.

We conclude that, when $\dim_{\mathbb{C}}X=3$, the exact sequence (4.172) splits holomorphically if and only if ω is Hermitian-symplectic.

4.6.4 An example of SKT manifold: $S^3 \times S^3$ equipped with the Calabi-Eckmann complex structure

Most of the material in this subsection is taken from [TT17]. Recall that the *special unitary group* $SU(2)$ is diffeomorphic to the 3-sphere S^3 as a manifold, so $SU(2)$ is simply connected and S^3 has the structure of a compact connected Lie group. This is because $SU(2) = \{A \in M(2, \mathbb{C}) \mid A\,{}^t\bar{A} = I_2 \text{ and } \det A = 1\}$, so $SU(2)$ consists of the matrices

$$A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ satisfying the relations: $|\alpha|^2 + |\gamma|^2 = |\beta|^2 + |\delta|^2 = 1$, $\alpha\bar{\beta} + \gamma\bar{\delta} = 0$ and $\alpha\delta - \beta\gamma = 1$. Multiplying the equality $\alpha\bar{\beta} + \gamma\bar{\delta} = 0$ by β and using the other relations, we get:

$$\alpha|\beta|^2 + (\alpha\delta - 1)\,\bar{\delta} = 0 \iff \alpha\,(|\beta|^2 + |\delta|^2) = \bar{\delta} \iff \alpha = \bar{\delta},$$

while multiplying the equality $\alpha\bar{\beta} + \gamma\bar{\delta} = 0$ by δ and using the other relations, we get:

$$(1 + \beta\gamma)\,\bar{\beta} + \gamma|\delta|^2 = 0 \iff \bar{\beta} + \gamma\,(|\beta|^2 + |\delta|^2) = 0 \iff \gamma = -\bar{\beta}.$$

This leads to

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C} \text{ such that } |\alpha|^2 + |\beta|^2 = 1 \right\} \simeq S^3.$$

As is also standard, this proves that every element in $SU(2)$ maps isomorphically to the unit-norm quaternion $a\,\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$, where $\alpha := a + ib$ and $\beta := -c + id$ with $a, b, c, d \in \mathbb{R}$.

The *Lie algebra* of $SU(2)$ is $su(2) = \{A \in M(2, \mathbb{C}) \mid A +^t \bar{A} = O_2 \text{ and } \operatorname{Tr} A = 0\}$. Explicitly, $su(2)$ consists of the matrices

$$A = \begin{pmatrix} iy & -\bar{z} \\ z & -iy \end{pmatrix}$$

with $y \in \mathbb{R}$ and $z \in \mathbb{C}$. In particular, the Lie algebra $su(2)$ is generated as an $\mathbb{R}$-vector space by

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The basis $\{e_1, e_2, e_3\}$ of $su(2)$ is easily seen to satisfy the *quaternion relations*:

$$e_2\,e_3 = -e_3\,e_2 = e_1, \quad e_3\,e_1 = -e_1\,e_3 = e_2, \quad e_1\,e_2 = -e_2\,e_1 = e_3$$

and $e_1^2 = e_2^2 = e_3^2 = -I_2$. We infer that the *Lie bracket* of $su(2)$ is specified by the relations:

$$[e_1, e_2] = 2e_3, \quad [e_1, e_3] = -2e_2, \quad [e_2, e_3] = 2e_1. \tag{4.173}$$

The well-known *Cartan formula* for arbitrary 1-forms α reads:

$$(d\alpha)(\xi_0, \xi_1) = \xi_0.\alpha(\xi_1) - \xi_1.\alpha(\xi_0) - \alpha([\xi_0, \xi_1])$$

for all vector fields ξ_0, ξ_1. Together with (4.173), it yields the following *structure equations* for the *dual co-frame* $\{e^1, e^2, e^3\}$ associated with the basis $\{e_1, e_2, e_3\}$ of $su(2)$:

$$de^1 = -2e^2 \wedge e^3, \quad de^2 = 2e^1 \wedge e^3, \quad de^3 = -2e^1 \wedge e^2. \tag{4.174}$$

Now, consider the differentiable manifold $X = S^3 \times S^3 \simeq SU(2) \times SU(2)$. Let $\{e_1, e_2, e_3\}$, resp. $\{f_1, f_2, f_3\}$, be a basis of the first copy, resp. of the second copy, of $su(2)$. Let $\{e^1, e^2, e^3\}$ and $\{f^1, f^2, f^3\}$ be the corresponding dual co-frames.

Definition 4.6.18. *One defines a* **complex structure** J *on* X *by setting:*

$$Je_1 = e_2, \quad Jf_1 = f_2, \quad Je_3 = f_3.$$

It turns out (see [Par03]) that J coincides with the **Calabi-Eckmann complex structure** (introduced in [CE53]) or its conjugate. Therefore, we denote $J := J_{CE}$.

We now switch from vector fields to differential forms using the following action of J on 1-forms: for every 1-form α, we define the 1-form $J\alpha$ by requiring that, for every vector field ξ, we have:

$$(J\alpha)(\xi) = \alpha(J\xi).$$

Thus, the action of J on vector fields described in Definition 4.6.18 translates to the following relations at the level of 1-forms:

$$Je^1 = -e^2, \quad Jf^1 = -f^2, \quad Je^3 = -f^3. \tag{4.175}$$

A **complex co-frame** of $(1,\,0)$-*forms* on $(S^3 \times S^3,\, J_{CE})$ is given by

$$\varphi^1 := e^1 + ie^2, \quad \varphi^2 := f^1 + if^2, \quad \varphi^3 := e^3 + if^3. \tag{4.176}$$

(That φ^1, φ^2 and φ^3 are indeed i-eigenvectors for J, hence 1-forms of bidegree $(1,\,0)$, follows at once from (4.175). For example,

$$J\varphi^1 = Je^1 + iJe^2 = -e^2 + ie^1 = i\,(e^1 + ie^2) = i\,\varphi^1.)$$

Taking d in (4.176) and using (4.174), we get the following *complex structure equations* for $(S^3 \times S^3,\, J_{CE})$:

$$d\varphi^1 = i\varphi^1 \wedge \varphi^3 + i\varphi^1 \wedge \overline{\varphi}^3, \quad d\varphi^2 = \varphi^2 \wedge \varphi^3 - \varphi^2 \wedge \overline{\varphi}^3, \quad d\varphi^3 = -i\varphi^1 \wedge \overline{\varphi}^1 + \varphi^2 \wedge \overline{\varphi}^2. \tag{4.177}$$

These are equivalent to the following two groups of equations in bidegree $(2,\,0)$, resp. $(1,\,1)$:

$$\partial\varphi^1 = i\varphi^1 \wedge \varphi^3, \quad \partial\varphi^2 = \varphi^2 \wedge \varphi^3, \quad \partial\varphi^3 = 0, \tag{4.178}$$

$$\bar\partial\varphi^1 = i\varphi^1 \wedge \overline{\varphi}^3, \quad \bar\partial\varphi^2 = -\varphi^2 \wedge \overline{\varphi}^3, \quad \bar\partial\varphi^3 = -i\varphi^1 \wedge \overline{\varphi}^1 + \varphi^2 \wedge \overline{\varphi}^2. \tag{4.179}$$

Definition 4.6.19. *One defines a* **Hermitian metric** ω *on* $(S^3 \times S^3,\, J_{CE})$ *by setting:*

$$\omega := \frac{i}{2}\sum_{j=1}^{3} \varphi^j \wedge \overline{\varphi}^j.$$

We now get our first example of a higher-dimensional compact non-Kähler SKT manifold.

Proposition 4.6.20. *The Hermitian metric* ω *of Definition 4.6.19 has the property* $\partial\bar\partial\omega = 0$*. Hence, the Calabi-Eckmann manifold* $(S^3 \times S^3,\, J_{CE})$ *is an* **SKT manifold** *of complex dimension* 3.

Proof. Straightforward computations yield:

$$\begin{aligned}
\bar\partial\omega &= \frac{i}{2}\,(\bar\partial\varphi^1 \wedge \overline{\varphi}^1 - \varphi^1 \wedge \bar\partial\overline{\varphi}^1 + \bar\partial\varphi^2 \wedge \overline{\varphi}^2 - \varphi^2 \wedge \bar\partial\overline{\varphi}^2 + \bar\partial\varphi^3 \wedge \overline{\varphi}^3 - \varphi^3 \wedge \bar\partial\overline{\varphi}^3) \\
&= \frac{1}{2}\,\varphi^1 \wedge \overline{\varphi}^1 \wedge \overline{\varphi}^3 + \frac{i}{2}\,\varphi^2 \wedge \overline{\varphi}^2 \wedge \overline{\varphi}^3,
\end{aligned}$$

where (4.178) and (4.179) were used to get the last identity from the previous one.

Taking ∂, we further get:

$$\begin{aligned}\partial\bar\partial\omega &= \frac{1}{2}\,\partial\varphi^1\wedge\overline{\varphi}^1\wedge\overline{\varphi}^3 - \frac{1}{2}\,\varphi^1\wedge\partial\overline{\varphi}^1\wedge\overline{\varphi}^3 + \frac{1}{2}\,\varphi^1\wedge\overline{\varphi}^1\wedge\partial\overline{\varphi}^3\\ &+ \frac{i}{2}\,\partial\varphi^2\wedge\overline{\varphi}^2\wedge\overline{\varphi}^3 - \frac{i}{2}\,\varphi^2\wedge\partial\overline{\varphi}^2\wedge\overline{\varphi}^3 + \frac{i}{2}\,\varphi^2\wedge\overline{\varphi}^2\wedge\partial\overline{\varphi}^3\\ &= -\frac{1}{2}\,\varphi^1\wedge\overline{\varphi}^1\wedge\varphi^2\wedge\overline{\varphi}^2 + \frac{1}{2}\,\varphi^1\wedge\overline{\varphi}^1\wedge\varphi^2\wedge\overline{\varphi}^2 = 0,\end{aligned}$$

where (4.178) and (4.179) were used to get the first identity on the last line from the previous one. □

Following [TT17], let us now consider the following holomorphic family of *small deformations* J_t of the complex structure $J_0 := J_{CE}$ defined as:

$$\varphi_t^1 := \varphi^1, \qquad \varphi_t^2 := \varphi^2, \qquad \varphi_t^3 := \varphi^3 - t\overline{\varphi}^3, \tag{4.180}$$

for t varying in a small disc D about 0 in $\mathbb{C}$.

The last identity in (4.180) implies $\overline{\varphi}_t^3 = \overline{\varphi}^3 - \bar t\,\varphi^3$, so we get

$$\varphi^3 = \varphi_t^3 + t\,(\overline{\varphi}_t^3 + \bar t\,\varphi^3) = \varphi_t^3 + t\,\overline{\varphi}_t^3 + |t|^2\,\varphi^3,$$

yielding

$$\varphi^3 = \frac{\varphi_t^3 + t\,\overline{\varphi}_t^3}{1-|t|^2} \quad \text{and} \quad \overline{\varphi}^3 = \frac{\overline{\varphi}_t^3 + \bar t\,\varphi_t^3}{1-|t|^2}.$$

Using these identities, (4.180) and (4.177), straightforward computations yield:

$$\begin{aligned}d\varphi_t^1 &= \frac{i(\bar t+1)}{1-|t|^2}\,\varphi_t^1\wedge\varphi_t^3 + \frac{i(t+1)}{1-|t|^2}\,\varphi_t^1\wedge\overline{\varphi}_t^3\\ d\varphi_t^2 &= \frac{1-\bar t}{1-|t|^2}\,\varphi_t^2\wedge\varphi_t^3 + \frac{t-1}{1-|t|^2}\,\varphi_t^2\wedge\overline{\varphi}_t^3\\ d\varphi_t^3 &= i(t-1)\,\varphi_t^1\wedge\overline{\varphi}_t^1 + (t+1)\,\varphi_t^2\wedge\overline{\varphi}_t^2.\end{aligned} \tag{4.181}$$

For example, the last identity follows from:

$$d\varphi_t^3 = d\varphi^3 - td\overline{\varphi}^3 = -i\varphi^1\wedge\overline{\varphi}^1 + \varphi^2\wedge\overline{\varphi}^2 + it\varphi^1\wedge\overline{\varphi}^1 + t\varphi^2\wedge\overline{\varphi}^2.$$

Lemma 4.6.21. *On the complex manifold $(S^3\times S^3,\,J_t)$, the $(1,\,1)$-form $\varphi_t^3\wedge\overline{\varphi}_t^3$ has the property:*

$$\partial_t\bar\partial_t(\varphi_t^3\wedge\overline{\varphi}_t^3) = -4Im(t)\,i\varphi_t^1\wedge\overline{\varphi}_t^1\wedge i\varphi_t^2\wedge\overline{\varphi}_t^2, \qquad t\in D.$$

In particular, if $Im(t)<0$, $\partial_t\bar\partial_t(\varphi_t^3\wedge\overline{\varphi}_t^3)\geq 0$ as a non-zero C^∞ $(2,\,2)$-form on $(S^3\times S^3,\,J_t)$.

Proof. Straightforward computations yield:

$$\partial_t\bar\partial_t(\varphi_t^3\wedge\overline{\varphi}_t^3) = \partial_t\bar\partial_t\varphi_t^3\wedge\overline{\varphi}_t^3 + \bar\partial_t\varphi_t^3\wedge\partial_t\overline{\varphi}_t^3 - \partial_t\varphi_t^3\wedge\bar\partial_t\overline{\varphi}_t^3 + \varphi_t^3\wedge\partial_t\bar\partial_t\overline{\varphi}_t^3 = \bar\partial_t\varphi_t^3\wedge\partial_t\overline{\varphi}_t^3,$$

where the last identity follows from $\partial_t\varphi_t^3=0$ as a J_t-$(2,\,0)$-form, itself a consequence of the formula displaying $d\varphi_t^3$ as a J_t-$(1,\,1)$-form in (4.181). Since $\bar\partial_t\varphi_t^3 = i(t-1)\,\varphi_t^1\wedge\overline{\varphi}_t^1 + (t+1)\,\varphi_t^2\wedge\overline{\varphi}_t^2$ by the same formula, we further get:

$$\begin{aligned}\partial_t\bar\partial_t(\varphi_t^3\wedge\overline{\varphi}_t^3) &= [i(t-1)\,\varphi_t^1\wedge\overline{\varphi}_t^1 + (t+1)\,\varphi_t^2\wedge\overline{\varphi}_t^2]\wedge[i(\bar t-1)\,\varphi_t^1\wedge\overline{\varphi}_t^1 - (\bar t+1)\,\varphi_t^2\wedge\overline{\varphi}_t^2]\\ &= -i(t-1)(\bar t+1)\,\varphi_t^1\wedge\overline{\varphi}_t^1\wedge\varphi_t^2\wedge\overline{\varphi}_t^2 + i(\bar t-1)(t+1)\,\varphi_t^1\wedge\overline{\varphi}_t^1\wedge\varphi_t^2\wedge\overline{\varphi}_t^2\\ &= 2i(t-\bar t)\,i\varphi_t^1\wedge\overline{\varphi}_t^1\wedge i\varphi_t^2\wedge\overline{\varphi}_t^2 = -4\mathrm{Im}\,(t)\,i\varphi_t^1\wedge\overline{\varphi}_t^1\wedge i\varphi_t^2\wedge\overline{\varphi}_t^2.\end{aligned}$$

□

Corollary 4.6.22. *The complex manifold $(S^3 \times S^3,\, J_t)$ is* **not SKT** *if $Im(t) < 0$ with $t \in D$ close enough to 0.*

In particular, the SKT property of compact complex manifolds is **not open** *under holomorphic deformations of complex structures.*

Proof. By Lemma 4.6.21, the non-zero C^∞ $(2,\, 2)$-form $\partial_t\bar\partial_t(\varphi_t^3 \wedge \overline{\varphi}_t^3) \geq 0$ is a non-zero $(\partial\bar\partial)$-exact semi-positive $(n-1,\, n-1) = (2,\, 2)$-current on $(S^3 \times S^3,\, J_t)$ when $\text{Im}(t) < 0$. By Proposition 4.6.16, the complex manifold $(S^3 \times S^3,\, J_t)$ is not SKT in this case.

Since, moreover, $(S^3 \times S^3,\, J_0) = (S^3 \times S^3,\, J_{CE})$ is an SKT manifold by Proposition 4.6.20, we get the last conclusion of Corollary 4.6.22. $\square$

4.6.5 Behaviour of the Frölicher spectral sequence on SKT manifolds

The material in this subsection is taken from [Pop16] where the following conjecture was proposed.

Conjecture 4.6.23. *([Pop16, Conjecture 1.3]) The Frölicher spectral sequence of an* **SKT** *compact complex manifold X degenerates at E_2.*

If confirmed, this would be the first known non-Kähler metric hypothesis implying a degeneration property of the Frölicher spectral sequence. It would thus provide a link between the analytic, metrical side of the theory of compact complex manifolds and the algebraic, Hodge-theoretical side. While the general case of Conjecture 4.6.23 is still open, it was proved to hold in [Pop16] under two different and independent groups of extra assumptions on the metric that we shall now discuss separately.

(I) First sufficient metric condition for Frölicher E_2 degeneration

Conjecture 4.6.23 was proved to hold in [Pop16] when an SKT metric with *small torsion* exists in the sense that we now describe. Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. With every Hermitian metric ω on X we associate the following zero-order operators of type $(0,\, 0)$ depending only on the torsion of ω: $\bar{S}_\omega := [\bar\partial\omega \wedge \cdot,\, (\bar\partial\omega \wedge \cdot)^\star] \geq 0$,

$$Z_\omega := [\tau_\omega,\, \tau_\omega^\star] + [\partial\omega \wedge \cdot,\, (\partial\omega \wedge \cdot)^\star] \geq 0 \quad \text{and} \quad \bar{R}_\omega := [\bar\tau_\omega,\, \bar\tau_\omega^\star] - [\bar\partial\omega \wedge \cdot,\, (\bar\partial\omega \wedge \cdot)^\star], \tag{4.182}$$

where the notation is the standard one: $[A,\, B] := AB - (-1)^{a\,b} BA$ denotes the graded commutator of any pair of endomorphisms A, B of respective degrees a, b of the graded algebra $C^\infty_{\bullet,\,\bullet}(X,\, \mathbb{C})$ of smooth differential forms on X, while $\tau = \tau_\omega := [\Lambda,\, \partial\omega \wedge \cdot]$ is the torsion operator of order zero and bidegree $(1,\, 0)$ associated with ω (see Proposition 4.5.11) and $\Lambda = \Lambda_\omega$ is the formal adjoint of the Lefschetz operator $L := \omega \wedge \cdot$ w.r.t. the L^2 inner product induced by ω on differential forms.

By the torsion of a Hermitian metric ω being *small* we mean that the upper bound of the torsion operator Z_ω (which is bounded) is dominated by a certain fixed multiple of the smallest positive eigenvalue of the non-negative self-adjoint elliptic operator $\Delta' + \Delta''$ in every bidegree $(p,\, q)$.

Theorem 4.6.24. *Let X be a compact complex n-dimensional manifold. If X carries an SKT metric ω whose torsion satisfies the condition*

$$\sup\limits_{u \in C^\infty_{p,\,q}(X,\,\mathbb{C}),\; ||u||=1} \langle\langle Z_\omega u,\, u\rangle\rangle \leq \frac{1}{3}\, min\left(Spec\,(\Delta' + \Delta'')^{p,\,q} \cap (0,\, +\infty)\right) \tag{4.183}$$

for all $p, q \in \{0, \dots, n\}$, then the Frölicher spectral sequence of X degenerates at E_2.

By $(\Delta' + \Delta'')^{p,\,q}$ we mean the operator $\Delta' + \Delta''$ acting on $(p,\,q)$-forms, while $\mathrm{Spec}\,(\Delta' + \Delta'')^{p,\,q}$ stands for its spectrum and $||\cdot||, \langle\langle\cdot,\,\cdot\rangle\rangle$ denote the L^2-norm, resp. the L^2-inner product induced by ω on differential forms. Thus, the r.h.s. in (4.183) is a third of the size of the *spectral gap* of $\Delta' + \Delta''$, an important quantity standardly associated with a given metric ω.

Proof of Theorem 4.6.24.

Two of the main tools are the pseudo-differential Laplacian $\widetilde{\Delta}$ and the Hodge isomorphism for the spaces $E_2^{p,\,q}$ on the 2-nd page of the FSS of X it induces that were discussed in §.3.1.

Throughout this discussion, $(X,\,\omega)$ will be a compact Hermitian manifold with $\dim_{\mathbb{C}}X = n$. Recall that for every $k\in\{0,\dots,2n\}$, the d-Laplacian $\Delta : C^\infty_k(X,\,\mathbb{C})\longrightarrow C^\infty_k(X,\,\mathbb{C})$ is defined by $\Delta = dd^\star + d^\star d$. If we denote by $\mathcal{H}^k_\Delta(X,\,\mathbb{C})\subset C^\infty_k(X,\,\mathbb{C})$ the kernel of Δ acting on smooth forms of degree k, we have the Hodge isomorphism $\mathcal{H}^k_\Delta(X,\,\mathbb{C})\simeq H^k_{DR}(X,\,\mathbb{C})$ with the De Rham cohomology group of degree k.

We start with the following very simple observation.

Lemma 4.6.25. (a) *If for every $p,q\in\{0,1,\dots,n\}$ the following map induced by the identity*

$$J^{p,\,q} : \widetilde{\mathcal{H}}^{p,\,q}_{\widetilde\Delta}(X,\,\mathbb{C})\longrightarrow \mathcal{H}^{p+q}_\Delta(X,\,\mathbb{C}),\qquad \gamma\longmapsto\gamma, \tag{4.184}$$

is well defined, then the Frölicher spectral sequence of X degenerates at E_2.

(b) *A sufficient condition for the map $J^{p,\,q}$ to be well defined is that the following inequality hold*

$$\Delta' - \Delta'_{p''} \leq \Delta'' + (C\Delta'' + (1-\varepsilon)\Delta')\qquad \textit{on } (p,\,q)\textit{-forms}, \tag{4.185}$$

for some constants $C\geq 0$ and $0<\varepsilon\leq 1$ depending only on X, ω and $(p,\,q)$. (Recall that $\Delta' - \Delta'_{p''} = \Delta'_{p''_\perp}\geq 0$.)

Thus, (4.185) implies the degeneration at E_2 of the Frölicher spectral sequence of X.

Proof. (a) Well-definedness for $J^{p,\,q}$ means that for every smooth $(p,\,q)$-form γ we have $\Delta\gamma = 0$ whenever $\widetilde\Delta\gamma = 0$. It is clear that $J^{p,\,q}$ is automatically injective if it is well defined, hence in that case $\dim\widetilde{\mathcal{H}}^{p,\,q}_{\widetilde\Delta}(X,\,\mathbb{C})\leq\dim\mathcal{H}^{p+q}_\Delta(X,\,\mathbb{C})$. Therefore, if all the maps $J^{p,\,q}$ are well defined, then

$$\sum_{p+q=k}\dim E_2^{p,\,q}\leq b_k := \dim H^k_{DR}(X,\,\mathbb{C})\quad\text{for all }\ k\in\{0,\dots,2n\} \tag{4.186}$$

since $\dim E_2^{p,\,q} = \dim\widetilde{\mathcal{H}}^{p,\,q}_{\widetilde\Delta}(X,\,\mathbb{C})$ by the Hodge isomorphism (3.20) and the images $J^{p,\,q}(\widetilde{\mathcal{H}}^{p,\,q}_{\widetilde\Delta}(X,\,\mathbb{C}))$ in $\mathcal{H}^k_\Delta(X,\,\mathbb{C})$ have pairwise intersections reduced to zero for $p+q=k$ for bidegree reasons. Inequality (4.186) is precisely the degeneration condition at E_2.

(b) Clearly, a sufficient condition for $J^{p,\,q}$ to be well defined is that the following inequality hold

$$\langle\langle\Delta\gamma,\,\gamma\rangle\rangle\leq C\,\langle\langle\widetilde\Delta\gamma,\,\gamma\rangle\rangle\qquad\text{for all }\ \gamma\in C^\infty_{p,\,q}(X,\,\mathbb{C}) \tag{4.187}$$

since $\Delta,\widetilde\Delta\geq 0$. Now, by definition of $\widetilde\Delta$ (cf. (3.9)), $\langle\langle\widetilde\Delta\gamma,\,\gamma\rangle\rangle = \langle\langle\Delta'_{p''}\gamma,\,\gamma\rangle\rangle + \langle\langle\Delta''\gamma,\,\gamma\rangle\rangle$. Meanwhile, for every $(p,\,q)$-form γ, we have $\langle\langle\Delta\gamma,\,\gamma\rangle\rangle = ||\partial\gamma+\bar\partial\gamma||^2 + ||\partial^\star\gamma+\bar\partial^\star\gamma||^2 = ||\partial\gamma||^2 + ||\partial^\star\gamma||^2 + ||\bar\partial\gamma||^2 + ||\bar\partial^\star\gamma||^2 = \langle\langle\Delta'\gamma,\,\gamma\rangle\rangle + \langle\langle\Delta''\gamma,\,\gamma\rangle\rangle$ since $\partial\gamma$ is orthogonal to $\bar\partial\gamma$ and $\partial^\star\gamma$ is orthogonal to $\bar\partial^\star\gamma$ for bidegree reasons. (This argument breaks down if γ is not of pure type.) Thus

$$\langle\langle\Delta\gamma,\,\gamma\rangle\rangle = \langle\langle\Delta'\gamma,\,\gamma\rangle\rangle + \langle\langle\Delta''\gamma,\,\gamma\rangle\rangle\qquad\text{for all }\ \gamma\in C^\infty_{p,\,q}(X,\,\mathbb{C}). \tag{4.188}$$

It is now clear that (4.185) implies (4.187) with a possibly different constant C, so (4.185) implies the well-definedness of $J^{p,q}$. $\square$

Concerning inequality (4.185), note that the stronger inequality $\langle\langle\Delta'\gamma,\,\gamma\rangle\rangle \leq C\,\langle\langle\Delta''\gamma,\,\gamma\rangle\rangle$ for all $(p,\,q)$-forms γ and all bidegrees $(p,\,q)$ implies the degeneration at E_1 of the Frölicher spectral sequence, but we shall not pursue this here.

Use of (b) *of Lemma 4.6.25*

We shall now concentrate on proving inequality (4.185) under the SKT assumption coupled with a torsion assumption on the metric ω.

Lemma 4.6.26. *A sufficient condition for (4.185) to hold (hence for $E_2(X)=E_\infty(X)$) is that there exist constants $0<\delta<1-\varepsilon<1$ and $C\geq 0$ such that the following inequality holds*

$$\begin{aligned}(1-\varepsilon-\delta)\left(||p''_\perp\partial u||^2+||p''_\perp\partial^\star u||^2\right)+(1-\varepsilon)\,(||p''\partial u||^2+||p''\partial^\star u||^2)+C\,\langle\langle\Delta''u,\,u\rangle\rangle \geq \\ \left(\frac{1}{\delta}-1\right)\left(||p''_\perp\tau u||^2+||p''_\perp\tau^\star u||^2\right)+\langle\langle[\partial\omega\wedge\cdot,\,(\partial\omega\wedge\cdot)^\star]\,u,\,u\rangle\rangle-\langle\langle[\Lambda,\,[\Lambda,\,\frac{i}{2}\,\partial\bar\partial\omega]]\,u,\,u\rangle\rangle\end{aligned} \tag{4.189}$$

for every form $u\in C^\infty_{p,\,q}(X,\,\mathbb{C})$ and every bidegree $(p,\,q)$. (Note that all the terms on the r.h.s. of (4.189) are of order zero, hence bounded, while the last and only signless term vanishes if ω is SKT.)

Proof. By Demailly's non-Kähler Bochner-Kodaira-Nakano identity $\Delta''=\Delta'_\tau+T_\omega$ (cf. (4.81)), inequality (4.185) is equivalent to each of the following inequalities:

$$\begin{aligned}\Delta'-\Delta'_{p''} &\leq \Delta'+[\tau,\,\partial^\star]+[\partial,\,\tau^\star]+[\tau,\,\tau^\star]+C\,\Delta''+(1-\varepsilon)\,\Delta'+T_\omega \iff \\ 0 &\leq \left(\Delta'_{p''}+(\tau p''\partial^\star+\partial^\star p''\tau)+(\partial p''\tau^\star+\tau^\star p''\partial)+(\tau p''\tau^\star+\tau^\star p''\tau)\right) \\ &+ (1-\varepsilon)\,\Delta'+C\,\Delta''+(\tau p''_\perp\partial^\star+\partial^\star p''_\perp\tau)+(\partial p''_\perp\tau^\star+\tau^\star p''_\perp\partial)+(\tau p''_\perp\tau^\star+\tau^\star p''_\perp\tau)+T_\omega.\end{aligned} \tag{4.190}$$

Since $\Delta'_{p''}+(\tau p''\partial^\star+\partial^\star p''\tau)+(\partial p''\tau^\star+\tau^\star p''\partial)+(\tau p''\tau^\star+\tau^\star p''\tau)=(\partial+\tau)p''(\partial^\star+\tau^\star)+(\partial^\star+\tau^\star)p''(\partial+\tau)\geq 0$, inequality (4.190) holds if the following inequality holds

$$\begin{aligned}(1-\varepsilon)\,\langle\langle\Delta'u,\,u\rangle\rangle &+ C\,\langle\langle\Delta''u,\,u\rangle\rangle+||p''_\perp\tau u||^2+||p''_\perp\tau^\star u||^2 \\ &\geq -2\,\mathrm{Re}\,\langle\langle p''_\perp\partial^\star u,\,p''_\perp\tau^\star u\rangle\rangle-2\,\mathrm{Re}\,\langle\langle p''_\perp\partial u,\,p''_\perp\tau u\rangle\rangle-\langle\langle T_\omega u,\,u\rangle\rangle.\end{aligned} \tag{4.191}$$

Now, suppose that $0<\varepsilon<1$ and choose any $0<\delta<1-\varepsilon$. The Cauchy-Schwarz inequality gives

$$\left|2\,\mathrm{Re}\,\langle\langle p''_\perp\partial u,\,p''_\perp\tau u\rangle\rangle\right|\leq\delta\,||p''_\perp\partial u||^2+\frac{1}{\delta}\,||p''_\perp\tau u||^2,\ \left|2\,\mathrm{Re}\,\langle\langle p''_\perp\partial^\star u,\,p''_\perp\tau^\star u\rangle\rangle\right|\leq\delta\,||p''_\perp\partial^\star u||^2+\frac{1}{\delta}\,||p''_\perp\tau^\star u||^2.$$

Thus, for (4.191) to hold, it suffices that the following inequality hold:

$$\begin{aligned}(1-\varepsilon)\,\langle\langle\Delta'u,\,u\rangle\rangle+C\,\langle\langle\Delta''u,\,u\rangle\rangle &\geq \delta\,(||p''_\perp\partial u||^2+||p''_\perp\partial^\star u||^2)+\left(\frac{1}{\delta}-1\right)(||p''_\perp\tau u||^2+||p''_\perp\tau^\star u||^2) \\ &+ \langle\langle[\partial\omega\wedge\cdot,\,(\partial\omega\wedge\cdot)^\star]\,u,\,u\rangle\rangle-\langle\langle[\Lambda,\,[\Lambda,\,\frac{i}{2}\,\partial\bar\partial\omega]]\,u,\,u\rangle\rangle.\end{aligned} \tag{4.192}$$

This is equivalent to (4.189) since

$$\langle\langle\Delta' u,\, u\rangle\rangle = ||p''\partial u + p''_\perp\partial u||^2 + ||p''\partial^\star u + p''_\perp\partial^\star u||^2 = (||p''\partial u||^2 + ||p''\partial^\star u||^2) + (||p''_\perp\partial u||^2 + ||p''_\perp\partial^\star u||^2)$$

thanks to the obvious orthogonality relations $p''\partial u \perp p''_\perp\partial u$ and $p''\partial^\star u \perp p''_\perp\partial^\star u$. $\square$

To apply Lemma 4.6.26, we start with a very simple elementary observation.

Lemma 4.6.27. *Let $\mathcal{H}$ be a Hilbert space and let $A, B : \mathcal{H} \to \mathcal{H}$ be closed linear operators such that $A, B \geq 0$, $A = A^\star$ and $B = B^\star$.*

If $\ker A \subset \ker B$ and if $B \leq A$ on $(\ker A)^\perp$, then $B \leq A$.

Proof. We have to prove that $\langle Bu,\, u\rangle \leq \langle Au,\, u\rangle$ for all u. Since A is closed, $\ker A$ is closed in $\mathcal{H}$, so every $u \in \mathcal{H}$ splits uniquely as $u = u_A + u_A^\perp$ with $u_A \in \ker A$ and $u_A^\perp \in (\ker A)^\perp$. Moreover,

$$A((\ker A)^\perp) \subset (\ker A)^\perp. \tag{4.193}$$

Indeed, for every $u_A^\perp \in (\ker A)^\perp$ and every $v \in \ker A$, we have: $\langle A(u_A^\perp),\, v\rangle = \langle u_A^\perp,\, Av\rangle = 0$ since $A^\star v = Av = 0$. Therefore, for every u, we get:

$$\langle Au,\, u\rangle = \langle Au_A^\perp,\, u_A + u_A^\perp\rangle = \langle Au_A^\perp,\, u_A^\perp\rangle \geq \langle Bu_A^\perp,\, u_A^\perp\rangle = \langle Bu_A^\perp,\, u\rangle = \langle Bu,\, u\rangle.$$

The second identity above followed from (4.193), the inequality followed from the hypothesis and the last two identities followed from the next relations:

$$(i)\;\; B((\ker A)^\perp) \subset (\ker A)^\perp \quad \text{and} \quad (ii)\;\; B(\ker A) = 0.$$

To prove (i), let $u_A^\perp \in (\ker A)^\perp$ and $v \in \ker A \subset \ker B$. We have: $\langle B(u_A^\perp),\, v\rangle = \langle u_A^\perp,\, Bv\rangle = 0$ since $B^\star v = Bv = 0$. Identity (ii) follows from the hypothesis $\ker A \subset \ker B$. $\square$

We shall now apply Lemma 4.6.27 to the non-negative self-adjoint operators

$$B := \Delta'_{p''_\perp} = \Delta' - \Delta'_{p''} \geq 0 \quad \text{and} \quad A := (C+1)\,\Delta'' + (1-\varepsilon)\,\Delta' \geq 0$$

for which we obviously have $\ker B = \ker \Delta'_{p''_\perp} \supset \ker \Delta' \supset \ker \Delta' \cap \ker \Delta'' = \ker A$. The choice of constants $C > 0$ and $0 < \varepsilon < 1$ will be specified later on.

We know from (b) of Lemma 4.6.25 that a sufficient condition for $E_2(X) = E_\infty(X)$ in the Frölicher spectral sequence is the validity of inequality (4.185), i.e. of the inequality $B \leq A$. By Lemma 4.6.27, this is equivalent to having $B \leq A$ on $(\ker A)^\perp = (\ker \Delta' \cap \ker \Delta'')^\perp$. Now, the proof of Lemma 4.6.26 shows that for this to hold, it suffices for the inequality (4.192) to hold on $(\ker \Delta' \cap \ker \Delta'')^\perp$. If we assume $\partial\bar\partial\omega = 0$, after bounding above $||p''_\perp v||$ by $||v||$ for $v \in \{\partial u, \partial^\star u, \tau u, \tau^\star u\}$ in the r.h.s. of (4.192), we see that it suffices to have

$$(1-\varepsilon-\delta)\,\langle\langle\Delta' u,\, u\rangle\rangle + C\,\langle\langle\Delta'' u,\, u\rangle\rangle \geq \left(\frac{1}{\delta} - 1\right)\langle\langle[\tau,\, \tau^\star]\, u,\, u\rangle\rangle + \langle\langle[\partial\omega\wedge\cdot,\, (\partial\omega\wedge\cdot)^\star]\, u,\, u\rangle\rangle \tag{4.194}$$

for all $u \in (\ker \Delta' \cap \ker \Delta'')^\perp$ and some fixed constants $C > 0$, $0 < \delta < 1 - \varepsilon < 1$.

Now, we choose the constants such that $\delta = 1 - 2\varepsilon > 0$ (so $0 < \varepsilon < \frac{1}{2}$) and $C = 1 - \varepsilon - \delta = \varepsilon$. Thus, $(1/\delta) - 1 = 2\varepsilon/(1-2\varepsilon)$. If, moreover, we choose ε such that $2/(1-2\varepsilon) < 3$ (i.e. such that $0 < \varepsilon < 1/6$), (4.194) holds with these choices of constants whenever the following inequality holds:

$$\langle\langle(\Delta' + \Delta'')\, u,\, u\rangle\rangle \geq 3\,\langle\langle([\tau,\, \tau^\star] + [\partial\omega\wedge\cdot,\, (\partial\omega\wedge\cdot)^\star])\, u,\, u\rangle\rangle \quad \text{for all } u \in (\ker \Delta' \cap \ker \Delta'')^\perp \tag{4.195}$$

For all $p, q \in \{0, \dots, n\}$, the non-negative self-adjoint differential operator $\Delta'+\Delta'' : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C})$ is elliptic. Therefore, since X is compact, it has a discrete spectrum contained in $[0,\,+\infty)$ with $+\infty$ as its only accumulation point. In particular, it has a smallest positive eigenvalue that we denote by

$$\rho^{p,\,q}_\omega := \min\bigg(\mathrm{Spec}\,(\Delta' + \Delta'')^{p,\,q} \cap (0,\,+\infty)\bigg) > 0. \tag{4.196}$$

Thus, $\rho^{p,\,q}_\omega$ is the size of the spectral gap of $\Delta' + \Delta''$ acting on $(p,\,q)$-forms. We get

$$\langle\langle(\Delta' + \Delta'')\,u,\,u\rangle\rangle \geq \rho^{p,\,q}_\omega\,||u||^2 \quad \text{for all } u \in C^\infty_{p,\,q}(X,\,\mathbb{C}) \cap (\ker\Delta' \cap \ker\Delta'')^\perp, \tag{4.197}$$

since $\ker(\Delta'+\Delta'') = \ker\Delta' \cap \ker\Delta''$. On the other hand, the non-negative torsion operator $[\tau,\,\tau^\star] + [\partial\omega\wedge\cdot,\,(\partial\omega\wedge\cdot)^\star]$ is of order zero, hence bounded, hence

$$\langle\langle([\tau,\,\tau^\star] + [\partial\omega\wedge\cdot,\,(\partial\omega\wedge\cdot)^\star])\,u,\,u\rangle\rangle \leq C^{p,\,q}_\omega\,||u||^2 \quad \text{for all } u \in C^\infty_{p,\,q}(X,\,\mathbb{C}), \tag{4.198}$$

where $C^{p,\,q}_\omega := \sup\limits_{u\in C^\infty_{p,\,q}(X,\mathbb{C}),\;||u||=1} \langle\langle([\tau,\,\tau^\star] + [\partial\omega\wedge\cdot,\,(\partial\omega\wedge\cdot)^\star])\,u,\,u\rangle\rangle$.

We conclude from (4.197) and (4.198) that (4.195) holds if $\rho^{p,\,q}_\omega \geq 3\,C^{p,\,q}_\omega$. We have thus proved the following statement which is nothing but Theorem 4.6.24.

Theorem 4.6.28. *Let X be a compact complex n-dimensional manifold. If X carries an SKT metric ω whose torsion satisfies the condition*

$$C^{p,\,q}_\omega \leq \frac{1}{3}\,\rho^{p,\,q}_\omega \tag{4.199}$$

for all $p, q \in \{0, \dots, n\}$, then the Frölicher spectral sequence of X degenerates at E_2.

The proof of Theorem 4.6.24 is complete.

(II) Second group of sufficient metric conditions for Frölicher E_2 degeneration

We shall now give a different kind of metric conditions ensuring that $E_2(X) = E_\infty(X)$ in the Frölicher spectral sequence. To this end, we shall use (a) of Lemma 4.6.25.

Lemma 4.6.29. *Let X be a compact complex manifold with $\dim_{\mathbb{C}}X = n$. If X admits a Hermitian metric ω whose induced operators $\Delta', \Delta'', \Delta'_{p''} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C})$ satisfy the condition*

$$\ker\Delta'_{p''} \cap \ker\Delta'' \subset \ker\Delta' \quad \textit{in every bidegree } (p,\,q), \tag{4.200}$$

the Frölicher spectral sequence of X degenerates at E_2.

Proof. As noticed in (3.10), we always have $\ker\Delta'_{p''} \supset \ker\Delta'$. Recall that $\ker\Delta'_{p''} \cap \ker\Delta'' = \ker\widetilde\Delta$ and that this space is denoted by $\widetilde{\mathcal{H}}^{p,\,q}_{\widetilde\Delta}(X,\,\mathbb{C})$ in bidegree $(p,\,q)$. For every $u \in \widetilde{\mathcal{H}}^{p,\,q}_{\widetilde\Delta}(X,\,\mathbb{C})$, we have $\Delta'u = 0$ thanks to (4.200), hence from (4.188) we get

$$\langle\langle\Delta u,\,u\rangle\rangle = \langle\langle\Delta' u,\,u\rangle\rangle + \langle\langle\Delta'' u,\,u\rangle\rangle = 0 + 0 = 0.$$

This shows that the identity map induces a well-defined linear map $\widetilde{\mathcal{H}}^{p,\,q}_{\widetilde\Delta}(X,\,\mathbb{C}) \longrightarrow \mathcal{H}^{p+q}_{\Delta}(X,\,\mathbb{C})$ for all $(p,\,q)$, hence $E_2(X) = E_\infty(X)$ by (a) of Lemma 4.6.25. $\square$

We now use Lemma 4.6.29 to give two sufficient metric conditions ensuring that $E_2(X) = E_\infty(X)$ in the Frölicher spectral sequence.

Theorem 4.6.30. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.*
(i) For any Hermitian metric ω on X, the following three conditions are equivalent:

(a) $p''\partial = \partial p''$ on all $(p,\,q)$-forms for all bidegrees $(p,\,q)$;
(b) $[\partial,\,\bar\partial^\star](\ker\Delta'') = 0$ and $[\partial,\,\bar\partial^\star](Im\,\bar\partial \oplus Im\,\bar\partial^\star) \subset Im\,\bar\partial \oplus Im\,\bar\partial^\star$;
(c) $[\partial,\,\bar\tau^\star](\ker\Delta'') = 0$ and $[\partial,\,\bar\tau^\star](Im\,\bar\partial \oplus Im\,\bar\partial^\star) \subset Im\,\bar\partial \oplus Im\,\bar\partial^\star$.

Moreover, if X carries a Hermitian metric ω satisfying one of the equivalent conditions $(a), (b), (c)$, the Frölicher spectral sequence of X degenerates at E_2.

(ii) If X carries an SKT metric ω (i.e. such that $\partial\bar\partial\omega = 0$) which moreover satisfies the identity

$$\langle\langle[\bar\tau,\,\bar\tau^\star]\,u,\,u\rangle\rangle = \langle\langle[\bar\partial\omega\wedge\cdot,\,(\bar\partial\omega\wedge\cdot)^\star]\,u,\,u\rangle\rangle \quad \textit{for all } u \in \ker\Delta'_{p''} \cap \ker\Delta'', \tag{4.201}$$

the Frölicher spectral sequence of X degenerates at E_2.

Proof. (i) Let ω be any Hermitian metric on X and let u be any smooth $(p,\,q)$-form. Then $u = u_0 + \bar\partial v + \bar\partial^\star w$ with $u_0 \in \ker\Delta''$ and v, w smooth forms of bidegrees $(p,\,q-1)$, resp. $(p,\,q+1)$. (Note that we can choose $v \in \mathrm{Im}\,\bar\partial^\star$ and $w \in \mathrm{Im}\,\bar\partial$ if these forms are chosen to have minimal L^2 norms.) Thus $p''u = u_0$, so the following equivalences hold:

$$\begin{aligned} p''\partial u = \partial p''u \quad &\iff \quad p''\partial u_0 + p''\partial\bar\partial v + p''\partial\bar\partial^\star w = \partial u_0 \iff p''\partial\bar\partial^\star w = p''_\perp\partial u_0 \\ &\iff \quad p''\partial\bar\partial^\star w = 0 \text{ and } p''_\perp\partial u_0 = 0 \iff \partial u_0 \in \ker\Delta'' \text{ and } \partial\bar\partial^\star w \in \mathrm{Im}\,\bar\partial \oplus \mathrm{Im}\,\bar\partial^\star \\ &\iff \quad \partial u_0 \in \ker\bar\partial^\star \text{ and } \partial\bar\partial^\star w \in \mathrm{Im}\,\bar\partial \oplus \mathrm{Im}\,\bar\partial^\star. \end{aligned} \tag{4.202}$$

We have successively used the following facts: $p''\partial\bar\partial v = -p''\bar\partial\partial v = 0$ because $\ker\Delta'' \perp \mathrm{Im}\,\bar\partial$, $1 - p'' = p''_\perp$, $\mathrm{Im}\,p'' = \ker\Delta'' \perp \mathrm{Im}\,\bar\partial \oplus \mathrm{Im}\,\bar\partial^\star = \mathrm{Im}\,p''_\perp$, $\partial u_0 \in \ker\bar\partial = \ker\Delta'' \oplus \mathrm{Im}\,\bar\partial$ (because $u_0 \in \ker\Delta'' \subset \ker\bar\partial$), hence the equivalence $\partial u_0 \in \ker\Delta'' \iff \partial u_0 \in \ker\bar\partial^\star$.

Now, $\bar\partial^\star\partial w \in \mathrm{Im}\,\bar\partial^\star$ and $\partial\bar\partial^\star u_0 = 0$ because $u_0 \in \ker\Delta'' = \ker\bar\partial \cap \ker\bar\partial^\star \subset \ker\bar\partial^\star$, so (4.202) is equivalent to

$$[\partial,\,\bar\partial^\star]\,u_0 = 0 \text{ and } [\partial,\,\bar\partial^\star]\,w \in \mathrm{Im}\,\bar\partial \oplus \mathrm{Im}\,\bar\partial^\star.$$

On the other hand, $[\partial,\,\bar\partial^\star] = -[\partial,\,\bar\tau^\star]$ by (4.83). Since $u_0 \in \ker\Delta''$ and $w \in \mathrm{Im}\,\bar\partial \oplus \mathrm{Im}\,\bar\partial^\star$ are arbitrary, the equivalences stated under (i) are proved.

To prove the last statement of (i), let ω be a metric satisfying condition (a). We are going to show that the inclusion (4.200) holds, hence by Lemma 4.6.29 we shall have $E_2(X) = E_\infty(X)$ in the Frölicher spectral sequence of X. Let $u \in \ker\Delta'_{p''} \cap \ker\Delta''$ of an arbitrary bidegree $(p,\,q)$. Then

$$0 = \langle\langle\Delta'_{p''}\,u,\,u\rangle\rangle = \langle\langle\Delta'(p''u),\,u\rangle\rangle = \langle\langle\Delta'\,u,\,u\rangle\rangle,$$

where the second identity followed from $p''\partial = \partial p''$ (which also implies $p''\partial^\star = \partial^\star p''$) and the last identity followed from $u \in \ker\Delta''$ (which amounts to $p''u = u$). Thus $\Delta' u = 0$, i.e. $u \in \ker\Delta'$. This proves (4.200), so Lemma 4.6.29 applies.

(ii) We prove that inclusion (4.200) holds under the assumptions made. Let $u \in \ker\Delta'_{p''} \cap \ker\Delta''$.

Note that the conjugate of Demailly's non-Kähler Bochner-Kodaira-Nakano identity $\Delta'' = \Delta'_\tau + T_\omega$ (cf. (4.81)) is

$$\Delta' = \Delta''_\tau + \overline{T}_\omega, \tag{4.203}$$

where $\Delta''_\tau := [\bar\partial + \bar\tau,\, \bar\partial^\star + \bar\tau^\star]$ and $\overline{T}_\omega = [\Lambda,\, [\Lambda,\, \frac{i}{2}\,\partial\bar\partial\omega\wedge\cdot]] - [\bar\partial\omega\wedge\cdot,\, (\bar\partial\omega\wedge\cdot)^\star]$. Thanks to formula (4.203), we have

$$\begin{aligned}\langle\langle\Delta'\,u,\,u\rangle\rangle &= \langle\langle(\Delta'' + [\bar\partial,\,\bar\tau^\star] + [\bar\tau,\,\bar\partial^\star])\,u,\,u\rangle\rangle + \langle\langle[\bar\tau,\,\bar\tau^\star]\,u,\,u\rangle\rangle - \langle\langle[\bar\partial\omega\wedge\cdot,\,(\bar\partial\omega\wedge\cdot)^\star]\,u,\,u\rangle\rangle\\ &= \langle\langle[\bar\tau,\,\bar\tau^\star]\,u,\,u\rangle\rangle - \langle\langle[\bar\partial\omega\wedge\cdot,\,(\bar\partial\omega\wedge\cdot)^\star]\,u,\,u\rangle\rangle,\end{aligned} \tag{4.204}$$

where we have used the SKT assumption on ω to have $\bar T_\omega$ reduced to $-[\bar\partial\omega\wedge\cdot,\,(\bar\partial\omega\wedge\cdot)^\star]$ in formula (4.203) and the argument below to infer that $\langle\langle[\bar\partial,\,\bar\tau^\star]\,u,\,u\rangle\rangle = \langle\langle[\bar\tau,\,\bar\partial^\star]\,u,\,u\rangle\rangle = 0$ from the assumption $u\in\ker\Delta'' = \ker\bar\partial\cap\ker\bar\partial^\star$:

$$\begin{aligned}\langle\langle[\bar\partial,\,\bar\tau^\star]\,u,\,u\rangle\rangle &= \langle\langle\bar\tau^\star\,u,\,\bar\partial^\star u\rangle\rangle + \langle\langle\bar\partial u,\,\bar\tau u\rangle\rangle = 0+0=0,\\ \langle\langle[\bar\tau,\,\bar\partial^\star]\,u,\,u\rangle\rangle &= \langle\langle\bar\partial^\star\,u,\,\bar\tau^\star u\rangle\rangle + \langle\langle\bar\tau u,\,\bar\partial u\rangle\rangle = 0+0=0.\end{aligned} \tag{4.205}$$

Now, $\Delta' = \Delta'_{p''} + \Delta'_{p''_\perp}$, so the assumption $u\in\ker\Delta'_{p''}$ reduces (4.204) to

$$\langle\langle\Delta'_{p''_\perp}\,u,\,u\rangle\rangle = \langle\langle[\bar\tau,\,\bar\tau^\star]\,u,\,u\rangle\rangle - \langle\langle[\bar\partial\omega\wedge\cdot,\,(\bar\partial\omega\wedge\cdot)^\star]\,u,\,u\rangle\rangle. \tag{4.206}$$

The r.h.s. of (4.206) vanishes thanks to the hypothesis (4.201), so $\Delta'_{p''_\perp}\,u = 0$, hence also $\Delta' u = 0$. $\square$

Remark 4.6.31. *The proof of* (*ii*) *of the above Theorem 4.6.30 shows that if X carries an SKT metric ω whose torsion satisfies the condition $[\tau,\,\tau^\star] = [\partial\omega\wedge\cdot,\,(\partial\omega\wedge\cdot)^\star]$, then the Frölicher spectral sequence of X degenerates at E_1.*

Proof. To get E_1 degeneration, it suffices for the inclusion $\mathcal{H}^{p,\,q}_{\Delta''}(X,\,\mathbb{C})\subset\mathcal{H}^{p+q}_{\Delta}(X,\,\mathbb{C})$ of Δ''-, resp. Δ-harmonic spaces to hold for all p,q. (The argument is analogous to the one for (a) of Lemma 4.6.25.) Now, (4.204) holds for all $u\in\mathcal{H}^{p,\,q}_{\Delta''}(X,\,\mathbb{C})$ if ω is SKT, hence $\Delta' u = 0$ whenever $\Delta'' u = 0$ under the present assumptions. Then, by (4.188), we get $\Delta u = 0$ for all $(p,\,q)$-forms u satisfying $\Delta'' u = 0$ and for all p,q. This proves the above inclusion, hence the contention. $\square$

Alternative expression for the torsion operator $\bar R_\omega$

We shall now compute the operator $\bar R_\omega := [\bar\tau,\,\bar\tau^\star] - [\bar\partial\omega\wedge\cdot,\,(\bar\partial\omega\wedge\cdot)^\star]$ featuring in (ii) of Theorem 4.6.30 in terms of the non-negative operator $\bar S_\omega$ (cf. (4.182)).

Lemma 4.6.32. *Let $(X,\,\omega)$ be an arbitrary compact Hermitian manifold of dimension n. Put $\bar S_\omega := [\bar\partial\omega\wedge\cdot,\,(\bar\partial\omega\wedge\cdot)^\star]\geq 0$. The following formula holds:*

$$[\bar\tau,\,\bar\tau^\star] - \bar S_\omega = 2\bar S_\omega + [[\Lambda,\,\bar S_\omega],\,L], \tag{4.207}$$

where, as usual, $L = L_\omega := \omega\wedge\cdot$. Moreover, for any bidegree $(p,\,q)$, $[[\Lambda,\,\bar S_\omega],\,L]$ is given by

$$\begin{aligned}\langle\langle[[\Lambda,\,\bar S_\omega],\,L]\,u,\,u\rangle\rangle = \langle\langle\bar S_\omega(\omega\wedge u),\,\omega\wedge u\rangle\rangle &+ \langle\langle\bar S_\omega(\Lambda\,u),\,\Lambda\,u\rangle\rangle + (p+q-n)\,\langle\langle\bar S_\omega u,\,u\rangle\rangle\\ &- 2Re\,\langle\langle\Lambda(\bar S_\omega u),\,\Lambda\,u\rangle\rangle,\qquad u\in C^\infty_{p,\,q}(X,\,\mathbb{C})\end{aligned} \tag{4.208}$$

Proof. Since $\tau = [\Lambda,\,\partial\omega\wedge\cdot]$, we get

$$[\bar\tau,\,\bar\tau^\star] = \bigg[[\Lambda,\,\bar\partial\omega\wedge\cdot],\,[(\bar\partial\omega\wedge\cdot)^\star,\,L]\bigg] = \bigg[[[(\bar\partial\omega\wedge\cdot)^\star,\,L],\,\Lambda],\,\bar\partial\omega\wedge\cdot\bigg] - \bigg[\bar\partial\omega\wedge\cdot,\,[(\bar\partial\omega\wedge\cdot)^\star,\,L]],\,\Lambda\bigg]. \tag{4.209}$$

where the last identity followed from Jacobi's identity applied to the operators $[(\bar\partial\omega\wedge\cdot)^\star,\, L]$, Λ and $\bar\partial\omega\wedge\cdot$.

To compute the first factor in the first term on the r.h.s. of (4.209), we apply again Jacobi's identity:

$$\bigg[[(\bar\partial\omega\wedge\cdot)^\star,\, L],\, \Lambda\bigg] = -\bigg[[L,\, \Lambda],\, (\bar\partial\omega\wedge\cdot)^\star\bigg] - \bigg[[\Lambda,\, (\bar\partial\omega\wedge\cdot)^\star],\, L\bigg]. \tag{4.210}$$

Using the standard fact that $[L,\, \Lambda] = (p+q-n)\,\mathrm{Id}$ on $(p,\, q)$-forms, for any $(p,\, q)$-form u we get

$$\begin{aligned}\bigg[[L,\, \Lambda],\, (\bar\partial\omega\wedge\cdot)^\star\bigg]\,u &= [L,\, \Lambda]\bigg((\bar\partial\omega\wedge\cdot)^\star u\bigg) - (\bar\partial\omega\wedge\cdot)^\star\bigg([L,\, \Lambda],\, u\bigg)\\ &= (p+q-3-n)\,(\bar\partial\omega\wedge\cdot)^\star u - (\bar\partial\omega\wedge\cdot)^\star((p+q-n)\,u) = -3(\bar\partial\omega\wedge\cdot)^\star u.\end{aligned}$$

Thus $\bigg[[L,\, \Lambda],\, (\bar\partial\omega\wedge\cdot)^\star\bigg] = -3(\bar\partial\omega\wedge\cdot)^\star$. On the other hand, $[\Lambda,\, (\bar\partial\omega\wedge\cdot)^\star] = [\bar\partial\omega\wedge\cdot,\, L]^\star = 0$ since, clearly, $[\bar\partial\omega\wedge\cdot,\, L]\,u = \bar\partial\omega\wedge\omega\wedge u - \omega\wedge\bar\partial\omega\wedge u = 0$ for any u. Therefore, (4.210) reduces to

$$\bigg[[(\bar\partial\omega\wedge\cdot)^\star,\, L],\, \Lambda\bigg] = 3(\bar\partial\omega\wedge\cdot)^\star. \tag{4.211}$$

Similarly, to compute the first factor in the second term on the r.h.s. of (4.209), we start by applying Jacobi's identity:

$$\begin{aligned}\bigg[\bar\partial\omega\wedge\cdot,\, [(\bar\partial\omega\wedge\cdot)^\star,\, L]\bigg] &= \bigg[(\bar\partial\omega\wedge\cdot)^\star,\, [L,\, \bar\partial\omega\wedge\cdot]\bigg] - \bigg[L,\, [\bar\partial\omega\wedge\cdot,\, (\bar\partial\omega\wedge\cdot)^\star]\bigg]\\ &= \bigg[[\bar\partial\omega\wedge\cdot,\, (\bar\partial\omega\wedge\cdot)^\star],\, L\bigg] = [\bar S_\omega,\, L],\end{aligned} \tag{4.212}$$

where the last but one identity followed from $[L,\, \bar\partial\omega\wedge\cdot] = 0$ seen above.

Putting together (4.209), (4.211) and (4.212), we get:

$$[\bar\tau,\, \bar\tau^\star] = 3\bar S_\omega - [[\bar S_\omega,\, L],\, \Lambda]. \tag{4.213}$$

A new application of Jacobi's identity spells

$$[[\bar S_\omega,\, L],\, \Lambda] + [[L,\, \Lambda],\, \bar S_\omega] + [[\Lambda,\, \bar S_\omega],\, L] = 0, \quad \text{which gives} \quad -[[\bar S_\omega,\, L],\, \Lambda] = [[\Lambda,\, \bar S_\omega],\, L]. \tag{4.214}$$

Indeed, since $[L,\, \Lambda] = (p+q-n)\,\mathrm{Id}$ on $(p,\, q)$-forms and $\bar S_\omega$ is an operator of type $(0,\, 0)$, we get $[[L,\, \Lambda],\, \bar S_\omega] = 0$ which accounts for the last statement in (4.214).

It is now clear that the combined (4.213) and (4.214) prove (4.207).

To prove (4.208), we start by computing

$$\begin{aligned}\langle\langle[[\Lambda,\, \bar S_\omega],\, L]\,u,\, u\rangle\rangle = \langle\langle[\Lambda,\, \bar S_\omega]\,(\omega\wedge u),\, u\rangle\rangle \; &- \; \langle\langle\omega\wedge[\Lambda,\, \bar S_\omega]\,u,\, u\rangle\rangle\\ = \langle\langle\bar S_\omega(\omega\wedge u),\, \omega\wedge u\rangle\rangle \; &- \; \langle\langle\omega\wedge u,\, \omega\wedge\bar S_\omega u\rangle\rangle + \langle\langle\bar S_\omega(\Lambda\,u),\, \Lambda\,u\rangle\rangle\\ &- \; \langle\langle\Lambda(\bar S_\omega u),\, \Lambda\,u\rangle\rangle.\end{aligned} \tag{4.215}$$

Then we notice the general fact that for every $(p,\,q)$-forms u, v we have:

$$\langle\langle\omega\wedge u,\,\omega\wedge v\rangle\rangle = \langle\langle\Lambda\,u,\,\Lambda\,v\rangle\rangle - (p+q-n)\,\langle\langle u,\,v\rangle\rangle. \tag{4.216}$$

Indeed, $\langle\langle\omega\wedge u,\,\omega\wedge v\rangle\rangle = \langle\langle\Lambda(\omega\wedge u),\,v\rangle\rangle$ and $\Lambda(\omega\wedge u) = \omega\wedge\Lambda\,u - (p+q-n)\,u$. Now, applying (4.216), we get

$$\langle\langle\omega\wedge u,\,\omega\wedge\bar{S}_\omega u\rangle\rangle = \langle\langle\Lambda\,u,\,\Lambda(\bar{S}_\omega u)\rangle\rangle - (p+q-n)\,\langle\langle u,\,\bar{S}_\omega u\rangle\rangle. \tag{4.217}$$

It is now clear that the combined (4.215) and (4.217) prove (4.208) because $\langle\langle\Lambda\,u,\,\Lambda(\bar{S}_\omega u)\rangle\rangle$ is the conjugate of $\langle\langle\Lambda(\bar{S}_\omega u),\,\Lambda\,u\rangle\rangle$ and $\langle\langle u,\,\bar{S}_\omega u\rangle\rangle = \langle\langle\bar{S}_\omega u,\,u\rangle\rangle$. $\square$

Putting the hypothesis $\partial p'' = p''\partial$ in context

We now reinterpret the commutation of ∂ with p'' (the simplest sufficient condition for $E_2(X) = E_\infty(X)$ found so far, cf. Theorem 4.6.30).

Lemma 4.6.33. *Let $(X,\,\omega)$ be a compact Hermitian manifold. The following implication and equivalence hold:*

$$\partial\Delta'' = \Delta''\partial \implies \partial p'' = p''\partial \iff \partial(\ker\Delta'')\subset\ker\Delta'' \text{ and } \partial^\star(\ker\Delta'')\subset\ker\Delta''. \tag{4.218}$$

Proof. Suppose that $\partial\Delta'' = \Delta''\partial$. Then, taking adjoints, we also have $\Delta''\partial^\star = \partial^\star\Delta''$. These identities immediately imply

$$\partial(\ker\Delta'')\subset\ker\Delta'' \text{ and } \partial^\star(\ker\Delta'')\subset\ker\Delta''. \tag{4.219}$$

Now suppose that (4.219) holds. We shall prove that $\partial p'' = p''\partial$. Let u be an arbitrary smooth form. Then u splits as $u = u_0 + \bar\partial v + \bar\partial^\star w$ with $u_0\in\ker\Delta''$. Thus $\partial p'' u = \partial u_0$ and

$$p''\partial u = p''\partial u_0 + p''\partial\bar\partial v + p''\partial\bar\partial^\star w = \partial u_0 + p''\partial\bar\partial^\star w$$

because $\partial u_0\in\ker\Delta''$ by (4.219) and $p''\partial\bar\partial v = -p''\bar\partial\partial v = 0$ since $\operatorname{Im}\bar\partial\perp\ker\Delta''$. We now prove that $p''\partial\bar\partial^\star w = 0$ and this will show that $\partial p'' u = p''\partial u$, as desired. Proving that $p''\partial\bar\partial^\star w = 0$ is equivalent to proving that $\partial\bar\partial^\star w\in(\ker\Delta'')^\perp$. Let $\zeta\in\ker\Delta''$, arbitrary. We have

$$\langle\langle\zeta,\,\partial\bar\partial^\star w\rangle\rangle = \langle\langle\partial^\star\zeta,\,\bar\partial^\star w\rangle\rangle = 0$$

because $\partial^\star\zeta\in\ker\Delta''$ thanks to (4.219), $\bar\partial^\star w\in\operatorname{Im}\bar\partial^\star$ and $\ker\Delta''\perp\operatorname{Im}\bar\partial^\star$.

It remains to prove that if $\partial p'' = p''\partial$, then (4.219) holds. Note the general fact that for any form u, $u\in\ker\Delta''$ iff $p''u = u$. Let us now suppose that $\partial p'' = p''\partial$. Then, taking adjoints, we also have $\partial^\star p'' = p''\partial^\star$, so (4.219) holds. $\square$

4.6.6 A generalised volume invariant for Aeppli cohomology classes of Hermitian-symplectic metrics

The material in this subsection is taken from [DP20]. Suppose X is a compact Hermitian-symplectic manifold. We investigate Question 4.6.9 by introducing a functional F on the open convex subset $\mathcal{S}_{\{\omega_0\}}\subset\{\omega_0\}_A\cap C^\infty_{1,1}(X,\,\mathbb{R})$ of all the Hermitian-symplectic metrics ω lying in the Aeppli cohomology class $\{\omega_0\}_A\in H^{1,1}_A(X,\,\mathbb{R})$ of a given Hermitian-symplectic metric ω_0. We then go on to show that, when $\dim_{\mathbb{C}} X = 3$, the critical points of this functional, if any, are precisely the Kähler metrics in $\{\omega_0\}_A$. We go on to exhibit these critical points as maximisers of the volume of the metric in its Aeppli class and propose a Monge-Ampère-type equation to study their existence. Our functional is further utilised to define a numerical invariant for any Aeppli cohomology class of Hermitian-symplectic metrics that generalises the volume of a Kähler class. We obtain two cohomological interpretations of this invariant. Meanwhile, we construct an invariant in the form of an E_2-cohomology class, that we call the E_2-torsion class, associated with every Aeppli class of Hermitian-symplectic metrics and show that its vanishing is a necessary condition for the existence of a Kähler metric in the given Hermitian-symplectic Aeppli class.

- **The energy functional: case of H-S metrics on compact complex manifolds**

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ such that X admits *Hermitian-symplectic* metrics.. Recall that these are C^∞ positive definite $(1,1)$-forms $\omega > 0$ for which there exists $\rho^{2,0} \in C^\infty_{2,0}(X, \mathbb{C})$ such that

$$d(\rho^{2,0} + \omega + \rho^{0,2}) = 0, \tag{4.220}$$

where $\rho^{0,2} := \overline{\rho^{2,0}}$. Alternatively, we say that $\widetilde{\omega} := \rho^{2,0} + \omega + \rho^{0,2}$ is a Hermitian-symplectic 2-form.

Lemma and Definition 4.6.34. *For every Hermitian-symplectic metric ω on X, there exists a unique smooth $(2,0)$-form $\rho^{2,0}_\omega$ on X such that*

$$(i)\ \partial\rho^{2,0}_\omega = 0 \quad \text{and} \quad (ii)\ \bar\partial\rho^{2,0}_\omega = -\partial\omega \quad \text{and} \quad (iii)\ \rho^{2,0}_\omega \in Im\,\partial^\star_\omega + Im\,\bar\partial^\star_\omega. \tag{4.221}$$

Moreover, property (iii) ensures that $\rho^{2,0}_\omega$ has **minimal L^2_ω norm** *among all the $(2,0)$-forms satisfying properties (i) and (ii).*

*We call $\rho^{2,0}_\omega$ the $(2,0)$-***torsion form** *and its conjugate $\rho^{0,2}_\omega$ the $(0,2)$-***torsion form** *of the Hermitian-symplectic metric ω. One has the explicit* **Neumann-type formula***:*

$$\rho^{2,0}_\omega = -\Delta^{-1}_{BC}[\bar\partial^\star\partial\omega + \bar\partial^\star\partial\partial^\star\partial\omega], \tag{4.222}$$

where Δ^{-1}_{BC} is the Green operator of the Bott-Chern Laplacian Δ_{BC} induced by ω, while $\partial^\star = \partial^\star_\omega$ and $\bar\partial^\star = \bar\partial^\star_\omega$ are the formal adjoints of ∂, resp. $\bar\partial$, w.r.t. the L^2 inner product defined by ω.

Proof. Condition (4.220) is equivalent to the vanishing of each of the components of pure types $(3,0)$, $(2,1)$, $(1,2)$ and $(0,3)$ of the real 3-form $d(\rho^{2,0} + \omega + \rho^{0,2})$. Since the $(3,0)$- and $(2,1)$-components are the conjugates of the $(0,3)$- and resp. $(1,2)$-components, these vanishings are equivalent to conditions (i) and (ii) of (4.221) being satisfied by $\rho^{2,0}$ in place of $\rho^{2,0}_\omega$.

Now, the forms $\rho^{2,0}$ satisfying equations (i) and (ii) of (4.221) are unique modulo $\ker\partial \cap \ker\bar\partial$. On the other hand, considering the 3-space decomposition (1.10) of $C^\infty_{2,0}(X,\mathbb{C})$ induced by the Bott-Chern Laplacian $\Delta_{BC} : C^\infty_{2,0}(X, \mathbb{C}) \to C^\infty_{2,0}(X, \mathbb{C})$ associated with the metric ω, we see that the form $\rho^{2,0}$ with minimal L^2_ω norm satisfying equations (i) and (ii) of (4.221) is the unique such form lying in the orthogonal complement of $\ker\partial \cap \ker\bar\partial = \ker\Delta_{BC} \oplus \mathrm{Im}\,\partial\bar\partial$ in $C^\infty_{2,0}(X,\mathbb{C})$, which is $\mathrm{Im}\,\partial^\star_\omega + \mathrm{Im}\,\bar\partial^\star_\omega$.

For the proof of formula (4.222), see Lemma 4.6.35 below with $v = -\partial\omega$. □

The following result gives a Neumann-type formula for the minimal L^2_ω-norm solution of a $\bar\partial$-equation with an extra constraint.

Lemma 4.6.35. *Let $(X,\,\omega)$ be a compact Hermitian manifold. For every $p,q=0,\dots,n=dim_{\mathbb{C}}X$ and every form $v\in C^\infty_{p,\,q}(X,\,\mathbb{C})$, consider the following $\bar\partial$-equation problem:*

$$\bar\partial u = v \quad \textit{subject to the condition} \quad \partial u = 0. \tag{4.223}$$

If problem (4.223) is solvable for u, the solution of minimal L^2_ω-norm is given by the Neumann-type formula:

$$u = \Delta_{BC}^{-1}[\bar\partial^\star v + \bar\partial^\star\partial\partial^\star v]. \tag{4.224}$$

Proof. The solution u of problem (4.223) is unique up to $\ker\partial\cap\ker\bar\partial = \ker\Delta_{BC}\oplus\operatorname{Im}\partial\bar\partial$. Thanks to (1.10), the minimal L^2_ω-norm solution of problem (4.223) is uniquely determined by the condition $u\in\operatorname{Im}\partial^\star+\operatorname{Im}\bar\partial^\star$. In other words, there exist forms ξ and η such that

$$u=\partial^\star\xi+\bar\partial^\star\eta, \quad \text{hence} \quad \partial^\star u = -\bar\partial^\star\partial^\star\eta, \quad \bar\partial^\star u = -\partial^\star\bar\partial^\star\xi \quad \text{and} \quad (\partial\bar\partial)^\star u = 0.$$

Applying Δ_{BC}, we get

$$\Delta_{BC}u = \bar\partial^\star(\bar\partial u) + \bar\partial^\star\partial\partial^\star(\bar\partial u),$$

since the first, third (after writing $\partial\bar\partial=-\bar\partial\partial$) and sixth (after writing $(\partial^\star\bar\partial)^\star=\bar\partial^\star\partial$) terms in Δ_{BC} end with ∂ and $\partial u=0$, while the fourth term in Δ_{BC} ends with $(\partial\bar\partial)^\star$ and $(\partial\bar\partial)^\star u=0$.

Now, the restriction of Δ_{BC} to the orthogonal complement of $\ker\Delta_{BC}$ is an isomorphism onto this same orthogonal complement, so using the inverse Δ_{BC}^{-1} of this restriction (= the Green operator of Δ_{BC}), we get

$$u = \Delta_{BC}^{-1}[\partial^\star(\partial u) + \partial^\star\bar\partial\bar\partial^\star(\partial u)],$$

since both u and $\partial^\star(\partial u)+\partial^\star\bar\partial\bar\partial^\star(\partial u)$ lie in $(\ker\Delta_{BC})^\perp$.

Since $\partial u = v$, the last formula for u is precisely (4.224). $\square$

Going back to the (2, 0)-torsion form, we notice a simplification in dimension 3.

Observation 4.6.36. *When $dim_{\mathbb{C}}X=3$, formula (4.222) for the $(2,\,0)$-torsion form $\rho^{2,\,0}_\omega$ of any Hermitian-symplectic metric ω simplifies to*

$$\rho^{2,\,0}_\omega = -\Delta''^{-1}\bar\partial^\star(\partial\omega), \tag{4.225}$$

where $\Delta''^{-1}=\Delta''^{-1}_\omega$ is the Green operator of the $\bar\partial$-Laplacian $\Delta''=\Delta''_\omega:=\bar\partial\bar\partial^\star+\bar\partial^\star\bar\partial$ induced by ω via $\bar\partial^\star=\bar\partial^\star_\omega$.

Proof. It is a standard and easily-verified fact that on any compact complex n-dimensional manifold, any $\bar\partial$-closed $(n-1,\,0)$-form is ∂-closed. Now, the $(2,\,0)$-form $\rho^{2,\,0}$ satisfying $\partial\rho^{2,\,0}=0$ and $\bar\partial\rho^{2,\,0}=-\partial\omega$ (cf. (4.221)) is unique up to the addition of an arbitrary $(2,\,0)$-form $\zeta\in\ker\partial\cap\ker\bar\partial$. When $n=3$, $n-1=2$, so $\ker\partial\cap\ker\bar\partial=\ker\bar\partial$ in bidegree $(2,\,0)$. Therefore, $\rho^{2,\,0}_\omega\in\operatorname{Im}\bar\partial^\star_\omega$, i.e. $\rho^{2,\,0}_\omega=\bar\partial^\star\xi$ for some $(2,\,1)$-form ξ. We get $\Delta''\rho^{2,\,0}_\omega=\bar\partial^\star\bar\partial(\bar\partial^\star\xi)=-\bar\partial^\star(\partial\omega)$. This is equivalent to (4.225). $\square$

If ω_0 is a Hermitian-symplectic metric on X, any C^∞ positive definite $(1,\,1)$-form ω lying in the Aeppli cohomology class of ω_0 is a Hermitian-symplectic metric. Indeed, by (i) and (ii) of (4.221), $\partial\omega_0=-\bar\partial\rho_0^{2,\,0}$ for some ∂-closed $(2,\,0)$-form $\rho_0^{2,\,0}$ on X. Meanwhile, $\omega=\omega_0+\partial\bar u+\bar\partial u$ for some $(1,\,0)$-form u, so $\partial\omega=\partial\omega_0+\partial\bar\partial u=-\bar\partial(\rho_0^{2,\,0}+\partial u)$. Meanwhile, $\rho_0^{2,\,0}+\partial u$ is ∂-closed since $\rho_0^{2,\,0}$

is. Therefore, ω is Hermitian-symplectic (cf. (i) and (ii) of (4.221) which characterise the H-S property).

By a *Hermitian-symplectic (H-S) Aeppli class* $\{\omega\}_A \in H^{1,1}_A(X, \mathbb{R})$ we shall mean a real Aeppli cohomology class of bidegree $(1, 1)$ that contains an H-S metric ω. The set of all H-S classes

$$\mathcal{HS}_X := \left\{\{\omega\}_A \in H^{1,1}_A(X, \mathbb{R}) \mid \omega \text{ is an H-S metric on } X\right\} \subset H^{1,1}_A(X, \mathbb{R})$$

is an open convex cone. Moreover, for every Hermitian-symplectic Aeppli class $\{\omega\}_A$, we denote by

$$\mathcal{S}_{\{\omega\}} := \left\{\omega + \partial\bar{u} + \bar\partial u \mid u \in C^\infty_{1,0}(X, \mathbb{C}) \text{ such that } \omega + \partial\bar{u} + \bar\partial u > 0\right\} \subset \{\omega\}_A \cap C^\infty_{1,1}(X, \mathbb{R})$$

the set of all (necessarily H-S) metrics in $\{\omega\}_A$. It is an *open convex subset* of the real affine space $\{\omega\}_A \cap C^\infty_{1,1}(X, \mathbb{R}) = \{\omega + \partial\bar{u} + \bar\partial u \mid u \in C^\infty_{1,0}(X, \mathbb{C})\}$.

Definition 4.6.37. *Let X be a compact complex Hermitian-symplectic manifold with $\dim_{\mathbb{C}} X = n$. For the Aeppli cohomology class $\{\omega_0\}_A \in \mathcal{HS}_X$ of any Hermitian-symplectic metric ω_0, we define the following* **energy functional***:*

$$F : \mathcal{S}_{\{\omega_0\}} \to [0, +\infty), \qquad F(\omega) = \int\limits_X |\rho^{2,0}_\omega|^2_\omega \, dV_\omega = ||\rho^{2,0}_\omega||^2_\omega, \tag{4.226}$$

where $\rho^{2,0}_\omega$ is the $(2, 0)$-torsion form of the Hermitian-symplectic metric $\omega \in \mathcal{S}_{\{\omega_0\}}$ defined in Lemma and Definition 4.6.34, while $|\;|_\omega$ is the pointwise norm and $||\;||_\omega$ is the L^2 norm induced by ω.

The first trivial observation that justifies the introduction of the functional F is the following.

Lemma 4.6.38. *Let $\{\omega_0\}_A \in \mathcal{S}_X$ and $\omega \in \mathcal{S}_{\{\omega_0\}}$. Then, the following equivalence holds:*

$$\omega \text{ is Kähler} \iff F(\omega) = 0. \tag{4.227}$$

Proof. If ω is Kähler, $\partial\omega = 0$ and the minimal L^2-norm solution of the equation $\bar\partial\rho = 0$ vanishes. Thus $\rho^{2,0}_\omega = 0$, hence $F(\omega) = 0$. Conversely, if $F(\omega) = 0$, then $\rho^{2,0}_\omega$ vanishes identically on X, hence $\partial\omega = -\bar\partial\rho^{2,0}_\omega = 0$, so ω is Kähler. □

We now compute the critical points of the energy functional F.

Note that definition (4.248) of F translates to

$$F(\omega) = \int\limits_X \rho^{2,0}_\omega \wedge \star\overline{\rho^{2,0}_\omega} = \int\limits_X \rho^{2,0}_\omega \wedge \rho^{0,2}_\omega \wedge \frac{\omega^{n-2}}{(n-2)!}. \tag{4.228}$$

Indeed, $\overline{\rho^{2,0}_\omega} = \rho^{0,2}_\omega$ is primitive since it is of bidegree $(0, 2)$, so $\star\overline{\rho^{2,0}_\omega} = \overline{\rho^{2,0}_\omega} \wedge \omega^{n-2}/(n-2)!$ by (4.68).

We now fix a Hermitian-symplectic metric ω on X and we vary it in its Aeppli class along the path $\omega + t\gamma$, where $\gamma = \partial\bar{u} + \bar\partial u \in C^\infty_{1,1}(X, \mathbb{R})$ is a fixed real $(1, 1)$-form chosen to be Aeppli cohomologous to zero. Recall that the $(2, 0)$-torsion form $\rho^{2,0}_\omega$ satisfies the condition $\bar\partial\rho^{2,0}_\omega = -\partial\omega$ and has minimal L^2_ω-norm with this property. We get

$$\bar\partial(\rho^{2,0}_\omega + t\partial u) = -\partial(\omega + t\gamma), \tag{4.229}$$

although $\rho_\omega^{2,\,0} + t\partial u$ need not be of minimal $L^2_{\omega+t\gamma}$-norm with this property. For every $t\in\mathbb{R}$ close to 0, we define the new functional:

$$\begin{aligned}\widetilde{F}_t(\omega) \;&:=\; \int\limits_X |\rho_\omega^{2,\,0} + t\partial u|^2_{\omega+t\gamma}\,\frac{(\omega+t\gamma)^n}{n!} = \int\limits_X (\rho_\omega^{2,\,0} + t\partial u)\wedge\star_{\omega+t\gamma}\,(\overline{\rho_\omega^{2,\,0}} + t\,\bar\partial\bar{u})\\ &= \int\limits_X (\rho_\omega^{2,\,0} + t\partial u)\wedge(\overline{\rho_\omega^{2,\,0}} + t\bar\partial\bar{u})\wedge\frac{(\omega+t\gamma)^{n-2}}{(n-2)!}.\end{aligned} \tag{4.230}$$

The properties of $\widetilde{F}_t$ are summed up in the following statement.

Proposition 4.6.39. (i) *The two energy functionals are related by the inequality:*

$$\widetilde{F}_t(\omega) \geq F(\omega + t\gamma) \quad \textit{for all } t\in\mathbb{R} \textit{ close to } 0. \tag{4.231}$$

(ii) *The differential at ω of F is given by the formula:*

$$\begin{aligned}(d_\omega F)(\gamma) = \frac{d}{dt}_{|t=0}\widetilde{F}_t(\omega) \;&=\; -2\,Re\,\langle\langle u,\,\bar\partial^\star\omega\rangle\rangle_\omega + 2\,Re\int\limits_X u\wedge\rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\bar\partial\bigg(\frac{\omega^{n-3}}{(n-3)!}\bigg)\\ &=\; -\langle\langle\gamma\,,\omega\rangle\rangle + 2\,Re\int\limits_X u\wedge\rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\bar\partial\bigg(\frac{\omega^{n-3}}{(n-3)!}\bigg),\end{aligned} \tag{4.232}$$

for every $(1,\,1)$*-form* $\gamma = \partial\bar{u} + \bar\partial u$.

Proof. (i) If t is sufficiently close to 0, $\omega + t\gamma > 0$, hence $\omega + t\gamma$ is a Hermitian-symplectic metric. By (ii) in Lemma and Definition 4.6.34, we have $\bar\partial\rho^{2,\,0}_{\omega+t\gamma} = -\partial(\omega + t\gamma)$ and $\rho^{2,\,0}_{\omega+t\gamma}$ has minimal $L^2_{\omega+t\gamma}$-norm with this property. Since $\rho_\omega^{2,\,0} + t\partial u$ solves the same equation as $\rho^{2,\,0}_{\omega+t\gamma}$ (cf. (4.229)), we conclude that

$$\widetilde{F}_t(\omega) \geq \int\limits_X |\rho^{2,\,0}_{\omega+t\gamma}|^2_{\omega+t\gamma}\,\frac{(\omega+t\gamma)^n}{n!} = F(\omega+t\gamma), \quad t\in\mathbb{R} \text{ close to } 0.$$

(ii) Since $\widetilde{F}_t(\omega) - F(\omega+t\gamma) \geq 0$ for all $t\in\mathbb{R}$ close to 0 and since $\widetilde{F}_0(\omega) = F(\omega)$, the smooth function $t\mapsto \widetilde{F}_t(\omega) - F(\omega+t\gamma)$ achieves a minimum at $t=0$. Hence, its derivative vanishes at $t=0$. We get:

$$\frac{d}{dt}_{|t=0}\widetilde{F}_t(\omega) = \frac{d}{dt}_{|t=0}F(\omega+t\gamma) = (d_\omega F)(\gamma),$$

which is precisely the first identity in (4.232).

We now prove the second identity in (4.232) starting from (4.230). For all $t\in\mathbb{R}$ close to 0, we get:

$$\begin{aligned}\frac{d}{dt}_{|t=0}\widetilde{F}_t(\omega) \;&=\; \frac{d}{dt}_{|t=0}\int\limits_X (\rho_\omega^{2,\,0} + t\partial u)\wedge(\overline{\rho_\omega^{2,\,0}} + t\bar\partial\bar{u})\wedge\frac{(\omega+t\gamma)^{n-2}}{(n-2)!}\\ &= \int\limits_X \partial u\wedge\overline{\rho_\omega^{2,\,0}}\wedge\frac{\omega^{n-2}}{(n-2)!} + \int\limits_X \bar\partial\bar{u}\wedge\rho_\omega^{2,\,0}\wedge\frac{\omega^{n-2}}{(n-2)!}\\ &\quad + \int\limits_X \rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\frac{\omega^{n-3}}{(n-3)!}\wedge(\partial\bar{u} + \bar\partial u).\end{aligned}$$

Applying Stokes's theorem in each integral to remove the derivatives from u and $\bar{u}$, we get:

$$\begin{aligned}\frac{d}{dt}_{|t=0}\widetilde{F}_t(\omega) &= \int\limits_X u\wedge\partial\overline{\rho_\omega^{2,\,0}}\wedge\frac{\omega^{n-2}}{(n-2)!} + \int\limits_X u\wedge\overline{\rho_\omega^{2,\,0}}\wedge\partial\bigg(\frac{\omega^{n-2}}{(n-2)!}\bigg)\\ &+ \int\limits_X \bar{u}\wedge\bar\partial\rho_\omega^{2,\,0}\wedge\frac{\omega^{n-2}}{(n-2)!} + \int\limits_X \bar{u}\wedge\rho_\omega^{2,\,0}\wedge\bar\partial\bigg(\frac{\omega^{n-2}}{(n-2)!}\bigg)\\ &+ \int\limits_X \bar{u}\wedge\partial\rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\frac{\omega^{n-3}}{(n-3)!} + \int\limits_X \bar{u}\wedge\rho_\omega^{2,\,0}\wedge\partial\overline{\rho_\omega^{2,\,0}}\wedge\frac{\omega^{n-3}}{(n-3)!}\\ &+ \int\limits_X \bar{u}\wedge\rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\partial\bigg(\frac{\omega^{n-3}}{(n-3)!}\bigg) + \int\limits_X u\wedge\rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\bar\partial\bigg(\frac{\omega^{n-3}}{(n-3)!}\bigg)\\ &+ \int\limits_X u\wedge\bar\partial\rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\frac{\omega^{n-3}}{(n-3)!} + \int\limits_X u\wedge\rho_\omega^{2,\,0}\wedge\bar\partial\overline{\rho_\omega^{2,\,0}}\wedge\frac{\omega^{n-3}}{(n-3)!}.\end{aligned}$$

Grouping the terms on the r.h.s. according to whether the integrands are divisible by u or by $\bar{u}$ and using the identities $\partial\overline{\rho_\omega^{2,\,0}} = -\bar\partial\omega$ and $\bar\partial\rho_\omega^{2,\,0} = -\partial\omega$, we get:

$$\begin{aligned}\frac{d}{dt}_{|t=0}\widetilde{F}_t(\omega) &= -\int\limits_X u\wedge\bigg[\bar\partial\omega\wedge\frac{\omega^{n-2}}{(n-2)!} + \bigg(-\overline{\rho_\omega^{2,\,0}}\wedge\partial\frac{\omega^{n-2}}{(n-2)!} + \partial\omega\wedge\overline{\rho_\omega^{2,\,0}}\wedge\frac{\omega^{n-3}}{(n-3)!}\bigg)\bigg]\\ &+ \int\limits_X u\wedge\bigg[\rho_\omega^{2,\,0}\wedge\bar\partial\overline{\rho_\omega^{2,\,0}}\wedge\frac{\omega^{n-3}}{(n-3)!} + \rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\bar\partial\frac{\omega^{n-3}}{(n-3)!}\bigg]\\ &- \int\limits_X \bar{u}\wedge\bigg[\partial\omega\wedge\frac{\omega^{n-2}}{(n-2)!} + \bigg(-\rho_\omega^{2,\,0}\wedge\bar\partial\frac{\omega^{n-2}}{(n-2)!} + \bar\partial\omega\wedge\rho_\omega^{2,\,0}\wedge\frac{\omega^{n-3}}{(n-3)!}\bigg)\bigg]\\ &+ \int\limits_X \bar{u}\wedge\bigg[\overline{\rho_\omega^{2,\,0}}\wedge\partial\rho_\omega^{2,\,0}\wedge\frac{\omega^{n-3}}{(n-3)!} + \rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\partial\frac{\omega^{n-3}}{(n-3)!}\bigg].\end{aligned}$$

Now, the terms on the first two lines on the r.h.s. above are respectively conjugated to the terms on the third and fourth lines, while the two inner large paratheses on lines 1 and 3 vanish since $\partial\omega^{n-2}/(n-2)! = \partial\omega\wedge\omega^{n-3}/(n-3)!$. On the other hand, we recall that $\partial\rho_\omega^{2,\,0} = 0$, hence also $\bar\partial\overline{\rho_\omega^{2,\,0}} = 0$. Thus, the two integrals containing these factors on the r.h.s. above vanish. We are reduced to

$$\begin{aligned}\frac{d}{dt}_{|t=0}\widetilde{F}_t(\omega) &= -\int\limits_X u\wedge\bigg[\bar\partial\frac{\omega^{n-1}}{(n-1)!} - \rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\bar\partial\frac{\omega^{n-3}}{(n-3)!}\bigg]\\ &- \int\limits_X \bar{u}\wedge\bigg[\partial\frac{\omega^{n-1}}{(n-1)!} - \rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\partial\frac{\omega^{n-3}}{(n-3)!}\bigg],\end{aligned} \tag{4.233}$$

or equivalently, to

$$\frac{d}{dt}_{|t=0}\widetilde{F}_t(\omega) = -2\,\mathrm{Re}\int\limits_X u\wedge\bar\partial\bigg(\frac{\omega^{n-1}}{(n-1)!}\bigg) + 2\,\mathrm{Re}\int\limits_X u\wedge\rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\bar\partial\bigg(\frac{\omega^{n-3}}{(n-3)!}\bigg). \tag{4.234}$$

Now, from $\star_\omega\omega = \omega^{n-1}/(n-1)!$ and $\partial^\star = -\star\bar\partial\star$, we get:

$$\bar\partial\bigg(\frac{\omega^{n-1}}{(n-1)!}\bigg) = \bar\partial\star\omega = \star(-\star\bar\partial\star)\,\omega = \star\partial^\star\omega = \star\,\overline{\bar\partial^\star\omega},$$

hence

$$u\wedge\bar\partial\bigg(\frac{\omega^{n-1}}{(n-1)!}\bigg) = \langle u,\,\bar\partial^\star\omega\rangle_\omega\,dV_\omega. \tag{4.235}$$

Thus, (4.234) and (4.235) prove the second identity in (4.232). The third identity in (4.232) is obvious. □

Corollary 4.6.40. *Suppose $n=3$. Then a Hermitian-symplectic metric ω on a compact complex manifold X of dimension 3 is a* **critical point** *of the energy functional F* **if and only if** *ω is* **Kähler***.*

Proof. It is obvious that every Kähler metric ω is a critical point for F since $\partial\omega=0$, hence $\rho_\omega^{2,\,0}=0$.

If $n=3$, $\bar\partial\omega^{n-3}=0$, so (4.232) reduces to $(d_\omega F)(\gamma) = -2\,\mathrm{Re}\,\langle\langle u,\,\bar\partial^\star\omega\rangle\rangle_\omega$.

Now, a metric ω is a critical point of F if and only if $(d_\omega F)(\gamma)=0$ for every $\gamma=\partial\bar u+\bar\partial u$. By the above discussion, this amounts to $\mathrm{Re}\,\langle\langle u,\,\bar\partial^\star\omega\rangle\rangle_\omega=0$ for every $(1,\,0)$-form u. Thus, if ω is a critical point of F, by taking $u=\bar\partial^\star\omega$ we get $\bar\partial^\star\omega=0$. This is equivalent to ω being balanced. However, ω is already SKT since it is Hermitian-symplectic, so ω must be Kähler by Proposition 4.6.11. □

Corollary 4.6.41. *Let X be a compact complex manifold of dimension $n=3$ admitting Hermitian-symplectic metrics. Then, for every Aeppli-cohomologous Hermitian-symplectic metrics ω and ω_η:*

$$\omega_\eta = \omega+\partial\bar\eta+\bar\partial\eta>0, \qquad \textit{with } \eta\in C^\infty_{1,\,0}(X,\,\mathbb{C}), \tag{4.236}$$

the respective $(2,\,0)$-torsion forms $\rho_\omega^{2,\,0}$ and $\rho_\eta^{2,\,0}:=\rho_{\omega_\eta}^{2,\,0}$ satisfy the identity:

$$||\rho_\eta^{2,\,0}||^2_{\omega_\eta} + \int\limits_X\frac{\omega_\eta^3}{3!} = ||\rho_\omega^{2,\,0}||^2_\omega + \int\limits_X\frac{\omega^3}{3!} \tag{4.237}$$

and are related by

$$\rho_\eta^{2,\,0} = \rho_\omega^{2,\,0}+\partial\eta. \tag{4.238}$$

In particular, if $\partial\eta=0$ (a condition that is equivalent to $\omega_\eta-\omega$ being d-exact), we are reduced to $\rho_\eta^{2,\,0}=\rho_\omega^{2,\,0}$ and

$$||\rho_\omega^{2,\,0}||^2_{\omega_\eta} = ||\rho_\omega^{2,\,0}||^2_\omega + \int\limits_X\rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge(\omega_\eta-\omega). \tag{4.239}$$

Proof. In arbitrary dimension n, we compute the differential of the map

$$\mathcal{S}_{\{\omega_0\}}\ni\omega\mapsto\int\limits_X\frac{\omega^n}{n!} := \mathrm{Vol}_\omega(X)$$

when the metric ω varies in its Aeppli cohomology class $\{\omega_0\}_A$. For any real, Aeppli null-cohomologous $(1,\,1)$-form $\gamma=\partial\bar{u}+\bar\partial u$ (with $u\in C^\infty_{1,\,0}(X,\,\mathbb{C})$), we have

$$\begin{aligned}\frac{d}{dt}_{|t=0}\int\limits_X\frac{(\omega+t\gamma)^n}{n!} &= \frac{1}{(n-1)!}\int\limits_X\omega^{n-1}\wedge\gamma = 2\,\mathrm{Re}\int\limits_X\bar\partial u\wedge\frac{\omega^{n-1}}{(n-1)!} = 2\,\mathrm{Re}\int\limits_X u\wedge\bar\partial\star\omega\\ &= 2\,\mathrm{Re}\int\limits_X u\wedge\star\Big(-\star\bar\partial\star\omega\Big) = 2\,\mathrm{Re}\int\limits_X u\wedge\star\partial^\star\omega = 2\,\mathrm{Re}\int\limits_X u\wedge\star\overline{\bar\partial^\star\omega}\\ &= 2\,\mathrm{Re}\,\langle\langle u,\,\bar\partial^\star\omega\rangle\rangle.\end{aligned}$$

Together with (4.232) (recall that $n=3$ here), this identity shows that the differential at ω of the map

$$\mathcal{S}_{\{\omega_0\}}\ni\omega\mapsto||\rho_\omega^{2,\,0}||^2_\omega+\int\limits_X\frac{\omega^3}{3!}$$

vanishes identically. Therefore, this map is constant on the Hermitian-symplectic metrics lying in a same Aeppli cohomology class $\{\omega_0\}_A$. This proves (4.237).

To prove (4.238), recall that definition (4.221) of the $(2,\,0)$-torsion forms implies the following relations:

$$(i)\,\bar\partial(\rho_\eta^{2,\,0}-\partial\eta)=-\partial\omega\quad\text{and}\quad(ii)\,||\rho_\eta^{2,\,0}-\partial\eta||_\omega\geq||\rho_\omega^{2,\,0}||_\omega,\tag{4.240}$$

where (ii) follows from (i) and from the L^2_ω-norm minimality of $\rho_\omega^{2,\,0}$ among the $(2,\,0)$-forms ρ solving the equation $\bar\partial\rho=-\partial\omega$.

Now, (4.237) gives the first of the following identities:

$$||\rho_\omega^{2,\,0}||^2_\omega+\int\limits_X\frac{\omega^3}{3!} = ||\rho_\eta^{2,\,0}||^2_{\omega_\eta}+\int\limits_X\frac{\omega_\eta^3}{3!}=\int\limits_X\rho_\eta^{2,\,0}\wedge\overline{\rho_\eta^{2,\,0}}\wedge\omega_\eta+\int\limits_X\frac{\omega_\eta^3}{3!}.\tag{4.241}$$

On the other hand, we have:

$$\begin{aligned}(\omega_\eta+\rho_\eta^{2,\,0}+\overline{\rho_\eta^{2,\,0}})^3 &= \omega_\eta^3+3\,\omega_\eta^2\wedge(\rho_\eta^{2,\,0}+\overline{\rho_\eta^{2,\,0}})+3\,\omega_\eta\wedge(\rho_\eta^{2,\,0}+\overline{\rho_\eta^{2,\,0}})^2+(\rho_\eta^{2,\,0}+\overline{\rho_\eta^{2,\,0}})^3\\ &= \omega_\eta^3+6\,\omega_\eta\wedge\rho_\eta^{2,\,0}\wedge\overline{\rho_\eta^{2,\,0}},\end{aligned}\tag{4.242}$$

where the last identity follows from the cancellation of several terms for bidegree reasons. Putting (4.241) and (4.242) together, we get:

$$\begin{aligned}||\rho_\omega^{2,\,0}||^2_\omega+\frac{1}{3!}\int\limits_X\omega^3 &= \frac{1}{3!}\int\limits_X(\omega_\eta+\rho_\eta^{2,\,0}+\overline{\rho_\eta^{2,\,0}})^3=\frac{1}{3!}\int\limits_X[\omega+(\rho_\eta^{2,\,0}-\partial\eta)+(\overline{\rho_\eta^{2,\,0}}-\bar\partial\bar\eta)+d(\eta+\bar\eta)]^3\\ &\stackrel{(a)}{=} \frac{1}{3!}\int\limits_X[\omega+(\rho_\eta^{2,\,0}-\partial\eta)+(\overline{\rho_\eta^{2,\,0}}-\bar\partial\bar\eta)]^3\stackrel{(b)}{=}\frac{1}{3!}\int\limits_X\omega^3+\int\limits_X(\rho_\eta^{2,\,0}-\partial\eta)\wedge(\overline{\rho_\eta^{2,\,0}}-\bar\partial\eta)\wedge\omega\\ &= ||\rho_\eta^{2,\,0}-\partial\eta||^2_\omega+\frac{1}{3!}\int\limits_X\omega^3\stackrel{(c)}{\geq}||\rho_\omega^{2,\,0}||^2_\omega+\frac{1}{3!}\int\limits_X\omega^3.\end{aligned}\tag{4.243}$$

· Identity (a) followed from Stokes's theorem and the d-closedness of the form $\omega+(\rho_\eta^{2,\,0}-\partial\eta)+(\overline{\rho_\eta^{2,\,0}}-\bar\partial\bar\eta)$ that is seen through the following very simple computation:

$$\begin{aligned}d[\omega+(\rho_\eta^{2,\,0}-\partial\eta)+(\overline{\rho_\eta^{2,\,0}}-\bar\partial\bar\eta)] &= \partial\omega+\bar\partial\omega+\bar\partial\rho_\eta^{2,\,0}-\bar\partial\partial\eta+\partial\overline{\rho_\eta^{2,\,0}}-\partial\bar\partial\bar\eta\\ &= \partial\omega+\bar\partial\omega-\partial(\omega+\partial\bar\eta+\bar\partial\eta)-\bar\partial\partial\eta-\bar\partial(\omega+\bar\partial\eta+\partial\bar\eta)-\partial\bar\partial\bar\eta\\ &= -(\partial\bar\partial\eta+\bar\partial\partial\eta)-(\bar\partial\partial\bar\eta+\partial\bar\partial\bar\eta)=0,\end{aligned}$$

where the second identity followed from $\bar\partial\rho_\eta^{2,\,0} = -\partial\omega_\eta = -\partial(\omega + \bar\partial\eta + \partial\bar\eta)$ and from the conjugated expression.

· Identity (b) in (4.243) followed from the analogue of (4.242) in this context, while inequality (c) in (4.243) followed from part (ii) of (4.240).

We see that the first and the last terms in (4.243) are equal. This forces (c) to be an equality, hence part (ii) of (4.240) must be an equality. This means that $\rho_\eta^{2,\,0} - \partial\eta$ and $\rho_\omega^{2,\,0}$ are both solutions of the equation $\bar\partial\rho^{2,\,0} = -\partial\omega$ (see part (i) of (4.240) and part (ii) of (4.221)) and have *equal* L^2_ω-norms. Since $\rho_\omega^{2,\,0}$ is the minimal L^2_ω-norm solution, we infer that $\rho_\eta^{2,\,0} - \partial\eta = \rho_\omega^{2,\,0}$ by the uniqueness of the minimal L^2_ω-norm solution. This proves (4.238).

Finally, we write $\omega_\eta = \omega + (\omega_\eta - \omega)$ and

$$||\rho_\omega^{2,\,0}||^2_{\omega_\eta} = \int\limits_X \rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\omega_\eta = ||\rho_\omega^{2,\,0}||^2_\omega + \int\limits_X \rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge(\omega_\eta - \omega).$$

This is (4.239). □

The main takeaway from Corollary 4.6.41 is that the sum $F(\omega) + \mathrm{Vol}_\omega(X)$ (where $\mathrm{Vol}_\omega(X) := \int_X \omega^3/3!$) remains **constant** when ω ranges over the (necessarily Hermitian-symplectic) metrics in the Aeppli cohomology class of a fixed Hermitian-symplectic metric ω_0. This invariant attached to any Aeppli class of Hermitian-symplectic metrics generalises the classical volume of a Kähler class and constitutes one of our main findings in this work.

Definition 4.6.42. *Let X be a 3-dimensional compact complex manifold supposed to carry Hermitian-symplectic metrics. For any such metric ω on X, the constant*

$$A = A_{\{\omega\}_A} := F(\omega) + Vol_\omega(X) > 0 \tag{4.244}$$

depending only on $\{\omega\}_A$ is called the **generalised volume** *of the Hermitian-symplectic Aeppli class $\{\omega\}_A$.*

• The energy functional: case of SKT metrics on compact $\partial\bar\partial$-manifolds

We now discuss an analogous functional in a special case.

Lemma 4.6.43. *For every SKT metric ω on a compact $\partial\bar\partial$-manifold X, there exists a unique smooth $(2,\,0)$-form Γ_ω on X such that*

$$(i)\ \bar\partial\Gamma_\omega = -\partial\omega \quad and \quad (ii)\ \Gamma_\omega \in Im\,\bar\partial^\star_\omega, \tag{4.245}$$

where the subscript ω indicates that the formal adjoint is computed w.r.t. the L^2 inner product defined by ω.

The form Γ_ω will be called the $(2,\,0)$-**torsion form** *of the SKT metric ω. It is given by the von Neumann-type formula:*

$$\Gamma_\omega = -\Delta''^{-1}\bar\partial^\star(\partial\omega), \tag{4.246}$$

where Δ''^{-1} is the Green operator of the Laplacian $\Delta'' = \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial$ induced by the metric ω.

Proof. The $(2,\,1)$-form $\partial\omega$ is d-closed (thanks to the SKT assumption on ω) and ∂-exact, hence by the $\partial\bar\partial$-assumption on X it is also $\bar\partial$-exact. This means that the equation $\bar\partial\Gamma = -\partial\omega$ is solvable. Its solutions Γ are unique up to the addition of any element in $\ker\bar\partial$, so the minimal L^2_ω-norm solution is the unique solution lying in the orthogonal complement of $\ker\bar\partial$, which is $\mathrm{Im}\,\bar\partial^\star_\omega$. The von Neumann formula is well known and can be easily proved: $\bar\partial(-\Delta''^{-1}\bar\partial^\star(\partial\omega)) = -\partial\omega$ (immediate verification) and $\Delta''^{-1}\bar\partial^\star(\partial\omega) = \bar\partial^\star\Delta''^{-1}(\partial\omega) \in \mathrm{Im}\,\bar\partial^\star$. □

The following is a very simple observation.

Lemma 4.6.44. *The* $(2,\,0)$*-torsion form* Γ_ω *of any SKT metric* ω *on a compact* $\partial\bar\partial$*-manifold* X *has the property:*

$$\partial\Gamma_\omega = 0. \tag{4.247}$$

Proof. The $(3,\,0)$-form $\partial\Gamma_\omega$ is ∂-exact (obviously) and d-closed (since $\bar\partial(\partial\Gamma_\omega) = -\partial(\bar\partial\Gamma_\omega) = \partial^2\omega = 0$), hence it must be $\partial\bar\partial$-exact thanks to the $\partial\bar\partial$ assumption on X. This means that there exists a $(2,\,-1)$-form ζ (which must vanish for bidegree reasons) such that $\partial\bar\partial\zeta = \partial\Gamma_\omega$. Then $\partial\Gamma_\omega$ vanishes since $\zeta = 0$. □

We now define a new energy functional by the L^2-norm of the $(2,\,0)$-torsion form Γ_ω.

Definition 4.6.45. *Let* X *be a compact SKT* $\partial\bar\partial$*-manifold with* $\dim_{\mathbb{C}}X = n$*. For every Aeppli cohomology class* $\{\omega_0\}_A$ *representable by an SKT metric, we define the following* **energy functional***:*

$$F : \mathcal{S}_{\{\omega_0\}} \to [0,\,+\infty), \qquad F(\omega) = \int\limits_X |\Gamma_\omega|^2_\omega\,dV_\omega = ||\Gamma_\omega||^2_\omega, \tag{4.248}$$

where Γ_ω *is the* $(2,\,0)$*-torsion form of the SKT metric* $\omega \in \mathcal{S}_{\{\omega_0\}}$ *defined in Lemma 4.6.43.*

The remaining arguments are identical to those given in the H-S case if we replace $\rho^{2,\,0}_\omega$ with Γ_ω. Recall that by (4.247) we have $\partial\Gamma_\omega = 0$ (cf. (i) of (4.221)).

The first variation of F can be computed as in the H-S case discussed above. We get

Proposition 4.6.46. *The differential of* F *at any SKT metric* ω *is given by the formula:*

$$\begin{aligned}(d_\omega F)(\gamma) &= -2\,Re\,\langle\langle u,\,\bar\partial^\star\omega\rangle\rangle_\omega + 2\,Re\int\limits_X u\wedge\Gamma_\omega\wedge\overline{\Gamma}_\omega\wedge\bar\partial\bigg(\frac{\omega^{n-3}}{(n-3)!}\bigg) \\ &= -\langle\langle\gamma\,,\omega\rangle\rangle + 2\,Re\int\limits_X u\wedge\Gamma_\omega\wedge\overline{\Gamma}_\omega\wedge\bar\partial\bigg(\frac{\omega^{n-3}}{(n-3)!}\bigg)\end{aligned} \tag{4.249}$$

for every $(1,\,1)$*-form* $\gamma = \partial\bar u + \bar\partial u$.

In particular, **if** $n=3$*, an SKT metric* ω *on* X *is a* **critical point** *of the energy functional* F **if and only if** ω *is* **Kähler***.*

• Variation of the $(2,\,0)$-torsion form for $\partial\bar\partial$-cohomologous metrics

We first show that the $(2,\,0)$-torsion form of a Hermitian-symplectic metric does not change when the metric changes only by an element in $\mathrm{Im}\,\partial\bar\partial$. The next statement can be compared with Corollary 4.6.41: it supposes more and achieves more.

Proposition 4.6.47. *Let* X *be a compact complex manifold with* $\dim_{\mathbb{C}}X = 3$*. Suppose that* $\omega > 0$ *and* $\widetilde\omega = \omega + i\partial\bar\partial\varphi > 0$ *are* **SKT** *metrics on* X*.*

(i) For every form $\rho^{2,0}\in C^\infty_{2,0}(X,\,\mathbb{C})$ such that $\partial\rho^{2,0}=0$ and $\bar\partial\rho^{2,0}=-\partial\omega$, the L^2-norms of $\rho^{2,0}$ w.r.t. $\widetilde\omega$ and ω are related in the following way:

$$||\rho^{2,0}||^2_{\widetilde\omega}=||\rho^{2,0}||^2_\omega-\frac{1}{2}\int\limits_X(\widetilde\omega-\omega)\wedge\omega^2. \tag{4.250}$$

This relation is equivalent to

$$||\rho^{2,0}||^2_{\widetilde\omega}+\int\limits_X\frac{\widetilde\omega^3}{3!}=||\rho^{2,0}||^2_\omega+\int\limits_X\frac{\omega^3}{3!}. \tag{4.251}$$

(ii) If $\omega>0$ and $\widetilde\omega=\omega+i\partial\bar\partial\varphi>0$ are **Hermitian-symplectic** *metrics, their $(2,\,0)$-torsion forms coincide, i.e.*

$$\rho^{2,0}_{\widetilde\omega}=\rho^{2,0}_\omega. \tag{4.252}$$

Proof. (i) From the assumptions, we get the following identities:

$$\begin{aligned}||\rho^{2,0}||^2_{\widetilde\omega} &= \int\limits_X\rho^{2,0}\wedge\overline{\rho^{2,0}}\wedge\widetilde\omega=\int\limits_X\rho^{2,0}\wedge\overline{\rho^{2,0}}\wedge\omega+\int\limits_X\rho^{2,0}\wedge\overline{\rho^{2,0}}\wedge i\partial\bar\partial\varphi\\ &\overset{(a)}{=} ||\rho^{2,0}||^2_\omega-i\int\limits_X\rho^{2,0}\wedge\partial\overline{\rho^{2,0}}\wedge\bar\partial\varphi\\ &\overset{(b)}{=} ||\rho^{2,0}||^2_\omega-i\int\limits_X\varphi\,\bar\partial\rho^{2,0}\wedge\partial\overline{\rho^{2,0}}\overset{(c)}{=}||\rho^{2,0}||^2_\omega-i\int\limits_X\varphi\,\partial\omega\wedge\bar\partial\omega,\end{aligned} \tag{4.253}$$

where (a) and (b) follow from Stokes combined with the identity $\partial\rho^{2,0}=0$ and its conjugate $\bar\partial\overline{\rho^{2,0}}=0$, while (c) follows from the identity $\bar\partial\rho^{2,0}=-\partial\omega$ and its conjugate $\partial\overline{\rho^{2,0}}=-\bar\partial\omega$.

Now, the SKT property of ω implies that $\partial\omega\wedge\bar\partial\omega=\partial(\omega\wedge\bar\partial\omega)=(1/2)\,\partial\bar\partial\omega^2$, so two further applications of Stokes yield the second identity below:

$$i\int\limits_X\varphi\,\partial\omega\wedge\bar\partial\omega=\frac{i}{2}\int\limits_X\varphi\,\partial\bar\partial\omega^2=\frac{1}{2}\int\limits_X i\partial\bar\partial\varphi\wedge\omega^2=\frac{1}{2}\int\limits_X(\widetilde\omega-\omega)\wedge\omega^2. \tag{4.254}$$

We now see that (4.253) and (4.254) prove (4.250) between them.

To prove the equivalence of (4.251) and (4.250), we have to show that

$$(1/6)\int_X(\widetilde\omega^3-\omega^3)=(1/2)\int_X(\widetilde\omega-\omega)\wedge\omega^2.$$

Now, since $\widetilde\omega^2=\omega^2+2\,i\partial\bar\partial\varphi\wedge\omega+(i\partial\bar\partial\varphi)^2$, we get:

$$\begin{aligned}\frac{1}{6}\int\limits_X(\widetilde\omega^3-\omega^3) &= \frac{1}{6}\int\limits_X(\widetilde\omega-\omega)\wedge(\widetilde\omega^2+\widetilde\omega\wedge\omega+\omega^2)\\ &= \frac{1}{6}\int\limits_X(\widetilde\omega-\omega)\wedge\omega^2+\frac{1}{3}\int\limits_X(\widetilde\omega-\omega)\wedge i\partial\bar\partial\varphi\wedge\omega+\frac{1}{6}\int\limits_X(\widetilde\omega-\omega)\wedge(i\partial\bar\partial\varphi)^2\\ &+ \frac{1}{6}\int\limits_X(\widetilde\omega-\omega)\wedge\omega^2+\frac{1}{6}\int\limits_X(\widetilde\omega-\omega)\wedge i\partial\bar\partial\varphi\wedge\omega+\frac{1}{6}\int\limits_X(\widetilde\omega-\omega)\wedge\omega^2\\ &= 3\cdot\frac{1}{6}\int\limits_X(\widetilde\omega-\omega)\wedge\omega^2,\end{aligned}$$

since all the other terms vanish by Stokes, the identities $\partial(\widetilde{\omega}-\omega)=0$ and $\bar\partial(\widetilde{\omega}-\omega)=0$ and the SKT assumption on ω.

(ii) The stronger H-S assumption on $\widetilde{\omega}$ and ω is only made to ensure the existence of the (2, 0)-torsion forms $\rho^{2,\,0}_{\widetilde{\omega}}$ and $\rho^{2,\,0}_{\omega}$. The assumption $\widetilde{\omega}=\omega+i\partial\bar\partial\varphi$ implies $\partial\widetilde{\omega}=\partial\omega$, so $\rho^{2,\,0}_{\widetilde{\omega}}$ and $\rho^{2,\,0}_{\omega}$ are the minimal $L^2_{\widetilde{\omega}}$-norm solution, resp. the minimal L^2_{ω}-norm solution, of the same equation $\bar\partial\rho^{2,\,0}=-\partial\omega$.

However, (4.250) shows that when $\rho^{2,\,0}$ ranges over the set of smooth (2, 0)-forms $\rho^{2,\,0}$ satisfying the conditions $\partial\rho^{2,\,0}=0$ and $\bar\partial\rho^{2,\,0}=-\partial\omega$ (both of which are satisfied by both $\rho^{2,\,0}_{\widetilde{\omega}}$ and $\rho^{2,\,0}_{\omega}$), $||\rho^{2,\,0}||_{\widetilde{\omega}}$ is minimal if and only if $||\rho^{2,\,0}||_{\omega}$ is minimal since the discrepancy term

$$-\frac{1}{2}\int\limits_X(\widetilde{\omega}-\omega)\wedge\omega^2$$

is independent of $\rho^{2,\,0}$ (depending only on the given metrics $\widetilde{\omega}$ and ω). This means that the same $\rho^{2,\,0}$ achieves the minimal L^2 norm w.r.t. either of the metrics $\widetilde{\omega}$ and ω. By uniqueness of the minimal L^2-norm solution of the $\bar\partial$ equation, we get $\rho^{2,\,0}_{\widetilde{\omega}}=\rho^{2,\,0}_{\omega}$. $\square$

- **Volume form and Monge-Ampère-type equation associated with an H-S metric**

We now digress briefly to point out another possible future use of the new invariant defined by the generalised volume. In fact, a new volume form that seems better suited to featuring in the right-hand term of complex Monge-Ampère equations can be associated with every Hermitian-symplectic metric on a 3-dimensional compact complex manifold.

Definition 4.6.48. *If ω is a Hermitian-symplectic metric on a compact complex manifold X with $dim_{\mathbb{C}}X=3$ and $\rho^{2,\,0}_{\omega}$ is the $(2,\,0)$-torsion form of ω, we define the following volume form on X:*

$$d\widetilde{V}_\omega:=(1+|\rho^{2,\,0}_{\omega}|^2_{\omega})\,dV_\omega.$$

The main interest in this volume form stems from the fact that its volume is independent of the choice of metric in a given Hermitian-symplectic Aeppli class, as follows from Corollary 4.6.41:

$$\int\limits_X d\widetilde{V}_{\omega_1}=\int\limits_X d\widetilde{V}_{\omega_2}=A, \qquad \text{for all metrics } \omega_1,\omega_2\in\{\omega\}_A,$$

where $A=A_{\{\omega\}_A}>0$ is the *generalised volume* of the H-S Aeppli class $\{\omega\}_A$.

Now, if ω is a Hermitian-symplectic metric on a manifold X as above, it seems natural to consider the Monge-Ampère equation

$$\frac{(\omega+i\partial\bar\partial\varphi)^3}{3!}=b\,d\widetilde{V}_\omega,$$

subject to the condition $\omega+i\partial\bar\partial\varphi>0$, where $b>0$ is a given constant. By [TW10, Corollary 1], there exists a unique b such that this equation is solvable. Moreover, for that b, the solution $\omega+i\partial\bar\partial\varphi>0$ is unique. Note that

$$b=\frac{\text{Vol}_{\omega+i\partial\bar\partial\varphi}(X)}{A_{\{\omega\}_A}}\in(0,\,1]$$

since $A_{\{\omega\}_A}=F(\omega+i\partial\bar\partial\varphi)+\text{Vol}_{\omega+i\partial\bar\partial\varphi}(X)\geq\text{Vol}_{\omega+i\partial\bar\partial\varphi}(X)$. We hope that this can shed some light on the mysterious constant b in this context.

- **The E_2-torsion class**

We now point out an obstruction to the Aeppli cohomology class of a given Hermitian-symplectic metric containing a Kähler metric.

Lemma and Definition 4.6.49. *Suppose that ω is a* **Hermitian-symplectic metric** *on a compact complex n-dimensional manifold X.*

(i) The $(0,\,2)$-torsion form $\rho_\omega^{0,\,2}\in C^\infty_{0,\,2}(X,\,\mathbb{C})$ of ω represents an E_2-cohomology class $\{\rho_\omega^{0,\,2}\}_{E_2}\in E_2^{0,\,2}(X)$. Moreover, $\{\rho_\omega^{0,\,2}\}_{E_2}\in\ker(d_2 : E_2^{0,\,2}(X)\to E_2^{2,\,1}(X))$.

(ii) Suppose that $n=3$. Then, the class $\{\rho_\omega^{0,\,2}\}_{E_2}\in E_2^{0,\,2}(X)$ is constant when the Hermitian-symplectic metric ω varies in a fixed Aeppli cohomology class.

The class $\{\rho_\omega^{0,\,2}\}_{E_2}\in E_2^{0,\,2}(X)$ will be called the **E_2-torsion class** *of the Hermitian-symplectic Aeppli class $\{\omega\}_A$.*

Proof. (i) By construction, the $(0,\,2)$-torsion form $\rho_\omega^{0,\,2}$ has the properties:

$$\bar\partial\rho_\omega^{0,\,2}=0\quad\text{ and }\quad\partial\rho_\omega^{0,\,2}\in\operatorname{Im}\bar\partial\quad(\text{since }\partial\rho_\omega^{0,\,2}=-\bar\partial\omega),$$

which translate precisely to $\rho_\omega^{0,\,2}$ being E_2-closed (see terminology in [Pop19, Proposition 3.1]), namely to $\rho_\omega^{0,\,2}$ representing an E_2-cohomology class.

Moreover, the class $d_2(\{\rho_\omega^{0,\,2}\}_{E_2})\in E_2^{2,\,1}(X)$ is represented by $-\partial\omega$ since $-\omega$ is such that $\bar\partial(-\omega)=\partial\rho_\omega^{0,\,2}$. (See Definition 1.2.9 and Theorem 1.2.10.) However, $\partial\omega$ is $\bar\partial$-exact, so, in particular, $\{\partial\omega\}_{E_2}=0$. We get

$$d_2(\{\rho_\omega^{0,\,2}\}_{E_2})=-\{\partial\omega\}_{E_2}=0\in E_2^{2,\,1}(X).$$

(ii) When $n=3$, Corollary 4.6.41 tells us that the respective $(2,\,0)$-torsion forms $\rho_\omega^{2,\,0}$ and $\rho_\eta^{2,\,0}:=\rho_{\omega_\eta}^{2,\,0}$ of any two Aeppli-cohomologous Hermitian-symplectic metrics ω and $\omega_\eta=\omega+\partial\bar\eta+\bar\partial\eta>0$, with $\eta\in C^\infty_{1,\,0}(X,\,\mathbb{C})$, are related by $\rho_\eta^{2,\,0}=\rho_\omega^{2,\,0}+\partial\eta$. Hence, for their conjugates, we get:

$$\rho_\eta^{0,\,2}=\rho_\omega^{0,\,2}+\bar\partial\bar\eta,\qquad\text{so}\qquad\{\rho_\eta^{0,\,2}\}_{\bar\partial}=\{\rho_\omega^{0,\,2}\}_{\bar\partial},\qquad\text{hence also}\qquad\{\rho_\eta^{0,\,2}\}_{E_2}=\{\rho_\omega^{0,\,2}\}_{E_2}\in E_2^{0,\,2}(X).$$

□

Since ω is Kähler if and only if $\rho_\omega^{0,\,2}=0$, we get the following necessary condition for a given Hermitian-symplectic Aeppli class $\{\omega\}_A$ to contain a Kähler metric.

Corollary 4.6.50. *Suppose that $n=3$. If a given Hermitian-symplectic Aeppli class $\{\omega\}_A$ contains a Kähler metric, then its E_2-torsion class $\{\rho_\omega^{0,\,2}\}_{E_2}\in E_2^{0,\,2}(X)$ vanishes.*

Moreover, the condition $\{\rho_\omega^{0,\,2}\}_{E_2}=0$ in $E_2^{0,\,2}(X)$ is equivalent to $\rho_\omega^{0,\,2}\in Im\,\bar\partial$ for some (hence every) metric ω lying in $\{\omega\}_A$.

Proof. Only the latter statement still needs a proof. The E_2-exactness condition on $\rho_\omega^{0,\,2}$ is equivalent to the existence of a $(0,\,1)$-form ξ and of a $(-1,\,2)$-form ζ such that $\rho_\omega^{0,\,2}=\partial\zeta+\bar\partial\xi$ and $\bar\partial\zeta=0$. (See Definition 1.2.9.) However, for bidegree reasons, every $(-1,\,2)$-form ζ is trivially the zero form, so the E_2-exactness condition on $(0,\,2)$-forms is equivalent to the $\bar\partial$-exactness condition. □

Therefore, we are led to restricting attention to Hermitian-symplectic Aeppli classes of vanishing torsion class on 3-dimensional manifolds. In this case, $\rho_\omega^{0,\,2}$ is $\bar\partial$-exact and we let

$$\xi_\omega^{0,\,1}=\Delta''^{-1}\bar\partial^\star\rho_\omega^{0,\,2}\in\operatorname{Im}\bar\partial^\star\subset C^\infty_{0,\,1}(X,\,\mathbb{C})\tag{4.255}$$

be the minimal L^2_ω-norm solution of the equation $\bar\partial\xi=\rho_\omega^{0,\,2}$. Our functional $F:\mathcal{S}_{\{\omega_0\}}\to[0,\,+\infty)$ of Definition 4.6.37 takes the form:

$$F(\omega)=\int\limits_X|\rho_\omega^{2,\,0}|^2_\omega\,dV_\omega=\int\limits_X\rho_\omega^{2,\,0}\wedge\rho_\omega^{0,\,2}\wedge\omega=\int\limits_X\partial\xi_\omega^{1,\,0}\wedge\bar\partial\xi_\omega^{0,\,1}\wedge\omega,\tag{4.256}$$

where $\xi_\omega^{1,\,0}$ is the conjugate of $\xi_\omega^{0,\,1}$.

- **Approach via a Monge-Ampère-type equation**

Ideally, if a Monge-Ampère-type equation with solutions in a given H-S **Aeppli class** could be solved, its solutions would be Kähler metrics. Specifically, we get the following result.

Proposition 4.6.51. *Let X be a compact complex Hermitian-symplectic manifold with $\dim_{\mathbb{C}}X = 3$. Fix an arbitrary Hermitian metric γ and an H-S metric ω on X. Let $A = A_{\{\omega\}_A} > 0$ be the generalised volume of the class $\{\omega\}_A$ and let $c = c_{\omega,\,\gamma} > 0$ be the constant defined by the requirement*

$$\frac{(\int\limits_X \omega\wedge\gamma^2/2!)^3}{(\int\limits_X \gamma^3/3!)^2} = \frac{6A}{c}.$$

If there exists a solution $\eta\in C^\infty_{1,\,0}(X,\,\mathbb{C})$ of the Monge-Ampère-type equation

$$(\omega + \partial\bar\eta + \bar\partial\eta)^3 = c\,(\Lambda_\gamma\omega)^3\,\frac{\gamma^3}{3!} \qquad\qquad (\star)$$

such that $\omega_\eta := \omega + \partial\bar\eta + \bar\partial\eta > 0$, then ω_η is a **Kähler metric** *lying in the Aeppli cohomology class $\{\omega\}_A$ of ω.*

Proof. As usual, let $A = A_{\{\omega\}_A} := F(\omega) + \text{Vol}_\omega(X) > 0$ be the generalised volume of the Aeppli cohomology class $\{\omega\}_A\in H^{1,\,1}_A(X,\,\mathbb{R})$. Let $c>0$ be the constant defined by

$$\bigg(\int_X\omega\wedge\frac{\gamma^2}{2!}\bigg)^3\bigg/\bigg(\int_X\frac{\gamma^3}{3!}\bigg)^2 = 6A/c. \tag{4.257}$$

Now, for any η such that $\omega + \partial\bar\eta + \bar\partial\eta>0$, the Hölder inequality with conjugate exponents $p=3$ and $q=3/2$ gives:

$$\int\limits_X\sqrt[3]{\frac{(\omega + \partial\bar\eta + \bar\partial\eta)^3}{\gamma^3}}\,\frac{\gamma^3}{3!} \leq \bigg(\int\limits_X\frac{(\omega + \partial\bar\eta + \bar\partial\eta)^3}{3!}\bigg)^{\frac{1}{3}}\bigg(\int\limits_X\frac{\gamma^3}{3!}\bigg)^{\frac{2}{3}}. \tag{4.258}$$

On the other hand, if the form η solves equation $(\star)$, we have

$$\int\limits_X\sqrt[3]{\frac{(\omega + \partial\bar\eta + \bar\partial\eta)^3}{\gamma^3}}\,\frac{\gamma^3}{3!} = (c/3!)^{\frac{1}{3}}\int\limits_X\Lambda_\gamma\omega\,\frac{\gamma^3}{3!} = (c/3!)^{\frac{1}{3}}\int\limits_X\omega\wedge\frac{\gamma^2}{2!}. \tag{4.259}$$

Putting together (4.257), (4.258) and (4.259), we get, whenever η solves equation $(\star)$:

$$\begin{aligned}A \geq \int\limits_X\frac{(\omega + \partial\bar\eta + \bar\partial\eta)^3}{3!} \geq \frac{1}{(\text{Vol}_\gamma(X))^2}\bigg(\int\limits_X\sqrt[3]{\frac{(\omega + \partial\bar\eta + \bar\partial\eta)^3}{\gamma^3}}\,\frac{\gamma^3}{3!}\bigg)^3 & = \frac{c}{3!}\,\frac{\bigg(\int\limits_X\omega\wedge\frac{\gamma^2}{2!}\bigg)^3}{(\text{Vol}_\gamma(X))^2} \\ & = \frac{c}{3!}\,\frac{6A}{c} = A,\end{aligned}$$

where $\mathrm{Vol}_\gamma(X) := \int_X \gamma^3/3!$. This implies, thanks to (4.244), that $F(\omega_\eta) = 0$ in this case, which is equivalent to ω_η being a Kähler metric. $\square$

- **Stratification of the Aeppli class**

Unfortunately, little is known about the solvability of Monge-Ampère-type equations like ($\star$). Therefore, we now consider the special case of equation ($\star$) where the solution η is of the shape $\eta = -(i/2)\,\partial\varphi$, with $\varphi : X \to \mathbb{R}$ a C^∞ function. Equation ($\star$) becomes:

$$(\omega + i\partial\bar\partial\varphi)^3 = c\,(\Lambda_\gamma\omega)^3\,\frac{\gamma^3}{3!}, \qquad (\star\star)$$

subject to the condition $\omega + i\partial\bar\partial\varphi > 0$, where γ is an arbitrary Hermitian metric fixed on X, A is the generalised volume of $\{\omega\}_A$ defined in (4.244) and the constant $c > 0$ is defined in (4.257). The advantage is that we are now dealing with a scalar equation and the existence theory is much more developed in this set-up (c.f. [Che87], [GL09], [TW10]). The drawback is that the perturbation of ω by $i\partial\bar\partial\varphi$ is non-generic within its Aeppli class and this forces us to break $\{\omega\}_A$ into subclasses (to be defined below) and study equation ($\star\star$) in each subclass.

A conformal rescaling of γ by a C^∞ function $f : X \to (0, +\infty)$ will change the constant $c > 0$ to some constant $c_f > 0$ defined by the analogue of (4.257):

$$\frac{6A}{c_f} = \frac{\left(\int\limits_X f^2\,\omega\wedge\frac{\gamma^2}{2!}\right)^3}{\left(\int\limits_X \frac{f^3\,\gamma^3}{3!}\right)^2} = \frac{\left(\int\limits_X f^2\,(\Lambda_\gamma\omega)\,dV_\gamma\right)^3}{\left(\int\limits_X f^3\,dV_\gamma\right)^2} \leq \int\limits_X (\Lambda_\gamma\omega)^3\,dV_\gamma,$$

where the last inequality is Hölder's inequality applied to the functions f^2 and $\Lambda_\gamma\omega$ with the conjugate exponents $p = 3/2$ and $q = 3$. This translates to the following eligibility condition for c_f:

$$c_f \geq \frac{6A}{\int\limits_X (\Lambda_\gamma\omega)^3\,dV_\gamma}.$$

Now, Hölder's inequality is an equality if $f = \Lambda_\gamma\omega$. In particular, if the metric $\gamma > 0$ is chosen such that $\Lambda_\gamma\omega \equiv 1$, no conformal rescaling of γ is necessary (i.e. we can choose $f \equiv 1$) to get the *minimal* constant

$$c = \frac{6A}{\int\limits_X dV_\gamma}.$$

Definition 4.6.52. *Let ω be a fixed Hermitian metric on a compact complex manifold X. A Hermitian metric γ on X is said to be ω-***normalised** *if $\Lambda_\gamma\omega = 1$ at every point of X.*

The following observation is trivial.

Lemma 4.6.53. *Every conformal class of Hermitian metrics on X* **contains** *a* **unique** *ω-normalised representative.*

Proof. Let γ be an arbitrary Hermitian metric on X. We are looking for C^∞ functions $f : X \to (0, +\infty)$ such that $\Lambda_{f\gamma}\omega = 1$ on X. Since $\Lambda_{f\gamma}\omega = (1/f)\,\Lambda_\gamma\omega$, the only possible choice for f is $f = \Lambda_\gamma\omega$. $\square$

We saw above that if equality is achieved in Hölder's inequality (i.e. if the constant $c>0$ assumes its *minimal* value computed above) and if equation $(\star\star)$ is solvable with this minimal constant c on the right, then its solution is a Kähler metric. In other words, Proposition 4.6.51 and the above considerations lead to

Conclusion 4.6.54. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = 3$. Suppose there exists a* **Hermitian-symplectic** *metric ω on X and fix an arbitrary ω-***normalised** *Hermitian metric γ on X. Let $A>0$ be the generalised volume of $\{\omega\}_A$ defined in (4.244).*

If there exists a C^∞ solution $\varphi : X \to \mathbb{R}$ of the equation

$$\frac{(\omega + i\partial\bar\partial\varphi)^3}{3!} = A\,\frac{dV_\gamma}{\int\limits_X dV_\gamma} \qquad (\star\star)$$

such that $\omega_\varphi := \omega + i\partial\bar\partial\varphi > 0$, then ω_φ is a Kähler metric lying in the Aeppli cohomology class $\{\omega\}_A$.

(1) The strata

A Hermitian-symplectic metric ω need not be d-closed, but let us still call the affine space $\{\omega\}_{BC} := \{\omega + i\partial\bar\partial\varphi \mid \varphi \in C^\infty(X,\mathbb{R})\}$ the **Bott-Chern subclass** (or **stratum**) of ω. It is a subspace of the Aeppli class $\{\omega\}_A$ of ω. Similarly, by analogy with the open convex subset

$$\mathcal{S}_{\{\omega\}} := \{\omega' > 0 \mid \omega' \in \{\omega\}_A\} \subset \{\omega\}_A \cap C^\infty_{1,1}(X,\mathbb{R})$$

of metrics in the Aeppli class of a given H-S metric ω, we define the open convex subset

$$\mathcal{D}_{[\omega']} := \{\omega'' > 0 \mid \omega'' \in \{\omega'\}_{BC}\} \subset \{\omega'\}_{BC}$$

of metrics in the Bott-Chern subclass of a given H-S metric ω'.

If we fix a Hermitian-symplectic metric ω, we can **partition** $\mathcal{S}_{\{\omega\}}$ as

$$\mathcal{S}_{\{\omega\}} = \bigcup_{j\in J} \mathcal{D}_{[\omega_j]}, \tag{4.260}$$

where $(\omega_j)_{j\in J}$ is a system of representatives of the Bott-Chern subclasses $\mathcal{D}_{[\omega']}$ when ω' ranges over $\mathcal{S}_{\{\omega\}}$. Moreover, for every $j\in J$, let γ_j be an ω_j-normalised Hermitian metric on X and let us consider the equation:

$$\frac{(\omega_j + i\partial\bar\partial\varphi)^3}{3!} = A\,\frac{dV_{\gamma_j}}{\int\limits_X dV_{\gamma_j}} \qquad (\star\star_j)$$

such that $\omega_j + i\partial\bar\partial\varphi > 0$. (No other condition is imposed at this point on γ_j.)

By the Tosatti-Weinkove theorem [TW10, Corollary 1], there exists a unique constant $b_j>0$ such that the equation

$$\frac{(\omega_j + i\partial\bar\partial\varphi)^3}{3!} = b_j A\,\frac{dV_{\gamma_j}}{\int\limits_X dV_{\gamma_j}} \qquad (\star\star\star_j),$$

subject to the extra condition $\omega_j + i\partial\bar\partial\varphi > 0$, is solvable. Integrating and using the inequality $\int_X (\omega_j + i\partial\bar\partial\varphi)^3/3! \leq A$, which follows from (4.244), we get:

$$b_j \leq 1, \qquad j\in J.$$

From this and from Conclusion 4.6.54, we infer Proposition ?? stated in the Introduction.

The next observation is that, within Bott-Chern subclasses of Hermitian-symplectic metrics that contain a *Gauduchon* metric, the volume remains *constant* and all the metrics are *Gauduchon*. These Bott-Chern subclasses will be called *Gauduchon strata*.

Lemma 4.6.55. *Let $\dim_{\mathbb{C}} X = 3$. Suppose that a metric ω on X is both* **SKT** *and* **Gauduchon**. *Then, for every $\varphi : X \longrightarrow \mathbb{R}$, we have*

$$(a)\ \int\limits_X (\omega + i\partial\bar\partial\varphi)^3 = \int\limits_X \omega^3 \quad \textit{and} \quad (b)\ \partial\bar\partial(\omega + i\partial\bar\partial\varphi)^2 = 0.$$

Proof. (a) Straightforward calculations give:

$$\begin{aligned}\int\limits_X (\omega + i\partial\bar\partial\varphi)^3 &= \int\limits_X \omega^3 + 3\int\limits_X \omega^2\wedge i\partial\bar\partial\varphi + 3\int\limits_X \omega\wedge(i\partial\bar\partial\varphi)^2 + \int\limits_X (i\partial\bar\partial\varphi)^3 \\ &= \int\limits_X \omega^3 + 3i\int\limits_X \varphi\,\partial\bar\partial\omega^2 - 3\int\limits_X \varphi\,\partial\bar\partial\omega\wedge\partial\bar\partial\varphi + \int\limits_X \partial(i\bar\partial\varphi\wedge(i\partial\bar\partial\varphi)^2) = \int\limits_X \omega^3,\end{aligned}$$

where $\partial\bar\partial\omega = 0$ since ω is SKT, while $\partial\bar\partial\omega^2 = 0$ since ω is Gauduchon.

(b) Straightforward calculations give:

$$\partial\bar\partial(\omega + i\partial\bar\partial\varphi)^2 = \partial\bar\partial\omega^2 + 2\,\partial\bar\partial\omega\wedge(i\partial\bar\partial\varphi) + \partial\bar\partial(i\partial\bar\partial\varphi)^2 = 0,$$

since $\partial\bar\partial\omega = 0$ and $\partial\bar\partial\omega^2 = 0$ for the same reasons as in (a). □

(2) Volume comparison within a Bott-Chern stratum

We have seen that for any SKT metric ω on a 3-dimensional compact complex manifold X, we have:

$$\begin{aligned}\int\limits_X \frac{(\omega + i\partial\bar\partial\varphi)^3}{3!} &= \int\limits_X \frac{\omega^3}{3!} + \int\limits_X \omega^2\wedge\frac{i}{2}\partial\bar\partial\varphi = \int\limits_X \frac{\omega^3}{3!} + \int\limits_X \varphi\,\frac{i}{2}\partial\bar\partial\omega^2 \\ &= \int\limits_X \frac{\omega^3}{3!} + \int\limits_X \varphi\, i\partial\omega\wedge\bar\partial\omega,\end{aligned} \tag{4.261}$$

where the SKT hypothesis on ω is used only to get the last identity. Thus, to understand the variation of $\text{Vol}_{\omega_\varphi}(X) = \int_X(\omega + i\partial\bar\partial\varphi)^3/3!$ when φ ranges over the C^∞ real-valued functions on X such that $\omega + i\partial\bar\partial\varphi > 0$, the following observation will come in handy.

Lemma 4.6.56. *Let X be a 3-dimensional complex manifold and let ω be an arbitrary Hermitian metric on X. If*

$$\partial\omega = (\partial\omega)_{prim} + \alpha^{1,0}\wedge\omega$$

is the Lefschetz decomposition of $\partial\omega$ into a primitive part (w.r.t. ω) and a part divisible by ω, with $\alpha^{1,0} \in C^\infty_{1,0}(X, \mathbb{C})$, then

$$i\partial\omega\wedge\bar\partial\omega = \left(|\alpha^{1,0}\wedge\omega|^2_\omega - |(\partial\omega)_{prim}|^2_\omega\right) dV_\omega. \tag{4.262}$$

Proof. From the Lefschetz decomposition, we get:

$$i\partial\omega\wedge\bar\partial\omega = i(\partial\omega)_{prim}\wedge(\bar\partial\omega)_{prim} + i\alpha^{1,\,0}\wedge\alpha^{0,\,1}\wedge\omega^2,$$

where $\alpha^{0,\,1}=\overline{\alpha^{1,\,0}}$. This is because $(\partial\omega)_{prim}\wedge\omega=0$ and $(\bar\partial\omega)_{prim}\wedge\omega=0$. Indeed, $(\partial\omega)_{prim}$ and $(\bar\partial\omega)_{prim}$ are primitive 3-forms on a 3-dimensional complex manifold, so they lie in the kernel of $\omega\wedge\cdot$.

From the general formula (4.68), we get:

$$(\bar\partial\omega)_{prim} = i\,\star(\bar\partial\omega)_{prim} \quad \text{and} \quad \alpha^{0,\,1}\wedge\frac{\omega^2}{2!} = -i\,\star\alpha^{0,\,1},$$

where $\star$ is the Hodge star operator induced by ω. Hence,

$$\begin{aligned} i(\partial\omega)_{prim}\wedge(\bar\partial\omega)_{prim} &= -(\partial\omega)_{prim}\wedge\star(\bar\partial\omega)_{prim} = -|(\partial\omega)_{prim}|^2\,dV_\omega \\ i\alpha^{1,\,0}\wedge\alpha^{0,\,1}\wedge\omega^2 &= 2\,\alpha^{1,\,0}\wedge\star\alpha^{0,\,1} = 2\,|\alpha^{1,\,0}|^2_\omega\,dV_\omega. \end{aligned}$$

Formula (4.262) follows from these computations after further noticing that

$$\begin{aligned} |\alpha^{1,\,0}\wedge\omega|^2_\omega &= \langle\alpha^{1,\,0}\wedge\omega,\,\alpha^{1,\,0}\wedge\omega\rangle_\omega = \langle\Lambda_\omega(\alpha^{1,\,0}\wedge\omega),\,\alpha^{1,\,0}\rangle_\omega \\ &= \langle[\Lambda_\omega,\,\omega\wedge\cdot](\alpha^{1,\,0}),\,\alpha^{1,\,0}\rangle_\omega = 2\,|\alpha^{1,\,0}|^2_\omega, \end{aligned}$$

where the last identity follows from the well-known formula $[\Lambda_\omega,\,\omega\wedge\cdot]=(n-k)\,\mathrm{Id}$ on k-forms on an n-dimensional complex manifold. (In our case, $k=1$ and $n=3$.) □

Notice that, in the setting of Lemma 4.6.56, ω is *balanced* if and only if $\partial\omega=(\partial\omega)_{prim}$, while ω is *lck* (i.e. *locally conformally Kähler*) if and only if $\partial\omega=\alpha^{1,\,0}\wedge\omega$. This accounts for the terminology used in the next

Corollary 4.6.57. *Let X be a 3-dimensional compact complex manifold equipped with an* **SKT metric** *ω.*

(a) If $|\alpha^{1,\,0}\wedge\omega|_\omega\geq|(\partial\omega)_{prim}|_\omega$ at every point of X (we will say in this case that ω is **almost lck***), then ω is* **Gauduchon** *and $|\alpha^{1,\,0}\wedge\omega|_\omega=|(\partial\omega)_{prim}|_\omega$ at every point of X.*

(b) If $|(\partial\omega)_{prim}|_\omega\geq|\alpha^{1,\,0}\wedge\omega|_\omega$ at every point of X (we will say in this case that ω is **almost balanced***), then ω is* **Gauduchon** *and $|\alpha^{1,\,0}\wedge\omega|_\omega=|(\partial\omega)_{prim}|_\omega$ at every point of X.*

(c) ω is **almost lck** $\iff$ *ω is* **almost balanced** $\iff$ *ω is* **Gauduchon** $\iff$ *$|\alpha^{1,\,0}\wedge\omega|_\omega=|(\partial\omega)_{prim}|_\omega$ at every point of X.*

Proof. The SKT assumption on ω implies that $i\partial\omega\wedge\bar\partial\omega=\frac{i}{2}\,\partial\bar\partial\omega^2$. Integrating this identity and using the Stokes theorem and formula (4.262), we get:

$$\int\limits_X|\alpha^{1,\,0}\wedge\omega|^2_\omega\,dV_\omega = \int\limits_X|(\partial\omega)_{prim}|^2_\omega\,dV_\omega. \tag{4.263}$$

Therefore, if $|(\partial\omega)_{prim}|^2_\omega-|\alpha^{1,\,0}\wedge\omega|^2_\omega$ has constant sign on X, it must vanish identically. This is equivalent to $\frac{i}{2}\,\partial\bar\partial\omega^2$ vanishing identically, hence to ω being Gauduchon. □

Based on these observations, let us introduce the following

Notation 4.6.58. *For any SKT metric ω on a 3-dimensional compact complex manifold X, we put:*

$$\begin{aligned} U_\omega : &= \bigg\{x\in X \;\Big|\; |\alpha^{1,0}\wedge\omega|_\omega(x) < |(\partial\omega)_{prim}|_\omega(x)\bigg\}, \\ V_\omega : &= \bigg\{x\in X \;\Big|\; |\alpha^{1,0}\wedge\omega|_\omega(x) > |(\partial\omega)_{prim}|_\omega(x)\bigg\}, \\ Z_\omega : &= \bigg\{x\in X \;\Big|\; |\alpha^{1,0}\wedge\omega|_\omega(x) = |(\partial\omega)_{prim}|_\omega(x)\bigg\}. \end{aligned}$$

Clearly, U_ω and V_ω are open subsets of X, while Z_ω is closed. The three of them form a partition of X. Moreover, Corollary 4.6.57 ensures that ω is Gauduchon if and only if $U_\omega = V_\omega = \emptyset$. This happens if and only if either $U_\omega = \emptyset$ or $V_\omega = \emptyset$.

Returning to the variation of the volume of $\omega_\varphi := \omega + i\partial\bar\partial\varphi$, we now observe a stark contrast between the non-Gauduchon strata dealt with below and the Gauduchon ones treated in Lemma 4.6.55.

Lemma 4.6.59. *Let X be a 3-dimensional compact complex manifold. Suppose that ω is an* **SKT non-Gauduchon** *metric on X. Then, the map*

$$\bigg\{\varphi\in C^\infty(X)\,\Big|\,\omega + i\partial\bar\partial\varphi>0\bigg\}\ni\varphi\longmapsto\int\limits_X\frac{(\omega+i\partial\bar\partial\varphi)^3}{3!} := Vol_{\omega_\varphi}(X)\in(0,\,+\infty)$$

does not achieve any local extremum.

Proof. Suppose this map achieves, say, a local maximum at some metric $\omega_0 = \omega + i\partial\bar\partial\varphi_0 > 0$. Without loss of generality, we may assume that $\omega_0 = \omega$ (and $\varphi_0 \equiv 1$). Since ω is not Gauduchon, both U_ω and V_ω are not empty. Thanks to (4.261) and (4.262), the local maximality of ω translates to

$$\int\limits_X\varphi\,|\alpha^{1,0}\wedge\omega|^2_\omega\,dV_\omega \leq \int\limits_X\varphi\,|(\partial\omega)_{prim}|^2_\omega\,dV_\omega \tag{4.264}$$

for every $\varphi\in C^\infty(X,\,\mathbb{R})$ such that $\omega + i\partial\bar\partial\varphi > 0$ and φ is close enough to $\varphi_0\equiv 1$ in C^2 norm.

Now, (4.263) translates to

$$\begin{aligned}\int\limits_{U_\omega}(|\alpha^{1,0}\wedge\omega|^2_\omega - |(\partial\omega)_{prim}|^2_\omega)\,dV_\omega \;&+\; \int\limits_{V_\omega}(|\alpha^{1,0}\wedge\omega|^2_\omega - |(\partial\omega)_{prim}|^2_\omega)\,dV_\omega \\ &+\; \int\limits_{Z_\omega}(|\alpha^{1,0}\wedge\omega|^2_\omega - |(\partial\omega)_{prim}|^2_\omega)\,dV_\omega = 0. \end{aligned}\tag{4.265}$$

Thus, if we can find a $\varphi\in C^\infty(X,\,\mathbb{R})$ sufficiently close to $\varphi_0\equiv 1$ in C^2 norm (this will also imply that $\omega + i\partial\bar\partial\varphi > 0$) such that

$$\varphi\equiv 1 \text{ on } U_\omega\cup Z_\omega, \qquad \varphi\equiv 1+\varepsilon \text{ on } V_\omega'\Subset V_\omega, \qquad \text{and} \qquad 1\leq\varphi\leq 1+\varepsilon \text{ on } V_\omega\setminus V_\omega',$$

for some constant $\varepsilon > 0$, where V_ω' is a pregiven relatively compact open subset of V_ω, we will have

$$\int\limits_{V_\omega}(\varphi - 1)\,(|\alpha^{1,0}\wedge\omega|^2_\omega - |(\partial\omega)_{prim}|^2_\omega)\,dV_\omega > 0.$$

Thanks to (4.265), this will imply that

$$\int\limits_X \varphi\,|\alpha^{1,\,0}\wedge\omega|^2_\omega\,dV_\omega > \int\limits_X \varphi\,|(\partial\omega)_{prim}|^2_\omega\,dV_\omega,$$

which will contradict (4.264).

Now, if $\varepsilon > 0$ is chosen small enough, it is obvious that a function $\varphi \in C^\infty(X,\,\mathbb{R})$ with the above properties exists. □

Summing up, the volume of $\omega_\varphi := \omega + i\partial\bar\partial\varphi$ is *constant* on the *Gauduchon strata* (if any), while it *achieves no local extremum* on the *non-Gauduchon strata.*

- **Cohomological interpretations of the generalised volume**

Before turning to cohomological interpretations of the invariant A in (2) and (3) below, we first display A in the context of Hermitian-symplectic and strongly Gauduchon metrics in (1).

(1) sG metrics induced by H-S metrics

From Proposition 4.6.5, we infer the following construction. Let $\dim_{\mathbb{C}} X = 3$. With any Hermitian-symplectic metric ω on X, we uniquely associate the C^∞ positive definite $(2,\,2)$-form

$$\Omega_\omega := \omega^2 + 2\rho_\omega^{2,\,0}\wedge\rho_\omega^{0,\,2}, \tag{4.266}$$

where $\rho_\omega^{2,\,0}$ is the $(2,\,0)$-*torsion form* of ω and $\rho_\omega^{0,\,2} = \overline{\rho_\omega^{2,\,0}}$. As is well known (see e.g. [Mic83]), there exists a unique positive definite $(1,\,1)$-form γ_ω such that

$$\gamma_\omega^2 = \Omega_\omega.$$

By construction and the proof of Proposition 4.6.5, γ_ω is a *strongly Gauduchon* metric on X that will be called the **sG metric associated with** ω. Of course, $\gamma_\omega = \omega$ if and only if ω is Kähler. Since γ_ω^2 and Ω_ω determine each other uniquely, we will often identify them. In particular, we will also refer to Ω_ω as the *sG metric associated with* ω. We get:

$$\frac{1}{3!}\,\Omega_\omega\wedge\omega = \frac{1}{3!}\,\omega^3 + \frac{1}{3}\,|\rho_\omega^{2,\,0}|^2_\omega\,dV_\omega.$$

Hence,

$$\frac{1}{6}\int\limits_X \Omega_\omega\wedge\omega = \frac{2}{3}\,\text{Vol}_\omega(X) + \frac{1}{3}\,A, \tag{4.267}$$

where $A = \text{Vol}_\omega(X) + F(\omega) > 0$ is the generalised volume of the H-S Aeppli class $\{\omega\}_A$.

Thus, the problem of maximising $\text{Vol}_\omega(X)$ when ω ranges over the metrics in $\{\omega\}_A$ is equivalent to maximising the quantity $\int_X \Omega_\omega\wedge\omega$.

(2) The first cohomological interpretation of the generalised volume A

We first observe that the Aeppli cohomology class of Ω_ω depends only on the Aeppli class of ω.

Lemma 4.6.60. *Suppose that* $\dim_{\mathbb{C}} X = 3$*. For any* **Aeppli cohomologous** *Hermitian-symplectic metrics* ω *and* $\omega_\eta = \omega + \partial\bar\eta + \bar\partial\eta$ *on* X*, with* $\eta \in C^\infty_{1,\,0}(X,\,\mathbb{C})$*, the associated sG metrics* Ω_ω *and* Ω_{ω_η} *are again* **Aeppli cohomologous**.

Specifically, we have:

$$\begin{aligned}\Omega_{\omega_\eta} - \Omega_\omega &= \partial(\bar\eta\wedge\partial\bar\eta) + \bar\partial(\eta\wedge\bar\partial\eta) + 2\,\partial(\bar\eta\wedge\bar\partial\eta) + 2\,\bar\partial(\partial\eta\wedge\bar\eta)\\ &+ 2\,\partial(\eta\wedge\rho_\omega^{0,\,2}) + 2\,\bar\partial(\bar\eta\wedge\rho_\omega^{2,\,0}) + 2\,\partial(\bar\eta\wedge\omega) + 2\,\bar\partial(\eta\wedge\omega),\end{aligned} \tag{4.268}$$

so $\Omega_{\omega_\eta} - \Omega_\omega \in Im\,\partial + Im\,\bar\partial$.

Proof. We know from Corollary 4.6.41 that the (2, 0)-torsion forms of ω_η and ω are related by $\rho_\eta^{2,\,0} = \rho_\omega^{2,\,0} + \partial\eta$. We get:

$$\begin{aligned}\Omega_{\omega_\eta} &= \omega_\eta^2 + 2\,\rho_{\omega_\eta}^{2,\,0}\wedge\rho_{\omega_\eta}^{0,\,2} = (\omega + \partial\bar\eta + \bar\partial\eta)^2 + 2\,(\rho_\omega^{2,\,0} + \partial\eta)\wedge(\rho_\omega^{0,\,2} + \bar\partial\bar\eta)\\ &= \omega^2 + (\partial\bar\eta + \bar\partial\eta)^2 + 2\,\omega\wedge(\partial\bar\eta + \bar\partial\eta) + 2\,\rho_\omega^{2,\,0}\wedge\rho_\omega^{0,\,2} + 2\,\partial\eta\wedge\bar\partial\bar\eta + 2\,\rho_\omega^{2,\,0}\wedge\bar\partial\bar\eta + 2\,\partial\eta\wedge\rho_\omega^{0,\,2}\\ &= \Omega_\omega + \partial(\bar\eta\wedge\partial\bar\eta) + \bar\partial(\eta\wedge\bar\partial\eta) + 2\,\partial(\bar\eta\wedge\bar\partial\eta) + 2\,\bar\eta\wedge\partial\bar\partial\eta\\ &+ 2\,\partial\bar\eta\wedge\omega + 2\,\bar\partial\eta\wedge\omega + 2\,\bar\partial(\partial\eta\wedge\bar\eta) + 2\,\partial\bar\partial\eta\wedge\bar\eta\\ &+ 2\,\bar\partial(\bar\eta\wedge\rho_\omega^{2,\,0}) + 2\,\partial(\bar\eta\wedge\omega) - 2\,\partial\bar\eta\wedge\omega + 2\,\partial(\eta\wedge\rho_\omega^{0,\,2}) + 2\,\bar\partial(\eta\wedge\omega) - 2\,\bar\partial\eta\wedge\omega.\end{aligned}$$

This proves (4.268) since all the terms that are neither in $\mathrm{Im}\,\partial$ nor in $\mathrm{Im}\,\bar\partial$ reoccur with the opposite sign and cancel. □

We will need the following

Lemma 4.6.61. *(Proposition 6.2 in [PSU20b]) Let X be a compact complex manifold with $dim_{\mathbb{C}}X = n$ and let ω be a Hermitian metric on X.*

(i) The metric ω is **strongly Gauduchon (sG)** *if and only if ω^{n-1} is $E_2\overline{E}_2$*-**closed**.

(ii) The metric ω is **Hermitian-symplectic (H-S)** *if and only if ω is $E_3\overline{E}_3$*-**closed**.
If $n=3$, ω is **Hermitian-symplectic (H-S)** *if and only if ω is $E_2\overline{E}_2$*-**closed**.

Proof. (i) The sG condition on ω is defined by requiring $\partial\omega^{n-1}$ to be $\bar\partial$-exact. By conjugation, this is equivalent to $\bar\partial\omega^{n-1}$ being ∂-exact. These conditions are equivalent to ω^{n-1} being $E_2\overline{E}_2$-**closed**.

(ii) The H-S condition on ω is equivalent to the existence of a form $\rho^{2,\,0}\in C^\infty_{2,\,0}(X,\,\mathbb{C})$ such that $\partial\omega = -\bar\partial\rho^{2,\,0}$ and $\partial\rho^{2,\,0} = 0$. By conjugation, these conditions are equivalent to the existence of a form $\rho^{0,\,2}\in C^\infty_{0,\,2}(X,\,\mathbb{C})$ such that $\bar\partial\omega = -\partial\rho^{0,\,2}$ and $\bar\partial\rho^{0,\,2} = 0$. These four conditions express the fact that ω satisfies the two towers of 2 equations in (i) of Definition 3.4.1, or equivalently that ω is $E_3\overline{E}_3$-closed.

When $n=3$, the condition $\partial\rho^{2,\,0}=0$ is automatic since

$$i\partial\rho^{2,\,0}\wedge\bar\partial\rho^{0,\,2} = \partial\rho^{2,\,0}\wedge\star\bar\partial\rho^{0,\,2} = |\partial\rho^{2,\,0}|^2_\omega\,dV_\omega \geq 0 \quad \text{and} \quad \int\limits_X i\partial\rho^{2,\,0}\wedge\bar\partial\rho^{0,\,2} = \int\limits_X \partial(i\rho^{2,\,0}\wedge\bar\partial\rho^{0,\,2}) = 0,$$

where the primitivity (trivial for bidegree reasons) of the (0, 3)-form $\bar\partial\rho^{0,\,2}$ was used to infer the identity $\star\bar\partial\rho^{0,\,2} = i\bar\partial\rho^{0,\,2}$ from the general formula (4.68). □

In our case, the consequence of Proposition 4.6.5 and of Lemmas 4.6.60 and 4.6.61 is the following

Lemma 4.6.62. *Let X be a compact complex manifold with $dim_{\mathbb{C}}X = 3$. For any* **Aeppli cohomologous** *Hermitian-symplectic metrics ω and $\omega_\eta = \omega + \partial\bar\eta + \bar\partial\eta$ on X, with $\eta\in C^\infty_{1,\,0}(X,\,\mathbb{C})$, the corresponding sG metrics Ω_ω and Ω_{ω_η} represent the* **same E_2-Aeppli class**:

$$\{\Omega_{\omega_\eta}\}_{E_2,\,A} = \{\Omega_\omega\}_{E_2,\,A}\in E_{2,\,A}^{2,\,2}(X).$$

Proof. We know from Proposition 4.6.5 that Ω_ω and Ω_{ω_η} are sG metrics, so by (i) of Lemma 4.6.61 they represent E_2-Aeppli classes. Meanwhile, by Lemma 4.6.60, Ω_ω and Ω_{ω_η} are Aeppli cohomologous, hence also E_2-Aeppli cohomologous. $\square$

In our case, as a consequence of Theorem 3.4.17, we get a *unique lift* $\mathfrak{c}_\omega \in E_{2,\,BC}^{2,\,2}(X)$ of $\{\Omega_\omega\}_{E_2,\,A} \in E_{2,\,A}^{2,\,2}(X)$ under the appropriate assumption on X.

Corollary 4.6.63. *Let X be a* **page-1-$\partial\bar\partial$-manifold** *with $\dim_{\mathbb{C}} X = 3$. For any* **Aeppli cohomologous** *Hermitian-symplectic metrics ω and $\omega_\eta = \omega + \partial\bar\eta + \bar\partial\eta$ on X, with $\eta \in C^\infty_{1,\,0}(X,\,\mathbb{C})$, there exists a unique E_2-***Bott-Chern class** *$\mathfrak{c}_\omega \in E_{2,\,BC}^{2,\,2}(X)$ such that*

$$(S_2^{2,\,2}\circ T_2^{2,\,2})(\mathfrak{c}_\omega) = \{\Omega_{\omega_\eta}\}_{E_2,\,A} = \{\Omega_\omega\}_{E_2,\,A} \in E_{2,\,A}^{2,\,2}(X),$$

where Ω_ω and Ω_{ω_η} are the sG metrics associated with ω, resp. ω_η.

*In particular, the E_2-***Bott-Chern class** *$\mathfrak{c}_\omega \in E_{2,\,BC}^{2,\,2}(X)$ depends only on the E_2-***Aeppli class** *$\{\omega\}_{E_2,\,A} \in E_{2,\,A}^{1,\,1}(X)$.*

We can now state and prove the main result of this discussion. It will use the **duality** between the E_r-Bott-Chern cohomology of any bidegree $(p,\,q)$ and the E_r-Aeppli cohomology of the complementary bidegree $(n-p,\,n-q)$ proved in Theorem 3.4.11. In our case, $n=3$, $r=2$ and $(p,\,q)=(2,\,2)$.

Theorem 4.6.64. *Let X be a* **page-1-$\partial\bar\partial$-manifold** *with $\dim_{\mathbb{C}} X = 3$. Suppose there exists a Hermitian-symplectic metric ω on X whose E_2-***torsion class vanishes** *(i.e. $\{\rho_\omega^{0,\,2}\}_{E_2} = 0 \in E_2^{0,\,2}(X)$).*

Then, the generalised volume $A = F(\omega) + Vol_\omega(X)$ of $\{\omega\}_A$ is given as the following intersection number in cohomology:

$$A = \frac{1}{6}\,\mathfrak{c}_\omega.\{\omega\}_{E_2,\,A}. \tag{4.269}$$

Proof. • We will first construct a smooth d-closed $(2,\,2)$-form $\widetilde\Omega_\omega$ that represents the E_2-Bott-Chern class $\mathfrak{c}_\omega \in E_{2,\,BC}^{2,\,2}(X)$ in the most economical way possible. We will proceed in two stages that correspond to lifting the E_2-Aeppli class $\{\Omega_\omega\}_{E_2,\,A} \in E_{2,\,A}^{2,\,2}(X)$ to $E_2^{2,\,2}(X)$ under the isomorphism $S_2^{2,\,2} : E_2^{2,\,2}(X) \longrightarrow E_{2,\,A}^{2,\,2}(X)$ induced by the identity, respectively to lifting the resulting E_2-class in $E_2^{2,\,2}(X)$ to $E_{2,\,BC}^{2,\,2}(X)$ under the isomorphism $T_2^{2,\,2} : E_{2,\,BC}^{2,\,2}(X) \longrightarrow E_2^{2,\,2}(X)$ induced by the identity.

Stage 1. To lift $\{\Omega_\omega\}_{E_2,\,A} \in E_{2,\,A}^{2,\,2}(X)$ to $E_2^{2,\,2}(X)$ under the isomorphism $S_2^{2,\,2} : E_2^{2,\,2}(X) \longrightarrow E_{2,\,A}^{2,\,2}(X)$, we need to find a $(2,\,2)$-form $\Gamma^{2,\,2}$ such that $\Gamma^{2,\,2} \in \mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial$ (because we need $\Omega_\omega + \Gamma^{2,\,2}$ to represent the same $E_{2,\,A}$-class as the original Ω_ω) and such that

$$\bar\partial(\Omega_\omega + \Gamma^{2,\,2}) = 0 \quad\text{ and }\quad \partial(\Omega_\omega + \Gamma^{2,\,2}) \in \mathrm{Im}\,\bar\partial,$$

(because we need $\Omega_\omega + \Gamma^{2,\,2}$ to represent an E_2-class). The last two conditions are equivalent to

$$\bar\partial\Gamma^{2,\,2} = -\bar\partial\Omega_\omega \quad\text{ and }\quad \partial\Gamma^{2,\,2} \in \mathrm{Im}\,\bar\partial, \tag{4.270}$$

because, for the last condition, we already have $\partial\Omega_\omega \in \mathrm{Im}\,\bar\partial$ by the sG property of Ω_ω.

If we denote by $\mathcal{Z}_{2\bar{2}}^{2,2}$ the space of smooth $E_2\overline{E}_2$-closed $(2, 2)$-forms on X, we have $\Omega_\omega \in \mathcal{Z}_{2\bar{2}}^{2,2}$, hence $-\bar\partial\Omega_\omega \in \bar\partial(\mathcal{Z}_{2\bar{2}}^{2,2})$. On the other hand, Theorem 3.4.15 ensures that $\bar\partial(\mathcal{Z}_{2\bar{2}}^{2,2}) \subset \mathrm{Im}\,(\partial\bar\partial)$ because X is a *page*-1-$\partial\bar\partial$-*manifold*. (Actually, this inclusion is equivalent to the surjectivity of the map $S_2^{2,2}$.) We conclude that $-\bar\partial\Omega_\omega \in \mathrm{Im}\,(\partial\bar\partial)$, so the equation

$$\partial\bar\partial u^{1,2} = \bar\partial\Omega_\omega \tag{4.271}$$

admits solutions $u^{1,2} \in C^\infty_{1,2}(X, \mathbb{C})$. Let $u^{1,2}_\omega$ be the minimal L^2_ω-norm such solution and put $\Gamma^{2,2}_\omega := \partial u^{1,2}_\omega$.

Thus, $\Gamma^{2,2}_\omega = \partial u^{1,2}_\omega$ satisfies conditions (4.270) and $\Gamma^{2,2}_\omega \in \mathrm{Im}\,\partial \subset \mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial$. So, we have got the minimal lift $\{\Omega_\omega + \partial u^{1,2}_\omega\}_{E_2}$ of $\{\Omega_\omega\}_{E_2,A} \in E^{2,2}_{2,A}(X)$ to $E^{2,2}_2(X)$, i.e.

$$S_2^{2,2}(\{\Omega_\omega + \partial u^{1,2}_\omega\}_{E_2}) = \{\Omega_\omega\}_{E_2,A}.$$

Stage 2. To lift $\{\Omega_\omega + \partial u^{1,2}_\omega\}_{E_2} \in E^{2,2}_2(X)$ to $E^{2,2}_{2,BC}(X)$ under the isomorphism $T_2^{2,2} : E^{2,2}_{2,BC}(X) \longrightarrow E^{2,2}_2(X)$, we need to find a $(2, 2)$-form $V^{2,2}$ such that $V^{2,2} \in \partial(\ker\bar\partial) + \mathrm{Im}\,\bar\partial$ (because we need $\Omega_\omega + \partial u^{1,2}_\omega + V^{2,2}$ to represent the same E_2-class as $\Omega_\omega + \partial u^{1,2}_\omega$) and such that

$$\partial(\Omega_\omega + \partial u^{1,2}_\omega + V^{2,2}) = 0 \quad \text{and} \quad \bar\partial(\Omega_\omega + \partial u^{1,2}_\omega + V^{2,2}) = 0,$$

(because we need $\Omega_\omega + \partial u^{1,2}_\omega + V^{2,2}$ to represent an $E_{2,BC}$-class). The last two conditions are equivalent to

$$\partial V^{2,2} = -\partial(\Omega_\omega + \partial u^{1,2}_\omega) \quad \text{and} \quad \bar\partial V^{2,2} = 0, \tag{4.272}$$

because, for the last condition, we already have $\bar\partial(\Omega_\omega + \partial u^{1,2}_\omega) = 0$.

Now, $\Omega_\omega + \partial u^{1,2}_\omega \in \mathcal{Z}_{2\bar{2}}^{2,2}$, hence $-\partial(\Omega_\omega + \partial u^{1,2}_\omega) \in \partial(\mathcal{Z}_{2\bar{2}}^{2,2}) \subset \mathrm{Im}\,(\partial\bar\partial)$, the last inclusion being a consequence of Theorem 3.4.15 and of X being a *page*-1-$\partial\bar\partial$-*manifold*. (Actually, this inclusion is equivalent to the surjectivity of the map $T_2^{2,2}$.)

We conclude that $-\partial(\Omega_\omega + \partial u^{1,2}_\omega) \in \mathrm{Im}\,(\partial\bar\partial)$, so the equation

$$\partial\bar\partial u^{2,1} = -\partial(\Omega_\omega + \partial u^{1,2}_\omega) \tag{4.273}$$

admits solutions $u^{2,1} \in C^\infty_{2,1}(X, \mathbb{C})$. Let $u^{2,1}_\omega$ be the minimal L^2_ω-norm such solution and put $V^{2,2}_\omega := \bar\partial u^{2,1}_\omega$. Clearly, $u^{2,1}_\omega = \overline{u^{1,2}_\omega}$ since the equations whose minimal solutions are $u^{2,1}_\omega$ and $u^{1,2}_\omega$ are conjugated to each other.

Thus, $V^{2,2}_\omega = \bar\partial u^{2,1}_\omega$ satisfies conditions (4.272) and $V^{2,2}_\omega \in \partial(\ker\bar\partial) + \mathrm{Im}\,\bar\partial$.

The upshot of the above construction is that the form

$$\widetilde\Omega_\omega := \Omega_\omega + \partial u^{1,2}_\omega + \bar\partial u^{2,1}_\omega \in C^\infty_{2,2}(X, \mathbb{C})$$

is the minimal completion of Ω_ω to a *d-closed* pure-type form of bidegree $(2, 2)$. Moreover, the class $\{\widetilde\Omega_\omega\}_{E_2,BC} \in E^{2,2}_{2,BC}(X)$ has the property that

$$(S_2^{2,2} \circ T_2^{2,2})(\{\widetilde\Omega_\omega\}_{E_2,BC}) = \{\Omega_\omega\}_{E_2,A}.$$

Hence, $\{\widetilde\Omega_\omega\}_{E_2,BC} = \mathfrak{c}_\omega$ since the map $S_2^{2,2} \circ T_2^{2,2} : E^{2,2}_{2,BC}(X) \longrightarrow E^{2,2}_{2,A}(X)$ is bijective (thanks to the *page*-1-$\partial\bar\partial$-*assumption* on X).

• We will now use the representative $\widetilde{\Omega}_\omega$ of the class $\mathfrak{c}_\omega$ to relate the intersection number in (4.269) to the generalised volume of $\{\omega\}_A$. We have:

$$\mathfrak{c}_\omega.\{\omega\}_{E_2,\,A} = \int\limits_X \widetilde{\Omega}_\omega\wedge\omega = \int\limits_X \Omega_\omega\wedge\omega + \int\limits_X \partial u_\omega^{1,\,2}\wedge\omega + \int\limits_X \bar\partial u_\omega^{2,\,1}\wedge\omega.$$

Since $\rho_\omega^{2,\,0} = \partial\xi_\omega^{1,\,0}$ thanks to the hypothesis $\{\rho_\omega^{0,\,2}\}_{E_2} = 0 \in E_2^{0,\,2}(X)$ (see Corollary 4.6.50 and the minimal choice (4.255) of $\xi_\omega^{0,\,1} = \overline{\xi_\omega^{1,\,0}}$), we get:

$$\begin{aligned}\int\limits_X \partial u_\omega^{1,\,2}\wedge\omega &= \int\limits_X u_\omega^{1,\,2}\wedge\partial\omega = -\int\limits_X u_\omega^{1,\,2}\wedge\bar\partial\rho_\omega^{2,\,0} = \int\limits_X u_\omega^{1,\,2}\wedge\partial\bar\partial\xi_\omega^{1,\,0}\\ &= \int\limits_X \partial\bar\partial u_\omega^{1,\,2}\wedge\xi_\omega^{1,\,0} \overset{(a)}{=} \int\limits_X \bar\partial\Omega_\omega\wedge\xi_\omega^{1,\,0} \overset{(b)}{=} -2\int\limits_X \partial(\rho_\omega^{0,\,2}\wedge\omega)\wedge\xi_\omega^{1,\,0}\\ &= 2\int\limits_X \rho_\omega^{0,\,2}\wedge\omega\wedge\partial\xi_\omega^{1,\,0} = 2\int\limits_X \rho_\omega^{2,\,0}\wedge\rho_\omega^{0,\,2}\wedge\omega = 2\,||\rho_\omega^{2,\,0}||_\omega^2 = 2F(\omega),\end{aligned}$$

where (a) follows from (4.271) and (b) follows from the formula

$$\bar\partial\Omega_\omega = -2\,\partial(\rho_\omega^{0,\,2}\wedge\omega) \tag{4.274}$$

which in turn follows at once from (4.266).

By conjugation, we also have $\int_X \bar\partial u_\omega^{2,\,1}\wedge\omega = 2F(\omega)$. Putting the various pieces of information together, we get:

$$\mathfrak{c}_\omega.\{\omega\}_{E_2,\,A} = \int\limits_X \Omega_\omega\wedge\omega + 4F(\omega) = 4\,\mathrm{Vol}_\omega(X) + 2A + 4F(\omega) = 6A,$$

where the second identity follows from (4.267).

The proof of Theorem 4.6.64 is complete. □

(3) The second cohomological interpretation of the generalised volume A

We will now work in the general case (i.e. without the extra assumptions made in Theorem 4.6.64). The result will show, yet again, that the generalised volume $A = A_{\{\omega\}_A} > 0$ of a Hermitian-symplectic Aeppli class $\{\omega\}_A$ is a natural analogue in this more general context of the volume of a Kähler class.

Definition 4.6.65. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = 3$. For any Hermitian-symplectic metric ω on X, the d-closed real 2-form*

$$\widetilde{\omega} = \rho_\omega^{2,\,0} + \omega + \rho_\omega^{0,\,2}$$

is called the **minimal completion** *of ω, where $\rho_\omega^{2,\,0}$, resp. $\rho_\omega^{0,\,2}$, is the $(2,\,0)$-torsion form, resp. the $(0,\,2)$-torsion form, of ω.*

We will now notice the following consequence of Corollary 4.6.41 It gives a new cohomological interpretation of the generalised volume $A = A_{\{\omega\}_A} > 0$.

Proposition 4.6.66. *Let X be a compact complex Hermitian-symplectic manifold of dimension $n=3$.*

(a) For any Hermitian-symplectic metric ω on X, its minimal completion 2-form $\widetilde{\omega}$ has the property:

$$\int\limits_X \frac{\widetilde{\omega}^3}{3!} = Vol_\omega(X) + F(\omega) = A_{\{\omega\}_A}. \tag{4.275}$$

(b) For any Aeppli-cohomologous Hermitian-symplectic metrics ω and ω_η

$$\omega_\eta = \omega + \partial\bar\eta + \bar\partial\eta > 0 \quad (\textit{where } \eta \in C^\infty_{1,0}(X,\,\mathbb{C})), \tag{4.276}$$

the respective minimal completion 2-forms $\widetilde{\omega}_\eta$ and $\widetilde{\omega}$ lie in the same De Rham cohomology class.

Thus, $A_{\{\omega\}_A} = \{\widetilde{\omega}\}^3_{DR}/3!$.

Proof. (a) Using (4.267) for identity (a) below and the above notation, we get:

$$\begin{aligned}\int\limits_X \widetilde{\omega}^3 &= \int\limits_X \widetilde{\omega}^2 \wedge (\rho^{2,\,0}_\omega + \omega + \rho^{0,\,2}_\omega) = \int\limits_X \Omega_\omega \wedge (\rho^{2,\,0}_\omega + \omega + \rho^{0,\,2}_\omega)\\ &+ 2\int\limits_X \rho^{2,\,0}_\omega \wedge \omega \wedge \rho^{0,\,2}_\omega + 2\int\limits_X \rho^{0,\,2}_\omega \wedge \omega \wedge \rho^{2,\,0}_\omega\\ &= \int\limits_X \Omega_\omega \wedge \omega + 4F(\omega) \stackrel{(a)}{=} 4\mathrm{Vol}_\omega(X) + 2\mathrm{Vol}_\omega(X) + 2F(\omega) + 4F(\omega) = 6A.\end{aligned}$$

(b) We know from Corollary 4.6.41 that the $(2,\,0)$-torsion forms of ω_η and ω are related by $\rho^{2,\,0}_\eta = \rho^{2,\,0}_\omega + \partial\eta$. We get:

$$\widetilde{\omega}_\eta = \rho^{2,\,0}_\eta + \omega_\eta + \rho^{0,\,2}_\eta = \widetilde{\omega} + d(\eta + \bar\eta).$$

This proves the contention. ∎

Chapter 5

Co-polarised Deformations of Balanced Calabi-Yau $\partial\bar\partial$-Manifolds

This chapter is taken from [Pop13] where the notion of **co-polarisation** by a balanced De Rham cohomology class of some of the fibres of a holomorphic family of balanced Calabi-Yau $\partial\bar\partial$- manifolds was introduced. It generalises the classical notion of **polarisation** by a Kähler class.

5.1 Small deformations of balanced $\partial\bar\partial$-manifolds

Recall that, by Definition 2.4.1, an n-dimensional compact complex manifold X is said to be a *Calabi-Yau manifold* if its canonical bundle K_X is trivial. This condition is deformation open when the Hodge number $h^{n,\,0}$ does not jump in the neighbourhood of the given fibre.

Specifically, let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds $X_t := \pi^{-1}(t)$ with $\dim_{\mathbb{C}} X_t = n$ for all $t \in B$, where $B \subset \mathbb{C}^N$ is a small open ball about the origin for some $N \in \mathbb{N}^\star$. If some fibre X_0 is a Calabi-Yau manifold and if $h^{n,\,0}_{\bar\partial}(t) := \dim_{\mathbb{C}} H^{n,\,0}_{\bar\partial}(X_t,\,\mathbb{C})$ is independent of t varying in a small neighbourhood B_ε of 0 in B, then the fibre X_t is again a Calabi-Yau manifold for every $t \in B_\varepsilon$.

We saw in Theorem 2.6.3 that, if X_0 is supposed to be a $\partial\bar\partial$*-manifold*, all the Hodge numbers $h^{p,\,q}_{\bar\partial}(t) := \dim_{\mathbb{C}} H^{p,\,q}_{\bar\partial}(X_t,\,\mathbb{C})$ are independent of t varying in a small neighbourhood of 0 in B. In particular, since the $\partial\bar\partial$-property of compact complex manifolds is **open** under holomorphic deformations of complex structures by Theorem 2.6.4, we get that, if X_0 is a *Calabi-Yau* $\partial\bar\partial$*-manifold*, all the X_t's with t sufficiently close to 0 are again *Calabi-Yau* $\partial\bar\partial$*-manifolds*. Moreover, when combined with Wu's Theorem 4.5.46, this fact implies the following

Corollary 5.1.1. *Let $(X_t)_{t\in B}$ be a holomorphic family of compact complex manifolds over an open ball B containing the origin in some $\mathbb{C}^N$.*

If the fibre X_0 is a **balanced Calabi-Yau $\partial\bar\partial$-manifold***, the fibre X_t is again a* **balanced Calabi-Yau $\partial\bar\partial$-manifold** *for every $t \in B$ sufficiently close to 0.*

If Conjecture 4.5.56 turns out to be true, the balanced condition can be dropped from the above statement, but in the current state of play, it has to be kept.

Since all the fibres X_t with t close enough to 0 are $\partial\bar\partial$-manifolds, the De Rham cohomology space $H^{2n-2}_{DR}(X,\,\mathbb{C})$ (which is independent of the complex structure J_t) admits, for all $t \in B$ close to 0, a Hodge decomposition:

$$H^{2n-2}_{DR}(X,\,\mathbb{C}) = H^{n,\,n-2}_{\bar\partial}(X_t,\,\mathbb{C}) \oplus H^{n-1,\,n-1}_{\bar\partial}(X_t,\,\mathbb{C}) \oplus H^{n-2,\,n}_{\bar\partial}(X_t,\,\mathbb{C}) \tag{5.1}$$

depending on the complex structure J_t and satisfying the Hodge symmetry:

$$H^{n,\,n-2}_{\bar\partial}(X_t,\,\mathbb{C}) \simeq \overline{H^{n-2,\,n}_{\bar\partial}(X_t,\,\mathbb{C})}.$$

Definition 5.1.2. *A cohomology class $\{\omega^{n-1}\}_{DR} \in H^{2n-2}_{DR}(X,\,\mathbb{C})$ is said to be of type $(n-1,\,n-1)$ for the complex structure J_t of X_t if its components of types $(n,\,n-2)$ and $(n-2,\,n)$ in the Hodge decomposition (5.1) vanish.*

If $\{\omega^{n-1}\}_{DR} \in H^{2n-2}_{DR}(X,\,\mathbb{C})$ is a balanced class, it is real, so $\{\omega^{n-1}\}_{DR}$ is of J_t-type $(n-1,\,n-1)$ if and only if either of its components of J_t-types $(n,\,n-2)$ and $(n-2,\,n)$ vanishes. The condition is still equivalent to the De Rham class $\{\omega^{n-1}\}_{DR} \in H^{2n-2}_{DR}(X,\,\mathbb{C})$ being representable by a form of J_t-pure type $(n-1,\,n-1)$.

The starting point of the discussion in this chapter is the following observation. While independent of Wu's approach to Theorem 4.5.46, the proof uses similar techniques and, in particular, reproves Theorem 4.5.46.

Observation 5.1.3. *([Pop13, Observation 7.2]) Let $(X_t)_{t\in B}$ be a holomorphic family of n-dimensional compact complex manifolds such that the fibre X_0 is a* **balanced $\partial\bar\partial$-manifold**. *We denote by X the differentiable manifold underlying the fibres X_t (after possibly shrinking B about 0.)*

Let ω_0 be a **balanced metric** *on X_0 and suppose that the De Rham class $\{\omega_0^{n-1}\}_{DR} \in H^{2n-2}_{DR}(X,\,\mathbb{C})$ is of type $(n-1,\,n-1)$ for the complex structure J_t of X_t for all t close to zero and lying on a path through 0 in B.*

Then, the De Rham class $\{\omega_0^{n-1}\}_{DR}$ contains a **J_t-balanced metric** *for every t as above sufficiently close to 0.*

Proof. Since X_t is a $\partial\bar\partial$-manifold for every t close to 0, there are canonical isomorphisms $H^{p,\,q}_{BC}(X_t,\,\mathbb{C}) \simeq H^{p,\,q}_{A}(X_t,\,\mathbb{C})$ (for every $(p,\,q)$) and

$$\begin{aligned} H^{2n-2}_{DR}(X,\,\mathbb{C}) &\simeq H^{n,\,n-2}_{BC}(X_t,\,\mathbb{C}) \oplus H^{n-1,\,n-1}_{BC}(X_t,\,\mathbb{C}) \oplus H^{n-2,\,n}_{BC}(X_t,\,\mathbb{C}) \\ &\simeq H^{n,\,n-2}_{A}(X_t,\,\mathbb{C}) \oplus H^{n-1,\,n-1}_{A}(X_t,\,\mathbb{C}) \oplus H^{n-2,\,n}_{A}(X_t,\,\mathbb{C}). \end{aligned}$$

Now, let $\omega_0^{n-1} = \Omega_t^{n,\,n-2} + \Omega_t^{n-1,\,n-1} + \Omega_t^{n-2,\,n}$ be the splitting of ω_0^{n-1} into components of pure J_t-types. In particular, $\Omega_t^{n-1,\,n-1}$ is a *real* J_t-type $(n-1,\,n-1)$-form that varies in a C^∞ way with t and is positive definite for every t sufficiently close to 0 since $\Omega_0^{n-1,\,n-1} = \omega_0^{n-1} > 0$.

Meanwhile, since $d\omega_0^{n-1} = 0$, it is easy to see (cf. e.g. [Pop15a]) that $\partial_t\bar\partial_t\Omega_t^{n-1,\,n-1} = 0$ and that the Aeppli cohomology class $[\Omega_t^{n-1,\,n-1}]_A$ is the image of $\{\omega_0^{n-1}\}_{DR}$ under the projection $H^{2n-2}_{DR}(X,\,\mathbb{C}) \longrightarrow H^{n-1,\,n-1}_{A}(X_t,\,\mathbb{C})$ defined by the latter cohomology splitting above.

To construct the image of $[\Omega_t^{n-1,\,n-1}]_A \in H^{n-1,\,n-1}_{A}(X_t,\,\mathbb{C})$ in $H^{n-1,\,n-1}_{BC}(X_t,\,\mathbb{C})$ under the canonical isomorphism $H^{n-1,\,n-1}_{A}(X_t,\,\mathbb{C}) \simeq H^{n-1,\,n-1}_{BC}(X_t,\,\mathbb{C})$, we can proceed as in [Pop15a] and look for the "most economic choice" of a J_t-$(n-2,\,n-1)$-form u_t and a J_t-$(n-1,\,n-2)$-form v_t such that the J_t-$(n-1,\,n-1)$-form

$$\widetilde{\Omega}_t^{n-1,\,n-1} := \Omega_t^{n-1,\,n-1} + \partial_t u_t + \bar\partial_t v_t$$

is d-closed. This amounts to $\partial_t\bar\partial_t u_t = \bar\partial_t\Omega_t^{n-1,\,n-1}$ and $\partial_t\bar\partial_t v_t = -\partial_t\Omega_t^{n-1,\,n-1}$. If we choose $v_t := \bar u_t$, the latter equation becomes redundant, while the minimal $L^2_{\gamma_t}$-norm solution of the former equation (which is solvable since X_t is a $\partial\bar\partial$-manifold) is given by the following Neumann-type formula (see Theorem 4.5.47):

$$u_t=(\partial_t\bar\partial_t)^\star\Delta_{BC}^{-1}(t)\bar\partial_t\Omega_t^{n-1,\,n-1},\qquad t\in B,$$

after possibly shrinking B about 0 to ensure that X_t is a $\partial\bar\partial$-manifold. (As usual, we have fixed an arbitrary C^∞ family $(\gamma_t)_{t\in B}$ of Hermitian metrics on the fibres $(X_t)_{t\in B}$.)

Then, for all t close to 0, we get

$$\widetilde\Omega_t^{n-1,\,n-1}:=\Omega_t^{n-1,\,n-1}+\partial_t(\partial_t\bar\partial_t)^\star\Delta_{BC}^{-1}(t)\bar\partial_t\Omega_t^{n-1,\,n-1}+\bar\partial_t(\bar\partial_t\partial_t)^\star\overline{\Delta_{BC}^{-1}(t)}\partial_t\Omega_t^{n-1,\,n-1}.$$

When $t=0$, $\partial_0\bar\partial_0u_0=\bar\partial_0\Omega_0^{n-1,\,n-1}=\bar\partial_0\omega_0^{n-1}=0$ (the last identity being a consequence of ω_0 being balanced), so the minimal L^2-norm solution of this equation is $u_0=0$. Note that u_t, hence also $\widetilde\Omega_t^{n-1,\,n-1}$, depends in a C^∞ way on t for the same reason as in the proof of Wu's Theorem 4.5.46: the $\partial\bar\partial$-assumption implies the invariance w.r.t. t of the Bott-Chern numbers $h_{BC}^{p,\,q}(t)$, which implies the smooth dependence on t of $\Delta_{BC}^{-1}(t)$.

We have thus constructed a C^∞ family of real d-closed J_t-$(n-1,\,n-1)$-forms $\widetilde\Omega_t^{n-1,\,n-1}$ such that $\widetilde\Omega_0^{n-1,\,n-1}=\omega_0^{n-1}>0$. By continuity, we must have $\widetilde\Omega_t^{n-1,\,n-1}>0$, hence $\widetilde\Omega_t^{n-1,\,n-1}$ defines a balanced metric on X_t, for all t close to 0. (In particular, this gives another proof of Wu's Theorem 4.5.46.)

Moreover, $[\widetilde\Omega_t^{n-1,\,n-1}]_{BC}$ is the image in $H_{BC}^{n-1,\,n-1}(X_t,\,\mathbb{C})$ of $[\Omega_t^{n-1,\,n-1}]_A$ under the canonical isomorphism $H_A^{n-1,\,n-1}(X_t,\,\mathbb{C})\to H_{BC}^{n-1,\,n-1}(X_t,\,\mathbb{C})$. Since $[\Omega_t^{n-1,\,n-1}]_A$ is the image of $\{\omega_0^{n-1}\}_{DR}$ under the canonical projection of $H_{DR}^{2n-2}(X,\,\mathbb{C})$ onto $H_A^{n-1,\,n-1}(X_t,\,\mathbb{C})$, we infer that $[\widetilde\Omega_t^{n-1,\,n-1}]_{BC}$ is the image in $H_{BC}^{n-1,\,n-1}(X_t,\,\mathbb{C})$ of $\{\omega_0^{n-1}\}_{DR}$ under the canonical projection of $H_{DR}^{2n-2}(X,\,\mathbb{C})$ onto $H_{BC}^{n-1,\,n-1}(X_t,\,\mathbb{C})$. Meanwhile, if the class $\{\omega_0^{n-1}\}_{DR}\in H_{DR}^{2n-2}(X,\,\mathbb{C})$ is supposed to be of J_t-type $(n-1,\,n-1)$, it coincides with its projection $[\widetilde\Omega_t^{n-1,\,n-1}]_{BC}$ (after the obvious canonical identification of $H_{BC}^{n-1,\,n-1}(X_t,\,\mathbb{C})$ with its image in $H_{DR}^{2n-2}(X,\,\mathbb{C})$). This means that $\{\omega_0^{n-1}\}_{DR}=\{\widetilde\Omega_t^{n-1,\,n-1}\}_{DR}$ for all t sufficiently close to 0 and lying on the path through 0 in Δ along which $\{\omega_0^{n-1}\}_{DR}$ is assumed to be of J_t-type $(n-1,\,n-1)$. Thus, the class $\{\omega_0^{n-1}\}_{DR}$ contains the J_t-balanced metric $\widetilde\Omega_t^{n-1,\,n-1}$ for all these t's. $\square$

5.1.1 The balanced Ricci-flat Bochner principle

We collect here essentially known facts that will come in handy later on. A preliminary observation is the following addition to Corollary 4.5.21.

Proposition 5.1.4. *([Gau77b, Proposition 1, p.120, Proposition 7, p.128]) Let $(E,\,h_E)\to(X,\,\omega)$ be a complex Hermitian C^∞ vector bundle of rank $r\geq 1$ equipped with a Hermitian connection $D_E=D'_E+D''_E$. If ω is* **balanced***, then*

$$\Delta''_E\sigma=\Delta'_E\sigma+[i\Theta(E)^{1,\,1},\,\Lambda](\sigma)\quad \textit{for all } \textbf{sections } \sigma\in C^\infty_{0,\,0}(X,\,E),\tag{5.2}$$

where $\Delta''_E=D''_E\,D''^\star_E+D''^\star_E\,D''_E$ and $\Delta'_E=D'_E\,D'^\star_E+D'^\star_E\,D'_E$, while $i\Theta(E)^{1,\,1}$ stands for the (multiplication operator by the) component of type $(1,\,1)$ of the curvature form of E and $\Lambda:=\Lambda_\omega\otimes Id_E$.

Proof. The proof is similar to that of Corollary 4.5.21. It involves using the Hermitian commutation relations:

$$[\Lambda,\,D'_E]=i\,(D''^\star_E+\bar\tau^\star)\quad\text{and}\quad[\Lambda,\,D''_E]=-i\,(D'^\star_E+\tau^\star)\tag{5.3}$$

and the fact that $D'_ED''_E\sigma+D''_ED'_E\sigma=\Theta(E)^{1,\,1}\wedge\sigma$. $\square$

We go on to notice the following

Proposition 5.1.5. *Let X be a compact complex manifold ($\dim_{\mathbb{C}} X = n$) endowed with an arbitrary Hermitian metric ω. Let $|\cdot|_\omega$ denote the pointwise norm of sections of K_X w.r.t. the metric induced by ω.*

(i) For every $u \in C^\infty_{n,0}(X, \mathbb{C}) \simeq C^\infty(X, K_X)$, we have

$$i^{n^2}\, u \wedge \bar u = |u|^2_\omega\, \omega^n. \tag{5.4}$$

(ii) Equip K_X with the metric induced by ω and denote by $D = D' + \bar\partial$ the corresponding Chern connection of K_X. If ω is **balanced** *and $Ric(\omega) = 0$, then every $u \in C^\infty_{n,0}(X, \mathbb{C}) \simeq C^\infty(X, K_X)$ satisfies*

$$||\bar\partial u||^2 = ||D'u||^2,$$

where $||\cdot||$ denotes the L^2 norm of K_X-valued forms w.r.t. the metric induced by ω. In particular, every holomorphic n-form u on X (i.e. $u \in H^0(X, K_X)$) is **parallel** *(i.e. $Du = 0$) and satisfies*

$$|u|^2_\omega = C \quad (\text{hence also} \quad i^{n^2}\, u \wedge \bar u = C\,\omega^n) \quad \text{on } X \tag{5.5}$$

for some constant $C \geq 0$.

Proof. (i) Fix an arbitrary point $x_0 \in X$ and choose local holomorphic coordinates $z_1, \dots, z_n$ about x_0. If we write $u = f\, dz_1 \wedge \dots \wedge dz_n$ on some open subset $U \subset X$, where f is a C^∞ function on U, we have

$$i^n\, u \wedge \bar u = (-1)^{\frac{n(n-1)}{2}}\, |f|^2\, i\, dz_1 \wedge d\bar z_1 \wedge \dots \wedge i\, dz_n \wedge d\bar z_n. \tag{5.6}$$

On the other hand, if we write

$$\omega = i \sum_{\alpha,\beta} \omega_{\alpha\beta}\, dz_\alpha \wedge d\bar z_\beta \quad \text{on } U,$$

we have $\omega^n = \det(\omega_{\alpha\beta})\, i\, dz_1 \wedge d\bar z_1 \wedge \dots \wedge i\, dz_n \wedge d\bar z_n$ on U. Thus, if $h = \exp(-\varphi)$ is the fibre metric induced by ω on $\Lambda^n T^{1,0}X = \det(T^{1,0}X) = -K_X$, we have

$$\det(\omega_{\alpha\beta}) = e^{-\varphi} \ \text{ on } U.$$

If we regard u as a section of K_X, we have

$$|u|^2_\omega = |f|^2\, |dz_1 \wedge \dots \wedge dz_n|^2_\omega = e^{\varphi}\, |f|^2 = \frac{|f|^2}{\det(\omega_{\alpha\beta})} \ \text{ on } U.$$

Since $\omega^n = \det(\omega_{\alpha\beta})\, i\, dz_1 \wedge d\bar z_1 \wedge \dots \wedge i\, dz_n \wedge d\bar z_n$ on U, we get

$$|u|^2_\omega\, \omega^n = |f|^2\, i\, dz_1 \wedge d\bar z_1 \wedge \dots \wedge i\, dz_n \wedge d\bar z_n \ \text{ on } U. \tag{5.7}$$

Since $(-1)^{\frac{n(n-1)}{2}}\, i^n = i^{n^2}$, we see that (5.6) and (5.7) add up to (5.4).

(ii) If ω is balanced, (b) of Proposition 5.1.4 applied to $E = K_X$ reads:

$$\Delta''_{K_X} u = \Delta'_{K_X} u + [i\Theta_\omega(K_X),\, \Lambda](u).$$

Now, $i\Theta_\omega(K_X) = -\mathrm{Ric}\,(\omega) = 0$, while

$$\langle\langle\Delta''_{K_X}u,\,u\rangle\rangle = ||\bar\partial u||^2 \quad \text{and} \quad \langle\langle\Delta'_{K_X}u,\,u\rangle\rangle = ||D'u||^2.$$

Thus we get the former part of (ii). To get the latter part of (ii), it suffices to notice that if u is holomorphic, then $\bar\partial u = 0$, hence $D'u = 0$ by the above identity, hence $Du = 0$ at every point of X. Meanwhile, for every holomorphic n-form u, we have

$$d\,|u|^2_\omega = d\,\langle u,\,u\rangle_\omega = \{Du,\,u\} + \{u,\,Du\} = 0$$

thanks to $Du = 0$, hence $|u|^2_\omega$ is constant on X. (Here $\{\ ,\ \}$ stands for the sesquilinear pairing of K_X-valued forms combining the exterior product of scalar-valued forms, conjugation in the second factor and the pointwise scalar product $\langle\ ,\ \rangle_\omega$ induced by ω on the fibres of K_X.) $\square$

It turns out that the balanced assumption is unnecessary in the last statement of Proposition 5.1.5. Indeed, every holomorphic section of a flat line bundle is parallel. In particular, (5.5) holds in Proposition 5.1.5 for any Hermitian metric ω such that $\text{Ric}(\omega) = 0$.

Observation 5.1.6. [1] *Let $(L,\,h) \to X$ be a Hermitian holomorphic line bundle over a compact complex manifold such that the curvature form $i\Theta_h(L)$ vanishes identically on X. Then any global holomorphic section $\sigma \in H^0(X,\,L)$ satisfies $D\sigma = 0$, where D is the Chern connection of $(L,\,h)$.*

Proof. Let $n := \dim_{\mathbb{C}}X$. Pick any Gauduchon metric ω on X. Thus ω is a C^∞ positive definite $(1,\,1)$-form such that $\partial\bar\partial\omega^{n-1} = 0$ on X (cf. [Gau77a]). If $|\sigma|^2_h$ is the pointwise squared norm of σ w.r.t. h, Stokes' theorem implies

$$\int\limits_X i\partial\bar\partial|\sigma|^2_h \wedge \omega^{n-1} = 0. \tag{5.8}$$

On the other hand, computing the real C^∞ $(1,\,1)$-form $i\partial\bar\partial|\sigma|^2_h$, we get:

$$\begin{aligned} i\partial\bar\partial|\sigma|^2_h &= i\partial\bar\partial\langle\sigma,\,\sigma\rangle_h = i\partial\,\{\sigma,\,D'\sigma\}_h \\ &= i\,\{D'\sigma,\,D'\sigma\}_h + i\,\{\sigma,\,\bar\partial D'\sigma\}_h = i\,\{D'\sigma,\,D'\sigma\}_h \geq 0 \end{aligned} \tag{5.9}$$

at every point in X, where $\langle\ ,\ \rangle_h$ stands for the pointwise scalar product defined by h on the fibres of L and $\{\ ,\ \}$ denotes the sesquilinear pairing of L-valued forms combining the exterior product of scalar-valued forms, conjugation in the second factor and $\langle\ ,\ \rangle_h$, while $D = D' + \bar\partial$ is the splitting of D into components of respective types $(1,\,0)$ and $(0,\,1)$. Since σ is holomorphic, $\bar\partial\sigma = 0$, while the flatness assumption $i\Theta_h(L) = 0$ translates to $D'\bar\partial + \bar\partial D' = 0$. Hence $\bar\partial D'\sigma = 0$, accounting for the last identity in (5.9). To justify the nonnegativity inequality in (5.9), pick an arbitrary point $x \in X$, a local holomorphic frame e of L and write $D'\sigma = \alpha \otimes e$ for a scalar-valued $(1,\,0)$-form α in a neighbourhood of x. Then

$$i\,\{D'\sigma,\,D'\sigma\}_h = |e|^2_h \cdot i\alpha \wedge \bar\alpha \geq 0, \quad \text{hence} \quad i\partial\bar\partial|\sigma|^2_h \wedge \omega^{n-1} \geq 0, \tag{5.10}$$

at every point near x since $i\alpha\wedge\bar\alpha$ is a semi-positive $(1,\,1)$-form whenever α is a $(1,\,0)$-form. Moreover, $i\alpha \wedge \bar\alpha \wedge \omega^{n-1}$ vanishes at a given point x if and only if α vanishes at x.

[1] Both the statement and the proof of this observation have been kindly pointed out to the author by J.-P. Demailly. These facts are actually well known, cf. e.g. [KW70] and [Gau77b].

By (5.8), (5.9) and (5.10), $|D'\sigma|_h = 0$ at every point in X. Thus $D'\sigma = 0$, hence $D\sigma = 0$ on X. $\square$

The following simple and well-known fact will be used further down.

Lemma 5.1.7. *Let X be a compact complex manifold ($\dim_{\mathbb{C}} X = n$) endowed with a Hermitian metric ω such that $Ric(\omega) = 0$. If K_X is trivial and if $u \in C^\infty_{n,0}(X, \mathbb{C})$ such that $\bar\partial u = 0$, u has no zeroes and*

$$i^{n^2}\int\limits_X u\wedge\bar u = \int\limits_X dV_\omega, \qquad (where\ dV_\omega := \frac{\omega^n}{n!}) \tag{5.11}$$

then the Calabi-Yau isomorphism $T_u : C^\infty_{0,1}(X, T^{1,0}X) \to C^\infty_{n-1,1}(X, \mathbb{C})$ (see (2.46) with $q = 1$) is an isometry w.r.t. the **pointwise** *(hence also the L^2) scalar products induced by ω on the vector bundles involved.*

Proof. Fix an arbitrary point $x_0 \in X$ and choose local holomorphic coordinates $z_1, \dots, z_n$ about x_0 such that

$$\omega(x_0) = i\sum_{j=1}^n \lambda_j\, dz_j\wedge d\bar z_j \quad \text{ and } \quad u(x_0) = f\, dz_1\wedge\cdots\wedge dz_n.$$

A simple calculation shows that for any $\theta, \eta \in C^\infty_{0,1}(X, T^{1,0}X)$, the pointwise scalar products at x_0 are related by

$$\langle\theta,\, \eta\rangle = \frac{\lambda_1\dots\lambda_n}{|f|^2}\,\langle\theta\lrcorner u,\, \eta\lrcorner u\rangle.$$

Thus having $\langle\theta,\, \eta\rangle = \langle\theta\lrcorner u,\, \eta\lrcorner u\rangle$ at x_0 is equivalent to having $|f|^2 = \lambda_1\dots\lambda_n$. On the other hand, the identity $i^{n^2}\, u\wedge\bar u = |u|^2_\omega\,\omega^n$ implies that

$$|f|^2 = (n!)\,|u|^2_\omega\,(\lambda_1\dots\lambda_n).$$

Thus T_u is an isometry w.r.t. the pointwise scalar products induced by ω if and only if

$$|u|^2_\omega = \frac{1}{n!} \quad \text{at every point of } X. \tag{5.12}$$

Since we know from (5.5) of Proposition 5.1.5 and from Observation 5.1.6 that $|u|^2_\omega$ is constant on X, we see from the identity $i^{n^2}\, u\wedge\bar u = |u|^2_\omega\,\omega^n$ that the normalisation (5.11) of u is equivalent to (5.12), i.e. to T_u being an isometry w.r.t. the pointwise scalar products induced by ω on the vector bundles involved. $\square$

Finally, let us mention the following addition to Lemma and Definition 2.4.4.

Lemma 5.1.8. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Suppose that K_X is* **trivial** *and let u be a* **Calabi-Yau form** *on X.*

If ω is any Hermitian metric on X such that $Ric(\omega) = 0$, the isomorphism

$$T_u : C^\infty_{0,1}(X, T^{1,0}X) \xrightarrow{\cdot\lrcorner u} C^\infty_{n-1,1}(X, T^{1,0}X)$$

of (2.46) for $q = 1$ satisfies:

$$T_u(Im\,\bar\partial^\star) = Im\,\bar\partial^\star \quad and \quad T_u(\ker\Delta'') = \ker\Delta''. \tag{5.13}$$

Proof. We know from (5.5) of Proposition 5.1.5 and from Observation 5.1.6 that $|u|^2_\omega$ is constant on X whenever $\mathrm{Ric}(\omega)=0$. Then the proof of Lemma 5.1.7 shows that $\langle\theta,\,\eta\rangle = \mathrm{Const}\cdot\langle\theta\lrcorner u,\,\eta\lrcorner u\rangle$ for all $\theta,\eta\in C^\infty_{0,1}(X,\,T^{1,0}X)$, hence $\theta\perp\eta$ if and only if $\theta\lrcorner u\perp\eta\lrcorner u$. (The notation is the obvious one.) This fact suffices to deduce (5.13) from the pairwise orthogonality of $\ker\Delta''$, $\mathrm{Im}\,\bar\partial$ and $\mathrm{Im}\,\bar\partial^\star$ in the three-space decompositions of $C^\infty_{0,1}(X,\,T^{1,0}X)$ and $C^\infty_{n-1,1}(X,\,\mathbb{C})$ and from the identities (2.47). $\square$

5.2 Examples of non-Kähler, balanced Calabi-Yau $\partial\bar\partial$-manifolds

We now pause to point out a few examples of manifolds of the above type in support of the theory that will be developed in the next sections.

(1) We have seen that all *class* $\mathcal{C}$ manifolds are both balanced and $\partial\bar\partial$. However, the implications are strict and even the simultaneous occurence of the balanced and $\partial\bar\partial$ conditions does not ensure the *class* $\mathcal{C}$ property. The following observation is a reinforcement of Observation 2.6.15.

Observation 5.2.1. *There exist compact balanced $\partial\bar\partial$-manifolds that are not of class $\mathcal{C}$. In other words, the class of compact* **balanced $\partial\bar\partial$-manifolds** *strictly contains Fujiki's class $\mathcal{C}$.*

Proof. It suffices to put together the proofs of Theorem 2.6.13 and of Observation 2.6.15 and to remember that, by a result of Gauduchon [Gau91], all twistor spaces are balanced. Indeed, [Cam91a] and [LP92] exhibit holomorphic families of twistor spaces $(X_t)_{t\in B}$ in which the central fibre X_0 is Moishezon (hence is also a $\partial\bar\partial$-manifold), while, for every $t\in B\setminus\{0\}$ sufficiently close to 0, the fibre X_t has vanishing algebraic dimension (hence is non-Moishezon, hence is not of *class* $\mathcal{C}$ since, by another result of Campana [Cam91b], the Moishezon and class $\mathcal{C}$ properties of twistor spaces are equivalent). Thus, any of the fibres X_t with $t\neq 0$ but t close to 0 provides an example as stated. $\square$

Notice that the above examples are not Calabi-Yau manifolds since the restriction of the canonical bundle of any twistor space to any twistor line is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-4)$, hence it cannot be trivial.

(2) On the other hand, examples of compact **non-Kähler, class $\mathcal{C}$, holomorphic symplectic manifolds** were constructed by Yoshioka in [Yos01, section 4.4]. In particular, Yoshioka's manifolds are compact, non-Kähler, balanced Calabi-Yau $\partial\bar\partial$-manifolds. Thus they fall into the category of manifolds that will be investigated in this chapter. While Yoshioka's manifolds are of *class* $\mathcal{C}$, parts (3) and (4) below show that compact, *non-class* $\mathcal{C}$, balanced Calabi-Yau $\partial\bar\partial$-manifolds (i.e. manifolds as in Observation 5.2.1 having, in addition, a trivial canonical bundle) exist.

(3) We will now point out a first class of examples of compact **balanced Calabi-Yau $\partial\bar\partial$-manifolds** that are **not of class $\mathcal{C}$**. In [FOU14, Theorem 5.2] (see also [Kas13]), a (compact) solvmanifold M of real dimension 6 and a holomorphic family of complex structures $(J_a)_{a\in D}$ on M are constructed (where $D:=\{a\in\mathbb{C}\,;\,|a|<1\}$) such that $X_a:=(M,\,J_a)$ is a balanced Calabi-Yau $\partial\bar\partial$-manifold for every $a\in D\setminus\{0\}$. Furthermore, it can be easily checked that $X_a=(M,\,J_a)$ is not of class $\mathcal{C}$ for any $a\in D$ by either of the next two arguments.

(a) A direct calculation shows the existence of a C^∞ positive definite $(1,\,1)$-form ω on X_a such that $i\partial\bar\partial\omega\geq 0$. Then, by Theorem 2.3 in [Chi14], if X_a were of class $\mathcal{C}$, it would have to be Kähler. However, a direct calculation shows that no Kähler metrics exist on any X_a. This argument has kindly been communicated to the author by L. Ugarte.

(b) Since the fundamental group is a bimeromorphic invariant of compact complex manifolds, if X_a were of class $\mathcal{C}$, its fundamental group would also occur as the fundamental group of a compact

Kähler manifold. However, this is impossible as follows from [Cam04] (where it is proved that the Albanese morphism $\alpha_X : X \to \mathrm{Alb}(X)$ of any Calabi-Yau class $\mathcal{C}$ manifold X is *surjective*) combined with [Cam95] (where $\pi_1(X)$ is studied when α_X is surjective). This argument has kindly been communicated to the author by F. Campana.

(4) In [Fri17], a large class of compact, *non-class* $\mathcal{C}$, balanced Calabi-Yau $\partial\bar\partial$-manifolds obtained via a construction of Clemens's (that was subsequently used by many authors, including Friedman himself, [Rei86] and [FLY12]) was produced.

This construction, termed *conifold transition* and also outlined in §4.5.6 just above Example 4.5.63, runs as follows. One starts off with a compact Kähler Calabi-Yau manifold X, contracts X under a crepant map to some *singular* non-*class* $\mathcal{C}$) variety Y and then smoothes Y by slightly deforming it to some *non-class* $\mathcal{C}$, but balanced, Calabi-Yau $\partial\bar\partial$) manifold Y_t.

In [Fri17], the original compact 3-fold X with trivial K_X is only assumed to be $\partial\bar\partial$ (so not necessarily Kähler), to have $h^{0,1} = h^{0,2} = 0$ (hence also $h^{1,0} = h^{2,0} = 0$ thanks to the Hodge symmetry that holds on any $\partial\bar\partial$-manifold) and to have disjoint smooth rational $(-1,-1)$-curves $C_1, \dots, C_r$ whose classes $[C_i] \in H^4(X, \mathbb{C})$ satisfy a linear dependence relation but generate $H^4(X, \mathbb{C})$. Friedman shows in [Fri17] that the singular compact 3-fold Y obtained from any such X by contracting the C_i's has smooth small deformations that are $\partial\bar\partial$-manifolds, but are not of *class* $\mathcal{C}$. They are not even deformation equivalent to any *class* $\mathcal{C}$ manifold. Actually, all the small deformations lying in an open dense subset of the moduli space are shown in [Fri17] to be $\partial\bar\partial$-manifolds, while all small smoothings are conjectured to be. They are balanced by [FLY12].

5.3 Co-polarisations by balanced classes

In this section, we give the main new construction of this chapter.

5.3.1 Definitions

Let (X, ω) be a compact **balanced Calabi-Yau $\partial\bar\partial$-manifold** ($n = \dim_{\mathbb{C}} X$). Denote by $\pi : \mathcal{X} \to B$ the Kuranishi family of X. Thus π is a proper holomorphic submersion from a complex manifold $\mathcal{X}$, while the fibres X_t with $t \in B \setminus \{0\}$ can be seen as deformations of the given manifold $X_0 = X$. The base space B is smooth and can be viewed as an open subset of $H^{0,1}(X, T^{1,0}X)$ (or as a ball containing the origin in $\mathbb{C}^N$, where $N = \dim_{\mathbb{C}} H^{0,1}(X, T^{1,0}X)$) by Theorem 2.4.7. Hence the tangent space at 0 is

$$T_0 B \simeq H^{0,1}(X, T^{1,0}X).$$

By Corollary 5.1.1, X_t is a **balanced Calabi-Yau $\partial\bar\partial$-manifold** for all $t \in B$ close to 0.

Recall that, by Ehresmann's Theorem, for every degree k, the k-th De Rham cohomology space $H^k_{DR}(X_t, \mathbb{C})$ of X_t can be identified with a fixed $\mathbb{C}$-vector space $H^k_{DR}(X, \mathbb{C})$ for all $t \in B$. (We also denote by X the C^∞ manifold underlying all the fibres X_t.) By a *balanced class* $[\omega_t^{n-1}] \in H^{n-1,n-1}_{\bar\partial}(X_t, \mathbb{C}) \subset H^{2n-2}_{DR}(X, \mathbb{C})$ on X_t we shall mean the Dolbeault cohomology class of type $(n-1, n-1)$ (or the De Rham cohomology class of degree $2n-2$ that is the image of the former under the above canonical inclusion of vector spaces which holds thanks to the $\partial\bar\partial$-property of X_t and (5.1)) of the $(n-1)^{st}$ power of a balanced metric ω_t on X_t.

Recall that in the special case where the metric ω is Kähler (and thus defines a Kähler class $[\omega]_{\bar\partial} \in H^{1,1}_{\bar\partial}(X, \mathbb{C}) \subset H^2_{DR}(X, \mathbb{C})$ that is, furthermore, often required to be integral, but we deal with arbitrary, possibly non-rational classes here), it is standard to define the deformations of $X_0 = X$ *polarised* by $[\omega]_{\bar\partial}$ as those nearby fibres X_t on which the De Rham class $\{\omega\}_{DR} \in H^2_{DR}(X, \mathbb{C})$ is still a Kähler class (hence, in particular, of type (1, 1)) for the complex structure J_t of X_t. In the more general balanced case treated here, ω need not define a class, but ω^{n-1} does. Taking our cue from the standard Kähler case, we propose the following dual notion in the balanced context.

Definition 5.3.1. *Having fixed a balanced class*

$$[\omega^{n-1}]_{\bar\partial} \in H^{n-1,\,n-1}_{\bar\partial}(X,\,\mathbb{C}) \subset H^{2n-2}_{DR}(X,\,\mathbb{C}),$$

on $X_0 = X$, we say that a fibre X_t is **co-polarised** *by $[\omega^{n-1}]_{\bar\partial}$ if the De Rham class*

$$\{\omega^{n-1}\}_{DR} \in H^{2n-2}_{DR}(X,\,\mathbb{C})$$

is of type $(n-1,\,n-1)$ for the complex structure J_t of X_t.

The restricted family $\pi : \mathcal{X}_{[\omega^{n-1}]} \to B_{[\omega^{n-1}]}$ will be called the universal family of deformations of X that are co-polarised by the balanced class $[\omega^{n-1}]_{\bar\partial}$, where $B_{[\omega^{n-1}]}$ is the set of $t \in B$ such that X_t is co-polarised by $[\omega^{n-1}]_{\bar\partial}$ and $\mathcal{X}_{[\omega^{n-1}]} = \pi^{-1}(B_{[\omega^{n-1}]}) \subset \mathcal{X}$.

After possibly shrinking $B_{[\omega^{n-1}]}$ about 0, we may assume that $\{\omega^{n-1}\}_{DR} \in H^{2n-2}_{DR}(X,\,\mathbb{C})$ is a balanced class (i.e. contains the $(n-1)$-st power of a balanced metric) for the complex structure J_t of the fibre X_t for every $t \in B_{[\omega^{n-1}]}$ (cf. Observation 5.1.3).

Note that in the special case where ω is Kähler on $X_0 = X$, the $(2n-2)$-class $\{\omega^{n-1}\}_{DR}$ is a balanced class for J_t whenever the 2-class $\{\omega\}_{DR}$ is a Kähler class for J_t. We shall see further down that the converse also holds, meaning that in the special Kähler case the notion of co-polarised deformations of X coincides with that of polarised deformations.

When ω is Kähler, it is a standard fact that the deformations of X polarised by $[\omega]_{\bar\partial}$ are parametrised by the following subspace of $H^{0,\,1}(X,\,T^{1,\,0}X)$:

$$H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega]} := \left\{[\theta] \in H^{0,\,1}(X,\,T^{1,\,0}X) \;\mid\; [\theta\lrcorner\omega] = 0 \in H^{0,\,2}_{\bar\partial}(X,\,\mathbb{C})\right\} \tag{5.14}$$

which is isomorphic under $T_{[u]}$ (cf. (2.48)) to the space of *primitive* Dolbeault classes of type $(n-1,\,1)$:

$$H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega]} \xrightarrow{T_{[u]}} H^{n-1,\,1}_{prim}(X,\,\mathbb{C}). \tag{5.15}$$

We shall now see that the co-polarised deformations of X are parametrised by an analogous subspace.

Lemma 5.3.2. *For a given balanced class $[\omega^{n-1}]_{\bar\partial} \in H^{n-1,\,n-1}_{\bar\partial}(X,\,\mathbb{C})$, consider the following vector subspace of $H^{0,\,1}(X,\,T^{1,\,0}X)$:*

$$H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]} := \left\{[\theta] \in H^{0,\,1}(X,\,T^{1,\,0}X)\;;\; [\theta\lrcorner\omega^{n-1}] = 0 \in H^{n-2,\,n}_{\bar\partial}(X,\,\mathbb{C})\right\}. \tag{5.16}$$

Then:
(a) the space $H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]}$ is well-defined (i.e. the class $[\theta\lrcorner\omega^{n-1}]_{\bar\partial} \in H^{n-2,\,n}_{\bar\partial}(X,\,\mathbb{C})$ is independent of the choice of representative θ in the class $[\theta] \in H^{0,\,1}(X,\,T^{1,\,0}X)$ and of the choice of representative ω^{n-1} in the class $[\omega^{n-1}]_{\bar\partial} \in H^{n-1,\,n-1}_{\bar\partial}(X,\,\mathbb{C})$). We can therefore put:

$$[\theta]\lrcorner[\omega^{n-1}] := [\theta\lrcorner\omega^{n-1}]. \tag{5.17}$$

(b) the open subset $B\subset H^{0,\,1}(X,\,T^{1,\,0}X)$ relates to $B_{[\omega^{n-1}]}$ as follows:

$$B_{[\omega^{n-1}]}=B\cap H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]}.$$

Implicitly, $T_0B_{[\omega^{n-1}]}\simeq H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]}$.

Proof. (a) follows from Lemma 5.3.3 below. Indeed, if $\theta+\bar\partial\xi$ is another representative of the class $[\theta]$ for some vector field $\xi\in C^\infty(X,\,T^{1,\,0}X)$, then

$$(\theta+\bar\partial\xi)\lrcorner\omega^{n-1}=\theta\lrcorner\omega^{n-1}+\bar\partial(\xi\lrcorner\omega^{n-1})$$

since ω is balanced. Hence $[(\theta+\bar\partial\xi)\lrcorner\omega^{n-1}]_{\bar\partial}=[\theta\lrcorner\omega^{n-1}]_{\bar\partial}$. Similarly, if $\omega^{n-1}+\bar\partial\lambda$ is another representative of the Dolbeault class $[\omega^{n-1}]$ for some $(n-1,\,n-2)$-form λ, then

$$\theta\lrcorner(\omega^{n-1}+\bar\partial\lambda)=\theta\lrcorner\omega^{n-1}+\bar\partial(\theta\lrcorner\lambda)$$

since $\bar\partial\theta=0$. Hence $[\theta\lrcorner(\omega^{n-1}+\bar\partial\lambda)]_{\bar\partial}=[\theta\lrcorner\omega^{n-1}]_{\bar\partial}$.

(b) Since X_t is a $\partial\bar\partial$-manifold for every t close to 0, it admits a Hodge decomposition which in degree $2n-2$ spells:

$$H^{2n-2}_{DR}(X,\,\mathbb{C})=H^{n,\,n-2}_{\bar\partial}(X_t,\,\mathbb{C})\oplus H^{n-1,\,n-1}_{\bar\partial}(X_t,\,\mathbb{C})\oplus H^{n-2,\,n}_{\bar\partial}(X_t,\,\mathbb{C}),$$

with $H^{n-2,\,n}_{\bar\partial}(X_t,\,\mathbb{C})\simeq\overline{H^{n,\,n-2}_{\bar\partial}(X_t,\,\mathbb{C})}$. In our case, the real De Rham class $\{\omega^{n-1}\}_{DR}\in H^{2n-2}_{DR}(X,\,\mathbb{R})$ splits accordingly as

$$\{\omega^{n-1}\}_{DR}=\{\omega^{n-1}\}^{n,\,n-2}_t+\{\omega^{n-1}\}^{n-1,\,n-1}_t+\{\omega^{n-1}\}^{n-2,\,n}_t,$$

with $\{\omega^{n-1}\}^{n-2,\,n}_t=\overline{\{\omega^{n-1}\}^{n,\,n-2}_t}$ and $\{\omega^{n-1}\}^{n-1,\,n-1}_t$ real. Thus, the definition of $B_{[\omega^{n-1}]}$ translates to

$$B_{[\omega^{n-1}]}=\bigg\{t\in B\;;\;\{\omega^{n-1}\}^{n-2,\,n}_t=0\in H^{n-2,\,n}_{\bar\partial}(X_t,\,\mathbb{C})\bigg\}.$$

Moreover, $\{\omega^{n-1}\}_{DR}$ is of type $(n-1,\,n-1)$ for J_0, so $\{\omega^{n-1}\}^{n-2,\,n}_0=0$ and $\{\omega^{n-1}\}^{n,\,n-2}_0=0$. Let $t_1,\dots,t_N$ be local holomorphic coordinates about 0 in Δ. So $t=(t_1,\dots,t_N)\in B$ identifies with $[\theta]$ varying in an open subset of $H^{0,\,1}(X,\,T^{1,\,0}X)$. Let $[\theta]\in H^{0,\,1}(X,\,T^{1,\,0}X)$ be the image of $\frac{\partial}{\partial t_i}_{|t_i=0}$ under the Kodaira-Spencer map $\rho:T_0B\stackrel{\simeq}{\longrightarrow}H^{0,\,1}(X,\,T^{1,\,0}X)$. Then, under the Gauss-Manin connection on the Hodge bundle $B\ni t\mapsto H^{2n-2}_{DR}(X_t,\,\mathbb{C})$, the derivative of the class $[\omega^{n-1}]^{n-2,\,n}_t\in H^{n-2,\,n}_{\bar\partial}(X_t,\,\mathbb{C})$ in the direction of t_i at $t_i=0$ is the class $[\theta\lrcorner\omega^{n-1}]\in H^{n-2,\,n}_{\bar\partial}(X,\,\mathbb{C})$. $\square$

Here is the lemma that has been used in the proof of (a) above.

Lemma 5.3.3. *Let X be a compact complex manifold ($\dim_{\mathbb{C}}X=n$) equipped with an arbitrary Hermitian metric ω. Then:*

(i) $\bar\partial(\xi\lrcorner\omega^{n-1})=(\bar\partial\xi)\lrcorner\omega^{n-1}-\xi\lrcorner\bar\partial\omega^{n-1}$, *for every $\xi\in C^\infty(X,\,T^{1,\,0}X)$.*

Hence, if ω is balanced, we have $\bar\partial(\xi\lrcorner\omega^{n-1})=(\bar\partial\xi)\lrcorner\omega^{n-1}$.

(ii) $\bar\partial(\theta\lrcorner\omega)=(\bar\partial\theta)\lrcorner\omega+\theta\lrcorner\bar\partial\omega$, *for every $\theta\in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$.*

Analogous identities hold for forms of any type in place of ω or ω^{n-1}. However, the analogous identities for ∂ in place of $\bar\partial$ fail (intuitively because ∂ increases the holomorphic degree of forms, while the contraction by a vector field of type $(1, 0)$ decreases the same holomorphic degree).

Proof. Fix an arbitrary point $x_0 \in X$ and let $z_1, \dots, z_n$ be local holomorphic coordinates about x_0. If we denote

$$\omega^{n-1} = i^{n-1}\sum_{\alpha,\,\beta}\gamma_{\alpha\beta}\, \widehat{dz_\alpha\wedge d\bar z_\beta} \quad \text{and} \quad \xi = \sum_j \xi_j\,\frac{\partial}{\partial z_j},$$

where $\widehat{dz_\alpha\wedge d\bar z_\beta} := dz_1\wedge\cdots\wedge\widehat{dz_\alpha}\wedge\cdots\wedge dz_n\wedge d\bar z_1\wedge\cdots\wedge\widehat{d\bar z_\beta}\wedge\cdots\wedge d\bar z_n$, we get:

$$\begin{aligned}\xi\lrcorner\omega^{n-1} &= i^{n-1}\sum_{\substack{\beta\\ j<\alpha}}(-1)^{j-1}\,\xi_j\gamma_{\alpha\beta}\,(dz_j\wedge\widehat{dz_\alpha}\wedge d\bar z_\beta) + i^{n-1}\sum_{\substack{\beta\\ j>\alpha}}(-1)^{j}\,\xi_j\gamma_{\alpha\beta}\,(dz_\alpha\wedge\widehat{dz_j}\wedge d\bar z_\beta)\\ &= i^{n-1}\sum_{\substack{\beta\\ j<\alpha}}\bigg((-1)^{j-1}\,\xi_j\gamma_{\alpha\beta} + (-1)^{\alpha}\,\xi_\alpha\gamma_{j\beta}\bigg)\,(dz_j\wedge\widehat{dz_\alpha}\wedge d\bar z_\beta),\end{aligned}$$

where we have used the notation:

$$(dz_j\wedge\widehat{dz_\alpha}\wedge d\bar z_\beta) := dz_1\wedge\cdots\wedge\widehat{dz_j}\wedge\cdots\wedge\widehat{dz_\alpha}\wedge\cdots\wedge dz_n\wedge d\bar z_1\wedge\cdots\wedge\widehat{d\bar z_\beta}\wedge\cdots\wedge d\bar z_n.$$

Hence, by applying $\bar\partial$, we get:

$$\begin{aligned}\bar\partial(\xi\lrcorner\omega^{n-1}) = i^{n-1}\sum_{\substack{\beta\\ j<\alpha}}(-1)^{n+\beta-1}\bigg[(-1)^{j-1}\,\xi_j\frac{\partial\gamma_{\alpha\beta}}{\partial\bar z_\beta} &+ (-1)^{j-1}\,\frac{\partial\xi_j}{\partial\bar z_\beta}\gamma_{\alpha\beta} + (-1)^{\alpha}\xi_\alpha\frac{\partial\gamma_{j\beta}}{\partial\bar z_\beta}\\ &+ (-1)^{\alpha}\frac{\partial\xi_\alpha}{\partial\bar z_\beta}\gamma_{j\beta}\bigg]\,\widehat{dz_j\wedge dz_\alpha}. \qquad (5.18)\end{aligned}$$

Similar calculations yield:

$$(\bar\partial\xi)\lrcorner\omega^{n-1} = (-1)^n i^{n-1}\sum_{\substack{\beta\\ j<\alpha}}(-1)^{\beta}\bigg((-1)^{j}\frac{\partial\xi_j}{\partial\bar z_\beta}\gamma_{\alpha\beta} - (-1)^{\alpha}\frac{\partial\xi_\alpha}{\partial\bar z_\beta}\gamma_{j\beta}\bigg)\widehat{dz_j\wedge dz_\alpha}, \qquad (5.19)$$

showing that $(\bar\partial\xi)\lrcorner\omega^{n-1}$ equals the sum of the second and fourth groups of terms in the expression (5.18) for $(\bar\partial\xi)\lrcorner\omega^{n-1}$. On the other hand, we get:

$$\bar\partial\omega^{n-1} = (-1)^n i^{n-1}\sum_{\alpha,\beta}(-1)^{\beta}\frac{\partial\gamma_{\alpha\beta}}{\partial\bar z_\beta}\,\widehat{dz_\alpha},$$

leading to

$$\begin{aligned}\xi\lrcorner\bar\partial\omega^{n-1} &= (-1)^n i^{n-1}\sum_{\substack{\beta\\ j<\alpha}}(-1)^{j+\beta-1}\xi_j\frac{\partial\gamma_{\alpha\beta}}{\partial\bar z_\beta}\,\widehat{dz_j\wedge dz_\alpha} + (-1)^n i^{n-1}\sum_{\substack{\beta\\ j>\alpha}}(-1)^{j+\beta}\xi_j\frac{\partial\gamma_{\alpha\beta}}{\partial\bar z_\beta}\,\widehat{dz_\alpha\wedge dz_j}\\ &= (-1)^n i^{n-1}\sum_{\substack{\beta\\ j<\alpha}}(-1)^{\beta}\bigg((-1)^{j-1}\xi_j\frac{\partial\gamma_{\alpha\beta}}{\partial\bar z_\beta} + (-1)^{\alpha}\xi_\alpha\frac{\partial\gamma_{j\beta}}{\partial\bar z_\beta}\bigg)\,\widehat{dz_j\wedge dz_\alpha}.\end{aligned}$$

Thus $\xi\lrcorner\bar\partial\omega^{n-1}$ equals the sum multiplied by (-1) of the first and third groups of terms in the expression (5.18) for $\bar\partial(\xi\lrcorner\omega^{n-1})$. Combining with (5.18) and (5.19), we get the identity claimed in (i). Similar calculations prove (ii). □

5.3.2 Comparison to polarisations of the Kähler case

We now pause to observe that in the special case of a Kähler class $[\omega] \in H^{1,1}(X, \mathbb{C})$, co-polarised deformations of X coincide with polarised deformations. Thus, although the space $H^{0,1}(X, T^{1,0}X)_{[\omega]}$ of (5.14) no longer makes sense for a non-Kähler ω, $H^{0,1}(X, T^{1,0}X)_{[\omega^{n-1}]}$ defined in (5.16) naturally extends its meaning to the case of a balanced class $[\omega^{n-1}]$.

Proposition 5.3.4. *Let (X, ω) be a compact Kähler manifold ($n = dim_{\mathbb{C}}X$) such that K_X is trivial. Then the following identity holds:*

$$H^{0,1}(X, T^{1,0}X)_{[\omega]} = H^{0,1}(X, T^{1,0}X)_{[\omega^{n-1}]}. \tag{5.20}$$

Proof. We start by noticing that for any Hermitian metric ω (no assumption is necessary on ω here) and any $\theta \in C^\infty_{0,1}(X, T^{1,0}X)$, we have

$$\theta\lrcorner\omega^k = k\,\omega^{k-1}\wedge(\theta\lrcorner\omega) \quad \text{for any} \quad k. \tag{5.21}$$

This follows from the property $\theta\lrcorner(\omega\wedge\omega^{k-1}) = (\theta\lrcorner\omega)\wedge\omega^{k-1} + \omega\wedge(\theta\lrcorner\omega^{k-1})$.

Suppose now that ω is Kähler and let $[\theta] \in H^{0,1}(X, T^{1,0}X)_{[\omega]}$, i.e. $\theta\lrcorner\omega$ is $\bar\partial$-exact. Writing $\theta\lrcorner\omega = \bar\partial v$ for some $(0, 1)$-form v, from (5.21) we get:

$$\theta\lrcorner\omega^{n-1} = (n-1)\,\omega^{n-2}\wedge\bar\partial v = (n-1)\,\bar\partial(\omega^{n-2}\wedge v)$$

since $\bar\partial\omega^{n-2} = 0$ by the Kähler assumption on ω. Thus $\theta\lrcorner\omega^{n-1}$ is $\bar\partial$-exact, proving that $[\theta] \in H^{0,1}(X, T^{1,0}X)_{[\omega^{n-1}]}$. This proves the inclusion "$\subset$".

Proving the reverse inclusion "$\supset$" in (5.20) takes more work. Let us consider the Lefschetz operator

$$L^{n-2}_\omega : C^\infty_{0,2}(X, \mathbb{C}) \to C^\infty_{n-2,n}(X, \mathbb{C}), \qquad \alpha \mapsto \omega^{n-2}\wedge\alpha, \tag{5.22}$$

of multiplication by ω^{n-2} which is well known to be an isomorphism for any Hermitian (even non-Kähler or non-balanced) metric ω (see e.g. [Voi02, lemma 6.20, p. 146]). We clearly have $\theta\lrcorner\omega^{n-1} = (n-1)\,L^{n-2}_\omega(\theta\lrcorner\omega)$ by (5.21).

The next lemma explains how the three-space decomposition (w.r.t. ω)

$$C^\infty_{0,2}(X, \mathbb{C}) = \ker\Delta'' \oplus \mathrm{Im}\,\bar\partial \oplus \mathrm{Im}\,\bar\partial^\star$$

transforms under L^{n-2}_ω and compares to the analogous decomposition of $C^\infty_{n-2,n}(X, \mathbb{C})$. Note that in $C^\infty_{n-2,n}(X, \mathbb{C})$ the subspace $\mathrm{Im}\,\bar\partial^\star$ is reduced to zero for bidegree reasons.

Lemma 5.3.5. *If ω is a Kähler metric on a compact complex manifold X with $n = dim_{\mathbb{C}}X$, then the operator (5.22) satisfies:*

$$L^{n-2}_\omega(\ker\Delta'') = \ker\Delta'' \quad \textit{and} \quad L^{n-2}_\omega(Im\,\bar\partial \oplus Im\,\bar\partial^\star) = Im\,\bar\partial. \tag{5.23}$$

This will follow from two formulae that have an interest of their own.

Lemma 5.3.6. *If ω is Kähler, then for every $\alpha \in C^\infty_{0,2}(X, \mathbb{C})$ we have:*

$$\bar\partial^\star(\omega^{n-2}\wedge\alpha) = \omega^{n-2}\wedge\bar\partial^\star\alpha + (n-2)\,\omega^{n-3}\wedge i\partial\alpha. \tag{5.24}$$

Proof. Using the Kähler commutation relation $\bar\partial^\star = -i\,[\Lambda,\,\partial]$, we get:

$$\bar\partial^\star(\omega^{n-2}\wedge\alpha) = -i\,\Lambda(\omega^{n-2}\wedge\partial\alpha) + i\,\partial(\Lambda(\omega^{n-2}\wedge\alpha)). \tag{5.25}$$

In the first term on the right-hand side of (5.25), we have:

$$\Lambda(\omega^{n-2}\wedge\partial\alpha) = [\Lambda,\,L^{n-2}](\partial\alpha) + \omega^{n-2}\wedge\Lambda(\partial\alpha) = \omega^{n-2}\wedge\Lambda(\partial\alpha). \tag{5.26}$$

The last identity follows from the well-known formula (cf. [Voi02, p. 148]):

$$[L^r,\,\Lambda] = r(k-n+r-1)L^{r-1} \quad \text{on } k\text{-forms, for every } r, \tag{5.27}$$

which, when applied with $r=n-2$ to the 3-form $\partial\alpha$, gives $[\Lambda,\,L^{n-2}](\partial\alpha)=0$.

In the second term on the right-hand side of (5.25), we have:

$$\Lambda(\omega^{n-2}\wedge\alpha) = [\Lambda,\,L^{n-2}](\alpha) + \omega^{n-2}\wedge\Lambda(\alpha) = (n-2)\,\omega^{n-3}\wedge\alpha + \omega^{n-2}\wedge\Lambda(\alpha),$$

where the last identity follows again from (5.27) applied with $r=n-2$ to the 2-form α. (Note that $\Lambda\alpha=0$, but we ignore this here.) Taking ∂ on either side of the above identity and using the Kähler assumption on ω, we get:

$$\partial(\Lambda(\omega^{n-2}\wedge\alpha)) = (n-2)\,\omega^{n-3}\wedge\partial\alpha + \omega^{n-2}\wedge\partial\Lambda(\alpha), \tag{5.28}$$

Thus, putting (5.26) and (5.28) together, we see that (5.25) transforms to

$$\begin{aligned}\bar\partial^\star(\omega^{n-2}\wedge\alpha) &= -i\,\omega^{n-2}\wedge\Lambda(\partial\alpha) + (n-2)\,\omega^{n-3}\wedge i\partial\alpha + \omega^{n-2}\wedge i\partial\Lambda(\alpha)\\ &= \omega^{n-2}\wedge i\,[\partial,\,\Lambda](\alpha) + (n-2)\,\omega^{n-3}\wedge i\partial\alpha\\ &= \omega^{n-2}\wedge\bar\partial^\star\alpha + (n-2)\,\omega^{n-3}\wedge i\partial\alpha.\end{aligned}$$

This is what we had set out to prove. Note that we have used again the Kähler commutation relation $i\,[\partial,\,\Lambda] = -i\,[\Lambda,\,\partial] = \bar\partial^\star$. $\square$

The next formula we need is the following.

Lemma 5.3.7. *If ω is Kähler, then for every $\alpha\in C^\infty_{0,\,2}(X,\,\mathbb{C})$ we have:*

$$\Delta''_\omega(\omega^{n-2}\wedge\alpha) = \omega^{n-2}\wedge\Delta''_\omega\alpha. \tag{5.29}$$

Proof. This is an immediate consequence of the commutation property

$$[L_\omega,\,\Delta''_\omega]=0, \qquad \text{hence} \qquad [L^k_\omega,\,\Delta''_\omega]=0 \quad \text{for all } k,$$

which in turn follows from the Kähler identities. Alternatively, we can use Lemma 5.3.6 and the Kähler identities to give a direct proof as follows. Since $\bar\partial(\omega^{n-2}\wedge\alpha)=0$ for bidegree reasons, $\Delta''_\omega(\omega^{n-2}\wedge\alpha)$ reduces to its first term, so using (5.24) we get:

$$\begin{aligned}\Delta''_\omega(\omega^{n-2}\wedge\alpha) &= \bar\partial\bar\partial^\star(\omega^{n-2}\wedge\alpha) = \bar\partial(\omega^{n-2}\wedge\bar\partial^\star\alpha + (n-2)\,\omega^{n-3}\wedge i\partial\alpha)\\ &= \omega^{n-2}\wedge\bar\partial\bar\partial^\star\alpha + (n-2)\,\omega^{n-3}\wedge i\bar\partial\partial\alpha.\end{aligned} \tag{5.30}$$

Now, using the Kähler identity $\bar\partial^\star = -i\,[\Lambda,\,\partial]$, we get:

$$\begin{aligned}\omega^{n-2}\wedge\bar\partial^\star\bar\partial\alpha &= -i\,\omega^{n-2}\wedge[\Lambda,\,\partial]\bar\partial\alpha = -i\,\omega^{n-2}\wedge\Lambda(\partial\bar\partial\alpha) + i\,\omega^{n-2}\wedge\partial\Lambda(\bar\partial\alpha)\\ &= -i\,\omega^{n-2}\wedge\Lambda(\partial\bar\partial\alpha)\end{aligned} \tag{5.31}$$

because $\bar\partial\alpha$ is of type $(0,\,3)$, so $\Lambda(\bar\partial\alpha)=0$ for bidegree reasons. Meanwhile,

$$\begin{aligned}\omega^{n-2}\wedge\Lambda(\partial\bar\partial\alpha) &= [L^{n-2},\,\Lambda](\partial\bar\partial\alpha)+\Lambda(\omega^{n-2}\wedge\partial\bar\partial\alpha)=[L^{n-2},\,\Lambda](\partial\bar\partial\alpha)\\ &= (n-2)\,\omega^{n-3}\wedge\partial\bar\partial\alpha.\end{aligned}\tag{5.32}$$

The second identity on the top line above follows from $\omega^{n-2}\wedge\partial\bar\partial\alpha=0$ for bidegree reasons (since $\omega^{n-2}\wedge\partial\bar\partial\alpha$ is of type $(n-1,\,n+1)$, hence vanishes), while the last identity follows from formula (5.27) with $r=n-2$ and $k=4$.

The combined identities (5.31) and (5.32) yield:

$$\omega^{n-2}\wedge\bar\partial^\star\bar\partial\alpha=-(n-2)\,\omega^{n-3}\wedge i\partial\bar\partial\alpha=(n-2)\,\omega^{n-3}\wedge i\bar\partial\partial\alpha.$$

This last identity combines with (5.30) to prove the claim. □

We need yet another observation.

Lemma 5.3.8. *For any Hermitian metric ω on X, the normalised Lefschetz operator*

$$\frac{1}{(n-2)!}\,L_\omega^{n-2}:C^\infty_{0,\,2}(X,\,\mathbb{C})\to C^\infty_{n-2,\,n}(X,\,\mathbb{C})$$

is an isometry w.r.t. the L^2 scalar product induced by ω on scalar-valued forms.

Proof. We will show that for every $l=3,\dots,n$, the following formula holds:

$$\langle\langle\omega^{n-2}\wedge\alpha,\,\omega^{n-2}\wedge\beta\rangle\rangle=(n-2)!\,\frac{(l-2)!}{(n-l)!}\,\langle\langle\omega^{n-l}\wedge\alpha,\,\omega^{n-l}\wedge\beta\rangle\rangle\tag{5.33}$$

for all forms $\alpha,\beta\in C^\infty_{0,\,2}(X,\,\mathbb{C})$. We have:

$$\begin{aligned}\langle\langle\omega^{n-2}\wedge\alpha,\,\omega^{n-2}\wedge\beta\rangle\rangle &= \langle\langle\Lambda(\omega^{n-2}\wedge\alpha),\,\omega^{n-3}\wedge\beta\rangle\rangle\\ &= \langle\langle[\Lambda,\,L^{n-2}]\,\alpha,\,\omega^{n-3}\wedge\beta\rangle\rangle\\ &= (n-2)\,\langle\langle\omega^{n-3}\wedge\alpha,\,\omega^{n-3}\wedge\beta\rangle\rangle,\end{aligned}$$

where in going from the first to the second line, we have used the identities $[\Lambda,\,L^{n-2}]\,\alpha=\Lambda(\omega^{n-2}\wedge\alpha)-\omega^{n-2}\wedge\Lambda\,\alpha=\Lambda(\omega^{n-2}\wedge\alpha)$ since $\Lambda\alpha=0$ for bidegree reasons, while in going from the second to the third line we have used formula (5.27) with $r=n-2$ and the anti-commutation $[\Lambda,\,L^{n-2}]=-[L^{n-2},\,\Lambda]$. This proves (5.33) for $l=3$. We can now continue by induction on l. Suppose that (5.33) has been proved for l. We have:

$$\begin{aligned}\langle\langle\omega^{n-l}\wedge\alpha,\,\omega^{n-l}\wedge\beta\rangle\rangle &= \langle\langle\Lambda(\omega^{n-l}\wedge\alpha),\,\omega^{n-l-1}\wedge\beta\rangle\rangle\\ &= \langle\langle[\Lambda,\,L^{n-l}]\,\alpha,\,\omega^{n-l-1}\wedge\beta\rangle\rangle\\ &= (n-l)(l-1)\,\langle\langle\omega^{n-l-1}\wedge\alpha,\,\omega^{n-l-1}\wedge\beta\rangle\rangle\end{aligned}$$

by arguments similar to those above, where formula (5.27) has been used with $r=n-l$. We thus obtain (5.33) with $l+1$ in place of l.

It is now clear that (5.33) for $l=n$ proves the contention. □

End of proof of Lemma 5.3.5. Since the map L_ω^{n-2} of (5.22) is an isomorphism, it follows from Lemma 5.3.7 that $L_\omega^{n-2}(\ker\Delta''_\omega)=\ker\Delta''_\omega$. Since L_ω^{n-2} maps any pair of orthogonal forms in $C^\infty_{0,\,2}(X,\,\mathbb{C})$ to a pair of orthogonal forms in $C^\infty_{n-2,\,n}(X,\,\mathbb{C})$ by Lemma 5.3.8, it follows that the orthogonal complement of $\ker\Delta''_\omega$ in $C^\infty_{0,2}(X,\,\mathbb{C})$ (i.e. $\mathrm{Im}\,\bar\partial\oplus\mathrm{Im}\,\bar\partial^\star$) is isomorphic under L^{n-2}_ω to the orthogonal complement of $\ker\Delta''_\omega$ in $C^\infty_{n-2,\,n}(X,\,\mathbb{C})$ (i.e. $\mathrm{Im}\,\bar\partial$). Note that $\mathrm{Im}\,\bar\partial^\star=0$ in $C^\infty_{n-2,\,n}(X,\,\mathbb{C})$ for type reasons. The proof is complete. $\square$

End of proof of Proposition 5.3.4. Recall that we have yet to prove the inclusion "$\supset$" in (5.20). Let $[\theta]\in H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]}$. This means that $\theta\lrcorner\omega^{n-1}\in\mathrm{Im}\,\bar\partial\subset C^\infty_{n-2,\,n}(X,\,\mathbb{C})$ (cf. (5.16)). Since $\theta\lrcorner\omega^{n-1}=(n-1)\,L^{n-2}_\omega(\theta\lrcorner\omega)$ (cf. (5.21)) and $\theta\lrcorner\omega$ is of type $(0,\,2)$, we get from Lemma 5.3.5 that

$$\theta\lrcorner\omega\in\mathrm{Im}\,\bar\partial\oplus\mathrm{Im}\,\bar\partial^\star\subset C^\infty_{0,\,2}(X,\,\mathbb{C}). \tag{5.34}$$

On the other hand, $\bar\partial\theta=0$ (since θ represents a class $[\theta]\in H^{0,\,1}(X,\,T^{1,\,0}X)$) and $\bar\partial\omega=0$ (since ω is assumed Kähler). Hence (ii) of Lemma 5.3.3 gives:

$$\bar\partial(\theta\lrcorner\omega)=0, \qquad \text{i.e.} \qquad \theta\lrcorner\omega\in\ker\bar\partial=\ker\Delta''_\omega\oplus\mathrm{Im}\,\bar\partial\subset C^\infty_{0,\,2}(X,\,\mathbb{C}). \tag{5.35}$$

Since the three subspaces in the decomposition $C^\infty_{0,\,2}(X,\,\mathbb{C})=\ker\Delta''_\omega\oplus\mathrm{Im}\,\bar\partial\oplus\mathrm{Im}\,\bar\partial^\star$ are mutually orthogonal, (5.34) and (5.35) imply that $\theta\lrcorner\omega\in\mathrm{Im}\,\bar\partial$, i.e. $[\theta]\in H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega]}$ (cf. (5.14)). $\square$

5.3.3 Primitive $(n-1,\,1)$-classes on balanced manifolds

In the case of a Kähler class $[\omega]$, primitive Dolbeault cohomology classes of type $(n-1,\,1)$ (for $[\omega]$) play a pivotal role in the theory of deformations of X that are polarised by $[\omega]$ thanks to the isomorphism (5.15) induced by the Calabi-Yau isomorphism. However, if $[\omega]$ is replaced by a balanced class $[\omega^{n-1}]$, primitive classes can no longer be defined in the standard way except in the case of $(1,\,1)$-classes or, more generally, in that of De Rham 2-classes (since the definition uses then the $(n-1)^{st}$ power of ω that is closed by the balanced assumption). In particular, defining an $(n-1,\,1)$-class $[\alpha]$ as *primitive* by requiring that $\omega\wedge\alpha$ be $\bar\partial$-exact would be meaningless if ω is not closed since this definition would depend on the choice of representative α of the class $[\alpha]$. However, since the space $H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]}$ carries over the meaning of $H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega]}$ to the balanced case, it is natural to make the following ad hoc definition in the balanced case.

Definition 5.3.9. *Let X be a compact balanced Calabi-Yau $\partial\bar\partial$-manifold ($n:=\dim_{\mathbb{C}}X$). Fix a non-vanishing holomorphic $(n,\,0)$-form u and a balanced class $[\omega^{n-1}]$ on X. The space of* **primitive** *classes of type $(n-1,\,1)$ (for $[\omega^{n-1}]$) is defined as the image under the Calabi-Yau isomorphism*

$$T_{[u]}:H^{0,\,1}(X,\,T^{1,\,0}X)\xrightarrow{\cdot\lrcorner[u]}H^{n-1,\,1}(X,\,\mathbb{C})$$

in (2.48) of the subspace $H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]}\subset H^{0,\,1}(X,\,T^{1,\,0}X)$, i.e.

$$H^{n-1,\,1}_{prim}(X,\,\mathbb{C}):=T_{[u]}\bigg(H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]}\bigg)\subset H^{n-1,\,1}(X,\,\mathbb{C}).$$

Explicitly, given the definition (5.16) of $H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]}$, this means:

$$[\theta\lrcorner u]\in H^{n-1,\,1}_{prim}(X,\,\mathbb{C}) \quad \textit{iff} \quad [\theta\lrcorner\omega^{n-1}]=0\in H^{n-2,\,n}(X,\,\mathbb{C}) \tag{5.36}$$

for any class $[\theta]\in H^{0,\,1}(X,\,T^{1,\,0}X)$.

It is clear that $H^{n-1,\,1}_{prim}(X,\,\mathbb{C})$ does not depend on the choice of u (which is unique up to a constant factor), but depends on the choice of balanced class $[\omega^{n-1}]$. When ω is Kähler, the ad hoc definition of $H^{n-1,1}_{prim}(X,\,\mathbb{C})$ coincides with the standard definition thanks to the isomorphism (5.15) and to Proposition 5.3.4.

Recall that unlike cohomology classes, primitive forms can be defined in the standard way for any Hermitian metric ω: for any $k\leq n$, a k-form α on X is primitive for ω if $\omega^{n-k+1}\wedge\alpha=0$. This condition is well known to be equivalent to $\Lambda_\omega\alpha=0$. No closedness assumption on ω is needed.

In the rest of this subsection we shall investigate the extent to which the ad hoc primitive $(n-1,\,1)$-classes defined by a balanced class retain the properties of primitive classes standardly defined by a Kähler class. We start with the form analogue of (5.36). By the Calabi-Yau isomorphism (2.46), all $(n-1,\,1)$-forms are of the shape $\theta\lrcorner u$ for some $\theta\in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$.

Lemma 5.3.10. *Let $(X,\,\omega)$ be an arbitrary Hermitian compact complex manifold ($n:=\dim_{\mathbb{C}}X$) with K_X trivial. Fix a non-vanishing holomorphic $(n,0)$-form u. Then for any $\theta\in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$, the following equivalences hold:*

$$\theta\lrcorner u \text{ is primitive for } \omega \quad\iff \theta\lrcorner\omega=0 \quad\iff \theta\lrcorner\omega^{n-1}=0. \tag{5.37}$$

Proof. By the definition of primitiveness, the $(n-1,\,1)$-form $\theta\lrcorner u$ is primitive for ω if and only if $\omega\wedge(\theta\lrcorner u)=0$. Meanwhile,

$$0=\theta\lrcorner(\omega\wedge u)=(\theta\lrcorner\omega)\wedge u+\omega\wedge(\theta\lrcorner u),$$

where the first identity holds for type reasons since the form $\omega\wedge u$ is of type $(n+1,\,1)$, hence vanishes. Thus the vanishing of $\omega\wedge(\theta\lrcorner u)$ is equivalent to the vanishing of $(\theta\lrcorner\omega)\wedge u$ which, in turn, is equivalent to the vanishing of $\theta\lrcorner\omega$ as can be easily checked using the property $u\neq 0$ at every point of X. This proves the first equivalence in (5.37). The second equivalence follows from

$$\theta\lrcorner\omega^{n-1}=(n-1)\,\omega^{n-2}\wedge(\theta\lrcorner\omega)$$

(cf. (5.21)) and from the map (5.22) being an isomorphism. $\square$

We saw in (a) of Theorem 1.3.2 that every Dolbeault cohomology class on a $\partial\bar\partial$-manifold can be represented by a d-closed form (which is, of course, not unique). The question we will now address is the following.

Question 5.3.11. *Is it true that on a balanced Calabi-Yau $\partial\bar\partial$-manifold, every* **primitive** *$(n-1,\,1)$-class (in the sense of the ad hoc Definition 5.3.9) can be represented by a form that is both* **primitive** *and d-***closed***?*

Should the answer to this question be affirmative, it would bear significantly on the discussion of Weil-Petersson metrics in §.5.4.2. It is clear that in the Kähler case the answer is affirmative: the Δ''-harmonic representative of any *primitive* (in the standard sense defined by the Kähler class = the ad hoc sense in the case of $(n-1,\,1)$-classes) $(p,\,q)$-class is both *primitive* and *d-closed.* We shall now see that the balanced case is far more complicated.

Lemma 5.3.12. *Let $(X,\,\omega)$ be a compact Hermitian manifold ($n:=\dim_{\mathbb{C}}X$) and let v be an arbitrary primitive form of type $(n-1,\,1)$ on X. Then, the following equivalences hold:*

$$\bar\partial^\star v=0\iff\partial v=0\quad \text{and}\quad \partial^\star v=0\iff\bar\partial v=0. \tag{5.38}$$

Proof. It is well-known (cf. e.g. [Dem97, VI, §. 5.1]) that $\bar\partial^\star = -\star\partial\star$ and $\partial^\star = -\star\bar\partial\star$, where $\star : \Lambda^{p,\,q}T^\star X \longrightarrow \Lambda^{n-q,n-p}T^\star X$ is the Hodge star operator associated with ω. On the other hand, the standard formula (4.68) yields:

$$\star\, v = i^{n^2+2n-2}\, v \quad \text{ for all } v \in C^\infty_{n-1,\,1}(X,\,\mathbb{C})_{prim}. \tag{5.39}$$

Since $\star$ is an isomorphism, we see that the identity $\bar\partial^\star v = 0$ is equivalent to $\partial(\star\, v) = 0$, hence to $\partial v = 0$ by (5.39). The equivalence for $\partial^\star v = 0$ is inferred similarly. $\square$

Corollary 5.3.13. *Under the assumptions of Lemma 5.3.12, we have:*

(i) if $v \in C^\infty_{n-1,\,1}(X,\,\mathbb{C})_{prim}$ and $\bar\partial v = 0$, then

$$dv = 0 \iff \Delta'' v = 0.$$

(ii) if $v \in C^\infty_{n-1,\,1}(X,\,\mathbb{C})_{prim}$ and $\partial v = 0$, then

$$dv = 0 \iff \Delta' v = 0.$$

(iii) if $v \in C^\infty_{n-1,\,1}(X,\,\mathbb{C})_{prim}$ and $dv = 0$, then

$$\Delta' v = 0, \;\; \Delta'' v = 0 \;\textit{ and }\; \Delta v = 0.$$

Proof. Since X is compact, we have $\ker\Delta'' = \ker\bar\partial \cap \ker\bar\partial^\star$ and $\ker\Delta' = \ker\partial\cap\ker\partial^\star$. Since for any *pure-type* form v, the equivalence

$$dv = 0 \iff \partial v = 0 \text{ and } \bar\partial v = 0$$

holds, (i) and (ii) follow immediately from the two equivalences in (5.38). Now (i) and (ii) obviously give $\Delta' v = 0$ and $\Delta'' v = 0$ under the assumptions of (iii). To infer that $\Delta v = 0$, it suffices to notice that for any *pure-type* form v on a compact *Hermitian* manifold $(X,\,\omega)$, we have:

$$\langle\langle\Delta v,\, v\rangle\rangle = \langle\langle\Delta' v,\, v\rangle\rangle + \langle\langle\Delta'' v,\, v\rangle\rangle \tag{5.40}$$

since $\langle\langle\Delta v,\, v\rangle\rangle = ||d\,v||^2 + ||d^\star v||^2$, $\langle\langle\Delta' v,\, v\rangle\rangle = ||\partial v||^2 + ||\partial^\star v||^2$ and $\langle\langle\Delta'' v,\, v\rangle\rangle = ||\bar\partial v||^2 + ||\bar\partial^\star v||^2$, while $||dv||^2 = ||\partial v||^2 + ||\bar\partial v||^2$ (because ∂v and $\bar\partial v$ are pure-type forms of different types, hence orthogonal) and similarly $||d^\star v||^2 = ||\partial^\star v||^2 + ||\bar\partial^\star v||^2$ (because $\partial^\star v$ and $\bar\partial^\star v$ are orthogonal for the same reason). Since $\Delta' v = 0$ and $\Delta'' v = 0$, from (5.40) we get $\langle\langle\Delta v,\, v\rangle\rangle = 0$ which amounts to $dv = 0$ and $d^\star v = 0$, hence to $\Delta v = 0$. $\square$

The conclusion (iii) of the above Corollary 5.3.13 is that if an $(n-1,\,1)$-form is both *primitive* and *d-closed*, it must be harmonic for each of the Laplacians Δ', Δ'' and Δ. Thus, if a representative that is both *primitive* and *d-closed* of a primitive $(n-1,\,1)$-class exists, it can only be the Δ''-harmonic representative. Fortunately we have

Lemma 5.3.14. *Let $(X,\,\omega)$ be a compact Hermitian manifold ($n := \dim_{\mathbb{C}}X$). Suppose v is a primitive $(n-1,\,1)$-form such that $\Delta'' v = 0$. Then $\Delta' v = 0$ and $\Delta v = 0$. In particular, $dv = 0$.*

Proof. The assumption $\Delta'' v = 0$ means that $\bar\partial v = 0$ and $\bar\partial^\star v = 0$. Then (i) of Corollary 5.3.13 implies that $dv = 0$, i.e. $\partial v = 0$. Then (ii) of Corollary 5.3.13 ensures that $\Delta' v = 0$. Then (5.40) ensures that $\Delta v = 0$. $\square$

Thus Question 5.3.11 reduces to whether on a balanced Calabi-Yau $\partial\bar\partial$-manifold (X,ω), the Δ''-harmonic representative of any *primitive* $(n-1,\,1)$-class (in the sense of the ad hoc Definition 5.3.9) is a *primitive* form. It will then also be d-closed by Lemma 5.3.14. Fix therefore a *primitive* $(n-1,\,1)$-class $[\theta\lrcorner u]$ on X, where $[\theta]\in H^{0,\,1}(X,\,T^{1,\,0}X)$. By (5.36), this means that

$$\theta\lrcorner\omega^{n-1}\in\operatorname{Im}\bar\partial. \tag{5.41}$$

Suppose, furthermore, that $\Delta''(\theta\lrcorner u)=0$. The question is whether $\theta\lrcorner u$ is primitive, or equivalently (cf. (5.37)) whether $\theta\lrcorner\omega^{n-1}=0$. Since $\ker\Delta''$ and $\operatorname{Im}\bar\partial$ are orthogonal subspaces of $C^\infty_{n-2,\,n}(X,\,\mathbb{C})$, (5.41) reduces the question to determining whether

$$\Delta''(\theta\lrcorner\omega^{n-1})=0,\quad\text{or equivalently whether}\quad\bar\partial^\star(\theta\lrcorner\omega^{n-1})=0, \tag{5.42}$$

since $\bar\partial(\theta\lrcorner\omega^{n-1})=0$ (trivially since $\theta\lrcorner\omega^{n-1}$ is of type $(n-2,\,n)$).

The next lemma transforms identity (5.42) whose validity we are trying to determine.

Lemma 5.3.15. *Let X be a compact complex manifold ($\dim_{\mathbb{C}}X=n$) equipped with an arbitrary Hermitian metric ω. Fix any $\theta\in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$. The following equivalence holds:*

$$\bar\partial^\star(\theta\lrcorner\omega^{n-1})=0\iff\partial(\theta\lrcorner\omega)=0.$$

Proof. Formula (4.68) applied to the (primitive) $(0,\,2)$-form $v:=\theta\lrcorner\omega$ reads:

$$\star(\theta\lrcorner\omega)=\frac{\omega^{n-2}}{(n-2)!}\wedge(\theta\lrcorner\omega)=\theta\lrcorner\frac{\omega^{n-1}}{(n-1)!},\quad\text{i.e.}\quad\star\left(\theta\lrcorner\frac{\omega^{n-1}}{(n-1)!}\right)=\theta\lrcorner\omega, \tag{5.43}$$

having also used the property $\star^2=\mathrm{Id}$ on 2-forms. Now, $\bar\partial^\star=-\star\partial\star$, hence the condition $\bar\partial^\star(\theta\lrcorner\omega^{n-1})=0$ is equivalent to $\partial(\star(\theta\lrcorner\omega^{n-1}))=0$ which in turn is equivalent to $\partial(\theta\lrcorner\omega)=0$ by (5.43). This proves the contention. $\square$

However, we can see no reason why the desired condition $\partial(\theta\lrcorner\omega)=0$ should hold even if we exploit the assumption $\Delta''(\theta\lrcorner u)=0$. Note that if $\operatorname{Ric}(\omega)=0$, by (5.13) this assumption means that $\Delta''\theta=0$, i.e. $\bar\partial^\star\theta=0$ since we always have $\bar\partial\theta=0$. The most we can make of the property $\bar\partial^\star\theta=0$ is expressed in part *(ii)* of the following lemma. Parts *(i)* and *(iii)* show that more can be said about *scalar-valued* $(0,\,1)$-forms v, although even if that information applied to the $T^{1,\,0}X$-valued $(0,\,1)$-form θ, it would not suffice to deduce that $\partial(\theta\lrcorner\omega)=0$.

Lemma 5.3.16. *Let X be a compact complex manifold ($\dim_{\mathbb{C}}X=n$) supposed to carry a* **balanced** *metric ω.*

(i) For every $v\in C^\infty_{0,\,1}(X,\,\mathbb{C})$, the following equivalence holds:

$$\bar\partial^\star v=0\iff\partial v\ \textit{ is primitive.}$$

(ii) For every $\theta\in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$, the following equivalence holds:

$$\bar\partial^\star\theta=0\iff(D'\theta)\wedge\omega^{n-1}=0\in C^\infty_{n,\,n}(X,\,T^{1,\,0}X).$$

(iii) Suppose, furthermore, that X is a $\partial\bar\partial$-manifold. Then, for every $v\in C^\infty_{0,\,1}(X,\,\mathbb{C})\cap\ker\bar\partial$, the following equivalence holds:

$$\Delta''v=0\iff\partial v=0\quad(\iff\Delta'v=0).$$

Proof. Since any $(0,\,1)$-form is primitive, for $\star : C^\infty_{0,\,1}(X,\,\mathbb{C}) \to C^\infty_{n-1,\,n}(X,\,\mathbb{C})$ formula (4.68) reads

$$\star\, v = i\, v \wedge \frac{\omega^{n-1}}{(n-1)!}, \qquad v \in C^\infty_{0,\,1}(X,\,\mathbb{C}). \tag{5.44}$$

Since $\bar\partial^\star = -\star\partial\star$, we see that the condition $\bar\partial^\star v = 0$ is equivalent to $\partial(v \wedge \omega^{n-1}) = 0$. Since $\partial\omega^{n-1} = 0$ (by the balanced assumption), the last identity is equivalent to $(\partial v) \wedge \omega^{n-1} = 0$, which is precisely the condition that the $(1,\,1)$-form ∂v be primitive. This proves (i).

The proof of (ii) runs along the same lines as that of (i) using the formula $\bar\partial^\star = -\star D'\star$ when $\bar\partial^\star$ acts on $T^{1,\,0}X$-valued forms and D' is the $(1,\,0)$-component of the Chern connection D of $(T^{1,\,0}X,\,\omega)$. Indeed, formula (5.44) still holds for $T^{1,\,0}X$-valued $(0,\,1)$-forms θ in place of v and

$$D'(\theta \wedge \omega^{n-1}) = (D'\theta) \wedge \omega^{n-1} - \theta \wedge \partial\omega^{n-1} = (D'\theta) \wedge \omega^{n-1},$$

where the last identity follows from ω being balanced.

To prove (iii), fix an arbitrary form $v \in C^\infty_{0,\,1}(X,\,\mathbb{C}) \cap \ker\bar\partial$. Since $\ker\Delta'' = \ker\bar\partial \cap \ker\bar\partial^\star$, the condition $\Delta'' v = 0$ is equivalent for this v to $\bar\partial^\star v = 0$, which is equivalent to ∂v being primitive by (i). We are thus reduced to proving for this v the equivalence: ∂v is primitive $\iff$ $\partial v = 0$.

Notice that $\bar\partial(\partial v) = 0$ thanks to the assumption $\bar\partial v = 0$. Hence the pure-type form ∂v is d-closed and ∂-exact, so by the $\partial\bar\partial$-lemma it must be $\partial\bar\partial$-exact:

$$\partial v = i\partial\bar\partial\varphi \quad \text{for some } C^\infty \text{ function } \varphi : X \to \mathbb{C}.$$

Then we have the equivalences:

$$\partial v \text{ is primitive} \iff \Lambda_\omega(i\partial\bar\partial\varphi) = 0 \iff \Delta_\omega\varphi = 0 \iff \varphi \text{ is constant },$$

where the last equivalence follows by the maximum principle from X being compact. Meanwhile, φ being constant is equivalent to the vanishing of $i\partial\bar\partial\varphi$, hence to the vanishing of ∂v. □

The conclusion of these considerations is that Question 5.3.11 may have a negative answer in general in the balanced case. Let us now notice that even the answer to the following weaker question may be negative in the balanced case.

Question 5.3.17. *Is it true that on a balanced Calabi-Yau $\partial\bar\partial$-manifold, every* **primitive** $(n-1,\,1)$-*class (in the sense of the ad hoc Definition 5.3.9) can be represented by a* **primitive** *form?*

Let $[\theta\lrcorner u] \in H^{n-1,\,1}_{prim}(X,\,\mathbb{C})$ be a *primitive* class in the ad hoc sense. This means that $\theta\lrcorner\omega^{n-1}$ is $\bar\partial$-exact (for any representative θ of the class $[\theta] \in H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]}$). Pick any representative θ and any $\bar\partial$-potential $w \in C^\infty_{n-2,\,n-1}(X,\,\mathbb{C})$ of $\theta\lrcorner\omega^{n-1}$, i.e. $\bar\partial w = \theta\lrcorner\omega^{n-1}$. Since

$$L^{n-3}_\omega : C^\infty_{1,\,2}(X,\,\mathbb{C}) \to C^\infty_{n-2,\,n-1}(X,\,\mathbb{C}), \qquad \alpha \mapsto \omega^{n-3} \wedge \alpha,$$

is an isomorphism (see e.g. [Voi02, lemma 6.20, p. 146]), since there is a Lefschetz decomposition (cf. [Voi02, proposition 6.22, p. 147])

$$\Lambda^{1,\,2} = \Lambda^{1,\,2}_{prim} \oplus \left(\omega \wedge \Lambda^{0,\,1}\right)$$

and since every C^∞ $(0,\,1)$-form can be written as $(n-1)\,\xi\lrcorner\omega$ for a unique vector field $\xi \in C^\infty(X,\,T^{1,\,0}X)$ (because ω is non-degenerate), we see that there is a unique *primitive* C^∞ form α_0 of type $(1,\,2)$ and a unique C^∞ vector field ξ of type $(1,\,0)$ such that

$$w = \omega^{n-3}\wedge\bigg(\alpha_0 + (n-1)\,\omega\wedge(\xi\lrcorner\omega)\bigg) = \omega^{n-3}\wedge\alpha_0 + \xi\lrcorner\omega^{n-1}. \tag{5.45}$$

Consequently, $\theta\lrcorner\omega^{n-1} = \bar\partial w = \bar\partial(\omega^{n-3}\wedge\alpha_0) + (\bar\partial\xi)\lrcorner\omega^{n-1}$ since $\bar\partial(\xi\lrcorner\omega^{n-1}) = (\bar\partial\xi)\lrcorner\omega^{n-1} - \xi\lrcorner(\bar\partial\omega^{n-1})$ (cf. (i) of Lemma 5.3.3) and here $\bar\partial\omega^{n-1}=0$ by the balanced assumption on ω. Thus we get

$$(\theta-\bar\partial\xi)\lrcorner\omega^{n-1} = \bar\partial(\omega^{n-3}\wedge\alpha_0).$$

We see that $\theta-\bar\partial\xi$ represents the class $[\theta]\in H^{0,1}(X,\,T^{1,0}X)_{[\omega^{n-1}]}$, so $(\theta-\bar\partial\xi)\lrcorner u$ represents the class $[\theta\lrcorner u]\in H^{n-1,1}_{prim}(X,\,\mathbb{C})$. We know from Lemma 5.3.10 that the primitivity condition on the form $(\theta-\bar\partial\xi)\lrcorner u$ is equivalent to $(\theta-\bar\partial\xi)\lrcorner\omega^{n-1}=0$, i.e. to $\bar\partial(\omega^{n-3}\wedge\alpha_0)=0$ in this case. However, we can see no reason why this vanishing should occur, part of the obstruction being the primitive $(1,\,2)$-form α_0.

Thus in the balanced, non-Kähler case, the answer to Question 5.3.17 may be negative in general.

5.4 Period map and Weil-Petersson metrics

We now fix an arbitrary balanced Calabi-Yau $\partial\bar\partial$-manifold X, $\dim_{\mathbb{C}}X=n$. All the fibres $(X_t)_{t\in B}$ in the Kuranishi family of $X=X_0$ are again balanced Calabi-Yau $\partial\bar\partial$-manifolds if t is sufficiently close to $0\in B$. This follows from Wu's theorem in [Wu06] and from the deformation openness of the triviality of the canonical bundle K_{X_t} when the dimension of $H^{n,0}(X_t,\,\mathbb{C})$ is locally independent of t (as the $\partial\bar\partial$ assumption ensures this to be the case here). Thus $H^{n,0}(X_t,\,\mathbb{C})$ is a complex line varying holomorphically with t inside the fixed complex vector space $H^n(X,\,\mathbb{C})$. The canonical injection $H^{n,0}(X_t,\,\mathbb{C})\subset H^n(X,\,\mathbb{C})$ is induced by the $\partial\bar\partial$-lemma property of X_t (cf. Lemma 1.3.2 and comments thereafter). The associated period map $B\ni t\mapsto H^{n,0}(X_t,\,\mathbb{C})$ takes values in the complex projective space $\mathbb{P}H^n(X,\,\mathbb{C})$ after identifying each complex line $H^{n,0}(X_t,\,\mathbb{C})$ with the point it defines therein.

5.4.1 Period domain and the local Torelli theorem

Most of the material in this subsection before Theorem 5.4.4 is essentially known, but we take this oportunity to stress that only minimal assumptions are needed and to fix the notation for the rest of the paper.

Let ω be a Hermitian metric on X. All the formal adjoint operators and Laplacians will be calculated w.r.t. ω. The Hodge $\star$-operator defined by ω on n-forms

$$\star\;:\;C^\infty_n(X,\,\mathbb{C})\longrightarrow C^\infty_n(X,\,\mathbb{C})$$

satisfies $\star^2=(-1)^n$, so it induces a decomposition

$$C^\infty_n(X,\,\mathbb{C}) = \Lambda^n_+\oplus\Lambda^n_-, \tag{5.46}$$

where $\Lambda^n_\pm$ stand for the eigenspaces of $\star$ corresponding to the eigenvalues ±1 (if n is even), $\pm i$ (if n is odd). This decomposition is easily seen to be orthogonal for the L^2 scalar product induced by ω: for any $u\in\Lambda^n_+$ and any $v\in\Lambda^n_-$, one easily checks that $\langle\langle u,\,v\rangle\rangle = -\langle\langle u,\,v\rangle\rangle$ by writing $u=\star u$ (if n is even) and $u = -i\,(\star u)$ (if n is odd) and using the easy-to-check identity $\langle\langle\star u,\, v\rangle\rangle = (-1)^n\,\langle\langle u,\, \star v\rangle\rangle$ for any n-forms u, v.

When $\star$ is restricted to Δ-harmonic forms, it assumes Δ-harmonic values:

$$\star\ :\ \mathcal{H}^n_\Delta(X,\,\mathbb{C}) \longrightarrow \mathcal{H}^n_\Delta(X,\,\mathbb{C})$$

since $\Delta := dd^\star + d^\star d$ commutes with $\star$ as is well known to follow from the standard formula $d^\star = -\star\, d\star$. Thus the Hodge isomorphism $H^n(X,\,\mathbb{C}) \simeq \mathcal{H}^n_\Delta(X,\,\mathbb{C})$ mapping any De Rham class to its Δ-harmonic representative extends the definition of $\star$ to the De Rham cohomology of degree n:

$$\star\ :\ H^n(X,\,\mathbb{C}) \longrightarrow H^n(X,\,\mathbb{C}) \tag{5.47}$$

and we get a decomposition in cohomology analogous to (5.46):

$$H^n(X,\,\mathbb{C}) = H^n_+(X,\,\mathbb{C}) \oplus H^n_-(X,\,\mathbb{C}), \tag{5.48}$$

where $H^n_\pm(X,\,\mathbb{C})$ are the eigenspaces of $\star$ corresponding to the eigenvalues ± 1 (if n is even), $\pm i$ (if n is odd). Thus $H^n_+(X,\,\mathbb{C})$ (resp. $H^n_-(X,\,\mathbb{C})$) consists of the De Rham classes $\{\alpha\}$ of degree n whose Δ-harmonic representative α lies in Λ^n_+ (resp. Λ^n_-). Note that no assumption whatsoever (either Kähler or balanced) is needed on the Hermitian metric ω.

On the other hand, the Hodge-Riemann bilinear form can always be defined on the De Rham cohomology of degree n:

$$\begin{aligned} Q\ &:\ H^n(X,\,\mathbb{C}) \times H^n(X,\,\mathbb{C}) \longrightarrow \mathbb{C}, \\ &(\{\alpha\}, \{\beta\}) \longmapsto (-1)^{\frac{n(n-1)}{2}} \int\limits_X \alpha\wedge\beta := Q(\{\alpha\},\, \{\beta\}). \end{aligned} \tag{5.49}$$

It is clear that $Q(\cdot,\,\cdot)$ is independent of the choice of representatives α and β of the respective De Rham classes of degree n since no power of ω is involved in the definition of Q, so no Kähler or balanced or any other assumption is needed on ω unlike the case of the De Rham cohomology in degree $k < n$. Thus Q is independent of ω and of the complex structure of X, depending only on the differential structure of X. It is also clear that Q is non-degenerate since for any Δ-harmonic n-form α, $\star\bar\alpha$ is again Δ-harmonic and

$$\begin{aligned} Q(\{\alpha\},\, \{\star\bar\alpha\}) &= (-1)^{\frac{n(n-1)}{2}} \int\limits_X \alpha\wedge\star\bar\alpha = (-1)^{\frac{n(n-1)}{2}} \int\limits_X \langle\alpha,\, \alpha\rangle_\omega\, dV_\omega \\ &= (-1)^{\frac{n(n-1)}{2}}\, ||\alpha||^2_\omega \neq 0 \quad \text{if } \alpha\neq 0. \end{aligned}$$

Hence the associated sesquilinear form

$$\begin{aligned} H\ &:\ H^n(X,\,\mathbb{C}) \times H^n(X,\,\mathbb{C}) \longrightarrow \mathbb{C}, \\ &(\{\alpha\}, \{\beta\}) \longmapsto (-1)^{\frac{n(n+1)}{2}}\, i^n \int\limits_X \alpha\wedge\bar\beta = (-i)^n\, Q(\{\alpha\},\, \{\bar\beta\}) \end{aligned} \tag{5.50}$$

is non-degenerate.

Lemma 5.4.1. *(a) $H(\{\alpha\},\,\{\alpha\})>0$ for every class $\{\alpha\}\in H^n_+(X,\,\mathbb{C})\setminus\{0\}$. Hence H defines a positive definite sesquilinear form (i.e. a Hermitian metric) on $H^n_+(X,\,\mathbb{C})$.*
(b) $H(\{\alpha\},\,\{\alpha\})<0$ for every class $\{\alpha\}\in H^n_-(X,\,\mathbb{C})\setminus\{0\}$.
(c) $H(\{\alpha\},\,\{\beta\})=0$ for every class $\{\alpha\}\in H^n_+(X,\,\mathbb{C})$ and every class $\{\beta\}\in H^n_-(X,\,\mathbb{C})$. Hence the decomposition (5.48) is orthogonal for H.

Proof. (a) Let α be a Δ-harmonic n-form such that the class $\{\alpha\}\in H^n_+(X,\,\mathbb{C})$.
If n is even, $\star\,\alpha=\alpha$, hence taking conjugates we get $\star\,\bar\alpha=\bar\alpha$. Thus

$$H(\{\alpha\},\,\{\alpha\})=(-1)^{\frac{n(n+1)}{2}}\,i^n\int\limits_X\alpha\wedge\star\bar\alpha=\int\limits_X|\alpha|^2_\omega\,dV_\omega=||\alpha||^2_\omega>0$$

if $\alpha\neq0$, since $(-1)^{\frac{n(n+1)}{2}}\,i^n=i^{n^2+2n}=1$ when n is even. (Indeed, $n^2+2n\in4\mathbb{Z}$ when n is even.)
If n is odd, $\star\,\alpha=i\,\alpha$, hence taking conjugates we get $\star\,\bar\alpha=-i\,\bar\alpha$. Equivalently, $\bar\alpha=i\star\bar\alpha$. On the other hand, $(-1)^{\frac{n(n+1)}{2}}\,i^n=i^{n^2+2n}=-i$ when n is odd since $n^2+2n\in4\mathbb{Z}+3$ in this case. We then get as above that again $H(\{\alpha\},\,\{\alpha\})=||\alpha||^2_\omega>0$ if $\alpha\neq0$. This proves (a). The proof of (b) is very similar and is left to the reader.

(c) Let α and β be Δ-harmonic n-forms such that $\{\alpha\}\in H^n_+(X,\,\mathbb{C})$ and $\{\beta\}\in H^n_-(X,\,\mathbb{C})$. If n is even, this means that $\star\,\alpha=\alpha$ and $\star\,\beta=-\beta$. Using the property $\star\,\beta=-\beta$, we get

$$H(\{\alpha\},\,\{\beta\})=-(-1)^{\frac{n(n+1)}{2}}\,i^n\int\limits_X\alpha\wedge\star\bar\beta=-(-1)^{\frac{n(n+1)}{2}}\,i^n\,\langle\langle\alpha,\,\beta\rangle\rangle_\omega,\tag{5.51}$$

while using the property $\star\,\alpha=\alpha$, we get

$$\begin{aligned}H(\{\alpha\},\,\{\beta\})&=(-1)^{\frac{n(n+1)}{2}}\,i^n\int\limits_X\star\,\alpha\wedge\bar\beta=(-1)^{n^2}(-1)^{\frac{n(n+1)}{2}}\,i^n\int\limits_X\bar\beta\wedge\star\,\alpha\\&=(-1)^{\frac{n(n+1)}{2}}\,i^n\int\limits_X\overline{\langle\beta,\,\alpha\rangle_\omega}\,dV_\omega=(-1)^{\frac{n(n+1)}{2}}\,i^n\,\langle\langle\alpha,\,\beta\rangle\rangle_\omega,\end{aligned}\tag{5.52}$$

having used the fact $(-1)^{n^2}=1$ since n is even and the identity $\overline{\langle\beta,\,\alpha\rangle_\omega}=\langle\alpha,\,\beta\rangle_\omega$. The expressions (5.51) and (5.52) for $H(\{\alpha\},\,\{\beta\})$ are now seen to differ only by a sign, hence $H(\{\alpha\},\,\{\beta\})=0$.
When n is odd, we have $\star\,\alpha=i\,\alpha$ (hence $\alpha=-i\star\alpha$) and $\star\,\beta=-i\,\beta$ (hence $\bar\beta=-i\star\bar\beta$). Using the former and then the latter of these two pieces of information, we get as above two expressions for $H(\{\alpha\},\,\{\beta\})$ that differ only by a sign. Hence $H(\{\alpha\},\,\{\beta\})=0$. □

We now bring in the complex structure of X (that is supposed to have the $\partial\bar\partial$ property which induces the inclusion $H^{n,\,0}(X,\,\mathbb{C})\subset H^n(X,\,\mathbb{C})$).

Lemma 5.4.2. *Let X be a compact complex $\partial\bar\partial$-manifold ($\dim_{\mathbb{C}}X=n$). Then the following inclusions hold:*

$H^{n,\,0}(X,\,\mathbb{C})\subset H^n_+(X,\,\mathbb{C})$ if n is **even**, *$H^{n,\,0}(X,\,\mathbb{C})\subset H^n_-(X,\,\mathbb{C})$ if n is* **odd**.

In particular, the restriction $H:H^{n,\,0}(X,\,\mathbb{C})\times H^{n,\,0}(X,\,\mathbb{C})\to\mathbb{C}$ of H to $H^{n,\,0}(X,\,\mathbb{C})$ is **positive definite if n is even** *and is* **negative definite if n is odd** *thanks to Lemma 5.4.1 (hence we get a Hermitian metric on $H^{n,\,0}(X,\,\mathbb{C})$ defined by the scalar product induced by H when n is even and by $-H$ when n is odd).*

Before proving this statement, we make a trivial but useful observation.

Lemma 5.4.3. *Let $(X,\,\omega)$ be any compact complex Hermitian manifold ($\dim_{\mathbb{C}}X=n$). For every $(n,\,0)$-form α, the following equivalence and implication hold:*

$$\Delta''\alpha=0\iff\Delta'\alpha=0\implies\Delta\alpha=0.$$

Proof. Since X is compact, $\ker\Delta''=\ker\bar\partial\cap\ker\bar\partial^\star$ and $\ker\Delta'=\ker\partial\cap\ker\partial^\star$. However, $\partial\alpha=0$ and $\bar\partial^\star\alpha=0$ for any $(n,\,0)$-form α for trivial bidegree reasons. Hence, for any $\alpha\in C^\infty_{n,\,0}(X,\,\mathbb{C})$, the following equivalences hold:

$$\Delta'\alpha=0\Leftrightarrow\partial^\star\alpha=0\quad\text{ and }\quad\Delta''\alpha=0\Leftrightarrow\bar\partial\alpha=0.$$

Consequently, from the identity $\partial^\star=-\star\bar\partial\star$ (cf. e.g. [Dem97, VI, §.5.1]) and from the fact that $\star$ is an isomorphism, we get the equivalence: $\Delta'\alpha=0\;\Leftrightarrow\;\bar\partial(\star\alpha)=0$. Since α is of type $(n,\,0)$, it is primitive (w.r.t. any metric, hence also w.r.t. ω), so formula (4.68) applied to α reads: $\star\,\alpha=(-1)^{n(n+1)/2}\,i^n\,\alpha$. Thus the previous equivalence implies the following equivalence:

$$\Delta'\alpha=0\;\;\Leftrightarrow\;\;\bar\partial\alpha=0,$$

while the equivalence $\bar\partial\alpha=0\;\;\Leftrightarrow\;\;\Delta''\alpha=0$ has already been observed. We have thus proved the equivalence claimed in the statement. The implication claimed in the statement now follows from identity (5.40) applied to the pure-type form α and the fact that $\langle\langle\Delta\alpha,\,\alpha\rangle\rangle\geq0$ with equality if and only if $\Delta\alpha=0$. $\square$

Proof of Lemma 5.4.2. Let $[\alpha]\in H^{n,\,0}(X,\,\mathbb{C})$ be an arbitrary Dolbeault cohomology class of type $(n,\,0)$. Since the only $\bar\partial$-exact form of type $(n,\,0)$ is the zero form, the class $[\alpha]$ contains a unique representative α. Clearly, α is of type $(n,\,0)$ and Δ''-harmonic, so from Lemma 5.4.3 we get $\Delta\alpha=0$. On the other hand, formula (4.68) applied to α (which is primitive since it is of type $(n,\,0)$) reads: $\star\,\alpha=(-1)^{n(n+1)/2}\,i^n\alpha=i^{n(n+2)}\,\alpha$. Hence, if n is even, $\alpha\in\Lambda^n_+$ since $i^{n(n+2)}=1$, while if n is odd, $\alpha\in\Lambda^n_-$ since $i^{n(n+2)}=-i$. Therefore the De Rham cohomology class $\{\alpha\}\in H^n(X,\,\mathbb{C})$ represented by the Δ-harmonic form α must belong to $H^n_+(X,\mathbb{C})$ when n is even, resp. to $H^n_-(X,\mathbb{C})$ when n is odd. $\square$

Let us now consider a holomorphic family $(J_t)_{t\in B}$ of Calabi-Yau $\partial\bar\partial$ complex structures on a compact differential manifold X. We set $X_t:=(X,\,J_t)$ and let $n:=\dim_{\mathbb{C}}X_t$ for all $t\in B$. Notice that Q and H (cf. (6.34) and (6.35)) depend only on the differential structure of X. Thus,

$$C_+:=\bigg\{\{\alpha\}\in H^n(X,\,\mathbb{C})\;/\;H(\{\alpha\},\,\{\alpha\})>0\bigg\}\subset H^n(X,\,\mathbb{C}),$$

and

$$C_-:=\bigg\{\{\alpha\}\in H^n(X,\,\mathbb{C})\;/\;H(\{\alpha\},\,\{\alpha\})<0\bigg\}\subset H^n(X,\,\mathbb{C})$$

are *open* subsets of $H^n(X,\,\mathbb{C})$ and depend only on the differential structure of X. Furthermore, if we equip the fibres X_t with a C^∞ family of arbitrary Hermitian metrics $(\omega_t)_{t\in B}$, the corresponding Hodge $\star$ operator $\star=\star_t$ has eigenspaces $H^n_+(X_t,\,\mathbb{C})$ and $H^n_-(X_t,\,\mathbb{C})$ (cf. (5.48)) depending on the complex structure J_t via the metric ω_t (which is in particular a J_t-type $(1,\,1)$-form). Lemma 5.4.1 ensures that

$$H^n_+(X_t,\,\mathbb{C})\setminus\{0\}\subset C_+\quad\text{ and }\quad H^n_-(X_t,\,\mathbb{C})\setminus\{0\}\subset C_-\quad\text{ for all }t\in B.$$

Moreover, Lemmas 5.4.1 and 5.4.2 imply the following inclusions:

$$\begin{aligned} H^{n,\,0}(X_t,\,\mathbb{C})\setminus\{0\} &\subset H^n_+(X_t,\,\mathbb{C})\setminus\{0\}\subset C_+\subset H^n(X,\,\mathbb{C}) \quad \text{if } n \text{ is even},\\ H^{n,\,0}(X_t,\,\mathbb{C})\setminus\{0\} &\subset H^n_-(X_t,\,\mathbb{C})\setminus\{0\}\subset C_-\subset H^n(X,\,\mathbb{C}) \quad \text{if } n \text{ is odd}. \end{aligned} \tag{5.53}$$

It is clear that for any class $\varphi_t=[\alpha_t]\in H^{n,\,0}(X_t,\,\mathbb{C})$, $Q(\varphi_t,\,\varphi_t)=0$ since $\alpha_t\wedge\alpha_t=0$ for any form of J_t-type $(n,\,0)$. Thus the **period domain**, containing the complex lines $H^{n,\,0}(X_t,\,\mathbb{C})$ varying inside $H^n(X,\,\mathbb{C})$ when J_t varies, can be defined as in the standard (i.e. Kähler) case as

$D=\{\mathbb{C}\text{-line } l\subset H^n(X,\,\mathbb{C})\ ;\ \forall\varphi\in l\setminus\{0\},\ Q(\varphi,\,\varphi)=0 \text{ and } H(\varphi,\,\varphi)>0\}$

if n is even (so, in particular, $l\subset C_+$ whenever $l\in D$), and as

$D=\{\mathbb{C}\text{-line } l\subset H^n(X,\,\mathbb{C})\ ;\ \forall\varphi\in l\setminus\{0\},\ Q(\varphi,\,\varphi)=0 \text{ and } H(\varphi,\,\varphi)<0\}$

if n is odd (so, in particular, $l\subset C_-$ whenever $l\in D$). Given the natural holomorphic embedding $D\subset\mathbb{P}H^n(X,\,\mathbb{C})$, the complex manifold D is projective and is contained in the quadric defined by Q in $\mathbb{P}H^n(X,\,\mathbb{C})$.

We can now show that the **local Torelli theorem** holds in this context.

Theorem 5.4.4. *Let X be a compact Calabi-Yau $\partial\bar\partial$-manifold, $\dim_{\mathbb{C}}X=n$, and let $\pi:\mathcal{X}\longrightarrow\Delta$ be its Kuranishi family. Then the associated period map*

$$\mathcal{P}\ :\ B\longrightarrow D\subset\mathbb{P}H^n(X,\,\mathbb{C}),\qquad B\ni t\mapsto H^{n,\,0}(X_t,\,\mathbb{C}),$$

is a local holomorphic immersion.

Proof. As usual, we denote by $(X_t)_{t\in B}$ the fibres of the Kuranishi family of $X=X_0$. They are all C^∞-diffeomorphic to X and the holomorphic family $(X_t)_{t\in B}$ can be seen as a fixed C^∞ manifold X equipped with a holomorphic family of complex structures $(J_t)_{t\in B}$. Let $(u_t)_{t\in B}$ be a holomorphic family of nowhere vanishing n-forms on X such that for every $t\in B$, u_t is of type $(n,\,0)$ for the complex structure J_t and $\bar\partial_t u_t=0$. The form u_t identifies with the class $[u_t]$ it defines in $H^{n,\,0}(X_t,\,\mathbb{C})$, hence with the whole space $H^{n,\,0}(X_t,\,\mathbb{C})=\mathbb{C}\,u_t$. Thus the period map identifies with the map

$$B\ni t\mapsto u_t.$$

It suffices to prove that $\mathcal{P}$ is a local immersion at $t=0$. Recall that in the present situation the Kodaira-Spencer map $\rho:T_0B\rightarrow H^{0,\,1}(X,\,T^{1,\,0}X)$ is an isomorphism (thanks to Theorem 2.4.7) and that for any tangent vector $\partial/\partial t\in T_0B$, the choice of a representative θ in the class $\rho(\partial/\partial t)=[\theta]\in H^{0,\,1}(X,\,T^{1,\,0}X)$ determines a C^∞ trivialisation $\Phi\ :\ \mathcal{X}\longrightarrow B\times X_0$ (after possibly shrinking B about 0), which in turn determines about any pre-given point $x\in X$ a choice of local J_t-holomorphic coordinates $z_1(t),\dots,z_n(t)$ for every $t\in B$.

Denote $u=u_0$. Fix an arbitrary tangent vector $\partial/\partial t\in T_0B\setminus\{0\}$ and choose a representative θ of the class $\rho(\partial/\partial t)\in H^{0,\,1}(X,\,T^{1,\,0}X)$ such that the representative $\theta\lrcorner u$ of the class $[\theta\lrcorner u]\in H^{n-1,\,1}(X,\,\mathbb{C})$ is d-closed. This is possible by the $\partial\bar\partial$-assumption on X, by (a) of Theorem 1.3.2 and by Lemma and Definition 2.4.4. The associated local C^∞ trivialisation $\Phi:\mathcal{X}\rightarrow B\times X_0$ induces C^∞ diffeomorphisms $\Phi_t^{-1}:X_0\rightarrow X_t$, $t\in B$, so the differential of the period map at $t=0$ in the $\partial/\partial t$-direction identifies with

$$\frac{\partial(\Phi_t^{-1})^\star u_t}{\partial t}_{|t=0} = \theta\lrcorner u + v \quad \text{on } X, \tag{5.54}$$

where v is some $(n,\,0)$-form on $X = X_0$. The identity in (5.54) can be proved in the usual way (see e.g. [Tia87, proof of Lemma 7.2]): having fixed an arbitrary point $x \in X$, one writes

$$u_t = f_t\, dz_1(t) \wedge \cdots \wedge dz_n(t) \tag{5.55}$$

where f_t is a holomorphic function in a neighbourhood of x in X_t and $z_1(t), \dots, z_n(t)$ are the local J_t-holomorphic coordinates about x determined by the choice of θ in the class $\rho(\partial/\partial t)$. Taking $\partial/\partial t$ at $t=0$ in (5.55), one finds on the right-hand side the sum of the form $v = (\partial f_t/\partial t)_{|t=0}\, dz_1(0) \wedge \cdots \wedge dz_n(0)$ of J_0-type $(n,\,0)$ with the form $\theta\lrcorner u$ of J_0-type $(n-1,\,1)$. The latter form is easily seen to be the sum of the terms obtained by deriving one of the $dz_j(t)$ in (5.55) since, with the above choices of θ and $z_1(t), \dots, z_n(t)$, we have

$$\frac{\partial}{\partial t}(dz_j(t))_{|t=0} = \theta\lrcorner\, dz_j(0), \qquad j = 1, \dots, n.$$

Now, $du_t = 0$ for all t, hence the left-hand term in (5.54) is a d-closed n-form on X. Thus $d(\theta\lrcorner u + v) = 0$. By our choice of θ (based on a key application of the $\partial\bar\partial$ lemma), $d(\theta\lrcorner u) = 0$, hence $dv = 0$. In particular, v is a $\bar\partial_0$-closed form of J_0-type $(n,\,0)$, so $v = c\,u$ for some constant $c \in \mathbb{C}$.

It is now clear that if $(d\mathcal{P})_0(\partial/\partial t) = 0$, then $\theta\lrcorner u = 0$ and $v = c\,u = 0$, so $\theta = 0$ (since $T_u(\theta) = \theta\lrcorner u$ and T_u is an isomorphism – see (2.46)), hence $\partial/\partial t = 0$ (since the Kodaira-Spencer map is an isomorphism here). This last vanishing contradicts the choice of $\partial/\partial t \neq 0$. We have thus shown that $\mathcal{P}$ is a local immersion at $t = 0$. $\square$

5.4.2 Weil-Petersson metrics on B

We start with a refinement of (a) Theorem 1.3.2 singling out a particular d-closed representative of a given Dolbeault cohomology class on a $\partial\bar\partial$-manifold.

Definition 5.4.5. *Let X be a compact $\partial\bar\partial$-manifold equipped with an arbitrary Hermitian metric ω. Given any Dolbeault cohomology class $[\alpha] \in H^{p,\,q}(X,\,\mathbb{C})$, let α be its Δ''_ω-harmonic representative and let $v_{min} \in Im(\partial\bar\partial)^\star \subset C^\infty_{p,\,q-1}(X,\,\mathbb{C})$ be the solution of minimal L^2 norm (w.r.t. ω) of the equation*

$$\partial\bar\partial v = -\partial\alpha. \tag{5.56}$$

*The d-closed $(p,\,q)$-form $\alpha_{min} := \alpha + \bar\partial v_{min}$ will be called the ω-**minimal d-closed representative** of the class $[\alpha]$. (It coincides with the Δ''_ω-harmonic representative if ω is Kähler.)*

A word of explanation is in order. Recall the elliptic fourth-order Aeppli Laplacian $\Delta_A^{p,\,q} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C})$ of Definition 1.1.11 that induces the L^2_ω-orthogonal three-space decomposition (1.11) of Corollary 1.1.13 in which

$$\ker(\partial\bar\partial) = \ker\Delta_A^{p,\,q} \oplus (\mathrm{Im}\partial + \mathrm{Im}\bar\partial),$$

yielding the Hodge isomorphism $H_A^{p,\,q}(X,\,\mathbb{C}) \simeq \ker\Delta_A^{p,\,q}$. Since the solution v of equation (5.56) is unique only modulo $\ker(\partial\bar\partial)$, the solution of minimal L^2 norm is the unique solution lying in

$\ker(\partial\bar\partial)^\perp = \mathrm{Im}(\partial\bar\partial)^\star$. Note that if the Δ''-harmonic representative α of the class $[\alpha]$ happens to be d-closed (for example, this is the case if the metric ω is Kähler), then $\partial\alpha = 0$ and $v_{min} = 0$, so $\alpha_{min} = \alpha$. Thus, α_{min} can be seen as the minimal d-closed correction in a given Dolbeault class of the Δ''-harmonic representative of that class.

Recall that if we fix a compact balanced Calabi-Yau $\partial\bar\partial$-manifold $(X,\,\omega)$ ($\dim_{\mathbb{C}}X = n$), the base space $B_{[\omega^{n-1}]}$ of the local universal family $(X_t)_{t\in B_{[\omega^{n-1}]}}$ of deformations of X that are co-polarised by the balanced class $[\omega^{n-1}]\in H^{n-1,\,n-1}(X,\,\mathbb{C})$ identifies to an open subset of $H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]}$ and

$$T_tB_{[\omega^{n-1}]} \simeq H^{0,\,1}(X_t,\,T^{1,\,0}X_t)_{[\omega^{n-1}]} \simeq H^{n-1,\,1}_{prim}(X_t,\,\mathbb{C}), \qquad t\in B_{[\omega^{n-1}]}.$$

We shall now define two Weil-Petersson metrics on $B_{[\omega^{n-1}]}$ induced by pre-given balanced metrics on the fibres X_t whose $(n-1)^{st}$ powers lie in the co-polarising balanced class.

Definition 5.4.6. *Fix any holomorphic family of nonvanishing holomorphic n-forms $(u_t)_{t\in B}$ on the fibres $(X_t)_{t\in B}$. Let $(\omega_t)_{t\in B_{[\omega^{n-1}]}}$ be a C^∞ family of balanced metrics on the fibres $(X_t)_{t\in B_{[\omega^{n-1}]}}$ such that $\omega_t^{n-1}\in\{\omega^{n-1}\}$ for all t and $\omega_0=\omega$. The associated* **Weil-Petersson metrics** *$G^{(1)}_{WP}$ and $G^{(2)}_{WP}$ on $B_{[\omega^{n-1}]}$ are defined as follows. For any $t\in B_{[\omega^{n-1}]}$ and any $[\theta_t],[\eta_t]\in H^{0,\,1}(X_t,\,T^{1,\,0}X_t)_{[\omega^{n-1}]}$, let*

$$G^{(1)}_{WP}([\theta_t],\,[\eta_t]) := \frac{\langle\langle\theta_t,\,\eta_t\rangle\rangle}{\int\limits_{X_t} dV_{\omega_t}}, \qquad (\textit{where } dV_{\omega_t} := \frac{\omega_t^n}{n!}) \tag{5.57}$$

$$G^{(2)}_{WP}([\theta_t],\,[\eta_t]) := \frac{\langle\langle\theta_t\lrcorner u_t,\,\eta_t\lrcorner u_t\rangle\rangle}{i^{n^2}\int_{X_t}u_t\wedge\bar u_t}, \tag{5.58}$$

where θ_t (resp. η_t) is chosen in its class $[\theta_t]$ (resp. $[\eta_t]$) such that $\theta_t\lrcorner u_t$ (resp. $\eta_t\lrcorner u_t$) is the ω_t-minimal d-closed representative of the class $[\theta_t\lrcorner u_t]\in H^{n-1,\,1}(X_t,\,\mathbb{C})$ (resp. $[\eta_t\lrcorner u_t]\in H^{n-1,\,1}(X_t,\,\mathbb{C})$), while $\langle\langle\ ,\ \rangle\rangle$ stands for the L^2 scalar product induced by ω_t on the spaces involved.

The C^∞ positive definite $(1,\,1)$-forms on $B_{[\omega^{n-1}]}$ associated with $G^{(1)}_{WP}$ and $G^{(2)}_{WP}$ are denoted by

$$\omega^{(1)}_{WP}>0 \quad \textit{and} \quad \omega^{(2)}_{WP}>0 \quad \textit{on } B_{[\omega^{n-1}]}.$$

Since every u_t is unique up to a constant factor, the definition of $G^{(2)}_{WP}$ is independent of the choice of the family $(u_t)_{t\in B}$. From Lemma 5.1.7 we infer

Observation 5.4.7. *If the balanced metrics can be chosen such that $Ric(\omega_t)=0$ for all $t\in B_{[\omega^{n-1}]}$, then*

$$\omega^{(1)}_{WP} = \omega^{(2)}_{WP} \quad \textit{on } B_{[\omega^{n-1}]}.$$

5.4.3 Metric on B induced by the period map

Let $L=\mathcal{O}_{\mathbb{P}H^n(X,\,\mathbb{C})}(-1)$ be the tautological line bundle on $\mathbb{P}H^n(X,\,\mathbb{C})$. We will endow the restrictions of L to two open subsets of $\mathbb{P}H^n(X,\,\mathbb{C})$ with Hermitian fibres metrics induced by H. We set:

$$U^n_+ := \bigg\{[l]\in\mathbb{P}H^n(X,\,\mathbb{C})\ /\ l \text{ is a }\mathbb{C}\text{-line such that } l\subset C_+\bigg\}\subset\mathbb{P}H^n(X,\,\mathbb{C}),$$

and

$$U_-^n := \left\{[l] \in \mathbb{P}H^n(X,\,\mathbb{C}) \;/\; l \text{ is a } \mathbb{C}\text{-line such that } l \subset C_-\right\} \subset \mathbb{P}H^n(X,\,\mathbb{C}),$$

where $[l]$ denotes the point in $\mathbb{P}H^n(X,\,\mathbb{C})$ defined by the line $l \subset H^n(X,\,\mathbb{C})$. It follows from the discussion of C_+ and C_- in §.5.4.1 that U_+^n and U_-^n are *open* subsets of $\mathbb{P}H^n(X,\,\mathbb{C})$ and depend only on the differential structure of X.

Moreover, for every $[l] \in U_+^n$, the fibre $L_{[l]} = l \subset C_+$ is endowed with the scalar product defined by the restriction of H. Thus $L_{|U_+^n}$ has a Hermitian fibre metric h_L^+ induced by H. The (negative) curvature form $i\,\Theta_{h_L^+}(L_{|U_+^n})$ defines the associated Fubini-Study metric on U_+^n by

$$\omega_{FS}^+ = -i\,\Theta_{h_L^+}(L_{|U_+^n}) > 0 \quad \text{ on } U_+^n \subset \mathbb{P}H^n(X,\,\mathbb{C}).$$

Likewise, for every $[l] \in U_-^n$, the fibre $L_{[l]} = l \subset C_-$ is endowed with the scalar product defined by the restriction of $-H$. Thus $L_{|U_-^n}$ has a Hermitian fibre metric h_L^- induced by $-H$. The (negative) curvature form $i\,\Theta_{h_L^-}(L_{|U_-^n})$ defines the associated Fubini-Study metric on U_-^n by

$$\omega_{FS}^- = -i\,\Theta_{h_L^-}(L_{|U_-^n}) > 0 \quad \text{ on } U_-^n \subset \mathbb{P}H^n(X,\,\mathbb{C}).$$

It follows from the above discussion that ω_{FS}^+ and ω_{FS}^- depend only on the differential structure of X. Composing the period map with the holomorphic embedding $D \stackrel{\iota}{\hookrightarrow} \mathbb{P}H^n(X,\,\mathbb{C})$, we obtain a local holomorphic immersion $\iota\circ\mathcal{P} : B \to \mathbb{P}H^n(X,\,\mathbb{C})$ (cf. Theorem 5.4.4). From (5.53), we get:

$$\text{Im}\,(\iota\circ\mathcal{P}) \subset U_+^n \quad \text{if } n \text{ is even}, \qquad \text{Im}\,(\iota\circ\mathcal{P}) \subset U_-^n \quad \text{if } n \text{ is odd}.$$

Taking the inverse image of ω_{FS}^+ when n is even, resp. of ω_{FS}^- when n is odd, we get a Hermitian metric (i.e. a positive definite C^∞ $(1,\,1)$-form) γ on B which is actually Kähler:

$$\gamma := (\iota\circ\mathcal{P})^\star(\omega_{FS}^+) > 0 \quad \text{if } n \text{ is even}, \qquad \gamma := (\iota\circ\mathcal{P})^\star(\omega_{FS}^-) > 0 \quad \text{if } n \text{ is odd}.$$

Computation of γ. We can compute γ at any point $t\in B$ (e.g. at $t=0$) in the same way as in [Tia87, §.7]. We spell out the details for the reader's convenience. Let $(u_t)_{t\in B}$ be a holomorphic family of nonvanishing holomorphic n-forms on the fibres $(X_t)_{t\in B}$. Recall that a tangent vector $(\partial/\partial t)_{|t=0}$ to B at 0 identifies via the Kodaira-Spencer map with a class $[\theta] \in H^{0,\,1}(X,\,T^{1,\,0}X)$. Fix any such class $[\theta]$. We will compute $\gamma_0([\theta],\,[\theta])$.

We have: $L_{u_t} = \mathbb{C}\cdot u_t = H^{n,\,0}(X_t,\,\mathbb{C})$. Thus:

(i) if n is *even*, then $L_{u_t} \subset H_+^n(X_t,\,\mathbb{C})$ and $(-i)^n\,Q(u_t,\,\bar{u}_t) = H(u_t,\,u_t) = |u_t|^2_{h_L^+} = e^{-\rho(t)}$, where ρ denotes the local weight function of the fibre metric h_L^+ of $L_{|U_+^n}$. We get $\rho(t) = -\log((-i)^n\,Q(u_t,\,\bar{u}_t))$;

(ii) if n is *odd*, then $L_{u_t} \subset H_-^n(X_t,\,\mathbb{C})$ and $-(-i)^n\,Q(u_t,\,\bar{u}_t) = -H(u_t,\,u_t) = |u_t|^2_{h_L^-} = e^{-\rho(t)}$, where ρ denotes the local weight function of the fibre metric h_L^- of $L_{|U_-^n}$. We get $\rho(t) = -\log(-(-i)^n\,Q(u_t,\,\bar{u}_t))$.

Now suppose that n is *even*. The curvature form of $(L,\,h_L^+)$ on a $\mathbb{C}$-line $\mathbb{C}\cdot t$ in a small neighbourhood of 0 equals $i\partial_t\bar\partial_t\rho(t)$, which in turn equals:

$$-i\partial_t\bar\partial_t\log((-i)^n\,Q(u_t,\,\bar{u}_t)) = -i\,\frac{\partial^2\log((-i)^n\,Q(u_t,\,\bar{u}_t))}{\partial t\,\partial\bar{t}}\,dt\wedge d\bar{t},$$

This means that for $[\theta] = \rho(\partial/\partial t_{|t=0})$, using the fact that $\frac{\partial u_t}{\partial\bar{t}} = 0$ (since u_t varies holomorphically with t), we get:

$$\begin{aligned}\gamma_0([\theta],\,[\theta]) &= -\frac{\partial^2\log((-i)^n\,Q(u_t,\,\bar{u}_t))}{\partial t\,\partial\bar{t}}_{|t=0} = -\frac{\partial}{\partial t}\left((-1)^n\,\frac{Q(u_t,\,\frac{\partial\bar{u}_t}{\partial\bar{t}})}{Q(u_t,\,\bar{u}_t)}\right)_{|t=0}\\ &= (-1)^{n+1}\left[\frac{Q(\frac{\partial u_t}{\partial t}_{|t=0},\,\frac{\partial\bar{u}_t}{\partial\bar{t}}_{|t=0})}{Q(u_0,\,\bar{u}_0)} - \frac{Q(\frac{\partial u_t}{\partial t}_{|t=0},\,\bar{u}_0)\cdot Q(u_0,\,\frac{\partial\bar{u}_t}{\partial\bar{t}}_{|t=0})}{Q(u_0,\,\bar{u}_0)^2}\right].\end{aligned}$$

Now recall that in the proof of Theorem 5.4.4 a key application of the $\partial\bar\partial$ lemma enabled us to choose the representative θ of the class $[\theta]$ such that $d(\theta\lrcorner u)=0$. With this choice, if $u:=u_0$, in formula (5.54) we had $v=c\,u$ and

$$\frac{\partial u_t}{\partial t}_{|t=0} = \theta\lrcorner u + c\,u,$$

where $c\in\mathbb{C}$ is a constant, if we identify u_t with $(\Phi_t^{-1})^\star u_t$ when $\Phi_t:X_t\to X_0$ $(t\in B)$ denote the C^∞ isomorphisms induced by the choice of θ in $[\theta]$. Using this, the above formula for $\gamma_0([\theta],\,[\theta])$ translates to

$$\begin{aligned}\gamma_0([\theta],\,[\theta]) &= (-1)^{n+1}\,\frac{Q(u,\,\bar{u})\cdot Q(\theta\lrcorner u,\,\overline{\theta\lrcorner u}) + |c|^2\,Q(u,\,\bar{u})^2 - |c|^2\,Q(u,\,\bar{u})^2}{Q(u,\,\bar{u})^2}\\ &= (-1)^{n+1}\,\frac{Q(\theta\lrcorner u,\,\overline{\theta\lrcorner u})}{Q(u,\,\bar{u})} = \frac{-H(\{\theta\lrcorner u\},\,\{\theta\lrcorner u\})}{i^{n^2}\int\limits_X u\wedge\bar{u}}.\end{aligned}$$

In the case when n is odd, the formula for $\gamma_0([\theta],\,[\theta])$ gets an extra (-1) factor. The conclusion of these calculations is summed up in the following

Lemma 5.4.8. *The Kähler metric γ defined on B by $\gamma:=(\iota\circ\mathcal{P})^\star(\omega_{FS}^+)>0$ when n is even and by $\gamma:=(\iota\circ\mathcal{P})^\star(\omega_{FS}^-)>0$ when n is odd, is independent of the choice of any metrics on $(X_t)_{t\in B}$ and is explicitly given by the formula:*

$$\gamma_t([\theta_t],\,[\theta_t]) = \frac{-\int\limits_X(\theta_t\lrcorner u_t)\wedge\overline{(\theta_t\lrcorner u_t)}}{i^{n^2}\int\limits_X u_t\wedge\bar{u}_t} = \frac{-H(\{\theta_t\lrcorner u_t\},\,\{\theta_t\lrcorner u_t\})}{i^{n^2}\int\limits_X u_t\wedge\bar{u}_t},\quad \textit{if } n \textit{ is even},$$

$$\gamma_t([\theta_t],\,[\theta_t]) = \frac{-i\,\int\limits_X(\theta_t\lrcorner u_t)\wedge\overline{(\theta_t\lrcorner u_t)}}{i^{n^2}\int\limits_X u_t\wedge\bar{u}_t} = \frac{H(\{\theta_t\lrcorner u_t\},\,\{\theta_t\lrcorner u_t\})}{i^{n^2}\int\limits_X u_t\wedge\bar{u}_t},\quad \textit{if } n \textit{ is odd},$$

for every $t\in B$ and every $[\theta_t]\in H^{0,1}(X_t,\,T^{1,0}X_t)$.
In particular, we see that $\gamma_t([\theta_t],\,[\theta_t])$ is independent of the choice of representative θ_t in the class $[\theta_t]\in H^{0,1}(X_t,\,T^{1,0}X_t)$ such that $\theta_t\lrcorner u_t$ is d-closed. Since for every $t\in B$, u_t is unique up to a constant factor, γ is independent of the choice of holomorphic family $(u_t)_{t\in B}$ of J_t-holomorphic n-forms.

Notice that $i^{n^2}\,u_t\wedge\bar{u}_t>0$ at every point of X_t for any non-vanishing $(n,\,0)$-form u_t. On the other hand, it follows from Lemma 5.4.9 below that $H(\{\theta_t\lrcorner u_t\},\,\{\theta_t\lrcorner u_t\})<0$ when n is *even* and that $H(\{\theta_t\lrcorner u_t\},\,\{\theta_t\lrcorner u_t\})>0$ when n is *odd* if a d-closed representative $\theta_t\lrcorner u_t$ of the class $[\theta_t\lrcorner u_t]\in H^{n-1,1}(X_t,\,\mathbb{C})\subset H^n(X,\,\mathbb{C})$ can be chosen to be *primitive*. This reproves that $\gamma_t([\theta_t],\,[\theta_t])>0$ in this case (which does occur if primitivess is taken w.r.t. a Kähler metric).

5.4.4 Comparison of metrics on B

We shall now compare the Weil-Petersson metric $\omega_{WP}^{(2)}$ with the period-map metric γ on $B_{[\omega^{n-1}]}$. We need a general fact first.

Let X be a compact complex manifold ($\dim_{\mathbb{C}} X = n$) equipped with a Hermitian metric ω and let $\star : \Lambda^{n-1,\,1} \to \Lambda^{n-1,\,1}$ be the Hodge $\star$ operator defined by ω on $(n-1,\,1)$-forms. (Here $\Lambda^{n-1,\,1}$ stands for the space $C^\infty_{n-1,\,1}(X,\,\mathbb{C})$ of global smooth forms of bidegree $(n-1,\,1)$ on X although $\star$ acts even pointwise on forms.) Since $\star^2 = (-1)^n$, $\star$ induces a decomposition that is orthogonal for the L^2 scalar product defined by ω on X (cf. §.5.4.1):

$$\Lambda^{n-1,\,1} = \Lambda^{n-1,\,1}_- \oplus \Lambda^{n-1,\,1}_+ \qquad \text{(the duality decomposition)}, \tag{5.59}$$

where $\Lambda^{n-1,\,1}_\pm$ stand for the eigenspaces of $\star$ corresponding to the eigenvalues ± 1 (if n is even), $\pm i$ (if n is odd). On the other hand, the Hermitian metric ω induces the Lefschetz decomposition (cf. [Voi02, proposition 6.22, p. 147])

$$\Lambda^{n-1,\,1} = \Lambda^{n-1,\,1}_{prim} \oplus \bigg(\omega\wedge\Lambda^{n-2,\,0}\bigg), \tag{5.60}$$

which is again orthogonal for the L^2 scalar product defined by ω on X, where $\Lambda^{n-1,\,1}_{prim}$ denotes the space of *primitive* $(n-1,\,1)$-forms u (i.e. those $u\in\Lambda^{n-1,\,1}$ for which $\omega\wedge u = 0$ or, equivalently, $\Lambda u = 0$), while $\omega\wedge\Lambda^{n-2,\,0}$ denotes the space of forms $\omega\wedge v$ with v an arbitrary form of bidegree $(n-2,\,0)$.

Lemma 5.4.9. *The decompositions (5.59) and (5.60) coincide up to order, i.e.*

$$\Lambda^{n-1,\,1}_- = \Lambda^{n-1,\,1}_{prim} \quad \textit{and} \quad \Lambda^{n-1,\,1}_+ = \omega\wedge\Lambda^{n-2,\,0} \quad \textit{if } n \textit{ is even},$$
$$\Lambda^{n-1,\,1}_+ = \Lambda^{n-1,\,1}_{prim} \quad \textit{and} \quad \Lambda^{n-1,\,1}_- = \omega\wedge\Lambda^{n-2,\,0} \quad \textit{if } n \textit{ is odd}.$$

Proof. It suffices to prove the inclusions:

$$(A)\quad \Lambda^{n-1,\,1}_{prim}\subset\Lambda^{n-1,\,1}_- \quad\text{and}\quad (B)\quad \omega\wedge\Lambda^{n-2,\,0}\subset\Lambda^{n-1,\,1}_+ \quad \text{if } n \text{ is even},$$
$$(A)\quad \Lambda^{n-1,\,1}_{prim}\subset\Lambda^{n-1,\,1}_+ \quad\text{and}\quad (B)\quad \omega\wedge\Lambda^{n-2,\,0}\subset\Lambda^{n-1,\,1}_- \quad \text{if } n \text{ is odd}.$$

Let $u\in\Lambda^{n-1,\,1}_{prim}$. Formula (4.68) gives $\star u = (-1)^{n(n+1)/2}\,i^{n-2}\,u = i^{n^2+2n-2}\,u$. If n is even, $n^2+2n-2\in 4\mathbb{Z}-2$, hence $i^{n^2+2n-2} = i^{-2} = -1$, so $u\in\Lambda^{n-1,\,1}_-$. If n is odd, $n^2+2n-2\in 4\mathbb{Z}+1$, hence $i^{n^2+2n-2} = i$, so $u\in\Lambda^{n-1,\,1}_+$. This proves inclusions (A).

To prove inclusions (B), we first prove the following formula

$$\star(\omega\wedge v) = i^{n(n-2)}\,\omega\wedge v \qquad \text{for all } v\in\Lambda^{n-2,\,0}. \tag{5.61}$$

Pick any $v\in\Lambda^{n-2,\,0}$. Then $\omega\wedge v\in\Lambda^{n-1,\,1}$. For every $u\in\Lambda^{n-1,\,1}$, we have

$$\int\limits_X u\wedge\star\overline{(\omega\wedge v)} = \int\limits_X \langle u,\,\omega\wedge v\rangle\,dV_\omega = \langle\langle u,\,\omega\wedge v\rangle\rangle = \langle\langle\Lambda u,\,v\rangle\rangle. \tag{5.62}$$

On the other hand, the following formula holds

$$\omega\wedge u = \frac{\omega^2}{2!}\wedge\Lambda u \quad \text{for all } u\in\Lambda^{n-1,\,1}. \tag{5.63}$$

Indeed, $\omega^2\wedge\Lambda u = [L^2,\,\Lambda]\,u = 2(n-n+2-1)\,Lu = 2\,\omega\wedge u$, where for the first identity we have used the fact that $L^2u=0$ since L^2u is of type $(n+1,\,3)$, while for the second identity we have used the standard formula (5.27) with $r=2$ and $k=n$.
Applying (5.63) on the top line below, for every $u\in\Lambda^{n-1,\,1}$ we get

$$\begin{aligned}\int\limits_X u\wedge\overline{(\omega\wedge v)} &= \int\limits_X(\omega\wedge u)\wedge\bar v = \int\limits_X\left(\frac{\omega^2}{2!}\wedge\Lambda u\right)\wedge\bar v\\ &= \int\limits_X(\Lambda u)\wedge\overline{\left(\frac{\omega^2}{2!}\wedge v\right)} = i^{n(n-2)}\int\limits_X(\Lambda u)\wedge\star\bar v\\ &= i^{n(n-2)}\int\limits_X\langle\Lambda u,\,v\rangle\,dV_\omega = i^{n(n-2)}\,\langle\langle\Lambda u,\,v\rangle\rangle,\end{aligned}\tag{5.64}$$

where the last identity on the second line above has followed from the formula

$\star v = i^{n(n-2)}\,\frac{\omega^2}{2!}\wedge v, \quad v\in\Lambda^{n-2,\,0}$ (cf. (4.68) with $(p,\,q)=(n-2,\,0)$).

It is clear that the combination of (5.62) and (5.64) proves formula (5.61).

With (5.61) in place, inclusions (B) follow immediately. Indeed, if n is even, $n(n-2)\in 4\mathbb{Z}$, hence $i^{n(n-2)}=1$, so $\omega\wedge v\in\Lambda_+^{n-1,\,1}$ for all $v\in\Lambda^{n-2,\,0}$. If n is odd, $n(n-2)\in 4\mathbb{Z}-1$, hence $i^{n(n-2)}=-i$, so $\omega\wedge v\in\Lambda_-^{n-1,\,1}$ for all $v\in\Lambda^{n-2,\,0}$. □

For any $\theta\in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$, we denote by

$$\theta\lrcorner u = \theta'\lrcorner u + \omega\wedge\zeta \tag{5.65}$$

the decomposition of $\theta\lrcorner u\in\Lambda^{n-1,\,1}$ induced by the Lefschetz decomposition (5.60). Thus $\theta'\lrcorner u\in\Lambda^{n-1,\,1}_{prim}$ and $\zeta\in\Lambda^{n-2,\,0}$. By orthogonality we have $||\theta\lrcorner u||^2 = ||\theta'\lrcorner u||^2 + ||\omega\wedge\zeta||^2$. Now

$$||\omega\wedge\zeta||^2 = \langle\langle\Lambda(\omega\wedge\zeta),\,\zeta\rangle\rangle = \langle\langle[\Lambda,\,L]\,\zeta,\,\zeta\rangle\rangle = 2||\zeta||^2,$$

since $\Lambda\zeta=0$ for bidegree reasons (hence $[\Lambda,\,L]\,\zeta = \Lambda(\omega\wedge\zeta)-\omega\wedge\Lambda\zeta = \Lambda(\omega\wedge\zeta)$) and $[\Lambda,\,L]\,\zeta = 2\zeta$ (by formula (5.27) with $r=1$ and $k=n-2$).

Theorem 5.4.10. *Let X be a compact balanced Calabi-Yau $\partial\bar\partial$-manifold of complex dimension n. Then the metrics $G^{(2)}_{WP}$ and γ on the base space $B_{[\omega^{n-1}]}$ of the local universal family of deformations of X that are co-polarised by a given balanced class $[\omega^{n-1}]\in H^{n-1,\,n-1}(X,\,\mathbb{C})\subset H^{2n-2}(X,\,\mathbb{C})$ are given at every point $t\in B_{[\omega^{n-1}]}$ by the formulae (see notation (5.65)):*

$$G^{(2)}_{WP,\,t}([\theta_t],\,[\theta_t]) = \frac{||\theta'_t\lrcorner u_t||^2 + 2||\zeta_t||^2}{i^{n^2}\int_X u_t\wedge\bar u_t}, \quad [\theta_t]\in H^{0,\,1}(X_t,\,T^{1,\,0}X_t)_{[\omega^{n-1}]}, \tag{5.66}$$

$$\gamma_t([\theta_t],\,[\theta_t]) = \frac{||\theta'_t\lrcorner u_t||^2 - 2||\zeta_t||^2}{i^{n^2}\int_X u_t\wedge\bar u_t}, \quad [\theta_t]\in H^{0,\,1}(X_t,\,T^{1,\,0}X_t)_{[\omega^{n-1}]}. \tag{5.67}$$

Here θ_t is chosen in its class $[\theta_t]$ such that $\theta_t\lrcorner u_t$ is the ω_t-minimal d-closed representative of the class $[\theta_t\lrcorner u_t]\in H^{n-1,\,1}(X_t,\,\mathbb{C})$ (where the $\omega_t\in\{\omega^{n-1}\}$ are balanced metrics in the co-polarising balanced class given beforehand).

Proof. We may assume that $t=0$. Formula (5.66) follows immediately from (5.58) and from the above considerations. To get (5.67), notice that Lemma 5.4.9 shows that if n is even, then $\theta\lrcorner u=\star(-\theta'\lrcorner u+\omega\wedge\zeta)$, from which we get

$$\int\limits_X(\theta\lrcorner u)\wedge\overline{(\theta\lrcorner u)}=\int\limits_X\bigg(\theta'\lrcorner u+\omega\wedge\zeta\bigg)\wedge\bigg(-\star\overline{(\theta'\lrcorner u)}+\star\overline{(\omega\wedge\zeta)}\bigg)=-||\theta'\lrcorner u||^2+2\,||\zeta||^2,$$

while if n is odd, then $\theta\lrcorner u=\star(-i\,\theta'\lrcorner u+i\,\omega\wedge\zeta)$, from which we get

$$\int\limits_X(\theta\lrcorner u)\wedge\overline{(\theta\lrcorner u)}=\int\limits_X\bigg(\theta'\lrcorner u+\omega\wedge\zeta\bigg)\wedge\bigg(i\star\overline{(\theta'\lrcorner u)}-i\star\overline{(\omega\wedge\zeta)}\bigg)=i\,||\theta'\lrcorner u||^2-2i\,||\zeta||^2.$$

Now (5.67) follows from these expressions and from Lemma 5.4.8. $\square$

Corollary 5.4.11. *For all $[\theta_t]\in H^{0,\,1}(X_t,\,T^{1,\,0}X_t)_{[\omega^{n-1}]}\setminus\{0\}$, we have*

$$(G^{(2)}_{WP}-\gamma)_t([\theta_t],\,[\theta_t])=\frac{4\,||\zeta_t||^2}{i^{n^2}\int_{X_t}u_t\wedge\bar u_t}\geq 0,\qquad t\in B_{[\omega^{n-1}]},$$

hence the Hermitian metric $\omega^{(2)}_{WP}$ on $B_{[\omega^{n-1}]}$ defined by $G^{(2)}_{WP}$ is bounded below by the Kähler metric γ.

It is now clear that the obstruction to the metrics $\omega^{(2)}_{WP}$ and γ coinciding on $B_{[\omega^{n-1}]}$ is the possible negative answer to Question 5.3.11 in the case of balanced, non-Kähler fibres. Indeed, if every class in $H^{n-1,\,1}_{prim}(X_t,\,\mathbb{C})$ could be represented by a form $\eta_t\lrcorner u_t$ that is both *primitive* and *d-closed*, we would have, thanks to Lemma 5.4.9, that $\star\overline{(\eta_t\lrcorner u_t)}=c\,\overline{(\eta_t\lrcorner u_t)}$ with $c=-1$ (if n is even), $c=-i$ (if n is odd). Hence, from Lemma 5.4.8, we would get $\omega^{(2)}_{WP}=\gamma$ as in the case of Kähler polarised deformations of [Tia87] since formula (5.58) can be re-written in the following obvious way:

$$G^{(2)}_{WP}([\theta_t],\,[\eta_t])=\frac{\int\limits_{X_t}(\theta_t\lrcorner u_t)\wedge\star\overline{(\eta_t\lrcorner u_t)}}{i^{n^2}\int\limits_{X_t}u_t\wedge\bar u_t}$$

5.5 Balanced holomorphic symplectic $\partial\bar\partial$-manifolds

5.5.1 Primitive $(1,\,1)$-classes on balanced manifolds

Let $(X,\,\omega)$ be a compact, balanced manifold ($\dim_{\mathbb{C}}X=n$). The balanced class $[\omega^{n-1}]\in H^{n-1,\,n-1}(X,\,\mathbb{C})$ enables one to define the notion of *primitive* 2-classes on X in the same way as in the standard Kähler case. Indeed, at the level of Dolbeault cohomology, the linear operator

$$L^{n-1}_\omega:H^{1,\,1}(X,\,\mathbb{C})\to H^{n,\,n}(X,\,\mathbb{C})\simeq\mathbb{C},\qquad[\alpha]\mapsto[\omega^{n-1}\wedge\alpha],\tag{5.68}$$

is well defined because, thanks to the *balanced* assumption on ω, $\bar\partial(\omega^{n-1}\wedge\alpha)=0$ whenever $\bar\partial\alpha=0$ and $\omega^{n-1}\wedge\alpha=\bar\partial(\omega^{n-1}\wedge\beta)$ whenever $\alpha=\bar\partial\beta$ is $\bar\partial$-exact. We can then call *primitive* those classes that are in the kernel of L^{n-1}_ω, i.e.

$$H^{1,\,1}_{prim}(X,\,\mathbb{C}):=\{[\alpha]\in H^{1,\,1}(X,\,\mathbb{C})\ ;\ \omega^{n-1}\wedge\alpha\ \text{is}\ \bar\partial-\text{exact}\}. \tag{5.69}$$

Analogous definitions can be made for De Rham 2-classes and Dolbeault $(2,\,0)$ and $(0,\,2)$-classes, but all $(2,\,0)$ and $(0,\,2)$-classes are primitive for trivial bidegree reasons. Thus, if the $\partial\bar\partial$-lemma is supposed to hold on X, the Hodge decomposition $H^2(X,\,\mathbb{C})=H^{2,\,0}(X,\,\mathbb{C})\oplus H^{1,\,1}(X,\,\mathbb{C})\oplus H^{0,\,2}(X,\,\mathbb{C})$ shows that only the $H^{1,\,1}(X,\,\mathbb{C})$ component supports a nontrivial notion of primitivity. Notice that for $k>2$, there is no corresponding notion of primitive k-classes if ω is only balanced since ω^{n-k+1} is not closed unless ω is Kähler. It had to be replaced in bidegree $(n-1,\,1)$ by the ad-hoc definition 5.3.9 using the Calabi-Yau isomorphism when K_X was assumed to be trivial.

Lemma 5.5.1. *Let $(X,\,\omega)$ be a compact, balanced manifold ($\dim_{\mathbb{C}}X=n$). Then a class $[\alpha]\in H^{1,\,1}(X,\,\mathbb{C})$ is primitive if and only if it can be represented by a primitive form.*

Proof. By the standard definition (applicable to any Hermitian metric ω), a $(1,\,1)$-form α is primitive if $\omega^{n-1}\wedge\alpha=0$. It is thus obvious that any class representable by a primitive form is primitive. To see the converse, pick any class $[\alpha]\in H^{1,\,1}_{prim}(X,\,\mathbb{C})$ and any representative α. We have to prove the existence of a $(1,\,0)$-form u such that the representative $\alpha+\bar\partial u$ of $[\alpha]$ is primitive. This amounts to $\omega^{n-1}\wedge(\alpha+\bar\partial u)=0$, which is equivalent to $\bar\partial(\omega^{n-1}\wedge u)=-\omega^{n-1}\wedge\alpha$ thanks to the balanced assumption $\bar\partial\omega^{n-1}=0$. Now, $\omega^{n-1}\wedge\alpha$ is $\bar\partial$-exact by the primitivity assumption on the class $[\alpha]$. Pick any $w\in C^\infty_{n,\,n-1}(X,\,\mathbb{C})$ such that $\bar\partial w=-\omega^{n-1}\wedge\alpha$. It thus suffices to prove the existence of a $(1,\,0)$-form u such that $\omega^{n-1}\wedge u=w$. The linear operator

$$L^{n-1}_\omega: C^\infty_{1,\,0}(X,\,\mathbb{C})\to C^\infty_{n,\,n-1}(X,\,\mathbb{C}),\qquad u\mapsto\omega^{n-1}\wedge u, \tag{5.70}$$

is an isomorphism (for any Hermitian metric ω), so there is a unique $(1,\,0)$-form u such that $\omega^{n-1}\wedge u=w$. □

The primitive representative of a primitive class $[\alpha]\in H^{1,\,1}(X,\,\mathbb{C})$ need not be unique, but we can single out a particular one that is uniquely determined by the metric ω in the given primitive class in the following way.

Choice of a primitive representative: *given a primitive $(1,\,1)$-class, let α be its Δ''_ω-harmonic representative. Then choose $w\in C^\infty_{n\,n-1}(X,\,\mathbb{C})$ to be the solution of minimal L^2-norm (w.r.t. ω) of the equation $\bar\partial w=-\omega^{n-1}\wedge\alpha$. Since the map (5.70) is an isomorphism, the $(1,\,0)$-form u such that $\omega^{n-1}\wedge u=w$ is uniquely determined by w. Since the above choices of α and w make them unique, the primitive representative $\alpha+\bar\partial u$ of the primitive class $[\alpha]$ is uniquely determined in this way by ω and $[\alpha]\in H^{1,\,1}_{prim}(X,\,\mathbb{C})$.* $(\star)$

When ω is Kähler, the Δ''_ω-harmonic representative α of a primitive class is a *primitive* form, a standard fact that follows from Δ''_ω and L_ω commuting (as can be easily seen from the Kähler identities). Thus $\omega^{n-1}\wedge\alpha=0$, hence $w=0$ is the minimal L^2-norm solution of equation $\bar\partial w=-\omega^{n-1}\wedge\alpha$. Consequently, $u=0$ and $\alpha+\bar\partial u=\alpha$, showing that our choice $(\star)$ of primitive representative coincides with the standard Δ''_ω-harmonic choice when ω is Kähler. However, when ω is only balanced, it is not clear whether the Δ''_ω-harmonic representative of a primitive class is a *primitive* form. This accounts for the need of introducing the choice $(\star)$.

5.5.2 Co-polarised deformations of holomorphic symplectic manifolds

Let $(X,\,\omega)$ be a compact, balanced $\partial\bar\partial$-manifold ($\dim_{\mathbb{C}}X=n$). Suppose there exists a C^∞ $\bar\partial$-closed $(2,\,0)$-form σ that is non-degenerate at every point of X and that such a σ is unique up to a nonzero constant factor. Thus $H^{2,\,0}(X,\,\mathbb{C})\simeq\mathbb{C}$ and σ defines a holomorphic symplectic structure on X. The form σ naturally identifies with the class $[\sigma]\in H^{2,\,0}(X,\,\mathbb{C})$.

It follows from the $\partial\bar\partial$-assumption on X that σ is actually d-closed by the following observation which is standard when X is Kähler (and probably also under the weaker $\partial\bar\partial$-assumption). The standard Kähler-case proof, using the Laplacian equality $\Delta'=\Delta''$, no longer holds in the $\partial\bar\partial$-case for which we spell out the argument below for the sake of completeness.

Lemma 5.5.2. *Every holomorphic p-form is d-closed on any compact complex $\partial\bar\partial$-manifold X for any $0\leq p\leq n=\dim_{\mathbb{C}}X$.*

Proof. Fix any p and let $\alpha\in C^\infty_{p,\,0}(X,\,\mathbb{C})$ be $\bar\partial$-closed. To show that $d\alpha=0$, it suffices to show that $\partial\alpha=0$. Now, $\partial\alpha$ is $\bar\partial$-closed since α is, while ∂ and $\bar\partial$ anti-commute. Thus $\partial\alpha$ is a d-closed, ∂-exact form of pure type $(p+1,\,0)$. By the $\partial\bar\partial$-lemma, $\partial\alpha$ must be $\partial\bar\partial$-exact, i.e. $\partial\alpha=\partial\bar\partial\beta$ for some $(p,\,-1)$-form β. Since β must vanish for type reasons, $\partial\alpha$ vanishes. $\square$

We are now ready to connect the primitive $(1,\,1)$-cohomology to the parameter space of co-polarised deformations defined by a balanced class via the natural isomorphism associated with the holomorphic symplectic structure.

Lemma 5.5.3. *Let X be a compact complex manifold ($\dim_{\mathbb{C}}X=n$) admitting a holomorphic symplectic structure σ that is unique up to a constant factor.*
(i) The linear map defined by σ as

$$T_\sigma\ :\ C^\infty_{0,\,1}(X,\,T^{1,\,0}X)\stackrel{\cdot\lrcorner\sigma}{\longrightarrow}C^\infty_{1,\,1}(X,\,\mathbb{C}),\qquad\theta\mapsto T_\sigma(\theta):=\theta\lrcorner\sigma,\tag{5.71}$$

is an isomorphism satisfying the following properties:

$$T_\sigma(\ker\bar\partial)=\ker\bar\partial\quad\text{ and }\quad T_\sigma(Im\,\bar\partial)=Im\,\bar\partial.\tag{5.72}$$

Consequently, T_σ induces an isomorphism in cohomology

$$T_{[\sigma]}\ ;\ H^{0,\,1}(X,\,T^{1,\,0}X)\stackrel{\cdot\lrcorner[\sigma]}{\longrightarrow}H^{1,\,1}(X,\,\mathbb{C})\tag{5.73}$$

defined by $T_{[\sigma]}([\theta])=[\theta\lrcorner\sigma]$ for all $[\theta]\in H^{0,\,1}(X,\,T^{1,\,0}X)$.

(ii) If ω is a **balanced** *metric on X, then the image under $T_{[\sigma]}$ of the subspace $H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]}\subset H^{0,\,1}(X,\,T^{1,\,0}X)$ defined in (5.16) is the subspace $H^{1,\,1}_{prim}(X,\,\mathbb{C})\subset H^{1,\,1}(X,\,\mathbb{C})$ of primitive $(1,\,1)$-classes defined in (5.69), i.e.*

$$T_{[\sigma]}\ :\ H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]}\stackrel{\simeq}{\longrightarrow}H^{1,\,1}_{prim}(X,\,\mathbb{C}).\tag{5.74}$$

Proof. It is clear that T_σ is an isomorphism. As in the proof of Lemma **??**, the rest of (i) follows from the easy-to-check formulae

$$\bar\partial(\theta\lrcorner\sigma) = (\bar\partial\theta)\lrcorner\sigma + \theta\lrcorner(\bar\partial\sigma) = (\bar\partial\theta)\lrcorner\sigma, \;\; \bar\partial(\xi\lrcorner\sigma) = (\bar\partial\xi)\lrcorner\sigma - \xi\lrcorner(\bar\partial\sigma) = (\bar\partial\xi)\lrcorner\sigma \tag{5.75}$$

for all $\theta \in C^\infty_{0,1}(X,\, T^{1,0}X)$ and all $\xi \in C^\infty(X,\, T^{1,0}X)$ which readily imply the inclusions $T_\sigma(\ker\bar\partial) \subset \ker\bar\partial$ and $T_\sigma(\mathrm{Im}\,\bar\partial) \subset \mathrm{Im}\,\bar\partial$.

Let us prove, for example, the identity $\mathrm{Im}\,\bar\partial = T_\sigma(\mathrm{Im}\,\bar\partial)$. This amounts to proving that θ is $\bar\partial$-exact if and only if $\theta\lrcorner\sigma$ is $\bar\partial$-exact. Having fixed local holomorphic coordinates $z_1, \dots, z_n$ on some open subset $U \subset X$, let

$$\theta = \sum_{\alpha,\,\beta}\theta^\alpha_\beta\,\frac{\partial}{\partial z^\alpha}\,d\bar z^\beta \quad \text{and} \quad \sigma = \sum_{\alpha,\,\delta}\sigma_{\alpha,\,\delta}\,dz^\alpha\wedge dz^\delta,$$

where the coefficients $\sigma_{\alpha,\,\delta}$ are holomorphic functions (since σ is holomorphic) and the matrix $(\sigma_{\alpha,\,\delta})_{\alpha,\,\delta}$ is invertible at every point since σ is non-degenerate at every point. Then $\theta\lrcorner\sigma = \sum\limits_{\alpha,\,\beta,\,\delta}\theta^\alpha_\beta\,(\sigma_{\alpha,\,\delta} - \sigma_{\delta,\,\alpha})\,d\bar z^\beta\wedge dz^\delta$. Thus $\theta\lrcorner\sigma$ is $\bar\partial$-exact if and only if there exists a $(1,\,0)$-form $v = \sum\limits_\delta v_\delta\,dz^\delta$ such that $\theta\lrcorner\sigma = \bar\partial v$, which amounts to

$$\sum_{\alpha,\,\beta,\,\delta}\theta^\alpha_\beta\,(\sigma_{\alpha,\,\delta} - \sigma_{\delta,\,\alpha})\,d\bar z^\beta\wedge dz^\delta = \sum_{\delta,\,\beta}\frac{\partial v_\delta}{\partial\bar z^\beta}\,d\bar z^\beta\wedge dz^\delta \;\;\Leftrightarrow\;\; \sum_\alpha\theta^\alpha_\beta\,(\sigma_{\alpha,\,\delta} - \sigma_{\delta,\,\alpha}) = \frac{\partial v_\delta}{\partial\bar z^\beta}$$

for all $\beta,\,\delta$. The last identity is equivalent to $\theta^\alpha_\beta = \sum\limits_\delta\frac{\partial v_\delta}{\partial\bar z^\beta}\,(\sigma^{\delta,\,\alpha} - \sigma^{\alpha,\,\delta}) = \frac{\partial}{\partial\bar z^\beta}(\sum\limits_\delta(\sigma^{\delta,\,\alpha} - \sigma^{\alpha,\,\delta})\,v_\delta)$ for all $\alpha,\,\beta$, where the matrix $(\sigma^{\delta,\,\alpha})_{\alpha,\,\delta}$ is the inverse of $(\sigma_{\alpha,\,\delta})_{\alpha,\,\delta}$. (We have used the fact that the $\sigma^{\delta,\,\alpha}$'s are holomorphic functions since the $\sigma_{\alpha,\,\delta}$'s are.) This, in turn, is equivalent to $\theta = \bar\partial(\sum\limits_\alpha(\sum\limits_\delta(\sigma^{\delta,\,\alpha} - \sigma^{\alpha,\,\delta})\,v_\delta)\,\frac{\partial}{\partial z^\alpha})$, i.e. to θ being $\bar\partial$-exact. We have thus proved that $\theta\lrcorner\sigma$ is $\bar\partial$-exact if and only if θ is $\bar\partial$-exact, i.e. the latter identity in (5.72).

The remaining inclusion in the former identity of (5.72) is proved in a similar way.

The proof of (ii) will run in two steps. First we prove the inclusion

$$T_{[\sigma]}\left(H^{0,\,1}(X,\,T^{1,\,0}X)_{[\omega^{n-1}]}\right) \subset H^{1,\,1}_{prim}(X,\,\mathbb{C}), \tag{5.76}$$

which amounts to proving that for every class $[\theta] \in H^{0,\,1}(X,\,T^{1,\,0}X)$ for which $\theta\lrcorner\omega^{n-1}$ is $\bar\partial$-exact, $\omega^{n-1}\wedge(\theta\lrcorner\sigma)$ is also $\bar\partial$-exact. Now, we always have

$$0 = \theta\lrcorner(\omega^{n-1}\wedge\sigma) = (\theta\lrcorner\omega^{n-1})\wedge\sigma + \omega^{n-1}\wedge(\theta\lrcorner\sigma),$$

where the first identity follows from the fact that $\omega^{n-1}\wedge\sigma$ is of type $(n+1,\,n-1)$, hence vanishes. Thus

$$(\theta\lrcorner\omega^{n-1})\wedge\sigma = -\omega^{n-1}\wedge(\theta\lrcorner\sigma) \quad \text{for all } \theta \in C^\infty_{0,\,1}(X,\,T^{1,\,0}X).$$

Now, if $\theta\lrcorner\omega^{n-1}$ is supposed to be $\bar\partial$-exact, then $(\theta\lrcorner\omega^{n-1})\wedge\sigma$ is $\bar\partial$-exact, too, since σ is $\bar\partial$-closed. Hence $\omega^{n-1}\wedge(\theta\lrcorner\sigma)$ is $\bar\partial$-exact whenever $\theta\lrcorner\omega^{n-1}$is, proving the inclusion (5.76).

Since $T_{[\sigma]}$ is injective by (i), it suffices to prove the dimension equality

$$\dim H^{0,1}(X,\, T^{1,0}X)_{[\omega^{n-1}]} = \dim H^{1,1}_{prim}(X,\, \mathbb{C}) \tag{5.77}$$

to be able to conclude that the inclusion (5.76) is actually an identity.

By definition (5.69), we have

$$H^{1,1}_{prim}(X,\, \mathbb{C}) = \ker\left(L_\omega^{n-1}\ :\ H^{1,1}(X,\, \mathbb{C}) \to H^{n,n}(X,\, \mathbb{C}) \simeq \mathbb{C}\right).$$

The linear map (5.68) cannot vanish identically, so it is surjective. Hence

$$\dim H^{1,1}_{prim}(X,\, \mathbb{C}) = h^{1,1} - 1, \tag{5.78}$$

where $h^{1,1} := \dim H^{1,1}(X,\, \mathbb{C})$. Meanwhile, definition (5.16) translates to

$$H^{0,1}(X,\, T^{1,0}X)_{[\omega^{n-1}]} = \ker\left(H^{0,1}(X,\, T^{1,0}X) \ni [\theta] \stackrel{T_{[\omega^{n-1}]}}{\longmapsto} [\theta \lrcorner \omega^{n-1}] \in H^{n-2,n}(X,\, \mathbb{C})\right),$$

while $H^{n-2,n}(X,\, \mathbb{C}) \simeq H^{2,0}(X,\, \mathbb{C}) \simeq \mathbb{C}$ by Serre duality and the uniqueness (up to a constant factor) assumption on the holomorphic symplectic structure $[\sigma] \in H^{2,0}(X,\, \mathbb{C})$. It is clear that the linear map $T_{[\omega^{n-1}]}$ does not vanish identically, so it must be surjective. Thus we get

$$\dim H^{0,1}(X,\, T^{1,0}X)_{[\omega^{n-1}]} = \dim H^{0,1}(X,\, T^{1,0}X) - 1 = h^{1,1} - 1, \tag{5.79}$$

where the last identity follows from the isomorphism (5.73) dealt with under (i). It is now clear that the dimension equality (5.77) is a consequence of the combined identities (5.78) and (5.79). The proof is complete. $\square$

The use of the isomorphism $T_{[\sigma]}$ in (5.73) in the holomorphic symplectic case may be an alternative to the use of the isomorphism $T_{[u]}$ in (2.48) of the more general Calabi-Yau case while running the construction of the Weil-Petersson metrics of section 5.4.

Chapter 6

Non-Kähler Mirror Symmetry: the Iwasawa Manifold and Beyond

This chapter is mostly taken from [Pop18a] where a new approach to the Mirror Symmetry Conjecture extended to the possibly non-Kähler setting was proposed. Further developments of the theory are presented in section 6.8 (taken from [PSU20c]) and in section 6.9 (taken from [Pop18b]).

Our general methods, some of which have been presented in earlier chapters of this book, are then applied by proving that the Iwasawa manifold X is *its own mirror dual* to the extent that its Gauduchon cone, replacing the classical Kähler cone that is empty in this case, corresponds to what we call the local universal family of *essential deformations* of X. These are obtained by removing from the Kuranishi family the two "superfluous" dimensions of complex parallelisable deformations that have a similar geometry to that of the Iwasawa manifold. The remaining four dimensions are shown to have a clear geometric meaning including in terms of the degeneration at E_2 of the Frölicher spectral sequence. On the local moduli space of *essential* complex structures, we obtain a canonical Hodge decomposition of weight 3 and a variation of Hodge structures, construct coordinates and Yukawa couplings, while implicitly proving a local Torelli theorem. On the metric side of the mirror, we construct a variation of Hodge structures parametrised by a subset of the complexified Gauduchon cone of the Iwasawa manifold using the sGG property of all the small deformations of this manifold proved in Corollary 4.3.5. Finally, we define a mirror map linking the two variations of Hodge structures and we highlight its properties.

One of the main ideas in this chapter is to overcome the double whammy of a possible non-existence of both Kähler metrics and rational curves on a given C-Y manifold X by using the Gauduchon cone $\mathcal{G}_X$ (see Definition 4.1.14 and §.4.1.3-§.4.1.6 for other roles it plays) of X. This furnishes both an alternative to the classical Kähler cone (that is empty on a non-Kähler manifold) and a transcendental substitute for cohomology classes of (currents of integration on) curves (e.g. by virtue of its elements' bidegree $(n-1,\, n-1)$, but also in a far deeper sense). We stress that the Gauduchon cone is relevant even on projective and on Kähler non-projective manifolds where it might be preferable to the Kähler cone in certain circumstances (for example, when it is strictly bigger, allowing for more flexibility).

6.1 Broad outline of the goals and the concepts

Before going into the details, we will use this section to give a very brief outline of the classical approach to mirror symmetry, followed by a similar overview of our approach.

6.1.1 Standard approach

The standard mirror symmetry conjecture predicts that the Calabi-Yau (C-Y) threefolds, defined as compact Kähler manifolds of complex dimension 3 whose canonical bundle is trivial, ought to occur in pairs $(X, \widetilde{X})$ such that the local universal family of deformations of the complex structure (i.e. the Kuranishi family) of X is isomorphic to the moduli space of Kähler structures enriched with B-fields (i.e. the complexified Kähler cone) of $\widetilde{X}$, and vice-versa.

As is well known, there is an obvious cohomological obstruction to some Kähler C-Y threefolds X having Kähler mirror duals $\widetilde{X}$. The Kuranishi family $(X)_{t\in B}$ of a given Kähler C-Y manifold $X = X_0$ is unobstructed (i.e. its base space B is **smooth**, hence can be viewed as an open ball in the classifying space $H^{0,1}(X, T^{1,0}X))$ by the Bogomolov-Tian-Todorov theorem ([Bog78], [Tia87], [Tod89]). The triviality of the canonical bundle K_X implies the isomorphism $H^{0,1}(X, T^{1,0}X) \simeq H^{n-1,1}(X, \mathbb{C}) = H^{2,1}(X, \mathbb{C})$, where the last identity follows from the assumption $\dim_{\mathbb{C}} X := n = 3$. On the other hand, the complexified Kähler cone $\widetilde{\mathcal{K}}_{\widetilde{X}}$ of $\widetilde{X}$ is an open subset of $H^{1,1}(\widetilde{X}, \mathbb{C})$. So a necessary condition for X and $\widetilde{X}$ to be mirror dual is that the tangent space to Δ at 0 (i.e. $H^{2,1}(X, \mathbb{C})$) be isomorphic to the tangent space to the complexified Kähler cone $\widetilde{\mathcal{K}}_{\widetilde{X}}$ at some point (i.e. $H^{1,1}(\widetilde{X}, \mathbb{C})$), and vice-versa. It is thus necessary to have

$$h^{2,1}(X) = h^{1,1}(\widetilde{X}) \quad \text{and} \quad h^{2,1}(\widetilde{X}) = h^{1,1}(X).$$

However, there exist Kähler C-Y threefolds X such that $h^{2,1}(X) = 0$ (the so-called *rigid* such threefolds, those that do not deform). Consequently, the mirror dual $\widetilde{X}$, if it exists, cannot be Kähler since $h^{1,1}(\widetilde{X}) = 0$.

This standard observation has prompted many authors so far to conjecture the mirror symmetry only for *generic* Kähler C-Y threefolds so that the discussion is confined to the Kähler realm. The idea of investigating the possible existence of a mirror symmetry phenomenon beyond the Kähler world was loosely suggested in [Rei87] and received attention recently in [LTY15]. This investigation is our main motivation in the present work. Our methods and point of view are very different from those in [LTY15].

The standard approach to the study of the Kähler side of the mirror is to use Gromov-Witten invariants attached to pseudo-holomorphic curves and to count rational curves. However, on many non-Kähler compact complex threefolds with trivial canonical bundle, there exist no rational curves.

6.1.2 Outline of our approach

We present in this chapter the new approach to mirror symmetry introduced in [Pop18a] by means of transcendental methods in the general, possibly non-Kähler context of compact complex manifolds whose canonical bundle is trivial (that we still call **Calabi-Yau (C-Y) manifolds** as in §.2.4.1 and subsequently). We test our new point of view on the Iwasawa manifold by taking full advantage of the explicit nature of extensive computations for this particular manifold found in the works [Nak75] (cf. §.4.5.3), [Ang11] and [Ang14] of Nakamura and Angella.

We hope that these methods will apply to larger classes of C-Y manifolds in the future. One of the new ideas in this approach is the notion of local universal family of *essential deformations*, viewed as a subfamily of the Kuranishi family, of the Iwasawa manifold X. Three equivalent definitions are given: by removing the complex parallelisable small deformations from the Kuranishi family; by selecting the small deformations that have a kind of *polarisation* by the holomorphic non-closed 1-form γ associated with X (cf. Definition 6.2.2); and by selecting the vector subspace of the Dolbeault cohomology space $H^{n-1,1}(X, \mathbb{C})$ (known to parametrise all the small deformations of a C-Y manifold

X, while the complex dimension of X is $n = 3$ here) that is naturally isomorphic to the vector space $E_2^{n-1,\,1}(X)$ featuring in bidegree $(n-1,\,1)$ on the second page of the Frölicher spectral sequence of X.

Looking ahead beyond the special case of the Iwasawa manifold treated in this chapter, we come up against the question of what makes a deformation of a general, possibly non-Kähler, C-Y manifold *essential*. The notion of essential deformations was extended from the case of the Iwasawa manifold to the general case of page-1-$\partial\bar\partial$-manifolds (discussed in §.3.3) in the very recent work [PSU20c] that is presented at the end of this chapter.

We now flesh out this outline of our approach with a few more preliminary observations.

(I) On the **complex-structure side of the mirror**, the starting point of our method is the observation that a natural Hodge decomposition of weight 3 exists on the Iwasawa manifold X if $H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$ is "pared down" to a 4-dimensional vector subspace $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) \subset H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$ (induced by the *vertical* 1-form γ defined alongside the *horizontal* 1-forms α and β in (1.54)) that injects canonically into $H^3_{DR}(X,\,\mathbb{C})$ and parametrises what we call the *essential deformations* of X. Specifically, recalling that $\Delta \subset H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$ is an open ball, if we put

$$B_{[\gamma]} := B \cap H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}),$$

we implicitly remove from the Kuranishi family $(X_t)_{t\in B}$ the two dimensions corresponding to complex parallelisable deformations X_t of X (that have a similar geometry to that of X, so no geometric information is lost) and we are left with a family $(X_t)_{t\in B_{[\gamma]}}$ of non-complex parallelisable deformations that we call *essential*. This description of the local deformations of X is made possible by Nakamura's explicit calculations in [Nak75]. The holomorphic tangent space to $B_{[\gamma]}$ at any of its points t is isomorphic via the Kodaira-Spencer map to the analogue $H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C})$ at t of $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) = H^{2,\,1}_{[\gamma]}(X_0,\,\mathbb{C})$. We get a Hodge decomposition of weight 3 for every $t \in B_{[\gamma]}$ (cf. Proposition 6.3.3) in the following form.

Proposition 6.1.1. *There exist canonical isomorphisms*

$$H^3_{DR}(X,\,\mathbb{C}) \simeq H^{3,\,0}_{\bar\partial}(X_t,\,\mathbb{C}) \oplus H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C}) \oplus H^{1,\,2}_{[\gamma]}(X_t,\,\mathbb{C}) \oplus H^{0,\,3}_{\bar\partial}(X_t,\,\mathbb{C}), \qquad t \in B_{[\gamma]}, \tag{6.1}$$

(where $H^{1,\,2}_{[\gamma]}(X_t,\,\mathbb{C}) \subset H^{1,\,2}_{\bar\partial}(X_t,\,\mathbb{C})$ is defined by analogy with $H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C})$) and

$$H^{3,\,0}_{\bar\partial}(X_t,\,\mathbb{C}) \simeq \overline{H^{0,\,3}_{\bar\partial}(X_t,\,\mathbb{C})} \quad \textit{and} \quad H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) \simeq \overline{H^{1,\,2}_{[\gamma]}(X,\,\mathbb{C})}, \qquad t \in B_{[\gamma]}. \tag{6.2}$$

We go on to show that $B_{[\gamma]} \ni t \mapsto H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C})$ is a C^∞ vector bundle of rank 4 (cf. Proposition 6.3.4) and that (6.1) and (6.2) define a Hodge filtration

$$F^2\mathcal{H}^3_{[\gamma]} \supset F^3\mathcal{H}^3$$

of *holomorphic* vector subbundles over $B_{[\gamma]}$ of the constant bundle $\mathcal{H}^3$ of fibre $H^3_{DR}(X,\,\mathbb{C})$. This induces a variation of Hodge structures (VHS) endowed with a Gauss-Manin connection satisfying the Griffiths transversality condition (cf. Theorem 6.3.10).

Thus, after restricting attention to the *essential deformations* of the non-$\partial\bar\partial$ Iwasawa manifold, we get a picture similar to the one described in §.1.3.1 for $\partial\bar\partial$-manifolds.

Two further crucial observations cement the role played by the space $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C})$ in this approach and its *canonical* nature. (By an isomorphism being canonical, we will mean that it is defined in an obvious way, not involving arbitrary choices, by the three standard holomorphic 1-forms α, β, γ that generate the whole cohomology of the Iwasawa manifold and are induced by the canonical basis of $\mathbb{C}^3$ as specified in (1.54).)

The first observation (cf. Proposition 6.3.9, (c)) is the following

Proposition 6.1.2. *There exists a canonical isomorphism*

$$H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) \simeq E^{2,\,1}_2(X,\,\mathbb{C}) \tag{6.3}$$

where $E^{2,\,1}_2(X,\,\mathbb{C})$ is the space featuring at the second step of the Frölicher spectral sequence of X (known to degenerate at E_2 as do its counterparts for all the small deformations X_t).

Moreover, the Hodge decomposition (6.1) reflects precisely this E_2 degeneration since there exist isomorphisms (cf. (6.29))

$$H^3_{DR}(X,\,\mathbb{C}) \simeq E^{3,\,0}_2(X_t,\,\mathbb{C}) \oplus E^{2,\,1}_2(X_t,\,\mathbb{C}) \oplus E^{1,\,2}_2(X_t,\,\mathbb{C}) \oplus E^{0,\,3}_2(X_t,\,\mathbb{C}), \qquad t \in B_{[\gamma]}, \tag{6.4}$$

in which each of the four spaces on the r.h.s. is isomorphic to the corresponding space on the r.h.s. of (6.1).

The second observation (cf. Observation 6.5.11) is the following

Proposition 6.1.3. *There exists a canonical isomorphism*

$$H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) \simeq H^{2,\,2}_A(X,\,\mathbb{C}). \tag{6.5}$$

The isomorphism (6.5) justifies us in choosing the *essential deformations* of X on the complex-structure side of the mirror and the *Gauduchon cone* of X on the metric side of the mirror as the two main structures mirroring each other. Indeed, $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C})$ is the tangent space to $B_{[\gamma]}$ at 0, while $H^{2,\,2}_A(X,\,\mathbb{C})$ is the tangent space to the complexified Gauduchon cone (see Definition 6.6.2) at any of its points. The Aeppli-Gauduchon class $[\omega_0^2] \in \mathcal{G}_{X_0}$ of a natural Gauduchon metric ω_0 induced on X_0 by the complex parallalisable structure of X_0 will be the privileged point chosen in the Gauduchon cone. It is the image of $0 \in B_{[\gamma]}$ under the mirror map that will be defined in Defintions 6.6.1 and 6.6.2. Isomorphism (6.5) is the single most powerful piece of initial motivating evidence in favour of the new mirror symmetry phenomenon that we highlight in this paper.

(II) On the **metric side of the mirror**, we start off by constructing a C^∞ family $(\omega_t)_{t\in B_{[\gamma]}}$ of Gauduchon metrics on the fibres $(X_t)_{t\in B_{[\gamma]}}$ (cf. Lemma 6.5.1) and a C^∞ family $(\omega_t^{1,\,1})_{t\in B}$ of Gauduchon metrics on X_0 (cf. Lemma 6.5.2). The $\omega_t^{1,\,1}$'s are the $(1,\,1)$-components of the ω_t's w.r.t. the complex structure J_0 of X_0.

Then we prove (cf. Corollary 6.5.6) that the Aeppli cohomology groups of bidegree $(2,\,2)$ of the local essential deformations X_t of the Iwasawa manifold $X = X_0$, namely the vector spaces

$$B_{[\gamma]} \ni t \mapsto H^{2,\,2}_A(X_t,\,\mathbb{C}),$$

define a C^∞ vector bundle $\mathcal{H}^{2,\,2}_A$ of rank 4 that injects as a C^∞ vector subbundle of the constant bundle $\mathcal{H}^4 \to B_{[\gamma]}$ of fibre given by the De Rham cohomology group $H^4_{DR}(X_t,\,\mathbb{C}) = H^4(X,\,\mathbb{C})$. This injection is proved by using in a crucial way the **sGG property** (cf. [PU14]) of all the fibres X_t

and the family $(\omega_t)_{t\in B_{[\gamma]}}$ of Gauduchon metrics thereon. Denoting by $\widetilde{H^{2,\,2}_{\omega_t}}$ the image of $H^{2,\,2}_A(X_t,\,\mathbb{C})$ into $H^4(X,\,\mathbb{C})$ under this ω_t-induced injection, we get a C^∞ vector bundle $\widetilde{\mathcal{H}^{2,\,2}_\omega}$ of rank 4

$$\mathcal{G}_{X_0}\ni[(\omega_t^{1,\,1})^2]_A\mapsto\widetilde{H^{2,\,2}_{\omega_t}}\subset H^4(X,\,\mathbb{C})$$

after suitable identifications of certain spaces depending on ω_t with spaces depending on $\omega_t^{1,\,1}$ (cf. Conclusion 6.5.12).

This produces a Hodge filtration

$$F_{\mathcal{G}}\mathcal{H}^4:=\mathcal{H}^{2,\,0}(B)\oplus\widetilde{\mathcal{H}^{2,\,2}_\omega}\supset F'_{\mathcal{G}}\mathcal{H}^4:=\mathcal{H}^{2,\,0}(B)$$

of *holomorphic* vector bundles over the complexification $\widetilde{\mathcal{G}_0}$ of the subset $\mathcal{G}_0$ of the Gauduchon cone $\mathcal{G}_{X_0}$ consisting of the classes $[(\omega_t^{1,\,1})^2]_A$ with $t\in B_{[\gamma]}$, where $\mathcal{H}^{2,\,0}(B)$ is a holomorphic line bundle over $B_{[\gamma]}$ induced by the Albanese tori B_t of the fibres X_t.

(III) The **link between the two sides of the mirror** is provided by the holomorphic family $(B_t)_{t\in B}$ of 2-dimensional complex Albanese tori $B_t=\mathrm{Alb}(X_t)$ of the small deformations X_t of the Iwasawa manifold $X=X_0$. Indeed, every small deformation X_t of X is a locally trivial holomorphic fibration $\pi_t: X_t\to B_t$ over its Albanese torus B_t. We get a holomorphic vector bundle of rank 5

$$\widetilde{\mathcal{G}_0}\ni[(\omega_t^{1,\,1})^2]_A\mapsto H^{2,\,0}(B_t,\,\mathbb{C})\oplus\widetilde{H^{2,\,2}_{\omega_t}}\subset H^3(X,\,\mathbb{C})\oplus H^4(X,\,\mathbb{C})\tag{6.6}$$

and a VHS parametrised by the complexification $\widetilde{\mathcal{G}_0}$ of the subset

$$\mathcal{G}_0:=\{[(\omega_t^{1,\,1})^2]_A\ \mid\ t\in B_{[\gamma]}\}$$

of the Gauduchon cone $\mathcal{G}_{X_0}$ of X_0 (cf. Conclusion 6.5.12).

The VHS (6.6), constructed on the *metric side of the mirror*, is then proved to be C^∞ *isomorphic* to the VHS induced by (6.1) and (6.2) on the *complex-structure side of the mirror*. This C^∞ isomorphism is actually *holomorphic* at the level of the 1-dimensional parts of the two VHS's and *anti-holomorphic* at the level of the 4-dimensional parts. This regularity meshes with the *sesquilinear* self-duality of the Iwasawa manifold highlighted in the next work [Pop17] of the author. This isomorphism will be obtained by proving (cf. Corollary 6.3.11) that each of the two Hodge filtrations is C^∞ *isomorphic* to the Hodge filtration $F^1\mathcal{H}^2(B)\supset F^2\mathcal{H}^2(B)$ of holomorphic vector bundles induced by the family of tori $(B_t)_{t\in B_{[\gamma]}}$ over the moduli space $B_{[\gamma]}$ of *essential deformations* of the Iwasawa manifold.

We also define explicitly (cf. Definition 6.6.2) a **mirror map**

$$\widetilde{\mathcal{M}}: B_{[\gamma]}\to\widetilde{\mathcal{G}}_X.$$

It has the property of taking the point $0\in B_{[\gamma]}$ (i.e. the Iwasawa manifold $X=X_0$, the marked point in $B_{[\gamma]}$) to the Aeppli cohomology class $[\omega_0^2]_A\in\mathcal{G}_X$ of the canonical Gauduchon metric ω_0 on X (the marked point in the Gauduchon cone $\mathcal{G}_X$). The mirror map $\widetilde{\mathcal{M}}$ is then proved in Theorem 6.6.3 to be a *local biholomorphism* whose differential at $0\in B_{[\gamma]}$ is the canonical isomorphism $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C})\simeq H^{2,\,2}_A(X,\,\mathbb{C})$ of Proposition 6.1.3. The analogous statement holds at every $t\in B_{[\gamma]}$ after we observe a canonical isomorphism $H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C})\simeq H^{2,\,2}_A(X_t,\,\mathbb{C})$ (cf. Observation 6.5.11).

The mirror map is defined by "complexification" of what we call the **positive mirror map** defined (cf. Definition 6.6.1) by

$$\mathcal{M} : B_{[\gamma]} \to \mathcal{G}_X, \quad t \mapsto \Big[(\omega_t^{1,1})^2\Big]_A.$$

We hope that these methods can be extended to other classes of compact complex manifolds. The ultimate goal is to get a general mirror symmetry theory asserting that every compact complex n-dimensional *sGG manifold* X (possibly, but not necessarily, assumed to be $\partial\bar\partial$) whose canonical bundle K_X is *trivial* and having some other familiar properties (e.g. *unobstructedness* of its Kuranishi family, degeneration at E_2 of its Frölicher spectral sequence, etc) admits a *mirror dual* $\widetilde{X}$ such that the moduli space $EssDef(X)$ of *essential deformations of the complex structure* of X (defined, e.g. using the space $E_2^{n-1,\,1}$ on the second page of the Frölicher spectral sequence of X) corresponds via a local biholomorphism to the complexified Gauduchon cone $\widetilde{\mathcal{G}}_{\widetilde{X}}$ of $\widetilde{X}$ and vice versa. This local biholomorphism ought to induce an isomorphism of variations of Hodge structures parametrised respectively by $EssDef(X)$ and $\widetilde{\mathcal{G}}_{\widetilde{X}}$. This isomorphism may turn out to be *holomorphic* at the level of certain parts of the two VHS's and *anti-holomorphic* for the other parts. Certain non-linear PDEs (e.g. of the Monge-Ampère or Hessian type) are expected to produce canonical metrics representing Aeppli cohomology classes in the Gauduchon cone. Some classes of nilmanifodls and solvmanifolds provide a fertile testing ground for this conjecture.

6.2 Essential deformations of the Iwasawa manifold

6.2.1 The Calabi-Yau isomorphism

Since $T^{1,0}X$ is trivial, the Iwasawa manifold X is, in particular, a Calabi-Yau manifold. Since its Kuranishi family $(X_t)_{t\in B}$ is *unobstructed* (by Nakamura [Nak75], see Theorem 4.5.39), its base B can be identified with an open ball in the Dolbeault cohomology group $H^{0,\,1}(X,\,T^{1,\,0}X)$ of classes of smooth $\bar\partial$-closed $(0,\,1)$-forms with values in the holomorphic tangent bundle $T^{1,\,0}X$. In particular, the holomorphic tangent space $T_0^{1,\,0}B$ to B at 0 is isomorphic, via the Kodaira-Spencer map ρ, to $H^{0,\,1}(X,\,T^{1,\,0}X)$.

On the other hand, the Calabi-Yau structure of X is defined by any nowhere-vanishing holomorphic $(3,\,0)$-form Ω on X. All such forms are equal up to a multiplicative constant, so we may choose, for example, $\Omega := \alpha\wedge\beta\wedge\gamma$. We get the following isomorphisms, the second of which will be called the *Calabi-Yau isomorphism*:

$$\begin{array}{rcl} T_0^{1,\,0}B \underset{\simeq}{\xrightarrow{\rho}} H^{0,\,1}(X,\,T^{1,\,0}X) & \underset{\simeq}{\xrightarrow{T_\Omega}} & H_{\bar\partial}^{2,\,1}(X,\,\mathbb{C}), \\ {[\theta]_{\bar\partial}} & \longmapsto & [\theta\lrcorner\Omega]_{\bar\partial}. \end{array} \tag{6.7}$$

The Calabi-Yau isomorphism can be described explicitly in the case of the Iwasawa manifold. Let $\xi_\alpha, \xi_\beta, \xi_\gamma \in H^0(X,\,T^{1,\,0}X)$ be the frame of holomorphic vector fields of type $(1,\,0)$ dual to the frame $\{\alpha,\,\beta,\,\gamma\}$. Thus,

$$\xi_\alpha = p_\star\bigg(\frac{\partial}{\partial z_1}\bigg), \xi_\beta = p_\star\bigg(\frac{\partial}{\partial z_2} + z_1\frac{\partial}{\partial z_3}\bigg) \quad \text{and} \quad \xi_\gamma = p_\star\bigg(\frac{\partial}{\partial z_3}\bigg)$$

where $p_\star$ stands for the differential of the quotient map $p : G \to X$.

Now, $T^{1,0}X$ being trivial, $H^{0,1}(X,\,T^{1,0}X) = H^{0,1}(X,\,\mathbb{C}) \otimes H^0(X,\,T^{1,0}X)$ is generated (cf. [Nak75]) by the Dolbeault cohomology classes

$$H^{0,1}(X,\,T^{1,0}X) = \left\langle [\overline{\alpha}\otimes\xi_\alpha], [\overline{\alpha}\otimes\xi_\beta], [\overline{\alpha}\otimes\xi_\gamma], [\overline{\beta}\otimes\xi_\alpha], [\overline{\beta}\otimes\xi_\beta], [\overline{\beta}\otimes\xi_\gamma] \right\rangle. \tag{6.8}$$

In particular, $\dim_{\mathbb{C}} H^{0,1}(X,\,T^{1,0}X) = 6$, so the Kuranishi family of X is 6-dimensional. The images under the Calabi-Yau isomorphism T_Ω of these generators are $[(\overline{\alpha}\otimes\xi_\alpha)\lrcorner(\alpha\wedge\beta\wedge\gamma)]_{\bar\partial} = [\beta\wedge\gamma\wedge\overline{\alpha}]_{\bar\partial}$ and its analogues for the remaining five generators, hence the description of $H^{2,1}_{\bar\partial}(X,\,\mathbb{C})$ in (1.57).

For future reference, we recall the following standard piece of notation. We let $\alpha_1 = \alpha, \alpha_2 = \beta, \xi_1 = \xi_\alpha, \xi_2 = \xi_\beta, \xi_3 = \xi_\gamma$ and denote by $t_{i\lambda}$, with $1\leq\lambda\leq 2$ and $1\leq i\leq 3$, the coordinates induced on $H^{0,1}(X,T^{1,0}X)$ by the basis $([\overline{\alpha}_\lambda\otimes\xi_i])_{\substack{1\leq\lambda\leq 2\\ 1\leq i\leq 3}}$ (cf. (6.8)). Since Δ is an open ball about the origin in $H^{0,1}(X,T^{1,0}X)$, we can view $(t_{11},\,t_{12},\,t_{21},\,t_{22},\,t_{31},\,t_{32})$ as coordinates on Δ. Thus, the points $t\in B\subset H^{0,1}(X,T^{1,0}X)$ can be written uniquely as

$$t = \sum_{\substack{1\leq\lambda\leq 2\\ 1\leq i\leq 3}} t_{i\lambda}\,\overline{\alpha}_\lambda\xi_i \in H^{0,1}(X,T^{1,0}X).$$

6.2.2 The essential deformations

The sequence of low-degree terms in the Leray spectral sequence induced by π and TX (the sheaf associated with the holomorphic tangent bundle $T^{1,0}X$) whose second page is given by $E_2^{p,q} = H^p(B,\mathcal{R}^q\pi_\star TX)$, together with the cohomologies of the short exact sequence

$$0\to T\pi\to TX\to \pi^\star TB\to 0$$

defining the relative tangent bundle to the submersion π, reads [1]

$$\begin{tikzcd}
0 \arrow[r] & H^1(B,\pi_\star T\pi) \arrow[r] \arrow[d] & H^1(X,T\pi) \arrow[r] \arrow[d] & H^0(B,\mathcal{R}^1\pi_\star T\pi) \arrow[d] \\
0 \arrow[r] & H^1(B,\pi_\star TX) \arrow[r, "iso"] \arrow[d] & H^1(X,TX) \arrow[r] \arrow[d] & H^0(B,\mathcal{R}^1\pi_\star TX) \arrow[d] \\
0 \arrow[r] & H^1(B,TB) \arrow[r, "iso"] & H^1(X,\pi^\star TB) \arrow[r] \arrow[u, bend right, "L"'] & H^0(B,\mathcal{R}^1\pi_\star\mathcal{O}_X\otimes TB).
\end{tikzcd}$$

As TX is trivial and as all $(0,1)$-Dolbeault cohomology classes on X come from classes on B (i.e. in terms of the Leray filtration, we have $H^1(X,\mathcal{O}_X) = \pi^\star H^1(B,\mathcal{O}_B) = F^1H^1(X,\mathcal{O}_X)$), the horizontal map $H^1(B,\pi_\star TX)\to H^1(X,TX)$ is an isomorphism. As $\bar\gamma\otimes\xi_.$ is $\bar\partial_\pi$-closed (i.e. $\bar\partial(\bar\gamma\otimes\xi_.) = -\bar\alpha\wedge\bar\beta\otimes\xi_.$ vanishes on the fibres of π), it defines an element in $H^0(B,\mathcal{R}^1\pi_\star T\pi)$, i.e. a deformation of the fibres of π. However, since $\bar\gamma\otimes\xi_.$ is not $\bar\partial$-closed, this does not lift to a global deformation of X.

Now, consider the quotient map

$$H^{0,1}(X,T^{1,0}X) = H^{0,1}(X)\otimes H^0(X,TX)\to H^{0,1}(X)\otimes H^0(X,\pi^\star TB) = H^{0,1}(X,\pi^\star T^{1,0}B)$$

[1]The notation used here refers to sheaves. We shall often use in the sequel the vector-bundle notation. For example, $H^1(B,\,TB)$ (in sheaf notation) coincides with $H^{0,1}(B,\,T^{1,0}B)$ (in vector-bundle notation).

given by the differential of the submersion π and choose its lift $L : H^{0,1}(X, \pi^\star T^{1,0}B) \to H^{0,1}(X, T^{1,0}X)$ defined by

$$L : (\pi \circ p)_\star\left(\frac{\partial}{\partial z_1}\right) \mapsto \xi_\alpha, \qquad (\pi \circ p)_\star\left(\frac{\partial}{\partial z_2}\right) \mapsto \xi_\beta.$$

Consider the subspace of $H^{0,\,1}(X,\, T^{1,\,0}X)$ defined by

$$H^{0,\,1}_{[\gamma]}(X,\, T^{1,\,0}X) := LH^{0,\,1}(X,\, \pi^\star T^{1,\,0}B) = \left\langle [\overline{\alpha} \otimes \xi_\alpha],\, [\overline{\alpha} \otimes \xi_\beta],\, [\overline{\beta} \otimes \xi_\alpha],\, [\overline{\beta} \otimes \xi_\beta] \right\rangle \tag{6.9}$$

This amounts to singling out, for every first-order deformation of B (i.e. for every element of $H^{0,1}(B, T^{1,0}B)$), a suitable first-order automorphism in $H^1(B, \pi_\star T\pi)$ of the fibres of π.

Lemma 6.2.1. *The map* $H^{0,\,1}(X,\, T^{1,\,0}X) \xrightarrow{\cdot\lrcorner[\gamma]_{\bar\partial}} H^{0,\,1}_{\bar\partial}(X,\, \mathbb{C})$, $\quad [\theta] \mapsto [\theta\lrcorner\gamma]_{\bar\partial}$, *is well defined and its kernel is precisely* $H^{0,\,1}_{[\gamma]}(X,\, T^{1,\,0}X)$, *i.e.*

$$H^{0,\,1}_{[\gamma]}(X,\, T^{1,\,0}X) = \left\{[\theta] \in H^{0,\,1}(X,\, T^{1,\,0}X) \Big/ [\theta\lrcorner\gamma] = 0 \in H^{0,\,1}_{\bar\partial}(X,\, \mathbb{C})\right\}. \tag{6.10}$$

Proof. For every $[\theta] \in H^{0,\,1}(X,\, T^{1,\,0}X)$, we have $\bar\partial(\theta\lrcorner\gamma) = (\bar\partial\theta)\lrcorner\gamma + \theta\lrcorner(\bar\partial\gamma) = 0$ since $\bar\partial\theta = 0$ (where $\bar\partial$ is the canonical $(0,\,1)$-connection of the holomorphic vector bundle $T^{1,\,0}X$ and θ is viewed as a $\bar\partial$-closed $(0,\,1)$-form with values in this bundle) and $\bar\partial\gamma = 0$. Thus, $\theta\lrcorner\gamma$ defines indeed a Dolbeault cohomology class of type $(0,\,1)$ which furthermore is independent of the choice of representative θ of the class $[\theta] \in H^{0,\,1}(X,\, T^{1,\,0}X)$. To see this last point, take two cohomologous θ_1, θ_2. Then, $\theta_1 - \theta_2 = \bar\partial\xi$ for some $\xi \in C^\infty(X,\, T^{1,\,0}X)$. We have $\bar\partial(\xi\lrcorner\gamma) = (\bar\partial\xi)\lrcorner\gamma - \xi\lrcorner(\bar\partial\gamma) = (\bar\partial\xi)\lrcorner\gamma$. This proves the well-definedness of the map $\cdot\lrcorner[\gamma]_{\bar\partial}$. Identity (6.10) follows at once from (6.8) and (6.9). □

Definition 6.2.2. *Bearing in mind that* $B \subset H^{0,\,1}(X,\, T^{1,\,0}X)$ *is an open subset, let*

$$B_{[\gamma]} := B \cap H^{0,\,1}_{[\gamma]}(X,\, T^{1,\,0}X).$$

So formally, thanks to (6.10) and by analogy with polarising $(1,\,1)$-classes [2], the family of deformations $(X_t)_{t\in B_{[\gamma]}}$ is "polarised" by the $(1,\,0)$-class $[\gamma]_{\bar\partial} \in H^{1,\,0}_{\bar\partial}(X,\, \mathbb{C})$.

It follows from Nakamura's description of the Kuranishi family of the Iwasawa manifold ([Nak75, p. 96]) that the manifolds X_t with $t \in B_{[\gamma]} \setminus \{0\}$ are contained in the union of Nakamura's classes (ii) and (iii). They are not complex parallelisable. Meanwhile, the removed deformations X_t with $t \in B \setminus \{0\}$ corresponding to $[\theta\lrcorner\Omega] \in \langle[\alpha\wedge\beta\wedge\overline{\alpha}]_{\bar\partial},\, [\alpha\wedge\beta\wedge\overline{\beta}]_{\bar\partial}\rangle \subset H^{2,\,1}_{\bar\partial}(X,\, \mathbb{C})$ make up Nakamura's class (i). They are all complex parallelisable (and, in a sense, have the *same* geometry as the Iwasawa manifold $X = X_0$). So, no geometric information is lost by these removals. For this reason, we call $(X_t)_{t\in B_{[\gamma]}}$ the local universal family of **essential deformations** of X.

In terms of coordinates, we see that $(t_{11},\, t_{12},\, t_{21},\, t_{22})$ define coordinates on $B_{[\gamma]}$. Consequently, the points $t \in B_{[\gamma]} \subset H^{0,1}_{[\gamma]}(X, T^{1,0}X)$ can be written uniquely as

$$t = \sum_{\substack{1\leq\lambda\leq 2\\ 1\leq i\leq 2}} t_{i\lambda}\,\overline{\alpha}_\lambda\xi_i \in H^{0,1}_{[\gamma]}(X, T^{1,0}X).$$

[2]Recall that in the standard case of a Kähler class $[\omega]$ on X_0, the fibres X_t *polarised* by $[\omega]$, i.e. the fibres X_t for which $[\omega]$ remains of J_t-type $(1,\,1)$, are precisely those corresponding to $[\theta] \in H^{0,\,1}(X_0,\, T^{1,\,0}X_0)$ satisfying the condition $[\theta\lrcorner\omega] = 0$ in $H^{0,\,2}(X_0,\, \mathbb{C})$.

6.3 Weight-three Hodge decomposition for the Iwasawa manifold

6.3.1 The $(3,0)$-part

We start with a simple general observation.

Lemma 6.3.1. *Let Y be an arbitrary compact complex manifold with $\dim_{\mathbb{C}} Y = n$. Then, there is a canonical injection $H^{n,\,0}_{\bar\partial}(Y,\,\mathbb{C}) \hookrightarrow H^n_{DR}(Y,\,\mathbb{C})$.*

Proof. It is clear that $H^{n,\,0}_{\bar\partial}(Y,\,\mathbb{C}) = C^\infty_{n,\,0}(Y,\,\mathbb{C}) \cap \ker\bar\partial$ since every Dolbeault cohomology class $[u]_{\bar\partial}$ of bidegree $(n,\,0)$ has a unique representative u. Indeed, zero is the only $\bar\partial$-exact $(n,\,0)$-form. Moreover, every such $(n,0)$-form u is d-closed since $\partial u = 0$ for bidegree reasons. Therefore, the following map is well defined :

$$H^{n,\,0}_{\bar\partial}(Y,\,\mathbb{C}) \longrightarrow H^n_{DR}(Y,\,\mathbb{C}), \qquad [u]_{\bar\partial} \longmapsto \{u\}_{DR}.$$

It remains to prove that this map is injective, i.e. that $u = 0$ whenever u is d-exact. Suppose that for a $\bar\partial$-closed $(n,\,0)$-form u, we have $u = dv$. Then $u = \partial v$ for bidegree reasons. Hence

$$0 \leq \int_Y i^{n^2}\, u\wedge\bar u = \int_Y i^{n^2}\, u\wedge\bar\partial\bar v = (-1)^n\, i^{n^2} \int_Y \bar\partial(u\wedge\bar v) = (-1)^n\, i^{n^2} \int_Y d(u\wedge\bar v) = 0,$$

where the last identity follows from Stokes's theorem.

Since the smooth $(n,\,n)$-form $i^{n^2}\, u\wedge\bar u$ is non-negative at every point, this can only happen if $i^{n^2}\, u\wedge\bar u = 0$ at every point. We get $u = 0$ on X. Indeed, writing $u = f\, dz_1\wedge\cdots\wedge dz_n$ in local coordinates, we see that $i^{n^2}\, u\wedge\bar u = |f|^2\, idz_1\wedge d\bar z_1\wedge\cdots\wedge idz_n\wedge d\bar z_n$, hence $f = 0$ in our situation. $\square$

6.3.2 The $(2,1)$-part: definition of $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C})$

As above, X will stand for the Iwasawa manifold.

The space $H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$ does not inject canonically into $H^3_{DR}(X,\,\mathbb{C})$ as can be seen from (1.56) and (1.57), so there is no standard Hodge decomposition for $H^3_{DR}(X,\,\mathbb{C})$ on the Iwasawa manifold X. This can also be seen by a simple dimension count: $b_3 = 10$, while $h^{3,\,0} + h^{2,\,1} + h^{1,\,2} + h^{0,\,3} = 1+6+6+1 = 14 > 10$. However, we shall shrink the Dolbeault cohomology group of bidegree $(2,\,1)$ in order to make it *fit* into $H^3_{DR}(X,\,\mathbb{C})$ and shall thus obtain a corresponding Hodge decomposition of weight 3 that will be seen to have a precise geometric meaning in terms of the essential deformations of X defined in §.6.2.2.

Definition 6.3.2. *The 4-dimensional subspace $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C})$ of $H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$ is defined as the image of $H^{0,\,1}_{[\gamma]}(X,\,T^{1,\,0}X) = LH^{0,\,1}(X,\,\pi^\star T^{1,\,0}B)$ under the Calabi-Yau isomorphism $T_\Omega : H^{0,\,1}(X,\,T^{1,\,0}X) \to H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$.*

Thus, by (6.9), we get

$$H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) = \left\langle [\alpha\wedge\gamma\wedge\overline\alpha]_{\bar\partial},\, [\alpha\wedge\gamma\wedge\overline\beta]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline\alpha]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline\beta]_{\bar\partial}\right\rangle = [\gamma\wedge\pi^\star H^{1,1}(B,\mathbb{C})]_{\bar\partial}. \quad (6.11)$$

We see from (1.56) that $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C})$ injects into $H^3_{DR}(X,\mathbb{C})$. Note that, since $[\xi_\gamma\lrcorner\Omega] = [\alpha\wedge\beta] = [-d\gamma] = 0 \in H^2_{DR}(X,\mathbb{C})$ while $\bar\alpha$ and $\bar\beta$ are closed, the image of $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C})$ in $H^3_{DR}(X,\mathbb{C})$ does depend neither on the choice of the lift L [3] nor on the choice of $[\gamma]$ in $\frac{H^0(X,\Omega^1_X)}{H^0(\pi^\star\Omega^1_B)}$.

[3]It is also the image of the map $H^0(X,\Omega^1_{X/B}) \times \pi^\star H^{1,1}_{\bar\partial}(B,\mathbb{C}) \to H^3_{DR}(X,\mathbb{C}), ([\gamma],[u]_{\bar\partial}) \mapsto \{u\wedge\gamma\}_{DR}$

We get isomorphisms

$$T_0^{1,0}B_{[\gamma]} \underset{\simeq}{\xrightarrow{\rho}} H^{0,1}_{[\gamma]}(X,\,T^{1,0}X) \underset{\simeq}{\xrightarrow{T_\Omega}} T_\Omega(H^{0,1}_{[\gamma]}(X,\,T^{1,0}X)) =: H^{2,1}_{[\gamma]}(X,\,\mathbb{C}).$$

6.3.3 The $(1,2)$-part

Recall now that if a Hermitian metric ω has been fixed on an arbitrary compact complex n-dimensional manifold Y, the corresponding Hodge star operator $\star$ (defined by $u\wedge\star\bar{v} = \langle u,v\rangle_\omega\,dV_\omega$) leads to the following isomorphisms for every bidegree $(p,\,q)$:

$$\iota : H^{p,\,q}_{\bar\partial}(Y,\,\mathbb{C}) \xrightarrow{\simeq} H^{n-q,\,n-p}_{\partial}(Y,\,\mathbb{C}) \xrightarrow{\simeq} H^{n-p,\,n-q}_{\bar\partial}(Y,\,\mathbb{C}).$$

Indeed, the first isomorphism is given by $\star$ since $\star\,\Delta'' = \Delta'\,\star$, while the second one, which is $\mathbb{C}$-anti-linear, is defined by conjugation.

In our case, $n=3$ and the Iwasawa manifold X_0 is endowed with the canonical metric

$$\omega = \omega_0 := i\alpha\wedge\bar\alpha + i\beta\wedge\bar\beta + i\gamma\wedge\bar\gamma, \tag{6.12}$$

so we get

$$H^{3,\,0}_{\bar\partial}(X,\,\mathbb{C}) \xrightarrow{\simeq} H^{0,\,3}_{\bar\partial}(X,\,\mathbb{C}) \quad\text{and}\quad H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C}) \xrightarrow{\simeq} H^{1,\,2}_{\bar\partial}(X,\,\mathbb{C}).$$

Accordingly, we define

$$\begin{aligned} H^{1,\,2}_{[\gamma]}(X,\,\mathbb{C}) : &= \iota H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) = \Big\langle [\star(\beta\wedge\overline\beta\wedge\overline\gamma)]_{\bar\partial},\,[\star(\alpha\wedge\overline\beta\wedge\overline\gamma)]_{\bar\partial},\,[\star(\beta\wedge\overline\alpha\wedge\overline\gamma)]_{\bar\partial},\,[\star(\alpha\wedge\overline\alpha\wedge\overline\gamma)]_{\bar\partial}\Big\rangle \\ &= \Big\langle [\beta\wedge\overline\beta\wedge\overline\gamma]_{\bar\partial},\,[\alpha\wedge\overline\beta\wedge\overline\gamma]_{\bar\partial},\,[\beta\wedge\overline\alpha\wedge\overline\gamma]_{\bar\partial},\,[\alpha\wedge\overline\alpha\wedge\overline\gamma]_{\bar\partial}\Big\rangle \subset H^3_{DR}(X,\,\mathbb{C}), \end{aligned} \tag{6.13}$$

where $\star=\star_\omega$ is the Hodge star operator associated with ω. The fact that $\star$ can be dropped from the above definition of $H^{1,\,2}_{[\gamma]}(X,\,\mathbb{C})$ to give the expression on the second line follows from Lemma 6.4.2 below.

Proposition 6.3.3. *Let X be the Iwasawa manifold.*
There are canonical injections $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) \hookrightarrow H^3_{DR}(X,\,\mathbb{C})$ and $H^{1,\,2}_{[\gamma]}(X,\,\mathbb{C}) \hookrightarrow H^3_{DR}(X,\,\mathbb{C})$ giving rise to a canonical isomorphism

$$H^3_{DR}(X,\,\mathbb{C}) \simeq H^{3,\,0}_{\bar\partial}(X,\,\mathbb{C}) \oplus H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) \oplus H^{1,\,2}_{[\gamma]}(X,\,\mathbb{C}) \oplus H^{0,\,3}_{\bar\partial}(X,\,\mathbb{C}) \tag{6.14}$$

that will be called the **essential weight-three Hodge decomposition** *of the Iwasawa manifold. Moreover, there are canonical isomorphisms given by conjugation*

$$H^{3,\,0}_{\bar\partial}(X,\,\mathbb{C}) \simeq \overline{H^{0,\,3}_{\bar\partial}(X,\,\mathbb{C})} \quad\textit{and}\quad H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}) \simeq \overline{H^{1,\,2}_{[\gamma]}(X,\,\mathbb{C})}\ {}^{4} \tag{6.15}$$

that will be called the **essential weight-three Hodge symmetry** *of the Iwasawa manifold.*

Proof. The canonical injections follow obviously from the descriptions (6.11), (6.13) and (1.57) of $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C})$, $H^{1,\,2}_{[\gamma]}(X,\,\mathbb{C})$ and resp. $H^3_{DR}(X,\,\mathbb{C})$. On the other hand, $H^{3,\,0}_{\bar\partial}(X,\,\mathbb{C})$ injects canonically into $H^3_{DR}(X,\,\mathbb{C})$ by Lemma 6.3.1, while $H^{0,\,3}_{\bar\partial}(X,\,\mathbb{C})$ injects canonically thanks to its explicit description in (1.57). Since the images in $H^3_{DR}(X,\,\mathbb{C})$ of $H^{3,\,0}_{\bar\partial}(X,\,\mathbb{C}), H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C}), H^{1,\,2}_{[\gamma]}(X,\,\mathbb{C}), H^{0,\,3}_{\bar\partial}(X,\,\mathbb{C})$ are mutually transversal by the explicit description of the injections and since $10 = \dim H^3 = \dim H^{3,\,0} + \dim H^{2,\,1}_{[\gamma]} + \dim H^{1,\,2}_{[\gamma]} + \dim H^{0,\,3} = 1+4+4+1$, we get the isomorphism (6.14). The isomorphisms (6.15) follow from (1.57), (6.11) and (6.13). $\square$

[4] Note from the explicit descriptions in (1.57) that this isomorphism does not hold for the full Dolbeault cohomology

6.3.4 Hodge decomposition for small essential deformations of X

Recall that $B_{[\gamma]} = \{t \in B \mid t_{31} = t_{32} = 0\}$, so $(t_{11},\, t_{12},\, t_{21},\, t_{22})$ are coordinates on $B_{[\gamma]}$.

Proposition 6.3.4. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold $X = X_0$. Then the space $H^{2,1}_{[\gamma]}(X,\,\mathbb{C}) = H^{2,1}_{[\gamma]}(X_0,\,\mathbb{C})$ described in (6.11) is the fibre over $t = 0$ of a C^∞ vector bundle $B_{[\gamma]} \ni t \mapsto H^{2,1}_{[\gamma]}(X_t,\,\mathbb{C})$ of rank 4 on $B_{[\gamma]}$ that will be denoted by $\mathcal{H}^{2,1}_{[\gamma]}$.*

Proof. Recall that by [Nak75, p. 95], for $t = \sum_{\substack{1\leq\lambda\leq 2\\ 1\leq i\leq 3}} t_{i\lambda}\,\overline{\alpha}_\lambda \xi_i \in H^{0,1}(X_0, T^{1,0}X_0)$ [5], a system of local holomorphic coordinates $(\zeta_1(t),\, \zeta_2(t),\, \zeta_3(t))$ on $X_t = \mathbb{C}^3/\Gamma_t$ is given in terms of a system of local holomorphic coordinates $(z_1,\, z_2,\, z_3)$ on $X = X_0$ by the formulae

$$\zeta_1(t) = z_1 + \sum_{\lambda=1}^{2} t_{1\lambda}\,\bar{z}_\lambda, \quad \zeta_2(t) = z_2 + \sum_{\lambda=1}^{2} t_{2\lambda}\,\bar{z}_\lambda, \quad \zeta_3(t) = z_3 + \sum_{\lambda=1}^{2}(t_{3\lambda} + t_{2\lambda}\,z_1)\,\bar{z}_\lambda + A_t(\bar{z}) - D(t)\,\bar{z}_3, \tag{6.16}$$

where

$$\begin{aligned} A_t(\bar{z}) &:= \frac{1}{2}\,[t_{11}\,t_{21}\,\bar{z}_1^2 + 2\,t_{11}\,t_{22}\,\bar{z}_1\,\bar{z}_2 + t_{12}\,t_{22}\,\bar{z}_2^2] \\ \text{and} \qquad D(t) &:= t_{11}\,t_{22} - t_{12}\,t_{21}. \end{aligned}$$

Note that the $\zeta_j(t)$'s depend holomorphically on t. The projection map given in coordinates by

$$(\zeta_1(t),\, \zeta_2(t),\, \zeta_3(t)) \overset{\pi_t}{\mapsto} (\zeta_1(t),\, \zeta_2(t))$$

displays X_t as fibred over an *Abelian surface* $B_t = \mathrm{Alb}(X_t)$, the Albanse torus of X_t. These coordinates induce ([Ang11, §.4.3]), for every $t \in B$ close to 0, the co-frame

$$\alpha_t := d\zeta_1(t), \quad \beta_t := d\zeta_2(t), \quad \gamma_t := d\zeta_3(t) - z_1\,d\zeta_2(t) - (t_{21}\,\bar{z}_1 + t_{22}\,\bar{z}_2)\,d\zeta_1(t) \tag{6.17}$$

of $(1,\,0)$-forms on X_t (i.e. a Γ_t-invariant co-frame of $(1,\,0)$-forms on $\mathbb{C}^3$) varying in a holomorphic way with t. Note that $\alpha_t, \beta_t, \gamma_t$ are linearly independent at every point of X_t if t is sufficiently close to 0 by mere continuity of their dependence on t since $\alpha_0 = \alpha$, $\beta_0 = \beta$ and $\gamma_0 = \gamma$ are linearly independent at every point of X_0. Also note that γ_t need not be $\bar\partial_t$-closed when $t \neq 0$. Actually, the complex structure of X_t is complex parallelisable iff $\bar\partial_t\gamma_t = 0$ iff X_t is in Nakamura's class (i) (see [Nak75, p. 94-96]).

Moreover, for t in one of Nakamura's classes (ii) or (iii) (in particular, for $t \in B_{[\gamma]}$), the **structure equations** for γ_t (cf. [Ang11, §.4.3]) read

$$\begin{aligned} \bar\partial_t\gamma_t &= \sigma_{1\bar{1}}(t)\,\alpha_t \wedge \bar\alpha_t + \sigma_{1\bar{2}}(t)\,\alpha_t \wedge \bar\beta_t + \sigma_{2\bar{1}}(t)\,\beta_t \wedge \bar\alpha_t + \sigma_{2\bar{2}}(t)\,\beta_t \wedge \bar\beta_t, \\ \partial_t\gamma_t &= \sigma_{12}(t)\,\alpha_t \wedge \beta_t, \end{aligned} \tag{6.18}$$

where σ_{12} and $\sigma_{i\bar{j}}$ are C^∞ functions of $t \in B_{[\gamma]}$ that depend only on t (so $\sigma_{12}(t)$ and $\sigma_{i\bar{j}}(t)$ are complex numbers for every fixed $t \in B_{[\gamma]}$) and satisfy $\sigma_{12}(0) = -1$ and $\sigma_{i\bar{j}}(0) = 0$ for all i, j.

Now, for every $t \in B$ close to 0, the J_t-$(1,\,1)$-form

$$\omega_t := i\alpha_t \wedge \overline{\alpha}_t + i\beta_t \wedge \overline{\beta}_t + i\gamma_t \wedge \overline{\gamma}_t \tag{6.19}$$

[5] where $\alpha_1 = \alpha, \alpha_2 = \beta, \xi_1 = \xi_\alpha, \xi_2 = \xi_\beta, \xi_3 = \xi_\gamma$

is positive definite, hence it defines a Hermitian metric on X_t that varies in a C^∞ way with t. Note that ω_0 is *canonically* induced by the complex parallelisable structure of the Iwasawa manifold X_0. This feature will play a key role further down.

Let $\Delta''_t = \bar\partial_t\bar\partial_t^\star + \bar\partial_t^\star\bar\partial_t$ be the $\bar\partial$-Laplacian on X_t defined by ω_t. According to [Ang14, p. 80], for every t in one of Nakamura's classes (ii) or (iii) (in particular, for every $t \in B_{[\gamma]}$), the following J_t-(2, 1)-forms

$$\Gamma_1(t) := \alpha_t \wedge \gamma_t \wedge \bar\alpha_t - \frac{\sigma_{2\bar 2}(t)}{\bar\sigma_{12}(t)}\,\alpha_t \wedge \beta_t \wedge \bar\gamma_t, \qquad \Gamma_2(t) := \alpha_t \wedge \gamma_t \wedge \bar\beta_t - \frac{\sigma_{2\bar 1}(t)}{\bar\sigma_{12}(t)}\,\alpha_t \wedge \beta_t \wedge \bar\gamma_t,$$
$$\Gamma_3(t) := \beta_t \wedge \gamma_t \wedge \bar\alpha_t - \frac{\sigma_{1\bar 2}(t)}{\bar\sigma_{12}(t)}\,\alpha_t \wedge \beta_t \wedge \bar\gamma_t, \qquad \Gamma_4(t) := \beta_t \wedge \gamma_t \wedge \bar\beta_t - \frac{\sigma_{1\bar 1}(t)}{\bar\sigma_{12}(t)}\,\alpha_t \wedge \beta_t \wedge \bar\gamma_t \quad (6.20)$$

are linearly independent Δ''_t-harmonic forms. So, their Dolbeault classes are linearly independent.

Definition 6.3.5. *We define*

$$H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C}) := \left\langle [\Gamma_1(t)]_{\bar\partial},\,[\Gamma_2(t)]_{\bar\partial},\,[\Gamma_3(t)]_{\bar\partial},\,[\Gamma_4(t)]_{\bar\partial}\right\rangle \subset H^{2,\,1}_{\bar\partial}(X_t,\,\mathbb{C}) \quad \textit{for every } t \in B_{[\gamma]}. \quad (6.21)$$

The families $(\Gamma_k(t))_{t\in B_{[\gamma]}}$ are C^∞ families of Δ''_t-harmonic (2, 1)-forms (inducing C^∞ families $([\Gamma_k(t)]_{\bar\partial})_{t\in B_{[\gamma]}}$ of $\bar\partial$-cohomology classes) on the fibres of $(X_t)_{t\in B_{[\gamma]}}$ such that $\Gamma_1(0) = \alpha\wedge\gamma\wedge\bar\alpha$, $\Gamma_2(0) = \alpha\wedge\gamma\wedge\bar\beta$, $\Gamma_3(0) = \beta\wedge\gamma\wedge\bar\alpha$, $\Gamma_4(0) = \beta\wedge\gamma\wedge\bar\beta$. Note that the $\Gamma_k(t)$'s do not depend holomorphically on t.

Therefore, we get a C^∞ vector bundle $\mathcal{H}^{2,\,1}_{[\gamma]} \longrightarrow B_{[\gamma]}$ of rank 4, $B_{[\gamma]} \ni t \mapsto H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C}) = \mathcal{H}^{2,\,1}_{[\gamma],\,t}$ whose fibre above $t=0$ is $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C})$ defined in (6.11) [6]. □

Remark 6.3.6. By analogy with §.6.2.1, for every $t \in B_{[\gamma]}$ we consider the J_t-(3, 0)-form

$$\Omega_t := \alpha_t \wedge \beta_t \wedge \gamma_t.$$

Then Ω_t depends holomorphically on t, hence (by continuity) it is non-vanishing on X_t for all t sufficiently close to zero since Ω_0 is non-vanishing. Moreover, Ω_t is holomorphic since $\bar\partial_t\Omega_t = \alpha_t \wedge \beta_t \wedge \bar\partial_t\gamma_t = 0$, the last identity being a consequence of the special shape of the structure equations (6.18) (displaying the form $\bar\partial_t\gamma_t$ as lying in $\pi_t^\star\mathcal{C}^\infty_{1,1}(B_t,\mathbb{C})$). This shows again that the canonical bundle of X_t is trivial. By analogy with (6.7), for every $t \in B_{[\gamma]}$ we define the *Calabi-Yau isomorphism* of X_t by

$$H^{0,\,1}(X_t,\,T^{1,\,0}X_t) \xrightarrow[\simeq]{T_{\Omega_t}} H^{2,\,1}_{\bar\partial}(X_t,\,\mathbb{C}), \qquad [\theta] \longmapsto [\theta \lrcorner \Omega_t], \quad (6.22)$$

and finally, using the subspace $H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C}) \subset H^{2,\,1}_{\bar\partial}(X_t,\,\mathbb{C})$ introduced in Definition 6.3.5, we put

$$H^{0,\,1}_{[\gamma]}(X_t,\,T^{1,\,0}X_t) := T^{-1}_{\Omega_t}(H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C})) \subset H^{0,\,1}(X_t,\,T^{1,\,0}X_t), \qquad t \in B_{[\gamma]}. \quad (6.23)$$

In particular, the family $(T_{\Omega_t})_{t\in B_{[\gamma]}}$ of Calabi-Yau isomorphisms is holomorphic and $T^{1,\,0}_t B_{[\gamma]} \simeq H^{0,\,1}_{[\gamma]}(X_t,\,T^{1,\,0}X_t)$ for all $t \in B_{[\gamma]}$.

[6] Alternatively, we could have displayed $\mathcal{H}^{2,\,1}_{[\gamma],\,t}$ as the bundle of kernels of a smooth family of elliptic differential operators involving a zeroth-order perturbation by the γ_t.

The following statement follows from definitions (6.20) and the structure equations (6.18). For $t=0$, it overlaps with Lemma 6.4.1.

Lemma 6.3.7. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold $X = X_0$. Then, for every $t \in B_{[\gamma]}$, the J_t-$(2,\,1)$-forms $\Gamma_1(t), \Gamma_2(t), \Gamma_3(t), \Gamma_4(t)$ of (6.20) are d-closed and $\bar\partial_t^\star$-closed, where $\bar\partial_t^\star$ is the formal adjoint of $\bar\partial_t$ w.r.t. the metric ω_t defined in (6.19). When $t=0$, they are also $\partial_0^\star$-closed.*

Proof. Thanks to (6.17), we have $d\alpha_t = d\beta_t = 0$. Meanwhile, $\partial_t\gamma_t = \sigma_{12}(t)\,\alpha_t\wedge\beta_t$ comes from a form of type $(2,0)$ on B_t by (6.18). Hence,

$$\partial_t(\alpha_t\wedge\gamma_t\wedge\bar\alpha_t) \;=\; -\alpha_t\wedge\partial_t\gamma_t\wedge\bar\alpha_t = 0$$

and also

$$\partial_t(\beta_t\wedge\gamma_t\wedge\bar\alpha_t) = \partial_t(\alpha_t\wedge\gamma_t\wedge\bar\beta_t) = \partial_t(\beta_t\wedge\gamma_t\wedge\bar\beta_t) = 0.$$

From (6.18), we get

$$\partial_t\bar\gamma_t = \overline{\bar\partial_t\gamma_t} = \overline{\sigma_{1\bar1}(t)}\,\bar\alpha_t\wedge\alpha_t + \overline{\sigma_{1\bar2}(t)}\,\bar\alpha_t\wedge\beta_t + \overline{\sigma_{2\bar1}(t)}\,\bar\beta_t\wedge\alpha_t + \overline{\sigma_{2\bar2}(t)}\,\bar\beta_t\wedge\beta_t,$$

hence

$$\partial_t(\alpha_t\wedge\beta_t\wedge\bar\gamma_t) = \alpha_t\wedge\beta_t\wedge\partial_t\bar\gamma_t = 0$$

since all the terms in the resulting sum contain a product $\alpha_t\wedge\alpha_t = 0$ or $\beta_t\wedge\beta_t = 0$. These identities, together with (6.20), prove that $\partial_t\Gamma_j(t) = 0$ for all $t\in B_{[\gamma]}$ and all $j = 1,2,3,4$.

On the other hand, $\bar\partial_t\Gamma_j(t) = 0$ and $\bar\partial_t^\star\Gamma_j(t) = 0$ since the forms $\Gamma_j(t)$ are Δ_t''-harmonic ([Ang14, p. 80]). Therefore, they are all d-closed and $\bar\partial_t^\star$-closed.

Thanks to (6.20), checking whether or not the forms $\Gamma_j(t)$ lie in the kernel of $\partial_t^\star$ involves computing the quantities $\langle\langle\partial_t^\star(\alpha_t\wedge\gamma_t\wedge\bar\alpha_t),\,u\rangle\rangle$, $\langle\langle\partial_t^\star(\beta_t\wedge\gamma_t\wedge\bar\alpha_t),\,u\rangle\rangle$, $\langle\langle\partial_t^\star(\alpha_t\wedge\gamma_t\wedge\bar\beta_t),\,u\rangle\rangle$, $\langle\langle\partial_t^\star(\beta_t\wedge\gamma_t\wedge\bar\beta_t),\,u\rangle\rangle$, $\langle\langle\partial_t^\star(\alpha_t\wedge\beta_t\wedge\bar\gamma_t),\,u\rangle\rangle$ for all forms $u\in C^\infty_{1,1}(X_t,\,\mathbb{C})$ in a system of generators. Now, among the generators $\alpha_t\wedge\bar\alpha_t$, $\alpha_t\wedge\bar\beta_t$, $\alpha_t\wedge\bar\gamma_t$, $\beta_t\wedge\bar\alpha_t$, $\beta_t\wedge\bar\beta_t$, $\beta_t\wedge\bar\gamma_t$, $\gamma_t\wedge\bar\alpha_t$, $\gamma_t\wedge\bar\beta_t$, $\gamma_t\wedge\bar\gamma_t$ of the space $C^\infty_{1,1}(X_t,\,\mathbb{C})$, only those containing γ_t or $\bar\gamma_t$ are not ∂_t-closed. Moreover, when u is one of these except $\gamma_t\wedge\bar\gamma_t$, $\partial_t u$ is a sum of factors none of which is either γ_t or $\bar\gamma_t$, so the above $L^2_{\omega_t}$ inner products vanish.

Indeed, for example, if $u = \alpha_t\wedge\bar\gamma_t$, then

$$\partial_t u = -\alpha_t\wedge\partial_t\bar\gamma_t = -\alpha_t\wedge[\overline{\sigma_{1\bar2}(t)}\,\bar\alpha_t\wedge\beta_t + \overline{\sigma_{2\bar2}(t)}\,\bar\beta_t\wedge\beta_t],$$

where the last identity follows from (6.18). We get

$$\langle\langle\partial_t^\star(\alpha_t\wedge\gamma_t\wedge\bar\alpha_t),\,u\rangle\rangle = \langle\langle\alpha_t\wedge\gamma_t\wedge\bar\alpha_t,\,\partial_t u\rangle\rangle = 0$$

since $\alpha_t\wedge\gamma_t\wedge\bar\alpha_t$ is $L^2_{\omega_t}$-orthogonal onto $\alpha_t\wedge\bar\alpha_t\wedge\beta_t$ and onto $\alpha_t\wedge\bar\beta_t\wedge\beta_t$. This orthogonality follows from the basis of $(1,\,0)$-forms $\alpha_t,\beta_t,\gamma_t$ being $L^2_{\omega_t}$-orthonormal.

However, when $u = \gamma_t\wedge\bar\gamma_t$, we get

$$\begin{aligned}\partial_t u = \partial_t\gamma_t\wedge\bar\gamma_t - \gamma_t\wedge\partial_t\bar\gamma_t \;&=\; \sigma_{12}(t)\,\alpha_t\wedge\beta_t\wedge\bar\gamma_t + \overline{\sigma_{1\bar1}(t)}\,\gamma_t\wedge\alpha_t\wedge\bar\alpha_t + \overline{\sigma_{1\bar2}(t)}\,\gamma_t\wedge\beta_t\wedge\bar\alpha_t\\ &+\; \overline{\sigma_{2\bar1}(t)}\,\gamma_t\wedge\alpha_t\wedge\bar\beta_t + \overline{\sigma_{2\bar2}(t)}\,\gamma_t\wedge\beta_t\wedge\bar\beta_t.\end{aligned}$$

Hence $\langle\langle\alpha_t\wedge\gamma_t\wedge\bar\alpha_t,\,\partial_t u\rangle\rangle = -\sigma_{1\bar1}(t)$ and $\langle\langle\frac{\sigma_{2\bar2}(t)}{\sigma_{12}(t)}\,\alpha_t\wedge\beta_t\wedge\bar\gamma_t,\,\partial_t u\rangle\rangle = \sigma_{2\bar2}(t)$, so $\partial_t^\star\Gamma_1(t) = 0$ if and only if $\sigma_{2\bar2}(t) = -\sigma_{1\bar1}(t)$. There is no reason for this to happen when $t\neq 0$, but it does happen at $t=0$ since $\sigma_{i\bar j}(0) = 0$ for all i,j.

The forms $\Gamma_2(t), \Gamma_3(t), \Gamma_4(t)$ can be treated in a similar way. $\square$

Corollary 6.3.8. *For every $t\in B_{[\gamma]}$ sufficiently close to 0, we have a linear* **injection**

$$H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C})\longrightarrow H^3_{DR}(X,\,\mathbb{C}),\qquad [\Gamma_j(t)]_{\bar\partial}\mapsto\{\Gamma_j(t)\}_{DR}\quad \textit{for } j=1,\dots,4, \tag{6.24}$$

where X is the C^∞ manifold underlying the fibres X_t.

Proof. The Δ_0-harmonicity of the linearly independent forms $\Gamma_1(0),\Gamma_2(0),\Gamma_3(0),\Gamma_4(0)$ implies that the De Rham classes they define are linearly independent in $H^3_{DR}(X,\,\mathbb{C})$. Thus, the linear map defined in (6.24) is an injection when $t=0$. Then, by continuity, it remains an injection for $t\in B_{[\gamma]}$ sufficiently close to 0. $\square$

As earlier on, we define

$$H^{1,\,2}_{[\gamma]}(X_t,\,\mathbb{C}):=\iota_t(H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C}))=\bigg\langle[\star_t\overline\Gamma_1(t)]_{\bar\partial},\,[\star_t\overline\Gamma_2(t)]_{\bar\partial},\,[\star_t\overline\Gamma_3(t)]_{\bar\partial},\,[\star_t\overline\Gamma_4(t)]_{\bar\partial}\bigg\rangle\subset H^{1,\,2}_{\bar\partial}(X_t,\,\mathbb{C}) \tag{6.25}$$

where $\star_t:=\star_{\omega_t}$ is the Hodge star operator associated with the metric ω_t defined in (6.19) on X_t.

6.3.5 Identification of $H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C})$ with $E^{2,\,1}_2(X_t)$

We shall now give a cohomological interpretation of the spaces $H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C})$ in terms of the groups $E^{2,\,1}_2(X_t,\,\mathbb{C})$ featuring at the second step of the Frölicher spectral sequence of each small deformation X_t of the Iwasawa manifold $X=X_0$. At least the first conclusion of the following statement was observed in [COUV16]. Nakamura's classes (i), (ii) and (iii) into which the small deformations of the Iwasawa manifold are divided were described in Theorem and Definition 4.5.40.

Proposition 6.3.9. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold $X=X_0$. Then:*

(a) the Frölicher spectral sequence of X_t degenerates at E_2 for every $t\in B$ sufficiently close to 0;

(b) at the second step of the Frölicher spectral sequence, we have $\dim E^{2,\,1}_2(X_t,\,\mathbb{C})=4$ for $t=0$ and for every X_t in any of Nakamura's classes (ii) and (iii) (in particular, for every $t\in B_{[\gamma]}$);

(c) there is a canonical isomorphism $E^{2,\,1}_2(X_t,\,\mathbb{C})\simeq H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C})$ for $t=0$ and for every X_t in any of Nakamura's classes (ii) and (iii) (in particular, for every $t\in B_{[\gamma]}$).

Proof. (a) This follows from Theorem 5.6 in [COUV16]. Indeed, the X_t's are nilmanifolds of real dimension 6 endowed with invariant complex structures and admitting sG metrics. This last property follows from the Iwasawa manifold X_0 being balanced, hence sG, and from the sG property being deformation open ([Pop14, Theorem 3.1]).

(b) and (c) For $X=X_0$, the part of the E_1 page of the Frölicher spectral sequence relevant to us is

$$\cdots\xrightarrow{\partial}H^{1,\,1}_{\bar\partial}(X,\,\mathbb{C})\xrightarrow{\partial}H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})=H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C})\oplus\bigg\langle[\alpha\wedge\beta\wedge\bar\alpha]_{\bar\partial},\,[\alpha\wedge\beta\wedge\bar\beta]_{\bar\partial}\bigg\rangle\xrightarrow{\partial}H^{3,\,1}_{\bar\partial}(X,\,\mathbb{C})\xrightarrow{\partial}0,$$

where ∂ is defined in cohomology by $\partial([u]_{\bar\partial})=[\partial u]_{\bar\partial}$ and the direct-sum splitting follows from (1.57) and (6.11). Now, we see that much like $\alpha\wedge\beta\wedge\bar\alpha$ and $\alpha\wedge\beta\wedge\bar\beta$, the representatives $\alpha\wedge\gamma\wedge\bar\alpha$, $\alpha\wedge\gamma\wedge\bar\beta$, $\beta\wedge\gamma\wedge\bar\alpha$ and $\beta\wedge\gamma\wedge\bar\beta$ of the four $(2,\,1)$-classes generating $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C})$ (cf. (6.11)) are ∂-closed. Indeed, for example, $\partial(\alpha\wedge\gamma\wedge\bar\alpha)=-\alpha\wedge\partial\gamma\wedge\bar\alpha=\alpha\wedge(\alpha\wedge\beta)\wedge\bar\alpha=0$. Hence, the whole of $H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$

is contained in the kernel of ∂. Using the explicit description (1.57) of $H^{1,1}_{\bar\partial}(X,\mathbb{C})$ and the structure equation $\partial\gamma = -\alpha\wedge\beta$ of, we infer that the image of the map $\partial : H^{1,1}_{\bar\partial}(X,\mathbb{C}) \longrightarrow H^{2,1}_{\bar\partial}(X,\mathbb{C})$ is $\langle[\alpha\wedge\beta\wedge\bar\alpha]_{\bar\partial},\, [\alpha\wedge\beta\wedge\bar\beta]_{\bar\partial}\rangle$. This proves that

$$\begin{aligned} E_2^{2,1}(X) &= \ker\Big(\partial : H^{2,1}_{\bar\partial}(X,\mathbb{C}) \longrightarrow H^{3,1}_{\bar\partial}(X,\mathbb{C})\Big)\Big/\mathrm{Im}\Big(\partial : H^{1,1}_{\bar\partial}(X,\mathbb{C}) \longrightarrow H^{2,1}_{\bar\partial}(X,\mathbb{C})\Big) \\ &= H^{2,1}_{\bar\partial}(X,C)\Big/\Big\langle[\alpha\wedge\beta\wedge\bar\alpha]_{\bar\partial},\, [\alpha\wedge\beta\wedge\bar\beta]_{\bar\partial}\Big\rangle \simeq H^{2,1}_{[\gamma]}(X,\mathbb{C}) \end{aligned}$$

which is (c) for $t=0$. In particular, $\dim E_2^{2,1}(X) = 4$ since $H^{2,1}_{[\gamma]}(X,\mathbb{C})$ has dimension 4 by construction.

We now analyse the case when X_t is in Nakamura's class (iii) and show that the Frölicher spectral sequence degenerates even at E_1. Indeed, the Betti numbers (deformation invariant) and the Hodge numbers of any such X_t computed in [Nak75] read

$$b_1 = 4 = 2+2 = h^{1,0}(t) + h^{0,1}(t), \quad b_2 = 8 = 1+5+2 = h^{2,0}(t) + h^{1,1}(t) + h^{0,2}(t),$$

$$b_3 = 10 = 1+4+4+1 = h^{3,0}(t) + h^{2,1}(t) + h^{1,2}(t) + h^{0,3}(t).$$

By Poincaré and Serre duality, we also get $b_4 = 8 = 2+5+1 = h^{3,1}(t) + h^{2,2}(t) + h^{1,3}(t)$ and $b_5 = 4 = 2+2 = h^{3,2}(t) + h^{2,3}(t)$. These identities amount to $E_1(X_t) = E_\infty(X_t)$ for every X_t in Nakamura's class (iii). In particular, $E_2^{2,1}(X_t) = E_1^{2,1}(X_t) = H^{2,1}_{\bar\partial}(X_t,\mathbb{C})$ whose dimension is $h^{2,1}(t) = 4$. Since the vector subspace $H^{2,1}_{[\gamma]}(X_t,\mathbb{C}) \subset H^{2,1}_{\bar\partial}(X_t,\mathbb{C})$ has the same dimension 4 (cf. (6.11)), we get $E_2^{2,1}(X_t) = H^{2,1}_{\bar\partial}(X_t,\mathbb{C}) = H^{2,1}_{[\gamma]}(X_t,\mathbb{C})$. This proves (b) and (c) for X_t in Nakamura's class (iii).

Suppose now that X_t is in Nakamura's class (ii). Using the description (cf. [Ang11, Appendix A])

$$H^{2,1}_{\bar\partial}(X_t,\mathbb{C}) = H^{2,1}_{[\gamma]}(X_t,\mathbb{C}) \oplus \Big\langle[\alpha_t\wedge\beta_t\wedge\bar\alpha_t]_{\bar\partial},\, [\alpha_t\wedge\beta_t\wedge\bar\beta_t]_{\bar\partial}\Big\rangle,$$

where $\dim\langle[\alpha_t\wedge\beta_t\wedge\bar\alpha_t]_{\bar\partial},\, [\alpha_t\wedge\beta_t\wedge\bar\beta_t]_{\bar\partial}\rangle = 1$, and Lemma 6.3.7, we find that the map $\partial_t : H^{2,1}_{\bar\partial}(X_t,\mathbb{C}) \longrightarrow H^{3,1}_{\bar\partial}(X_t,\mathbb{C})$ is identically zero.

Recall that, thanks to [Ang11], we have the splitting

$$H^{1,1}_{\bar\partial}(X_t,\mathbb{C}) = \pi_t^\star H^{1,1}(B_t,\mathbb{C}) \oplus H^{1,1}_{vert}(X_t,\mathbb{C})$$

in which $H^{1,1}_{vert}(X_t,\mathbb{C})$ is of dimension 2 and is generated by classes represented by forms (containing the vertical form γ) of the shape $E\,\alpha_t\wedge\bar\gamma_t + F\,\beta_t\wedge\bar\gamma_t + G\,\gamma_t\wedge\bar\alpha_t + H\,\gamma_t\wedge\bar\beta_t$, where E,F,G,H are constants. Since $d\alpha_t = d\beta_t = 0$, $\partial_t(\pi_t^\star H^{1,1}(B_t,\mathbb{C})) = 0$. Meanwhile, immediate computations and the use of (6.18) give

$$\begin{aligned} \partial_t(\alpha_t\wedge\bar\gamma_t) &= -\alpha_t\wedge\overline{\bar\partial_t\gamma_t} = -\alpha_t\wedge(\overline{\sigma_{1\bar2}(t)}\,\bar\alpha_t\wedge\beta_t + \overline{\sigma_{2\bar2}(t)}\,\bar\beta_t\wedge\beta_t), \\ \partial_t(\beta_t\wedge\bar\gamma_t) &= -\beta_t\wedge\overline{\bar\partial_t\gamma_t} = -\beta_t\wedge(\overline{\sigma_{1\bar1}(t)}\,\bar\alpha_t\wedge\alpha_t + \overline{\sigma_{2\bar1}(t)}\,\bar\beta_t\wedge\alpha_t), \\ \partial_t(\gamma_t\wedge\bar\alpha_t) &= \partial_t\gamma_t\wedge\bar\alpha_t = \sigma_{12}(t)\,\alpha_t\wedge\beta_t\wedge\bar\alpha_t, \quad \partial_t(\gamma_t\wedge\bar\beta_t) = \partial_t\gamma_t\wedge\bar\beta_t = \sigma_{12}(t)\,\alpha_t\wedge\beta_t\wedge\bar\beta_t. \end{aligned}$$

Thus, $\partial_t(H^{1,1}_{\bar\partial}(X_t,\mathbb{C})) = \langle[\alpha_t\wedge\beta_t\wedge\bar\alpha_t]_{\bar\partial},\, [\alpha_t\wedge\beta_t\wedge\bar\beta_t]_{\bar\partial}\rangle$. This settles the case of Nakamura's class (ii). □

The conclusion of these considerations is summed up in the following

Theorem 6.3.10. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold $X = X_0$.*

(i) There exists over $B_{[\gamma]}$ a **variation of Hodge structures (VHS)** *of weight 3*

$$\mathcal{H}^3 = \mathcal{H}^{3,\,0} \oplus \mathcal{H}^{2,\,1}_{[\gamma]} \oplus \mathcal{H}^{1,\,2}_{[\gamma]} \oplus \mathcal{H}^{0,\,3}, \tag{6.26}$$

where $\mathcal{H}^3$ is the local system of fibre $H^3_{DR}(X,\,\mathbb{C})$, $\mathcal{H}^{3,\,0}$ is the holomorphic line bundle $B_{[\gamma]} \ni t \mapsto H^{3,\,0}_{\bar\partial}(X_t,\,\mathbb{C})$, $\mathcal{H}^{2,\,1}_{[\gamma]}$ is the C^∞ vector bundle $B_{[\gamma]} \ni t \mapsto H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C}) \simeq E_2^{2,\,1}(X_t,\,\mathbb{C})$ of rank 4, while $\mathcal{H}^{1,\,2}_{[\gamma]} \simeq \overline{\mathcal{H}^{2,\,1}_{[\gamma]}}$ and $\mathcal{H}^{0,\,3} = \overline{\mathcal{H}^{3,\,0}}$.

(ii) The vector subbundles $F^3\mathcal{H}^3 := \mathcal{H}^{3,\,0} \subset \mathcal{H}^3$ and $F^2\mathcal{H}^3_{[\gamma]} := \mathcal{H}^{3,\,0} \oplus \mathcal{H}^{2,\,1}_{[\gamma]} \subset \mathcal{H}^3$ are **holomorphic**.

The C^∞ vector subbundle $F^1\mathcal{H}^3_{[\gamma]} := \mathcal{H}^{3,\,0} \oplus \mathcal{H}^{2,\,1}_{[\gamma]} \oplus \mathcal{H}^{1,\,2}_{[\gamma]} \subset \mathcal{H}^3$ is **not holomorphic**. *This is one of two possible deviations from the behaviour of a standard Hodge filtration.*

(iii) As in the standard case, there is a flat connection $\nabla : \mathcal{H}^3 \longrightarrow \mathcal{H}^3 \otimes \Omega_{B_{[\gamma]}}$ (the **Gauss-Manin connection***) satisfying the* **Griffiths transversality condition**

$$\nabla F^3\mathcal{H}^3 \subset F^2\mathcal{H}^3_{[\gamma]} \otimes \Omega_{B_{[\gamma]}}. \tag{6.27}$$

Moreover, in the case of $F^1\mathcal{H}^3_{[\gamma]}$, the **orthogonality relations** *derived from a possible transversality statement remain true:*

$$Q(\nabla F^1\mathcal{H}^3_{[\gamma]},\, F^0\mathcal{H}^3_{[\gamma]}) = 0. \tag{6.28}$$

It is unclear whether the transversality condition $\nabla F^p\mathcal{H}^3_{[\gamma]} \subset F^{p-1}\mathcal{H}^3_{[\gamma]} \otimes \Omega_{B_{[\gamma]}}$ holds for $p = 2$ or $p = 1$ (the second possible deviation from the behaviour of a standard Hodge filtration).

Proof. (i) The injection $\mathcal{H}^{3,\,0} \hookrightarrow \mathcal{H}^3$ is a consequence of Lemma 6.3.1, while the injection $\mathcal{H}^{2,\,1}_{[\gamma]} \hookrightarrow \mathcal{H}^3$ follows from Corollary 6.3.8.

Moreover, the property $E_2(X_t) = E_\infty(X_t)$ (cf. (a) of Proposition 6.3.9) gives an isomorphism

$$H^3_{DR}(X,\,\mathbb{C}) \simeq E_2^{3,\,0}(X_t,\,\mathbb{C}) \oplus E_2^{2,\,1}(X_t,\,\mathbb{C}) \oplus E_2^{1,\,2}(X_t,\,\mathbb{C}) \oplus E_2^{0,\,3}(X_t,\,\mathbb{C}) \quad \text{for every } t \in B. \tag{6.29}$$

We have (cf. (c) of Proposition 6.3.9) a canonical isomorphism $E_2^{2,\,1}(X_t,\,\mathbb{C}) \simeq H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C})$, while it is easy to prove that $E_2^{3,\,0}(X_t,\,\mathbb{C}) = H^{3,\,0}_{\bar\partial}(X_t,\,\mathbb{C})$ for every $t \in B_{[\gamma]}$. Indeed, to see this last point, recall that

$$E_2^{3,\,0}(X_t,\,\mathbb{C}) = \ker\left(\partial_t : H^{3,\,0}_{\bar\partial}(X_t,\,\mathbb{C}) \longrightarrow 0\right)\Big/\operatorname{Im}\left(\partial_t : H^{2,\,0}_{\bar\partial}(X_t,\,\mathbb{C}) \longrightarrow H^{3,\,0}_{\bar\partial}(X_t,\,\mathbb{C})\right). \tag{6.30}$$

The map ∂_t acting on $H^{3,\,0}_{\bar\partial}(X_t,\,\mathbb{C})$ arrives in $H^{4,\,0}_{\bar\partial}(X_t,\,\mathbb{C}) = 0$, while $H^{2,\,0}_{\bar\partial}(X_t,\,\mathbb{C})$ is generated by $[\alpha_t \wedge \beta_t]_{\bar\partial}$ when X_t is in Nakamura's class (iii) and by $[\alpha_t \wedge \beta_t]_{\bar\partial}$ and either $[\alpha_t \wedge \gamma_t]_{\bar\partial}$ or $[\beta_t \wedge \gamma_t]_{\bar\partial}$ when X_t is in Nakamura's class (ii). Now, all the three forms $\alpha_t \wedge \beta_t, \alpha_t \wedge \gamma_t, \beta_t \wedge \gamma_t$ are ∂_t-closed since α_t and β_t are ∂_t-closed and $\partial_t\gamma_t$ is a multiple of $\alpha_t \wedge \beta_t$. Therefore, $\partial_t(H^{2,\,0}_{\bar\partial}(X_t,\,\mathbb{C})) = 0$. Thus, we get from (6.30) that $E_2^{3,\,0}(X_t,\,\mathbb{C}) = H^{3,\,0}_{\bar\partial}(X_t,\,\mathbb{C})$, as stated.

It can then be proved from this that $E_2^{2,\,1}(X_t,\,\mathbb{C}) \xrightarrow{\simeq} \overline{E_2^{1,\,2}(X_t,\,\mathbb{C})}$ for every $t \in B_{[\gamma]}$. Now, (6.26) follows by combining these facts with Proposition 6.3.4.

(ii) In the first statement, only the fact that the C^∞ vector subbundle $F^2\mathcal{H}^3_{[\gamma]} \subset \mathcal{H}^3$ is actually holomorphic still needs a proof. We have to show that the holomorphic structure of $F^2\mathcal{H}^3_{[\gamma]}$ is the restriction of the holomorphic structure of $\mathcal{H}^3$. In other words, we have to show that for any C^∞ section s of $F^2\mathcal{H}^3_{[\gamma]}$, the a priori $\mathcal{H}^3$-valued $(0, 1)$-form $D''s$ is actually $F^2\mathcal{H}^3_{[\gamma]}$-valued, where D'' is the canonical $(0, 1)$-connection of the constant bundle $\mathcal{H}^3$. We are thus reduced to showing that all the anti-holomorphic first-order derivatives of the $[\Gamma_j(t)]_{\bar\partial}$'s lie in $F^2H^3_{[\gamma]}(X_t, \mathbb{C})$, i.e. that

$$\frac{\partial[\Gamma_j]_{\bar\partial}}{\partial\bar t_{i\lambda}}(t) \in H^{3,0}(X_t, \mathbb{C}) \oplus H^{2,1}_{[\gamma]}(X_t, \mathbb{C}) = F^2H^3_{[\gamma]}(X_t, \mathbb{C}) \quad \text{for all } t \in B_{[\gamma]} \text{ all } i, \lambda. \tag{6.31}$$

By way of example, we will show this for the derivatives at $t = 0$.

To this end, we will make use of the explicit formula for $\Gamma_1(t)$ and its analogues for $\Gamma_2(t), \Gamma_3(t), \Gamma_4(t)$ obtained in Lemma 6.7.1 and also of Lemma 6.7.2. Only the terms on the r.h.s. of that formula that are linear in the $\bar t_{i\lambda}$'s give a non-trivial contribution to $(\partial\Gamma_1(t)/\partial\bar t_{i\lambda})(0)$. Now, in each of the formulae for $\Gamma_1(t), \Gamma_2(t), \Gamma_3(t), \Gamma_4(t)$, the only such term featuring on the r.h.s. is, respectively,

$$-\bar t_{12}\,\alpha\wedge\beta\wedge\gamma, \quad -\bar t_{22}\,\alpha\wedge\beta\wedge\gamma, \quad \bar t_{11}\,\alpha\wedge\beta\wedge\gamma, \quad \bar t_{21}\,\alpha\wedge\beta\wedge\gamma,$$

whose derivative in the $\bar t_{12}$-direction (respectively the $\bar t_{22}$-, $\bar t_{11}$-, $\bar t_{21}$-direction) is obviously $-\alpha\wedge\beta\wedge\gamma$ (respectively $-\alpha\wedge\beta\wedge\gamma$, $\alpha\wedge\beta\wedge\gamma$, $\alpha\wedge\beta\wedge\gamma$). Thus, for $j \in \{1, 2, 3, 4\}$, the only non-vanishing first-order anti-holomorphic derivatives of the $[\Gamma_j]_{\bar\partial}$'s at 0 are

$$\frac{\partial[\Gamma_j]_{\bar\partial}}{\partial\bar t_{i\lambda}}(0) = \pm\,[\alpha\wedge\beta\wedge\gamma]_{\bar\partial} \in H^{3,0}(X_0, \mathbb{C}) \subset H^{3,0}(X_0, \mathbb{C}) \oplus H^{2,1}_{[\gamma]}(X_0, \mathbb{C}) = F^2H^3_{[\gamma]}(X_0, \mathbb{C}).$$

This proves the contention. Note that this also shows that the C^∞ vector subbundle $\mathcal{H}^{2,1}_{[\gamma]}$ of $\mathcal{H}^3$ is not a holomorphic subbundle, so the analogy with the standard, Kähler, case is preserved.

The second statement under (ii) is proved under (B) in the comments that follow the end of this proof.

(iii) The transversality statement is an immediate consequence of the fact that the $(-1, +1)$-component of the connection $\nabla_{[\theta]}$ coincides at any point $[\theta] \in T^{1,0}_tB_{[\gamma]} \simeq H^{0,1}_{[\gamma]}(X_t, T^{1,0}X_t)$ (for $t \in B_{[\gamma]}$) with the contraction operator $[\theta]\lrcorner\cdot$ (see (6.22) and (6.23)). Note that the relation $[\bar\alpha\wedge\bar\beta]_{\bar\partial} = [-\bar\partial\bar\gamma]_{\bar\partial} = 0$ implies that the contraction of the forms of (6.11) by the elements of (6.9) vanishes, hence we get transversality at 0: for all $[\theta] \in T^{1,0}_0B_{[\gamma]} \simeq H^{0,1}_{[\gamma]}(X, T^{1,0}X)$,

$$\begin{aligned} \nabla_{[\theta]}H^{3,0}(X, \mathbb{C}) \quad &\subset \quad H^{3,0}(X, \mathbb{C}) \oplus H^{2,1}_{[\gamma]}(X, \mathbb{C}), \\ \nabla_{[\theta]}H^{2,1}_{[\gamma]}(X, \mathbb{C}) \quad &\subset \quad H^{2,1}_{[\gamma]}(X, \mathbb{C}) \subset H^{2,1}_{[\gamma]}(X, \mathbb{C}) \oplus H^{1,2}_{[\gamma]}(X, \mathbb{C}). \end{aligned}$$

$\square$

We end this discussion with further comments about the Hodge filtration of Theorem 6.3.10. We notice (cf. Corollary 6.3.11) that the Hodge filtration $F^2\mathcal{H}^3_{[\gamma]} \supset F^3\mathcal{H}^3$ of holomorphic vector bundles over $B_{[\gamma]}$ constructed on the *complex-structure side of the mirror* is C^∞ *isomorphic* to the Hodge filtration $F^1\mathcal{H}^2(B) \supset F^2\mathcal{H}^2(B)$ of holomorphic vector bundles over $B_{[\gamma]}$ determined by the holomorphic family $(B_t)_{t\in B_{[\gamma]}}$ of Albanese tori $B_t = \mathrm{Alb}(X_t)$ of the fibres X_t. The latter Hodge filtration will be proved to be C^∞ isomorphic to a Hodge filtration that we shall construct on the *metric side of the mirror* in section 6.5, providing thus the link between the two sides.

(A) Recall that the fibres X_t are locally trivial holomorphic fibrations $\pi_t : X_t \to B_t$ over complex tori B_t (the Albanese tori of the X_t's) of dimension 2 varying in a holomorphic family $(B_t)_{t\in B}$. Implicit in the definition of $H^{2,1}_{[\gamma]}(X_t,\,\mathbb{C}) \subset H^{2,1}_{\bar\partial}(X_t,\,\mathbb{C})$ (cf. Definition 6.3.5) are the isomorphisms of complex vector spaces

$$H^{3,0}(X_t,\,\mathbb{C}) \simeq [\gamma_t\wedge\pi_t^\star H^{2,0}(B_t,\,\mathbb{C})]_{\bar\partial} \quad \text{and} \quad H^{2,1}_{[\gamma]}(X_t,\,\mathbb{C}) \simeq [\gamma_t\wedge\pi_t^\star H^{1,1}(B_t,\,\mathbb{C})]_{\bar\partial}, \quad t\in B_{[\gamma]}, \tag{6.32}$$

defined by the descriptions $H^{2,0}(B_t,\,\mathbb{C}) = \mathbb{C}\,[\alpha_t\wedge\beta_t]_{\bar\partial}$ and $H^{1,1}(B_t,\,\mathbb{C}) = \langle[\alpha_t\wedge\bar\alpha_t]_{\bar\partial},\,[\alpha_t\wedge\bar\beta_t]_{\bar\partial},\,[\beta_t\wedge\bar\alpha_t]_{\bar\partial},\,[\beta_t\wedge\bar\beta_t]_{\bar\partial}\rangle$ of these vector spaces.

Corollary 6.3.11. *The vector space isomorphisms (6.32) induce C^∞ isomorphisms of vector bundles over $B_{[\gamma]}$*

$$F^3\mathcal{H}^3 \simeq F^2\mathcal{H}^2(B) \quad \textit{and} \quad F^2\mathcal{H}^3_{[\gamma]} \simeq F^1\mathcal{H}^2(B), \tag{6.33}$$

where $F^2\mathcal{H}^2(B)$ stands for the vector bundle $B_{[\gamma]}\ni t\mapsto H^{2,0}(B_t,\,\mathbb{C})$ and $F^1\mathcal{H}^2(B)$ stands for the vector bundle $B_{[\gamma]}\ni t\mapsto H^{2,0}(B_t,\,\mathbb{C})\oplus H^{1,1}(B_t,\,\mathbb{C})$.

Although the first isomorphism in (6.33) is holomorphic (because γ_t and π_t depend holomorphically on t), it is unclear whether the second one is holomorphic since the pullback under π_t and the subsequent exterior multiplication by γ_t are followed by the subtraction of a multiple of $\alpha_t\wedge\beta\wedge\bar\gamma_t$ in the definition (6.20) of the $\Gamma_j(t)$'s that need not depend holomorphically on t.

Now, $(B_t)_{t\in B}$ is a holomorphic family of compact Kähler manifolds, so its Hodge filtration $F^p\mathcal{H}^2(B)$ consists of *holomorphic* subbundles of the constant bundle $B_{[\gamma]}\ni t\mapsto H^2(B_t)$ (denoted henceforth by $\mathcal{H}^2(B)$). On the other hand, we know from the conclusion (ii) of Theorem 6.3.10 that the subbundles $F^3\mathcal{H}^3\longrightarrow B_{[\gamma]}$ and $F^2\mathcal{H}^3_{[\gamma]}\longrightarrow B_{[\gamma]}$ of the constant bundle $\mathcal{H}^3\longrightarrow B_{[\gamma]}$ are *holomorphic*.

(B) We now prove the last statement in part (ii) of Theorem 6.3.10. We know from (6.25) that the vector bundle $\mathcal{H}^{1,2}_{[\gamma]}$ is trivialised in a neighbourhood of $0\in B_{[\gamma]}$ by the Dolbeault cohomology classes of the forms $\star_t\overline{\Gamma}_j(t)$ with $j=1,\dots,4$.

It will be seen in Lemma 6.4.2 that $\star(\alpha\wedge\beta\wedge\overline{\gamma}) = i\,\alpha\wedge\beta\wedge\overline{\gamma}$. This also applies at an arbitrary t as do all the identities in Lemma 6.4.2, so $\star_t(\alpha_t\wedge\beta_t\wedge\overline{\gamma}_t) = i\,\alpha_t\wedge\beta_t\wedge\overline{\gamma}_t$ for all $t\in B$. Therefore, using (6.20) for the first line below and (6.42) for the second line, we get for all $t\in B$

$$\begin{aligned}
\star_t\Gamma_1(t) &= -i\,\beta_t\wedge\gamma_t\wedge\bar\beta_t - i\,\frac{\sigma_{2\bar2}(t)}{\bar\sigma_{12}(t)}\,\alpha_t\wedge\beta_t\wedge\bar\gamma_t \\
&= -i\,(\beta+t_{21}\,\bar\alpha+t_{22}\,\bar\beta)\wedge(\gamma+t_{31}\,\bar\alpha+t_{32}\,\bar\beta-D(t)\,\bar\gamma)\wedge(\bar\beta+\bar t_{21}\,\alpha+\bar t_{22}\,\beta) \\
&- \;i\,\frac{\sigma_{2\bar2}(t)}{\bar\sigma_{12}(t)}\,(\alpha+t_{11}\,\bar\alpha+t_{12}\,\bar\beta)\wedge(\beta+t_{21}\,\bar\alpha+t_{22}\,\bar\beta)\wedge(\bar\gamma+\bar t_{31}\,\alpha+\bar t_{32}\,\beta-\overline{D(t)}\,\gamma).
\end{aligned}$$

Thus, the terms of $\overline{\star_t\Gamma_1(t)}$ that are linear in the $\bar t_{i\lambda}$'s are contained in

$$i\,\bar t_{21}\,\alpha\wedge\bar\gamma\wedge\beta + i\,\bar t_{31}\,\bar\beta\wedge\alpha\wedge\beta + i\,\frac{\overline{\sigma_{2\bar2}(t)}}{\sigma_{12}(t)}\,\bar\alpha\wedge\bar\beta\wedge\gamma.$$

Deriving at $t=0$, we get

$$\frac{\partial\,\overline{\star_t\Gamma_1(t)}}{\partial\bar{t}_{21}}_{|t=0} = i\,\alpha\wedge\bar\gamma\wedge\beta + i\,\frac{\partial}{\partial\bar{t}_{21}}\left(\frac{\overline{\sigma_{2\bar{2}}(t)}}{\sigma_{12}(t)}\right)_{|t=0}\bar\alpha\wedge\bar\beta\wedge\gamma.$$

However, although the form $\bar\alpha\wedge\bar\beta\wedge\gamma$ is $\bar\partial_0$-closed, the form $\alpha\wedge\bar\gamma\wedge\beta$ is not (since $\bar\partial_0(\alpha\wedge\bar\gamma\wedge\beta) = -\alpha\wedge\bar\alpha\wedge\beta\wedge\bar\beta\neq 0$), so the form $(\partial\,\overline{\star_t\Gamma_1(t)})/(\partial\bar{t}_{21})_{|t=0}$ defines no Dolbeault cohomology class for $\bar\partial_0$. In particular, the C^∞ section

$$B_{[\gamma]}\ni t\mapsto[\star_t\,\overline{\Gamma}_1(t)]_{\bar\partial}$$

of the C^∞ vector subbundle $B_{[\gamma]}\ni t\mapsto H^{1,2}_{[\gamma]}(X_t,\,\mathbb{C})$ of $\mathcal{H}^3\longrightarrow B_{[\gamma]}$ does not remain a section of this bundle after derivation in the direction $\bar{t}_{21}$.

We conclude that $B_{[\gamma]}\ni t\mapsto H^{1,2}_{[\gamma]}(X_t,\,\mathbb{C})$ is not a holomorphic subbundle of $\mathcal{H}^3\longrightarrow B_{[\gamma]}$.

6.4 Coordinates on the base $B_{[\gamma]}$ of essential deformations

6.4.1 Signature of the intersection form on $F^2_{[\gamma]}H^3(X,\,\mathbb{C})$

Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold $X = X_0$. Recall that the Hodge-Riemann bilinear intersection form Q can always be canonically defined on $H^n_{DR}(X,\,\mathbb{C})$ for any compact complex n-dimensional manifold X. It is *non-degenerate* and depends only on the differential structure of X. When $\dim_{\mathbb{C}}X = 3$, Q is *alternating* and reads

$$Q: H^3_{DR}(X,\,\mathbb{C})\times H^3_{DR}(X,\,\mathbb{C})\longrightarrow\mathbb{C},\qquad (\{u\},\{v\})\longmapsto -\int_X u\wedge v\ ^{7}. \tag{6.34}$$

The associated sesquilinear form

$$H: H^3_{DR}(X,\,\mathbb{C})\times H^3_{DR}(X,\,\mathbb{C})\longrightarrow\mathbb{C},\qquad (\{u\},\{v\})\longmapsto -i\int_X u\wedge\bar v = i\,Q(\{u\},\,\{\bar v\})\ ^{8} \tag{6.35}$$

is non-degenerate.

Also recall that if a Hermitian metric ω has been fixed on an arbitrary compact complex n-dimensional manifold Y, the corresponding Hodge star operator $\star$ maps Δ-harmonic n-forms to Δ-harmonic n-forms (where $\Delta := dd^\star + d^\star d$ is the usual d-Laplacian), hence defines in conjunction with the Hodge isomorphism $H^n_{DR}(Y,\,\mathbb{C})\simeq\ker(\Delta: C^\infty_n(Y,\,\mathbb{C})\to C^\infty_n(Y,\,\mathbb{C}))$ a linear map $\star: H^n_{DR}(Y,\,\mathbb{C})\longrightarrow H^n_{DR}(Y,\,\mathbb{C})$ satisfying $\star^2 = (-1)^n$. When $n=3$, the eigenvalues of the operator $\star$ are $-i, i$ and we get a decomposition

$$H^3_{DR}(X,\,\mathbb{C}) = H^3_+(X,\,\mathbb{C})\oplus H^3_-(X,\,\mathbb{C}), \tag{6.36}$$

where $H^3_\pm(X,\,\mathbb{C})$ are the eigenspaces of $\star$ corresponding to the eigenvalues $+i$, resp. $-i$.

Suppose now that $\dim_{\mathbb{C}}X = 3$. It was shown in [Pop13b, Lemmas 5.1 and 5.2] that for any Hermitian metric ω on X, $H(\cdot,\,\cdot)$ is positive definite on $H^3_+(X,\,\mathbb{C})$, negative definite on $H^3_-(X,\,\mathbb{C})$ and $H^3_+(X,\,\mathbb{C})$ is H-orthogonal to $H^3_-(X,\,\mathbb{C})$. Moreover,

$$H^{3,0}(X,\,\mathbb{C})\subset H^3_-(X,\,\mathbb{C}). \tag{6.37}$$

[7]In dimension n, the coefficient of the integral is $(-1)^{\frac{n(n-1)}{2}}$.

[8]For arbitrary n, the coefficient of the integral is $(-1)^{\frac{n(n+1)}{2}}\,i^n$.

Similar statements hold in arbitrary dimension n after adjusting for the parity of n.

Finally, recall that any compact *complex parallelisable* manifold X has a natural inner product defined on its space $C^\infty_{p,\,q}(X,\,\mathbb{C})$ of smooth differential forms of any bidegree $(p,\,q)$ (cf. [Nak75, §.4] for a construction going back to Kodaira). Indeed, if $n=\dim_{\mathbb{C}}X$, the hypothesis on X amounts to the existence of n holomorphic 1-forms $\varphi_1,\dots,\varphi_n\in C^\infty_{1,\,0}(X,\,\mathbb{C})$ that are linearly independent at every point in X. If $\xi_1,\dots,\xi_n\in H^0(X,\,T^{1,\,0}X)$ form the dual basis of holomorphic vector fields, every form $\varphi\in C^\infty_{0,\,1}(X,\,\mathbb{C})$ can be written uniquely as $\varphi=\sum\limits_{\lambda=1}^n f_\lambda\,\overline{\varphi}_\lambda$, where the f_λ's are smooth functions globally defined on X. One defines the L^2 **inner product** on $C^\infty_{0,\,1}(X,\,\mathbb{C})$ by

$$\langle\langle\varphi,\,\psi\rangle\rangle:=\int\limits_X\left(\sum\limits_{\lambda=1}^n f_\lambda\,\overline{g}_\lambda\right)i^{n^2}\,\varphi_1\wedge\dots\wedge\varphi_n\wedge\overline{\varphi}_1\wedge\dots\wedge\overline{\varphi}_n \tag{6.38}$$

for any smooth $(0,\,1)$-forms $\varphi=\sum\limits_{\lambda=1}^n f_\lambda\,\overline{\varphi}_\lambda$ and $\psi=\sum\limits_{\lambda=1}^n g_\lambda\,\overline{\varphi}_\lambda$. Note that $dV:=i^{n^2}\,\varphi_1\wedge\dots\wedge\varphi_n\wedge\overline{\varphi}_1\wedge\dots\wedge\overline{\varphi}_n>0$ is a C^∞ positive $(n,\,n)$-form on X that is used as volume form in (6.38). This means that $\langle\langle\varphi,\,\psi\rangle\rangle=\int_X\langle\varphi,\,\psi\rangle\,dV$, where the pointwise inner product $\langle\varphi,\,\psi\rangle$ on $(0,\,1)$-forms is defined by

$$\langle\overline{\varphi}_\lambda,\,\overline{\varphi}_\mu\rangle=\delta_{\lambda\,\mu}\quad\text{ for all }\ \lambda,\mu.$$

This induces a pointwise inner product on $C^\infty_{p,\,q}(X,\,\mathbb{C})$ for every p,q.

Now suppose that X is the Iwasawa manifold. Thus, $n=3$ and X is complex parallelisable, so with the notation of §.1.3.3 we can choose

$$\varphi_1=\alpha,\ \varphi_2=\beta,\ \varphi_3=\gamma\quad\text{and}\quad\xi_1=\xi_\alpha,\ \xi_2=\xi_\beta,\ \xi_3=\xi_\gamma.$$

The inner product defined above, induced by the complex parallelisable structure of X, coincides with the inner product induced by the canonical metric ω_0 on X defined in (6.12).

We can easily check that the $(2,\,1)$-forms $\alpha\wedge\gamma\wedge\overline{\alpha}$, $\alpha\wedge\gamma\wedge\overline{\beta}$, $\beta\wedge\gamma\wedge\overline{\alpha}$, $\beta\wedge\gamma\wedge\overline{\beta}$ representing the Dolbeault cohomology classes that generate $H^{2,\,1}_{[\gamma]}(X,\,\mathbb{C})$ (cf. (6.11)) are Δ-harmonic. Indeed, they are $\bar\partial$-closed since they are products of $\bar\partial$-closed forms. They are also ∂-closed (even if γ isn't), as can easily be checked. For example, we get $\partial(\alpha\wedge\gamma\wedge\overline{\alpha})=-\alpha\wedge\partial\gamma\wedge\overline{\alpha}=\alpha\wedge(\alpha\wedge\beta)\wedge\overline{\alpha}=0$ since $\alpha\wedge\alpha=0$. Thus, all these forms are d-closed. They are also both $\partial^\star$-closed and $\bar\partial^\star$-closed as shown in the next statement (cf. also Lemma 6.3.7).

Lemma 6.4.1. *The forms $\alpha\wedge\gamma\wedge\overline{\alpha}$, $\alpha\wedge\gamma\wedge\overline{\beta}$, $\beta\wedge\gamma\wedge\overline{\alpha}$, $\beta\wedge\gamma\wedge\overline{\beta}$ are all $\partial^\star$-closed and $\bar\partial^\star$-closed.*

Note furthermore that the forms $\alpha\wedge\beta\wedge\overline{\alpha}$, $\alpha\wedge\beta\wedge\overline{\beta}$ are $\bar\partial^\star$-closed but not $\partial^\star$-closed.

Proof. The identity $\partial^\star(\alpha\wedge\gamma\wedge\overline{\alpha})=0$ is equivalent to

$$\langle\langle\partial^\star(\alpha\wedge\gamma\wedge\overline{\alpha}),\,u\rangle\rangle=0,\quad\text{i.e. to}\quad\langle\langle\alpha\wedge\gamma\wedge\overline{\alpha},\,\partial u\rangle\rangle=0 \tag{6.39}$$

for every $(1,\,1)$-form u on X. Now, the space $C^\infty_{1,\,1}(X,\,\mathbb{C})$ of smooth $(1,\,1)$-forms on X is generated by $\alpha\wedge\bar\alpha$, $\alpha\wedge\bar\beta$, $\alpha\wedge\bar\gamma$, $\beta\wedge\bar\alpha$, $\beta\wedge\bar\beta$, $\beta\wedge\bar\gamma$, $\gamma\wedge\bar\alpha$, $\gamma\wedge\bar\beta$, $\gamma\wedge\bar\gamma$. Since $d\alpha=d\beta=0$ and $\bar\partial\gamma=0$, the only generators that are not d-closed are those containing γ. For them, since $\partial\gamma=-\alpha\wedge\beta$, we get:

1. if $u=\gamma\wedge\bar\alpha$, then $\partial u=-\alpha\wedge\beta\wedge\bar\alpha$, hence $\langle\langle\alpha\wedge\gamma\wedge\overline{\alpha},\,\partial u\rangle\rangle=-\langle\langle\alpha\wedge\gamma\wedge\overline{\alpha},\,\alpha\wedge\beta\wedge\bar\alpha\rangle\rangle=0$;
2. if $u=\gamma\wedge\bar\beta$, then $\partial u=-\alpha\wedge\beta\wedge\bar\beta$, hence $\langle\langle\alpha\wedge\gamma\wedge\overline{\alpha},\,\partial u\rangle\rangle=-\langle\langle\alpha\wedge\gamma\wedge\overline{\alpha},\,\alpha\wedge\beta\wedge\bar\beta\rangle\rangle=0$;

3. if $u = \gamma\wedge\bar\gamma$, then $\partial u = -\alpha\wedge\beta\wedge\bar\gamma$, hence $\langle\langle\alpha\wedge\gamma\wedge\overline{\alpha},\,\partial u\rangle\rangle = -\langle\langle\alpha\wedge\gamma\wedge\overline{\alpha},\,\alpha\wedge\beta\wedge\bar\gamma\rangle\rangle = 0$.

The three inner products above vanish since the forms α,β,γ are ω_0-orthonormal. We have thus proved identity (6.39). The identities $\partial^\star(\alpha\wedge\gamma\wedge\overline{\beta}) = \partial^\star(\beta\wedge\gamma\wedge\overline{\alpha}) = \partial^\star(\beta\wedge\gamma\wedge\overline{\beta}) = 0$ are proved in the same way: all the resulting inner products involve the pairing of a form containing γ with a form that does not contain γ, hence they vanish.

This argument does not hold for the forms $\alpha\wedge\beta\wedge\overline{\alpha}$ and $\alpha\wedge\beta\wedge\overline{\beta}$ since $\langle\langle\alpha\wedge\beta\wedge\overline{\alpha},\,\partial u\rangle\rangle\neq 0$ when $u=\gamma\wedge\bar\alpha$ and $\langle\langle\alpha\wedge\beta\wedge\overline{\beta},\,\partial u\rangle\rangle\neq 0$ when $u=\gamma\wedge\bar\beta$.

To prove the identities $\bar\partial^\star(\alpha\wedge\gamma\wedge\overline{\alpha}) = \bar\partial^\star(\alpha\wedge\gamma\wedge\overline{\beta}) = \bar\partial^\star(\beta\wedge\gamma\wedge\overline{\alpha}) = \bar\partial^\star(\beta\wedge\gamma\wedge\overline{\beta}) = 0$, we have to prove that for any form $v\in\{\alpha\wedge\gamma\wedge\overline{\alpha},\,\alpha\wedge\gamma\wedge\overline{\beta},\,\beta\wedge\gamma\wedge\overline{\alpha},\,\beta\wedge\gamma\wedge\overline{\beta}\}$ and any $w\in C^\infty_{2,0}(X,\,\mathbb{C})$, we have $\langle\langle v,\,\bar\partial w\rangle\rangle = 0$. This is obvious since $C^\infty_{2,0}(X,\,\mathbb{C})$ is generated by the $\bar\partial$-closed forms $\alpha\wedge\beta$, $\alpha\wedge\gamma$ and $\beta\wedge\gamma$. The same argument applies to yield the $\bar\partial^\star$-closedness of the forms $\alpha\wedge\beta\wedge\overline{\alpha}$ and $\alpha\wedge\beta\wedge\overline{\beta}$. $\square$

We now compute the Hodge star operator $\star$ induced by the pointwise inner product $\langle\cdot\,,\cdot\rangle$ defined by the complex parallelisable structure of X on the Δ''-harmonic representatives of the classes generating $H^{2,1}_{[\gamma]}(X,\,\mathbb{C})$.

Lemma 6.4.2. *On the Iwasawa manifold X, the following identities hold*

$$\begin{aligned}
\star(\alpha\wedge\gamma\wedge\overline{\alpha}) &= -i\,\beta\wedge\gamma\wedge\overline{\beta}, & \star(\beta\wedge\gamma\wedge\overline{\beta}) &= -i\,\alpha\wedge\gamma\wedge\overline{\alpha},\\
\star(\alpha\wedge\gamma\wedge\overline{\beta}) &= i\,\alpha\wedge\gamma\wedge\overline{\beta}, & \star(\beta\wedge\gamma\wedge\overline{\alpha}) &= i\,\beta\wedge\gamma\wedge\overline{\alpha},\\
\star(\alpha\wedge\beta\wedge\gamma) &= -i\alpha\wedge\beta\wedge\gamma, & \star(\alpha\wedge\beta\wedge\overline{\gamma}) &= i\,\alpha\wedge\beta\wedge\overline{\gamma}.
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
\star(\alpha\wedge\gamma\wedge\overline{\alpha}+\beta\wedge\gamma\wedge\overline{\beta}) &= -i\,(\alpha\wedge\gamma\wedge\overline{\alpha}+\beta\wedge\gamma\wedge\overline{\beta}),\\
\star(\alpha\wedge\gamma\wedge\overline{\alpha}-\beta\wedge\gamma\wedge\overline{\beta}) &= i\,(\alpha\wedge\gamma\wedge\overline{\alpha}-\beta\wedge\gamma\wedge\overline{\beta}).
\end{aligned}$$

Proof. From the definition of the Hodge star operator we know that

$$u\wedge\overline{\star(\alpha\wedge\gamma\wedge\overline{\alpha})} = \langle u,\,\alpha\wedge\gamma\wedge\overline{\alpha}\rangle\,dV$$

for every $(2,\,1)$-form u. Both sides of this identity vanish if u is the product of three forms chosen from $\alpha,\beta,\gamma,\overline{\alpha},\overline{\beta},\overline{\gamma}$, except if $u=\alpha\wedge\gamma\wedge\overline{\alpha}$. In this case, we get

$$(\alpha\wedge\gamma\wedge\overline{\alpha})\wedge\overline{\star(\alpha\wedge\gamma\wedge\overline{\alpha})} = \langle\alpha\wedge\gamma\wedge\overline{\alpha},\,\alpha\wedge\gamma\wedge\overline{\alpha}\rangle\,dV = i\,\alpha\wedge\overline{\alpha}\wedge i\,\beta\wedge\overline{\beta}\wedge i\,\gamma\wedge\overline{\gamma} = i\alpha\wedge\gamma\wedge\overline{\alpha}\wedge\beta\wedge\overline{\beta}\wedge\overline{\gamma}$$

hence $\overline{\star(\alpha\wedge\gamma\wedge\overline{\alpha})}$ must be the form complementary to $\alpha\wedge\gamma\wedge\overline{\alpha}$, i.e. $i\beta\wedge\overline{\beta}\wedge\overline{\gamma}$. We get $\star(\alpha\wedge\gamma\wedge\overline{\alpha}) = -i\,\beta\wedge\gamma\wedge\overline{\beta}$. The remaining identities are proved in a similar way. $\square$

We can now infer from these computations the signature of the sesquilinear intersection form H on $F^2_{[\gamma]}H^3(X,\,\mathbb{C})$. It is different from the one in the standard case of compact Kähler Calabi-Yau 3-folds with $h^{p,0}=0$ for $p=1,2$ (where the signature of H on the standard F^2H^3 is $(-,+,\dots,+)$ due to all classes in H^3 being primitive thanks to the assumption $h^{0,1}=0$ which implies $h^{3,2}=0$ by Serre duality). In our non-Kähler case of the Iwasawa manifold, primitivity is meaningless for classes in H^3 while $h^{0,1}=2\neq 0$. The different signature of H is a key feature of our situation compared to the standard one.

Corollary 6.4.3. *If X is the Iwasawa threefold, then $\{\alpha\wedge\gamma\wedge\overline{\alpha}+\beta\wedge\gamma\wedge\overline{\beta}\}_{DR}\in H^3_-(X,\,\mathbb{C})$, while*

$$\{\alpha\wedge\gamma\wedge\overline{\alpha}-\beta\wedge\gamma\wedge\overline{\beta}\}_{DR},\,\{\alpha\wedge\gamma\wedge\overline{\beta}\}_{DR},\,\{\beta\wedge\gamma\wedge\overline{\alpha}\}_{DR}\in H^3_+(X,\,\mathbb{C}).$$

Hence the signature of $H(\cdot\,,\,\cdot)$ on $H^{2,1}_{[\gamma]}(X,\,\mathbb{C})$ is $(-,\,+,\,+,\,+)$, while the signature of $H(\cdot\,,\,\cdot)$ on $F^2_{[\gamma]}H^3(X,\,\mathbb{C})$ is $(-,\,-,\,+,\,+,\,+)$.

Proof. We have argued above (cf. Lemma 6.4.1) that the forms $\alpha\wedge\gamma\wedge\overline{\alpha}+\beta\wedge\gamma\wedge\overline{\beta}$, $\alpha\wedge\gamma\wedge\overline{\alpha}-\beta\wedge\gamma\wedge\overline{\beta}$, $\alpha\wedge\gamma\wedge\overline{\beta}$ and $\beta\wedge\gamma\wedge\overline{\alpha}$ are all Δ-harmonic. Since the splitting (6.36) was defined by the analogous splitting of the space of Δ-harmonic 3-forms, the first statement follows from Lemma 6.4.2.

The second statement follows from (6.37), from the properties of $H(\cdot\,,\,\cdot)$ spelt out above (6.37) and from the fact that $\{[\alpha\wedge\gamma\wedge\overline{\alpha}+\beta\wedge\gamma\wedge\overline{\beta}],\,[\alpha\wedge\gamma\wedge\overline{\alpha}-\beta\wedge\gamma\wedge\overline{\beta}],\,[\alpha\wedge\gamma\wedge\overline{\beta}],\,[\beta\wedge\gamma\wedge\overline{\alpha}]\}$ is a basis of $H^{2,1}_{[\gamma]}(X,\,\mathbb{C})$. □

6.4.2 Construction of coordinates on $B_{[\gamma]}$

Abstract construction

Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold $X=X_0$. We know from [Nak75, table on p. 96] that $h^{3,\,0}_{\bar\partial}(X_t)=1$ for all $t\in B$. This implies that $B\ni t\mapsto H^{3,\,0}_{\bar\partial}(X_t,\,\mathbb{C})$ is a C^∞ line bundle by [KS60]. It is even holomorphic and denoted, as usual, by $\mathcal{H}^{3,\,0}$. Moreover, since K_{X_0} is trivial, the constancy of $h^{3,\,0}(X_t)$ also implies that K_{X_t} is trivial for all $t\in B$. Let us fix, after possibly shrinking Δ about 0, a holomorphic section $u=(u_t)_{t\in B}$ of the Hodge bundle $\mathcal{H}^{3,\,0}$ (i.e. a holomorphic family of holomorphic $(3,\,0)$-forms u_t on X_t) such that the form u_t is non-vanishing on X_t for every $t\in B$.

Put, for simplicity, $H^3(X,\,\mathbb{C}):=H^3_{DR}(X,\,\mathbb{C})$, where by X we mean the C^∞ manifold underlying the fibres X_t. We know from Lemma 6.3.1 that every space $H^{3,\,0}(X_t,\,\mathbb{C})$ injects canonically into $H^3(X,\,\mathbb{C})$, so u can be viewed as a holomorphic function $B\ni t\longmapsto u_t\in H^3(X,\,\mathbb{C})$.

Meanwhile, $(H^3(X,\,\mathbb{C}),\,Q(\cdot\,,\,\cdot))$ is a symplectic vector space (cf. (6.34)). We shall adapt to our context the presentation in [Voi96, lemme 3.1] to prove that a well-chosen symplectic basis $\{\eta_0,\eta_1,\dots,\eta_4,\,\nu_0,\nu_1,\dots,\nu_4\}$ (i.e. such that $Q(\eta_j,\,\eta_k)=Q(\nu_j,\,\nu_k)=0$ and $Q(\eta_j,\,\nu_k)=\delta_{jk}$ for all j,k) of $H^3(X,\,\mathbb{C})$ produces holomorphic coordinates $z_1,\dots,z_4$ near 0 on $B_{[\gamma]}$. We shall choose all the classes η_j and ν_k to be *real*, i.e. $\eta_j=\overline{\eta}_j$ and $\nu_k=\overline{\nu}_k$ for all j,k. Consider the following

Setup. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold $X=X_0$ on which we have fixed a non-vanishing holomorphic section $u=(u_t)_{t\in B}$ of $\mathcal{H}^{3,\,0}$. Let $\eta_0=\eta_0^{3,\,0}+\eta_0^{2,\,1}+\overline{\eta_0^{2,\,1}}+\overline{\eta_0^{3,\,0}}\in H^3(X,\,\mathbb{C})$ be a real class with $\eta_0^{3,\,0}\in H^{3,\,0}(X_0,\,\mathbb{C})$, $\eta_0^{2,\,1}\in H^{2,\,1}_{[\gamma]}(X_0,\,\mathbb{C})$ such that*

$$(i)\,Q(u_0,\,\eta_0)\neq 0\quad\text{and}\quad(ii)\,H(\eta_0^{2,\,1},\,\eta_0^{2,\,1})<0. \tag{6.40}$$

Complete η_0 to a symplectic basis $\{\eta_0,\eta_1,\dots,\eta_4,\,\nu_0,\nu_1,\dots,\nu_4\}$ of $(H^3(X,\,\mathbb{R}),\,Q(\cdot\,,\,\cdot))$. By continuity, we have $Q(u_t,\,\eta_0)\neq 0$ for all t in a neighbourhood of $0\in B$, so after replacing u_t by $u_t':=u_t/Q(u_t,\,\eta_0)$ we may assume that

$$Q(u_t,\,\eta_0)=1\quad\textit{for all } t\in B\quad\textit{sufficiently close to } 0.$$

We can now state the main result of this subsection.

Proposition 6.4.4. *In the setup described above, the functions*

$$z_i(t) := Q(u_t,\, \eta_i) \quad \text{for } t \in B_{[\gamma]} \quad \text{and } i \in \{1, \dots, 4\} \tag{6.41}$$

define **holomorphic coordinates** *on* $B_{[\gamma]}$ *in a neighbourhood of* 0.

Proof. Classes $\eta_0 \in H^3(X,\, \mathbb{C})$ satisfying (6.40) do exist. Indeed, for every 3-class η_0, $Q(u_0,\, \eta_0) = Q(u_0,\, \eta_0^{0,\,3})$ for bidegree reasons since u_0 is of type $(3,\, 0)$, so it suffices to choose a class $\eta_0^{0,\,3} \in H^{0,\,3}(X_0,\, \mathbb{C})$ such that $Q(u_0,\, \eta_0^{0,\,3}) \neq 0$ for (i) to be satisfied. This is possible since $u_0 \neq 0$. Classes $\eta_0^{2,\,1} \in H^{2,\,1}_{[\gamma]}(X_0,\, \mathbb{C})$ satisfying (ii) exist thanks to the signature of H on $H^{2,\,1}_{[\gamma]}(X_0,\, \mathbb{C})$ being $(-,\, +,\, +,\, +)$ (cf. Corollary 6.4.3). We can then put $\eta_0 := \overline{\eta_0^{0,\,3}} + \eta_0^{2,\,1} + \overline{\eta_0^{2,\,1}} + \eta_0^{0,\,3}$ to obtain a *real* class η_0 satisfying (6.40). Every class $\eta_0 \in H^3(X,\, \mathbb{C})$ automatically satisfies $Q(\eta_0,\, \eta_0) = 0$ since $Q(\eta_0,\, \eta_0) = Q(\eta_0^{3,\,0},\, \eta_0^{0,\,3}) + Q(\eta_0^{2,\,1},\, \eta_0^{1,\,2}) + Q(\eta_0^{1,\,2},\, \eta_0^{2,\,1}) + Q(\eta_0^{0,\,3},\, \eta_0^{3,\,0})$ while $Q(\eta_0^{3,\,0},\, \eta_0^{0,\,3}) = -Q(\eta_0^{0,\,3},\, \eta_0^{3,\,0})$ and $Q(\eta_0^{2,\,1},\, \eta_0^{1,\,2}) = -Q(\eta_0^{1,\,2},\, \eta_0^{2,\,1})$ since Q is alternating. So η_0 can be completed to a symplectic basis.

We have to prove that the holomorphic map

$$\Phi\, :\, B_{[\gamma]} \longrightarrow \mathbb{C}^4, \quad \Phi(t) := (z_1(t), \dots, z_4(t)),$$

is a local diffeomorphism at 0. Since Φ is the composition of the maps

$$B_{[\gamma]} \xrightarrow{u} H^3(X,\, \mathbb{C}) \xrightarrow{Q_{\eta_1,\dots,\eta_4}} \mathbb{C}^4, \quad \text{where } Q_{\eta_1,\dots,\eta_4}(\cdot) := \Big(Q(\cdot,\, \eta_1), \dots, Q(\cdot,\, \eta_4)\Big),$$

its differential map $d\Phi_0$ at 0 is the composition of the maps

$$T_0^{1,\,0}B_{[\gamma]} \underset{\simeq}{\xrightarrow{\rho}} H^{0,\,1}_{[\gamma]}(X_0,\, T^{1,\,0}X_0) \underset{\simeq}{\xrightarrow{\cdot \lrcorner u_0}} H^{2,\,1}_{[\gamma]}(X_0,\, \mathbb{C}) \hookrightarrow H^3(X,\, \mathbb{C}) \xrightarrow{Q_{\eta_1,\dots,\eta_4}} \mathbb{C}^4,$$

where ρ is the restriction to $B_{[\gamma]}$ of the Kodaira-Spencer map classifying the infinitesimal deformations of X_0 and the composition of the first three maps is the differential map $du_0\, :\, T_0^{1,\,0}B_{[\gamma]} \longrightarrow H^3(X,\, \mathbb{C})$ by [Gri68] and Proposition 6.3.3. Since $T_0^{1,\,0}B_{[\gamma]}$ and $\mathbb{C}^4$ have equal dimensions, it suffices to prove that $d\Phi_0$ is injective.

Reasoning by contradiction, suppose that $d\Phi_0$ is not injective. Then, there exists $\mu \in H^{2,\,1}_{[\gamma]}(X_0,\, \mathbb{C})$ such that $Q(\mu,\, \eta_1) = \cdots = Q(\mu,\, \eta_4) = 0$. Since $Q(u_t,\, \eta_0) = 1$ for all $t \in B_{[\gamma]}$ close to 0, $Q(du_0(\xi),\, \eta_0) = 0$ for every $\xi \in T_0^{1,\,0}B_{[\gamma]}$. Hence $Q(\mu,\, \eta_0) = 0$ because $\mu \in H^{2,\,1}_{[\gamma]}(X_0,\, \mathbb{C}) = (du_0)(T_0^{1,\,0}B_{[\gamma]})$. Therefore, $Q(\mu,\, \eta_0) = \cdots = Q(\mu,\, \eta_4) = 0$, so $\mu \in \langle \eta_0, \eta_1, \dots, \eta_4 \rangle$ since the basis $\{\eta_0, \eta_1, \dots, \eta_4,\, \nu_0, \nu_1, \dots, \nu_4\}$ is symplectic. This implies that $H(\mu,\, \mu) = 0$ since the subspace $\langle \eta_0, \eta_1, \dots, \eta_4 \rangle \subset H^3(X,\, \mathbb{C})$ is *real* and *totally Q-isotropic*.

On the other hand, $H(\mu,\, \eta_0) = 0$ because $Q(\mu,\, \eta_0) = 0$ and $\eta_0 = \overline{\eta_0}$. Thus, $0 = H(\mu,\, \eta_0) = H(\mu,\, \eta_0^{3,\,0}) + H(\mu,\, \eta_0^{2,\,1}) + H(\mu,\, \eta_0^{1,\,2}) + H(\mu,\, \eta_0^{0,\,3}) = H(\mu,\, \eta_0^{2,\,1})$, where the last identity holds trivially for bidegree reasons since μ is of type $(2,\, 1)$.

Summing up, we have the classes $\mu, \eta_0^{2,\,1} \in H^{2,\,1}_{[\gamma]}(X_0,\, \mathbb{C})$ with the properties $H(\mu,\, \mu) = 0$ and $H(\mu,\, \eta_0^{2,\,1}) = 0$. On the other hand, we know from Corollary 6.4.3 that the restriction of H to $H^{2,\,1}_{[\gamma]}(X_0,\, \mathbb{C})$ is non-degenerate of signature $(-,\, +,\, +,\, +)$, i.e. $H(\cdot\, ,\, \cdot)\, :\, H^{2,\,1}_{[\gamma]} \times H^{2,\,1}_{[\gamma]} \longrightarrow \mathbb{C}$ is a Lorentzian sesquilinear form.

Let $\rho_\varepsilon \in H^{2,\,1}_{[\gamma]}(X_0,\, \mathbb{C})$ such that $H(\rho_\varepsilon,\, \rho_\varepsilon) < 0$ for every $\varepsilon > 0$ and $\rho_\varepsilon \to \mu$ as $\varepsilon \to 0$ (i.e. $(\rho_\varepsilon)_{\varepsilon>0}$ is an approximation of μ, an element in the *lightlike cone* of H, by elements ρ_ε in the *timelike cone* of H). Let $\eta_{0,\,\varepsilon}^{2,\,1} \to \eta_0^{2,\,1}$ be an approximation of $\eta_0^{2,\,1}$ such that $H(\rho_\varepsilon,\, \eta_{0,\,\varepsilon}^{2,\,1}) = 0$ for every ε. Since ρ_ε

is timelike and the signature of H on $H^{2,1}_{[\gamma]}$ is $(-,+,+,+)$, the H-orthogonal complement $\langle\rho_\varepsilon\rangle^\perp$ in $H^{2,1}_{[\gamma]}$ of the line generated by ρ_ε is contained in the subspace $\{\zeta\in H^{2,1}_{[\gamma]}\,/\,H(\zeta,\zeta)\geq 0\}$. (This can be trivially checked by completing $\rho_\varepsilon/\sqrt{|H(\rho_\varepsilon,\rho_\varepsilon)|}$ to an orthonormal basis of $(H^{2,1}_{[\gamma]},H)$.) Thus, $H(\eta^{2,1}_{0,\varepsilon},\eta^{2,1}_{0,\varepsilon})\geq 0$ for every $\varepsilon>0$, hence for its limit as $\varepsilon\to 0$ we get $H(\eta^{2,1}_0,\eta^{2,1}_0)\geq 0$. This contradicts the assumption (ii) of (6.40). Therefore, $d\Phi_0$ must be injective. $\square$

Explicit computations

The construction of §.6.4.2 can be made explicit by choosing

$$\begin{aligned}
u_t &= \alpha_t\wedge\beta_t\wedge\gamma_t,\\
\eta_0^{3,0} &= \alpha\wedge\beta\wedge\gamma, \qquad\qquad \eta_0^{2,1}=i\,(\alpha\wedge\overline{\alpha}+\beta\wedge\overline{\beta})\wedge\gamma,\\
\eta_0 &= \alpha\wedge\beta\wedge\gamma+i\,(\alpha\wedge\bar\alpha+\beta\wedge\bar\beta)\wedge(\gamma+\bar\gamma)+\bar\alpha\wedge\bar\beta\wedge\bar\gamma,\\
\eta_1 &= \alpha\wedge\beta\wedge\gamma+\bar\alpha\wedge\bar\beta\wedge\bar\gamma, \qquad \eta_2 = i(\alpha\wedge\bar\alpha-\beta\wedge\bar\beta)\wedge(\gamma+\bar\gamma),\\
\eta_3 &= \alpha\wedge\bar\beta\wedge\gamma+\bar\alpha\wedge\beta\wedge\bar\gamma, \qquad \eta_4 = \bar\alpha\wedge\beta\wedge\gamma+\alpha\wedge\bar\beta\wedge\bar\gamma.
\end{aligned}$$

These forms satisfy condition (6.40). Indeed, for example, we have

$$H(\eta_0^{2,1},\eta_0^{2,1})=-\int\limits_X(i\,\alpha\wedge\bar\alpha+i\,\beta\wedge\bar\beta)^2\wedge i\,\gamma\wedge\bar\gamma<0.$$

The forms $\alpha_t,\beta_t,\gamma_t$ can be computed in terms of α,β,γ using relations (6.16) and (6.17). After recalling the notation $D(t):=t_{11}\,t_{22}-t_{12}\,t_{21}$, we get the following identities for all $t\in B$:

$$\begin{aligned}
\alpha_t &= d\zeta_1(t)=dz_1+(t_{11}\,\bar\alpha+t_{12}\,\bar\beta)=\alpha+t_{11}\,\bar\alpha+t_{12}\,\bar\beta\\
\beta_t &= d\zeta_2(t)=dz_2+(t_{21}\,\bar\alpha+t_{22}\,\bar\beta)=\beta+t_{21}\,\bar\alpha+t_{22}\,\bar\beta\\
\gamma_t &= d\zeta_3(t)-z_1\,d\zeta_2(t)-(t_{21}\,\bar z_1+t_{22}\,\bar z_2)\,d\zeta_1(t)\\
&= [dz_3+t_{21}\,dz_1\,\bar z_1+(t_{31}+t_{21}\,z_1)\,d\bar z_1+t_{22}\,dz_1\,\bar z_2+(t_{32}+t_{22}\,z_1)\,d\bar z_2+t_{11}\,t_{21}\,\bar z_1\,d\bar z_1+\\
&\quad +t_{11}\,t_{22}\,(\bar z_1\,d\bar z_2+\bar z_2\,d\bar z_1)+t_{12}\,t_{22}\,\bar z_2\,d\bar z_2-D(t)d\bar z_3]\\
&\quad -z_1\,(\beta+t_{21}\,\bar\alpha+t_{22}\,\bar\beta)-(t_{21}\,\bar z_1+t_{22}\,\bar z_2)\,(\alpha+t_{11}\,\bar\alpha+t_{12}\,\bar\beta) \qquad\qquad (6.42)\\
&\overset{(i)}{=} (\gamma+z_1\,\beta)+t_{21}\,\bar z_1\,\alpha+(t_{31}+t_{21}\,z_1)\,\bar\alpha+t_{22}\,\bar z_2\,\alpha+(t_{32}+t_{22}\,z_1)\,\bar\beta+t_{11}\,t_{21}\,\bar z_1\bar\alpha\\
&\quad +t_{11}\,t_{22}\,(\bar z_1\,\bar\beta+\bar z_2\,\bar\alpha)+t_{12}\,t_{22}\,\bar z_2\,\bar\beta-D(t)d\bar z_3\\
&\quad -z_1\,(\beta+t_{21}\,\bar\alpha+t_{22}\,\bar\beta)-(t_{21}\,\bar z_1+t_{22}\,\bar z_2)\,(\alpha+t_{11}\,\bar\alpha+t_{12}\,\bar\beta)\\
&= \gamma+t_{31}\,\bar\alpha+t_{32}\,\bar\beta-D(t)d\bar z_3+D(t)\,\bar z_1\,\bar\beta\\
&= \gamma+t_{31}\,\bar\alpha+t_{32}\,\bar\beta-D(t)\bar\gamma, \qquad\qquad t\in B,
\end{aligned}$$

where (i) followed from $dz_3=\gamma+z_1\,\beta$.

Consequently, we get

$$\begin{aligned}
u_t &= \big(\alpha+t_{11}\,\bar\alpha+t_{12}\,\bar\beta\big)\wedge\big(\beta+t_{21}\,\bar\alpha+t_{22}\,\bar\beta\big)\wedge\big(\gamma+t_{31}\,\bar\alpha+t_{32}\,\bar\beta-D(t)\bar\gamma\big)\\
&= \alpha\wedge\beta\wedge(\gamma-D(t)\bar\gamma)+D(t)\bar\alpha\wedge\bar\beta\wedge(\gamma-D(t)\bar\gamma)\\
&\quad +(t_{21}\alpha\wedge\bar\alpha-t_{12}\beta\wedge\bar\beta)\wedge(\gamma-D(t)\bar\gamma)+(t_{22}\alpha\wedge\bar\beta+t_{11}\bar\alpha\wedge\beta)\wedge(\gamma-D(t)\bar\gamma)\\
&\quad +\alpha\wedge\beta\wedge(t_{31}\bar\alpha+t_{32}\bar\beta)+(t_{21}t_{32}-t_{31}t_{22})\alpha\wedge\bar\alpha\wedge\bar\beta+(t_{11}t_{32}-t_{12}t_{31})\bar\alpha\wedge\beta\wedge\bar\beta.
\end{aligned}$$

Note that the terms are displayed according to their degree and type on the base B of $\pi:X\to B$. The part coming from the base (i.e. the terms on the last line, those containing neither γ nor $\bar\gamma$) vanishes on $B_{[\gamma]}$ since $t_{31}=t_{32}=0$ there.

We can now compute the resulting coordinates on $B_{[\gamma]}$. We get for $t \in B_{[\gamma]}$:

$$\begin{aligned} Q(u_t, \eta_0) &= -\int_X u_t \wedge \eta_0 = -\int_X u_t \wedge \Big(\alpha\wedge\beta\wedge\gamma + \bar\alpha\wedge\bar\beta\wedge\bar\gamma + i(\alpha\wedge\bar\alpha + \beta\wedge\bar\beta)\wedge(\gamma+\bar\gamma)\Big) \\ &= \Big(i(1+D(t)^2) + (t_{21}-t_{12})(1+D(t))\Big)\int_X i\alpha\wedge\bar\alpha\wedge i\beta\wedge\bar\beta\wedge i\gamma\wedge\bar\gamma = 1, \end{aligned}$$

where the last identity is the normalisation adopted in Proposition 6.4.4. We also get for $t \in B_{[\gamma]}$:

$$\begin{aligned} Q(u_t, \eta_1) &= \frac{i(1+D(t)^2)}{i(1+D(t)^2) + (t_{21}-t_{12})(1+D(t))}, \; Q(u_t,\eta_2) = -\frac{(t_{12}+t_{21})(1+D(t))}{i(1+D(t)^2) + (t_{21}-t_{12})(1+D(t))}, \\ Q(u_t, \eta_3) &= -i\,\frac{t_{11}\,D(t) + t_{22}}{i(1+D(t)^2) + (t_{21}-t_{12})(1+D(t))}, \; Q(u_t,\eta_4) = -i\,\frac{t_{22}\,D(t) + t_{11}}{i(1+D(t)^2) + (t_{21}-t_{12})(1+D(t))}. \end{aligned}$$

6.4.3 The B-Yukawa coupling

Definition 6.4.5. *Suppose we have fixed a non-vanishing holomorphic $(3,\,0)$-form u on the Iwasawa manifold X. It identifies with the class $[u] \in H^{3,\,0}_{\bar\partial}(X,\,\mathbb{C}) \simeq H^0(X,\,K_X) \simeq \mathbb{C}$. The* **Yukawa coupling** *associated with u is standardly defined as*

$$\begin{aligned} Y_2^{(u)} \,:\, H^{0,\,1}(X_0,\,T^{1,\,0}X_0)\times H^{0,\,1}(X_0,\,T^{1,\,0}X_0)\times H^{0,\,1}(X_0,\,T^{1,\,0}X_0) &\longrightarrow \mathbb{C} \\ ([\theta_1],\,[\theta_2],\,[\theta_3]) &\mapsto \Big\langle u^2,\,[\theta_1]\cdot[\theta_2]\cdot[\theta_3]\Big\rangle \end{aligned}$$

where u^2 is viewed as a section $u^2 \in H^0(X,\,K_X^{\otimes 2}) \simeq H^{3,\,0}(X,\,K_X)$, the cup product $[\theta_1]\cdot[\theta_2]\cdot[\theta_3] \in H^{0,\,3}(X,\,\Lambda^3T^{1,\,0}X) = H^{0,\,3}(X,\,K_X^{-1})$ and $\langle\cdot,\,\cdot\rangle \,:\, H^{3,\,0}(X,\,K_X)\times H^{0,\,3}(X,\,K_X^{-1}) \longrightarrow \mathbb{C}$ is the Serre duality.

We can now use the symplectic basis and the coordinates constructed in Proposition 6.4.4 to show, by the same method as in the standard Kähler case ([BG83]), that the Yukawa couplings Y_2 on $T^{1,\,0}_0 B_{[\gamma]} \simeq H^{0,\,1}_{[\gamma]}(X_0,\,T^{1,\,0}X_0)$ are defined by a potential.

Proposition 6.4.6. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold $X = X_0$ on which we have fixed a non-vanishing holomorphic section $u = (u_t)_{t\in B}$ of $\mathcal{H}^{3,\,0}$ normalised by the choice of a symplectic basis as in Proposition 6.4.4. Let $z_1, \dots, z_4$ be the induced holomorphic coordinates near 0 on $B_{[\gamma]}$ constructed in Proposition 6.4.4. Then, there exists a C^∞ function $F = F(z_1, \dots, z_4) : B_{[\gamma]} \longrightarrow \mathbb{C}$ such that*

$$Y_2^{(u)}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = \frac{\partial^3 F}{\partial z_i \partial z_j \partial z_k} \tag{6.43}$$

for all $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \in T^{1,\,0}_0 B_{[\gamma]} \simeq H^{0,\,1}_{[\gamma]}(X_0,\,T^{1,\,0}X_0)$.

Proof. The arguments are standard (see e.g. [Voi96, §.3.1.2]), but we spell them out for the reader's convenience and to show that they adapt to our non-standard situation.

Step 1. For all $i \in \{1, \dots, 4\}$, put $\Psi_i : B_{[\gamma]} \longrightarrow \mathbb{C}$, $\Psi_i(z_1(t), \dots, z_4(t)) := Q(u_t,\,\nu_i)$. Prove that

$$\frac{\partial \Psi_i}{\partial z_j} = \frac{\partial \Psi_j}{\partial z_i} \quad \text{for all } i, j \in \{1, \dots, 4\}. \tag{6.44}$$

This is proved by writing $u_t = a_0\nu_0 + \sum_{j=1}^4 a_j\nu_j + \sum_{j=1}^4 b_j\eta_j + b_0\eta_0$ and computing the coefficients a_j, b_j by using the relation $Q(u_t,\, \eta_0) = 1$ and the symplectic property of the basis $\{\eta_0, \eta_1, \dots, \eta_4,\, \nu_0, \nu_1, \dots, \nu_4\}$. We get $u_t = \nu_0 + \sum_{j=1}^4 z_j\nu_j - \sum_{j=1}^4 \Psi_j\eta_j - Q(u_t,\, \nu_0)\,\eta_0$. Taking the derivative $\partial/\partial z_i$, we get

$$\frac{\partial u}{\partial z_i} = \nu_i - \sum_{j=1}^4 \frac{\partial \Psi_j}{\partial z_i}\,\eta_j - \frac{\partial Q(u,\, \nu_0)}{\partial z_i}\,\eta_0.$$

From this and the symplectic property of the basis η_j, ν_k, we infer

$$Q\bigg(\frac{\partial u}{\partial z_i},\, \frac{\partial u}{\partial z_j}\bigg) = -\frac{\partial \Psi_i}{\partial z_j} + \frac{\partial \Psi_j}{\partial z_i}.$$

On the other hand, $\frac{\partial u}{\partial z_i} = \rho(\frac{\partial}{\partial z_i})\lrcorner u \in F^2_{[\gamma]}H^3(X,\, \mathbb{C})$ for all $\frac{\partial}{\partial z_i} \in T_0^{1,0}B_{[\gamma]} \overset{\rho}{\simeq} H^{0,1}_{[\gamma]}(X_0,\, T^{1,0}X_0)$ by Griffiths's transversality [Gri68] (see (6.27), our version of it), so for bidegree reasons we get: $0 = Q\bigg(\frac{\partial u}{\partial z_i},\, \frac{\partial u}{\partial z_j}\bigg)$. This proves (6.44).

It follows from (6.44) that there exists a C^∞ function $F = F(z_1, \dots, z_4) : B_{[\gamma]} \longrightarrow \mathbb{C}$ such that

$$\frac{\partial F}{\partial z_i} = \Psi_i \quad \text{for all } i \in \{1, \dots, 4\}.$$

Step 2. Prove (6.43) for this choice of F.

By the orthogonality relations (6.28), we have

$$\bigg\langle u,\, \frac{\partial^2 u}{\partial z_i \partial z_j}\bigg\rangle = 0$$

since $u_t \in H^{3,0}(X_t,\, \mathbb{C})$ for all t. Applying $\partial/\partial z_k$, we get

$$\bigg\langle \frac{\partial u}{\partial z_k},\, \frac{\partial^2 u}{\partial z_i \partial z_j}\bigg\rangle + \bigg\langle u,\, \frac{\partial^3 u}{\partial z_i \partial z_j \partial z_k}\bigg\rangle = 0, \quad \text{hence } Y_2^{(u)}\bigg(\frac{\partial}{\partial z_i},\, \frac{\partial}{\partial z_j},\, \frac{\partial}{\partial z_k}\bigg) = -\bigg\langle \frac{\partial u}{\partial z_k},\, \frac{\partial^2 u}{\partial z_i \partial z_j}\bigg\rangle.$$

On the other hand, from the identities $u_t = \nu_0 + \sum_{l=1}^4 z_l\,\nu_l - \sum_{l=1}^4 \Psi_l\,\eta_l - Q(u_t,\, \nu_0)\,\eta_0$ seen at Step 1, we compute

$$\begin{aligned}\bigg\langle \frac{\partial u}{\partial z_k},\, \frac{\partial^2 u}{\partial z_i \partial z_j}\bigg\rangle &= -\bigg\langle \nu_k - \sum_{l=1}^4 \frac{\partial \Psi_l}{\partial z_k}\,\eta_l - \frac{\partial Q(u,\, \nu_0)}{\partial z_k}\,\eta_0,\, \sum_{l=1}^4 \frac{\partial^2 \Psi_l}{\partial z_i \partial z_j}\,\eta_l + \frac{\partial^2 Q(u,\, \nu_0)}{\partial z_i \partial z_j}\,\eta_0\bigg\rangle \\ &= -\frac{\partial^2 \Psi_k}{\partial z_i \partial z_j} = -\frac{\partial^3 F}{\partial z_i \partial z_j \partial z_k}.\end{aligned}$$

The last two main identities combined prove (6.43). $\square$

6.5 The metric side of the mirror

As usual, we let $(X_t)_{t\in B_{[\gamma]}}$ stand for the Kuranishi family of the Iwasawa manifold $X = X_0$.

6.5.1 Constructing Gauduchon metrics

A smooth family $(\omega_t)_{t\in B_{[\gamma]}}$ of Gauduchon metrics on $(X_t)_{t\in B_{[\gamma]}}$

Recall that (6.19) provides us with a C^∞ family of canonical Hermitian metrics $(\omega_t)_{t\in B}$ on the fibres $(X_t)_{t\in B}$ after possibly shrinking Δ about 0. Simple calculations enable us to prove the following.

Lemma 6.5.1. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold $X = X_0$. Then, for every $t \in B_{[\gamma]}$, the metric $\omega_t = i\alpha_t \wedge \bar\alpha_t + i\beta_t \wedge \bar\beta_t + i\gamma_t \wedge \bar\gamma_t$ is a Gauduchon metric on X_t, hence $[\omega_t^2]_A$ defines an element in the Gauduchon cone $\mathcal{G}_{X_t}$ of X_t.*

Proof. Since $\dim_{\mathbb{C}} X_t = 3$, we have to show that $\partial_t\bar\partial_t\omega_t^2 = 0$ for $t \in B_{[\gamma]}$. For all $t \in B$,

$$\omega_t^2 = -2\,\alpha_t \wedge \bar\alpha_t \wedge \beta_t \wedge \bar\beta_t - 2\,\alpha_t \wedge \bar\alpha_t \wedge \gamma_t \wedge \bar\gamma_t - 2\,\beta_t \wedge \bar\beta_t \wedge \gamma_t \wedge \bar\gamma_t.$$

It now follows from lemma 6.5.10 that $\partial_t\bar\partial_t\omega_t^2 = 0$ for all $t \in B_{[\gamma]}$. □

A smooth family $(\omega_t^{1,1})_{t\in B_{[\gamma]}}$ of Gauduchon metrics on X_0

We will implicitly construct a smooth family of Aeppli-Gauduchon classes in $\mathcal{G}_{X_0}$ naturally induced by the structure of the family $(X_t)_{t\in B}$. Each Hermitian metric $\omega_t = i\alpha_t \wedge \bar\alpha_t + i\beta_t \wedge \bar\beta_t + i\gamma_t \wedge \bar\gamma_t$ (proved in Lemma 6.5.1 to be even a Gauduchon metric on X_t for $t \in B_{[\gamma]}$) can be viewed as a real 2-form on the C^∞ manifold X underlying the fibres X_t. As such, ω_t has a component of bidegree $(1,\,1)$ w.r.t. the complex structure J_0 of X_0. We denote it by $\omega_t^{1,\,1} \in C^\infty_{1,1}(X_0,\,\mathbb{R})$.

Proposition 6.5.2. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold X_0. Then, the J_0-$(1,\,1)$-form $\omega_t^{1,\,1}$ is a Gauduchon metric on X_0 for every $t \in B$ sufficiently close to 0. Moreover, $\omega_0^{1,\,1} = \omega_0$ and $\omega_t^{1,\,1}$ varies in a C^∞ way with t.*

Proof. Recall that $\omega_t = i\,\alpha_t \wedge \bar\alpha_t + i\,\beta_t \wedge \bar\beta_t + i\,\gamma_t \wedge \bar\gamma_t$. Hence, using the identities (6.42), we get

$$\begin{aligned}\omega_t &= i\,(\alpha + t_{11}\,\bar\alpha + t_{12}\,\bar\beta) \wedge (\bar\alpha + \bar t_{11}\,\alpha + \bar t_{12}\,\beta) + i\,(\beta + t_{21}\,\bar\alpha + t_{22}\,\bar\beta) \wedge (\bar\beta + \bar t_{21}\,\alpha + \bar t_{22}\,\beta)\\ &+ i\,[\gamma + t_{31}\,\bar\alpha + t_{32}\,\bar\beta - D(t)\,\bar\gamma] \wedge [\bar\gamma + \bar t_{31}\,\alpha + \bar t_{32}\,\beta - \overline{D(t)}\,\gamma].\end{aligned}$$

Hence, the J_0-type $(1,\,1)$-component of ω_t is

$$\omega_t^{1,\,1} = (1 + c_1(t))\,i\alpha \wedge \bar\alpha + (1 + c_2(t))\,i\beta \wedge \bar\beta + (1 + c_3(t))\,i\gamma \wedge \bar\gamma + d(t)\,i\alpha \wedge \bar\beta + \overline{d(t)}\,i\beta \wedge \bar\alpha, \quad (6.45)$$

where

$$\begin{aligned} c_1(t) &= -(|t_{11}|^2 + |t_{21}|^2 + |t_{31}|^2)\\ c_2(t) &= -(|t_{12}|^2 + |t_{22}|^2 + |t_{32}|^2),\\ c_3(t) &= -|D(t)|^2 = -|t_{11}\,t_{22} - t_{12}\,t_{21}|^2,\\ d(t) &= -(t_{12}\,\bar t_{11} + t_{22}\,\bar t_{21} + t_{32}\,\bar t_{31}). \end{aligned} \quad (6.46)$$

We see that $\omega_t^{1,\,1}$ varies in a C^∞ way with t and that $\omega_0^{1,\,1} = \omega_0$. In particular, since $\omega > 0$, by continuity we get $\omega_t^{1,\,1} > 0$ for all t sufficiently close to 0, so $(\omega_t^{1,\,1})_{t\in B}$ is a C^∞ family of Hermitian metrics on X_0 after possibly shrinking Δ about 0.

It remains to show that $\partial\bar\partial(\omega_t^{1,1})^2 = 0$, where $\partial = \partial_0$ and $\bar\partial = \bar\partial_0$, i.e. that each $\omega_t^{1,1}$ is a Gauduchon metric on X_0. Taking squares in (6.45), we get

$$\begin{aligned}(\omega_t^{1,1})^2 &= \omega_0^2 + 2c_1(t)\,\omega_0\wedge i\,\alpha\wedge\bar\alpha + 2c_2(t)\,\omega_0\wedge i\,\beta\wedge\bar\beta + 2c_3(t)\,\omega_0\wedge i\,\gamma\wedge\bar\gamma\\ &+ 2d(t)\,\omega_0\wedge i\,\alpha\wedge\bar\beta + 2\overline{d(t)}\,\omega_0\wedge i\,\beta\wedge\bar\alpha\\ &+ 2c_1(t)\,c_2(t)\,i\,\alpha\wedge\bar\alpha\wedge i\,\beta\wedge\bar\beta + 2c_2(t)\,c_3(t)\,i\,\beta\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma + 2c_1(t)\,c_3(t)\,i\,\alpha\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma\\ &+ 2c_1(t)\,d(t)\,i\,\alpha\wedge\bar\alpha\wedge i\,\alpha\wedge\bar\beta + 2c_1(t)\,\overline{d(t)}\,i\,\alpha\wedge\bar\alpha\wedge i\,\beta\wedge\bar\alpha\\ &+ 2c_2(t)\,d(t)\,i\,\beta\wedge\bar\beta\wedge i\,\alpha\wedge\bar\beta + 2c_2(t)\,\overline{d(t)}\,i\,\beta\wedge\bar\beta\wedge i\,\beta\wedge\bar\alpha - 2|d(t)|^2\,i\,\alpha\wedge\bar\alpha\wedge i\,\beta\wedge\bar\beta\\ &+ 2c_3(t)\,d(t)\,i\,\alpha\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma + 2c_3(t)\,\overline{d(t)}\,i\,\beta\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma.\end{aligned}$$

After removing the vanishing terms (that are products containing two equal factors chosen from $\alpha, \beta, \bar\alpha, \bar\beta$) and regrouping the remaining ones, we get

$$\begin{aligned}(\omega_t^{1,1})^2 &= \omega_0^2 + 2\,[c_1(t)+c_2(t)+c_1(t)\,c_2(t)-|d(t)|^2]\,i\,\alpha\wedge\bar\alpha\wedge i\,\beta\wedge\bar\beta\\ &+ 2\,[c_1(t)+c_3(t)+c_1(t)\,c_3(t)]\,i\,\alpha\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma + 2\,[c_2(t)+c_3(t)+c_2(t)\,c_3(t)]\,i\,\beta\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma\\ &+ 2\,d(t)\,[1+c_3(t)]\,i\,\alpha\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma + 2\,\overline{d(t)}\,[1+c_3(t)]\,i\,\beta\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma.\end{aligned}\tag{6.47}$$

We can now show, using the identities $d\alpha = d\beta = 0$, $\bar\partial\gamma = 0$ and $\partial\gamma = -\alpha\wedge\beta$ (cf. (1.55)), that every term on the r.h.s. of (6.47) is at least $\partial\bar\partial$-closed. We have already seen that $\partial\bar\partial\omega_0^2 = 0$. We get furthermore

$\bar\partial(i\,\alpha\wedge\bar\alpha\wedge i\,\beta\wedge\bar\beta) = 0$ since the forms α, $\bar\alpha$, β, $\bar\beta$ are all $\bar\partial$-closed,

$\bar\partial(i\,\alpha\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma) = -i\,\alpha\wedge\bar\alpha\wedge i\,\gamma\wedge\overline{\partial\gamma} = i\,\alpha\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\alpha\wedge\bar\beta = 0$ since $\bar\alpha\wedge\bar\alpha = 0$,

$\bar\partial(i\,\beta\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma) = -i\,\beta\wedge\bar\beta\wedge i\,\gamma\wedge\overline{\partial\gamma} = i\,\beta\wedge\bar\beta\wedge i\,\gamma\wedge\bar\alpha\wedge\bar\beta = 0$ since $\bar\beta\wedge\bar\beta = 0$,

$\bar\partial(i\,\alpha\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma) = -i\,\alpha\wedge\bar\beta\wedge i\,\gamma\wedge\overline{\partial\gamma} = i\,\alpha\wedge\bar\beta\wedge i\,\gamma\wedge\bar\alpha\wedge\bar\beta = 0$ since $\bar\beta\wedge\bar\beta = 0$,

$\bar\partial(i\,\beta\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma) = -i\,\beta\wedge\bar\alpha\wedge i\,\gamma\wedge\overline{\partial\gamma} = i\,\beta\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\alpha\wedge\bar\beta = 0$ since $\bar\alpha\wedge\bar\alpha = 0$.

We conclude from these identities and from (6.47) that $\partial\bar\partial(\omega_t^{1,1})^2 = 0$, so $\omega_t^{1,1}$ is indeed a Gauduchon metric on X_0 for all $t\in B$ close to 0. $\square$

We now observe that, in a certain sense, there are as "many" Aeppli-Gauduchon classes of the type $[(\omega_t^{1,1})^2]_A$ as elements in the Gauduchon cone $\mathcal{G}_{X_0}$.

Lemma 6.5.3. *For every $t\in B$ sufficiently close to 0, the Aeppli-Gauduchon class $[(\omega_t^{1,1})^2]_A\in\mathcal{G}_{X_0}$ satisfies the following identity*

$$\begin{aligned}\frac{1}{2}[(\omega_t^{1,1})^2]_A &= (1+c_1(t))(1+c_3(t))\,[i\,\alpha\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma]_A + (1+c_2(t))(1+c_3(t))\,[i\,\beta\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma]_A\\ &+ d(t)(1+c_3(t))\,[i\,\alpha\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma]_A + \overline{d(t)}(1+c_3(t))\,[i\,\beta\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma]_A.\end{aligned}\tag{6.48}$$

Note that since the classes $[i\,\alpha\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma]_A$, $[i\,\beta\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma]_A$, $[i\,\alpha\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma]_A$, $[i\,\beta\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma]_A$ generate $H_A^{2,2}(X_0,\mathbb{C})$ over $\mathbb{C}$, the real classes $[i\,\alpha\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma]_A$, $[i\,\beta\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma]_A$, $[i\,\alpha\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma + i\,\beta\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma]_A$ and $\frac{1}{2i}\,([i\,\alpha\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma - i\,\beta\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma]_A)$ generate $H_A^{2,2}(X_0,\mathbb{R})$ over $\mathbb{R}$.

Proof. Identity (6.48) follows from (6.47) after noticing that, since $\alpha\wedge\beta = -\partial\gamma$, we have

$$i\,\alpha\wedge\bar\alpha\wedge i\,\beta\wedge\bar\beta = \partial\gamma\wedge\bar\partial\bar\gamma = \partial(\gamma\wedge\bar\partial\bar\gamma)\in\operatorname{Im}\partial\subset\operatorname{Im}\partial+\operatorname{Im}\bar\partial,$$

hence $[i\,\alpha\wedge\bar\alpha\wedge i\,\beta\wedge\bar\beta]_A = 0$. $\square$

6.5.2 Use of the sGG property

Recall that the Iwasawa manifold X_0 and all its small deformations X_t are *sGG manifolds* ([PU14]). As such, there are *canonical surjections*

$$P_t : H^4_{DR}(X,\,\mathbb{R}) \twoheadrightarrow H^{2,\,2}_A(X_t,\,\mathbb{R}), \quad \{\Omega\}_{DR} \mapsto [\Omega^{2,\,2}_t]_A, \tag{6.49}$$

where $\Omega^{2,\,2}_t$ is the component of J_t-bidegree $(2,\,2)$ of Ω, while X is the C^∞ manifold underlying the fibres X_t. Moreover, for every fixed Hermitian metric ω_t on X_t, there is a lift of P_t naturally associated with ω_t, namely an injection

$$Q_{\omega_t} : H^{2,\,2}_A(X_t,\,\mathbb{R}) \hookrightarrow H^4_{DR}(X,\,\mathbb{R}), \quad [\Omega^{2,\,2}]_A \mapsto \{\Omega\}_{DR}, \tag{6.50}$$

such that $P_t \circ Q_{\omega_t} : H^{2,\,2}_A(X_t,\,\mathbb{R}) \longrightarrow H^{2,\,2}_A(X_t,\,\mathbb{R})$ is the identity map, defined in the following way (cf. [PU14, §.5.1). For every class $[\Omega^{2,\,2}]_A \in H^{2,\,2}_A(X_t,\,\mathbb{R})$, let $\Omega^{2,\,2}_A$ be the (unique) Aeppli-harmonic representative of $[\Omega^{2,\,2}]_A$ w.r.t. the Aeppli Laplacian $\Delta_{A,\,\omega_t}$ associated with the metric ω_t.[9] Let $\Omega^{3,\,1}_A$ be the (unique) minimal $L^2_{\omega_t}$-norm solution of the $\bar\partial$-equation

$$\bar\partial_t\Omega^{3,\,1}_A = -\partial_t\Omega^{2,\,2}_A. \tag{6.51}$$

This equation is solvable thanks to the sGG property of the manifold X_t for all $t \in B$ sufficiently close to 0. Indeed, n-dimensional sGG manifolds are characterised by the fact that every d-closed ∂-exact $(n,\,n-1)$-form is $\bar\partial$-exact ([PU14, Lemma 1.2]). Here $n = 3$, so $\partial_t\Omega^{2,\,2}_A$ is $\bar\partial_t$-exact. Thus, $\Omega^{3,\,1}_A$ exists and is given by the Neumann formula $\Omega^{3,\,1}_A = -\Delta''^{-1}_t\bar\partial^\star_t(\partial_t\Omega^{2,\,2}_A)$, where the formal adjoint $\bar\partial^\star_t$ of $\bar\partial_t$ and the Laplacian $\Delta''_t = \bar\partial_t\bar\partial^\star_t + \bar\partial^\star_t\bar\partial_t$ are computed w.r.t. the L^2 inner product induced by ω_t, while Δ''^{-1}_t is the Green operator of Δ''_t. Finally, we put

$$\Omega = \Omega_{\omega_t} := \Omega^{3,\,1}_A + \Omega^{2,\,2}_A + \overline{\Omega^{3,\,1}_A}$$

which is easily seen to be d-closed, to complete the definition (6.50) of Q_{ω_t} (cf. [PU14]).

Conclusion 6.5.4. *With every Hermitian metric ω_t on a small deformation X_t of the Iwasawa manifold $X = X_0$ there is associated a 4-dimensional real vector subspace of $H^4_{DR}(X,\,\mathbb{R})$ as follows*

$$\omega_t \mapsto Q_{\omega_t}(H^{2,\,2}_A(X_t,\,\mathbb{R})) \subset H^4_{DR}(X,\,\mathbb{R}). \tag{6.52}$$

Besides the metric-induced injections Q_{ω_t} of (6.50), there are canonical injections as follows.

Lemma 6.5.5. (*a*) *Let $X = X_0$ be the Iwasawa manifold. There is a* **canonical** *linear* **injection**

$$I_0 : H^{2,\,2}_A(X_0,\,\mathbb{C}) \longrightarrow H^4_{DR}(X,\,\mathbb{C}). \tag{6.53}$$

(*b*) *Let $\omega = \omega_0 := i\alpha\wedge\bar\alpha + i\beta\wedge\bar\beta + i\gamma\wedge\bar\gamma$ be the metric on the Iwasawa manifold $X = X_0$ canonically induced by the complex parallelisable structure of X (cf. (6.19)).*

The injection $Q_{\omega_0} : H^{2,\,2}_A(X_0,\,\mathbb{R}) \hookrightarrow H^4_{DR}(X,\,\mathbb{R})$ of (6.50) induced by ω_0 coincides with the canonical injection $I_0 : H^{2,\,2}_A(X_0,\,\mathbb{R}) \hookrightarrow H^4_{DR}(X,\,\mathbb{R})$ of (6.53).

[9] See [Sch07] for the definition of the Aeppli Laplacian.

Proof. (*a*) The contention follows from the explicit descriptions(1.56) and (6.58) of the cohomology groups involved. Specifically, I_0 is defined by letting

$$H^{2,2}_A(X_0,\,\mathbb{C})\ni[\Omega^{2,2}]_A\mapsto\{\Omega^{2,2}\}_{DR}:=I_0([\Omega^{2,2}]_A)\in H^4_{DR}(X,\,\mathbb{C}) \tag{6.54}$$

for every $\Omega^{2,2}\in\{\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma,\ \alpha\wedge\gamma\wedge\bar\beta\wedge\bar\gamma,\ \beta\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma,\ \beta\wedge\gamma\wedge\bar\beta\wedge\bar\gamma\}$ and extending by linearity. It is implicit that the forms $\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma$, $\alpha\wedge\gamma\wedge\bar\beta\wedge\bar\gamma$, $\beta\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma$, $\beta\wedge\gamma\wedge\bar\beta\wedge\bar\gamma$ are all d-closed, as can be readily checked.

(*b*) The representatives $\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma$, $\alpha\wedge\gamma\wedge\bar\beta\wedge\bar\gamma$, $\beta\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma$, $\beta\wedge\gamma\wedge\bar\beta\wedge\bar\gamma$ of the four Aeppli classes generating $H^{2,2}_A(X_0,\,\mathbb{C})$ are all in $\ker\partial^\star\cap\ker\bar\partial^\star$ when the adjoints $\partial^\star$ and $\bar\partial^\star$ are computed w.r.t. ω_0. Indeed,

(1) the identity $\partial^\star(\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma)=0$ is equivalent to $\langle\langle\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma,\,\partial u\rangle\rangle=0$ for all forms $u\in C^\infty_{1,2}(X,\,\mathbb{C})$. Now, the only generators of $C^\infty_{1,2}(X,\,\mathbb{C})$ that are not ∂-closed are $\gamma\wedge\bar\alpha\wedge\bar\beta$, $\gamma\wedge\bar\alpha\wedge\bar\gamma$ and $\gamma\wedge\bar\beta\wedge\bar\gamma$. When u is one of these forms, we have $\partial u=-\alpha\wedge\beta\wedge\bar\alpha\wedge\bar\beta$, or $\partial u=-\alpha\wedge\beta\wedge\bar\alpha\wedge\bar\gamma$, or $\partial u=-\alpha\wedge\beta\wedge\bar\beta\wedge\bar\gamma$ and the inner product of any of these forms against $\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma$ vanishes because they are all part of an ω_0-orthonormal basis and the ones do not contain γ while the other does. The same argument proves the $\partial^\star$-closedness of the remaining forms $\alpha\wedge\gamma\wedge\bar\beta\wedge\bar\gamma$, $\beta\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma$, $\beta\wedge\gamma\wedge\bar\beta\wedge\bar\gamma$ since they all contain γ.

(2) the identity $\bar\partial^\star(\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma)=0$ is equivalent to $\langle\langle\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma,\,\bar\partial v\rangle\rangle=0$ for all forms $v\in C^\infty_{2,1}(X,\,\mathbb{C})$. The only generators of $C^\infty_{2,1}(X,\,\mathbb{C})$ that are not $\bar\partial$-closed are $\alpha\wedge\beta\wedge\bar\gamma$, $\alpha\wedge\gamma\wedge\bar\gamma$ and $\beta\wedge\gamma\wedge\bar\gamma$. When v is one of these forms, we have $\bar\partial v=-\alpha\wedge\beta\wedge\bar\alpha\wedge\bar\beta$, or $\bar\partial v=-\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\beta$, or $\bar\partial v=-\beta\wedge\gamma\wedge\bar\alpha\wedge\bar\beta$ and the inner product of any of these forms against $\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma$ vanishes because they are all part of an ω_0-orthonormal basis and the ones do not contain $\bar\gamma$ while the other does. The same argument proves the $\bar\partial^\star$- closedness of the remaining forms $\alpha\wedge\gamma\wedge\bar\beta\wedge\bar\gamma$, $\beta\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma$, $\beta\wedge\gamma\wedge\bar\beta\wedge\bar\gamma$ since they all contain $\bar\gamma$.

Now, the forms $\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma$, $\alpha\wedge\gamma\wedge\bar\beta\wedge\bar\gamma$, $\beta\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma$, $\beta\wedge\gamma\wedge\bar\beta\wedge\bar\gamma$ are also $\partial\bar\partial$-closed (see lemma 6.5.10), so they must be **Aeppli-harmonic** [10] w.r.t. ω_0, i.e.

$$\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma,\ \alpha\wedge\gamma\wedge\bar\beta\wedge\bar\gamma,\ \beta\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma,\ \beta\wedge\gamma\wedge\bar\beta\wedge\bar\gamma\in\ker\Delta_{A,\,\omega_0}=\ker(\partial\bar\partial)\cap\ker\partial^\star_{\omega_0}\cap\ker\bar\partial^\star_{\omega_0}.$$

Thus, for any class $[\Omega^{2,2}]_A=c_1\,[\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma]_A+c_2\,[\alpha\wedge\gamma\wedge\bar\beta\wedge\bar\gamma]_A+c_3\,[\beta\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma]_A+c_4\,[\beta\wedge\gamma\wedge\bar\beta\wedge\bar\gamma]_A\in H^{2,2}_A(X_0,\,\mathbb{R})$ with coefficients $c_1,\dots,c_4\in\mathbb{R}$, the Aeppli-harmonic representative w.r.t. ω_0 is

$$\Omega^{2,2}_A=c_1\,\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma+c_2\,\alpha\wedge\gamma\wedge\bar\beta\wedge\bar\gamma+c_3\,\beta\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma+c_4\,\beta\wedge\gamma\wedge\bar\beta\wedge\bar\gamma.$$

Meanwhile, the forms $\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma$, $\alpha\wedge\gamma\wedge\bar\beta\wedge\bar\gamma$, $\beta\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma$, $\beta\wedge\gamma\wedge\bar\beta\wedge\bar\gamma$ are all d-closed, hence $d\Omega^{2,2}_A=0$. Since $\Omega^{2,2}_A$ is of pure type, this implies that $\partial_0\Omega^{2,2}_A=0$. Consequently, the minimal L^2-norm solution $\Omega^{3,1}_A$ of equation $\bar\partial_0\Omega^{3,1}_A=-\partial_0\Omega^{2,2}_A$ (cf. (6.51)) is the zero form. From (6.50) and (6.12) we get

$$Q_{\omega_0}([\Omega^{2,2}]_A)=Q_{\omega_0}([\Omega^{2,2}_A]_A)=\{\Omega^{2,2}_A\}_{DR}.$$

Comparing with (6.54), we see that $Q_{\omega_0}([\Omega^{2,2}]_A)=I_0([\Omega^{2,2}]_A)$. □

Corollary 6.5.6. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold $X=X_0$. Then*

$$B\ni t\mapsto H^{2,2}_A(X_t,\,\mathbb{C})$$

[10] For the definition of the Aeppli Laplacian Δ_A (an elliptic operator of order 4 whose kernel is isomorphic to the corresponding Aeppli cohomology group) and the description of its kernel used here, see [Sch07].

is a C^∞ vector bundle of rank 4 that we shall denote by $\mathcal{H}^{2,2}_A$. Moreover, $\mathcal{H}^{2,2}_A$ injects **canonically** *as a C^∞ vector subbundle of the constant bundle $\mathcal{H}^4 \to \Delta$ of fibre $H^4_{DR}(X, \mathbb{C})$ in the following way: for every $t \in B$ sufficiently close to 0, we define the* **canonical** *linear* **injection**

$$I_t : H^{2,2}_A(X_t, \mathbb{C}) \longrightarrow H^4_{DR}(X, \mathbb{C}) \quad by \quad I_t = Q_{\omega_t}, \tag{6.55}$$

the injection (6.50) induced by the canonical metric $\omega_t = i\alpha_t \wedge \bar\alpha_t + i\beta_t \wedge \bar\beta_t + i\gamma_t \wedge \bar\gamma_t$ of (6.19) on X_t.

Proof. Let $(\gamma_t)_{t\in B}$ be any C^∞ family of Hermitian metrics on the fibres $(X_t)_{t\in B}$ and let $(\Delta_{A,t})_{t\in B}$ be the associated C^∞ family of elliptic Aeppli Laplacians inducing Hodge isomorphisms $\ker \Delta_{A,t} \simeq H^{2,2}_A(X_t, \mathbb{C})$ for $t \in B$ ([Sch07]). Meanwhile, $\dim_{\mathbb{C}} H^{2,2}_A(X_t, \mathbb{C}) = 4$ for all $t \in B$ ([Ang11, §.4.3]). Since the dimension of the kernel of $\Delta_{A,t}$ is independent of $t \in B$, we infer by ellipticity from [KS60] that $B \ni t \mapsto \ker \Delta_{A,t} \simeq H^{2,2}_A(X_t, \mathbb{C})$ is a C^∞ vector bundle of rank 4.

The last statement follows from Lemma 6.5.5 and from the C^∞ dependence on t of the injections I_t (itself a consequence of the C^∞ dependence on t of each of the forms $\alpha_t, \beta_t, \gamma_t$). □

Remark 6.5.7. *Note that for $t \in B_{[\gamma]} \setminus \{0\}$, I_t cannot be defined by analogy with definition (6.54) of I_0 since the representatives $\alpha_t \wedge \gamma_t \wedge \bar\alpha_t \wedge \bar\gamma_t$, $\alpha_t \wedge \gamma_t \wedge \bar\beta_t \wedge \bar\gamma_t$, $\beta_t \wedge \gamma_t \wedge \bar\alpha_t \wedge \bar\gamma_t$, $\beta_t \wedge \gamma_t \wedge \bar\beta_t \wedge \bar\gamma_t$ of the classes generating $H^{2,2}_A(X_t, \mathbb{C})$ (cf. (6.58)) are not d-closed.*

For future reference, we notice the following trivialisation of the vector bundle $B \ni t \mapsto H^{2,2}_A(X_t, \mathbb{C})$. The following definition is meaningful thanks to Lemma 6.5.10 of the following subsection.

Definition 6.5.8. *For every $t \in B$, we consider the isomorphism of complex vector spaces*

$$B_t : H^{2,2}_A(X_t, \mathbb{C}) \longrightarrow H^{2,2}_A(X_0, \mathbb{C}) \tag{6.56}$$

defined by $[\alpha_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\gamma_t]_A \mapsto [\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma]_A$, $[\alpha_t\wedge\gamma_t\wedge\bar\beta_t\wedge\bar\gamma_t]_A \mapsto [\alpha\wedge\gamma\wedge\bar\beta\wedge\bar\gamma]_A$, $[\beta_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\gamma_t]_A \mapsto [\beta \wedge \gamma \wedge \bar\alpha \wedge \bar\gamma]_A$, $[\beta_t \wedge \gamma_t \wedge \bar\beta_t \wedge \bar\gamma_t]_A \mapsto [\beta \wedge \gamma \wedge \bar\beta \wedge \bar\gamma]_A$.

Corollary 6.5.9. *With every Aeppli-Gauduchon class of the shape $[(\omega_t^{1,1})^2]_A \in \mathcal{G}_{X_0}$ (for $t \in B$) on the Iwasawa manifold $X = X_0$ there is associated a 4-dimensional real vector subspace of $H^4_{DR}(X, \mathbb{R})$ as follows*

$$\mathcal{G}_{X_0} \ni [(\omega_t^{1,1})^2]_A \mapsto \widetilde{H_t^{2,2}} := Q_{\omega_t^{1,1}}\bigg(H^{2,2}_A(X_0, \mathbb{R})\bigg) \subset H^4_{DR}(X, \mathbb{R}), \tag{6.57}$$

where $Q_{\omega_t^{1,1}} : H^{2,2}_A(X_0, \mathbb{R}) \hookrightarrow H^4_{DR}(X, \mathbb{R})$ is the injective linear map of (6.50) defined by the metric $\omega_t^{1,1}$ on X_0.

6.5.3 The Hodge bundles $\mathcal{H}^{2,1}_{[\gamma]} \simeq \mathcal{H}^{2,2}_A$ and $\mathcal{H}^4$ over $B_{[\gamma]}$

The following description of the Aeppli cohomology groups of bidegree (2, 2) of the small deformations X_t with $t \in B$ of the Iwasawa manifold $X = X_0$ will be used several times in this section.

Lemma 6.5.10. *For every $t \in B$, the forms $\alpha_t \wedge \bar\alpha_t \wedge \gamma_t \wedge \bar\gamma_t, \beta_t \wedge \bar\beta_t \wedge \gamma_t \wedge \bar\gamma_t, \alpha_t \wedge \bar\beta_t \wedge \gamma_t \wedge \bar\gamma_t$ and $\beta_t \wedge \bar\alpha_t \wedge \gamma_t \wedge \bar\gamma_t$ are $\partial_t\bar\partial_t$-closed and*

$$H^{2,2}_A(X_t, \mathbb{C}) = \bigg\langle [\alpha_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\gamma_t]_A,\, [\alpha_t\wedge\gamma_t\wedge\bar\beta_t\wedge\bar\gamma_t]_A,\, [\beta_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\gamma_t]_A,\, [\beta_t\wedge\gamma_t\wedge\bar\beta_t\wedge\bar\gamma_t]_A\bigg\rangle. \tag{6.58}$$

Proof. We spell out the details of the pluriclosedness argument, that is similar for the four forms, when $t \in B_{[\gamma]}$. It goes

$$\begin{aligned}\partial_t(\alpha_t \wedge \gamma_t \wedge \bar\alpha_t \wedge \bar\gamma_t) &= -\alpha_t \wedge \partial_t\gamma_t \wedge \bar\alpha_t \wedge \bar\gamma_t - \alpha_t \wedge \gamma_t \wedge \bar\alpha_t \wedge \overline{\bar\partial_t\gamma_t} \\ &= -\sigma_{12}(t)\,\alpha_t \wedge (\alpha_t \wedge \beta_t) \wedge \bar\alpha_t \wedge \bar\gamma_t - \alpha_t \wedge \gamma_t \wedge \bar\alpha_t \wedge (\overline{\sigma_{2\bar 2}(t)}\,\bar\beta_t \wedge \beta_t) \\ &= -\overline{\sigma_{2\bar 2}(t)}\,\alpha_t \wedge \bar\alpha_t \wedge \beta_t \wedge \bar\beta_t \wedge \gamma_t,\end{aligned}$$

where we used the structure equations (6.18) to get the second line above. So, $\partial_t(\alpha_t \wedge \gamma_t \wedge \bar\alpha_t \wedge \bar\gamma_t) \neq 0$ when $t \neq 0$, but (6.18) implies that $\bar\partial_t\gamma_t$ comes from a 2-form on B_t, so applying $\bar\partial_t$ we get $\bar\partial_t\partial_t(\alpha_t \wedge \gamma_t \wedge \bar\alpha_t \wedge \bar\gamma_t) = 0$.

Now, it was shown in [Ang11, Remark 5.2] that

$$H^{1,1}_{BC}(X_t,\,\mathbb{C}) = \left\langle [\alpha_t \wedge \bar\alpha_t]_{BC},\, [\alpha_t \wedge \bar\beta_t]_{BC},\, [\beta_t \wedge \bar\alpha_t]_{BC},\, [\beta_t \wedge \bar\beta_t]_{BC} \right\rangle, \qquad t \in B.$$

This implies (6.58) via the non-degenerate duality $H^{1,1}_{BC}(X_t,\,\mathbb{C}) \times H^{2,2}_{A}(X_t,\,\mathbb{C}) \longrightarrow \mathbb{C}$, $([u]_{BC},\,[v]_A) \mapsto \int_X u \wedge v$. □

Observation 6.5.11. *(a) On the Iwasawa manifold X_0, there is a canonical isomorphism*

$$H^{2,1}_{[\gamma]}(X_0,\,\mathbb{C}) \xrightarrow{\simeq} H^{2,2}_{A}(X_0,\,\mathbb{C}) \tag{6.59}$$

defined by $[\Gamma]_{\bar\partial} \mapsto [\Gamma \wedge \bar\gamma]_A$ *for* $\Gamma \in \{\alpha \wedge \gamma \wedge \bar\alpha, \alpha \wedge \gamma \wedge \bar\beta, \beta \wedge \gamma \wedge \bar\alpha, \beta \wedge \gamma \wedge \bar\beta\}$.

(b) In the Kuranishi family $(X_t)_{t\in B}$ of the Iwasawa manifold X_0, there is a canonical isomorphism

$$A_t : H^{2,1}_{[\gamma]}(X_t,\,\mathbb{C}) \xrightarrow{\simeq} H^{2,2}_{A}(X_t,\,\mathbb{C}) \quad \textit{for every} \quad t \in B_{[\gamma]} \tag{6.60}$$

defined by $[\Gamma]_{\bar\partial} \mapsto [\Gamma \wedge \bar\gamma_t]_A$ *for* $\Gamma \in \{\Gamma_1(t),\, \Gamma_2(t),\, \Gamma_3(t),\, \Gamma_4(t)\}$ *(see (6.20) and (6.21)). (Note that A_t depends anti-holomorphically on t.)*

In particular, the rank-four C^∞ vector bundles $B_{[\gamma]} \ni t \mapsto H^{2,1}_{[\gamma]}(X_t,\,\mathbb{C})$ (of Definition 6.3.5) and $B_{[\gamma]} \ni t \mapsto H^{2,2}_{A}(X_t,\,\mathbb{C})$ (of Corollary 6.5.6) are canonically isomorphic, i.e. $\mathcal{H}^{2,1}_{[\gamma]} \simeq \mathcal{H}^{2,2}_{A}$.

Proof. Part (a) is a special case of part (b). To prove (b), we note that in conjunction with the description of $H^{2,1}_{[\gamma]}(X_t,\,\mathbb{C})$ given at the end of §.6.3.2 as $H^{2,1}_{[\gamma]}(X_t,\,\mathbb{C}) = \langle [\Gamma_1(t)]_{\bar\partial},\, [\Gamma_2(t)]_{\bar\partial},\, [\Gamma_3(t)]_{\bar\partial},\, [\Gamma_4(t)]_{\bar\partial}\rangle \subset H^{2,1}_{\bar\partial}(X_t,\,\mathbb{C})$ for all $t \in B_{[\gamma]}$, (6.58) proves the isomorphism (6.60). Indeed, $\Gamma_1(t)\wedge\bar\gamma_t = \alpha_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\gamma_t$, $\Gamma_2(t) \wedge \bar\gamma_t = \alpha_t \wedge \gamma_t \wedge \bar\beta_t \wedge \bar\gamma_t$, $\Gamma_3(t) \wedge \bar\gamma_t = \beta_t \wedge \gamma_t \wedge \bar\alpha_t \wedge \bar\gamma_t$, $\Gamma_4(t) \wedge \bar\gamma_t = \beta_t \wedge \gamma_t \wedge \bar\beta_t \wedge \bar\gamma_t$ and all these forms are $\partial\bar\partial$-closed as proved in Lemma 6.5.10.

□

6.5.4 Bringing the families of metrics $(\omega_t)_{t\in B_{[\gamma]}}$ and $(\omega_t^{1,1})_{t\in B_{[\gamma]}}$ together

We can now describe a VHS parametrised by Aeppli-Gauduchon classes on X_0. It is related to the VHS of weight 2 induced by the holomorphic family $(B_t)_{t\in B_{[\gamma]}}$ of 2-dimensional complex tori. Since the B_t's are Kähler, we get a weight-two Hodge decomposition

$$H^2(B,\,\mathbb{C}) \simeq H^{2,0}(B_t,\,\mathbb{C}) \oplus H^{1,1}(B_t,\,\mathbb{C}) \oplus H^{0,2}(B_t,\,\mathbb{C}), \qquad t \in B, \tag{6.61}$$

where B stands for the C^∞ manifold underlying the complex tori B_t and $H^2(B,\,\mathbb{C}) := H^2_{DR}(B_t,\,\mathbb{C})$ is the fibre of the constant bundle $\mathcal{H}^2(B)$ over Δ defined by the De Rham cohomology of degree 2 of the tori B_t with $t\in B$. As usual, we get holomorphic vector bundles

$$F^1\mathcal{H}^2(B) := \bigg(B\ni t\mapsto H^{2,\,0}(B_t,\,\mathbb{C})\oplus H^{1,\,1}(B_t,\,\mathbb{C})\bigg)\supset F^2\mathcal{H}^2(B) := \bigg(B\ni t\mapsto H^{2,\,0}(B_t,\,\mathbb{C})\bigg) \tag{6.62}$$

that constitute the Hodge filtration associated with the VHS (6.61). Let D be the Gauss-Manin connection of the constant bundle $\mathcal{H}^2(B)$. It satisfies the transversality condition

$$D_{[\theta]}F^2\mathcal{H}^2(B)_t\subset F^1\mathcal{H}^2(B)_t, \qquad t\in B_{[\gamma]}, \tag{6.63}$$

for all $[\theta]\in T^{1,\,0}_tB_{[\gamma]}\simeq H^{1,\,1}(B_t,\,\mathbb{C})\simeq H^{2,\,2}_A(X_t,\,\mathbb{C})\simeq H^{2,\,1}_{[\gamma]}(X_t,\,\mathbb{C})$.

The second isomorphism of vector spaces on the previous line is a consequence of the description of $H^{1,\,1}(B_t,\,\mathbb{C})$ as

$$\begin{aligned}
H^{1,\,1}(B_t,\,\mathbb{C}) &= \bigg\langle[\alpha_t\wedge\bar\alpha_t]_{\bar\partial},\,[\alpha_t\wedge\bar\beta_t]_{\bar\partial},\,[\beta_t\wedge\bar\alpha_t]_{\bar\partial},\,[\beta_t\wedge\bar\beta_t]_{\bar\partial}\bigg\rangle\\
&\simeq \bigg\langle[\alpha_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\gamma_t]_A,\,[\alpha_t\wedge\gamma_t\wedge\bar\beta_t\wedge\bar\gamma_t]_A,\,[\beta_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\gamma_t]_A,\,[\beta_t\wedge\gamma_t\wedge\bar\beta_t\wedge\bar\gamma_t]_A\bigg\rangle\\
&= H^{2,\,2}_A(X_t,\,\mathbb{C})\simeq Q_{\omega_t}(H^{2,\,2}_A(X_t,\,\mathbb{C})) := \widetilde{H^{2,\,2}_{\omega_t}}\subset H^4_{DR}(X,\,\mathbb{C}), \qquad t\in B_{[\gamma]},
\end{aligned} \tag{6.64}$$

where the first identity on the last line is (6.58) and $Q_{\omega_t}\,:\,H^{2,\,2}_A(X_t,\,\mathbb{C})\hookrightarrow H^4_{DR}(X,\,\mathbb{C})$ is the complexification of the injective linear map of (6.50) defined by the Gauduchon metric ω_t of (6.19) on X_t (and also denoted by I_t in (6.55)).

On the other hand,

$$H^{2,\,0}(B_t,\,\mathbb{C})\simeq H^{3,\,0}(X_t,\,\mathbb{C})\hookrightarrow H^3_{DR}(X,\,\mathbb{C}), \qquad t\in B, \tag{6.65}$$

since $H^{2,\,0}(B_t,\,\mathbb{C}) = \langle[\alpha_t\wedge\beta_t]_{\bar\partial}\rangle$ and $H^{3,\,0}(X_t,\,\mathbb{C}) = \langle[\alpha_t\wedge\beta_t\wedge\gamma_t]_{\bar\partial}\rangle$, while the $\mathbb{C}$-line $H^{3,\,0}(X_t,\,\mathbb{C})$ injects canonically into $H^3_{DR}(X,\,\mathbb{C})$ as observed in Lemma 6.3.1. We get a canonical injection of holomorphic vector bundles

$$j\,:\,F^2\mathcal{H}^2(B)\hookrightarrow\mathcal{H}^3$$

such that $j_t\,:\,H^{2,\,0}(B_t,\,\mathbb{C})\hookrightarrow H^3(X,\,\mathbb{C})$ is the composition of the maps (6.65) for every $t\in B$.

Together with (6.64), this gives an injection of holomorphic vector bundles

$$j\oplus Q\,:\,F^1\mathcal{H}^2(B)\hookrightarrow\mathcal{H}^3\oplus\mathcal{H}^4$$

such that $(j\oplus Q)_t = j_t\oplus Q_{\omega_t}$ for all $t\in B_{[\gamma]}$.

We now anticipate the definition of what will be called later the **complexified parameter set**:

$$\begin{aligned}
\widetilde{\mathcal{G}}_0 := \{[\omega_0^2]_A - t_{11}\,[i\beta\wedge\bar\alpha\wedge i\gamma\wedge\bar\gamma]_A + t_{22}\,[i\alpha\wedge\bar\beta\wedge i\gamma\wedge\bar\gamma]_A - t_{12}\,[i\beta\wedge\bar\beta\wedge i\gamma\wedge\bar\gamma]_A + t_{21}\,[i\alpha\wedge\bar\alpha\wedge i\gamma\wedge\bar\gamma]_A \;\mid\; t\in B_{[\gamma]}\}\\
\subset H^{2,\,2}_A(X_0,\,\mathbb{C}).
\end{aligned}$$

Recall the identification $B_{[\gamma]} = \{t = (t_{11}, t_{12}, t_{21}, t_{22}) \in H^{0,1}_{[\gamma]}(X_0, T^{1,0}X_0)\,;\, |t| < \varepsilon\}$ for some small $\varepsilon > 0$ when $H^{0,1}_{[\gamma]}(X_0, T^{1,0}X_0)$ is identified with $\mathbb{C}^4$ by the basis specified in (6.9). The set $\widetilde{\mathcal{G}}_0$ is a complexification of the **parameter set**

$$\mathcal{G}_0 := \{[(\omega_t^{1,1})^2]_A \mid t \in B_{[\gamma]}\} \subset \mathcal{G}_{X_0}. \tag{6.66}$$

Thus, $\widetilde{\mathcal{G}}_0$ is a subset of the complexified Gauduchon cone $\widetilde{\mathcal{G}_{X_0}} \subset H^{2,2}_A(X_0, \mathbb{C})$ (cf. Defintion 6.6.2) of the Iwasawa manifold $X = X_0$.

Conclusion 6.5.12. *Let $(X_t)_{t\in B_{[\gamma]}}$ be the local universal family of essential deformations of the Iwasawa manifold $X = X_0$.*

(i) Our discussion so far can be summed up in the following diagram for all $t \in B_{[\gamma]}$.

$$\begin{array}{ccc} H^{2,2}_A(X_t, \mathbb{C}) & \xrightarrow[B_t]{\simeq} & H^{2,2}_A(X_0, \mathbb{C}) \\ \simeq\Big\downarrow Q_{\omega_t} & & \simeq\Big\downarrow Q_{\omega_t^{1,1}} \\ H^4(X, \mathbb{C}) \supset H^{2,2}_A(X_t, \mathbb{C}) \simeq Q_{\omega_t}(H^{2,2}_A(X_t, \mathbb{C})) := \widetilde{H^{2,2}_{\omega_t}} & \xrightarrow{\simeq} & Q_{\omega_t^{1,1}}(H^{2,2}_A(X_0, \mathbb{C})) := \widetilde{H^{2,2}_t} \subset H^4(X, \mathbb{C}), \end{array}$$

where the isomorphism $\widetilde{H^{2,2}_{\omega_t}} \to \widetilde{H^{2,2}_t}$ is the composition $Q_{\omega_t^{1,1}} \circ B_t \circ Q_{\omega_t}^{-1}$.

(ii) Moreover, we get a C^∞ vector subbundle of rank 4 of the constant bundle $\mathcal{H}^4$:

$$B_{[\gamma]} \simeq \widetilde{\mathcal{G}}_0 \ni t \mapsto Q_{\omega_t}(H^{2,2}_A(X_t, \mathbb{C})) \subset H^4(X, \mathbb{C}), \tag{6.67}$$

denoted henceforth by $\widetilde{\mathcal{H}^{2,2}_\omega}$, and a **holomorphic** *vector subbundle of rank 1 of the constant bundle $\mathcal{H}^3(X)$:*

$$B_{[\gamma]} \simeq \widetilde{\mathcal{G}}_0 \ni t \mapsto H^{2,0}(B_t, \mathbb{C}) \xhookrightarrow{j_t} H^3(X, \mathbb{C}), \tag{6.68}$$

denoted henceforth by $\mathcal{H}^{2,0}(B) = F'_{\mathcal{G}}\mathcal{H}$, such that the following complex vector bundle of rank 5, denoted henceforth by $F_{\mathcal{G}}\mathcal{H}^4 := \mathcal{H}^{2,0}(B) \oplus \widetilde{\mathcal{H}^{2,2}_\omega}$,

$$B_{[\gamma]} \simeq \widetilde{\mathcal{G}}_0 \ni t \mapsto H^{2,0}(B_t, \mathbb{C}) \oplus Q_{\omega_t}(H^{2,2}_A(X_t, \mathbb{C})) \subset H^3(X, \mathbb{C}) \oplus H^4(X, \mathbb{C}) \tag{6.69}$$

is a **holomorphic** *subbundle of the constant bundle $\mathcal{H}^3 \oplus \mathcal{H}^4$ of fibre $H^3(X, \mathbb{C}) \oplus H^4(X, \mathbb{C})$ and is C^∞* **isomorphic** *to $F^1\mathcal{H}^2(B)$.*

(iii) In particular, the vector bundles (6.68) and (6.67) define a VHS parametrised by the subset

$$B_{[\gamma]} \simeq \widetilde{\mathcal{G}}_0 \subset \widetilde{\mathcal{G}_{X_0}} \tag{6.70}$$

whose corresponding Hodge filtration $F_{\mathcal{G}}\mathcal{H}^4 \supset F'_{\mathcal{G}}\mathcal{H}^4$ is C^∞ **isomorphic** *to the Hodge filtration $F^1\mathcal{H}^2(B) \supset F^2\mathcal{H}^2(B)$ associated with the holomorphic family $(B_t)_{t\in B_{[\gamma]}}$ of base tori of the family $(X_t)_{t\in B_{[\gamma]}}$.*

Only the *holomorphic* nature of the above vector bundle isomorphisms still needs a proof that is provided in the next subsection.

6.5.5 Holomorphicity of the Hodge filtration parametrised by $\mathcal{G}_0$

We prove in this subsection that the Hodge filtration

$$\mathcal{H}^3 \oplus \mathcal{H}^4 \supset F_{\mathcal{G}}\mathcal{H}^4 \supset F'_{\mathcal{G}}\mathcal{H}^4$$

constructed in the previous subsection (cf. Conclusion 6.5.12) consists of **holomorphic vector subbundles** of the constant bundle $\mathcal{H}^3 \oplus \mathcal{H}^4$ of fibre $H^3(X,\,\mathbb{C}) \oplus H^4(X,\,\mathbb{C})$ over $\overline{\mathcal{G}_0}$.

Our starting point is the following simple observation.

Lemma 6.5.13. *For every $t \in B$, there is a canonical linear injection*

$$H^{3,\,1}_{BC}(X_t,\,\mathbb{C}) \hookrightarrow H^{2,\,2}_A(X_t,\,\mathbb{C}). \tag{6.71}$$

Proof. From [Ang14, p. 83] we infer that $H^{3,1}_{BC}(X_t,\,\mathbb{C}) = \langle[\alpha_t \wedge \beta_t \wedge \gamma_t \wedge \bar{\alpha}_t]_{BC},\,[\alpha_t \wedge \beta_t \wedge \gamma_t \wedge \bar{\beta}_t]_{BC}\rangle$ for all $t \in B$. Coupled with (6.58), this allows us to explicitly define the canonical linear injection by

$$[\alpha_t \wedge \beta_t \wedge \gamma_t \wedge \bar{\alpha}_t]_{BC} \mapsto [\alpha_t \wedge \bar{\alpha}_t \wedge \gamma_t \wedge \bar{\gamma}_t]_A \quad \text{and} \quad [\alpha_t \wedge \beta_t \wedge \gamma_t \wedge \bar{\beta}_t]_{BC} \mapsto [\beta_t \wedge \bar{\beta}_t \wedge \gamma_t \wedge \bar{\gamma}_t]_A.$$

The forms $\alpha_t, \beta_t, \gamma_t$ are canonically associated with the complex structure of X_t, which makes the above linear injection canonical. □

Since $F'_{\mathcal{G}}\mathcal{H} = \mathcal{H}^{2,\,0}(B)$ is a holomorphic subbundle of $\mathcal{H}^3(X)$, we are reduced to proving the following

Lemma 6.5.14. *The holomorphic structure of the vector bundle $F_{\mathcal{G}}\mathcal{H}^4 := \mathcal{H}^{2,\,0}(B) \oplus \widetilde{\mathcal{H}^{2,\,2}_\omega}$ is the restriction of the holomorphic structure of the constant bundle $\mathcal{H}^3 \oplus \mathcal{H}^4$.*

Proof. We have to show that for any C^∞ section s of $\widetilde{\mathcal{H}^{2,\,2}_\omega}$, the a priori $\mathcal{H}^3(X) \oplus \mathcal{H}^4(X)$-valued $(0,\,1)$-form $D''s$ is actually $F_{\mathcal{G}}\mathcal{H}^4$-valued, where D'' is the canonical $(0,\,1)$-connection of the constant bundle $\mathcal{H}^3(X)\oplus\mathcal{H}^4(X)$. Thanks to (6.58), it suffices to prove that all the anti-holomorphic first-order derivatives of each of the classes $[\alpha_t\wedge\gamma_t\wedge\bar{\alpha}_t\wedge\bar{\gamma}_t]_A$, $[\alpha_t\wedge\gamma_t\wedge\bar{\beta}_t\wedge\bar{\gamma}_t]_A$, $[\beta_t\wedge\gamma_t\wedge\bar{\alpha}_t\wedge\bar{\gamma}_t]_A$, $[\beta_t\wedge\gamma_t\wedge\bar{\beta}_t\wedge\bar{\gamma}_t]_A$ lie in $F_{\mathcal{G}}\mathcal{H}^4$.

We now study these classes individually. By way of example, we compute derivatives at $t = 0$.

From (6.42), we infer that the only terms in

$$\alpha_t\wedge\gamma_t\wedge\bar{\alpha}_t\wedge\bar{\gamma}_t = (\alpha+t_{11}\,\bar{\alpha}+t_{12}\,\bar{\beta})\wedge(\gamma+t_{31}\,\bar{\alpha}+t_{32}\,\bar{\beta}-D(t)\,\bar{\gamma})\wedge(\bar{\alpha}+\bar{t}_{11}\,\alpha+\bar{t}_{12}\,\beta)\wedge(\bar{\gamma}+\bar{t}_{31}\,\alpha+\bar{t}_{32}\,\beta-\overline{D(t)}\,\gamma)$$

that are linear in the $\bar{t}_{i\lambda}$'s are

$$\bar{t}_{12}\,\alpha\wedge\gamma\wedge\beta\wedge\bar{\gamma} \quad \text{and} \quad \bar{t}_{32}\,\alpha\wedge\gamma\wedge\bar{\alpha}\wedge\beta.$$

So, the non-trivial anti-holomorphic first-order derivatives at $t = 0$ are

$$\frac{\partial(\alpha_t\wedge\gamma_t\wedge\bar{\alpha}_t\wedge\bar{\gamma}_t)}{\partial\bar{t}_{12}}_{|t=0} = -\alpha\wedge\beta\wedge\gamma\wedge\bar{\gamma} \quad \text{and} \quad \frac{\partial(\alpha_t\wedge\gamma_t\wedge\bar{\alpha}_t\wedge\bar{\gamma}_t)}{\partial\bar{t}_{32}}_{|t=0} = \alpha\wedge\beta\wedge\gamma\wedge\bar{\alpha}. \tag{6.72}$$

Note that $\alpha\wedge\beta\wedge\gamma\wedge\bar{\gamma}$ is not d-closed, so it defines no class in $H^{3,1}_{BC}(X_0,\,\mathbb{C})$. However, $\alpha\wedge\beta\wedge\gamma\wedge\bar{\gamma}$ is the image under the multiplication by $\gamma\wedge\bar{\gamma}$ of $\alpha\wedge\beta$ whose Dolbeault cohomology class $[\alpha\wedge\beta]$ is the (unique up to a multiplicative constant) generator of $H^{2,0}(B_0, \mathbb{C})$. Meanwhile, $\alpha\wedge\beta\wedge\gamma\wedge\bar\alpha$ is d-closed and its Bott-Chern cohomology class is one of the generators of $H^{3,1}_{BC}(X_0, \mathbb{C})$ (cf. proof of Lemma 6.5.13) which injects canonically into $H^{2,2}_A(X_0, \mathbb{C})$ by Lemma 6.5.13. Under this injection, $[\alpha\wedge\beta\wedge\gamma\wedge\bar\alpha]_{BC}$ identifies with its image $[\alpha\wedge\bar\alpha\wedge\gamma\wedge\bar\gamma]_A$ in $H^{2,2}_A(X_0, \mathbb{C})$, which in turn identifies with its image in $\widetilde{H^{2,2}_{\omega_0}} = Q_{\omega_0}(H^{2,2}_A(X_0, \mathbb{C}))$ under the canonical injection $Q_{\omega_0} = I_0 : H^{2,2}_A(X_0, \mathbb{C}) \hookrightarrow H^4(X, \mathbb{C})$ of Lemma 6.5.5.

The upshot is that after all these identifications, we have

$$\frac{\partial\,[\alpha_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\gamma_t]_A}{\partial\bar t_{i\lambda}}_{|t=0} \in (F_{\mathcal{G}}\mathcal{H}^4)_0 = H^{2,0}(B_0, \mathbb{C}) \oplus \widetilde{H^{2,2}_{\omega_0}}$$

for all indices i, λ.

Similarly, for the remaining 3 generators of $H^{2,2}_A(X_t, \mathbb{C})$, we get from (6.81) that the only terms linear in the $\bar t_{i\lambda}$'s in $\alpha_t\wedge\gamma_t\wedge\bar\beta_t\wedge\bar\gamma_t$ are $\bar t_{22}\,\alpha\wedge\gamma\wedge\beta\wedge\bar\gamma$ and $\bar t_{32}\,\alpha\wedge\gamma\wedge\bar\beta\wedge\beta$; in $\beta_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\gamma_t$ are $\bar t_{11}\,\beta\wedge\gamma\wedge\alpha\wedge\bar\gamma$ and $\bar t_{31}\,\beta\wedge\gamma\wedge\bar\alpha\wedge\alpha$; and in $\beta_t\wedge\gamma_t\wedge\bar\beta_t\wedge\bar\gamma_t$ are $\bar t_{21}\,\beta\wedge\gamma\wedge\alpha\wedge\bar\gamma$ and $\bar t_{31}\,\beta\wedge\gamma\wedge\bar\beta\wedge\alpha$. Thus, the only non-zero anti-holomorphic first-order derivatives at $t=0$ of these terms are

$$\pm\,\alpha\wedge\beta\wedge\gamma\wedge\bar\gamma, \quad \pm\,\alpha\wedge\beta\wedge\gamma\wedge\bar\alpha \quad \text{and} \quad \pm\,\alpha\wedge\beta\wedge\gamma\wedge\bar\beta.$$

Note that the only new quantity compared to (6.72) is $\alpha\wedge\beta\wedge\gamma\wedge\bar\beta$. It has the same properties as $\alpha\wedge\beta\wedge\gamma\wedge\bar\alpha$, i.e. it is d-closed and its Bott-Chern cohomology class is a generator of $H^{3,1}_{BC}(X_0, \mathbb{C})$ (cf. proof of Lemma 6.5.13). This vector space injects canonically into $H^{2,2}_A(X_0, \mathbb{C})$ by Lemma 6.5.13. So the above argument applies again and yields

$$\frac{\partial\,[\alpha_t\wedge\gamma_t\wedge\bar\beta_t\wedge\bar\gamma_t]_A}{\partial\bar t_{i\lambda}}_{|t=0}, \ \frac{\partial\,[\beta_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\gamma_t]_A}{\partial\bar t_{i\lambda}}_{|t=0}, \ \frac{\partial\,[\beta_t\wedge\gamma_t\wedge\bar\beta_t\wedge\bar\gamma_t]_A}{\partial\bar t_{i\lambda}}_{|t=0} \in (F_{\mathcal{G}}\mathcal{H}^4)_0$$

for all indices i, λ.

□

6.5.6 Construction of coordinates on the Gauduchon cone

Recall the isomorphisms

$$\begin{array}{ll}
H^{1,1}(B_0, \mathbb{C}) = & \Big\langle [\alpha\wedge\bar\alpha]_{\bar\partial},\, [\alpha\wedge\bar\beta]_{\bar\partial},\, [\beta\wedge\bar\alpha]_{\bar\partial},\, [\beta\wedge\bar\beta]_{\bar\partial} \Big\rangle \\
\quad \simeq \big\downarrow \cdot\wedge\gamma & \\
H^{2,1}_{[\gamma]}(X_0, \mathbb{C}) = & \Big\langle [\alpha\wedge\gamma\wedge\bar\alpha]_{\bar\partial},\, [\alpha\wedge\gamma\wedge\bar\beta]_{\bar\partial},\, [\beta\wedge\gamma\wedge\bar\alpha]_{\bar\partial},\, [\beta\wedge\gamma\wedge\bar\beta]_{\bar\partial} \Big\rangle \\
\quad \simeq \big\downarrow \cdot\wedge\bar\gamma & \\
H^{2,2}_A(X_0, \mathbb{C}) = & \Big\langle [\alpha\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma]_A,\, [\alpha\wedge\gamma\wedge\bar\beta\wedge\bar\gamma]_A,\, [\beta\wedge\gamma\wedge\bar\alpha\wedge\bar\gamma]_A,\, [\beta\wedge\gamma\wedge\bar\beta\wedge\bar\gamma]_A \Big\rangle.
\end{array}$$

On the other hand, on the vector space

$$H^{3,0}(X_t, \mathbb{C}) \oplus H^{2,1}_{[\gamma]}(X_t, \mathbb{C}) \simeq H^{2,0}(B_t, \mathbb{C}) \oplus H^{1,1}(B_t, \mathbb{C}), \qquad t\in B,$$

we have two sesquilinear intersection forms (the first of which was considered in (6.35)). The first one is obtained by restriction from $H^3_{DR}(X,\,\mathbb{C})\times H^3_{DR}(X,\,\mathbb{C})$ (where X is the differentiable manifold underlying the X_t's) when $H^{3,0}(X_t,\,\mathbb{C})\oplus H^{2,1}_{[\gamma]}(X_t,\,\mathbb{C})$ is viewed as a vector subspace of $H^3_{DR}(X,\,\mathbb{C})$:

$$\begin{aligned} H:\Big(H^{3,0}(X_t,\,\mathbb{C})\oplus H^{2,1}_{[\gamma]}(X_t,\,\mathbb{C})\Big) \quad &\times \quad \Big(H^{3,0}(X_t,\,\mathbb{C})\oplus H^{2,1}_{[\gamma]}(X_t,\,\mathbb{C})\Big)\longrightarrow\mathbb{C},\\ (\{u\},\,\{v\}) \quad &\mapsto \quad -i\int\limits_X u\wedge\bar{v}.\end{aligned}$$

Its signature is $(-,\,-,\,+,\,+,\,+)$ (cf. Corollary 6.4.3).
The second sesquilinear intersection form is obtained by restriction from $H^2_{DR}(B,\,\mathbb{C})$ (where B is the differentiable manifold underlying the tori B_t) when $H^{2,0}(B_t,\,\mathbb{C})\oplus H^{1,1}(B_t,\,\mathbb{C})$ is viewed as a vector subspace of $H^2_{DR}(B,\,\mathbb{C})$:

$$\begin{aligned} H_B:\Big(H^{2,0}(B_t,\,\mathbb{C})\oplus H^{1,1}(B_t,\,\mathbb{C})\Big) \quad &\times \quad \Big(H^{2,0}(B_t,\,\mathbb{C})\oplus H^{1,1}(B_t,\,\mathbb{C})\Big)\longrightarrow\mathbb{C},\\ (\{\xi\},\,\{\zeta\}) \quad &\mapsto \quad \int\limits_B \xi\wedge\bar{\zeta}.\end{aligned}\tag{6.73}$$

Indeed, the coefficient of the integral $\int_B\xi\wedge\bar\zeta$ in the defintion of the sesquilinear intersection form in degree n on an n-dimensional compact complex manifold is $(-1)^{\frac{n(n+1)}{2}}\,i^n$, so in the case of H_B, where $n=\dim_{\mathbb{C}}B_t=2$, this coefficient equals 1.

In particular, on the vector space

$$H^{2,0}(B_0,\,\mathbb{C})\oplus H^{1,1}(B_0,\,\mathbb{C})\overset{\cdot\wedge\gamma}{\simeq}H^{3,0}(X_0,\,\mathbb{C})\oplus H^{2,1}_{[\gamma]}(X_0,\,\mathbb{C}),\tag{6.74}$$

the two sesquilinear intersection forms are given by

$$H_B(\{u\},\,\{v\})=\int\limits_B u\wedge\bar{v}\quad\text{ and }\quad H(\{u\wedge\gamma\},\,\{v\wedge\gamma\})=-\int\limits_B(u\wedge\bar{v})\wedge(i\gamma\wedge\bar\gamma).$$

Proposition 6.5.15. *The signature of the sesquilinear intersection form H_B defined in (6.73) is $(+,\,+,\,-,\,-,\,-)$.*

Specifically, for any Hermitian metric ρ_t on B_t, $H^{2,0}(B_t,\,\mathbb{C})\subset H^2_+(B_t,\,\mathbb{C})$, while H_B has signature $(+,\,-,\,-,\,-)$ on $H^{1,1}(B_t,\,\mathbb{C})$.

Proof. Every class in $H^{2,0}(B_t,\,\mathbb{C})$ has a unique representative which, for bidegree reasons, is a *primitive* $(2,\,0)$-form w.r.t. any Hermitian metric we equip B_t with. Thus, the standard formula (4.68) applied with $p+q=n$ and $(p,\,q)=(2,\,0)$ yields $\star v=v$. Therefore, $H^{2,0}(B_t,\,\mathbb{C})\subset H^2_+(B_t,\,\mathbb{C})$.

Now, recall that $H^{1,1}(B_t,\,\mathbb{C})$ is generated by the classes $[\alpha_t\wedge\bar\alpha_t]_{\bar\partial}$, $[\alpha_t\wedge\bar\beta_t]_{\bar\partial}$, $[\beta_t\wedge\bar\alpha_t]_{\bar\partial}$, $[\beta_t\wedge\bar\beta_t]_{\bar\partial}$. Let us equip B_t with the Hermitian metric

$$\rho_t:=i\alpha_t\wedge\bar\alpha_t+i\beta_t\wedge\bar\beta_t,\qquad t\in B.$$

The associated volume form is $dV_{\rho_t}=\rho_t^2/2!=i\alpha_t\wedge\bar\alpha_t\wedge i\beta_t\wedge\bar\beta_t$. Denoting by $\star=\star_{\rho_t}$ the Hodge star operator induced by ρ_t, we can check as in Lemma 6.4.2 that the following identities hold

$$\begin{aligned}\star\,(i\alpha_t\wedge\bar\alpha_t)=i\beta_t\wedge\bar\beta_t,\qquad&\star\,(i\beta_t\wedge\bar\beta_t)=i\alpha_t\wedge\bar\alpha_t,\\ \star\,(i\alpha_t\wedge\bar\beta_t)=-i\alpha_t\wedge\bar\beta_t,\qquad&\star\,(i\beta_t\wedge\bar\alpha_t)=-i\beta_t\wedge\bar\alpha_t\end{aligned}\tag{6.75}$$

for every $t\in B$.

Indeed, from the definition of the Hodge star operator, we know that

$$u\wedge\overline{\star\,(i\alpha_t\wedge\bar\alpha_t)}=\langle u,\,i\alpha_t\wedge\bar\alpha_t\rangle\,dV_{\rho_t}.$$

When u is the product of a form chosen from α_t, β_t and a form chosen from $\bar\alpha_t$, $\bar\beta_t$, the two sides of this identity are non-zero only when $u=i\alpha_t\wedge\bar\alpha_t$. In this case, we get

$$(i\alpha_t\wedge\bar\alpha_t)\wedge\overline{\star\,(i\alpha_t\wedge\bar\alpha_t)}=i\alpha_t\wedge\bar\alpha_t\wedge i\beta_t\wedge\bar\beta_t,$$

so $\overline{\star\,(i\alpha_t\wedge\bar\alpha_t)}$ must be the form complementary to $i\alpha_t\wedge\bar\alpha_t$. We get $\star\,(i\alpha_t\wedge\bar\alpha_t)=i\beta_t\wedge\bar\beta_t$. The remaining identities in (6.75) are proved in an analogous way.

The last two identities in (6.75) show that $i\alpha_t\wedge\bar\beta_t$ and $i\beta_t\wedge\bar\alpha_t$ are eigenvectors of $\star$ corresponding to the eigenvalue -1, so they represent classes lying in $H^2_-(B_t,\,\mathbb{C})$. Meanwhile, the first two identities in (6.75) can be re-written as

$$\star\,(i\alpha_t\wedge\bar\alpha_t+i\beta_t\wedge\bar\beta_t)=i\alpha_t\wedge\bar\alpha_t+i\beta_t\wedge\bar\beta_t\quad\text{and}\quad\star\,(i\alpha_t\wedge\bar\alpha_t-i\beta_t\wedge\bar\beta_t)=-(i\alpha_t\wedge\bar\alpha_t-i\beta_t\wedge\bar\beta_t).$$

Therefore, $i\alpha_t\wedge\bar\alpha_t+i\beta_t\wedge\bar\beta_t$ represents a class lying in $H^2_+(B_t,\,\mathbb{C})$ and $i\alpha_t\wedge\bar\alpha_t-i\beta_t\wedge\bar\beta_t$ represents a class lying in $H^2_-(B_t,\,\mathbb{C})$. $\square$

A consequence of these considerations is that Proposition 6.4.4 can now be used to construct coordinates on the complexification $\widetilde{\mathcal{G}_0}\subset\widetilde{\mathcal{G}_{X_0}}$ of the parameter set $\mathcal{G}_0=\{[(\omega_t^{1,1})^2]_A\,|\,t\in B_{[\gamma]}\}\subset\mathcal{G}_{X_0}$ using the symplectic vector space $(H^2(B,\,\mathbb{C}),\,Q_B(\cdot\,,\,\cdot))$ equipped with the bilinear intersection form $Q_B:H^2(B,\,\mathbb{C})\times H^2(B,\,\mathbb{C})\to\mathbb{C}$ defined by $Q_B(\{u\},\,\{v\}):=-\int_B u\wedge v$. Consider the following

Setup. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold $X=X_0$ and let $(B_t)_{t\in B}$ be the associated family of 2-dimensional Albanese tori. Let $v=(v_t)_{t\in B_{[\gamma]}}$ be a holomorphic section of the vector bundle $B_{[\gamma]}\ni t\mapsto H^{2,0}(B_t,\,\mathbb{C})$ such that each $(2,\,0)$-form v_t is non-vanishing on B_t. (We may choose $v_t:=\alpha_t\wedge\beta_t$.)*

Let $\eta_0=\eta_0^{3,0}+\eta_0^{2,1}+\overline{\eta_0^{2,1}}+\overline{\eta_0^{3,0}}\in H^3(X,\,\mathbb{R})$ be a real class with $\eta_0^{3,0}\in H^{3,0}(X_0,\,\mathbb{C}),\eta_0^{2,1}\in H^{2,1}_{[\gamma]}(X_0,\,\mathbb{C})$ satisfying conditions (6.40) of Proposition 6.4.4. Thanks to isomorphism (6.74), there exist unique classes

$$\eta_{0,\,B}^{2,\,0}\in H^{2,\,0}(B_0,\,\mathbb{C})\quad\textit{and}\quad\eta_{0,\,B}^{1,\,1}\in H^{1,\,1}(B_0,\,\mathbb{C})$$

such that $\eta_0^{3,\,0}=\eta_{0,\,B}^{2,\,0}\wedge\gamma$ and $\eta_0^{2,\,1}=\eta_{0,\,B}^{1,\,1}\wedge\gamma$. Put

$$\eta_{0,\,B}:=\eta_{0,\,B}^{2,\,0}+\frac{\eta_{0,\,B}^{1,\,1}+\overline{\eta_{0,\,B}^{1,\,1}}}{2}+\overline{\eta_{0,\,B}^{2,\,0}}\in H^2(B_0,\,\mathbb{R}).$$

Complete $\eta_{0,\,B}$ to a symplectic basis $\{\eta_{0,\,B},\eta_{1,\,B},\dots,\eta_{4,\,B},\,\nu_{0,\,B},\nu_{1,\,B},\dots,\nu_{4,\,B}\}$ of $(H^2(B_0,\,\mathbb{R}),\,Q_B(\cdot\,,\,\cdot))$. Normalise such that

$$Q_B(v_t,\,\eta_{0,\,B})=1\quad\textit{for all } t\in B\quad\textit{sufficiently close to } 0.$$

We can now state the result we have been aiming at.

Proposition 6.5.16. *In the setup described above, the functions*

$$w_i(t) := Q_B(v_t,\, \eta_{i,\,B}) \quad \textit{for } t \in B_{[\gamma]} \quad \textit{and } i \in \{1, \dots , 4\}$$

define **holomorphic coordinates** *on $B_{[\gamma]}$ in a neighbourhood of 0 and implicitly on the complexified parameter set $\widetilde{\mathcal{G}}_0$, the complexification of*

$$\mathcal{G}_0 = \left\{[(\omega_t^{1,\,1})^2]_A \;\middle|\; t \in B_{[\gamma]}\right\} \subset \mathcal{G}_{X_0},$$

in a neighbourhood of $[\omega_0^2]_A$.

Proof. It runs along the lines of the proof of Proposition 6.4.4. □

6.6 The mirror map

We can now associate with every small deformation X_t of X_0 an element in the Gauduchon cone of X_0 in which the canonical class $[\omega_0^2]_A = [(\omega_0^{1,\,1})^2]_A$ is a marked point.

Definition 6.6.1. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold $X = X_0$ and let $(\omega_t^{1,\,1})_{t\in B_{[\gamma]}}$ be the smooth family of canonical Gauduchon metrics on X_0 constructed in Proposition 6.5.2. For every $t \in B_{[\gamma]}$, let $[(\omega_t^{1,\,1})^2]_A \in \mathcal{G}_{X_0} = \mathcal{G}_X$ be the associated Aeppli cohomology class.*

We define the **positive mirror map** *of $X = X_0$ by*

$$\mathcal{M} : B_{[\gamma]} \longrightarrow \mathcal{G}_X, \qquad t \mapsto [(\omega_t^{1,\,1})^2]_A, \tag{6.76}$$

where $\mathcal{G}_X$ is the Gauduchon cone of $X = X_0$ (i.e. the open subset of $H_A^{2,\,2}(X,\,\mathbb{R})$ consisting of real **positive** *classes). Thus, the parameter subset of the Gauduchon cone of X defined in (6.66) is $\mathcal{G}_0 = \mathcal{M}(B_{[\gamma]})$.*

From (6.46) and from Lemma 6.5.3 we get the following formula for the positive mirror map after recalling that $t_{3,\,1} = t_{3,\,2} = 0$ when $t \in B_{[\gamma]}$:

$$\begin{aligned}
\mathcal{M}(t) \;\; = \;\; & 2\left(1 - |t_{11}|^2 - |t_{21}|^2\right)\left(1 - |t_{11}\,t_{22} - t_{12}\,t_{21}|^2\right)\left[i\,\alpha\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma\right]_A \\
+ \;\; & 2\left(1 - |t_{12}|^2 - |t_{22}|^2\right)\left(1 - |t_{11}\,t_{22} - t_{12}\,t_{21}|^2\right)\left[i\,\beta\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma\right]_A \\
- \;\; & 2\left(t_{12}\,\bar t_{11} + t_{22}\,\bar t_{21}\right)\left(1 - |t_{11}\,t_{22} - t_{12}\,t_{21}|^2\right)\left[i\,\alpha\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma\right]_A \\
- \;\; & 2\left(t_{11}\,\bar t_{12} + t_{21}\,\bar t_{22}\right)\left(1 - |t_{11}\,t_{22} - t_{12}\,t_{21}|^2\right)\left[i\,\beta\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma\right]_A, \qquad t \in B_{[\gamma]}.
\end{aligned} \tag{6.77}$$

Alternatively, formula (6.47) yields for every $t \in B_{[\gamma]}$

$$\begin{aligned}\mathcal{M}(t) &= [\omega_0^2]_A \\ &+ 2\bigg(c_1(t)+c_3(t)+c_1(t)\,c_3(t)\bigg)\bigg[i\,\alpha\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma\bigg]_A \\ &+ 2\bigg(c_2(t)+c_3(t)+c_2(t)\,c_3(t)\bigg)\bigg[i\,\beta\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma\bigg]_A \\ &+ 2\,d(t)\bigg(1+c_3(t)\bigg)\bigg[i\,\alpha\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma\bigg]_A + 2\,\overline{d(t)}\bigg(1+c_3(t)\bigg)\bigg[i\,\beta\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma\bigg]_A\end{aligned} \tag{6.78}$$

where $c_j(t)$ and $d(t)$ are defined by (6.46) with $t_{3,1}=t_{3,2}=0$ when $t\in B_{[\gamma]}$.

Since $B_{[\gamma]}$ is an open subset in a vector space of complex dimension 4 (see Definition 6.2.2) while $\mathcal{G}_X$ is an open subset in a vector space of real dimension 4, we rebalance the two sides of (6.76) by complexifying the latter set.

Definition 6.6.2. *Let $X=X_0$ be the Iwasawa manifold.*

(i) We know from (6.58) that $H^{2,2}_A(X_0,\,\mathbb{C})$ injects canonically (and $\mathbb{C}$-linearly) into $H^4_{DR}(X,\,\mathbb{C})$. Similarly, $H^{2,2}_A(X_0,\,\mathbb{R})$ injects canonically (and $\mathbb{R}$-linearly) into $H^4_{DR}(X,\,\mathbb{R})$. On the other hand, we know that the image of $H^4(X,\,\mathbb{Z})$ in $H^4_{DR}(X,\,\mathbb{R})$ under the natural map $H^4(X,\,\mathbb{Z})\hookrightarrow H^4_{DR}(X,\,\mathbb{R})$ is a **lattice**. *We put*

$$H^{2,2}_A(X_0,\,\mathbb{Z}) := H^{2,2}_A(X_0,\,\mathbb{R})\cap H^4(X,\,\mathbb{Z})\subset H^{2,2}_A(X_0,\,\mathbb{R}).$$

Thus $H^{2,2}_A(X_0,\,\mathbb{Z})$ is a **lattice** *in $H^{2,2}_A(X_0,\,\mathbb{R})$.*

(ii) We define the **complexified Gauduchon cone** *of the Iwasawa manifold $X=X_0$ by*

$$\widetilde{\mathcal{G}}_{X_0} := \mathcal{G}_{X_0}\oplus H^{2,2}_A(X_0,\,\mathbb{R})/2\pi i\,H^{2,2}_A(X_0,\,\mathbb{Z}).$$

(iii) We define the **mirror map** *$\widetilde{\mathcal{M}}: B_{[\gamma]}\longrightarrow\widetilde{\mathcal{G}}_{X_0}$ of $X=X_0$ by*

$$\begin{aligned}\widetilde{\mathcal{M}}(t) &= [\omega_0^2]_A - t_{11}\,[i\,\beta\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma]_A + t_{22}\,[i\,\alpha\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma]_A \\ &- t_{12}\,[i\,\beta\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma]_A + t_{21}\,[i\,\alpha\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma]_A.\end{aligned}$$

Thus, the positive mirror map $\mathcal{M}$ is a kind of "squared absolute value" of $\widetilde{\mathcal{M}}$.

(iv) We define the **complexified parameter set** *by $\widetilde{\mathcal{G}}_0 := \widetilde{\mathcal{M}}(B_{[\gamma]})$. It contains the marked point $[\omega_0^2]_A$ of the Gauduchon cone $\mathcal{G}_X$.*

Thus, if the radius of $B_{[\gamma]}$ as an open ball about the origin in $H^{0,1}(X,\,T^{1,0}_X)$ is small enough, $\widetilde{\mathcal{M}}$ defines a biholomorphism between $B_{[\gamma]}$ and the open subset $\widetilde{\mathcal{G}}_0\subset\widetilde{\mathcal{G}}_X\subset H^{2,2}_A(X_0,\,\mathbb{C})$.

Our discussion can be summed up as follows.

Theorem 6.6.3. *The mirror map $\widetilde{\mathcal{M}}: B_{[\gamma]}\longrightarrow\widetilde{\mathcal{G}}_X$ of the Iwasawa manifold $X=X_0$ enjoys the following properties.*

(i) $\widetilde{\mathcal{M}}$ is holomorphic and defines a biholomorphism onto its image if the radius of $B_{[\gamma]}$ as an open ball in $H^{0,1}(X_0,\,T^{1,0}X_0)$ is small enough;

(ii) $\widetilde{\mathcal{M}}(0) = [\omega_0^2]_A \in \mathcal{G}_X$*, where* ω_0 *is the Gauduchon metric on* X *canonically induced by the complex parallelisable structure of* X *(cf. (6.19));*

(iii) The composition of the canonical isomorphism A_t *observed in (6.60) with* B_t *defined in (6.56) and with the Kodaira-Spencer and the Calabi-Yau isomorphisms is the following canonical isomorphism*

$$T_t^{1,0}B_{[\gamma]} \simeq H_{[\gamma]}^{2,1}(X_t,\,\mathbb{C}) \xrightarrow[A_t]{\simeq} H_A^{2,2}(X_t,\,\mathbb{C}) \xrightarrow[B_t]{\simeq} H_A^{2,2}(X_0,\,\mathbb{C}) = T^{1,0}_{\widetilde{\mathcal{M}}(t)}\widetilde{\mathcal{G}_X}, \qquad [\Gamma]_{\bar\partial} \mapsto [\Gamma\wedge\bar\gamma_t]_A = A_t([\Gamma]_{\bar\partial}),$$

that coincides at $t=0$ *with the differential map* $d\widetilde{\mathcal{M}}_0$ *of* $\widetilde{\mathcal{M}}$ *and depends anti-holomorphically on* t*;*

(iv) On the **metric side of the mirror***, there is a variation of Hodge structures (VHS)*

$$\mathcal{H}^3\oplus\mathcal{H}^4 \supset F_{\mathcal{G}}\mathcal{H}^4 = \mathcal{H}^{2,0}(B)\oplus\widetilde{\mathcal{H}^{2,2}_\omega} \supset F'_{\mathcal{G}}\mathcal{H}^4 = \mathcal{H}^{2,0}(B)$$

parametrised by $\widetilde{\mathcal{G}}_0 = \widetilde{\mathcal{M}}(B_{[\gamma]}) \simeq B_{[\gamma]}$ *whose* 4*-dimensional fibre over any point* $\widetilde{\mathcal{M}}(t)\in\widetilde{\mathcal{G}}_0$ *is the vector subspace* $\widetilde{H^{2,2}_{\omega_t}} := Q_{\omega_t}(H_A^{2,2}(X_t,\,\mathbb{C})) \subset H^4(X,\,\mathbb{C})$ *defined in Conclusion 6.5.12. Moreover, there exists a* C^∞ *isomorphism of VHS between this VHS and the VHS*

$$\mathcal{H}^3 \supset F^2\mathcal{H}^3_{[\gamma]} \supset F^3\mathcal{H}^3$$

parametrised by $B_{[\gamma]}$ *and defined on the* **complex-structure side of the mirror** *in Theorem 6.3.10.*

This isomorphism is **holomorphic** *between the* 1*-dimensional parts* $\mathcal{H}^{2,0}(B)$*, resp.* $F^3\mathcal{H}^3$ *(it is the multiplication by* γ_t*), while the isomorphism between the rank-*4 *vector bundles* $\mathcal{H}^{2,1}_{[\gamma]}$ *and* $\mathcal{H}^{2,2}_A$ *(defining, up to identifications, the* 4*-dimensional parts of these VHS's) is* **anti-holomorphic** *(given by the* A_t*'s, the multiplication by* $\bar\gamma_t$*).*

Moreover, each of the two Hodge filtrations $F^2\mathcal{H}^3_{[\gamma]}\supset F^3\mathcal{H}^3$ *and* $F_{\mathcal{G}}\mathcal{H}^4\supset F'_{\mathcal{G}}\mathcal{H}^4$ *is* C^∞ *isomorphic to the Hodge filtration* $F^1\mathcal{H}^2(B)\supset F^2\mathcal{H}^2(B)$ *associated with the family* $(B_t)_{t\in B_{[\gamma]}}$ *of Albanese tori of the small essential deformations* $(X_t)_{t\in B_{[\gamma]}}$ *of the Iwasawa manifold* $X=X_0$*.*

(v) There is a bijection

$$\Big(z_1(t),\,z_2(t),\,z_3(t),\,z_4(t)\Big)\mapsto\Big(w_1(t),\,w_2(t),\,w_3(t),\,w_4(t)\Big), \qquad t\in B_{[\gamma]} \tag{6.79}$$

depending holomorphically on t *between the holomorphic coordinates defined in Proposition 6.4.4 on* $B_{[\gamma]}$ *in a neighbourhood of* 0 *and the holomorphic coordinates defined in Proposition 6.5.16 on* $\{[(\omega_t^{1,1})^2]_A\,/\,t\in B_{[\gamma]}\}\subset\mathcal{G}_{X_0}$ *in a neighbourhood of* $[\omega_0^2]_A$*.*

Proof. (i) and (ii) follow from the construction. To prove (iii), we start by recalling that with the notation $\alpha_1:=\alpha$, $\alpha_2:=\beta$, $\xi_1:=\xi_\alpha$, $\xi_2:=\xi_\beta$, $\xi_3:=\xi_\gamma$, the space $H^{0,1}(X,\,T^{1,0}X)$ consists of the objects

$$\sum_{\substack{i=1,2,3\\ \lambda=1,2}} t_{i\lambda}\,\xi_i\otimes\bar\alpha_\lambda$$

where the $t_{i\lambda}$ define holomorphic coordinates on Δ. Also recall that $t_{31}=t_{32}=0$ on $B_{[\gamma]}$. Thus, the holomorphic tangent space to $B_{[\gamma]}$ at 0 is generated by $\partial/\partial t_{11}$, $\partial/\partial t_{12}$, $\partial/\partial t_{21}$, $\partial/\partial t_{22}$ and the images of these vector fields under the composition of the Kodaira-Spencer map ρ with the Calabi-Yau isomorphism T_Ω $(=\cdot\lrcorner(\alpha\wedge\beta\wedge\gamma))$

$$T_0^{1,0}B_{[\gamma]} \xrightarrow[\simeq]{\rho} H^{0,1}_{[\gamma]}(X,\,T^{1,0}X) \xrightarrow[\simeq]{T_\Omega} H^{2,1}_{[\gamma]}(X,\,\mathbb{C})$$

(cf. (6.7)) are spelt out as follows

$$\frac{\partial}{\partial t_{11}}\mapsto[\xi_1\otimes\bar\alpha_1=\xi_\alpha\otimes\bar\alpha]\mapsto-[\beta\wedge\gamma\wedge\bar\alpha]_{\bar\partial},\qquad\frac{\partial}{\partial t_{12}}\mapsto[\xi_1\otimes\bar\alpha_2=\xi_\alpha\otimes\bar\beta]\mapsto-[\beta\wedge\gamma\wedge\bar\beta]_{\bar\partial},$$
$$\frac{\partial}{\partial t_{21}}\mapsto[\xi_2\otimes\bar\alpha_1=\xi_\beta\otimes\bar\alpha]\mapsto[\alpha\wedge\gamma\wedge\bar\alpha]_{\bar\partial},\qquad\frac{\partial}{\partial t_{22}}\mapsto[\xi_2\otimes\bar\alpha_2=\xi_\beta\otimes\bar\beta]\mapsto[\alpha\wedge\gamma\wedge\bar\beta]_{\bar\partial}.\tag{6.80}$$

We get

$$\begin{aligned}
d\widetilde{\mathcal{M}}\Big(\frac{\partial}{\partial t_{11}}\Big) &= \frac{\partial\widetilde{\mathcal{M}}}{\partial t_{11}}=-[i\,\beta\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma]_A=A_0(-[\beta\wedge\gamma\wedge\bar\alpha]_{\bar\partial})\simeq A_0\Big(\frac{\partial}{\partial t_{11}}\Big),\\
d\widetilde{\mathcal{M}}\Big(\frac{\partial}{\partial t_{22}}\Big) &= \frac{\partial\widetilde{\mathcal{M}}}{\partial t_{22}}=[i\,\alpha\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma]_A=A_0([\alpha\wedge\gamma\wedge\bar\beta]_{\bar\partial})\simeq A_0\Big(\frac{\partial}{\partial t_{22}}\Big),\\
d\widetilde{\mathcal{M}}\Big(\frac{\partial}{\partial t_{12}}\Big) &= \frac{\partial\widetilde{\mathcal{M}}}{\partial t_{12}}=-\,[i\,\beta\wedge\bar\beta\wedge i\,\gamma\wedge\bar\gamma]_A=A_0(-[\beta\wedge\gamma\wedge\bar\beta]_{\bar\partial})\simeq A_0\Big(\frac{\partial}{\partial t_{12}}\Big),\\
d\widetilde{\mathcal{M}}\Big(\frac{\partial}{\partial t_{21}}\Big) &= \frac{\partial\widetilde{\mathcal{M}}}{\partial t_{21}}=[i\,\alpha\wedge\bar\alpha\wedge i\,\gamma\wedge\bar\gamma]_A=A_0([\alpha\wedge\gamma\wedge\bar\alpha]_{\bar\partial})\simeq A_0\Big(\frac{\partial}{\partial t_{21}}\Big),
\end{aligned}$$

where $\simeq$ stands for the identifications under (6.80).

We conclude that $d\widetilde{\mathcal{M}}_0=A_0$, so part (iii) is proved at $t=0$.

(iv) is contained in Theorem 6.3.10, Corollary 6.3.11 and Conclusion 6.5.12.

(v) is contained in Propositions 6.4.4 and 6.5.16.

□

6.7 Further computations in the case of the Iwasawa manifold

We spell out the details of the computations of the first-order anti-holomorphic partial derivatives of the forms $\Gamma_j(t)$ defined in (6.20) for $j\in\{1,2,3,4\}$.

Recall the following identities proved in (6.42):

$$\alpha_t=\alpha+t_{11}\,\bar\alpha+t_{12}\,\bar\beta,\quad\beta_t=\beta+t_{21}\,\bar\alpha+t_{22}\,\bar\beta,\quad\gamma_t=\gamma+t_{31}\,\bar\alpha+t_{32}\,\bar\beta-D(t)\,\bar\gamma.$$

So we get

$$\begin{aligned}
\Gamma_1(t) &= (\alpha+t_{11}\,\bar\alpha+t_{12}\,\bar\beta)\wedge(\gamma+t_{31}\,\bar\alpha+t_{32}\,\bar\beta-D(t)\,\bar\gamma)\wedge(\bar\alpha+\bar t_{11}\,\alpha+\bar t_{12}\,\beta)\\
&- \frac{\sigma_{2\bar2}(t)}{\bar\sigma_{12}(t)}\,(\alpha+t_{11}\,\bar\alpha+t_{12}\,\bar\beta)\wedge(\beta+t_{21}\,\bar\alpha+t_{22}\,\bar\beta)\wedge(\bar\gamma+\bar t_{31}\,\alpha+\bar t_{32}\,\beta-\overline{D(t)}\,\gamma)\\
&= -\Big[\alpha\wedge\bar\alpha+\bar t_{12}\,\alpha\wedge\beta-|t_{11}|^2\,\alpha\wedge\bar\alpha+t_{11}\,\bar t_{12}\,\bar\alpha\wedge\beta-t_{12}\,\bar\alpha\wedge\bar\beta-t_{12}\,\bar t_{11}\,\alpha\wedge\bar\beta-|t_{12}|^2\,\beta\wedge\bar\beta\Big]\\
&\qquad\qquad\wedge(\gamma+t_{31}\,\bar\alpha+t_{32}\,\bar\beta-D(t)\,\bar\gamma)\\
&- \frac{\sigma_{2\bar2}(t)}{\bar\sigma_{12}(t)}\Big[\alpha\wedge\beta+t_{21}\,\alpha\wedge\bar\alpha+t_{22}\,\alpha\wedge\bar\beta+t_{11}\,\bar\alpha\wedge\beta+t_{11}\,t_{22}\,\bar\alpha\wedge\bar\beta-t_{12}\,\beta\wedge\bar\beta-t_{12}\,t_{21}\,\bar\alpha\wedge\bar\beta\Big]\\
&\qquad\qquad\wedge(\bar\gamma+\bar t_{31}\,\alpha+\bar t_{32}\,\beta-\overline{D(t)}\,\gamma).
\end{aligned}$$

After expanding and grouping the terms, we get

Lemma 6.7.1. *For every $t \in B_{[\gamma]}$, the J_t-(2, 1)-form $\Gamma_1(t)$ of (6.20) is explicitly given by the following formula in terms of a basis of 3-forms generated by α, β, γ and their conjugates:*

$$\begin{aligned}
\Gamma_1(t) &= -\bar{t}_{12}\,\alpha\wedge\beta\wedge\gamma \\
&- D(t)\left(t_{12}+\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\right)\bar{\alpha}\wedge\bar{\beta}\wedge\bar{\gamma} - \left(1-|t_{11}|^2-\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,t_{21}\,\overline{D(t)}\right)\alpha\wedge\bar{\alpha}\wedge\gamma \\
&- \left[t_{32}\,(1-|t_{11}|^2)+t_{12}\,\bar{t}_{11}\,t_{31}-\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,D(t)\,\bar{t}_{31}\right]\alpha\wedge\bar{\alpha}\wedge\bar{\beta} \\
&- \left[\bar{t}_{12}\,t_{31}-\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,(t_{21}\,\bar{t}_{32}+t_{11}\,\bar{t}_{31})\right]\alpha\wedge\beta\wedge\bar{\alpha} - \left[(|t_{11}|^2-1)\,D(t)+\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,t_{21}\right]\alpha\wedge\bar{\alpha}\wedge\bar{\gamma} \\
&- \left[\bar{t}_{12}\,t_{32}-\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,(t_{22}\,\bar{t}_{32}+t_{12}\,\bar{t}_{31})\right]\alpha\wedge\beta\wedge\bar{\beta} + \left[\bar{t}_{12}\,D(t)-\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\right]\alpha\wedge\beta\wedge\bar{\gamma} \\
&- \left[t_{11}\,\bar{t}_{12}-\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,t_{11}\,\overline{D(t)}\right]\bar{\alpha}\wedge\beta\wedge\gamma - \left[t_{11}\,\bar{t}_{12}\,t_{32}-|t_{12}|^2\,t_{31}-\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,D(t)\,\bar{t}_{32}\right]\bar{\alpha}\wedge\beta\wedge\bar{\beta} \\
&+ \left[D(t)\,t_{11}\,\bar{t}_{12}-\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,t_{11}\right]\bar{\alpha}\wedge\beta\wedge\bar{\gamma} + \left[t_{12}+\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,|D(t)|^2\right]\bar{\alpha}\wedge\bar{\beta}\wedge\gamma \\
&+ \left[t_{12}\,\bar{t}_{11}+\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,t_{22}\,\overline{D(t)}\right]\alpha\wedge\bar{\beta}\wedge\gamma + \frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,\overline{D(t)}\,\alpha\wedge\beta\wedge\gamma + \left[t_{12}\,\bar{t}_{11}\,D(t)+\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,t_{22}\right]\alpha\wedge\bar{\beta}\wedge\bar{\gamma} \\
&+ \left[|t_{12}|^2-\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,t_{12}\,\overline{D(t)}\right]\beta\wedge\bar{\beta}\wedge\gamma - \left[|t_{12}|^2\,D(t)-\frac{\sigma_{2\bar{2}}(t)}{\bar{\sigma}_{12}(t)}\,t_{12}\right]\beta\wedge\bar{\beta}\wedge\bar{\gamma}.
\end{aligned}$$

Analogous formulae hold for the J_t-(2, 1)-forms $\Gamma_2(t)$, $\Gamma_3(t)$, $\Gamma_4(t)$ of (6.20). Each formula contains on the r.h.s. a single term featuring an isolated anti-holomorphic factor $\bar{t}_{i\lambda}$ (i.e. an anti-holomorphic factor $\bar{t}_{i\lambda}$ that is not multiplied by any other $t_{j\mu}$ or $\bar{t}_{j\mu}$). These terms are, respectively,

$$-\bar{t}_{22}\,\alpha\wedge\beta\wedge\gamma, \quad \bar{t}_{11}\,\alpha\wedge\beta\wedge\gamma, \quad \bar{t}_{21}\,\alpha\wedge\beta\wedge\gamma.$$

On the other hand, the dependence on t of the C^∞ functions $\sigma_{12}(t)$, $\sigma_{1\bar{1}}(t), \sigma_{1\bar{2}}(t), \sigma_{2\bar{1}}(t), \sigma_{2\bar{2}}(t)$ can be made explicit using computations from [Ang14]. Indeed, consider the following functions of t (cf. [Ang14, p. 76] where the notation α, β, γ was used instead of $a(t), b(t), c(t)$ featuring below):

$$\begin{aligned}
&a(t) = \frac{1}{1-|t_{22}|^2-t_{21}\,\bar{t}_{12}}, \quad b(t) = t_{21}\,\bar{t}_{11}+t_{22}\,\bar{t}_{21}, \quad c(t) = \frac{1}{1-|t_{11}|^2-a(t)\,b(t)\,(t_{11}\,\bar{t}_{12}+t_{12}\,\bar{t}_{22})-\bar{t}_{12}\,t_{21}}, \\
&\lambda_1(t) = -t_{11}\,(1+a(t)\,\bar{t}_{12}\,t_{21}+a(t)\,|t_{22}|^2), \quad \lambda_2(t) = a(t)\,(t_{11}\,\bar{t}_{12}+t_{12}\,\bar{t}_{22}), \\
&\lambda_3(t) = -t_{12}\,(1+a(t)\,\bar{t}_{12}\,t_{21}+a(t)\,|t_{22}|^2), \quad \mu_0(t) = b(t)\,c(t), \quad \mu_1(t) = \lambda_1(t)\,b(t)\,c(t)-t_{21}, \\
&\mu_2(t) = 1+\lambda_2(t)\,b(t)\,c(t), \quad \mu_3(t) = \lambda_3(t)\,b(t)\,c(t)-t_{22}.
\end{aligned}$$

Then, for all t in Nakamura's class (ii), we have the explicit formulae (cf. e.g. [Ang14, p.77]):

$$\sigma_{12}(t) = -c(t)+t_{21}\,\bar{\lambda}_3(t)\,\bar{c}(t)+t_{22}\,\bar{a}(t)\,\bar{\mu}_3(t),$$

and

$$\sigma_{1\bar{1}}(t) = t_{21}\,\overline{c(t)\,(1 + t_{21}\,\bar{t}_{12}\,a(t) + |t_{22}|^2\,a(t))}, \qquad \sigma_{1\bar{2}}(t) = t_{22}\,\overline{c(t)\,(1 + t_{21}\,\bar{t}_{12}\,a(t) + |t_{22}|^2\,a(t))},$$
$$\sigma_{2\bar{1}}(t) = -t_{11}\,c(t)\,(1 + t_{21}\,\bar{t}_{12}\,a(t) + |t_{22}|^2\,a(t)), \qquad \sigma_{2\bar{2}}(t) = -t_{12}\,c(t)\,(1 + t_{21}\,\bar{t}_{12}\,a(t) + |t_{22}|^2\,a(t)).$$

This explicitly yields

$$\sigma_{2\bar{2}}(t) = -t_{12}\,\frac{1 + \frac{t_{21}\,\bar{t}_{12}}{1-|t_{22}|^2-t_{21}\,\bar{t}_{12}} + \frac{|t_{22}|^2}{1-|t_{22}|^2-t_{21}\,\bar{t}_{12}}}{1 - |t_{11}|^2 - \frac{(t_{11}\,\bar{t}_{12}+t_{12}\,\bar{t}_{22})(t_{21}\,\bar{t}_{11}+t_{22}\,\bar{t}_{21})}{1-|t_{22}|^2-t_{21}\,\bar{t}_{12}} - t_{12}\,\bar{t}_{21}} \tag{6.81}$$

and analogous formulae for $\sigma_{1\bar{1}}(t)$, $\sigma_{1\bar{2}}(t)$, $\sigma_{2\bar{1}}(t)$ with a different holomorphic factor $\pm t_{i\lambda}$ and a possibly conjugated big fraction.

The conclusion is the following

Lemma 6.7.2. *For all t in Nakamura's class (ii) and for all i, λ, we have*

$$\frac{\partial\sigma_{1\bar{1}}}{\partial\bar{t}_{i\lambda}}(0) = \frac{\partial\sigma_{1\bar{2}}}{\partial\bar{t}_{i\lambda}}(0) = \frac{\partial\sigma_{2\bar{1}}}{\partial\bar{t}_{i\lambda}}(0) = \frac{\partial\sigma_{2\bar{2}}}{\partial\bar{t}_{i\lambda}}(0) = 0. \tag{6.82}$$

The same conclusion holds for all t in Nakamura's class (iii), hence in particular for all $t \in B_{[\gamma]}$, by very similar computations.

Proof. Whenever some $\bar{t}_{i\lambda}$ features in formula (6.81) for $\sigma_{2\bar{2}}(t)$ or in one of its analogues for $\sigma_{1\bar{1}}(t)$, $\sigma_{1\bar{2}}(t)$ and $\sigma_{2\bar{1}}(t)$, it is multiplied by a factor $t_{j\mu}$ or $\bar{t}_{j\mu}$ which vanishes at $t = 0$, while the denominators on the r.h.s. of (6.81) equal 1 at $t = 0$. $\square$

6.8 Essential deformations of page-1-$\partial\bar{\partial}$-manifolds

This section, in which the notion of essential deformations of the Iwasawa manifold is extended to the whole class of page-1-$\partial\bar{\partial}$-manifolds, is taken from [PSU20c]. It generalises the discussion of §.6.2.

The undertaking is motivated by the fact that unobstructedness of the Kuranishi family occurs for some well-known compact complex manifolds that are not $\partial\bar{\partial}$-manifolds but are page-1-$\partial\bar{\partial}$-manifolds, such as the Iwasawa manifold $I^{(3)}$ and its 5-dimensional counterpart $I^{(5)}$. The point we will make is that $I^{(3)}$ and $I^{(5)}$ are not isolated examples, but they are part of a pattern. The fairly large class of Calabi-Yau page-1-$\partial\bar{\partial}$-manifolds, that contains $I^{(3)}$ and $I^{(5)}$, seems well suited for an application of the generalised mirror symmetry theory presented in this chapter.

We will first discuss non-essential deformations, mainly the complex parallelisable deformations of a complex parallelisable nilmanifold, in order to emphasise the contrast with the essential deformations that are defined subsequently in this more general context.

6.8.1 Small non-essential deformations

A key point made in §.6.2.2 was: $I^{(3)}$ is a complex parallelisable nilmanifold, so removing from its Kuranishi family its complex parallelisable small deformations, which have the same geometry as $I^{(3)}$, does not induce any loss of geometric information. This point is now generalised to the context of arbitrary complex parallelisable nilmanifolds.

Theorem 6.8.1. *Let $X = G/\Gamma$ be a* **complex parallelisable nilmanifold***, where G is a simply connected nilpotent complex Lie group and $\Gamma \subset G$ is a lattice. The universal cover of any complex parallelisable small deformation of X is isomorphic to G as a Lie group with left-invariant complex structure.*

Note that this result does **not** state that the corresponding small deformations of X are themselves biholomorphic. For example, when X is a torus, we only recover the fact that the universal cover of each small deformation is $\mathbb{C}^n$ (while, of course, the lattice changes).

Before proving Theorem 6.8.1, we make a few preliminary observations.

For complex parallelisable nilmanifolds $X = G/\Gamma$, where G is a simply connected nilpotent complex Lie group and $\Gamma \subseteq G$ a lattice, the Dolbeault cohomology can be computed by left invariant forms (cf. [Sak76]). In particular, one has (cf. [Nak75]):

$$H^{0,1}(X, T^{1,0}X) \cong H^{0,\,1}(X, \mathbb{C}) \otimes \mathfrak{g}^{1,0} = (\ker \bar{\partial} \cap A^{0,1}_{\mathfrak{g}}) \otimes \mathfrak{g}^{1,0},$$

where $\mathfrak{g}$ is the Lie algebra of G.

Furthermore, $\mathfrak{g}$ is actually a *complex* Lie algebra and $\mathfrak{g}^{1,0} \subseteq \mathfrak{g}_{\mathbb{C}}$ is a *complex* subalgebra. In fact, one has an identification of complex Lie algebras $\mathfrak{g} \cong \mathfrak{g}^{1,0}$ given by $z \mapsto \frac{1}{2}(z - iJz)$. In what follows, we will always tacitly use the above identifications.

Of particular interest are the cohomology classes in

$$H_{par}(X) := H^{0,\,1}(X, \mathbb{C}) \otimes Z(\mathfrak{g}) = (\ker \bar{\partial} \cap A^{0,1}_{\mathfrak{g}}) \otimes Z(\mathfrak{g}) \subset H^{0,1}(X, T^{1,0}X),$$

where $Z(\mathfrak{g})$ is the centre of $\mathfrak{g}$ (which coincides with the Lie algebra of the centre $Z(G)$ of G since G is connected). They will be called *infinitesimally complex parallelisable deformations* of X due to the following

Theorem 6.8.2. *([Rol11]) Let $X = G/\Gamma$ be a complex parallelisable nilmanifold. Let $\mu \in H^{0,1}(X, T^{1,0}X)$. The following statements are equivalent.*

1. *$\mu \in H_{par}(X)$.*
2. *For all $X, Y \in \mathfrak{g}$, one has $[X, \mu\overline{Y}] = 0$.*
3. *$t\mu$ induces a 1-parameter family of complex parallelisable manifolds for t small enough.*

Moreover, for each such μ, the sequence of equations $(Eq.\ (\nu))_{\nu \geq 1}$ (equivalently, (2.15)) is solvable with $\psi = \psi_1 = \mu$.

We will show that the cohomology is the same for all the complex parallelisable small deformations of a given complex parallelisable nilmanifold $X = G/\Gamma$. This is a consequence of Theorem 6.8.1 that we now prove.

Proof of Theorem 6.8.1. It is known that all small deformations of a left-invariant complex structure on a complex parallelisable nilmanifold $X = G/\Gamma$ are again left-invariant (cf. [Rol11, sect. 4]).

On the other hand, the C^∞ manifold underlying the universal cover of any sufficiently small deformation X_t of $X = X_0$ is again G, since it depends only on the smooth structure of X_t which is the same as that of $X = X_0$ when t is sufficiently close to 0. Obviously, the C^∞ manifold G is determined entirely by $\mathfrak{g}$ through the Lie-group/Lie-algebra correspondence. However, the complex structure induced on G by the complex structure of X_t varies with μ. Since it remains left-invariant, it depends only on the complex structure on the Lie algebra $\mathfrak{g}$, so it is determined by the splitting $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{0,1}_{\mu} \oplus \mathfrak{g}^{1,0}_{\mu}$ into i- and $(-i)$-eigenspaces, which can be computed from the complex structure of the central fibre $X = X_0$ via $\mathfrak{g}^{0,1}_{\mu} = (\mathrm{Id} + \mu)\mathfrak{g}^{0,1}_0$ and $\mathfrak{g}^{1,0}_{\mu} = (\mathrm{Id} + \bar{\mu})\mathfrak{g}^{1,0}_0$.

Claim 6.8.3. *The linear map of vector spaces*

$$\alpha : \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathfrak{g}_{\mathbb{C}},$$

defined as $(\mathrm{Id} + \mu)$ *on* $\mathfrak{g}_0^{0,1}$ *and as* $(\mathrm{Id} + \overline{\mu})$ *on* $\mathfrak{g}_0^{1,0}$, *is an isomorphism of Lie algebras.*

Proof of Claim 6.8.3. Since μ is small, α is an isomorphism of vector spaces and the point is to show that it is also a morphism of Lie algebras. We use $[X, \bar{Y}] = 0$ for all $X \in \mathfrak{g}^{1,0}$ and $\bar{Y} \in \mathfrak{g}^{0,1}$. Since $\mu \in H^{0,1}(X, \mathbb{C}) \otimes Z(\mathfrak{g})$, one also has $[X, \mu\bar{Y}] = 0$, so $[Z, \mu\bar{Y}] = [Z, \bar{\mu}X] = 0$ for any $Z \in \mathfrak{g}_{\mathbb{C}}$. So, for $\bar{X}, \bar{Y} \in \mathfrak{g}^{0,1}$, we have:

$$\begin{aligned} [\alpha\bar{X}, \alpha\bar{Y}] &= [\bar{X}, \bar{Y}] + [\mu\bar{X}, \mu\bar{Y}] + [\mu\bar{X}, \bar{Y}] + [\bar{X}, \mu\bar{Y}] \\ &= [\bar{X}, \bar{Y}] = [\bar{X}, \bar{Y}] + \mu([\bar{X}, \bar{Y}]) = \alpha([\bar{X}, \bar{Y}]). \end{aligned}$$

Regarding the last-but-one equality, recall Cartan's formula $(\bar{\partial}\bar{\eta})(\bar{X}, \bar{Y}) = -\bar{\eta}([\bar{X}, \bar{Y}])$ that holds for any left-invariant $(0,1)$-form $\bar{\eta}$ and that $\mu \in \ker \bar{\partial} \cap A_{\mathfrak{g}}^{0,1} \otimes Z(\mathfrak{g})$. Therefore, $\mu([\bar{X}, \bar{Y}]) = -(\bar{\partial}\mu)(\bar{X}, \bar{Y}) = 0$. By a similar argument, $[\alpha X, \alpha Y] = \alpha([X, Y])$ for $X, Y \in \mathfrak{g}^{1,0}$.

Finally, for all $X \in \mathfrak{g}^{1,0}$ and all $\bar{Y} \in \mathfrak{g}^{0,1}$, we have:

$$\begin{aligned} [\alpha X, \alpha\bar{Y}] &= [X, \bar{Y}] + [\bar{\mu}X, \mu\bar{Y}] + [\bar{\mu}X, \bar{Y}] + [X, \mu\bar{Y}] \\ &= 0 = \alpha([X, \bar{Y}]). \end{aligned}$$

Summing up, α is an isomorphism of Lie algebras. Thus, we get an induced isomorphism $G \to G$ which by construction is compatible with the complex structures corresponding to 0 resp. μ.

This finishes the proof of Claim 6.8.3 and that of Theorem 6.8.1. □

The next statement is a general result in the context of non-essential complex parallelisable deformations. It says that for any complex parallelisable small deformation X_t of a complex parallelisable nilmanifold X_0, all types of cohomology discussed in this book (De Rham, Dolbeault, Bott-Chern, Aeppli and their higher-page analogues discussed in chapter 3) remain unchanged. Thus, much as in the special case where $X_0 = I^{(3)}$, the complex parallelisable small deformations have the same geometry as X_0.

Corollary 6.8.4. *Let X' be a complex parallelisable* **small deformation** *of a complex parallelisable nilmanifold X. Then, there exists an* **isomorphism** *between the double complexes of left invariant forms on X and X'.*

In particular, there exist **isomorphisms** $H(X) \cong H(X')$, *where H stands for any cohomology of one of the following types: Dolbeault, Frölicher E_r, De Rham, Bott-Chern, Aeppli and higher-page Bott-Chern and Aeppli.*

Proof. The first statement follows from Claim 6.8.3, since the double complex of left invariant forms can be computed in terms of the Lie-algebra with its complex structure, while the second follows from [Ste20, Prop. 12] and the fact that for any nilmanifold $X = G/\Gamma$, the inclusion of the double complex of left-invariant forms on G into all forms on X is an E_1**-isomorphism**. (This is conjectured to hold for all complex nilmanifolds and it is known for complex parallelisable ones, see [Sak76]). □

6.8.2 Small essential deformations of Calabi-Yau manifolds

The notion of *essential deformations* was introduced in [Pop18a] in the special case of the Iwasawa manifold $I^{(3)}$ (see §.6.2.2). We will now extend it to a larger class of Calabi-Yau manifolds.

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Recall that, for every integer $r \geq 1$ and every bidegree $(p,\, q)$, the vector space of smooth E_r-closed (resp. E_r-exact) $(p,\, q)$-forms on X is denoted by $\mathcal{Z}_r^{p,\,q}(X)$ (resp. $\mathcal{C}_r^{p,\,q}(X)$). Let us now define the following vector subspace of $E_1^{p,\,q}(X)$:

$$E_1^{p,\,q}(X)_0 := \frac{\{\alpha \in C^\infty_{p,\,q}(X) \mid \bar\partial\alpha = 0 \text{ and } \partial\alpha \in \operatorname{Im}\bar\partial\}}{\{\bar\partial\beta \mid \beta \in C^\infty_{p,\,q-1}(X)\}} = \frac{\mathcal{Z}_2^{p,\,q}(X)}{\mathcal{C}_1^{p,\,q}(X)} \subset E_1^{p,\,q}(X).$$

In other words, $E_1^{p,\,q}(X)_0 = \ker d_1$ consists of the E_1-cohomology classes (i.e. Dolbeault cohomology classes) representable by E_2-closed forms of type $(p,\, q)$.

Lemma 6.8.5. *For all p, q, the canonical linear map*

$$P^{p,\,q} : E_1^{p,\,q}(X)_0 \to E_2^{p,\,q}(X), \qquad \{\alpha\}_{E_1} \mapsto \{\alpha\}_{E_2},$$

is well defined and **surjective**. *Its kernel consists of the E_1-cohomology classes representable by E_2-exact forms of type $(p,\, q)$.*

In particular, $P^{p,\,q}$ is injective (hence an isomorphism) if and only if $\mathcal{C}_1^{p,\,q}(X) = \mathcal{C}_2^{p,\,q}(X)$.

Proof. Well-definedness means that $P^{p,\,q}(\{\alpha\}_{E_1})$ is independent of the choice of representative of the class $\{\alpha\}_{E_1} \in E_1^{p,\,q}(X)_0$. This follows from the inclusion $\mathcal{C}_1^{p,\,q}(X) \subset \mathcal{C}_2^{p,\,q}(X)$. The other three statements are obvious. $\square$

Let us now fix a Hermitian metric ω on X. By the Hodge theory for the E_2-cohomology introduced in [Pop16] (and used e.g. in [PSU20b]) and the standard Hodge theory for the Dolbeault cohomology, there are Hodge isomorphisms:

$$E_2^{n-1,\,1}(X) \simeq \mathcal{H}_2^{n-1,\,1} = \mathcal{H}_{2,\,\omega}^{n-1,\,1} \quad \text{and} \quad E_1^{n-1,\,1}(X) \simeq \mathcal{H}_1^{n-1,\,1} = \mathcal{H}_{1,\,\omega}^{n-1,\,1}$$

associating with every E_2- (resp. E_1-)class its unique E_2- (resp. E_1-)harmonic representative (w.r.t. ω), where the ω-dependent harmonic spaces are defined by

$$\mathcal{H}_2^{n-1,\,1} := \ker(\widetilde\Delta : C^\infty_{n-1,\,1}(X) \to C^\infty_{n-1,\,1}(X)) \subset \mathcal{H}_1^{n-1,\,1} := \ker(\Delta'' : C^\infty_{n-1,\,1}(X) \to C^\infty_{n-1,\,1}(X))$$

and $\widetilde\Delta = \partial p''\partial^\star + \partial^\star p''\partial + \Delta''$ is the pseudo-differential Laplacian introduced in [Pop16] and $\Delta'' = \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial$ is the standard $\bar\partial$-Laplacian, both associated with the metric ω. (Recall that p'' is the L^2_ω-orthogonal projection onto $\ker\Delta''$.)

Definition 6.8.6. *Let $(X,\, \omega)$ be an n-dimensional compact complex Hermitian manifold. The* ω**-lift** *of the canonical linear surjection $P^{n-1,\,1} : E_1^{n-1,\,1}(X)_0 \twoheadrightarrow E_2^{n-1,\,1}(X)$ introduced in Lemma 6.8.5 is the ω-dependent linear* **injection**

$$J_\omega^{n-1,\,1} : E_2^{n-1,\,1}(X) \hookrightarrow E_1^{n-1,\,1}(X)_0$$

induced by the inclusion $\mathcal{H}_{2,\,\omega}^{n-1,\,1} \subset \mathcal{H}_{1,\,\omega}^{n-1,\,1}$, namely the map $J_\omega^{n-1,\,1}$ that makes the following diagram **commutative**:

$$\begin{array}{ccc} E_2^{n-1,\,1}(X) & \xrightarrow{J_\omega^{n-1,\,1}} & E_1^{n-1,\,1}(X) \\ \simeq\downarrow & & \simeq\downarrow \\ \mathcal{H}_{2,\,\omega}^{n-1,\,1} & \xrightarrow{\text{inclusion}} & \mathcal{H}_{1,\,\omega}^{n-1,\,1}, \end{array}$$

where the vertical arrows are the Hodge isomorphisms.

It follows from the definitions that the image of the injection $J_\omega^{n-1,\,1} : E_2^{n-1,\,1}(X) \longrightarrow E_1^{n-1,\,1}(X)$ defined by the above commutative diagram is contained in $E_1^{n-1,\,1}(X)_0$ and we have

$$P^{n-1,\,1} \circ J_\omega^{n-1,\,1} = \mathrm{Id}_{E_2^{n-1,\,1}(X)}.$$

Thus, every Hermitian metric ω on X induces a natural injection $J_\omega^{n-1,\,1}$ of $E_2^{n-1,\,1}(X)$ into $E_1^{n-1,\,1}(X)$ (and even into $E_1^{n-1,\,1}(X)_0$). In particular, if a **canonical metric** ω_0 exists on X (in the sense that ω_0 depends only on the complex structure of X with no arbitrary choices involved in its definition), the associated map $J_{\omega_0}^{n-1,\,1}$ constitutes a **canonical injection** of $E_2^{n-1,\,1}(X)$ into $E_1^{n-1,\,1}(X)$.

We now specialize to page-1-$\partial\bar\partial$-manifolds. We refer to [PSU20a] for their definition and properties.

Definition 6.8.7. *Let X be a compact complex n-dimensional* **Calabi-Yau page-1-$\partial\bar\partial$-manifold**. *Suppose that X carries a* **canonical Hermitian metric** ω_0.

The space of **small essential deformations** *of X is defined as the image in $E_1^{n-1,\,1}(X)$ of the canonical injection $J_{\omega_0}^{n-1,\,1}$, namely*

$$E_1^{n-1,\,1}(X)_{ess} := J_{\omega_0}^{n-1,\,1}(E_2^{n-1,\,1}(X)) \subset E_1^{n-1,\,1}(X).$$

Remark 6.8.8. *If the page-1-$\partial\bar\partial$-assumption on X is replaced by a more general one (for example, the page-r-$\partial\bar\partial$-assumption for some $r \geq 2$ or merely the $E_r(X) = E_\infty(X)$ assumption for a specific $r \geq 2$), one can define a version of essential deformations using higher pages than the second one. The most natural choice is the degenerating page $E_r = E_\infty$ of the FSS if $r > 2$. Since at the moment we are mainly interested in page-1-$\partial\bar\partial$-manifolds, we confine ourselves to E_2.*

Example 6.8.9. *(The* **Iwasawa manifold***) If α, β, γ are the three canonical holomorphic $(1,\,0)$-forms induced on the complex 3-dimensional Iwasawa manifold $X = G/\Gamma$ by $dz_1, dz_2, dz_3 - z_1\, dz_2$ from $\mathbb{C}^3$ (the underlying complex manifold of the Heisenberg group G), it is well known that α and β are d-closed, while $d\gamma = \partial\gamma = -\alpha\wedge\beta \neq 0$. It is equally standard that the Dolbeault cohomology group of bidegree $(2,\,1)$ is generated as follows:*

$$E_1^{2,\,1}(X) = \left\langle [\alpha\wedge\gamma\wedge\overline\alpha]_{\bar\partial},\, [\alpha\wedge\gamma\wedge\overline\beta]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline\alpha]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline\beta]_{\bar\partial}\right\rangle \oplus \left\langle [\alpha\wedge\beta\wedge\overline\alpha]_{\bar\partial},\, [\alpha\wedge\beta\wedge\overline\beta]_{\bar\partial}\right\rangle.$$

In particular, we see that every E_1-class of bidegree $(2,\,1)$ can be represented by a d-closed form. Since every pure-type d-closed form is also E_2-closed (and, indeed, E_r-closed for every r), we get

$$E_1^{2,\,1}(X) = E_1^{2,\,1}(X)_0.$$

It is equally standard that the E_2-cohomology group of bidegree $(2,\,1)$ is generated as follows:

$$E_2^{2,\,1}(X) = \left\langle [\alpha\wedge\gamma\wedge\overline\alpha]_{E_2},\, [\alpha\wedge\gamma\wedge\overline\beta]_{E_2},\, [\beta\wedge\gamma\wedge\overline\alpha]_{E_2},\, [\beta\wedge\gamma\wedge\overline\beta]_{E_2}\right\rangle.$$

It identifies canonically with the vector subspace

$$H^{2,1}_{[\gamma]}(X) = \bigg\langle [\alpha\wedge\gamma\wedge\overline{\alpha}]_{\bar\partial},\, [\alpha\wedge\gamma\wedge\overline{\beta}]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline{\alpha}]_{\bar\partial},\, [\beta\wedge\gamma\wedge\overline{\beta}]_{\bar\partial}\bigg\rangle \simeq E_1^{2,1}(X)\bigg/\bigg\langle [\alpha\wedge\beta\wedge\bar\alpha]_{\bar\partial},\, [\alpha\wedge\beta\wedge\bar\beta]_{\bar\partial}\bigg\rangle$$

of $E_1^{2,\,1}(X)$ introduced in [Pop18a, §.4.2] as parametrising the **essential deformations** *defined there for the Iwasawa manifold X.*

On the other hand, let

$$\omega_0 := i\alpha\wedge\bar\alpha + i\beta\wedge\bar\beta + i\gamma\wedge\bar\gamma$$

be the Hermitian (even balanced) metric on X **canonically** *induced by the complex parallelisable structure of X. It can be easily seen that the vector space of* **small essential deformations** *coincides with the space $H^{2,\,1}_{[\gamma]}(X)$ of [Pop18a]:*

$$E_1^{2,\,1}(X)_{ess} = J^{2,\,1}_{\omega_0}(E_2^{2,\,1}(X)) = H^{2,\,1}_{[\gamma]}(X) \subset E_1^{2,\,1}(X).$$

Example 6.8.10. (**The Kuranishi family of the 5-dimensional Iwasawa-type manifold**)

Let us now consider the specific example of the complex parallelisable nilmanifold $X = I^{(5)}$ of complex dimension 5. Its complex structure is described by five holomorphic $(1,\,0)$-forms $\varphi_1,\dots,\varphi_5$ satisfying the equations:

$$d\varphi_1 = d\varphi_2 = 0, \quad d\varphi_3 = \varphi_1\wedge\varphi_2, \quad d\varphi_4 = \varphi_1\wedge\varphi_3, \quad d\varphi_5 = \varphi_2\wedge\varphi_3.$$

If $\theta_1,\dots,\,\theta_5$ form the dual basis of $(1,\,0)$-vector fields, then $[\theta_i,\,\theta_j] = 0$ except in the following cases:

$$[\theta_1,\,\theta_2] = -\theta_3, \qquad [\theta_1,\,\theta_3] = -\theta_4, \qquad [\theta_2,\,\theta_3] = -\theta_5,$$

hence also $\qquad [\theta_2,\,\theta_1] = \theta_3, \qquad [\theta_3,\,\theta_1] = \theta_4, \qquad [\theta_3,\,\theta_2] = \theta_5.$

In particular, $\quad H^{0,\,1}(X,\,T^{1.\,0}X) = \langle[\overline{\varphi}_1\otimes\theta_i],\,[\overline{\varphi}_2\otimes\theta_i]\ \mid\ i=1,\dots,5\rangle$, so $\dim_{\mathbb{C}}H^{0,\,1}(X,\,T^{1.\,0}X) = 10$.

This manifold is the 5-dimensional analogue of the 3-dimensional Iwasawa manifold $I^{(3)}$. The following fact was observed in [Rol11].

Proposition 6.8.11. *The Kuranishi family of the 5-dimensional nilmanifold $I^{(5)}$ is* **unobstructed**.

Proof. Consider any $\psi_1(t) := \sum\limits_{i=1}^5\sum\limits_{\lambda=1}^2 t_{i\lambda}\,\theta_i\,\overline{\varphi}_\lambda$ with arbitrary coefficients $t_{i\lambda}\in\mathbb{C}$ such that $|t|$ is close to 0. Then

$$[\psi_1(t),\,\psi_1(t)] = \sum_{i,\,j=1}^5\sum_{\lambda,\,\mu=1}^2 t_{i\lambda}\,t_{j\mu}\,[\theta_i,\,\theta_j]\,\overline{\varphi}_\lambda\wedge\overline{\varphi}_\mu.$$

Since $[\theta_i,\,\theta_j]=0$ except when $(i,\,j)\in\{(1,\,2),\,(2,\,1),\,(1,\,3),\,(3,\,1),(2,\,3),\,(3,\,2)\}$, we get

$$\begin{aligned}[\psi_1(t),\,\psi_1(t)] &= [-(t_{11}t_{22}-t_{21}t_{12})\,\theta_3 + (t_{12}t_{21}-t_{22}t_{11})\,\theta_3 - (t_{11}t_{32}-t_{31}t_{12})\,\theta_4 + (t_{12}t_{31}-t_{32}t_{11})\,\theta_4\\ &- (t_{21}t_{32}-t_{31}t_{22})\,\theta_5 + (t_{22}t_{31}-t_{32}t_{21})\,\theta_5]\,\overline{\varphi}_1\wedge\overline{\varphi}_2\\ &= 2\,[D_{21}(t)\,\theta_3 + D_{31}(t)\,\theta_4 + D_{32}(t)\,\theta_5]\ \overline{\varphi}_1\wedge\overline{\varphi}_2,\end{aligned}$$

where we set $D_{ji} := D^{12}_{ji}$ and

$$D^{\lambda\mu}_{ji} := \begin{vmatrix} t_{i\mu} & t_{j\lambda} \\ t_{j\mu} & t_{i\lambda} \end{vmatrix}$$

.

Since $\overline{\varphi}_1 \wedge \overline{\varphi}_2 = \bar\partial\overline{\varphi}_3$, equation (Eq. 2) reads

$$\bar\partial\psi_2(t) = (D_{21}(t)\,\theta_3 + D_{31}(t)\,\theta_4 + D_{32}(t)\,\theta_5)\;\bar\partial\overline{\varphi}_3,$$

so an obvious solution is $\psi_2(t) = (D_{21}(t)\,\theta_3 + D_{31}(t)\,\theta_4 + D_{32}(t)\,\theta_5)\;\overline{\varphi}_3$.

We now go on to compute

$$\begin{aligned}[\psi_1(t),\,\psi_2(t)] &= \sum_{i=1}^{5}\sum_{\lambda=1}^{2} t_{i\lambda}\,D_{21}(t)\,[\theta_i,\,\theta_3]\,\overline{\varphi}_\lambda\wedge\overline{\varphi}_3 \\ &+ \sum_{i=1}^{5}\sum_{\lambda=1}^{2} t_{i\lambda}\,D_{31}(t)\,[\theta_i,\,\theta_4]\,\overline{\varphi}_\lambda\wedge\overline{\varphi}_3 + \sum_{i=1}^{5}\sum_{\lambda=1}^{2} t_{i\lambda}\,D_{32}(t)\,[\theta_i,\,\theta_5]\,\overline{\varphi}_\lambda\wedge\overline{\varphi}_3\end{aligned}$$

All the terms on the second line above vanish since $[\theta_i,\,\theta_4] = [\theta_i,\,\theta_5] = 0$ for all i, and so do the terms with $i \notin \{1,\,2\}$ on the first line (see (6.83)), so using (6.83) we get

$$[\psi_1(t),\,\psi_2(t)] = -\sum_{\lambda=1}^{2} t_{1\lambda}\,D_{21}(t)\,\theta_4\,\overline{\varphi}_\lambda\wedge\overline{\varphi}_3 - \sum_{\lambda=1}^{2} t_{2\lambda}\,D_{21}(t)\,\theta_5\,\overline{\varphi}_\lambda\wedge\overline{\varphi}_3.$$

We infer that equation (Eq. 3) $\bar\partial\psi_3(t) = [\psi_1(t),\,\psi_2(t)]$ has the obvious solution

$$\psi_3(t) = -D_{21}(t)\,[(t_{11}\,\theta_4 + t_{21}\,\theta_5)\,\overline{\varphi}_4 + (t_{12}\,\theta_4 + t_{22}\,\theta_5)\,\overline{\varphi}_5].$$

To study equation (Eq. 4), namely

$$\bar\partial\psi_4(t) = [\psi_1(t),\,\psi_3(t)] + \frac{1}{2}\,[\psi_2(t),\,\psi_2(t)],$$

we notice that $[\psi_1(t),\,\psi_3(t)] = [\psi_2(t),\,\psi_2(t)] = 0$ because $[\theta_i,\,\theta_4] = [\theta_i,\,\theta_5] = 0$ for all i and $[\theta_3,\,\theta_3] = 0$. Consequently, equation (Eq. 4) is the trivial equation $\bar\partial\psi_4(t) = 0$ admitting the trivial solution $\psi_4(t) = 0$.

We conclude that the Kuranishi family of X is unobstructed and the deformations of its complex structure in any pregiven direction $\psi_1(t) := \sum_{i=1}^{5}\sum_{\lambda=1}^{2} t_{i\lambda}\,\theta_i\,\overline{\varphi}_\lambda$ are defined by the finite sum

$$\psi(t) = \sum_{i=1}^{5}\sum_{\lambda=1}^{2} t_{i\lambda}\,\theta_i\,\overline{\varphi}_\lambda + (D_{21}(t)\,\theta_3 + D_{31}(t)\,\theta_4 + D_{32}(t)\,\theta_5)\;\overline{\varphi}_3 - D_{21}(t)\,[(t_{11}\,\theta_4 + t_{21}\,\theta_5)\,\overline{\varphi}_4 + (t_{12}\,\theta_4 + t_{22}\,\theta_5)\,\overline{\varphi}_5].$$

So, no convergence issues are involved. □

Example 6.8.12. *(The* **manifold** $I^{(5)}$*) Let $X = I^{(5)}$ be the complex parallelisable nilmanifold of complex dimension 5 described in Example 6.8.10 (i.e. the 5-dimensional analogue of the Iwasawa manifold.) It is a page-1-$\partial\bar\partial$-manifold by [PSU20a, Thm. 4.7].*

We will use the standard notation $\varphi_{i_1\dots i_p\bar j_1\dots\bar j_q} := \varphi_{i_1}\wedge\cdots\wedge\varphi_{i_p}\wedge\overline{\varphi}_{j_1}\wedge\cdots\wedge\overline{\varphi}_{j_q}$.

For every $l \in \{3, 4, 5\}$, the linear map

$$T_l : H^{0,1}(X,\, T^{1,0}X) \longrightarrow H^{0,1}(X), \qquad [\theta] \mapsto [\theta \lrcorner \varphi_l],$$

is well defined. If we set

$$H^{0,1}_{ess}(X,\, T^{1,0}X) := \ker T_3 \cap \ker T_4 \cap \ker T_5 \subset H^{0,1}(X,\, T^{1,0}X),$$

and define $H^{4,1}_{ess}(X) \subset H^{4,1}(X)$ to be the image of $H^{0,1}_{ess}(X,\, T^{1,0}X)$ under the Calabi-Yau isomorphism $H^{0,1}(X,\, T^{1,0}X) \longrightarrow H^{4,1}(X)$ w.r.t. $u = \varphi_1 \wedge \ldots \wedge \varphi_5$, we get the following description:

$$H^{4,1}_{ess}(X) = \left\langle [\varphi_{2345\bar{1}}]_{\bar\partial},\, [\varphi_{1345\bar{1}}]_{\bar\partial},\, [\varphi_{2345\bar{2}}]_{\bar\partial},\, [\varphi_{1345\bar{2}}]_{\bar\partial} \right\rangle.$$

Moreover, we have the following identities of $\mathbb{C}$-vector spaces:

$$H^{4,1}_{ess}(X) = E_1^{4,1}(X)_{ess} := J^{4,1}_{\omega_0}(E_2^{4,1}(X)) \subset E_1^{4,1}(X),$$

where

$$\omega_0 := \sum_{j=1}^{5} i\varphi_j \wedge \overline{\varphi}_j,$$

is the canonical metric of $I^{(5)}$.

6.8.3 Deformation unobstructedness for page-1-$\partial\bar\partial$-manifolds

In this subsection, we prove a generalisation of the Bogomolov-Tian-Todorov Theorem 2.4.7. It says that, under certain cohomological conditions, the unobstructedness phenomenon described in the next definition holds when the $\partial\bar\partial$-assumption is weakened to the page-1-$\partial\bar\partial$-assumption.

Definition 6.8.13. *Let X be a Calabi-Yau page-1-$\partial\bar\partial$-manifold with $\dim_{\mathbb{C}} X = n$. Fix a non-vanishing holomorphic $(n,\, 0)$-form u on X.*

We say that the **essential Kuranishi family** *of X is* **unobstructed** *if every E_2-class in $E_2^{n-1,1}(X)$ admits a representative $\psi_1(t) \lrcorner u$ such that the integrability condition (2.15) is satisfied (i.e. all the equations (Eq. (ν)) in (2.22) of §.2.3.1 are solvable) when starting off with $\psi_1(t) \in C^\infty_{0,1}(X,\, T^{1,0}X)$.*

Meanwhile, for any bidegree $(p,\, q)$, we let $\mathcal{Z}_r^{p,\,q}(X)$ stand for the vector space of smooth E_r-closed $(p,\, q)$-forms on X. (These are the smooth $(p,\, q)$-forms on X that represent E_r-cohomology classes on the r-th page of the Frölicher spectral sequence. See e.g. Proposition 2.3 in [PSU20b] for a description of them.)

Theorem 6.8.14. *Let X be a compact* **Calabi-Yau page-1-$\partial\bar\partial$-manifold** *with $\dim_{\mathbb{C}} X = n$. Fix a non-vanishing holomorphic $(n,\, 0)$-form u on X and suppose that*

$$\psi_1(t) \lrcorner (\rho_1(s) \lrcorner u) \in \mathcal{Z}_2^{n-2,\,2} \tag{6.83}$$

for all $\psi_1(t), \rho_1(s) \in C^\infty_{0,1}(X,\, T^{1,0}X)$ such that $\psi_1(t) \lrcorner u, \rho_1(s) \lrcorner u \in \ker d \cup Im\, \partial$.

(i) Then, the **essential Kuranishi family** *of X is* **unobstructed***.*

(ii) If, moreover, $\mathcal{Z}_1^{n-1,1} = \mathcal{Z}_2^{n-1,1}$, the **Kuranishi family** *of X is* **unobstructed***.*

Before proving Theorem 6.8.14, we make a few comments. First, we notice an equivalent formulation for the assumption made in (ii). Needless to say, the inclusion $\mathcal{Z}_2^{n-1,\,1} \subset \mathcal{Z}_1^{n-1,\,1}$ always holds.

Lemma 6.8.15. *Let X be a compact complex* **page-1-$\partial\bar\partial$-manifold** *with $dim_{\mathbb{C}} X = n$. Then, $\mathcal{Z}_1^{n-1,\,1} = \mathcal{Z}_2^{n-1,\,1}$ if and only if every Dolbeault cohomology class of bidegree $(n-1,\,1)$ can be represented by a d-closed form.*

Proof. Let $\alpha \in C^\infty_{n-1,\,1}(X)$ be an arbitrary $\bar\partial$-closed form, i.e. $\alpha \in \mathcal{Z}_1^{n-1,\,1}$. The class $\{\alpha\}_{\bar\partial}$ can be represented by a d-closed form if and only if there exists β of bidegree $(n-1,\,0)$ such that $\partial(\alpha+\bar\partial\beta)=0$. This is equivalent to $\partial\alpha$ being $\partial\bar\partial$-exact, which implies that $\partial\alpha$ is $\bar\partial$-exact.

Conversely, since X is a page-1-$\partial\bar\partial$-manifold, the $\bar\partial$-exactness of $\partial\alpha$ implies its $\partial\bar\partial$-exactness. Indeed, $\bar\partial\alpha = 0$ and if $\partial\alpha$ is $\bar\partial$-exact, then $\alpha \in \mathcal{Z}_2^{n-1,\,1}$, so $\partial\alpha \in \partial(\mathcal{Z}_2^{n-1,\,1})$. Now, $\partial(\mathcal{Z}_2^{n-1,\,1}) = \mathrm{Im}\,(\partial\bar\partial)$ thanks to property (i) in characterisation (F) of the page-1-$\partial\bar\partial$-property given in [PSU20b, Thm. 4.3] (with $r=2$). Therefore, $\partial\alpha \in \mathrm{Im}\,(\partial\bar\partial)$ whenever $\alpha \in \mathcal{Z}_2^{n-1,\,1}$.

Summing up, the class $\{\alpha\}_{\bar\partial}$ can be represented by a d-closed form if and only if $\partial\alpha$ is $\bar\partial$-exact if and only if $\alpha \in \mathcal{Z}_2^{n-1,\,1}$. □

Second, we notice that both the Iwasawa manifold $I^{(3)}$ and the 5-dimensional Iwasawa manifold $I^{(5)}$ satisfy all the hypotheses of Theorem 6.8.14. Indeed, $I^{(3)}$ and $I^{(5)}$ are complex parallelisable nilmanifolds, so they are page-1-$\partial\bar\partial$-manifolds by Theorem [PSU20a, Thm. 4.7]. In particular they are also Calabi-Yau manifolds (actually, all nilmanifolds are). Moreover, we have

Lemma 6.8.16. *Let X be either $I^{(3)}$ or $I^{(5)}$ and let $n = dim_{\mathbb{C}} X \in \{3,\,5\}$. Let $u := \varphi_1\wedge\varphi_2\wedge\varphi_3 = \alpha\wedge\beta\wedge\gamma \in C^\infty_{3,\,0}(I^{(3)})$ or $u := \varphi_1\wedge\cdots\wedge\varphi_5 \in C^\infty_{5,\,0}(I^{(5)})$ according to whether $X = I^{(3)}$ or $X = I^{(5)}$, a non-vanishing holomorphic $(n,\,0)$-form on X.*

Then, for all $\psi_1(t), \rho_1(s) \in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$ such that $\psi_1(t)\lrcorner u, \rho_1(s)\lrcorner u \in \ker d \cup Im\,\partial$, we have

$$\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u) \in \mathcal{Z}_2^{n-2,\,2}.$$

Proof. It is given in §.6.8.4. □

Finally, let us mention that both manifolds $X = I^{(3)}$ and $X = I^{(5)}$ have the property that every Dolbeault cohomology class of type $(n-1,\,1)$ can be represented by a d-closed form. Indeed, as seen in the proof of Lemma 6.8.16 spelt out in §.6.8.4, $H^{n-1,\,1}_{\bar\partial}(X)$ is generated by the classes represented by the $\widehat{\varphi}_i\wedge\overline{\varphi}_1$'s and the $\widehat{\varphi}_i\wedge\overline{\varphi}_2$'s with $i\in\{1,2,3\}$ (in the case of $X = I^{(3)}$) and $i \in \{1,\dots,5\}$ (in the case of $X = I^{(5)}$), where $\widehat{\varphi}_i$ stands for $u = \varphi_1\wedge\ldots\wedge\varphi_5$ with φ_i omitted. All the forms $\widehat{\varphi}_i\wedge\overline{\varphi}_\lambda$, with $\lambda\in\{1,2\}$, are d-closed.

Note that the hypotheses of Theorem 6.8.14, all of which are satisfied by $X = I^{(3)}$ and $X = I^{(5)}$, have the advantage of being cohomological in nature, hence fairly general and not restricted to the class of nilmanifolds. Indeed, there is no mention of any structure equations in Theorem 6.8.14.

Proof of Theorem 6.8.14. Let $\{\eta_1\}_{E_2} \in E_2^{n-1,\,1}(X)$ be an arbitrary nonzero class. Pick any d-closed representative $\eta_1 \in C^\infty_{n-1,\,1}(X)$ of it. A d-closed representative exists thanks to the page-1-$\partial\bar\partial$-assumption on X. Under the extra assumption $\mathcal{Z}_1^{n-1,\,1} = \mathcal{Z}_2^{n-1,\,1}$ of (ii), there is even a d-closed representative η_1 in every Dolbeault class $\{\eta_1\}_{E_1} \in E_1^{n-1,\,1}(X)$, thanks to Lemma 6.8.15. So, we choose an arbitrary d-closed form $\eta_1 \in C^\infty_{n-1,\,1}(X)$ that represents an arbitrary nonzero class in either $E_2^{n-1,\,1}(X)$ or $E_1^{n-1,\,1}(X)$ depending on whether we are in case (i) or in case (ii). By the Calabi-Yau isomorphism, there exists a unique $\psi_1 \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$ such that

$$\psi_1 \lrcorner u = \eta_1.$$

We will prove the existence of forms $\psi_\nu \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$, with $\nu \in \mathbb{N}^\star$ and ψ_1 being the already fixed such form, that satisfy the equations

$$\bar\partial\psi_\nu = \frac{1}{2}\sum_{\mu=1}^{\nu-1}[\psi_\mu,\, \psi_{\nu-\mu}] \qquad (\text{Eq. } (\nu-1)), \qquad \nu \geq 2,$$

which, as recalled in §.**??**, are equivalent to the integrability condition $\bar\partial\psi(\tau) = (1/2)\,[\psi(\tau),\, \psi(\tau)]$ being satisfied by the form $\psi(\tau) := \psi_1\,\tau + \psi_2\,\tau^2 + \cdots + \psi_N\,\tau^N + \cdots \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$ for all $\tau \in \mathbb{C}$ with $|\tau|$ sufficiently small. The convergence in a Hölder norm of the series defining $\psi(\tau)$ for $|\tau|$ small enough is guaranteed by the general Kuranishi theory (cf. [Kur62]), while the resulting $\psi(\tau)$ defines a complex structure $\bar\partial_\tau$ on X that identifies on functions with $\bar\partial - \psi(\tau)$ and represents the infinitesimal deformation of the original complex structure $\bar\partial$ of X in the direction of $[\psi_1] \in H^{0,\,1}(X,\, T^{1,\,0}X)$.

Since $\partial(\psi_1 \lrcorner u) = \partial\eta_1 = 0$, the Tian-Todorov lemma ([Tia87], [Tod89]) guarantees that $[\psi_1,\, \psi_1] \lrcorner u \in \operatorname{Im}\partial$ and

$$[\psi_1,\, \psi_1] \lrcorner u = \partial(\psi_1 \lrcorner (\psi_1 \lrcorner u)).$$

On the other hand, $\bar\partial\eta_1 = 0$, hence $\bar\partial\psi_1 = 0$, hence $\psi_1 \lrcorner (\psi_1 \lrcorner u) \in \ker\bar\partial$. We even have the stronger property $\psi_1 \lrcorner (\psi_1 \lrcorner u) \in \mathcal{Z}_2^{n-2,\,2}$ thanks to assumption (6.83), since $\psi_1 \lrcorner u \in \ker d$. Therefore,

$$[\psi_1,\, \psi_1] \lrcorner u = \partial(\psi_1 \lrcorner (\psi_1 \lrcorner u)) \in \partial(\mathcal{Z}_2^{n-2,\,2}) = \operatorname{Im}(\partial\bar\partial),$$

the last identity being a consequence of the page-1-$\partial\bar\partial$-assumption on X. (See (i) of property (F) in [PSU20b, Thm. 4.3].)

Thus, there exists a form $\Phi_2 \in C^\infty_{n-2,\,1}(X)$ such that

$$\bar\partial\partial\Phi_2 = \frac{1}{2}\,[\psi_1,\, \psi_1] \lrcorner u.$$

If we fix an arbitrary Hermitian metric ω on X, we choose Φ_2 as the unique solution of the above equation with the extra property $\Phi_2 \in \operatorname{Im}(\partial\bar\partial)^\star$. This is the minimal L^2_ω-norm solution, as follows from the 3-space orthogonal decomposition of $C^\infty_{n-2,\,1}(X)$ induced by the Aeppli Laplacian (see [Sch07]). Let $\eta_2 := \partial\Phi_2 \in C^\infty_{n-1,\,1}(X)$. Thanks to the Calabi-Yau isomorphism, there exists a unique $\psi_2 \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$ such that $\psi_2 \lrcorner u = \eta_2$. In particular, $\partial(\psi_2 \lrcorner u) = 0$ and $(\bar\partial\psi_2) \lrcorner u = \bar\partial(\psi_2 \lrcorner u) = \bar\partial\eta_2 = (1/2)\,[\psi_1,\, \psi_1] \lrcorner u$. This means that

$$\bar\partial\psi_2 = \frac{1}{2}\,[\psi_1,\, \psi_1],$$

so ψ_2 is a solution of (Eq. 1). Moreover, by construction, ψ_2 has the extra key property that $\psi_2 \lrcorner u \in \operatorname{Im}\partial$.

Now, we continue inductively to construct the forms $(\psi_N)_{N\geq 3}$. Suppose the forms $\psi_1, \psi_2, \dots, \psi_{N-1} \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$ have been constructed as solutions of the equations (Eq. $(\nu-1)$) for all $\nu \in \{2, \dots, N-1\}$ with the further property $\psi_2 \lrcorner u, \dots, \psi_{N-1} \lrcorner u \in \operatorname{Im}\partial$. (Recall that $\psi_1 \lrcorner u \in \ker d$.) Since $\partial(\psi_1 \lrcorner u) = \partial(\psi_2 \lrcorner u) = \cdots = \partial(\psi_{N-1} \lrcorner u) = 0$, the Tian-Todorov lemma ([Tia87], [Tod89]) guarantees that $[\psi_\mu,\, \psi_{N-\mu}] \lrcorner u \in \operatorname{Im}\partial$ for all $\mu \in \{1, \dots, N-1\}$ and yields the first identity below:

$$\sum_{\mu=1}^{N-1}[\psi_\mu,\, \psi_{N-\mu}] \lrcorner u = \partial\bigg(\sum_{\mu=1}^{N-1}\psi_\mu \lrcorner (\psi_{N-\mu} \lrcorner u)\bigg) \in \partial(\mathcal{Z}_2^{n-2,\,2}) = \operatorname{Im}(\partial\bar\partial),$$

where the relation "$\in$" follows from assumption (6.83) and the last identity is a consequence of the page-1-$\partial\bar\partial$-assumption on X. (See (i) of property (F) in [PSU20b, Thm. 4.3].).

Thus, there exists a form $\Phi_N \in C^\infty_{n-2,\,1}(X)$ such that

$$\bar\partial\partial\Phi_N = \frac{1}{2}\sum_{\mu=1}^{N-1}[\psi_\mu,\,\psi_{N-\mu}]\lrcorner u.$$

We choose Φ_N to be the solution of minimal L^2_ω-norm of the above equation, so $\Phi_N \in \operatorname{Im}(\partial\bar\partial)^\star$. Let $\eta_N := \partial\Phi_N \in C^\infty_{n-1,\,1}(X)$. Thanks to the Calabi-Yau isomorphism, there exists a unique $\psi_N \in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$ such that $\psi_N\lrcorner u = \eta_N$. Hence, $(\bar\partial\psi_N)\lrcorner u = \bar\partial(\psi_N\lrcorner u) = \bar\partial\eta_N = \bar\partial\partial\Phi_N$, so

$$\bar\partial\psi_N = \frac{1}{2}\sum_{\mu=1}^{N-1}[\psi_\mu,\,\psi_{N-\mu}],$$

which means that ψ_N is a solution of (Eq. $(N-1)$). Moreover, by construction, ψ_N has the extra key property that $\psi_N\lrcorner u \in \operatorname{Im}\partial$.

This finishes the induction process and completes the proof of Theorem 6.8.14. $\square$

6.8.4 Explicit computations

In this subsection, we spell out the proof of Lemma 6.8.16.

- *Case where* $X = I^{(3)}$. We use the notation of Example 6.8.9, but also put $\varphi_1 := \alpha$, $\varphi_2 := \beta$ and $\varphi_3 := \gamma$. We have: $d\varphi_1 = d\varphi_2 = 0$ and $d\varphi_3 = -\varphi_1\wedge\varphi_2$. The dual basis of $(1,\,0)$-vector fields consists of

$$\theta_1 = \frac{\partial}{\partial z_1}, \quad \theta_2 = \frac{\partial}{\partial z_2} + z_1\frac{\partial}{\partial z_3}, \quad \theta_3 = \frac{\partial}{\partial z_3},$$

(actually of the vector fields induced by these ones on X by passage to the quotient) whose mutual Lie brackets are as follows:

$$[\theta_1,\,\theta_2] = -[\theta_2,\,\theta_1] = \theta_3 \quad \text{and} \quad [\theta_i,\,\theta_j] = 0 \quad \text{whenever} \quad \{i,\,j\} \neq \{1,\,2\}.$$

In particular, $H^{0,\,1}(X,\,T^{1.\,0}X) = \langle[\overline\varphi_1\otimes\theta_i],\,[\overline\varphi_2\otimes\theta_i] \mid i = 1,\ldots,3\rangle$, so $\dim_{\mathbb{C}}H^{0,\,1}(X,\,T^{1.\,0}X) = 6$.

Note that all the $(2,\,1)$-forms $(\overline\varphi_1\otimes\theta_i)\lrcorner u$ and $(\overline\varphi_2\otimes\theta_i)\lrcorner u$ are d-closed for $i\in\{1,\,2,\,3\}$, so every Dolbeault class in $H^{2,\,1}_{\bar\partial}(X)$ can be represented by a d-closed form.

(a) Let $\psi_1(t), \rho_1(s) \in C^\infty_{0,\,1}(X,\,T^{1,\,0}X)$ such that $\psi_1(t)\lrcorner u, \rho_1(s)\lrcorner u \in \ker d$. Then,

$$\psi_1(t) = \sum_{i=1}^{3}\sum_{\lambda=1}^{2}t_{i\lambda}\,\theta_i\overline\varphi_\lambda, \;\text{ so }\; \psi_1(t)\lrcorner u = \sum_{i=1}^{3}(-1)^{i-1}\sum_{\lambda=1}^{2}t_{i\lambda}\,\overline\varphi_\lambda\wedge\widehat\varphi_i,$$

$$\rho_1(s) = \sum_{j=1}^{3}\sum_{\mu=1}^{2}s_{j\mu}\,\theta_j\overline\varphi_\mu, \;\text{ so }\; \rho_1(s)\lrcorner u = \sum_{j=1}^{3}(-1)^{j-1}\sum_{\mu=1}^{2}s_{j\mu}\,\overline\varphi_\mu\wedge\widehat\varphi_j,$$

where $\widehat\varphi_j$ stands for $\varphi_1\wedge\varphi_2\wedge\varphi_3$ with φ_j omitted.

Since $\psi_1(t)\lrcorner u, \rho_1(s)\lrcorner u \in \ker\bar\partial$, $\psi_1(t)$ and $\rho_1(s)$ are $\bar\partial$-closed for the $\bar\partial$ of the holomorphic structure of $T^{1,\,0}X$, hence $\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u) \in \mathcal{Z}_1^{1,\,2}$. Moreover, since $\psi_1(t)\lrcorner u, \rho_1(s)\lrcorner u \in \ker\partial$, the so-called Tian-Todorov Lemma (see [Tia87], [Tod89]) ensures that

$$\partial(\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u)) = [\psi_1(t)\lrcorner u,\,\rho_1(s)\lrcorner u],$$

where $[\psi_1(t)\lrcorner u,\, \rho_1(s)\lrcorner u]$ is the scalar-valued $(n-1,\, 2)$-form defined by the identity $[\psi_1(t)\lrcorner u,\, \rho_1(s)\lrcorner u] = [\psi_1(t),\, \rho_1(s)]\lrcorner u$. So, we have to show that $[\psi_1(t)\lrcorner u,\, \rho_1(s)\lrcorner u]$ is $\bar\partial$-exact. We get:

$$[\psi_1(t),\, \rho_1(s)] = \sum_{1\leq i,\, j\leq 3}\, \sum_{1\leq \lambda,\, \mu\leq 2} t_{i\lambda}\, s_{j\mu}\, [\theta_i,\, \theta_j]\, \overline{\varphi}_\lambda\wedge\overline{\varphi}_\mu = D_{t,s}\, \theta_3\, \overline{\varphi}_1\wedge\overline{\varphi}_2,$$

where $D_{t,s} = (t_{11}\, s_{22} + t_{22}\, s_{11} - t_{12}\, s_{21} - t_{21}\, s_{12})$. Hence,

$$[\psi_1(t),\, \rho_1(s)]\lrcorner u = D_{t,s}\, \varphi_1\wedge\varphi_2\wedge\overline{\varphi}_1\wedge\overline{\varphi}_2 = \bar\partial(D_{t,s}\, \partial\varphi_3\wedge\overline{\varphi}_3) = \bar\partial\partial(D_{t,s}\, \varphi_3\wedge\overline{\varphi}_3) \in \mathrm{Im}\, \bar\partial,$$

as desired.

We conclude that $\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u) \in \mathcal{Z}_1^{1,\,2}$ and $\partial(\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u)) \in \mathrm{Im}\, \bar\partial$, hence $\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u) \in \mathcal{Z}_2^{1,\,2}$, as desired.

(b) Let $\psi_1(t), \rho_1(s) \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$ such that $\psi_1(t)\lrcorner u \in \ker d$ and $\rho_1(s)\lrcorner u \in \mathrm{Im}\, \partial$. Then, $\psi_1(t) = \sum_{1\leq i\leq 3}\sum_{1\leq\lambda\leq 2} t_{i\lambda}\, \theta_i\overline{\varphi}_\lambda$ and $\rho_1(s) = (\sum_{1\leq\mu\leq 3} s_\mu\, \overline{\varphi}_\mu)\, \theta_3$, so

$$\rho_1(s)\lrcorner u = \sum_{1\leq\mu\leq 3} s_\mu\, \overline{\varphi}_\mu\wedge\varphi_1\wedge\varphi_2 = \partial(-\sum_{1\leq\mu\leq 3} s_\mu\, \varphi_3\wedge\overline{\varphi}_\mu) \in \mathrm{Im}\, \partial.$$

On the one hand, we get $\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u) = \displaystyle\sum_{\lambda=1}^{2}\sum_{\mu=1}^{3} t_{1\lambda}\, s_\mu\, \overline{\varphi}_\lambda\wedge\overline{\varphi}_\mu\wedge\varphi_2 - \sum_{\lambda=1}^{2}\sum_{\mu=1}^{3} t_{2\lambda}\, s_\mu\, \overline{\varphi}_\lambda\wedge\overline{\varphi}_\mu\wedge\varphi_1$, hence $\bar\partial(\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u)) = -\displaystyle\sum_{\lambda=1}^{2} t_{1\lambda}\, s_3\, \overline{\varphi}_\lambda\wedge\bar\partial\overline{\varphi}_3\wedge\varphi_2 + \sum_{\lambda=1}^{2} t_{2\lambda}\, s_3\, \overline{\varphi}_\lambda\wedge\bar\partial\overline{\varphi}_3\wedge\varphi_1 = 0$ because $\bar\partial\overline{\varphi}_3 = -\overline{\varphi}_1\wedge\overline{\varphi}_2$. Thus, $\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u) \in \ker\bar\partial$.

On the other hand, since $[\theta_i,\, \theta_3] = 0$ for all i, we get

$$\partial(\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u)) = [\psi_1(t),\, \rho_1(s)]\lrcorner u = \sum_{i=1}^{3}\sum_{\lambda=1}^{2}\sum_{\mu=1}^{3} t_{i\lambda}\, s_\mu\, \overline{\varphi}_\lambda\wedge\overline{\varphi}_\mu\, [\theta_i,\, \theta_3] = 0$$

We conclude that $\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u) \in \mathcal{Z}_2^{1,\,2}$.

(c) If $\psi_1(t), \rho_1(s) \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$ are such that $\psi_1(t)\lrcorner u$ and $\rho_1(s)\lrcorner u$ both lie in $\mathrm{Im}\, \partial$, then $\psi_1(t) = (\sum_{1\leq\lambda\leq 3} t_\lambda\, \overline{\varphi}_\lambda)\, \theta_3$ and $\rho_1(s) = (\sum_{1\leq\mu\leq 3} s_\mu\, \overline{\varphi}_\mu)\, \theta_3$. We get

$$\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u) = -(\sum_{1\leq\lambda\leq 3} t_\lambda\, \overline{\varphi}_\lambda)\wedge\sum_{1\leq\mu\leq 3} s_\mu\, \overline{\varphi}_\mu\wedge[\theta_3\lrcorner(\varphi_1\wedge\varphi_2)] = 0$$

since $\theta_3\lrcorner\varphi_1 = \theta_3\lrcorner\varphi_2 = 0$. In particular, $\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u) \in \mathcal{Z}_2^{1,\,2}$.

- *Case where* $X = I^{(5)}$. We use the notation of Example 6.8.10.

(a) Let $\psi_1(t), \rho_1(s) \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$ such that $\psi_1(t)\lrcorner u, \rho_1(s)\lrcorner u \in \ker d$. Then,

$$\psi_1(t) = \sum_{i=1}^{5}\sum_{\lambda=1}^{2} t_{i\lambda}\, \theta_i\overline{\varphi}_\lambda, \;\text{ so }\; \psi_1(t)\lrcorner u = \sum_{i=1}^{5}(-1)^{i-1}\sum_{\lambda=1}^{2} t_{i\lambda}\, \overline{\varphi}_\lambda\wedge\widehat{\varphi}_i,$$

$$\rho_1(s) = \sum_{j=1}^{5}\sum_{\mu=1}^{2} s_{j\mu}\,\theta_j\overline{\varphi}_\mu, \ \text{ so } \ \rho_1(s)\lrcorner u = \sum_{j=1}^{5}(-1)^{j-1}\sum_{\mu=1}^{2} s_{j\mu}\,\overline{\varphi}_\mu\wedge\widehat{\varphi}_j,$$

where $\widehat{\varphi}_j$ stands for $\varphi_1\wedge\cdots\wedge\varphi_5$ with φ_j omitted.

Since $[\theta_i,\,\theta_j]=0$ unless $\{i,\,j\}\subset\{1,\,2,\,3\}$ and given the other values for $[\theta_i,\,\theta_j]$, we get:

$$\begin{aligned}[\psi_1(t),\,\rho_1(s)]\lrcorner u &= -D_3(t,\,s)\,\varphi_1\wedge\varphi_2\wedge\varphi_4\wedge\varphi_5\wedge\overline{\varphi}_1\wedge\overline{\varphi}_2 + D_2(t,\,s)\,\varphi_1\wedge\varphi_2\wedge\varphi_3\wedge\varphi_5\wedge\overline{\varphi}_1\wedge\overline{\varphi}_2\\ &- D_1(t,\,s)\,\varphi_1\wedge\varphi_2\wedge\varphi_3\wedge\varphi_4\wedge\overline{\varphi}_1\wedge\overline{\varphi}_2, \qquad \text{where}\end{aligned}$$

$$D_3(t,\,s)=\begin{vmatrix}t_{11} & t_{12}\\ s_{21} & s_{22}\end{vmatrix}-\begin{vmatrix}s_{11} & s_{12}\\ t_{21} & t_{22}\end{vmatrix},\ D_2(t,\,s)=\begin{vmatrix}t_{11} & t_{12}\\ s_{31} & s_{32}\end{vmatrix}-\begin{vmatrix}s_{11} & s_{12}\\ t_{31} & t_{32}\end{vmatrix},\ D_1(t,\,s)=\begin{vmatrix}t_{21} & t_{22}\\ s_{31} & s_{32}\end{vmatrix}-\begin{vmatrix}s_{21} & s_{22}\\ t_{31} & t_{32}\end{vmatrix}.$$

Now, since $\varphi_1\wedge\varphi_2=\partial\varphi_3$ and $\bar{\varphi}_1\wedge\overline{\varphi}_2=\bar\partial\overline{\varphi}_3$, using also the other properties of the φ_i's, we get

$$\begin{aligned}\varphi_1\wedge\varphi_2\wedge\varphi_4\wedge\varphi_5\wedge\overline{\varphi}_1\wedge\overline{\varphi}_2 &= \bar\partial\partial(\varphi_3\wedge\varphi_4\wedge\varphi_5\wedge\overline{\varphi}_3)\\ \varphi_1\wedge\varphi_2\wedge\varphi_3\wedge\varphi_5\wedge\overline{\varphi}_1\wedge\overline{\varphi}_2 &= \bar\partial\partial(\varphi_2\wedge\varphi_4\wedge\varphi_5\wedge\overline{\varphi}_3).\end{aligned}$$

Similarly, since $\varphi_2\wedge\varphi_3=\partial\varphi_5$ and $\bar{\varphi}_1\wedge\overline{\varphi}_2=\bar\partial\overline{\varphi}_3$, we get

$$\varphi_1\wedge\varphi_2\wedge\varphi_3\wedge\varphi_4\wedge\overline{\varphi}_1\wedge\overline{\varphi}_2=\bar\partial\partial(\varphi_1\wedge\varphi_4\wedge\varphi_5\wedge\overline{\varphi}_3).$$

We conclude that $\partial(\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u))=[\psi_1(t),\,\rho_1(s)]\lrcorner u\in\operatorname{Im}(\partial\bar\partial)\subset\operatorname{Im}\bar\partial$. Meanwhile, $\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u)$ is $\bar\partial$-closed (because $\psi_1(t)\lrcorner u$ and $\rho_1(s)\lrcorner u$ are), hence $\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u)\in\mathcal{Z}_2^{4,1}$, as desired.

(b) Let $\psi_1(t),\rho_1(s)\in C^\infty_{0,1}(X,\,T^{1,0}X)$ such that $\psi_1(t)\lrcorner u\in\ker d$ and $\rho_1(s)\lrcorner u\in\operatorname{Im}\partial$. Then,

$$\psi_1(t)=\sum_{i=1}^{5}\sum_{\lambda=1}^{2} t_{i\lambda}\,\theta_i\overline{\varphi}_\lambda, \ \text{ so } \ \psi_1(t)\lrcorner u=\sum_{i=1}^{5}(-1)^{i-1}\sum_{\lambda=1}^{2} t_{i\lambda}\,\overline{\varphi}_\lambda\wedge\widehat{\varphi}_i,$$

$$\rho_1(s)=\sum_{j=3}^{5} s_j\,\theta_j\overline{\varphi}_3, \ \text{ so } \ \rho_1(s)\lrcorner u=\sum_{j=3}^{5}(-1)^{j-1}\,s_j\,\overline{\varphi}_3\wedge\widehat{\varphi}_j.$$

Indeed, in the case of $\rho_1(s)\lrcorner u$, we have

$$\begin{aligned}&\widehat{\varphi}_3=\partial(\varphi_3\wedge\varphi_4\wedge\varphi_5),\ \text{so}\ \overline{\varphi}_3\wedge\widehat{\varphi}_3=-\partial(\overline{\varphi}_3\wedge\varphi_3\wedge\varphi_4\wedge\varphi_5),\\ &\widehat{\varphi}_4=\partial(\varphi_2\wedge\varphi_4\wedge\varphi_5),\ \text{so}\ \overline{\varphi}_3\wedge\widehat{\varphi}_4=-\partial(\overline{\varphi}_3\wedge\varphi_2\wedge\varphi_4\wedge\varphi_5),\\ &\widehat{\varphi}_5=\partial(\varphi_1\wedge\varphi_4\wedge\varphi_5),\ \text{so}\ \overline{\varphi}_3\wedge\widehat{\varphi}_5=-\partial(\overline{\varphi}_3\wedge\varphi_1\wedge\varphi_4\wedge\varphi_5)\end{aligned}$$

and every ∂-exact $(4,\,1)$-form is a linear combination of $\overline{\varphi}_3\wedge\widehat{\varphi}_3$, $\overline{\varphi}_3\wedge\widehat{\varphi}_4$ and $\overline{\varphi}_3\wedge\widehat{\varphi}_5$.

On the one hand, we get

$$\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u)=\sum_{i=1}^{5}\sum_{j=3}^{5}\sum_{\lambda=1}^{2}(-1)^{j-1}\,t_{i\lambda}\,s_j\,\overline{\varphi}_\lambda\wedge\overline{\varphi}_3\wedge(\theta_i\lrcorner\widehat{\varphi}_j).$$

Now, $\theta_i\lrcorner\widehat{\varphi}_j$ is always $\bar\partial$-closed because it vanishes when $i=j$, it equals $(-1)^{i-1}\widehat{\varphi}_{ij}$ when $i<j$ and it equals $(-1)^i\widehat{\varphi}_{ji}$ when $i>j$, where $\widehat{\varphi}_{ij}$ stands for $\varphi_1\wedge\cdots\wedge\varphi_5$ with φ_i and φ_j omitted and $i<j$. All the φ_i's being $\bar\partial$-closed, so are all the $\widehat{\varphi}_{ij}$'s. Meanwhile, $\bar\partial(\overline{\varphi}_\lambda\wedge\overline{\varphi}_3)=-\overline{\varphi}_\lambda\wedge\bar\partial\overline{\varphi}_3=0$ for all $\lambda\in\{1,\,2\}$, since $\bar\partial\overline{\varphi}_3=\overline{\varphi}_1\wedge\overline{\varphi}_2$. This proves that $\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u)\in\ker\bar\partial$.

On the other hand, we get

$$\begin{aligned}
\partial(\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u)) &= [\psi_1(t),\, \rho_1(s)]\lrcorner u = \sum_{i=1}^{5}\sum_{j=3}^{5}\sum_{\lambda=1}^{2} t_{i\lambda}\, s_j\, \overline{\varphi}_\lambda \wedge \overline{\varphi}_3 \wedge ([\theta_i,\, \theta_j]\lrcorner u) \\
&= -\sum_{\lambda=1}^{2} t_{1\lambda}\, s_3\, \overline{\varphi}_\lambda \wedge \overline{\varphi}_3 \wedge (\theta_4 \lrcorner u) - \sum_{\lambda=1}^{2} t_{2\lambda}\, s_3\, \overline{\varphi}_\lambda \wedge \overline{\varphi}_3 \wedge (\theta_5 \lrcorner u) \\
&= t_{11}\, s_3\, \overline{\varphi}_1 \wedge \overline{\varphi}_3 \wedge \widehat{\varphi}_4 + t_{12}\, s_3\, \overline{\varphi}_2 \wedge \overline{\varphi}_3 \wedge \widehat{\varphi}_4 - t_{21}\, s_3\, \overline{\varphi}_1 \wedge \overline{\varphi}_3 \wedge \widehat{\varphi}_5 - t_{22}\, s_3\, \overline{\varphi}_2 \wedge \overline{\varphi}_3 \wedge \widehat{\varphi}_5 \\
&= t_{11}\, s_3\, \bar\partial\overline{\varphi}_4 \wedge \widehat{\varphi}_4 + t_{12}\, s_3\, \bar\partial\overline{\varphi}_5 \wedge \widehat{\varphi}_4 - t_{21}\, s_3\, \bar\partial\overline{\varphi}_4 \wedge \widehat{\varphi}_5 - t_{22}\, s_3\, \bar\partial\overline{\varphi}_5 \wedge \widehat{\varphi}_5 \\
&= \bar\partial(t_{11}\, s_3\, \overline{\varphi}_4 \wedge \widehat{\varphi}_4 + t_{12}\, s_3\, \overline{\varphi}_5 \wedge \widehat{\varphi}_4 - t_{21}\, s_3\, \overline{\varphi}_4 \wedge \widehat{\varphi}_5 - t_{22}\, s_3\, \overline{\varphi}_5 \wedge \widehat{\varphi}_5) \in \operatorname{Im}\bar\partial,
\end{aligned}$$

where the second line followed from the fact that $[\theta_i,\, \theta_j] = 0$ unless $i, j \in \{1,\, 2,\, 3\}$ and $i \neq j$. Given the fact that the summation bears over $j \in \{3,\, 4,\, 5\}$, this forces $j = 3$ and $i \in \{1,\, 2\}$. Then, we get the second line from $[\theta_1,\, \theta_3] = -\theta_4$ and $[\theta_2,\, \theta_3] = -\theta_5$.

The facts that $\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u) \in \ker\bar\partial$ and $\partial(\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u)) \in \operatorname{Im}\bar\partial$ translate to $\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u) \in \mathcal{Z}_2^{3,\,2}$, as desired.

(c) Let $\psi_1(t), \rho_1(s) \in C^\infty_{0,\,1}(X,\, T^{1,\,0}X)$ such that $\psi_1(t)\lrcorner u, \rho_1(s)\lrcorner u \in \operatorname{Im}\partial$. Then,

$$\psi_1(t) = \sum_{i=3}^{5} t_i\, \theta_i \overline{\varphi}_3, \quad \rho_1(s) = \sum_{j=3}^{5} s_j\, \theta_j \overline{\varphi}_3, \ \text{ so } \ \rho_1(s)\lrcorner u = \sum_{j=3}^{5} (-1)^{j-1}\, s_j\, \overline{\varphi}_3 \wedge \widehat{\varphi}_j.$$

We get

$$\psi_1(t)\lrcorner(\rho_1(s)\lrcorner u) = \sum_{i=3}^{5}\sum_{j=3}^{5} (-1)^{j-1}\, t_i\, s_j\, \overline{\varphi}_3 \wedge \overline{\varphi}_3 \wedge (\theta_i \lrcorner \widehat{\varphi}_j) = 0 \in \mathcal{Z}_2^{3,\,2},$$

as desired.

This completes the proof of Lemma 6.8.16. □

6.9 Self-duality of the Iwasawa manifold in terms of the Albanese map

This section, taken from [Pop18b], is a complement from a different perspective to sections §.6.2-6.7 where a non-Kähler mirror symmetry of the Iwasawa manifold was described.

We now give another criterion of a different nature by which the Iwasawa manifold $I^{(3)}$ is self-dual in a sesquilinear way. It states that in the well-known description of $I^{(3)}$ as a locally holomorphically trivial fibration by elliptic curves over a two-dimensional complex torus (its *Albanese torus* $\mathrm{Alb}(I^{(3)})$), both the base and the fibre are self-dual tori. This means that the base torus $\mathrm{Alb}(I^{(3)})$ identifies canonically with its dual torus, the *Jacobian torus* of $I^{(3)}$, under a sesquilinear duality, while the fibre identifies with itself.

The *Albanese torus* and *map* of the Iwasawa manifold are manifestations of the Albanese torus and map (otherwise known to always be abstractly defined) we explicitly construct in full generality on any *sGG manifold* by means of Hodge theory duly adapted to the specific context of possibly non-Kähler sGG manifolds. This construction occupies subsections 6.9.1-6.9.3.

Our hope, motivating in part this section, is that the sesquilinear duality between the explicitly constructed Albanese and Jacobian tori of an arbitrary sGG manifold will show the way in the future to guessing the mirror dual of more general sGG manifolds that may not be mirror self-dual.

Our starting point in defining the *Jacobian* and *Albanese tori* and *maps* for arbitrary sGG manifolds will be Theorem 4.3.9. In particular, if X is an n-dimensional compact complex **sGG manifold**, the canonical splittings (4.30) and (4.31) of $H^1_{DR}(X,\,\mathbb{C})$ and resp. $H^{2n-1}_{DR}(X,\,\mathbb{C})$ are the weaker substitutes for the Hodge decomposition in degrees 1, resp. $2n-1$, afforded to sGG manifolds.

Corollary 6.9.1. *For every* **sGG manifold** X*, the Dolbeault cohomology group* $H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})$ **injects canonically** *into the De Rham cohomology group* $H^1_{DR}(X,\,\mathbb{C})$*. The canonical injection* $j:H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})\hookrightarrow H^1_{DR}(X,\,\mathbb{C})$ *is obtained as the composition of the injective linear maps*

$$H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})\hookrightarrow H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})\oplus\overline{H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})}\stackrel{F^{-1}}{\longrightarrow}H^1_{DR}(X,\,\mathbb{C}).$$

Proof. The sGG assumption ensures that the canonical linear map F defined in (4.30) is an isomorphism. Then so is its inverse F^{-1}. □

The canonical splittings (4.30) and (4.31) enable one to construct canonically and explicitly the *Jacobian variety* (cf. Definition 6.9.2) and the *Albanese variety* (cf. Definition 6.9.3) of any sGG manifold by imitating the classical constructions on compact Kähler (or merely $\partial\bar\partial$) manifolds with the necessary modifications. The details are spelt out in §.6.9.1 and §.6.9.2.

6.9.1 The Jacobian variety of an sGG manifold

Let X be an sGG manifold with $\dim_{\mathbb{C}}X=n$. The inclusions $\mathbb{Z}\subset\mathbb{R}\subset\mathbb{C}\subset\mathcal{O}$ induce morphisms

$$H^1(X,\,\mathbb{Z})\longrightarrow H^1(X,\,\mathbb{R})\longrightarrow H^1(X,\,\mathbb{C})\longrightarrow H^1(X,\,\mathcal{O})$$

where the image of $H^1(X,\,\mathbb{Z})$ is a lattice in $H^1(X,\,\mathbb{R})$. On the other hand, the map $H^1(X,\,\mathbb{R})\to H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})$ obtained by composing the maps $H^1(X,\,\mathbb{R})\to H^1(X,\,\mathbb{C})\to H^1(X,\,\mathcal{O})\simeq H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})$ identifies canonically with the composite map

$$H^1_{DR}(X,\,\mathbb{R})\stackrel{j_1}{\hookrightarrow}H^1_{DR}(X,\,\mathbb{C})\stackrel{p_1\circ F}{\longrightarrow}H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C}),$$

where j_1 is the natural injection and $p_1\;:\;H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})\oplus\overline{H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})}\longrightarrow H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})$ is the projection onto the first factor. Since F is an **isomorphism** (thanks to X being **sGG**), we get that

$$p_1\circ F\circ j_1\;:\;H^1_{DR}(X,\,\mathbb{R})\longrightarrow H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})$$

is an isomorphism. Hence $\operatorname{Im}H^1(X,\,\mathbb{Z})$ is a lattice in $H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})$. As a result, we can put

Definition 6.9.2. *The* **Jacobian variety** *of an n-dimensional sGG manifold X is defined exactly as in the Kähler (or merely $\partial\bar\partial$) case as the q-dimensional complex torus*

$$Jac(X):=H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})/Im\,H^1(X,\,\mathbb{Z}),\tag{6.84}$$

where $q:=h^{0,\,1}_{\bar\partial}(X)$ stands for the irregularity of X.

6.9.2 The Albanese variety of an sGG manifold

Let once again X be an sGG manifold with $\dim_{\mathbb{C}} X = n$. In a way similar to the above discussion, we have morphisms

$$H^{2n-1}(X,\,\mathbb{Z}) \longrightarrow H^{2n-1}(X,\,\mathbb{R}) \stackrel{j_{2n-1}}{\longrightarrow} H^{2n-1}(X,\,\mathbb{C}) \stackrel{(F^\star)^{-1}}{\longrightarrow} H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C}) \oplus \overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})},$$

where $\mathrm{Im}\,H^{2n-1}(X,\,\mathbb{Z})$ is a lattice in $H^{2n-1}(X,\,\mathbb{R})$ (a general feature of any compact complex manifold X) and $(F^\star)^{-1}$ is an **isomorphism** (thanks to X being **sGG**). If we denote by $p_2\ :\ H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C}) \oplus \overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})} \longrightarrow \overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})}$ the projection onto the second factor, then

$$p_2 \circ (F^\star)^{-1} \circ j_{2n-1}\ :\ H^{2n-1}_{DR}(X,\,\mathbb{R}) \longrightarrow \overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})}$$

is an isomorphism and therefore $\mathrm{Im}\,H^{2n-1}(X,\,\mathbb{Z})$ is a lattice in $\overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})} \simeq (\overline{H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})})^\star$.

Definition 6.9.3. *The* **Albanese variety** *of an n-dimensional sGG manifold X is the complex torus*

$$Alb(X) := \overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})}/Im\,H^{2n-1}(X,\,\mathbb{Z}) = \left(\overline{H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})}\right)^\star/Im\,H^1(X,\,\mathbb{Z})^\star. \tag{6.85}$$

The spaces $H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})$ and $H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})$ are dual under the Serre duality, while $H^{2n-1}(X,\,\mathbb{Z})$ and $H^1(X,\,\mathbb{Z})$ are Poincaré dual.

Recall that in the standard case when X is Kähler (or merely $\partial\bar\partial$), the Albanese torus of X is defined as the quotient

$$H^{n-1,\,n}(X,\,\mathbb{C})/\mathrm{Im}\,H^{2n-1}(X,\,\mathbb{Z}).$$

Since, by Hodge symmetry, the conjugation defines an isomorphism $H^{n-1,\,n}_{\bar\partial}(X,\,\mathbb{C}) \simeq \overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})}$ when X is Kähler (or merely $\partial\bar\partial$), our Definition 6.9.3 of the Albanese torus coincides with the standard defintion in the Kähler and $\partial\bar\partial$ cases.

Conclusion 6.9.4. *We can now conclude from Definitions 6.9.2 and 6.9.3 that the* **Jacobian torus** *and the* **Albanese torus** *of any sGG manifold X are* **dual tori** *in the sense of the following* **sesquilinear duality** *obtained by composing the bilinear Serre duality with the conjugation in the second factor:*

$$H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C}) \times \overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})} \longrightarrow \mathbb{C}, \qquad ([\alpha]_{\bar\partial},\,\overline{[\beta]_{\bar\partial}}) \mapsto \int\limits_X \alpha\wedge\beta. \tag{6.86}$$

6.9.3 The Albanese map of an sGG manifold

We can now easily adapt to the general context of sGG manifolds X the construction of the Albanese map $\alpha : X \longrightarrow \mathrm{Alb}(X)$ from the familiar Kähler (or merely $\partial\bar\partial$) case. We shall follow the presentation and use the notation of [Dem97, §.9.2].

Let X be an sGG manifold with $\dim_{\mathbb{C}} X = n$. The standard isomorphism

$$H_1(X,\,\mathbb{Z}) \longrightarrow H^{2n-1}(X,\,\mathbb{Z})$$

given by the Poincaré duality is induced by the map $[\xi] \mapsto \{I_\xi\}_{DR} \in H^{2n-1}_{DR}(X, \mathbb{R})$ associating with the homology class $[\xi]$ of every loop ξ in X the De Rham cohomology class of the current of integration I_ξ over ξ. Using this isomorphism, the expression (6.85) of the Albanese torus of X transforms to

$$\mathrm{Alb}(X) = \left(\overline{H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})}\right)^\star / \mathrm{Im}\, H_1(X,\,\mathbb{Z}), \tag{6.87}$$

where the map $H_1(X,\,\mathbb{Z}) \longrightarrow \overline{H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})}^\star$ is defined by

$$[\xi] \mapsto \widetilde{I}_\xi := \left(\overline{[v]} \mapsto \int\limits_\xi \overline{\{v\}}\right), \quad \text{where } \{v\} := j([v]) \in H^1_{DR}(X,\,\mathbb{C}). \tag{6.88}$$

We have used the canonical injection $j : H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C}) \hookrightarrow H^1_{DR}(X,\,\mathbb{C})$ defined in Corollary 6.9.1 and the fact that the integral $\int_\xi \overline{\{v\}}$ depends only on the homology class $[\xi]$ and on the cohomology class $\overline{\{v\}}$ (so not on the actual representatives of these classes).

Definition 6.9.5. *Let X be an sGG manifold. Fix a base point $a \in X$. For every point $x \in X$, let ξ be any path from a to x and let $\widetilde{I}_\xi \in \overline{H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})}^\star$ be the linear functional defined in (6.88).*

The canonical holomorphic map

$$\alpha : X \longrightarrow Alb(X) = \left(\overline{H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})}\right)^\star / Im\, H_1(X,\,\mathbb{Z}), \quad x \mapsto \widetilde{I}_\xi \quad mod\ Im\, H_1(X,\,\mathbb{Z}), \tag{6.89}$$

will be called the **Albanese map** *of the sGG manifold X.*

Note that the class of $\widetilde{I}_\xi$ modulo $\mathrm{Im}\, H_1(X,\,\mathbb{Z})$ does not depend on the choice of path ξ from a to x because for any other such path η, $\widetilde{I}_{\eta^{-1}\xi} \in \mathrm{Im}\, H_1(X,\,\mathbb{Z})$. Also note that definition (6.89) of the Albanese map for sGG manifolds X coincides with the standard definition when X is Kähler or just $\partial\bar\partial$. Indeed, in the Kähler and $\partial\bar\partial$ cases, $\overline{H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C})}$ is canonically isomorphic to $H^{1,\,0}_{\bar\partial}(X,\,\mathbb{C})$ by Hodge symmetry. Moreover, the role played by the canonical injection $j : H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C}) \hookrightarrow H^1_{DR}(X,\,\mathbb{C})$ defined in Corollary 6.9.1 when X is sGG is an apt substitute for the fact that every holomorphic 1-form (i.e. the unique representative of every element in $H^{1,\,0}_{\bar\partial}(X,\,\mathbb{C})$) is d-closed when X is Kähler or merely $\partial\bar\partial$.

As in the standard Kähler case, we have an alternative description of the Albanese map.

Observation 6.9.6. *Let X be an sGG manifold with $dim_{\mathbb{C}} X = n$. Using the expression (6.85) of the Albanese torus of X, the Albanese map of X is given by*

$$\alpha : X \longrightarrow Alb(X) = \overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})} / Im\, H^{2n-1}(X,\,\mathbb{Z}), \quad x \mapsto \overline{\{I_\xi\}^{n,\,n-1}} \quad mod\ Im\, H^{2n-1}(X,\,\mathbb{Z}),$$

where $\overline{\{I_\xi\}^{n,\,n-1}} \in \overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})}$ is the projection of the De Rham cohomology class $\{I_\xi\}_{DR} \in H^{2n-1}_{DR}(X,\,\mathbb{R})$ onto $\overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})}$ w.r.t. the isomorphism

$$(F^\star)^{-1} : H^{2n-1}_{DR}(X,\,\mathbb{C}) \xrightarrow{\simeq} H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C}) \oplus \overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})}$$

induced by (4.31). As usual, I_ξ stands for the current of integration over the path ξ from a to x in X.

Note that in Observation 6.9.6 the only difference in the sGG case compared with the standard Kähler (or $\partial\bar\partial$) case is the substitution of $\overline{H^{n,\,n-1}_{\bar\partial}(X,\,\mathbb{C})}$ for $H^{n-1,\,n}_{\bar\partial}(X,\,\mathbb{C})$. These spaces are isomorphic by Hodge symmetry when X is Kähler or merely $\partial\bar\partial$.

6.9.4 Application of the Albanese and Jacobian tori of sGG manifolds to the mirror self-duality of the Iwasawa manifold

In this section, we apply the above constructions to the Iwasawa manifold that is known to not be a $\partial\bar\partial$-manifold (cf. Proposition 1.3.22). However, the Iwasawa manifold $X = X_0$ and all its small deformations in its Kuranishi family $(X_t)_{t\in B}$ are sGG compact complex manifolds of dimension 3 (cf. Corollary 4.3.5). So, the extension to the sGG context of the classical constructions of the Albanese torus and map from the $\partial\bar\partial$-case, performed in §.6.9.2 and §.6.9.3, is key to our purposes here.

For the Iwasawa manifold $X = X_0$ and all its small deformations $(X_t)_{t\in B}$, the Albanese maps

$$\pi_t : X_t \longrightarrow \mathrm{Alb}(X_t) := B_t, \qquad t\in B,$$

have simple explicit descriptions and $\pi := \pi_0 : X_0 \to B_0$ is a locally holomorphically trivial fibration whose fibre $\pi^{-1}(s)$ is the Gauss elliptic curve $\mathbb{C}/\mathbb{Z}[i]$ and whose base is the 2-dimensional complex torus $\mathbb{C}/\mathbb{Z}[i]\times\mathbb{C}/\mathbb{Z}[i]$.

First, we show that the Albanese torus of every small deformation X_t of the Iwasawa manifold $X = X_0$ is **self-dual** in the context of the constructions of §.6.9.1 and §.6.9.2.

Lemma 6.9.7. *Let $(X_t)_{t\in B}$ be the Kuranishi family of the Iwasawa manifold $X = X_0$. Thus $n = \dim_{\mathbb{C}} X_t = 3$. For every $t\in B$ sufficiently close to 0, the dual Jacobian and Albanese tori $Jac(X_t)$ and $Alb(X_t)$ can be identified* **canonically** *in the following sense.*

There exist **canonical** *isomorphisms*

$$H^{0,\,1}_{\bar\partial}(X_t,\,\mathbb{C}) \simeq H^{3,\,2}_{\bar\partial}(X_t,\,\mathbb{C}) \quad \textit{and} \quad H^1(X_t,\,\mathbb{Z}) \simeq H^5(X_t,\,\mathbb{Z}), \qquad t\in B. \tag{6.90}$$

Proof. Dual finite-dimensional vector spaces are, of course, isomorphic, so the main feature of the isomorphisms (6.90) is their canonical nature. By "canonical" we mean "depending only on the complex or differential structure, independent of any choice of metric". As can be seen below, the canonical nature of these isomorphisms follows from the existence of canonical bases, defined by the structural differential forms $\alpha_t, \beta_t, \gamma_t$ mentioned in the introduction and their conjugates, in the vector spaces involved.

From [Sch07, p.6] and [Ang14, §.2.2.2, §.2.2.3], we gather that the vector spaces featuring in (6.90) are generated by the structural $(1,\,0)$-forms $\alpha_t, \beta_t, \gamma_t$ as follows:

$$\begin{aligned}
H^{0,\,1}_{\bar\partial}(X_t,\,\mathbb{C}) &= \left\langle [\bar\alpha_t]_{\bar\partial},\, [\bar\beta_t]_{\bar\partial}\right\rangle, \qquad H^{3,\,2}_{\bar\partial}(X_t,\,\mathbb{C}) = \left\langle [\alpha_t\wedge\beta_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\gamma_t]_{\bar\partial},\, [\alpha_t\wedge\beta_t\wedge\gamma_t\wedge\bar\beta_t\wedge\bar\gamma_t]_{\bar\partial}\right\rangle,\\
H^1_{DR}(X_t,\,\mathbb{C}) &= \left\langle \{\alpha_t\},\, \{\beta_t\},\, \{\bar\alpha_t\},\, \{\bar\beta_t\}\right\rangle, \qquad\qquad (6.91)\\
H^5_{DR}(X_t,\,\mathbb{C}) &= \Big\langle \{\alpha_t\wedge\beta_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\gamma_t\},\, \{\alpha_t\wedge\beta_t\wedge\gamma_t\wedge\bar\beta_t\wedge\bar\gamma_t\},\, \{\alpha_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\beta_t\wedge\bar\gamma_t\},\\
&\qquad \{\beta_t\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\beta_t\wedge\bar\gamma_t\}\Big\rangle,
\end{aligned}$$

where { } stands for De Rham cohomology classes.

Thus, the isomorphism $H^{0,\,1}_{\bar\partial}(X_t,\,\mathbb{C}) \simeq H^{3,\,2}_{\bar\partial}(X_t,\,\mathbb{C})$ of (6.90) is canonically defined by $[\bar\xi]_{\bar\partial} \mapsto [\bar\xi\wedge\alpha_t\wedge\beta_t\wedge\gamma_t\wedge\bar\gamma_t]_{\bar\partial}$ for $\xi\in\{\alpha_t,\,\beta_t\}$, while the isomorphism $H^1_{DR}(X_t,\,\mathbb{C})\simeq H^5_{DR}(X_t,\,\mathbb{C})$ is canonically defined by $\{\zeta\}\mapsto\{\zeta\wedge\alpha_t\wedge\beta_t\wedge\gamma_t\wedge\bar\gamma_t\}$ for $\zeta\in\{\bar\alpha_t,\,\bar\beta_t\}$ and by $\{\zeta\}\mapsto\{\zeta\wedge\gamma_t\wedge\bar\alpha_t\wedge\bar\beta_t\wedge\bar\gamma_t\}$ for $\zeta\in\{\alpha_t,\,\beta_t\}$. $\square$

Now, we recall two standard facts that prove between them that every elliptic curve (in particular, the fibre of the Albanese map $\pi := \pi_0 : X_0 \to B_0$) is **self-dual**.

Proposition 6.9.8. *(see e.g. [Dem97, §.10.2]) Let X be a compact complex manifold such that $dim_{\mathbb{C}}X = 1$ (i.e. X is a compact* **complex curve***).*

(i) The Jacobian torus $Jac(X)$ of X coincides with its Albanese torus $Alb(X)$. Moreover, for every point $a\in X$, the Jacobi map

$$\Phi_a : X \longrightarrow Jac(X), \qquad x\mapsto \mathcal{O}([x]-[a]),$$

coincides with the Albanese map

$$\alpha : X \longrightarrow Alb(X) = Jac(X).$$

(ii) If X is an **elliptic curve** *(i.e. $g=1$, where $g := h^{0,\,1}(X)$ is the genus of the complex curve X), then $\Phi_a = \alpha$ is an* **isomorphism***, i.e.*

$$X \simeq Jac(X) = Alb(X).$$

In particular, since the dual tori $Jac(X)$ and $Alb(X)$ coincide, X is self-dual.

We can now infer the main result of this paper showing that the Iwasawa manifold is its own dual in a simple sense pertaining to its Albanese torus and map. This self-duality point of view complements those considered in [Pop17].

Theorem 6.9.9. *The Iwasawa manifold $X = X_0$ is* **its own dual** *in the sense that in its Albanese map description*

$$\pi = \pi_0 : X_0 \longrightarrow B_0 := Alb(X_0)$$

as a locally holomorphically trivial fibration by elliptic curves $\mathbb{C}/\mathbb{Z}[i]$ over the 2-dimensional complex torus $\mathbb{C}/\mathbb{Z}[i]\times\mathbb{C}/\mathbb{Z}[i]$, both the base $Alb(X_0)$ and the fibre $\pi_0^{-1}(s)$ are (sesquilinearly) self-dual tori.

Proof. The self-duality of $\mathrm{Alb}(X_0)$ was proved in Lemma 6.9.7, while the self-duality of $\pi_0^{-1}(s)$ is the standard fact recalled in Proposition 6.9.8. $\square$

Chapter 7

Deformation Limits of Certain Classes of Compact Complex Manifolds

In this chapter, we discuss several **deformation closedness** results and conjectures in the sense of (ii) of Definition 2.6.1. Specifically, we investigate whether the properties of compact complex manifolds summed up in diagram ($\star$) of §.2.6.3 persist in the deformation limits and how they degenerate when they do not persist. Recall that a given fibre X_0 of a holomorphic family $\pi : \mathcal{X} \longrightarrow B$ of compact complex manifolds $(X_t = \pi^{-1}(t))_{t\in B}$ can be viewed as the limit, when the parameter $t \in B$ tends to $0 \in B$, of the nearby fibres X_t. Most of the material in this chapter is taken from [Pop09a], [Pop09b], [Pop10a] and [Pop19].

After Kodaira and Spencer proved the **deformation openness** of the Kähler condition (cf. Theorem 2.6.6), it became natural to wonder whether the Kähler property of compact complex manifolds is also deformation closed. This question was answered negatively by Hironaka for families of compact complex manifolds of dimensions ≥ 3.

Theorem 7.0.1. *([Hir62]) For every integer $n \geq 3$, there exists a holomorphic family of compact complex manifolds $\pi : \mathcal{X} \longrightarrow D$ over an open disc $D \subset \mathbb{C}$ centred at the origin such that the fibre $X_t := \pi^{-1}(t)$ is* **projective** *for every $t \in D \setminus \{0\}$, but the fibre $X_0 := \pi^{-1}(0)$ is* **not Kähler**.

(A) In the case of families of compact complex **surfaces**, the opposite conclusion holds: the Kähler property is **deformation closed**. Together with the deformation openness result of Kodaira and Spencer for the Kähler property in arbitrary dimension, this translates to the following

Theorem 7.0.2. *Let $\pi : \mathcal{X} \longrightarrow D$ be a holomorphic family of compact complex manifolds over an open disc $D \subset \mathbb{C}$ centred at the origin such that $dim_{\mathbb{C}} X_t = 2$ for all $t \in D$.*

If the fibre $X_0 := \pi^{-1}(0)$ is **Kähler**, *the fibre $X_t := \pi^{-1}(t)$ is again* **Kähler** *for every $t \in D$.*

This statement follows at once from the fact that the Kählerianity of compact complex *surfaces* is a *topological* property, depending solely on the parity of the first Betti number b_1, and from the C^∞ triviality of any family as above (cf. Ehresmann's Theorem 2.1.1) which obviously implies that b_1 is the same for all the fibres.

Theorem 7.0.3. *(Kodaira, Miyaoka, Siu) A compact complex* **surface** *X is Kähler if and only if its first Betti number $b_1(X)$ is even.*

This fact can be derived by putting together Kodaira's classification of surfaces, Miyaoka's result [Miy74] asserting that an elliptic surface is Kähler if and only if its first Betti number is even and

Siu's result [Siu83] asserting that every K3 surface is Kähler. Direct proofs of Theorem 7.0.3, which do not invoke Kodaira's classification of compact complex surfaces, were subsequently given by Buchdahl [Buc99] and Lamari [Lam99] independently. The reader will find further details on the history of this result in these references.

(B) Since, by Hironaka's Theorem 7.0.1, neither the projectivity, nor the Kählerianity of compact complex manifolds of dimension ≥ 3 is deformation closed, the question of what kind of manifolds can occur as limits X_0 of projective (or merely Moishezon) manifolds X_t, respectively of Kähler (or merely *class* $\mathcal{C}$) manifolds X_t, arises naturally. A series of conjectures[1] in this respect, that we now set about describing, were put forward in the 1970's.

Theorem 7.0.4. *([Pop10a] and again [Pop19]) Let $\pi : \mathcal{X} \to B$ be a holomorphic family of compact complex manifolds over an open ball $B \subset \mathbb{C}^N$ about the origin such that the fibre $X_t := \pi^{-1}(t)$ is a* **Moishezon** *manifold for every $t \in B \setminus \{0\}$. Then $X_0 := \pi^{-1}(0)$ is again a* **Moishezon** *manifold.*

In particular, any deformation limit X_0 of projective manifolds is Moishezon. The purpose of this chapter is to present two proofs of this result, the first of which appeared in [Pop09a], [Pop09b] and [Pop10a], the second of which appeared in [Pop19]. The second proof uses the machinery of the Frölicher spectral sequence and builds on material presented in chapters 1 and 3 to put on a more conceptual footing the first, ad hoc, proof.

Conjecture 7.0.5. *Let $\pi : \mathcal{X} \to B$ be a holomorphic family of compact complex manifolds over an open ball $B \subset \mathbb{C}^N$ about the origin such that the fibre $X_t := \pi^{-1}(t)$ is a* **class** $\mathcal{C}$ *manifold for every $t \in B \setminus \{0\}$. Then $X_0 := \pi^{-1}(0)$ is again a* **class** $\mathcal{C}$ *manifold.*

This conjecture, which can be regarded as a transcendental version of Theorem 7.0.4, is still wide open even when the stronger Kählerianity assumption is made on the fibres X_t with $t \in B \setminus \{0\}$. In particular, any degeneration of the Kähler property of compact complex manifolds in the deformation limit is expected to be only mild, in the form of the *class* $\mathcal{C}$ property at the worst.

A two-step strategy for an attack on Conjecture 7.0.5 was outlined in [Pop15a] (cf. §.4.3.1). The second step of this strategy was carried out in [PU18] where the class of **sGG manifolds** was introduced and studied for this purpose (cf. §.4.3).

The last conjecture in this series brings together the openness and the closedness points of view. It goes further than the Kodaira-Spencer Theorem 2.6.6 by predicting that the Kähler property of compact complex manifolds is **open** in the analytic Zariski topology of the base of the family of deformations of a complex structure.

Conjecture 7.0.6. *Let $\pi : \mathcal{X} \to B$ be a holomorphic family of compact complex manifolds over an open ball $B \subset \mathbb{C}^N$ about the origin. Suppose that the fibre $X_0 := \pi^{-1}(0)$ is a* **Kähler** *manifold.*

There exists a countable union $\Sigma = \bigcup\limits_{\nu \in \mathbb{N}} \Sigma_\nu \subset B$ of proper analytic subsets of B such that:

(a) X_t is **Kähler** *for all $t \in B \setminus \Sigma$;*

and

(b) X_t is of **class C** *for all $t \in \Sigma$.*

In other words, if one fibre is Kähler, not only are all the nearby fibres Kähler (Theorem 2.6.6), but almost all the fibres, even those lying far away from the originally Kähler one, are expected to be Kähler, except possibly some exceptional fibres that will nevertheless be of *class* C, as predicted by the previous Conjecture 7.0.5.

[1] These conjectures were brought to the author's attention by J.-P. Demailly.

(I) First proof of Theorem 7.0.4

We will present it in two stages over sections 7.1 and 7.2.

7.1 Limits of $\partial\bar\partial$-manifolds under holomorphic deformations

In this section, taken from [Pop09b], we prove the following key

Theorem 7.1.1. *([Pop09b, Theorem 1.4]) Let $\pi : \mathcal{X} \to B$ be a holomorphic family of compact complex manifolds such that the fibre $X_t := \pi^{-1}(t)$ is a* $\partial\bar\partial$**-manifold** *for every $t \in B^\star := B \setminus \{0\}$. Then, $X_0 := \pi^{-1}(0)$ is a* **strongly Gauduchon manifold***.*

This result is optimal since

-the $\partial\bar\partial$ assumption on the fibres X_t with $t \in B^\star$ cannot be weakened to the strongly Gauduchon assumption on these fibres (cf. [COUV16, Theorem 5.9]), so the strongly Gauduchon property of compact complex manifolds is not closed under deformations of the complex structure. (It is, however, open – see [Pop14].);

-the strongly Gauduchon conclusion on the limit fibre X_0 cannot be strengthened to the $\partial\bar\partial$ conclusion, so the $\partial\bar\partial$ property of compact complex manifolds is not closed under deformations (cf. [AK13] or [FOU15]).

Moreover, thanks to the $\partial\bar\partial$-property being satisfied by all compact Kähler and all *class* $\mathcal{C}$ manifolds, Theorem 7.1.1 is expected to play a key role in future attacks on Conjecture 7.0.5.

7.1.1 Notation and preliminary remarks

Throughout this chapter, we denote $n := \dim_{\mathbb{C}} X_t$ for $t \in B$.

The spaces of C^∞ forms of degree k, resp. of bidegree $(p,\, q)$, on X_t will be denoted by $C^\infty_k(X,\, \mathbb{C})$, resp. $C^\infty_{p,\,q}(X_t,\, \mathbb{C})$. Given a form u, its component of type (p, q) with respect to the complex structure J_t will be denoted by $u_t^{p,\,q}$.

The λ-eigenspace of a given elliptic differential operator $P_t : C^\infty_{p,\,q}(X_t,\, \mathbb{C}) \to C^\infty_{p,\,q}(X_t,\, \mathbb{C})$ will be denoted by $E^{p,\,q}_{P_t}(\lambda)$, where in most cases P_t will be taken to be one of the Laplace-Beltrami operators Δ'_t, Δ''_t associated with a given Hermitian metric on X_t.

As usual, we shall denote by $h^{p,\,q}(t) := \dim_{\mathbb{C}} H^{p,\,q}(X_t,\, \mathbb{C})$, resp. $b_k := \dim_{\mathbb{C}} H^k(X,\, \mathbb{C})$ the Hodge, resp. Betti numbers of X_t. Thanks to the $\partial\bar\partial$ assumption on X_t for every $t \neq 0$, every $h^{p,\,q}(t)$ is constant on $\Delta^\star$ after possibly shrinking B about 0. However, it may happen that $h^{p,\,q}(0) > h^{p,\,q}(t)$ for $t \neq 0$, although this case is *a posteriori* ruled out if X_t is projective for every $t \in B^\star$ by Theorem 7.0.4. But, of course, we have to contend with it until Theorem 7.0.4 has been proved. We stress that the weaker $\partial\bar\partial$-assumption for $t \neq 0$ in Theorem 7.1.1 need not rule out, even a posteriori, the jumping at $t = 0$ of $h^{p,\,q}(t)$.

Remark 7.1.2. One of the main difficulties one is faced with in trying to prove Theorems 7.0.4 and 7.1.1 is the possible jump at $t = 0$ of the Hodge numbers $h^{p,\,q}(t)$.

Specifically, suppose a C^∞ family $(\gamma_t)_{t\in B}$ of Hermitian metrics on the fibres $(X_t)_{t\in B}$ has been fixed. We get an associated C^∞ family $(\Delta''_t)_{t\in B}$ of Laplace-Beltrami operators acting on the J_t-$(p,\, q)$-forms of X for every bidegree $(p,\, q)$. For every $t \in B$ and every bidegree, the operator $\Delta''_t := \bar\partial_t\bar\partial_t^\star + \bar\partial_t^\star\bar\partial_t$ is elliptic and therefore has a compact resolvent and a discrete spectrum

$$0 = \lambda_0(t) \leq \lambda_1(t) \leq \cdots \leq \lambda_k(t) \leq \dots \tag{7.1}$$

with $\lambda_k(t) \to +\infty$ as $k \to +\infty$. Thanks to the Hodge isomorphism

$$H^{p,\,q}(X_t,\,\mathbb{C}) \simeq \ker(\Delta''_t : C^\infty_{p,\,q}(X_t,\,\mathbb{C}) \to C^\infty_{p,\,q}(X_t,\,\mathbb{C})), \quad t \in B,$$

the multiplicity of zero as an eigenvalue of Δ''_t equals $h^{p,q}(t)$. By Corollary 2.5.20, for every small $\varepsilon > 0$, the number $m \in \mathbb{N}^\star$ of eigenvalues (counted with multiplicities) of Δ''_t contained in the interval $[0,\,\varepsilon)$ is independent of t if $t \in B$ is sufficiently close to 0 (say δ_ε-close). If $\varepsilon > 0$ has been chosen so small that 0 is the only eigenvalue of Δ''_0 contained in $[0,\,\varepsilon)$, it follows that $m = h^{p,q}(0) \geq h^{p,q}(t)$ for t sufficiently close to 0 (the upper-semicontinuity property). Consequently, for t near 0, $h^{p,q}(0) = h^{p,q}(t)$ if and only if 0 is the only eigenvalue of Δ''_t lying in $[0,\,\varepsilon)$. In other words, if $h^{p,\,q}(0) > h^{p,q}(t)$ when $t(\neq 0)$ is near 0, choosing increasingly small $\varepsilon > 0$ gives eigenvalues of Δ''_t

$$0 < \lambda_{k_1}(t) \leq \lambda_{k_2(t)} \leq \cdots \leq \lambda_{k_N}(t) := \varepsilon_t < \varepsilon, \quad t \in B^\star, \tag{7.2}$$

that converge to zero (i.e. $\varepsilon_t \to 0$) when $t \to 0$, where $N = h^{p,\,q}(0) - h^{p,\,q}(t)$.

Now, we will have to solve on several occasions throughout this chapter, for C^∞ (up to $t = 0$) families of $\bar\partial_t$-exact forms $(v_t)_{t\in B}$, equations of the shape

$$\bar\partial_t u_t = v_t \quad \text{on } X_t, \quad \text{for } t \in B \setminus \{0\},$$

whose minimal $L^2_{\gamma_t}$-norm solution is given by the Neumann formula

$$u_t = \Delta''^{-1}_t \bar\partial^\star_t v_t, \quad t \in B \setminus \{0\},$$

that features the Green operator Δ''^{-1}_t of Δ''_t (i.e. the inverse of the restriction of Δ''_t to the orthogonal complement of its kernel). The inverses $1/\lambda_{k_j}(t)$ of the eigenvalues of Δ''_t are eigenvalues for Δ''^{-1}_t and $1/\lambda_{k_j}(t) \to +\infty$ when $t \to 0$ for every $k_j \in \{k_1, \dots, k_N\}$ (i.e. for every *small* eigenvalue) if there is a jump $h^{p,\,q}(0) > h^{p,\,q}(t)$. It follows that, if $\bar\partial^\star_t v_t$ has non-trivial projections onto the eigenspaces $E^{p,\,q}_{\Delta''_t}(\lambda_{k_j(t)})$ with $k_j \in \{k_1, \dots, k_N\}$, these projections get multiplied by $1/\lambda_{k_j}(t)$ when Δ''^{-1}_t is applied to $\bar\partial^\star_t v_t$. Then u_t need not be bounded as t approaches 0, unless the said projections can be proved to tend to zero sufficiently quickly to offset the growth $1/\lambda_{k_j}(t) \to +\infty$ when t approaches 0. This unboundedness may cause the family of forms $(u_t)_{t\in B\setminus\{0\}}$ to not extend across $t = 0$, i.e. to not have a limit u_0 on X_0 when $t \to 0$.

The same conclusion applies to the ∂_t-Laplacians $\Delta'_t := \partial_t\partial^\star_t + \partial^\star_t\partial_t$ because of the possible jump (upwards) at $t = 0$ of the dimensions of the cohomology groups $H^{p,\,q}_{\partial_t}(X_t, \mathbb{C})$ that depend on the complex structures J_t.

Remark 7.1.3. However, a simple observation that will play a major role in our approach to Theorem 7.1.1 is that the unboundedness phenomenon described in Remark 7.1.2 does not occur for the C^∞ family $(\Delta_t)_{t\in B}$ of d-Laplacians on the fibres $(X_t)_{t\in B}$.

Indeed, analogous families $(u_t = \Delta^{-1}_t d^\star_t v_t)_{t\in B\setminus\{0\}}$ of minimal $L^2_{\gamma_t}$-norm solutions of d-equations

$$du_t = v_t \quad \text{on } X_t, \quad t \in B \setminus \{0\},$$

for given C^∞ (up to $t = 0$) families of d-exact forms $(v_t)_{t\in B}$, always extend smoothly to a form u_0 on X_0 thanks to the De Rham cohomology of the fibres X_t being independent of $t \in B$. The reason is that the family of manifolds $(X_t)_{t\in B}$ is C^∞ trivial, so the Betti numbers b_k of the fibres X_t are constant. Therefore, there is no "jumping" phenomenon in this case.

7.1.2 Preliminaries to the proof of Theorem 7.1.1

We now start the proof of Theorem 7.1.1 that will occupy the rest of this section.

Reduction of the uniform boundedness problem to a positivity problem

Fix any C^∞ family $(\gamma_t)_{t\in B}$ of Gauduchon metrics on the respective fibres $(X_t)_{t\in B}$. (It is well known that such families exist, see e.g. Proposition 4.1.13.) For every $k\in\{0,\dots,2n\}$ and every $p,q\in\{0,\dots,n\}$, we denote by $\Delta_t := dd_t^\star + d_t^\star d : C_k^\infty(X,\mathbb{C})\to C_k^\infty(X,\mathbb{C})$ and by

$$\Delta'_t := \partial_t\partial_t^\star + \partial_t^\star\partial_t,\ \ \Delta''_t := \bar\partial_t\bar\partial_t^\star + \bar\partial_t^\star\bar\partial_t : C^\infty_{p,\,q}(X_t,\mathbb{C})\to C^\infty_{p,\,q}(X_t,\mathbb{C})$$

the d-, ∂_t- and $\bar\partial_t$-Laplace-Beltrami operators induced by the metrics γ_t on X_t. Let $(\lambda_j(t))_{j\in\mathbb{N}}$ be the eigenvalues, ordered non-increasingly and repeated as many times as the respective multiplicity, of

$$\Delta''_t : C^\infty_{n,n-1}(X_t,\mathbb{C})\longrightarrow C^\infty_{n,n-1}(X_t,\mathbb{C}),\qquad t\in B.$$

By Theorem A in §.2.5.1, each λ_j is a continuous function of $t\in B$. If there are eigenvalues such that $\lambda_j(t)>0$ for $t\neq 0$ and $\lambda_j(0)=0$, there are only finitely many of them numbering $h^{n,\,n-1}(0)-h^{n,\,n-1}(t) = h^{0,1}(0)-h^{0,1}(t)$ for any $t\neq 0$ close to 0. This number is, of course, independent of $t\neq 0$. For $t\neq 0$, let $\varepsilon''_t>0$ be the largest of these *small* eigenvalues, so $\varepsilon''_t\to 0$ as $t\to 0$. The remaining, infinitely many, eigenvalues are then bounded below (after possibly shrinking B about 0) by some $\varepsilon''>0$ independent of $t\in B$. Thus

$$\text{Spec}\,\Delta''_t\subset[0,\ \varepsilon''_t]\cup[\varepsilon'',\ +\infty),\qquad t\in B, \tag{7.3}$$

where we have set $\varepsilon''_0=0$. We get an $L^2_{\gamma_t}$-orthogonal eigenspace decomposition

$$C^\infty_{n,n-1}(X_t,\mathbb{C}) = \bigoplus_{\lambda\leq\varepsilon''_t} E^{n,n-1}_{\Delta''_t}(\lambda)\oplus\bigoplus_{\lambda\geq\varepsilon''} E^{n,n-1}_{\Delta''_t}(\lambda),\qquad t\in B. \tag{7.4}$$

Now, Δ''_t being an elliptic self-adjoint operator, it has a compact resolvent and there exists an orthonormal basis $(e_j^{n,n-1}(t))_{j\in\mathbb{N}}$ of $C^\infty_{n,n-1}(X_t,\mathbb{C})$ consisting of eigenvectors of Δ''_t :

$$\Delta''_t\, e_j^{n,n-1}(t) = \lambda_j(t)\, e_j^{n,n-1}(t),\qquad t\in B. \tag{7.5}$$

Furthermore, in the three-space orthogonal decomposition

$$C^\infty_{n,n-1}(X_t,\mathbb{C}) = \ker\Delta''_t\oplus\text{Im}\,\bar\partial_t\oplus\text{Im}\,\bar\partial_t^\star, \tag{7.6}$$

each subspace is Δ''_t-invariant due to Δ''_t commuting with $\bar\partial_t$ and $\bar\partial_t^\star$. This means that the eigenvectors $e_j^{n,n-1}(t)$ forming an orthonormal basis can be chosen such that each of them lies in one (and only one) of the three subspaces of (7.6). So none of the $e_j^{n,n-1}(t)$ straddles two or three subspaces. These simple reductions are valid for every $t\in B$ and we will henceforth suppose that the choices have been made as described above. The orthogonal decomposition of $\partial_t\gamma_t^{n-1}\in C^\infty_{n,n-1}(X_t,\mathbb{C})$ according to (7.4) has the shape:

$$\partial_t\gamma_t^{n-1} = \sum_{j\in J_1} c_j(t)\, e_j^{n,n-1}(t) + \sum_{j\in J_2} c_j(t)\, e_j^{n,n-1}(t) = U_t + V_t,\qquad t\in B, \tag{7.7}$$

where $U_t = \sum\limits_{j\in J_1} c_j(t)\, e_j^{n,n-1}(t)\in\bigoplus\limits_{\lambda\leq\varepsilon''_t} E^{n,n-1}_{\Delta''_t}(\lambda)$ and $V_t = \sum\limits_{j\in J_2} c_j(t)\, e_j^{n,n-1}(t)\in\bigoplus\limits_{\lambda\geq\varepsilon''} E^{n,n-1}_{\Delta''_t}(\lambda)$, with coefficients $c_j(t)\in\mathbb{C}^\star$ and index sets $J_1, J_2\subset\mathbb{N}$ such that $J_1\cap J_2=\emptyset$. As already noticed, by the Gauduchon condition, $\partial_t\gamma_t^{n-1}$ is d-closed for all $t\in B$ and, since it is ∂_t-exact, it must also be $\bar\partial_t$-exact for all $t\neq 0$ by the $\partial\bar\partial$-lemma. Since each eigenvector $e_j^{n,n-1}(t)$ belongs to one of the three orthogonal subspaces of (7.6), this means that only eigenvectors belonging to $\mathrm{Im}\,\bar\partial_t$ can have a non-trivial contribution to (7.7) for $t\neq 0$.

In particular, for every $t\neq 0$, both U_t and V_t are $\bar\partial_t$-exact. We can therefore find, for every $t\neq 0$, a smooth $J_t-(n,n-2)$-form w_t such that $V_t=\bar\partial w_t$. If we choose the form w_t of minimal L^2 norm (with respect to γ_t) with this property, the condition $V_t\in\bigoplus_{\lambda\geq\varepsilon''}E^{n,n-1}_{\Delta''_t}(\lambda)$ guarantees that the family of forms $(w_t)_{t\in B^\star}$ extends smoothly across $t=0$ to a family $(w_t)_{t\in B}$ varying in a C^∞ way with t up to $t=0$. This is because the eigenvalues λ contributing to V_t are uniformly bounded below by $\varepsilon''>0$.

As for $U_t\in\bigoplus_{\lambda\leq\varepsilon''_t}E^{n,n-1}_{\Delta''_t}(\lambda)$, we are unable to guarantee the boundedness near $t=0$ of its $\bar\partial_t$-potential because of the eigenvalues $\lambda_j(t)\leq\varepsilon''_t$ converging to 0. Therefore we will not consider the $\bar\partial_t$-potential. However, the $(n,n-1)$-form U_t is d-closed. Indeed, it is ∂_t-closed in a trivial way for bidegree reasons and is also $\bar\partial_t$-closed (even $\bar\partial_t$-exact, as it has been argued above). Thus, the $\partial\bar\partial$-lemma implies that U_t is d-exact for every $t\neq 0$. We can therefore find, for all $t\neq 0$, a form ξ_t of degree $2n-2$ such that $U_t=d\,\xi_t$. If we choose the form ξ_t of minimal L^2-norm (with respect to γ_t) with this property, we have

$$\xi_t=\Delta_t^{-1}\,d_t^\star U_t,\qquad t\neq 0,\tag{7.8}$$

where, for all $t\in B$ (including $t=0$), $\Delta_t=d\,d_t^\star+d_t^\star\,d:C^\infty_{2n-2}(X,\mathbb{C})\to C^\infty_{2n-2}(X,\mathbb{C})$ is the d-Laplacian associated with the metric γ_t and Δ_t^{-1} is the inverse of the restriction of Δ_t to the orthogonal complement of its kernel (the Green operator of Δ_t). Now, the Hodge isomorphism theorem gives:

$$\ker\Delta_t\simeq H^{2n-2}_{DR}(X_t,\mathbb{C})=H^{2n-2}(X,\mathbb{C}),\qquad t\in B,\tag{7.9}$$

and we know that all the De Rham cohomology groups $H^{2n-2}_{DR}(X_t,\mathbb{C})$ of the fibres X_t can be identified with a fixed space $H^{2n-2}(X,\mathbb{C})$. In particular, the dimension of $\ker\Delta_t$ is independent of $t\in B$, which means that the positive eigenvalues of Δ_t have a uniform positive (>0) lower bound for t close to 0 (cf. Kodaira-Spencer arguments recalled in Remarks 7.1.2 and 7.1.3 and applied to the C^∞ family of strongly elliptic operators $(\Delta_t)_{t\in B}$). Thus, in this respect, there is a sharp contrast between the d-Laplacian Δ_t and its $\bar\partial_t$-counterpart Δ''_t: unlike Δ''_t, Δ_t never displays the small eigenvalue phenomenon. In particular, the family of $(2n-2)$-forms $(\xi_t)_{t\in B^\star}$ extends smoothly across $t=0$ to a family $(\xi_t)_{t\in B}$ of forms varying in a C^∞ way with $t\in B$ (up to $t=0$).

Our discussion so far can be summed up as follows.

Lemma 7.1.4. *Given any family of Gauduchon metrics $(\gamma_t)_{t\in B}$ varying in a C^∞ way with $t\in B$ on the fibres of a family $(X_t)_{t\in B}$ in which X_t is a $\partial\bar\partial$-manifold for every $t\neq 0$, we can find a decomposition:*

$$\partial_t\gamma_t^{n-1}=d\,\xi_t+\bar\partial_t w_t,\qquad t\in B,\tag{7.10}$$

in such a way that

$$d\,\xi_t\in\bigoplus_{\lambda\leq\varepsilon''_t}E^{n,n-1}_{\Delta''_t}(\lambda),\qquad \bar\partial_t w_t\in\bigoplus_{\lambda\geq\varepsilon''}E^{n,n-1}_{\Delta''_t}(\lambda),\tag{7.11}$$

where $(w_t)_{t\in B}$ and $(\xi_t)_{t\in B}$ are families of $(2n-2)$-forms and respectively $(n,n-2)$-forms varying in a C^∞ way with $t\in B$ (up to $t=0$), $\varepsilon''>0$ is independent of t, $\varepsilon''_t>0$ for $t\neq 0$ and ε''_t converges to zero as t approaches $0\in B$ (thus $\varepsilon''_0=0$). Moreover, the following identity holds:

$$\partial_t(\gamma_t^{n-1}-\xi_t^{n-1,\,n-1})=\bar\partial_t(\xi_t^{n,\,n-2}+w_t),\qquad t\in B. \tag{7.12}$$

As the form $\xi_t^{n-1,\,n-1}$ need not be real, we find it more convenient to write:

$$\partial_t(\gamma_t^{n-1}-\xi_t^{n-1,\,n-1}-\overline{\xi_t^{n-1,\,n-1}})=\bar\partial_t(\xi_t^{n,\,n-2}+\overline{\xi_t^{n-2,\,n}}+w_t),\qquad t\in B. \tag{7.13}$$

To get (7.12) from (7.10), it suffices to write $d\,\xi_t=\partial_t\xi_t+\bar\partial_t\xi_t$ and to remember that $d\,\xi_t=U_t$ is a form of pure J_t-type $(n,n-1)$. Hence $d\,\xi_t=\partial_t\xi_t^{n-1,\,n-1}+\bar\partial_t\xi_t^{n,\,n-2}$. The vanishing of the $(n-1,n)$-component of $d\,\xi_t$ amounts to $\bar\partial_t\xi_t^{n-1,\,n-1}+\partial_t\xi_t^{n-2,\,n}=0$, or equivalently by conjugation to $\partial_t(-\overline{\xi_t^{n-1,\,n-1}})=\bar\partial_t\overline{\xi_t^{n-2,\,n}}$. Hence (7.13) follows from (7.12).

As all the forms involved in (7.13) vary in a C^∞ way with $t\in B$ (up to $t=0$), to finish the proof of Theorem 7.0.4 it clearly suffices to show that

$$\gamma_t^{n-1}-\xi_t^{n-1,\,n-1}-\overline{\xi_t^{n-1,\,n-1}}>0,\qquad \text{for all}\quad t\in B. \tag{7.14}$$

Indeed, if this positivity property has been proved, Michelsohn's observation in linear algebra of Lemma 4.0.1 enables one to extract the $(n-1)^{st}$ root of $\gamma_t^{n-1}-\xi_t^{n-1,\,n-1}-\overline{\xi_t^{n-1,\,n-1}}$ and to find, for all $t\in B$, a unique $J_t-(1,1)$-form $\rho_t>0$ such that

$$\gamma_t^{n-1}-\xi_t^{n-1,\,n-1}-\overline{\xi_t^{n-1,\,n-1}}=\rho_t^{n-1},\qquad t\in B. \tag{7.15}$$

By construction, ρ_t defines a *strongly Gauduchon* metric on X_t for every $t\in B$ thanks to (7.13). In particular, X_0 is a *strongly Gauduchon* manifold and Theorem 7.1.1 follows. It actually suffices to prove (7.14) for $t=0$.

Moreover, it would clearly suffice to prove the stronger property:

$$\xi_0^{n-1,n-1}=0. \tag{7.16}$$

If this has been proved, then identity (7.12) applied to $t=0$ reads $\partial_0\gamma_0^{n-1}=\bar\partial_0(\xi_0^{n,n-2}+w_0)$, hence γ_0 is a *strongly Gauduchon* metric on X_0 and Theorem 7.1.1 follows.

We have thus reduced our uniform boundedness problem for the *main quantity* I_t to the positivity problem (7.14) or the vanishing subproblem (7.16).

The positivity problem

Let $||\cdot||=||\cdot||_t$ and $\langle\langle\ ,\ \rangle\rangle=\langle\langle\ ,\ \rangle\rangle_t$ stand for the L^2-norm and respectively the L^2-scalar product defined by the Gauduchon metric γ_t on the forms of X_t.

Sufficiency of a small L^2 norm for the correcting form

As $\gamma_t^{n-1}>0$, we shall now see that in order to prove Theorem 7.1.1, it suffices to show that the L^2-norm $||\cdot||$ of $\xi_t^{n-1,\,n-1}$ can be made arbitrarily small (hence so can the L^2-norm of the real form $\xi_t^{n-1,\,n-1}+\overline{\xi_t^{n-1,\,n-1}}$) uniformly w.r.t. $t\in B$. It actually suffices to guarantee this property when $t=0$ as the following observation shows.

Lemma 7.1.5. *Suppose that for a constant $\varepsilon > 0$, we have:*

$$||\xi_0^{n-1,\,n-1}|| < \varepsilon. \tag{7.17}$$

Then, if ε is sufficiently small, there exists a C^∞ form $\rho_0 > 0$ that is positive definite and of type $(1,\,1)$ for J_0 on X_0 such that

$$\partial_0\rho_0^{n-1} - \partial_0\bigg(\gamma_0^{n-1} - \xi_0^{n-1,\,n-1} - \overline{\xi_0^{n-1,\,n-1}}\bigg) \in Im\,(\partial_0\bar\partial_0). \tag{7.18}$$

In particular, since $\partial_0(\gamma_0^{n-1} - \xi_0^{n-1,\,n-1} - \overline{\xi_0^{n-1,\,n-1}})$ is $\bar\partial_0$-exact by (7.13), we see that $\partial_0\rho_0^{n-1}$ is $\bar\partial_0$-exact, hence ρ_0 is a **strongly Gauduchon** *metric on X_0.*

Proof. To lighten the notation, we drop the indices and spell out the argument on an arbitrary compact complex $n-$fold X which will be taken to be X_0 in the end.

Having fixed the metric $\gamma(=\gamma_0)$ on $X(=X_0)$ and calculating all the formal adjoint operators w.r.t. γ, recall the following facts seen in §.1.1.1, including Corollary 1.1.13: the Aeppli Laplacian $\tilde\Delta_A^{p,\,q} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C})$ is elliptic and induces an L^2_γ-orthogonal three-space decomposition:

$$C^\infty_{p,\,q}(X,\mathbb{C}) = \ker\tilde\Delta_A^{p,\,q}\oplus(\mathrm{Im}\partial+\mathrm{Im}\bar\partial)\oplus\mathrm{Im}(\partial\bar\partial)^\star,$$

the orthogonal direct sum of the first two subspaces being the kernel of $\partial\bar\partial$:

$$\ker(\partial\bar\partial) = \ker\tilde\Delta_A^{p,\,q}\oplus(\mathrm{Im}\partial+\mathrm{Im}\bar\partial), \tag{7.19}$$

a decomposition proving the Hodge isomorphism $H^{p,\,q}_A(X,\,\mathbb{C})\simeq\ker\tilde\Delta_A^{p,\,q}$.

Taking $(p,\,q)=(n-1,\,n-1)$ in this general context, recall that we have (cf. (7.12) at $t=0$ with indices dropped, set $\xi:=\xi_0$):

$$\partial(\gamma^{n-1}-\xi^{n-1,\,n-1}) = \bar\partial(\xi^{n,\,n-2}+w).$$

Since $\partial\bar\partial\gamma^{n-1}=0$, taking $\bar\partial$ on both sides of the above identity, we get $\partial\bar\partial\xi^{n-1,\,n-1}=0$, hence the following decomposition according to (7.19):

$$\ker(\partial\bar\partial)\ni\xi^{n-1,\,n-1} = \xi^{n-1,\,n-1}_{\tilde\Delta_A}+(\partial\zeta+\bar\partial\eta), \tag{7.20}$$

where ζ and η are C^∞ forms of respective types $(n-2,\,n-1)$ and $(n-1,\,n-2)$, while $\xi^{n-1,\,n-1}_{\tilde\Delta_A}\in\ker\tilde\Delta_A^{n-1,\,n-1}$ is orthogonal onto the sum $\partial\zeta+\bar\partial\eta$. By orthogonality, we get:

$$0\leq||\xi^{n-1,\,n-1}_{\tilde\Delta_A}||\leq||\xi^{n-1,\,n-1}||<\varepsilon, \tag{7.21}$$

the last inequality being the hypothesis (7.17) (for $\xi^{n-1,\,n-1}:=\xi_0^{n-1,\,n-1}$).

Thus the $\tilde\Delta_A^{n-1,\,n-1}$-harmonic form $\xi^{n-1,\,n-1}_{\tilde\Delta_A}$ is small in L^2-norm by (7.21). However, the harmonicity w.r.t. an elliptic operator implies that $\xi^{n-1,\,n-1}_{\tilde\Delta_A}$ must be small in a much stronger norm. Indeed, applying the *fundamental a priori inequality* satisfied by elliptic operators to the fourth-order elliptic operator $\tilde\Delta_A^{n-1,\,n-1}$, we get for every $k\in\mathbb{N}$ and every L^2-form u of type $(n-1,\,n-1)$ such that $\tilde\Delta_A^{n-1,\,n-1}u$ is in the Sobolev space $W^k(X,\,\Lambda^{n-1,\,n-1}T^\star X)$ of $(n-1,\,n-1)$-forms on X whose derivatives up to order k are in L^2 :

$$||u||_{W^{k+4}}\leq C_k(||\tilde\Delta_A^{n-1,\,n-1}u||_{W^k}+||u||_{L^2}), \tag{7.22}$$

where $||\cdot||_{L^2} := ||\cdot||$ and $C_k > 0$ is a constant depending only on k. If $u = \xi^{n-1,\,n-1}_{\tilde\Delta_A}$, $\tilde\Delta^{n-1,\,n-1}_A u = 0$ and, by (7.21), $||u||_{L^2} < \varepsilon$. Thus (7.22) reduces to

$$||\xi^{n-1,\,n-1}_{\tilde\Delta_A}||_{W^{k+4}} \leq C_k\,\varepsilon, \qquad k \in \mathbb{N}. \tag{7.23}$$

Now by the well-known Sobolev Lemma, we have a continuous injection:

$$W^k(X, \Lambda^{n-1,\,n-1}T^\star X) \hookrightarrow C^l(X, \Lambda^{n-1,\,n-1}T^\star X), \qquad \forall k > l+n,$$

into the space of $(n-1,\,n-1)$-forms of class C^l on X. Choosing $l = 0$ and $k+4 > n$, we get, for a constant $C'_{k+4} > 0$ depending only on k:

$$||\xi^{n-1,\,n-1}_{\tilde\Delta_A}||_{C^0} \leq C'_{k+4}\,||\xi^{n-1,\,n-1}_{\tilde\Delta_A}||_{W^{k+4}} < C'_{k+4}\,C_k\,\varepsilon, \tag{7.24}$$

having used (7.23) for the last inequality.

Thus the C^0-norm of $\xi^{n-1,\,n-1}_{\tilde\Delta_A}$ can be made arbitrarily small by choosing ε small enough. Hence so can the C^0-norm of $\xi^{n-1,\,n-1}_{\tilde\Delta_A} + \overline{\xi^{n-1,\,n-1}_{\tilde\Delta_A}}$. Since $\gamma^{n-1} > 0$, it follows that $\gamma^{n-1} - \xi^{n-1,\,n-1}_{\tilde\Delta_A} - \overline{\xi^{n-1,\,n-1}_{\tilde\Delta_A}} > 0$ if $\varepsilon > 0$ is chosen small enough, achieving thus the desired positivity property (7.14). Extracting Michelsohn's $(n-1)^{st}$ root, we get a unique C^∞ $(1,1)$-form $\rho > 0$ (i.e. a Hermitian metric ρ on X) satisfying

$$\rho^{n-1} = \gamma^{n-1} - \xi^{n-1,\,n-1}_{\tilde\Delta_A} - \overline{\xi^{n-1,\,n-1}_{\tilde\Delta_A}} > 0.$$

On the other hand, it follows from (7.20) that

$$\begin{aligned}\gamma^{n-1} - \xi^{n-1,\,n-1} - \overline{\xi^{n-1,\,n-1}} &= \gamma^{n-1} - \xi^{n-1,\,n-1}_{\tilde\Delta_A} - \overline{\xi^{n-1,\,n-1}_{\tilde\Delta_A}} - \partial\zeta - \bar\partial\eta - \bar\partial\bar\zeta - \partial\bar\eta \\ &= \rho^{n-1} - \partial\zeta - \bar\partial\eta - \bar\partial\bar\zeta - \partial\bar\eta,\end{aligned}$$

hence, taking ∂ on either side of the above identity, we get

$$\partial\rho^{n-1} - \partial(\gamma^{n-1} - \xi^{n-1,\,n-1} - \overline{\xi^{n-1,\,n-1}}) = \partial\bar\partial(\eta + \bar\zeta),$$

proving contention (7.18) (indices have been dropped here). The proof is complete. $\square$

7.1.3 The iterative procedure and L^2 estimates

With Lemma 7.1.5 understood, the rest of the proof of Theorem 7.1.1 will focus on correcting the forms $\gamma_t^{n-1} > 0$ by subtracting real forms whose L^2-norms can be made arbitrarily small uniformly w.r.t. $t \in B$ such that the ∂_t of the difference is $\bar\partial_t$-exact for all $t \in B$. Actually, the case $t = 0$ will suffice. Thus, we will achieve the positivity posited in (7.14) thanks to Lemma 7.1.5.

However, we can see no reason that the L^2-norm of $\xi_t^{n-1,\,n-1}$ should be as small as needed in general. In other words, the forms $\xi_t^{n-1,\,n-1}$ constructed in Lemma 7.1.4 need not satisfy the hypothesis of Lemma 7.1.5. Therefore we will replace them by new forms $\widetilde\xi^{n-1,\,n-1}_{t,\,(p)}$ constructed by an inductive procedure that will be described below. The lower index (p) will indicate that $\widetilde\xi^{n-1,\,n-1}_{t,\,(p)}$ has been produced at step $p \in \mathbb{N}$ of the inductive procedure. This procedure is based on an iterative use of Lemma 7.1.4 in which $\partial_t\xi_t^{n-1,\,n-1}$ will be replaced by an appropriate form changing at each step p. Running the inductive procedure sufficiently many times $p \gg 1$, we shall get the L^2-norm

$||\widetilde{\xi}_{t,(p)}^{n-1,\,n-1}||$ to become arbitrarily small in a way that is uniform w.r.t. both $t\in B$ and the number $p\gg 1$ of iterations. Uniformity is of the essence in all that follows.

Nevertheless, an intermediate step is needed in passing from $(\xi_t^{n-1,\,n-1})_{t\in B}$ to $(\widetilde{\xi}_{t,(p)}^{n-1,\,n-1})_{t\in B}$. It will produce a family of forms $(\xi_{t,(p)}^{n-1,\,n-1})_{t\in B}$.

The inductive construction of the forms $(\xi_{t,(p)}^{n-1,\,n-1})_{t\in B}$

Smooth families of forms $(\xi_{t,(p)}^{n-1,\,n-1})_{t\in B}$ with $p\in\mathbb{N}$ will be constructed inductively. The main observation here is that the $\partial\bar\partial$-assumption enables the construction in Lemma 7.1.4 to run indefinitely. Identities (7.27) below compare to (7.12) and (7.28) to (7.13).

Lemma 7.1.6. *For $p\in\mathbb{N}$, let $(\xi_{t,(p)})_{t\in B}$ be the family of smooth $(2n-2)$-forms on the fibres $(X_t)_{t\in B}$ constructed inductively on $p\in\mathbb{N}$ by putting*

$$\xi_{t,(0)}:=\xi_t,\qquad t\in B,$$

and then defining $\xi_{t,(p+1)}$ as the minimal L^2-norm solution of the equation

$$d\,\xi_{t,(p+1)}=\partial_t\xi_{t,(p)}^{n-1,\,n-1},\qquad t\in B,\ p\in\mathbb{N},\tag{7.25}$$

where, as usual, $\xi_{t,(l)}^{r,\,s}$ denotes the component of J_t-type $(r,\,s)$ of $\xi_{t,(l)}$.

For every $t\in B$ and every $p\in\mathbb{N}$, let $\Omega_{t,(p)}^{n-1,\,n-1}$ be the smooth J_t-$(n-1,\,n-1)$-form on X_t defined as the minimal L^2-norm solution of the equation

$$\partial_t\Omega_{t,(p)}^{n-1,\,n-1}=\partial_t\xi_{t,(p)}^{n-1,\,n-1}.\tag{7.26}$$

Then, for every $p\in\mathbb{N}$, the family $(\xi_{t,(p)})_{t\in B}$ varies in a C^∞ way with t (up to $t=0$), the family $(\Omega_{t,(p)}^{n-1,\,n-1})_{t\in B}$ varies continuously with t (up to $t=0$) and, for all $t\in B$ and all $p\in\mathbb{N}$, we have:

$$\begin{aligned}\partial_t(\gamma_t^{n-1}-\Omega_{t,(p)}^{n-1,\,n-1}) &= \partial_t(\gamma_t^{n-1}-\xi_{t,(p)}^{n-1,\,n-1})\\ &= \bar\partial_t(\xi_{t,(p)}^{n,\,n-2}+\xi_{t,(p-1)}^{n,\,n-2}+\cdots+\xi_{t,(1)}^{n,\,n-2}+\xi_t^{n,\,n-2}+w_t).\end{aligned}\tag{7.27}$$

Equivalently, we have:

$$\partial_t(\gamma_t^{n-1}-\xi_{t,(p)}^{n-1,\,n-1}-\overline{\xi_{t,(p)}^{n-1,\,n-1}})=\bar\partial_t(\xi_{t,(p)}^{n,\,n-2}+\overline{\xi_{t,(p)}^{n-2,\,n}}+\xi_{t,(p-1)}^{n,\,n-2}+\cdots+\xi_t^{n,\,n-2}+w_t).\tag{7.28}$$

As the form $\Omega_{t,(p)}^{n-1,\,n-1}$ need not be real, (7.28) rather than (7.27) will prove useful to us later on. Note that $\Omega_{t,(p)}^{n-1,\,n-1}$ may be different from $\xi_{t,(p)}^{n-1,\,n-1}$ since, although $\xi_{t,(p)}$ is the minimal L^2-norm d-potential of $\partial_t\xi_{t,(p-1)}^{n-1,\,n-1}$, its $(n-1,\,n-1)$-component $\xi_{t,(p)}^{n-1,\,n-1}$ need not have minimal L^2-norm among the ∂_t-potentials of $\partial_t\xi_{t,(p)}^{n-1,\,n-1}$. Thus, the form $\Omega_{t,(p)}^{n-1,\,n-1}$ can be seen as a correction of $\xi_{t,(p)}^{n-1,\,n-1}$ if the latter does not have minimal L^2-norm. The forms $\Omega_{t,(p)}^{n-1,\,n-1}$ will only be used in some technical comparison arguments (e.g. in the proof of Lemma 7.1.7), but will eventually drop out of later statements.

Proof of Lemma 7.1.6. • The first thing we have to prove is that equation (7.25) is solvable (i.e. $\partial_t\xi_{t,(p)}^{n-1,\,n-1}$ is d-exact) for all $t\in B$ and all $p\in\mathbb{N}$. Equation (7.26) is obviously solvable.

Step $p=0$. We have already noticed that $\partial_t\gamma_t^{n-1}$ and its projections $d\,\xi_t$ and $\bar\partial_t w_t$ given in (7.10) are all d, ∂_t and $\bar\partial_t$-exact for all $t\neq 0$. Writing $d\,\xi_t = \partial_t\xi_t^{n-1,\,n-1} + \bar\partial_t\xi_t^{n,\,n-2}$, we see that $\bar\partial_t\xi_t^{n,\,n-2}$ is $\bar\partial_t$-closed (even $\bar\partial_t$-exact) and is also ∂_t-closed for bidegree reasons (being of pure type $(n, n-1)$). Thus $\bar\partial_t\xi_t^{n,\,n-2}$ is d-closed and of pure type. By the $\partial\bar\partial$-assumption, the $\bar\partial_t$-exactness of $\bar\partial_t\xi_t^{n,\,n-2}$ implies its d and ∂_t-exactness for all $t\neq 0$. Then $\partial_t\xi_t^{n-1,\,n-1}$ must also be d and ∂_t-exact for all $t\neq 0$ as a difference of two such forms. We can thus write:

$$\partial_t\xi_t^{n-1,\,n-1} = \partial_t\Omega_t^{n-1,\,n-1} = d\,\xi_{t,\,(1)}, \quad t\in B, \tag{7.29}$$

where $\Omega_t^{n-1,\,n-1}$ stands for the ∂_t-potential of minimal L^2-norm $||\cdot||$ and $\xi_{t,\,(1)}$ denotes the d-potential of minimal L^2-norm $||\cdot||$ of $\partial_t\xi_t^{n-1,\,n-1}$. In particular, equation (7.25) is solvable for all $t\in B$ and for $p=0$.

Identities (7.29) *a priori* hold only for $t\neq 0$ as the $\partial\bar\partial$-assumption is only made on X_t with $t\neq 0$. However, we have seen that in the Neumann formula for the minimal L^2-norm solution:

$$\xi_{t,\,(1)} = \Delta_t^{-1} d_t^\star(\partial_t\xi_t^{n-1,\,n-1}), \qquad t\in B^\star,$$

the family of Green's operators $(\Delta_t^{-1})_{t\in B}$ is a C^∞ family (up to $t=0$) by results of Kodaira-Spencer and the De Rham cohomology being constant on the fibres X_t, $t\in B$ (no *small eigenvalue* phenomenon for Δ_t). Thus $\partial_0\xi_0^{n-1,\,n-1}$ is d-exact and the family $(\xi_{t,\,(1)})_{t\in B}$ is defined and C^∞ up to $t=0$.

Meanwhile, as the minimal L^2-norm solution of the first equation in (7.29), $\Omega_t^{n-1,\,n-1}$ is given by the Neumann formula for a ∂_t-equation, namely

$$\Omega_t^{n-1,\,n-1} = \Delta_t'^{-1}\partial_t^\star(\partial_t\xi_t^{n-1,\,n-1}), \qquad t\in B^\star.$$

Thus, $\Omega_t^{n-1,\,n-1}$ is obtained by dividing by the eigenvalues of Δ_t' (some of which may tend to 0, hence their inverses may tend to $+\infty$, when t approaches $0\in B$ if there is a jump at $t=0$ of the dimension of $\ker\Delta_t'$) the coefficients of $\partial_t^\star(\partial_t\xi_t^{n-1,\,n-1})$ with respect to an orthonomal basis of $(n-1,\,n-1)$-forms that are eigenvectors of Δ_t'. (See the similar formula (7.37) further down.) However, the family of forms $(\partial_t^\star(\partial_t\xi_t^{n-1,\,n-1}))_{t\in B}$ depends in a C^∞ way with $t\in B$ (up to $t=0$). On the other hand, $||\Omega_t^{n-1,\,n-1}||\leq||\xi_t^{n-1,\,n-1}||$ for all $t\in B^\star$ by the L^2-norm minimality of $\Omega_t^{n-1,\,n-1}$. As $\xi_t^{n-1,\,n-1}$ is known to extend in a C^∞ way to X_0, the family $(\Omega_t^{n-1,\,n-1})_{t\in B^\star}$ is bounded near $t=0$. Since the eigenvalues of Δ_t' vary continuously with $t\in B$ by Kodaira-Spencer (see e.g. [Kod85, Theorem 7.2]), the boundedness w.r.t. t of $\Omega_t^{n-1,\,n-1}$ and the expression of $\Omega_t^{n-1,\,n-1}$ in terms of the inverses of the eigenvalues of Δ_t' imply that the family $(\Omega_t^{n-1,\,n-1})_{t\in B^\star}$ extends at least continuously across $0\in B$. (Elementarily, if f and g are continuous functions on a disc $\Delta\subset\mathbb{C}$ about 0, if $f(t)>0$ for all $t\neq 0$, $f(0)=0$ but $f(t)/g(t)$ is bounded near $t=0$, then $f(t_\nu)/g(t_\nu)$ converges to a finite limit for some sequence $t_\nu\to 0$.)

Thus identities (7.29) hold for all $t\in B$ (including $t=0$), while the families $(\Omega_t^{n-1,\,n-1})_{t\in B}$ and $(\xi_{t,\,(1)})_{t\in B}$ vary in a continuous, respectively C^∞, way with t. Set $\Omega_{t,\,0}^{n-1,\,n-1} := \Omega_t^{n-1,\,n-1}$.

The procedure described above can now be iterated indefinitely.

Step $p=1$. In view of (7.29), identity (7.12) becomes:

$$\partial_t(\gamma_t^{n-1} - \Omega_t^{n-1,\,n-1}) = \partial_t(\gamma_t^{n-1} - \xi_t^{n-1,\,n-1}) = \bar\partial_t(\xi_t^{n,\,n-2} + w_t), \quad t\in B. \tag{7.30}$$

Writing $d\,\xi_{t,\,(1)} = \partial_t\xi_{t,\,(1)}^{n-1,\,n-1} + \bar\partial_t\xi_{t,\,(1)}^{n,\,n-2}$ (recall that $d\,\xi_{t,\,(1)}$ is of J_t-type $(n,\,n-1)$) and using (7.29), we get:

$$\partial_t(\gamma_t^{n-1} - \xi_{t,\,(1)}^{n-1,\,n-1}) = \bar\partial_t(\xi_{t,\,(1)}^{n,\,n-2} + \xi_t^{n,\,n-2} + w_t), \quad t\in B. \tag{7.31}$$

The right-hand term in (7.31) is a d-closed and $\bar\partial_t$-exact $(n, n-1)$-form, hence it must be d, ∂_t and $\bar\partial_t$-exact for all $t\neq 0$ by the $\partial\bar\partial$-assumption. Then so is $\partial_t\xi_{t,(1)}^{n-1,n-1}$ as a difference of two such forms (i.e. $\partial_t\gamma_t^{n-1}$ and the right-hand term in (7.31)). We then get identities analogous to (7.29):

$$\partial_t\xi_{t,(1)}^{n-1,n-1} = \partial_t\Omega_{t,(1)}^{n-1,n-1} = d\,\xi_{t,(2)}, \quad t\in B,$$

where $\Omega_{t,(1)}^{n-1,n-1}$ and $\xi_{t,(2)}$ are the ∂_t and respectively d-potentials of $\partial_t\xi_{t,(1)}^{n-1,n-1}$ with minimal L^2-norms. They extend continuously, resp. smoothly to X_0 by the same arguments as above. In particular, equation (7.25) is solvable for all $t\in B$ and for $p=1$.

Moreover, writing $d\,\xi_{t,(2)} = \partial_t\xi_{t,(2)}^{n-1,n-1} + \bar\partial_t\xi_{t,(2)}^{n,n-2}$, (7.31) reads:

$$\partial_t(\gamma_t^{n-1} - \xi_{t,(2)}^{n-1,n-1}) = \bar\partial_t(\xi_{t,(2)}^{n,n-2} + \xi_{t,(1)}^{n,n-2} + \xi_t^{n,n-2} + w_t), \quad t\in B. \tag{7.32}$$

The $(n,\,n-1)$-form $\partial_t\xi_{t,(2)}^{n-1,n-1}$ is again d, ∂_t and $\bar\partial_t$-exact for all $t\neq 0$ by the $\partial\bar\partial$-assumption and the procedure can be repeated.

Step p. At step p, one gets:

$$\partial_t\xi_{t,(p)}^{n-1,n-1} = \partial_t\Omega_{t,(p)}^{n-1,n-1} = d\,\xi_{t,(p+1)}, \quad t\in B,\ p\in\mathbb{N}, \tag{7.33}$$

with $\Omega_{t,(p)}^{n-1,n-1}$ and $\xi_{t,(p+1)}$ the ∂_t and respectively d-potentials of minimal L^2-norms of $\partial_t\xi_{t,(p)}^{n-1,n-1}$.

• It is clear that the analogue for p of (7.30), (7.31), (7.32) and the definition of $\Omega_{t,(p)}^{n-1,n-1}$ in (7.33) add up to the identities (7.27) claimed in the statement. To get (7.28) from (7.27), recall that $\partial_t\xi_{t,(p-1)}^{n-1,n-1} = d\,\xi_{t,(p)}$ is of J_t-type $(n,\,n-1)$, hence its $(n-1,\,n)$-component $\partial_t\xi_{t,(p)}^{n-2,n} + \bar\partial_t\xi_{t,(p)}^{n-1,n-1}$ vanishes. Taking conjugates, one gets $\partial_t(-\overline{\xi_{t,(p)}^{n-1,n-1}}) = \bar\partial_t\overline{\xi_{t,(p)}^{n-2,n}}$ and this term can be added to (7.27) to get (7.28). □

The next observation is that the L^2-norm of $\xi_{t,(p)}^{n-1,n-1}$ can only decrease or stay constant when p increases, so successive iterations of the construction described in Lemma 7.1.6 bring us increasingly close to achieving our aim of rendering the L^2-norm of $\xi_{t,(p)}^{n-1,n-1}$ arbitrarily small when $p\gg 1$.

Lemma 7.1.7. *The C^∞ families of forms $(\xi_{t,(p)}^{n-1,n-1})_{t\in B}$, $p\in\mathbb{N}$, constructed in Lemma 7.1.6 obey the following L^2-norm inequalities:*

$$||\xi_{t,(p+1)}^{n-1,n-1}|| \leq ||\xi_{t,(p)}^{n-1,n-1}|| \ \textit{and}\ ||\xi_{t,(p+1)}|| \leq ||\xi_{t,(p)}|| \quad t\in B,\ p\in\mathbb{N}. \tag{7.34}$$

Proof. The minimal L^2-norm solutions of equations (7.33) are given by:

$$\xi_{t,(p+1)} = \Delta_t^{-1}d_t^\star(\partial_t\xi_{t,(p)}^{n-1,n-1}), \qquad \text{resp.} \quad \Omega_{t,(p)}^{n-1,n-1} = \Delta_t'^{-1}\partial_t^\star(\partial_t\xi_{t,(p)}^{n-1,n-1}). \tag{7.35}$$

Now it is easily seen that, for any ∂_t-exact $(r,\,s)$-form u on X_t, one has

$$||\Delta_t'^{-1}\partial_t^\star u|| = ||\Delta_t'^{-\frac{1}{2}}u||. \tag{7.36}$$

Indeed, if $(e_j^{r,s})_{j\in\mathbb{N}}$ is an orthonormal basis of $C^\infty_{r,s}(X_t,\,\mathbb{C})$ consisting of eigenvectors of Δ_t' such that $\Delta_t'e_j^{r,s} = \lambda_j\,e_j^{r,s}$ and if u splits as $u = \sum\limits_{j\in J_u} c_j\,e_j^{r,s}$ with $c_j\in\mathbb{C}$, then $e_j^{r,s}$ is ∂_t-exact for every $j\in J_u$ and

$$\Delta_t^{'-1}\partial_t^\star u = \sum_{j\in J_u}\frac{c_j}{\sqrt{\lambda_j}}\,e_j^{r-1,\,s}, \tag{7.37}$$

where $(e_j^{r-1,\,s})_{j\in J_u}$ is an orthonormal subset of $C^\infty_{r-1,\,s}(X_t,\,\mathbb{C})$ consisting of eigenvectors of Δ'_t corresponding to the same eigenvalues as for $(r,\,s)-$forms: $\Delta'_t e_j^{r-1,\,s} = \lambda_j\,e_j^{r-1,\,s}$. This is because

$$\partial^\star \;:\; \mathrm{Im}\,(\partial : C^\infty_{r-1,\,s}\to C^\infty_{r,\,s}) \longrightarrow \mathrm{Im}\,(\partial^\star : C^\infty_{r,\,s}\to C^\infty_{r-1,\,s})$$

is an angle-preserving isomorphism that maps any ∂-exact Δ'-eigenvector of type $(r,\,s)$ to a Δ'-eigenvector of type $(r-1,\,s)$ having the same eigenvalue λ and an L^2-norm multiplied by $\sqrt{\lambda}$. (We have suppressed indices t to ease the notation). A further application of $\Delta^{'-1}$ introduces divisions by the eigenvalues λ_j, hence the overall effect of applying $\Delta^{'-1}\partial^\star$ to u consists in multiplying the coefficients c_j by $\sqrt{\lambda_j}/\lambda_j = 1/\sqrt{\lambda_j}$ and replacing the orthonormal set of $(r,\,s)$-forms $\{e_j^{r,\,s},\,j\in J_u\}$ with an orthonormal set of $(r-1,\,s)$-forms $\{e_j^{r-1,\,s},\,j\in J_u\}$. Hence (7.37) follows.

On the other hand, $\Delta^{'-\frac{1}{2}}u = \sum\limits_{j\in J_u}\frac{c_j}{\sqrt{\lambda_j}}\,e_j^{r,\,s}$. Thus we get (7.36) since

$$||\Delta_t^{'-1}\partial_t^\star u||^2 = ||\Delta^{'-\frac{1}{2}}u||^2 = \sum_{j\in J_u}\frac{|c_j|^2}{\lambda_j}.$$

Similarly, for any d-exact k-form u on X_t, one has

$$||\Delta_t^{-1}d_t^\star u|| = ||\Delta^{-\frac{1}{2}}u||. \tag{7.38}$$

Thus in the light of (7.35), (7.36) and (7.38) with $u = \partial_t\xi_{t,\,(p)}^{n-1,\,n-1}$, we get

$$||\xi_{t,\,(p+1)}|| = ||\Delta_t^{-\frac{1}{2}}(\partial_t\xi_{t,\,(p)}^{n-1,\,n-1})||, \quad \text{resp.} \quad ||\Omega_{t,\,(p)}^{n-1,\,n-1}|| = ||\Delta_t^{'-\frac{1}{2}}(\partial_t\xi_{t,\,(p)}^{n-1,\,n-1})||. \tag{7.39}$$

We are thus led to compare the Laplacians Δ'_t and Δ_t for $t\in B$. We begin by noticing that for any *pure-type* (say $(r,\,s)$) form u on some X_t, we have:

$$\langle\langle\Delta_t u,\,u\rangle\rangle \geq \langle\langle\Delta'_t u,\,u\rangle\rangle, \tag{7.40}$$

while, if u is not Δ''_t-harmonic, we even have

$$\langle\langle\Delta_t u,\,u\rangle\rangle > \langle\langle\Delta'_t u,\,u\rangle\rangle. \tag{7.41}$$

Indeed, by compactness of X_t, any $(r,\,s)$-form u satisfies:

$$\begin{aligned}\langle\langle\Delta_t u,\,u\rangle\rangle &= ||d\,u||^2 + ||d_t^\star\,u||^2 = ||\partial_t u||^2 + ||\bar\partial_t u||^2 + ||\partial_t^\star u||^2 + ||\bar\partial_t^\star u||^2 \\ &= \langle\langle\Delta'_t u,\,u\rangle\rangle + \langle\langle\Delta''_t u,\,u\rangle\rangle \geq \langle\langle\Delta'_t u,\,u\rangle\rangle \geq 0,\end{aligned} \tag{7.42}$$

since $\langle\langle\Delta'_t u,\,u\rangle\rangle = ||\partial_t u||^2 + ||\partial_t^\star u||^2 \geq 0$ and $\langle\langle\Delta''_t u,\,u\rangle\rangle = ||\bar\partial_t u||^2 + ||\bar\partial_t^\star u||^2 \geq 0$, while the assumption that u is not Δ''_t-harmonic amounts to $\langle\langle\Delta''_t u,\,u\rangle\rangle > 0$. The equality between the top two lines follows from $d\,u = \partial_t u + \bar\partial_t u$ and the pure-type forms $\partial_t u$ and $\bar\partial_t u$ of distinct types $(r+1,\,s)$, resp. $(r,\,s+1)$, being orthogonal. Thus $||d\,u||^2 = ||\partial_t u||^2 + ||\bar\partial_t u||^2$ and the adjoints satisfy the analogous identity $||d_t^\star\,u||^2 = ||\partial_t^\star u||^2 + ||\bar\partial_t^\star u||^2$ for the same reasons.

Thus it follows from (7.39) and (7.42) that

$$||\xi_{t,\,(p+1)}|| \leq ||\Omega_{t,\,(p)}^{n-1,\,n-1}||. \tag{7.43}$$

Now $||\xi_{t,\,(p+1)}^{n-1,\,n-1}|| \leq ||\xi_{t,\,(p+1)}||$ by mutual orthogonality of the pure-type components of $\xi_{t,\,(p+1)}$. Similarly $||\xi_{t,\,(p)}^{n-1,\,n-1}|| \leq ||\xi_{t,\,(p)}||$, while

$$||\Omega_{t,\,(p)}^{n-1,\,n-1}|| \leq ||\xi_{t,\,(p)}^{n-1,\,n-1}|| \tag{7.44}$$

by L^2-norm minimality of $\Omega_{t,\,(p)}^{n-1,\,n-1}$ among the solutions of the equation $\partial_t\Omega_{t,\,(p)}^{n-1,\,n-1} = \partial_t\xi_{t,\,(p)}^{n-1,\,n-1}$ (cf. (7.33)). Thus we get

$$||\xi_{t,\,(p+1)}^{n-1,\,n-1}|| \leq ||\xi_{t,\,(p+1)}|| \leq ||\Omega_{t,\,(p)}^{n-1,\,n-1}|| \leq ||\xi_{t,\,(p)}^{n-1,\,n-1}|| \leq ||\xi_{t,\,(p)}||.$$

This sequence of inequalities contains (7.34). □

Taking our cue from the strict inequality (7.41), we now notice that inequality (7.34) can be improved in a way that is uniform w.r.t. $t\in B$ if the relevant forms $\partial_t\xi_{t,\,(p)}^{n-1,\,n-1}$ avoid the harmonic spaces $\ker\Delta''_t$ for all $t\in B$ (including $t=0$). This is not possible, however, if the non-Δ''_t-harmonicity assumption is only made at $t\neq 0$.

Observation 7.1.8. *(i) Let $(u_t)_{t\in B}$ be a family of J_t-$(r,\,s)$-forms varying continuously with t (up to $t=0$) such that $u_t\notin\ker\Delta''_t$ for all $t\in B$ (including $t=0$).*

Then there exists a constant $\varepsilon>0$ independent of $t\in B$ such that

$$\langle\langle\Delta_t u_t,\,u_t\rangle\rangle \geq (1+\varepsilon)\,\langle\langle\Delta'_t u_t,\,u_t\rangle\rangle \quad \textit{for all} \quad t\in B, \tag{7.45}$$

after possibly shrinking the base B about 0.

(ii) In particular, suppose that for a given $p\in\mathbb{N}$ we have

$$\partial_t\xi_{t,\,(p)}^{n-1,\,n-1}\notin\ker\Delta''_t, \qquad \textit{for all} \quad t\in B \quad (\textit{including} \quad t=0). \tag{7.46}$$

Then there exists a constant $\varepsilon>0$ independent of $t\in B$ such that

$$\langle\langle\Delta_t(\partial_t\xi_{t,\,(p)}^{n-1,\,n-1}),\,\partial_t\xi_{t,\,(p)}^{n-1,\,n-1}\rangle\rangle \geq (1+\varepsilon)\,\langle\langle\Delta'_t(\partial_t\xi_{t,\,(p)}^{n-1,\,n-1}),\,\partial_t\xi_{t,\,(p)}^{n-1,\,n-1}\rangle\rangle, \quad t\in B, \tag{7.47}$$

after possibly shrinking the base B about 0.

Implicitly, if hypothesis (7.46) is satisfied for a given $p\in\mathbb{N}$, we get

$$||\xi_{t,\,(p+1)}^{n-1,\,n-1}|| \leq \frac{1}{\sqrt{1+\varepsilon}}\,||\xi_{t,\,(p)}^{n-1,\,n-1}||, \qquad t\in B, \tag{7.48}$$

for a certain $\varepsilon>0$ independent of $t\in B$.

Proof. Since $u_0\notin\ker\Delta''_0$, inequality (7.41) applies to Δ_0, Δ'_0 and u_0 to give $\langle\langle\Delta_0 u_0,\,u_0\rangle\rangle > \langle\langle\Delta'_0 u_0,\,u_0\rangle\rangle$. Thus there exists a constant $\varepsilon>0$ such that this inequality strengthens to

$$\langle\langle\Delta_0 u_0,\,u_0\rangle\rangle > (1+\varepsilon)\,\langle\langle\Delta'_0 u_0,\,u_0\rangle\rangle. \tag{7.49}$$

Now $(\Delta_t)_{t\in B}$ and $(\Delta'_t)_{t\in B}$ are C^∞ families of operators since they are defined by metrics $(\gamma_t)_{t\in B}$ that vary in a C^∞ way with $t\in B$ (up to $t=0$). Since u_t also varies continuously with $t\in B$ (up to $t=0$), $\langle\langle\Delta_t u_t,\,u_t\rangle\rangle$ and $\langle\langle\Delta'_t u_t,\,u_t\rangle\rangle$ both vary continuously with $t\in B$ (up to $t=0$). By continuity, shrinking B about 0 if necessary, (7.49) extends to a small neighbourhood of 0 in B to give (7.45) and prove part (i).

The first statement of part (ii) is an immediate consequence of part (i). As for the second statement of part (ii), it follows from (7.47), (7.35), (7.36) and (7.38) that

$$||\xi_{t,\,(p+1)}|| \leq \frac{1}{\sqrt{1+\varepsilon}}\,||\Omega^{n-1,\,n-1}_{t,\,(p)}||,$$

while the easy comparison arguments given at the end of the proof of Lemma 7.1.7 further give the uniform estimate (7.48). The proof is complete. □

The forms $\xi^{n-1,\,n-1}_{t,\,(p)}$ produced iteratively in Lemma 7.1.6 may appear at first glance as the right substitute for the previous forms $\xi^{n-1,\,n-1}_t$ if $p \gg 1$. However, the L^2-norm of $\xi^{n-1,\,n-1}_{t,\,(p)}$ need not be small uniformly w.r.t. $t \in B$ and the number $p \gg 1$ of iterations due to the uncontrollable behaviour of $\partial_t\xi^{n-1,\,n-1}_{t,\,(p)}$ from which $\xi^{n-1,\,n-1}_{t,\,(p+1)}$ is constructed by solving equations (7.33). Indeed, $\partial_t\xi^{n-1,\,n-1}_{t,\,(p)}$ cannot be guaranteed to satisfy hypothesis (7.46) for all $t \in B$ and all $p \in \mathbb{N}$. Consequently, estimate (7.48) need not hold at all, let alone with a constant $\varepsilon > 0$ independent of both $t \in B$ and $p \in \mathbb{N}$. Even in the favourable case where $\partial_t\xi^{n-1,\,n-1}_{t,\,(p)} \notin \ker\Delta''_t$ for all t and p, the ε of (7.47) cannot not be guaranteed to be independent of $p \in \mathbb{N}$ since $\partial_t\xi^{n-1,\,n-1}_{t,\,(p)}$ may come arbitrarily close to $\ker\Delta''_t$ as $p \to +\infty$.

In other words, we cannot guarantee that inequality (7.34) does not become an identity for $p \gg 1$ or that the decrease of the L^2-norms $||\xi^{n-1,\,n-1}_{t,\,(p)}||$, should it occur, is uniform w.r.t. t and p as $p \to +\infty$. A further modification is needed to achieve uniformity in the L^2-estimates: the forms $\xi_{t,\,(p)}$ will be replaced by new inductively constructed forms $\widetilde{\xi}_{t,\,(p)}$ obtained in the following way. If $\widetilde{\xi}_{t,\,(p)}$ has been constructed at step p of the inductive procedure that will be described below, $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}$ will be altered to $\partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)})$ (for a suitably chosen form $\nu^{n-1,\,n-1}_{t,\,(p)}$ that will force $\partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)})$ to satisfy analogues of hypothesis (7.46) and of estimate (7.47)) before solving equations analogous to (7.33) and running step $(p+1)$ of the inductive procedure that will produce the next form $\widetilde{\xi}_{t,\,(p+1)}$. Thus we will "push" $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}$ away from $\ker\Delta''_t$ by adding some auxiliary form $\partial_t\nu^{n-1,\,n-1}_{t,\,(p)}$ changing with p. We stress that the auxiliary form must be changed at every step p to shift $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}$ beyond a uniform distance from $\ker\Delta''_t$. There is no "universal" choice of auxiliary form that would suit every p. The details are spelt out in the next sections.

The inductive construction of the forms $(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)})_{t\in B}$

Step 1. By (7.12) of Lemma 7.1.4 we get

$$\partial_t\gamma^{n-1}_t = \partial_t\xi^{n-1,\,n-1}_t + \bar\partial_t(\xi^{n,\,n-2}_t + w_t), \qquad t \in B. \tag{7.50}$$

Let $(\eta_t)_{t\in B}$ be a smooth family of J_t-$(n,\,n-1)$-forms (the *auxiliary forms at step* 1) satisfying the following three conditions $(\star)$:

(a) $\eta_t = \partial_t\nu^{n-1,\,n-1}_t = \bar\partial_t\vartheta^{n,\,n-2}_t$ for all $t \in B$ and for continuous families of forms $(\nu^{n-1,\,n-1}_t)_{t\in B}$, $(\vartheta^{n,\,n-2}_t)_{t\in B}$ of the shown types;

(b) $||\xi^{n-1,\,n-1}_t + \nu^{n-1,\,n-1}_t|| \leq ||\xi_t||, \qquad t \in B;$

(c) for all $t \in B$ and for some $\varepsilon_0 > 0$ independent of t we have

$$\frac{\langle\langle\Delta''_t(\partial_t\xi_t^{n-1,\,n-1}+\partial_t\nu_t^{n-1,\,n-1}),\,\partial_t\xi_t^{n-1,\,n-1}+\partial_t\nu_t^{n-1,\,n-1}\rangle\rangle}{\langle\langle\Delta'_t(\partial_t\xi_t^{n-1,\,n-1}+\partial_t\nu_t^{n-1,\,n-1}),\,\partial_t\xi_t^{n-1,\,n-1}+\partial_t\nu_t^{n-1,\,n-1}\rangle\rangle}\geq\varepsilon_0>0,$$

with the convention that if the denominator vanishes, any $\varepsilon_0>0$ will do.

Now using (a), (7.50) becomes:

$$\partial_t\gamma_t^{n-1}=\partial_t(\xi_t^{n-1,\,n-1}+\nu_t^{n-1,\,n-1})+\bar\partial_t(\xi_t^{n,\,n-2}+w_t-\vartheta_t^{n,\,n-2}),\qquad t\in B. \tag{7.51}$$

Let $\widetilde\Omega_t^{n-1,\,n-1}$ and $\widetilde\xi_{t,\,(1)}$ be the ∂_t-potential and respectively the d-potential of minimal L^2-norms of $\partial_t(\xi_t^{n-1,\,n-1}+\nu_t^{n-1,\,n-1})$:

$$\partial_t(\xi_t^{n-1,\,n-1}+\nu_t^{n-1,\,n-1})=\partial_t\widetilde\Omega_t^{n-1,\,n-1}=d\,\widetilde\xi_{t,\,(1)},\quad t\in B. \tag{7.52}$$

Notice that, since $d\,\widetilde\xi_{t,\,(1)}$ is of pure type $(n,\,n-1)$, we must have

$$d\,\widetilde\xi_{t,\,(1)}=\partial_t\widetilde\xi_{t,\,(1)}^{n-1,\,n-1}+\bar\partial_t\widetilde\xi_{t,\,(1)}^{n,\,n-2},\quad t\in B.$$

Using this and (7.52), (7.51) reads:

$$\partial_t\gamma_t^{n-1}=\partial_t\widetilde\xi_{t,\,(1)}^{n-1,\,n-1}+\bar\partial_t(\widetilde\xi_{t,\,(1)}^{n,\,n-2}+\xi_t^{n,\,n-2}+w_t-\vartheta_t^{n,\,n-2}),\qquad t\in B. \tag{7.53}$$

Step $p+1$**.** Suppose that Step p has been performed and has produced the following decomposition for all $t\in B$:

$$\begin{aligned}\partial_t\gamma_t^{n-1}=\partial_t\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}\quad & + \quad \bar\partial_t(\widetilde\xi_{t,\,(p)}^{n,\,n-2}+\cdots+\widetilde\xi_{t,\,(1)}^{n,\,n-2}+\xi_t^{n,\,n-2}+w_t\\ & - \quad \vartheta_t^{n,\,n-2}-\vartheta_{t,\,(1)}^{n,\,n-2}-\cdots-\vartheta_{t,\,(p-1)}^{n,\,n-2}).\end{aligned} \tag{7.54}$$

Let $(\eta_{t,\,(p)})_{t\in B}$ be a smooth family of J_t-$(n,\,n-1)$-forms (the *auxiliary forms at step* $p+1$) satisfying the following three conditions $(\star_p)$:

(a) $\eta_{t,\,(p)}=\partial_t\nu_{t,\,(p)}^{n-1,\,n-1}=\bar\partial_t\vartheta_{t,\,(p)}^{n,\,n-2}$ for all $t\in B$ and for continuous families of forms $(\nu_{t,\,(p)}^{n-1,\,n-1})_{t\in B}$, $(\vartheta_{t,\,(p)}^{n,\,n-2})_{t\in B}$ of the shown types;

(b) $||\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}+\nu_{t,(p)}^{n-1,\,n-1}||\leq||\widetilde\xi_{t,\,(p)}||,\qquad t\in B;$

(c) for all $t\in B$ and for some $\varepsilon_0>0$ independent of t and of $p\in\mathbb{N}$ we have

$$\frac{\langle\langle\Delta''_t(\partial_t\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}+\partial_t\nu_{t,\,(p)}^{n-1,\,n-1}),\,\partial_t\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}+\partial_t\nu_{t,\,(p)}^{n-1,\,n-1}\rangle\rangle}{\langle\langle\Delta'_t(\partial_t\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}+\partial_t\nu_{t,\,(p)}^{n-1,\,n-1}),\,\partial_t\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}+\partial_t\nu_{t,\,(p)}^{n-1,\,n-1}\rangle\rangle}\geq\varepsilon_0>0,$$

with the convention that if the denominator vanishes, any $\varepsilon_0>0$ will do.

Now using (a), (7.54) becomes for all $t\in B$:

$$\begin{aligned}\partial_t\gamma_t^{n-1}=\partial_t(\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}+\nu_{t,\,(p)}^{n-1,\,n-1})\quad & + \quad \bar\partial_t(\widetilde\xi_{t,\,(p)}^{n,\,n-2}+\cdots+\widetilde\xi_{t,\,(1)}^{n,\,n-2}+\xi_t^{n,\,n-2}+w_t\\ & - \quad \vartheta_t^{n,\,n-2}-\vartheta_{t,\,(1)}^{n,\,n-2}-\cdots-\vartheta_{t,\,(p)}^{n,\,n-2}).\end{aligned} \tag{7.55}$$

Let $\widetilde\Omega_{t,\,(p)}^{n-1,\,n-1}$ and $\widetilde\xi_{t,\,(p+1)}$ be the ∂_t-potential and respectively the d-potential of minimal L^2-norms of $\partial_t(\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}+\nu_{t,\,(p)}^{n-1,\,n-1})$:

$$\partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)}) = \partial_t\widetilde{\Omega}^{n-1,\,n-1}_{t,\,(p)} = d\,\widetilde{\xi}_{t,\,(p+1)}, \quad t \in B. \tag{7.56}$$

Notice that, since $d\,\widetilde{\xi}_{t,\,(p+1)}$ is of pure type $(n,\,n-1)$, we must have

$$d\,\widetilde{\xi}_{t,\,(p+1)} = \partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p+1)} + \bar\partial_t\widetilde{\xi}^{n,\,n-2}_{t,\,(p+1)}, \quad t \in B.$$

Using this and (7.56), (7.55) reads for all $t \in B$:

$$\begin{aligned}\partial_t\gamma^{n-1}_t = \partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p+1)} \quad &+ \quad \bar\partial_t(\widetilde{\xi}^{n,\,n-2}_{t,\,(p+1)} + \cdots + \widetilde{\xi}^{n,\,n-2}_{t,\,(1)} + \xi^{n,\,n-2}_t + w_t \\ &- \quad \vartheta^{n,\,n-2}_t - \vartheta^{n,\,n-2}_{t,\,(1)} - \cdots - \vartheta^{n,\,n-2}_{t,\,(p)}),\end{aligned} \tag{7.57}$$

completing the inductive construction of the families $(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)})_{t\in B}$, $p \in \mathbb{N}$.

Summing up: if we set $\widetilde{\xi}_{t,\,(0)} := \xi_t$ and $\widetilde{\xi}^{n-1,\,n-1}_{t,\,(0)} := \xi^{n-1,\,n-1}_t$ as well as $\widetilde{\Omega}^{n-1,\,n-1}_{t,\,(0)} := \widetilde{\Omega}^{n-1,\,n-1}_t$ and $\nu^{n-1,\,n-1}_{t,\,(0)} := \nu^{n-1,\,n-1}_t$, we get continuous families of forms $(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)})_{t\in B}$ and $(\widetilde{\Omega}^{n-1,\,n-1}_{t,\,(p)})_{t\in B}$ for each $p \in \mathbb{N}$.

Comment 7.1.9. It is clear that the forms $\eta_{t,\,(p)} = 0$ with $\nu^{n-1,\,n-1}_{t,\,(p)} = 0$ and $\vartheta^{n,\,n-2}_{t,\,(p)} = 0$ for all $t \in B$ trivially satisfy conditions (a) and (b) of $(\star_p)$, while they need not satisfy condition (c). Indeed, we have $||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}|| \leq ||\widetilde{\xi}_{t,\,(p)}||$ (hence (b) for $\nu^{n-1,\,n-1}_{t,\,(p)} = 0$) since the former form is the $(n-1,\,n-1)$-component of the latter and forms of distinct pure types are orthogonal. So, in general, the choices of these auxiliary forms are non-trivial. However, the trivial choice of identically zero auxiliary forms will do if it happens to satisfy (c) (see (7.65) below).

7.1.4 Proof of the existence of auxiliary forms

We now spell out the argument accounting for the existence of smooth families of forms $(\eta_{t,\,(p)})_{t\in B}$ satisfying conditions $(\star_p)$ for all $p \in \mathbb{N}$. The spectra of Δ'_t and Δ''_t acting on $(n,\,n-1)$-forms satisfy inclusions:

$$\text{Spec}\,\Delta'_t \subset [0,\,\varepsilon'_t] \cup [\varepsilon',\,+\infty), \quad \text{Spec}\,\Delta''_t \subset [0,\,\varepsilon''_t] \cup [\varepsilon'',\,+\infty), \quad t \in B, \tag{7.58}$$

where $\varepsilon', \varepsilon'' > 0$ are independent of t, while $\varepsilon'_t, \varepsilon''_t \to 0$ as $t \to 0$. (Thus $\varepsilon'_0 = \varepsilon''_0 = 0$.) Since the eigenspaces of Δ'_t and of Δ''_t are finite-dimensional and since there are at most finitely many eigenvalues of Δ'_t below ε' and of Δ''_t below ε'', each of the vector spaces $\oplus_{\mu\leq\varepsilon'_t} E^{n,\,n-1}_{\Delta'_t}(\mu)$ and $\oplus_{\lambda\leq\varepsilon''_t} E^{n,\,n-1}_{\Delta''_t}(\lambda)$ (which are the obstruction to what we are striving to achieve) has finite dimension. Hence their respective orthogonal complements $\oplus_{\mu\geq\varepsilon'} E^{n,\,n-1}_{\Delta'_t}(\mu)$ and $\oplus_{\lambda\geq\varepsilon''} E^{n,\,n-1}_{\Delta''_t}(\lambda)$ in the infinite-dimensional vector space $C^\infty_{n,\,n-1}(X_t,\,\mathbb{C})$ have both infinite dimension and so has their intersection with the infinite-dimensional subspace $\text{Im}\,\bar\partial_t$, i.e.

$$\mathcal{E}^{n,\,n-1}_t := \bigoplus_{\mu\geq\varepsilon'} E^{n,\,n-1}_{\Delta'_t}(\mu) \cap \bigoplus_{\lambda\geq\varepsilon''} E^{n,\,n-1}_{\Delta''_t}(\lambda) \cap \text{Im}\,\bar\partial_t \subset C^\infty_{n,\,n-1}(X_t,\,\mathbb{C}), \quad t \in B,$$

has infinite dimension. The infinite dimensionality of $\mathcal{E}^{n,\,n-1}_t$ will play a crucial role in the sequel: the auxiliary forms $\eta_{t,\,(p)}$ will be chosen in $\mathcal{E}^{n,\,n-1}_t$ and having plenty of "room for choice" will be a key factor. Moreover, $\Delta \ni t \mapsto \mathcal{E}^{n,\,n-1}_t$ defines an infinite-rank C^∞-subbundle of $\Delta \ni t \mapsto C^\infty_{n,\,n-1}(X_t,\,\mathbb{C})$. Notice the inclusion

$$\mathcal{E}_t^{n,\,n-1} \subset \operatorname{Im}\partial_t, \qquad t\in B. \tag{7.59}$$

Indeed, being of type $(n,\, n-1)$, every form $\eta_t \in \mathcal{E}_t^{n,\,n-1}$ is trivially ∂_t-closed, hence also d-closed since the $\bar\partial_t$-exactness assumption is implicit in the definition of $\mathcal{E}_t^{n,\,n-1}$. Then η_t is ∂_t-exact for all $t\neq 0$ by the $\partial\bar\partial$-lemma. Since any $\eta_t\in\mathcal{E}_t^{n,\,n-1}$ avoids the *small* eigenvalues of Δ'_t by definition of $\mathcal{E}_t^{n,\,n-1}$, it follows that η_0 must be again ∂_0-exact if η_0 stands in a C^∞ family $(\eta_t)_{t\in B}$ with $\eta_t\in\mathcal{E}_t^{n,\,n-1}$ for all $t\in B$.

Now fix $p\in\mathbb{N}$ and suppose that the induction has been performed up to *Step* p. In particular, the forms $(\widetilde\xi_{t,\,(p)})_{t\in B}$ have already been constructed. To run *Step* $(p+1)$, we have to show the existence of auxiliary forms $(\eta_{t,\,(p)})_{t\in B}$ adapted to the pre-existing forms $(\widetilde\xi_{t,\,(p)}^{n-1,\,n-1})_{t\in B}$ by satisfying conditions $(\star_p)$. To start with, pick any smooth family $(\eta_{t,\,(p)})_{t\in B}$ of non-zero J_t-$(n,\,n-1)$-forms such that

$$\eta_{t,\,(p)} = \partial_t\nu_{t,\,(p)}^{n-1,\,n-1} = \bar\partial_t\vartheta_{t,\,(p)}^{n,\,n-2}\in\mathcal{E}_t^{n,\,n-1}, \quad t\in B, \tag{7.60}$$

where the families of *minimal* L^2*-norm* ∂_t-potentials $(\nu_{t,\,(p)}^{n-1,\,n-1})_{t\in B}$ and $\bar\partial_t$-potentials $(\vartheta_{t,\,(p)}^{n,\,n-2})_{t\in B}$ vary continuously with $t\in B$ (up to $t=0$). We thus satisfy requirement (a) in the infinite-rank vector bundle $\Delta\ni t\mapsto\mathcal{E}_t^{n,\,n-1}\subset C^\infty_{n,\,n-1}(X_t,\,\mathbb{C})$. We have yet to satisfy the requirements (b) and (c).

Fo every $t\in B$, consider the map

$$\mathcal{E}_t^{n,\,n-1}\ni\eta_{t,\,(p)}\overset{S_t}{\longmapsto}\nu_{t,\,(p)}^{n-1,\,n-1}\in\operatorname{Im}\partial_t^\star\subset C^\infty_{n-1,\,n-1}(X_t,\,\mathbb{C}) \tag{7.61}$$

which associates with every $\eta_{t,\,(p)}\in\mathcal{E}_t^{n,\,n-1}$ its ∂_t-potential $\nu_{t,\,(p)}^{n-1,\,n-1}$ of *minimal* L^2*-norm* (i.e. the unique ∂_t-potential that lies in $\operatorname{Im}\partial_t^\star$). Since $\operatorname{Im}S_t\subset\operatorname{Im}\partial_t^\star$, we have

$$\operatorname{Im}S_t\perp\ker\partial_t. \tag{7.62}$$

It is clear that the map S_t is linear (because $\nu_{t,\,(p)}^{n-1,\,n-1}=\Delta_t^{'-1}\partial_t^\star\eta_{t,\,(p)}$ while $\Delta_t^{'-1}$ and $\partial_t^\star$ are linear operators) and injective (because $\partial_t\nu_{t,\,(p)}^{n-1,\,n-1}=\eta_{t,\,(p)}$). Hence $\operatorname{Im}S_t$ is an infinite-dimensional vector subspace of $\operatorname{Im}\partial_t^\star$.

Meanwhile, for every $t\in B$ and every $p\in\mathbb{N}$, let

$$\mathcal{U}_{t,\,(p)} := \overline{B}\bigg(-\widetilde\xi_{t,\,(p)}^{n-1,\,n-1},\,||\widetilde\xi_{t,\,(p)}||\bigg)\subset C^\infty_{n-1,\,n-1}(X_t,\,\mathbb{C})$$

be the closed ball (w.r.t. L^2-norm) centred at $-\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}$ and of radius $||\widetilde\xi_{t,\,(p)}||$ in $C^\infty_{n-1,\,n-1}(X_t,\,\mathbb{C})$. Clearly, $0\in\mathcal{U}_{t,\,(p)}$. Condition (b) of $(\star_p)$ translates to

$$\nu_{t,\,(p)}^{n-1,\,n-1}\in\mathcal{U}_{t,\,(p)}, \qquad t\in B, \tag{7.63}$$

so any form

$$\nu_{t,\,(p)}^{n-1,\,n-1}\in\operatorname{Im}S_t\cap\mathcal{U}_{t,\,(p)}, \qquad t\in B, \tag{7.64}$$

automatically satisfies conditions (a) (after setting $\eta_{t,\,(p)}:=\partial_t\nu_{t,\,(p)}^{n-1,\,n-1}$) and (b) of $(\star_p)$. Note that unless the form $\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}$ (given by the induction hypothesis) already satisfies the condition

$$\frac{\langle\langle\Delta''_t(\partial_t\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}),\,\partial_t\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}\rangle\rangle}{\langle\langle\Delta'_t(\partial_t\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}),\,\partial_t\widetilde\xi_{t,\,(p)}^{n-1,\,n-1}\rangle\rangle}\geq\varepsilon_0>0, \tag{7.65}$$

for the uniform ε_0 obtained from the previous induction steps $1,\dots,p$, the auxiliary form $\nu^{n-1,\,n-1}_{t,\,(p)}$ that we are now trying to construct cannot be chosen to be the zero form. Thus, unless $\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}$ satisfies (7.65), we must show that

$$\operatorname{Im} S_t \cap \mathcal{U}_{t,\,(p)} \supsetneq \{0\}, \qquad t \in B. \tag{7.66}$$

If we can manage to achieve (7.66), we will choose $0 \neq \nu^{n-1,\,n-1}_{t,\,(p)} \in \operatorname{Im} S_t \cap \mathcal{U}_{t,\,(p)}$ (cf. (7.64)) in a family varying in a continuous way with $t \in B$ and will set $\eta_{t,\,(p)} := \partial_t \nu^{n-1,\,n-1}_{t,\,(p)}$ for every $t \in B$. (Recall that $\operatorname{Im} S_t$ varies continuously with t up to $t = 0$ since the family $(\nu^{n-1,\,n-1}_{t,\,(p)})_{t\in B}$ of ∂_t-potentials does, as explained above.) Property (7.60) will then be satisfied and so will be (a) and (b) of $(\star_p)$.

The discussion of the possibility of enforcing the choice (7.64) falls into two cases that we now analyse.

Case 1*: if* $||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}|| < ||\widetilde{\xi}_{t,\,(p)}||$, then the origin 0 of $C^\infty_{n-1,\,n-1}(X_t,\,\mathbb{C})$ lies in the interior of the ball $\mathcal{U}_{t,\,(p)}$, so the vector subspace $\operatorname{Im} S_t$ meets the interior of $\mathcal{U}_{t,\,(p)}$. Hence (7.66) is guaranteed and we can choose $\nu^{n-1,\,n-1}_{t,\,(p)} \neq 0$ to satisfy (7.64). Conditions (a) and (b) of $(\star_p)$ are thus simultaneously fulfilled as explained above.

Case 2*: if* $||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}|| = ||\widetilde{\xi}_{t,\,(p)}||$, then $\widetilde{\xi}_{t,\,(p)} = \widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}$, hence $\widetilde{\xi}_{t,\,(p)}$ is of pure type $(n-1,\,n-1)$. (Recall that, in general, $||\widetilde{\xi}_{t,\,(p)}||^2 = ||\widetilde{\xi}^{n,\,n-2}_{t,\,(p)}||^2 + ||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}||^2 + ||\widetilde{\xi}^{n-2,\,n}_{t,\,(p)}||^2$ by mutual orthogonality of the pure-type components of a given form.) In this case the zero form 0 lies on the boundary of the ball $\mathcal{U}_{t,\,(p)}$.

Let $H_{t,\,(p)}$ denote the hyperplane of $C^\infty_{n-1,\,n-1}(X_t,\,\mathbb{C})$ that is orthogonal to the vector $\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}$ at 0. If the inclusion

$$\operatorname{Im} S_t \subset H_{t,\,(p)} \tag{7.67}$$

does not hold, then $\operatorname{Im} S_t$ meets the interior of the ball $\mathcal{U}_{t,\,(p)}$, (7.66) holds, we can choose $\nu^{n-1,\,n-1}_{t,\,(p)} \neq 0$ to satisfy (7.64) and we can proceed as in *Case* 1.

However, if the inclusion (7.67) happens to hold, then $\operatorname{Im} S_t$ does not meet the interior of $\mathcal{U}_{t,\,(p)}$ and $\operatorname{Im} S_t \cap \mathcal{U}_{t,\,(p)} = \{0\}$. Thus (7.66) does not hold. Meanwhile recall that $\widetilde{\xi}_{t,\,(p)}$ satisfies (by construction) the following induction hypothesis (cf. (7.56) with $p-1$ in place of p):

$$\partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p-1)} + \nu^{n-1,\,n-1}_{t,\,(p-1)}) = \partial_t \widetilde{\Omega}^{n-1,\,n-1}_{t,\,(p-1)} = d\,\widetilde{\xi}_{t,\,(p)}, \quad t \in B. \tag{7.68}$$

Since $d\,\widetilde{\xi}_{t,\,(p)} = \partial_t\,\widetilde{\xi}_{t,\,(p)} + \bar\partial_t\,\widetilde{\xi}_{t,\,(p)}$ is of pure type $(n,\,n-1)$ and since $\widetilde{\xi}_{t,\,(p)} = \widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}$ is of pure type $(n-1,\,n-1)$ here, we see that the $(n-1,\,n)$-form $\bar\partial_t\,\widetilde{\xi}_{t,\,(p)}$ must vanish for bidegree reasons. Thus (7.68) yields

$$\partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p-1)} + \nu^{n-1,\,n-1}_{t,\,(p-1)}) = \partial_t\,\widetilde{\xi}_{t,\,(p)} = \partial_t\,\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}, \quad t \in B. \tag{7.69}$$

Now recall that by the induction hypothesis the form $\partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p-1)} + \nu^{n-1,\,n-1}_{t,\,(p-1)})$ featuring in the left-hand side of (7.69) satisfies property (c) of $(\star_{p-1})$ with the uniform $\varepsilon_0 > 0$ obtained from the previous induction steps $1,\dots,p$. (The auxiliary forms $\nu^{n-1,\,n-1}_{t,\,(p-1)}$ were chosen as such at *Step* p of the induction process). Therefore (7.69) combined with (c) of $(\star_{p-1})$ shows that

$$\frac{\langle\langle\Delta''_t(\partial_t\,\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}),\,\partial_t\,\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}\rangle\rangle}{\langle\langle\Delta'_t(\partial_t\,\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}),\,\partial_t\,\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}\rangle\rangle}=\frac{\langle\langle\Delta''_t\partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p-1)}+\nu^{n-1,\,n-1}_{t,\,(p-1)}),\,\partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p-1)}+\nu^{n-1,\,n-1}_{t,\,(p-1)})\rangle\rangle}{\langle\langle\Delta'_t\partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p-1)}+\nu^{n-1,\,n-1}_{t,\,(p-1)}),\,\partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p-1)}+\nu^{n-1,\,n-1}_{t,\,(p-1)})\rangle\rangle}\geq\varepsilon_0>0,$$

which means that $\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}$ satisfies (7.65). Therefore we can make the trivial choice of auxiliary form $\nu^{n-1,\,n-1}_{t,\,(p)}$, i.e. we can (and will) choose

$$\nu^{n-1,\,n-1}_{t,\,(p)}=0\in\operatorname{Im}S_t\cap\mathcal{U}_{t,\,(p)}=\{0\}.$$

This guarantees (7.64), hence (a) and (b) of $(\star_p)$. This also guarantees (c) of $(\star_p)$ thanks to (7.65) (which holds as we have just seen). As explained in Comment 7.1.9, this choice meets our conditions in this case. (This is the only case where the choice of the *zero* form will do.)

Conclusion 7.1.10. *The choice (7.64) can always be enforced and we shall henceforth assume that* $\nu^{n-1,\,n-1}_{t,\,(p)}$ *has been chosen as in (7.64). This guarantees conditions* (a) *and* (b) *of* $(\star_p)$.

Moreover, in Case 2 *discussed above, condition* (c) *is satisfied simultaneously with* (a) *and* (b). *It remains to prove that, in Case* 1 *discussed above,* $\nu^{n-1,\,n-1}_{t,\,(p)}$ *can be chosen as in (7.64) to satisfy furthermore condition* (c) *of* $(\star_p)$.

Let us make the following observation. Since $\nu^{n-1,\,n-1}_{t,\,(p)}$ has been chosen as the minimal L^2-norm ∂_t-potential of $\eta_{t,\,(p)}$, it satisfies $\nu^{n-1,\,n-1}_{t,\,(p)}\perp\ker\partial_t$ in $C^\infty_{n-1,\,n-1}(X_t,\,\mathbb{C})$ (cf. (7.62)). Thus $\nu^{n-1,\,n-1}_{t,\,(p)}$ cannot have a non-trivial orthogonal projection on any of the eigenspaces $E^{n-1,\,n-1}_{\Delta'_t}(\mu)$ corresponding to eigenvalues $\mu\leq\varepsilon'_t$. Indeed, if $\delta_t\in E^{n-1,\,n-1}_{\Delta'_t}(\mu)\setminus\{0\}$ were such a projection, then $\partial_t\delta_t\in E^{n,\,n-1}_{\Delta'_t}(\mu)\setminus\{0\}$ would play the analogous role for $\eta_{t,\,(p)}=\partial_t\nu^{n-1,\,n-1}_{t,\,(p)}$ in bidegree $(n,\,n-1)$ since ∂_t and Δ'_t commute. However, the existence of such a component for $\eta_{t,\,(p)}$ is ruled out by (7.60) and the definition of $\mathcal{E}^{n,\,n-1}_t$. Therefore, any form $\nu^{n-1,\,n-1}_{t,\,(p)}\in\operatorname{Im}S_t$ satisfies

$$\nu^{n-1,\,n-1}_{t,\,(p)}\in\bigoplus_{\mu\geq\varepsilon'}E^{n-1,\,n-1}_{\Delta'_t}(\mu),\qquad t\in B. \tag{7.70}$$

We now explain how to choose a form $\nu^{n-1,\,n-1}_{t,\,(p)}$ as in (7.64) that also satisfies requirement (c) of $(\star_p)$ in *Case* 1.

Condition (c) essentially requires $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}+\partial_t\nu^{n-1,\,n-1}_{t,\,(p)}$ to stay away from $\ker\Delta''_t$ at an L^2-distance that is bounded below by a positive constant independent of both $t\in B$ and $p\in\mathbb{N}$ if simultaneously the behaviour of $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}+\partial_t\nu^{n-1,\,n-1}_{t,\,(p)}$ w.r.t. Δ'_t is kept under control relative to the behaviour w.r.t. Δ''_t.

The possibility that $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}$ be Δ''_t-harmonic cannot be ruled out and in this case condition (c) cannot not be fulfilled without correcting $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}$ by non-zero auxiliary forms $\eta_{t,\,(p)}$. Recall that the auxiliary form $\eta_{t,\,(p)}=\partial_t\nu^{n-1,\,n-1}_{t,\,(p)}$ is to be chosen among the forms that satisfy condition (7.60). Any such $\eta_{t,\,(p)}$ is $\bar\partial_t$-exact for all $t\in B$ by the choice (7.60), hence $\eta_{t,\,(p)}$ is orthogonal to $\ker\Delta''_t$ (since $\ker\Delta''_t\perp\operatorname{Im}\bar\partial_t$). Thus $\eta_{t,\,(p)}=\partial_t\nu^{n-1,\,n-1}_{t,\,(p)}$ is in a good position to "drive" $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}$ away from $\ker\Delta''_t$ and ensure that the corrected form $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)}+\partial_t\nu^{n-1,\,n-1}_{t,\,(p)}$ satisfies (c).

The discussion of the choice of a form $\nu^{n-1,\,n-1}_{t,\,(p)}$ as in (7.64) that also satisfies requirement (c) of $(\star_p)$ in *Case* 1 falls into two steps.

(I) Uniformly bounding the numerator of (c) in $(\star_p)$ from below in Case 1

It is clear that $\langle\langle\Delta''_t(\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \partial_t\nu^{n-1,\,n-1}_{t,\,(p)}),\, \partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \partial_t\nu^{n-1,\,n-1}_{t,\,(p)}\rangle\rangle$ has a uniform positive lower bound whenever the following three conditions are simultaneously met as $\Delta \ni t \to 0$ and $p \to +\infty$:

(i) the L^2-distance from $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \partial_t\nu^{n-1,\,n-1}_{t,\,(p)}$ to $\ker\Delta''_t$ does not become arbitrarily small;

(ii) the L^2-norm of $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \partial_t\nu^{n-1,\,n-1}_{t,\,(p)}$ does not become arbitrarily small;

(iii) $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \partial_t\nu^{n-1,\,n-1}_{t,\,(p)} \notin \oplus_{\lambda\leq\varepsilon''_t} E^{n,\,n-1}_{\Delta''_t}(\lambda)$ for $\varepsilon''_t \to 0$ as $t\to 0$.

In fact condition (iii) is related to condition (i): if $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \partial_t\nu^{n-1,\,n-1}_{t,\,(p)} \in \oplus_{\lambda\leq\varepsilon''_t} E^{n,\,n-1}_{\Delta''_t}(\lambda)$ for $\varepsilon''_t \to 0$ as $t\to 0$, then $\partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)} + \partial_0\nu^{n-1,\,n-1}_{0,\,(p)} \in \ker\Delta''_0$ in violation of (i).

Observation 7.1.11. *Without loss of generality we may make the following*

Assumption (A1): $\qquad \partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)} \in \ker\Delta''_0.$

Proof. There are three cases:

(1) if $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} \in \bigoplus_{\lambda\geq\varepsilon''} E^{n,n-1}_{\Delta''_t}(\lambda)$ for all $t\in B$, then $\partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)} \in \operatorname{Im}\bar\partial_0$. (Indeed, recall that $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} \in \operatorname{Im}\bar\partial_t$ for all $t\in B^\star$ by (7.54) and by the fact that, thanks to the $\partial\bar\partial$-lemma, $\partial_t\gamma_t^{n-1} \in \operatorname{Im}\bar\partial_t$ for $t\neq 0$. Recall moreover that the limit of $\bar\partial_t$-exact forms that avoid the *small* eigenvalues of Δ''_t is again $\bar\partial_0$-exact.) Hence γ_0 is *strongly Gauduchon* in this case and the proof of Theorem 7.1.1 ends here;

(2) if $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} \in \bigoplus_{\lambda\leq\varepsilon''_t} E^{n,n-1}_{\Delta''_t}(\lambda)$ for all $t\in B$, then $\partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)} \in \ker\Delta''_0$ as in the assumption (A1);

(3) if $\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} = u_t + v_t$ with $u_t \in \bigoplus_{\lambda\leq\varepsilon''_t} E^{n,n-1}_{\Delta''_t}(\lambda)$ and $v_t \in \bigoplus_{\lambda\geq\varepsilon''} E^{n,n-1}_{\Delta''_t}(\lambda)$ for all $t\in B$, then $u_t, v_t \in \operatorname{Im}\bar\partial_t$ for all $t\in B^\star$, while $u_0\in\ker\Delta''_0$ and $v_0\in\operatorname{Im}\bar\partial_0$. (In particular $u_0\perp v_0$, hence $||u_0|| \leq ||\partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)}||$.) Thus v_0 can be absorbed in the $\bar\partial_0$-exact part of $\partial_0\gamma_0^{n-1}$ in (7.54), while the new obstruction u_0 to $\partial_0\gamma_0^{n-1}$ being $\bar\partial_0$-exact is Δ''_0-harmonic, much as the former obstruction $\partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)}$ is supposed to be in assumption (A1). $\square$

Thus, after possibly replacing $\partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)}$ with u_0, we may (and will henceforth) make the assumption (A1). An immediate consequence of (A1) is

$$\ker\Delta''_0 \ni \partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)} \perp \partial_0\nu^{n-1,\,n-1}_{0,\,(p)}, \qquad \forall\ \nu^{n-1,\,n-1}_{0,\,(p)} \in (\operatorname{Im}S_0)\cap\mathcal{U}_{0,\,(p)}, \tag{7.71}$$

because $\partial_0\nu^{n-1,\,n-1}_{0,\,(p)} \in \operatorname{Im}\bar\partial_0$ by (7.60) and because $\ker\Delta''_0 \perp \operatorname{Im}\bar\partial_0$. We get

$\langle\langle\Delta''_0(\partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)} + \partial_0\nu^{n-1,\,n-1}_{0,\,(p)}),\, \partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)} + \partial_0\nu^{n-1,\,n-1}_{0,\,(p)}\rangle\rangle$

$$= \langle\langle \Delta''_0(\partial_0\nu^{n-1,\,n-1}_{0,\,(p)}),\, \partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)}\rangle\rangle + \langle\langle \Delta''_0(\partial_0\nu^{n-1,\,n-1}_{0,\,(p)}),\, \partial_0\nu^{n-1,\,n-1}_{0,\,(p)}\rangle\rangle$$

because $\Delta''_0(\partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)}) = 0$ by assumption (A1). Now $\Delta''_0(\partial_0\nu^{n-1,\,n-1}_{0,\,(p)}) \in \mathrm{Im}\,\bar\partial_0$ since $\partial_0\nu^{n-1,\,n-1}_{0,\,(p)} \in \mathrm{Im}\,\bar\partial_0$ by the choice (7.60) and since $\bar\partial_0$ and Δ''_0 commute. Meanwhile, $\partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)} \in \ker\Delta''_0$ by assumption (A1). Since $\ker\Delta''_0 \perp \mathrm{Im}\,\bar\partial_0$, the first term on the second line above vanishes. On the other hand, again by the choice (7.60) and the definition of $\mathcal{E}^{n,\,n-1}_0$, we have $\partial_0\nu^{n-1,\,n-1}_{0,\,(p)} \in \bigoplus\limits_{\lambda\geq\varepsilon''} E^{n,\,n-1}_{\Delta''_0}(\lambda)$. It follows that the second term on the second line above satisfies

$$\langle\langle \Delta''_0(\partial_0\nu^{n-1,\,n-1}_{0,\,(p)}),\, \partial_0\nu^{n-1,\,n-1}_{0,\,(p)}\rangle\rangle \geq \varepsilon''\,||\partial_0\nu^{n-1,\,n-1}_{0,\,(p)}||^2,$$

so we get

$$\begin{aligned}
&\langle\langle \Delta''_0(\partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)} + \partial_0\nu^{n-1,\,n-1}_{0,\,(p)}),\, \partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)} + \partial_0\nu^{n-1,\,n-1}_{0,\,(p)}\rangle\rangle \\
&\qquad \geq\ \varepsilon''\,||\partial_0\nu^{n-1,\,n-1}_{0,\,(p)}||^2 = \varepsilon''\,(||\partial_0\nu^{n-1,\,n-1}_{0,\,(p)}||^2 + ||\partial_0^\star\nu^{n-1,\,n-1}_{0,\,(p)}||^2) \\
&\qquad =\ \varepsilon''\,\langle\langle\Delta'_0\nu^{n-1,\,n-1}_{0,\,(p)}),\, \nu^{n-1,\,n-1}_{0,\,(p)}\rangle\rangle \geq \varepsilon'\,\varepsilon''\,||\nu^{n-1,\,n-1}_{0,\,(p)}||^2. \qquad (7.72)
\end{aligned}$$

The equality on the second line of (7.72) follows from $\partial_0^\star\nu^{n-1,\,n-1}_{0,\,(p)} = 0$ which in turn follows from $\nu^{n-1,\,n-1}_{0,\,(p)} \in \mathrm{Im}\,\partial_0^\star \subset \ker\partial_0^\star$. (Recall that $\nu^{n-1,\,n-1}_{0,\,(p)}$ has been chosen to have *minimal* L^2*-norm* among the ∂_0-potentials of $\eta_{0,\,(p)}$ in the definition (7.61) of the map S_0.) The last inequality on the third line in (7.72) follows from $\nu^{n-1,\,n-1}_{0,\,(p)} \in \bigoplus\limits_{\mu\geq\varepsilon'} E^{n-1,\,n-1}_{\Delta'_0}(\mu)$ (see (7.70) for $t=0$).

Conclusion 7.1.12. *Under the assumption (A1), we have:*

$$\langle\langle \Delta''_0(\partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)} + \partial_0\nu^{n-1,\,n-1}_{0,\,(p)}),\, \partial_0\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)} + \partial_0\nu^{n-1,\,n-1}_{0,\,(p)}\rangle\rangle \geq \varepsilon'\,\varepsilon''\,||\nu^{n-1,\,n-1}_{0,\,(p)}||^2 \qquad (7.73)$$

for all $\nu^{n-1,\,n-1}_{0,\,(p)} \in (Im\,S_0)\cap\mathcal{U}_{0,\,(p)}$.

Now recall that by Conclusion 7.1.10 it is only in *Case* 1 that condition (c) of $(\star_p)$ has yet to be obtained. (We have already argued that $(a), (b), (c)$ are simultaneously satisfied in *Case* 2 with the choices made so far.) Let

$$\alpha_{(p)} := ||\widetilde{\xi}_{0,\,(p)}|| - ||\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)}|| > 0 \qquad \text{in } \textit{Case } 1.$$

Lemma 7.1.13. *If* $\nu^{n-1,\,n-1}_{0,\,(p)} \in (Im\,S_0)\cap\mathcal{U}_{0,\,(p)}$ *is chosen of* **maximal** L^2**-norm** *among the forms in the intersection of the subspace* $Im\,S_0$ *with the ball* $\mathcal{U}_{0,\,(p)}$*, we have*

$$||\nu^{n-1,\,n-1}_{0,\,(p)}|| \geq \alpha_{(p)} \qquad (7.74)$$

and $\alpha_{(p)} > 0$ *in Case* 1.

Proof. It is clear that $\alpha_{(p)} > 0$ in *Case* 1 and $\alpha_{(p)} = 0$ in *Case* 2.

In the ball $\mathcal{U}_{0,\,(p)}$, the ray $R_{(p)}$ emanating from the centre $-\widetilde{\xi}^{n-1,\,n-1}_{0,\,(p)}$ of $\mathcal{U}_{0,\,(p)}$ and going through the origin $0 \in \mathcal{U}_{0,\,(p)}$ of the ambient vector space $C^\infty_{n-1,\,n-1}(X_0,\,\mathbb{C})$ cuts the boundary sphere of $\mathcal{U}_{0,\,(p)}$ in a point that we call $A_{(p)}$. If $d_{(p)}$ denotes the distance from 0 to $A_{(p)}$, then $d_{(p)} = \alpha_{(p)}$. Meanwhile, the hyperplane $H_{0,\,(p)}$ is orthogonal to the ray $R_{(p)}$ at 0 and the *maximal* L^2-norm that a vector $\nu^{n-1,\,n-1}_{0,\,(p)} \in (\mathrm{Im}\,S_0)\cap\mathcal{U}_{0,\,(p)}$ can have attains its *minimal* value when $\mathrm{Im}\,S_0$ is contained in

$H_{0,(p)}$. When $\mathrm{Im}\, S_0 \subset H_{0,(p)}$, the vector $\nu_{0,(p)}^{n-1,n-1}$ can be chosen in the intersection of $\mathrm{Im}\, S_0$ with the boundary sphere of $\mathcal{U}_{0,(p)}$ to attain the maximal value that the L^2-norm of a vector in $(\mathrm{Im}\, S_0)\cap\mathcal{U}_{0,(p)}$ can have in this case. Then in the right-angled triangle formed by the points 0, $\nu_{0,(p)}^{n-1,n-1}$ and $A_{(p)}$, the side joining 0 to $\nu_{0,(p)}^{n-1,n-1}$ (of length $||\nu_{0,(p)}^{n-1,n-1}||$) cannot be shorter than the side joining 0 to $A_{(p)}$ (of length $d_{(p)} = \alpha_{(p)}$) since the angle facing the former side is $\geq \pi/4$ while the angle facing the latter side is $\leq \pi/4$. $\square$

Now recall that in *Case* 2 we have $\alpha_{(p)} = 0$ and we can choose $\nu_{0,(p)}^{n-1,n-1} = 0$ because $\widetilde{\xi}_{0,(p)}$ already satisfies condition (c) of $(\star_p)$ with $\nu_{0,(p)}^{n-1,n-1} = 0$ for the uniform $\varepsilon_0 > 0$ obtained at the previous induction steps $1,\dots,p$. Therefore, if in *Case* 1 $\alpha_{(p)} \downarrow 0$ as $p\to+\infty$, we can satisfy condition (c) of $(\star_p)$ with the uniform $\varepsilon_0 > 0$ of $(\star_{p_0})(c)$ for all $p\geq p_0$ and for some $p_0\in\mathbb{N}$.

Thus it remains to treat the case covered by the following

Assumption (A2): $\qquad \alpha_{(p)} \geq \alpha_0 > 0, \quad \forall p\in\mathbb{N},$
for some $\alpha_0 > 0$ independent of $p\in\mathbb{N}$.

In this case, we get from the estimate (7.73) of Conclusion 7.1.12 and from the estimate (7.74) of Lemma 7.1.13 the following

Conclusion 7.1.14. *Under the assumptions (A1) and (A2), we have:*

$$\langle\langle\Delta_0''(\partial_0\widetilde{\xi}_{0,(p)}^{n-1,n-1} + \partial_0\nu_{0,(p)}^{n-1,n-1}),\, \partial_0\widetilde{\xi}_{0,(p)}^{n-1,n-1} + \partial_0\nu_{0,(p)}^{n-1,n-1}\rangle\rangle \geq \varepsilon'\,\varepsilon''\,\alpha_0^2 \tag{7.75}$$

for some $\nu_{0,(p)}^{n-1,n-1}\in(\mathrm{Im}\, S_0)\cap\mathcal{U}_{0,(p)}$ chosen to maximise the L^2-norm $||\nu_{0,(p)}^{n-1,n-1}||$.

We have thus achieved our purpose of proving the existence of auxiliary forms $\nu_{t,(p)}^{n-1,n-1}\in(\mathrm{Im}\, S_t)\cap\mathcal{U}_{t,(p)}$ (i.e. satisfying (7.64) which automatically guarantees (a) and (b) of $(\star_p)$) such that the numerator of (c) in $(\star_p)$ is uniformly bounded below by a positive constant in *Case* 1.

(II) Uniformly bounding the fraction of (c) in $(\star_p)$ from below in Case 1

Recall that under (I) above we have been working under the induction hypothesis that the induction steps $1,\dots,p$ had been run and have shown as a result the existence of auxiliary forms $\eta_{t,(p)} = \partial_t\nu_{t,(p)}^{n-1,n-1} = \bar\partial_t\vartheta_{t,(p)}^{n,n-2}$ satisfying conditions (a), (b) of $(\star_p)$ and (7.75). Thus the inductively constructed auxiliary forms satisfy (a) and (b) of $(\star_p)$ for all $p\in\mathbb{N}$ as well as the uniform lower bound:

$$\langle\langle\Delta_t''(\partial_t\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \partial_t\nu_{t,(p)}^{n-1,n-1}),\, \partial_t\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \partial_t\nu_{t,(p)}^{n-1,n-1}\rangle\rangle \geq \delta > 0, \tag{7.76}$$

for all $t\in B$ (after possibly shrinking B about 0) and all $p\in\mathbb{N}$, where we have denoted $\delta := \varepsilon'\,\varepsilon''\,\alpha_0^2 > 0$ (independent of t and p, cf. (7.75)).

Now we have:

$$\begin{aligned} A_{t,(p)} : &= \langle\langle\Delta_t(\partial_t\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \partial_t\nu_{t,(p)}^{n-1,n-1}),\, \partial_t\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \partial_t\nu_{t,(p)}^{n-1,n-1}\rangle\rangle \\ &= A'_{t,(p)} + \langle\langle\Delta_t''(\partial_t\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \partial_t\nu_{t,(p)}^{n-1,n-1}),\, \partial_t\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \partial_t\nu_{t,(p)}^{n-1,n-1}\rangle\rangle \\ &\geq A'_{t,(p)} + \delta, \qquad t\in B,\ p\in\mathbb{N}, \end{aligned} \tag{7.77}$$

where we have denoted $A'_{t,\,(p)} := \langle\langle\Delta'_t(\partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \partial_t\nu^{n-1,\,n-1}_{t,\,(p)}),\, \partial_t\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \partial_t\nu^{n-1,\,n-1}_{t,\,(p)}\rangle\rangle$ and by $A_{t,\,(p)}$ the analogous expression with Δ_t in place of Δ'_t. (To justify the identity between the top two lines in (7.77), recall that for any *pure-type* form u one has $\langle\langle\Delta_t u,\, u\rangle\rangle = \langle\langle\Delta'_t u,\, u\rangle\rangle + \langle\langle\Delta''_t u,\, u\rangle\rangle$ by (7.42).)

Recall that in order to guarantee condition (c) of $(\star_p)$ for all $p\in\mathbb{N}$ we need to prove the existence of an $\varepsilon_0>0$ independent of both $t\in B$ and $p\in\mathbb{N}$ such that

$$A_{t,\,(p)} \geq (1+\varepsilon_0)\,A'_{t,\,(p)}, \qquad t\in B,\ p\in\mathbb{N}. \tag{7.78}$$

Since (7.77) holds, it suffices to get a uniform $\varepsilon_0>0$ as above such that

$$A'_{t,\,(p)} + \delta \geq (1+\varepsilon_0)\,A'_{t,\,(p)} \quad \text{or equivalently} \quad A'_{t,\,(p)} \leq \frac{\delta}{\varepsilon_0}, \qquad t\in B,\ p\in\mathbb{N}. \tag{7.79}$$

The existence of such a uniform $\varepsilon_0>0$ is of course guaranteed if we can prove that $A'_{t,\,(p)}$ is uniformly bounded above. Since $A'_{t,\,(p)} \leq A_{t,\,(p)}$, it suffices to prove the existence of a uniform upper bound for the latter quantity.

Lemma 7.1.15. *In the above notation, the auxiliary forms $\eta_{t,\,(p)} = \partial_t\nu^{n-1,\,n-1}_{t,\,(p)} = \bar\partial_t\vartheta^{n,\,n-2}_{t,\,(p)} \in \mathcal{E}^{n,\,n-1}_t$ constructed by the induction procedure set up in the preceding paragraphs and with the choices made there satisfy*

$$A'_{t,\,(p)} \leq A_{t,\,(p)} \leq M < +\infty, \qquad t\in B,\ p\in\mathbb{N}, \tag{7.80}$$

for some M independent of both $t\in B$ and $p\in\mathbb{N}$.

Proof. Recall that in the induction process we solve the equations (cf. (7.56)):

$$d\widetilde{\xi}_{t,\,(p+1)} = \partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)}), \qquad t\in B,\ p\in\mathbb{N}, \tag{7.81}$$

and we choose $\widetilde{\xi}_{t,\,(p+1)}$ to be the minimal L^2-norm solution for every given $p\in\mathbb{N}$. Thus

$$\widetilde{\xi}_{t,\,(p+1)} = \Delta_t^{-1}d_t^\star\partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)}), \qquad t\in B,\ p\in\mathbb{N}, \tag{7.82}$$

and

$$||\widetilde{\xi}_{t,\,(p+1)}||^2 \ = \ ||\Delta_t^{-\frac{1}{2}}\partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)})||^2 = B_{t,\,(p)}, \qquad t\in B,\ p\in\mathbb{N},$$

where we have denoted

$$B_{t,\,(p)} := \langle\langle\Delta_t^{-1}\partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)}),\, \partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)})\rangle\rangle.$$

It is clear that if $B_{t,\,(p)}$ became arbitrarily small when $p\to+\infty$, then $||\widetilde{\xi}_{t,\,(p+1)}||$ would become arbitrarily small. This would give right away the conclusion of Corollary 7.1.18 below and the proof of Theorem 7.1.1 would follow as explained at the end of the section. This gives a hint that $A_{t,\,(p)}$ is likely to satisfy the uniform upper bound (7.80) at least in the complementary case (i.e. when $B_{t,\,(p)}$ is uniformly bounded below by a positive constant). Here are the details.

If we denote $\varpi_{t,\,(p)} := \partial_t(\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)})$, we know that

$$\varpi_{t,\,(p)} = d\,\widetilde{\xi}_{t,\,(p+1)}, \qquad \text{with} \quad \widetilde{\xi}_{t,\,(p+1)} \in \operatorname{Im} d_t^\star \subset \ker d_t^\star,\ t\in B,\ p\in\mathbb{N}.$$

So we get

$$\begin{aligned} A_{t,\,(p)} &= \langle\langle\Delta_t\varpi_{t,\,(p)},\,\varpi_{t,\,(p)}\rangle\rangle = ||d\,\varpi_{t,\,(p)}||^2 + ||d_t^\star\varpi_{t,\,(p)}||^2 = ||d_t^\star\varpi_{t,\,(p)}||^2 = ||d_t^\star d\,\widetilde{\xi}_{t,\,(p+1)}||^2 \\ &= ||d_t^\star d\,\widetilde{\xi}_{t,\,(p+1)} + d\,d_t^\star\widetilde{\xi}_{t,\,(p+1)}||^2 = ||\Delta_t\,\widetilde{\xi}_{t,\,(p+1)}||^2, \quad t\in B,\ p\in\mathbb{N}. \end{aligned} \tag{7.83}$$

Now observe that the proof of Lemma 7.1.7 shows that the families of forms $(\widetilde{\xi}_{t,\,(p)})_{t\in B}$ $(p\in\mathbb{N})$ defined by solving equations (7.56) for $p-1$ satisfy inequalities analogous to the inequalities (7.34) for $(\xi_{t,\,(p)})_{t\in B}$ $(p\in\mathbb{N})$:

$$||\widetilde{\xi}_{t,\,(p+1)}|| \leq ||\widetilde{\Omega}^{n-1,\,n-1}_{t,\,(p)}||, \quad t\in B,\ p\in\mathbb{N}, \quad (\text{cf. } (7.43)) \tag{7.84}$$

by comparison of the minimal d and ∂_t-potentials of the given form $\varpi_{t,\,(p)}$,

$$||\widetilde{\Omega}^{n-1,\,n-1}_{t,\,(p)}|| \leq ||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)}||, \quad t\in B,\ p\in\mathbb{N}, \quad (\text{cf. } (7.44)) \tag{7.85}$$

by minimality of $\widetilde{\Omega}^{n-1,\,n-1}_{t,\,(p)}$ among the ∂_t-potentials of $\varpi_{t,\,(p)}$, and

$$||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)}|| \leq ||\widetilde{\xi}_{t,\,(p)}||, \quad t\in B,\ p\in\mathbb{N}, \tag{7.86}$$

by (b) of $(\star_p)$. The last three inequalities add up to

$$||\widetilde{\xi}_{t,\,(p+1)}|| \leq ||\widetilde{\xi}_{t,\,(p)}||, \quad t\in B,\ p\in\mathbb{N}, \quad (\text{cf. } (7.34)). \tag{7.87}$$

The sequence $(||\widetilde{\xi}_{t,\,(p)}||)_{p\in\mathbb{N}}$ is thus non-increasing (hence bounded above) for each $t\in B$. After slightly shrinking B about 0, let

$$M_1 := \sup_{t\in B,\,p\in\mathbb{N}} ||\widetilde{\xi}_{t,\,(p)}|| = \sup_{t\in B} ||\widetilde{\xi}_{t,\,(0)}|| = \sup_{t\in B} ||\xi_t|| < +\infty. \tag{7.88}$$

Recall that in view of formula (7.83) we need to show that $||\Delta_t\widetilde{\xi}_{t,\,(p)}||$ is bounded above independently of $t\in B$ and $p\in\mathbb{N}$. Only the uniform boundedness w.r.t. p has yet to be justified. Note that Δ_t does not depend on p. We need a slight refinement of (7.87).

For every $t\in B$ and $p\in\mathbb{N}$ let

$$\widetilde{\xi}_{t,\,(p)} = \sum_{j=0}^{+\infty} u_j^{(p)}(t), \quad \text{with} \quad u_j^{(p)}(t)\in E_{\Delta_t}(\lambda_j), \tag{7.89}$$

be the decomposition of $\widetilde{\xi}_{t,\,(p)}$ w.r.t. the eigenspaces $E_{\Delta_t}(\lambda_j)$ of Δ_t. The eigenvalues $\lambda_j = \lambda_j(t)$ of Δ_t, ordered (without repetitions) increasingly, tend to $+\infty$ as j tends to $+\infty$. Inequality (7.87) translates to

$$||\widetilde{\xi}_{t,\,(p+1)}||^2 = \sum_{j=0}^{+\infty} ||u_j^{(p+1)}(t)||^2 \leq \sum_{j=0}^{+\infty} ||u_j^{(p)}(t)||^2 = ||\widetilde{\xi}_{t,\,(p)}||^2, \quad t\in B,\ p\in\mathbb{N}. \tag{7.90}$$

Meanwhile, we clearly have

$$||\Delta_t\widetilde{\xi}_{t,\,(p)}||^2 = \sum_{j=0}^{+\infty} \lambda_j^2\,||u_j^{(p)}(t)||^2, \quad t\in B,\ p\in\mathbb{N}. \tag{7.91}$$

The inductive process that produced the forms $(\widetilde{\xi}_{t,\,(p)})$ shows in effect that the norm inequality (7.87) occurs *component-wise*, i.e. for every $j \in \mathbb{N}$ we have:

$$||u_j^{(p+1)}(t)|| \leq ||u_j^{(p)}(t)||, \quad t \in B, \; p \in \mathbb{N}. \tag{7.92}$$

Indeed, recall that by (7.40) any pure-type form u satisfies $\langle\langle \Delta_t u,\, u\rangle\rangle \geq \langle\langle \Delta'_t u,\, u\rangle\rangle$, hence inequality (7.84) occurs *component-wise*. Inequality (7.85) occurs *component-wise* as well since $\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)}$ is obtained from $\widetilde{\Omega}^{n-1,\,n-1}_{t,\,(p)}$ by adding a form (lying in $\ker\partial_t$) that is orthogonal to the minimal L^2-norm ∂_t-potential $\widetilde{\Omega}^{n-1,\,n-1}_{t,\,(p)} \in \operatorname{Im}\partial_t^\star \perp \ker\partial_t$. On the other hand, $\nu^{n-1,\,n-1}_{t,\,(p)}$ is chosen to lie in $\operatorname{Im} S_t \cap \mathcal{U}_{t,\,(p)}$ (by (7.64)) and to have *maximal* L^2-norm among these forms (by Lemma 7.1.13) while S_t is independent of p and the radius of the ball $\mathcal{U}_{t,\,(p)}$ is non-increasing w.r.t. $p \in \mathbb{N}$ by (7.87). Hence we can choose the forms $\nu^{n-1,\,n-1}_{t,\,(p)}$ such that

$$||\nu^{n-1,\,n-1}_{t,\,(p)}|| \leq ||\nu^{n-1,\,n-1}_{t,\,(p-1)}|| \qquad \textit{component-wise}, \quad t \in B, \; p \in \mathbb{N}^\star.$$

Thus we obtain (7.92) inductively on $p \in \mathbb{N}$: if (7.92) has been shown for $p-1$, then for all $t \in B$ we have

$$||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p)} + \nu^{n-1,\,n-1}_{t,\,(p)}|| \leq ||\widetilde{\xi}^{n-1,\,n-1}_{t,\,(p-1)} + \nu^{n-1,\,n-1}_{t,\,(p-1)}|| \qquad \textit{component-wise},$$

which implies $||\widetilde{\xi}_{t,\,(p+1)}|| \leq ||\widetilde{\xi}_{t,\,(p)}||$ *component-wise*. This is nothing but (7.92).

Now (7.91) and (7.92) combine to show the existence of a uniform upper bound for the Laplacian of $\widetilde{\xi}_{t,\,(p)}$ (after slightly shrinking B about 0):

$$M := \sup_{t\in B,\, p\in\mathbb{N}} ||\Delta_t\, \widetilde{\xi}_{t,\,(p+1)}|| < +\infty, \tag{7.93}$$

which in view of (7.83) is nothing but (7.80).

Lemma 7.1.15 is proved. $\square$

We can now explicitly achieve (7.79), hence also (7.78) which is equivalent to condition (c) of $(\star_p)$. Indeed, estimate (7.80) obtained in Lemma 7.1.15 shows that the inductively constructed auxiliary forms fulfill condition (c) of $(\star_p)$ with the uniform $\varepsilon_0 > 0$ defined by

$$\varepsilon_0 := \frac{\delta}{M} = \frac{\varepsilon'\,\varepsilon''\,\alpha_0^2}{M} > 0,$$

where $\delta := \varepsilon'\,\varepsilon''\,\alpha_0^2 > 0$ is the uniform lower bound of (7.76) and $M < +\infty$ is the uniform upper bound of (7.93).

The existence of the auxiliary forms is thus accounted for. $\square$

7.1.5 Final arguments in proving Theorem 7.1.1

With the new inductive construction based on auxiliary forms in place, the identities of Lemma 7.1.6 obeyed by $\xi_{t,\,(p)}$ are transformed into the following identities obeyed by $\widetilde{\xi}_{t,\,(p)}$.

Lemma 7.1.16. *The family $(\widetilde{\xi}_{t,\,(p)})_{t\in B}$ of $(2n-2)$-forms constructed above varies in a C^∞ way with t (up to $t=0$) and satisfies for all $t \in B$ and all $p \in \mathbb{N}$:*

$$\partial_t(\gamma_t^{n-1} - \widetilde{\xi}_{t,(p)}^{n-1,n-1} - \overline{\widetilde{\xi}_{t,(p)}^{n-1,n-1}}) = \bar\partial_t(\widetilde{\xi}_{t,(p)}^{n,n-2} + \overline{\widetilde{\xi}_{t,(p)}^{n-2,n}} + \cdots + \widetilde{\xi}_{t,(1)}^{n,n-2} + \xi_t^{n,n-2} + w_t - \vartheta_t^{n,n-2} - \vartheta_{t,(1)}^{n,n-2} - \cdots - \vartheta_{t,(p-1)}^{n,n-2}). \tag{7.94}$$

Proof. It follows trivially from (7.57) with $p+1$ replaced by p and the fact that $d\,\widetilde{\xi}_{t,(p)} = \partial_t\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \bar\partial_t\widetilde{\xi}_{t,(p)}^{n,n-2}$ is of type $(n,\, n-1)$ (thus its $(n-1,\, n)$-component vanishes, hence $-\bar\partial_t\widetilde{\xi}_{t,(p)}^{n-1,n-1} = \partial_t\widetilde{\xi}_{t,(p)}^{n-2,n}$ and taking conjugates $-\partial_t\overline{\widetilde{\xi}_{t,(p)}^{n-1,n-1}} = \bar\partial_t\overline{\widetilde{\xi}_{t,(p)}^{n-2,n}}$) by arguments analogous to those of Lemma 7.1.6. □

The next, more substantial step is to show that the L^2-norm of $\widetilde{\xi}_{t,(p)}^{n-1,n-1}$ decreases strictly at each step p of the above inductive construction in a way that guarantees it to become arbitrarily small when p becomes large enough. The following lemma and its corollary provide the final argument to the proof of Theorem 7.1.1 and, implicitly, to that of Theorem 7.0.4.

Lemma 7.1.17. *There exists $\varepsilon > 0$ independent of $t \in B$ and of $p \in \mathbb{N}$ such that the minimal L^2-norm solutions $\widetilde{\Omega}_{t,(p)}^{n-1,n-1}$ and $\widetilde{\xi}_{t,(p+1)}$ of the equations*

$$\partial_t\widetilde{\Omega}_{t,(p)}^{n-1,n-1} = \partial_t(\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \nu_{t,(p)}^{n-1,n-1}) \quad \textit{and} \quad d\,\widetilde{\xi}_{t,(p+1)} = \partial_t(\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \nu_{t,(p)}^{n-1,n-1}) \tag{7.95}$$

satisfy the L^2-norm estimates:

$$||\widetilde{\xi}_{t,(p+1)}|| \leq \frac{1}{\sqrt{1+\varepsilon}}\,||\widetilde{\Omega}_{t,(p)}^{n-1,n-1}||, \quad t \in B,\ p \in \mathbb{N}. \tag{7.96}$$

Before proving this statement, we notice an immediate corollary.

Corollary 7.1.18. *The forms $\widetilde{\xi}_{t,(p)}$ obtained above satisfy*

$$||\widetilde{\xi}_{t,(p)}|| \leq \frac{1}{(\sqrt{1+\varepsilon})^p}\,||\xi_t||, \quad t \in B,\ \ p \in \mathbb{N}. \tag{7.97}$$

In particular, $||\widetilde{\xi}_{t,(p)}||$ (hence also $||\widetilde{\xi}_{t,(p)}^{n-1,n-1}||$ which is $\leq ||\widetilde{\xi}_{t,(p)}||$) becomes arbitrarily small, uniformly w.r.t. $t \in B$ and $p \gg 1$, if the number $p \in \mathbb{N}$ of iterations is sufficiently large.

Proof of Corollary 7.1.18. From Lemma 7.1.17 we get the following inequalities:

$$||\widetilde{\xi}_{t,(p+1)}|| \leq \frac{1}{\sqrt{1+\varepsilon}}\,||\widetilde{\Omega}_{t,(p)}^{n-1,n-1}|| \leq \frac{1}{\sqrt{1+\varepsilon}}\,||\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \nu_{t,(p)}^{n-1,n-1}||, \quad p \in \mathbb{N}.$$

The latter inequality follows from the L^2-norm minimality of $\widetilde{\Omega}_{t,(p)}^{n-1,n-1}$ among the solutions of the equation $\partial_t\widetilde{\Omega}_{t,(p)}^{n-1,n-1} = \partial_t(\widetilde{\xi}_{t,(p)}^{n-1,n-1} + \nu_{t,(p)}^{n-1,n-1})$. Combining with (b) of properties $(\star_p)$, we get

$$||\widetilde{\xi}_{t,(p+1)}|| \leq \frac{1}{\sqrt{1+\varepsilon}}\,||\widetilde{\xi}_{t,(p)}||, \quad t \in B,\ \ p \in \mathbb{N}.$$

Letting p run through $0, \dots, p-1$, these inequalities multiply up to (7.97). □

We now come to the key task of proving Lemma 7.1.17. However, the ground has been largely prepared by Lemma 7.1.7 and Observation 7.1.8 whose proofs outlined the difficulties and explained how to overcome them under certain hypotheses, as well as by the construction of auxiliary forms

$\eta_{t,(p)}$ satisfying conditions $(\star_p)$ which enable those hypotheses to be met. The remaining arguments are almost purely formal.

Proof of Lemma 7.1.17. Recall the notation $\widetilde{\xi}^{n-1,n-1}_{t,(0)} := \xi^{n-1,n-1}_t$, $\widetilde{\Omega}^{n-1,n-1}_{t,(0)} := \widetilde{\Omega}^{n-1,n-1}_t$ and $\nu^{n-1,n-1}_{t,(0)} := \nu^{n-1,n-1}_t$. Set $\varpi_{t,(p)} := \partial_t(\widetilde{\xi}^{n-1,n-1}_{t,(p)} + \nu^{n-1,n-1}_{t,(p)})$, the right-hand term of equations (7.95). The minimal L^2-norm solutions of equations (7.95) are explicitly given by the formulae:

$$\widetilde{\Omega}^{n-1,n-1}_{t,(p)} = \Delta'^{-1}_t\partial^\star_t\varpi_{t,(p)} \quad \text{and} \quad \widetilde{\xi}_{t,(p+1)} = \Delta^{-1}_t d^\star_t\varpi_{t,(p)}, \qquad t\in B,\ p\in\mathbb{N}. \tag{7.98}$$

Thus by (7.36) and (7.38) with $u=\varpi_{t,(p)}$, the proof of Lemma 7.1.17 reduces to proving that, for some $\varepsilon>0$ independent of $t\in B$ and $p\in\mathbb{N}$, we have:

$$||\Delta_t^{-\frac{1}{2}}\varpi_{t,(p)}|| \leq \frac{1}{\sqrt{1+\varepsilon}}\,||\Delta_t'^{-\frac{1}{2}}\varpi_{t,(p)}||, \qquad t\in B,\ p\in\mathbb{N}. \tag{7.99}$$

Now the forms $\eta_{t,(p)} = \partial_t\nu^{n-1,n-1}_{t,(p)}$ have been chosen to satisfy conditions $(\star_p)$ whose part (c) translates to:

$$0<\varepsilon_0\leq\frac{\langle\langle\Delta''_t\varpi_{t,(p)},\,\varpi_{t,(p)}\rangle\rangle}{\langle\langle\Delta'_t\varpi_{t,(p)},\,\varpi_{t,(p)}\rangle\rangle}, \qquad t\in B,\ p\in\mathbb{N}, \tag{7.100}$$

for an $\varepsilon_0>0$ independent of both $t\in B$ and $p\in\mathbb{N}$.

By the choice (a) of $(\star_p)$, we have $\eta_{t,(p)} = \partial_t\nu^{n-1,n-1}_{t,(p)} = \bar\partial_t\vartheta^{n,n-2}_{t,(p)}$, hence $\eta_{t,(p)}$ is $\bar\partial_t$-exact for all $t\in B$ and all $p\in\mathbb{N}$. It follows that:

(i) the form $\varpi_{t,(p)} = \partial_t\widetilde{\xi}^{n-1,n-1}_{t,(p)} + \eta_{t,(p)}$ is $\bar\partial_t$-exact for all $t\neq 0$, hence $\varpi_{t,(p)}$ is orthogonal to $\ker\Delta''_t$ for all $t\neq 0$;

(ii) when $t=0$, the form $\varpi_{0,(p)} = \partial_0\widetilde{\xi}^{n-1,n-1}_{0,(p)} + \eta_{0,(p)}$ cannot be Δ''_0-harmonic.

Indeed, otherwise the condition (c) of $(\star_p)$ would be violated (see (7.100) for $t=0$) unless we also have $\Delta'_0\varpi_{0,(p)}=0$. However, in this latter case the ∂_0-exact form $\varpi_{0,(p)} = \partial_0(\widetilde{\xi}^{n-1,n-1}_{0,(p)} + \nu^{n-1,n-1}_{0,(p)})$ would have to vanish (since $\operatorname{Im}\partial_0 \perp \ker\Delta'_0$) and $\partial_0\gamma_0^{n-1}$ would be $\bar\partial_0$-exact by (7.55) applied at $t=0$. Then γ_0 would be a *strongly Gauduchon* metric on X_0 and the proof of Theorem 7.1.1 would be complete.

We conclude that, for every fixed $p\in\mathbb{N}$, the family $(\varpi_{t,(p)})_{t\in B}$ satisfies the non-Δ''_t-harmonicity hypothesis (7.46), hence also estimate (7.47) uniformly w.r.t. $t\in B$.

Moreover, by (7.42) and by $\varpi_{t,(p)}$ being of pure type, we have

$$\langle\langle\Delta_t\varpi_{t,(p)},\,\varpi_{t,(p)}\rangle\rangle = \langle\langle\Delta'_t\varpi_{t,(p)},\,\varpi_{t,(p)}\rangle\rangle + \langle\langle\Delta''_t\varpi_{t,(p)},\,\varpi_{t,(p)}\rangle\rangle,$$

so the uniform estimate (7.100) amounts to

$$\langle\langle\Delta_t\varpi_{t,(p)},\,\varpi_{t,(p)}\rangle\rangle \geq (1+\varepsilon_0)\,\langle\langle\Delta'_t\varpi_{t,(p)},\,\varpi_{t,(p)}\rangle\rangle, \quad t\in B,\ p\in\mathbb{N}, \tag{7.101}$$

which provides unifomity w.r.t. $p\in\mathbb{N}$ besides the uniformity w.r.t. $t\in B$. This proves Lemma 7.1.17. $\square$

End of proof of Theorem 7.1.1. By Corollary 7.1.18, the L^2-norm $||\widetilde{\xi}^{n-1,n-1}_{t,(p)}||$ can be made arbitrarily small, uniformly with respect to $t\in B$ and $p\gg 1$, if p is chosen sufficiently large. In particular, so can the L^2-norm $||\widetilde{\xi}^{n-1,n-1}_{0,(p)}||$.

Thanks to Lemma 7.1.5, if p is sufficiently large, we get a C^∞ positive definite $J_0-(1,\,1)$-form $\rho_0 > 0$ such that

$$\partial_0\rho_0^{n-1} - \partial_0\bigg(\gamma_0^{n-1} - \widetilde{\xi}_{0,\,(p)}^{n-1,\,n-1} - \overline{\widetilde{\xi}_{0,\,(p)}^{n-1,\,n-1}}\bigg) \in \mathrm{Im}\,(\partial_0\bar\partial_0).$$

Since $\partial_0(\gamma_0^{n-1} - \widetilde{\xi}_{0,\,(p)}^{n-1,\,n-1} - \overline{\widetilde{\xi}_{0,\,(p)}^{n-1,\,n-1}})$ is known to be $\bar\partial_0$-exact by identity (7.94) of Lemma 7.1.16, we see that $\partial_0\rho_0^{n-1}$ must be $\bar\partial_0$-exact, hence ρ_0 is a *strongly Gauduchon* metric on X_0. The proof of Theorem 7.1.1 is complete. $\square$

7.2 Existence of limiting divisors in families whose generic fibre is a $\partial\bar\partial$-manifold

In this section, taken from [Pop10a], we provide the second and final main argument for the first proof of Theorem 7.0.4. Intuitively put, the main result of this section says that, if all the fibres X_t with $t\neq 0$ of a holomorphic family of compact complex manifolds over a small ball $B\subset\mathbb{C}^N$ about 0 are **$\partial\bar\partial$-manifolds**, the limiting fibre X_0 has *at least as many divisors* as the neighbouring fibres. This fact, stated in a precise way as Theorem 7.2.2 below, will be given a proof that critically relies on Theorem 7.1.1 of the previous section.

Recall that the *algebraic dimension* $a(X)$ of a compact complex n-dimensional manifold X is the maximal number of algebraically independent meromorphic functions on X. Equivalently, $a(X)$ is the transcendence degree over $\mathbb{C}$ of the field of meromorphic functions on X. It is standard that $a(X)\leq n$ and that $a(X)=n$ if and only if X is Moishezon ([Moi67]). Since every meromorphic function gives rise to its divisor of zeros and poles, Moishezon manifolds can be regarded as the compact complex manifolds that carry “many” divisors.

The algebraic dimension $a(X_t)$ does not, in general, depend upper-semicontinuously on the fibre X_t varying in a holomorphic family of compact complex manifolds. This is so even in families of surfaces, as shown by an example found by Fujiki and Pontecorvo in [FP09] of a family of compact non-Kähler complex surfaces of class VII in which the algebraic dimension drops from 1 on X_t with $t\neq 0$ to 0 on X_0. However, one key consequence of our results of this section is that the dependence of the algebraic dimension $a(X_t)$ on $t\in B$ becomes *upper-semicontinuous* if all the fibres, except possibly one, are supposed to be *$\partial\bar\partial$-manifolds*.

Theorem 7.2.1. *([Pop10a] and again [Pop19, Theorem 3.7.]) Let $\pi:\mathcal{X}\to B$ be a holomorphic family of compact complex manifolds over an open ball $B\subset\mathbb{C}^N$ about the origin such that the fibre $X_t:=\pi^{-1}(t)$ is a $\partial\bar\partial$-**manifold** for every $t\in B\setminus\{0\}$. Then $a(X_0)\geq a(X_t)$ for all $t\in B\setminus\{0\}$ sufficiently close to 0, where $a(X_t)$ is the algebraic dimension of X_t.*

Since any compact complex $\partial\bar\partial$-surface must be Kähler (because its first Betti number b_1 is necessarily even by Hodge decomposition and symmetry, so equal to $2\,h^{0,\,1}$, and thus the result of Buchdahl [Buc99] and Lamari [Lam99] can be applied to yield Kählerianity for any $\partial\bar\partial$-surface), the family exhibited in [FP09] does not satisfy the hypothesis of Theorem 7.2.1. Indeed, surfaces of class VII are very far from being Kähler. Thus, Theorem 7.2.1 shows once again the key role played by $\partial\bar\partial$-manifolds in deformation theory.

Proof of Theorem 7.0.4 assuming that Theorem 7.2.1 has been proved. Since X_t is assumed to be Moishezon for every $t\in B\setminus\{0\}$, $a(X_t)=n:=\dim_{\mathbb{C}}X_t$ and X_t is a $\partial\bar\partial$-manifold for every

$t \in B \setminus \{0\}$. In particular, Theorem 7.2.1 applies and implies that

$$a(X_0) \geq a(X_t) = n, \qquad t \in B \setminus \{0\}.$$

Hence, $a(X_0) = n = \dim_{\mathbb{C}} X_0$. Therefore, X_0 must be Moishezon. $\square$

The precise form of the main result of this section, which implies Theorem 7.2.1, can be stated in terms of the *relative Barlet space of divisors* $\mathcal{C}^{n-1}(\mathcal{X}/B)$ associated with a holomorphic family $\pi : \mathcal{X} \to B$ of compact complex manifolds X_t.

Theorem 7.2.2. *([Pop10a, Proposition 1.5]) Let $\pi : \mathcal{X} \to B$ be a holomorphic family of compact complex manifolds such that X_t is a* $\partial\bar\partial$**-manifold** *for every $t \in B \setminus \{0\}$. Then, the canonical holomorphic projection*

$$\mu_{n-1} : \mathcal{C}^{n-1}(\mathcal{X}/B) \to B, \qquad \mu_{n-1}(Z_t) = t,$$

mapping every divisor $Z_t \subset X_t$ contained in some fibre X_t to the base point $t \in B$, has the property that its restrictions to the irreducible components of $\mathcal{C}^{n-1}(\mathcal{X}/B)$ are **proper**.

We describe the barebones of Barlet's space of (relative) cycles in §.7.2.1 and refer the reader to [Bar75], [Bis64], [Cam80], [CP94] and [Lie78] for further details and proofs. Then, we give the proof of Theorem 7.2.2 in §.7.2.2.

7.2.1 Very brief reminder of a few fundamental facts in Barlet's theory of cyles

Let $\pi : \mathcal{X} \to B$ be a holomorphic family of compact complex manifolds $X_t := \pi^{-1}(t)$, with $t \in B \subset \mathbb{C}^N$, for some $N \in \mathbb{N}^\star$, and B a small open ball about the origin, and let n denote the complex dimension of the fibres X_t.

For every $p \in \{0, 1, \dots, n\}$, consider the **relative Barlet space** $\mathcal{C}^p(\mathcal{X}/B)$ of **effective analytic** p**-cycles** on $\mathcal{X}$ that are contained in the fibres X_t. It is a subspace of the **(absolute) Barlet space** $\mathcal{C}^p(\mathcal{X})$ of **compact** p**-cycles** on $\mathcal{X}$. Recall that $\mathcal{C}(\mathcal{X}) := \cup_p \mathcal{C}^p(\mathcal{X})$ is the **Chow scheme** of $\mathcal{X}$ (which, by definition, parametrises the compactly supported analytic cycles of $\mathcal{X}$) that Barlet endowed with a natural structure of **Banach analytic set** whose irreducible components are finite-dimensional analytic sets (cf. [Bar75]). Moreover, any irreducible component S of $\mathcal{C}(\mathcal{X})$ arises as an analytic family of compact cycles $(Z_s)_{s\in S}$ parametrised by S, while giving an analytic family $(Z_s)_{s\in S}$ of compact cycles of dimension p on $\mathcal{X}$ is equivalent to giving an analytic subset

$$\mathcal{Z} = \{(s,\, z) \in S \times \mathcal{X} \,/\, z \in |Z_s|\} \subset S \times \mathcal{X},$$

where $|Z_s|$ denotes the support of the cycle Z_s, such that the restriction to $\mathcal{Z}$ of the natural projection on S is proper, surjective and has fibres of pure dimension p (cf. [Bar75, Théorème 1, p. 38]). Recall finally Lieberman's strengthened form ([Lie78, Theorem 1.1]) of **Bishop's Theorem** [Bis64]: a subset $S \subset \mathcal{C}(\mathcal{X})$ is **relatively compact** if and only if the supports $|Z_s|$, $s \in S$, all lie in a same compact subset of $\mathcal{X}$ and the $\widetilde\omega$-volume of Z_s is **uniformly bounded** when $s \in S$ for some (hence any) Hermitian metric $\widetilde\omega$ on $\mathcal{X}$. Here, as usual, the $\widetilde\omega$**-volume** of a p-cycle $Z_s \subset \mathcal{X}$ is defined to be

$$v_{\widetilde\omega}(Z_s) := \int\limits_{\mathcal{X}} [Z_s] \wedge \widetilde\omega^p = \int\limits_{Z_s} \widetilde\omega^p,$$

where $[Z_s]$ is the current of integration on the cycle Z_s.

While the irreducible components of the Barlet space of cycles of arbitrary codimension $\mathcal{C}(X)$ need not be compact on a general compact complex manifold X (cf. [Lie78]), compactness of the irreducible components of the Barlet space $\mathcal{C}^{n-1}(X)$ of divisors of X always holds if X is compact (see e.g. [CP94, Remark 2.18.]). Thus the absolute case of Theorem 7.2.2 (i.e. when B is reduced to a point) is well-known and no special assumption is necessary.

It has been known since the work of Fujiki (see [Fuj78, Theorem 4.9.]) that the irreducible components of the Barlet space of cycles $\mathcal{C}(X)$ of a *class* $\mathcal{C}$ *manifold* X are compact. (They are even *class* $\mathcal{C}$ by [Cam80, Corollaire 3], but this extra property is immaterial to our purposes here.) As already mentioned, this last property fails if X is merely supposed to be compact (although it holds for divisors), while the *class* $\mathcal{C}$ assumption is the minimal requirement on X that we are aware of ensuring compactness of the irreducible components.

It thus appears natural to conjecture the (more general) relative case.

Conjecture 7.2.3. *Let $\pi : \mathcal{X} \to B$ be a complex analytic family of compact complex manifolds such that the fibre $X_t := \pi^{-1}(t)$ is a* **class $\mathcal{C}$ manifold** *for every $t \in B$. Then the irreducible components of the relative Barlet space $\mathcal{C}(\mathcal{X}/B)$ of cycles on $\mathcal{X}$ are* **proper** *over B.*

We have used the standard notation:

$$\mathcal{C}(\mathcal{X}/B) = \bigcup_{0\leq p\leq n} \mathcal{C}^p(\mathcal{X}/B),$$

where $\mathcal{C}^p(\mathcal{X}/B)$ stands for the relative Barlet space of effective analytic p-cycles contained in the fibres X_t. The special case of the above conjecture where all the fibres are supposed to be **Kähler** is well-known and quite easy to prove, but the general case of *class* $\mathcal{C}$ fibres is still elusive. We may even go so far as conjecture the same conclusion when the *class* $\mathcal{C}$ assumption is made to skip one of the fibres.

Conjecture 7.2.4. *Let $\pi : \mathcal{X} \to B$ be a holomorphic family of compact complex manifolds such that the fibre $X_t := \pi^{-1}(t)$ is a* **class $\mathcal{C}$ manifold** *for every $t \in B\setminus\{0\}$. Then, the irreducible components of the relative Barlet space of cycles $\mathcal{C}(\mathcal{X}/B)$ are* **proper** *over B.*

Theorem 7.2.2 answers affirmatively the stronger Conjecture 7.2.4 in the special case of divisors (and even under the weaker $\partial\bar\partial$-assumption which is satisfied by any *class* $\mathcal{C}$ manifold). A tantalising special case of Conjecture 7.2.4 is the one where the fibres X_t with $t \neq 0$ are supposed to be even Kähler. The central fibre X_0 is then expected to be *class* $\mathcal{C}$, but proving the compactness of the irreducible components of its Barlet space of cycles would be a first step towards confirming this expectation.

Proof of Theorem 7.2.1 assuming that Theorem 7.2.2 has been proved. The properness given by Theorem 7.2.2 guarantees that the images of the irreducible components of $\mathcal{C}^{n-1}(\mathcal{X}/B)$ under μ_{n-1} are *analytic* subsets of B thanks to Remmert's Proper Mapping Theorem. Let $\Sigma_\nu \subsetneq B$, for $\nu \in \mathbb{Z}$, be those such images (at most countably many) that are *strictly* contained in B. Each Σ_ν is thus a proper analytic subset of B. Bearing in mind the structure of the irreducible components of the (relative) Barlet space of cycles as described in [Bar75], we see that every irreducible component S of $\mathcal{C}^{n-1}(\mathcal{X}/B)$ gives rise to an analytic family (in the sense of [Bar75, Théorème 1, p. 38]) of relative effective divisors $(Z_s)_{s\in S}$ such that $Z_s \subset X_{\mu_{n-1}(s)}$ for all $s \in S$. We can either have

$$\mu_{n-1}(S) = B \quad \text{or} \tag{7.102}$$

$$\mu_{n-1}(S) = \Sigma_\nu \subsetneq B, \quad \text{for some } \nu \in \mathbb{Z}. \tag{7.103}$$

Let $\Sigma = \bigcup_\nu \Sigma_\nu \subsetneq B$. Thus, every divisor Z_{s_0} contained in a fibre X_{t_0} lying above some point $t_0 = \mu_{n-1}(s_0) \in B \setminus \Sigma$ (call such a fibre *generic*) stands in an analytic family of divisors $(Z_s)_{s\in S}$ covering the whole base B as in (7.102) (call these divisors *generic*), while the *exceptional* fibres X_t (i.e. those above points $t \in \Sigma$) may have extra divisors (those standing in *isolated* families satisfying (7.103)) besides the *generic* divisors that "sweep" B in families with the property (7.102).

In other words, properness of the irreducible components of $\mathcal{C}^{n-1}(\mathcal{X}/B)$ ensures that every fibre (in particular X_0) has at least as many divisors (the *generic* ones) as the *generic* fibres of the family. On the other hand, the algebraic dimension of any fibre X_t is the maximal number of effective prime divisors meeting transversally at a generic point of X_t (see e.g. [CP94, Remark 2.22]). It follows from the last two assertions that the algebraic dimension of X_0 is $\geq$ the algebraic dimension of the *generic* fibre. $\square$

7.2.2 Proof of Theorem 7.2.2.

Fix a family $(\gamma_t)_{t\in B}$ of Hermitian metrics, varying in a C^∞ way with t, on the respective fibres $(X_t)_{t\in B}$ of the given family of manifolds. As usual, J_t stands for the complex structure of X_t.

Also fix an arbitrary $p \in \{0, 1, \dots, n\}$. We start by running the proof of Theorem 7.2.2 as if we targeted the stronger conclusion of Conjecture 7.2.4 and will only assume that $p = 1$ towards the end.

Let $(Z_t)_{t\in B\setminus\{0\}}$ be a differentiable family of effective analytic $(n-p)$-cycles such that $Z_t \subset X_t$ for every $t \in B \setminus \{0\}$. The main difficulty in proving the properness predicted by Conjecture 7.2.4 is to ensure the uniform boundedness of the γ_t-volumes of the cycles Z_t:

$$v_{\gamma_t}(Z_t) = \int\limits_X [Z_t] \wedge \gamma_t^{n-p}, \qquad t \in B \setminus \{0\},$$

as t approaches $0 \in B$.

As every effective $(n-p)$-cycle $Z_t = \sum_j n_j(t)\, Z_j(t)$ on X_t is a finite linear combination with positive integers $n_j(t)$ of irreducible analytic subsets $Z_t \subset X_t$ of dimension $n-p$, the associated De Rham cohomology class $\{[Z_t]\} \in H^{2p}(X, \mathbb{R})$ is *integral*. Thus, the map

$$\Delta^\star \ni t \mapsto \{[Z_t]\} \in H^{2p}(X, \mathbb{Z}),$$

being continuous and integral-class-valued, must be constant. Fix any *real* (d-closed) differential $(2p)$-form α in this constant De Rham class. As $[Z_t]$ and α are d-cohomologous for every $t \in B\setminus\{0\}$, there exists a *real* current β'_t of degree $(2p-1)$ on X such that

$$\alpha = [Z_t] + d\beta'_t, \qquad t \in B \setminus \{0\}. \tag{7.104}$$

A double upper index r, s will denote throughout the component of pure type $(r,\, s)$ of the form or current to which the index is attached. Since the current $[Z_t]$ is of pure type $(p,\, p)$, identifying the pure-type components on either side of the equality, we see that identity (7.104) is equivalent to the following set of identities for all $t \in B \setminus \{0\}$:

$$\begin{array}{rcl}\alpha_t^{0,\,2p} & = & \bar\partial_t\beta_t^{\prime 0,\,2p-1},\\ \alpha_t^{1,\,2p-1}-\partial_t\beta_t^{\prime 0,\,2p-1}=\bar\partial_t\beta_t^{\prime 1,\,2p-2}, & \dots & ,\alpha_t^{p-1,\,p+1}-\partial_t\beta_t^{\prime p-2,\,p+1}=\bar\partial_t\beta_t^{\prime p-1,\,p},\\ \alpha_t^{p,\,p}-\partial_t\beta_t^{\prime p-1,\,p}-[Z_t] & = & \bar\partial_t\beta_t^{\prime p,\,p-1},\\ \alpha_t^{p+1,\,p-1}-\partial_t\beta_t^{\prime p,\,p-1}=\bar\partial_t\beta_t^{\prime p+1,\,p-2}, & \dots & ,\alpha_t^{2p-1,\,1}-\partial_t\beta_t^{\prime 2p-2,\,1}=\bar\partial_t\beta_t^{\prime 2p-1,\,0},\\ \alpha_t^{2p,\,0} & = & \partial_t\beta_t^{\prime 2p-1,\,0}.\end{array} \tag{7.105}$$

For all $t\in B\setminus\{0\}$, we also have $\beta'_t=\overline{\beta'_t}$ (as β'_t is real) which amounts to

$$\beta_t^{\prime l,\,2p-1-l}=\overline{\beta_t^{\prime 2p-1-l,\,l}},\qquad l=0,1,\dots,2p-1. \tag{7.106}$$

The current β'_t is determined only up to the kernel of d. We now proceed to construct a *real* C^∞ $(2p-1)$-form β_t, having the same properties as the current β'_t, by inductively choosing its pure-type components to be minimal L^2-norm solutions (w.r.t. γ_t) of the first half of equations (7.105) for all $t\in B\setminus\{0\}$.

Thus, for every $t\in B\setminus\{0\}$, let $\beta_t^{0,\,2p-1}$ be the form of J_t-type $(0,\,2p-1)$ which is the minimal L^2-norm solution of the equation (cf. first equation in (7.105)):

$$\alpha_t^{0,\,2p}=\bar\partial_t\beta_t^{0,\,2p-1},\qquad t\in B\setminus\{0\}. \tag{7.107}$$

In other words, $\beta_t^{0,\,2p-1}$ corrects $\beta_t^{\prime 0,\,2p-1}$ if the latter is not of minimal L^2-norm among the solutions of the above equation. It is explicitly given by the familiar Neumann formula:

$$\beta_t^{0,\,2p-1}=\Delta_t^{\prime\prime -1}\bar\partial_t^\star\alpha_t^{0,\,2p},\qquad t\in B\setminus\{0\}, \tag{7.108}$$

where $\Delta''_t:=\bar\partial_t\bar\partial_t^\star+\bar\partial_t^\star\bar\partial_t$ denotes the $\bar\partial_t$-Laplacian defined by the metric γ_t (involved in the adjoints) on the fibre X_t for all $t\in B$, while $\Delta_t^{\prime\prime -1}$ denotes the inverse of the restriction of Δ''_t to the orthogonal complement of the kernel of Δ''_t (i.e. $\Delta_t^{\prime\prime -1}$ is the Green operator of Δ''_t).

To continue, we first need to ensure that $\alpha_t^{1,\,2p-1}-\partial_t\beta_t^{0,\,2p-1}$ is $\bar\partial_t$-exact. Given that $\alpha_t^{1,\,2p-1}-\partial_t\beta_t^{\prime 0,\,2p-1}$ is $\bar\partial_t$-exact (see the second equation in (7.105)), the $\bar\partial_t$-exactness of the former form is equivalent to the $\bar\partial_t$-exactness of the difference of these two forms, i.e. the $\bar\partial_t$-exactness of:

$$(\alpha_t^{1,\,2p-1}-\partial_t\beta_t^{0,\,2p-1})-(\alpha_t^{1,\,2p-1}-\partial_t\beta_t^{\prime 0,\,2p-1})=\partial_t(\beta_t^{\prime 0,\,2p-1}-\beta_t^{0,\,2p-1}).$$

Now $d[\partial_t(\beta_t^{\prime 0,\,2p-1}-\beta_t^{0,\,2p-1})]=0$ because $\partial_t^2(\beta_t^{\prime 0,\,2p-1}-\beta_t^{0,\,2p-1})=0$ and

$$\bar\partial_t\partial_t(\beta_t^{\prime 0,\,2p-1}-\beta_t^{0,\,2p-1})=-\partial_t\bar\partial_t(\beta_t^{\prime 0,\,2p-1}-\beta_t^{0,\,2p-1})=-\partial_t(\alpha_t^{0,\,2p}-\alpha_t^{0,\,2p})=0,$$

thanks to the fact that $\bar\partial_t\beta_t^{\prime 0,\,2p-1}=\bar\partial_t\beta_t^{0,\,2p-1}$ as both $\beta_t^{0,\,2p-1}$ and $\beta_t^{\prime 0,\,2p-1}$ are solutions of equation (7.107) (see also the first equation in (7.105)). Thus, the pure type $(1,\,2p-1)$-form $\partial_t(\beta_t^{\prime 0,\,2p-1}-\beta_t^{0,\,2p-1})$ is d-closed and also, in an obvious way, ∂_t-exact for all $t\in B\setminus\{0\}$. Then, the $\partial\bar\partial$-assumption on X_t for $t\neq 0$ implies the $\bar\partial_t$-exactness of $\partial_t(\beta_t^{\prime 0,\,2p-1}-\beta_t^{0,\,2p-1})$ for all $t\neq 0$. This in turn implies, as has already been argued, that $\alpha_t^{1,\,2p-1}-\partial_t\beta_t^{0,\,2p-1}$ is $\bar\partial_t$-exact for all $t\in B\setminus\{0\}$.

Considering now the analogue of the second equation in (7.105), we define $\beta_t^{1,\,2p-2}$ to be the $(2p-1)$-form of pure J_t-type $(1,\,2p-2)$ which is the minimal L^2-norm solution of the equation:

$$\alpha_t^{1,\,2p-1}-\partial_t\beta_t^{0,\,2p-1}=\bar\partial_t\beta_t^{1,\,2p-2},\qquad t\in B\setminus\{0\}. \tag{7.109}$$

This equation does have solutions since we have proved that its left-hand side is $\bar\partial_t$-exact for all $t\in B\setminus\{0\}$. We can thus go on inductively to construct forms $\beta_t^{l,\,2p-1-l}$ of J_t-type $(l,\,2p-1-l)$ for all $l\in\{0,1,\dots,p-1\}$ and all $t\in B\setminus\{0\}$. Indeed, once $\beta_t^{l-1,\,2p-l}$ has been constructed as the minimal L^2-norm solution of the equation

$$\alpha_t^{l-1,\,2p-l+1}-\partial_t\beta_t^{l-2,\,2p-l+1}=\bar\partial_t\beta_t^{l-1,\,2p-l},\qquad t\in B\setminus\{0\},\tag{7.110}$$

the pure-type form $\alpha_t^{l,\,2p-l}-\partial_t\beta_t^{l-1,\,2p-l}$ is seen to be $\bar\partial_t$-exact by the same argument using the $\partial\bar\partial$-assumption on X_t ($t\neq 0$) as the one spelt out above for $l=1$. The form $\beta_t^{l,\,2p-l-1}$ is then defined to be the minimal L^2-norm solution of the equation

$$\alpha_t^{l,\,2p-l}-\partial_t\beta_t^{l-1,\,2p-l}=\bar\partial_t\beta_t^{l,\,2p-l-1},\qquad t\in B\setminus\{0\}.\tag{7.111}$$

In this case, the explicit Neumann formula for the minimal solution reads:

$$\beta_t^{l,\,2p-l-1}=\Delta_t''^{-1}\bar\partial_t^\star(\alpha_t^{l,\,2p-l}-\partial_t\beta_t^{l-1,\,2p-l}),\qquad t\in B\setminus\{0\},\;\; l=1,\dots p-1,\tag{7.112}$$

where $\Delta''_t: C^\infty_{l,\,2p-l-1}(X_t,\,\mathbb{C})\to C^\infty_{l,\,2p-l-1}(X_t,\,\mathbb{C})$ is the $\bar\partial_t$-Laplacian defined on the space of $(l,\,2p-l-1)$-forms of class C^∞ on X_t.

In this fashion, we have defined smooth forms $\beta_t^{0,\,2p-1},\beta_t^{1,\,2p-2},\dots,\beta_t^{p-1,\,p}$ for all $t\in B\setminus\{0\}$. They satisfy the first p equations (with β_t replacing β'_t) among the $(2p+1)$ equations in (7.105). We then go on to define, for all $t\in B\setminus\{0\}$, smooth forms $\beta_t^{p,\,p-1},\beta_t^{p+1,\,p-2},\dots,\beta_t^{2p-1,\,0}$ as the conjugates of the previous set of forms taken in reverse order:

$$\beta_t^{p+s,\,p-s-1}:=\overline{\beta_t^{p-s-1,\,p+s}},\qquad s=0,1,\dots,p-1,\;\; t\in B\setminus\{0\}.\tag{7.113}$$

Since the form α has been chosen to be real, we take conjugates and see that the forms $\beta_t^{p+s,\,p-s-1}$, $s=0,1,\dots,p-1$, satisfy the last p equations (with β_t replacing β'_t) among the $(2p+1)$ equations in (7.105). If we now set

$$\beta_t:=\beta_t^{0,\,2p-1}+\dots+\beta_t^{p-1,\,p}+\beta_t^{p,\,p-1}+\dots+\beta_t^{2p-1,\,0},\qquad t\in B\setminus\{0\},\tag{7.114}$$

we obtain a family $(\beta_t)_{t\in B\setminus\{0\}}$ of real C^∞ forms of degree $2p-1$ on X varying in a C^∞ way with $t\in B\setminus\{0\}$. Moreover, the $(2p)$-current $\alpha-[Z_t]-d\beta_t$ is of pure type $(p,\,p)$ for all $t\in B\setminus\{0\}$ as can be seen from the construction of β_t: its pure-type components satisfy the analogues for β_t (instead of β'_t) of equations (7.105), except the one involving $[Z_t]$, which amount to the vanishing of all the pure-type components of $\alpha-[Z_t]-d\beta_t$, except the one of type $(p,\,p)$ which is the only one to which $[Z_t]$ contributes. The current $\alpha-[Z_t]-d\beta_t$ is also d-exact in an obvious way (it equals $d(\beta'_t-\beta_t)$).

A final application of the $\partial\bar\partial$-assumption on every X_t with $t\neq 0$ shows that $\alpha-[Z_t]-d\beta_t$ is also $\partial_t\bar\partial_t$-exact for $t\neq 0$. Thus, there exists a family $(R_t)_{t\in B\setminus\{0\}}$ of $(2p-2)$-currents of respective J_t-types $(p-1,\,p-1)$ such that

$$\alpha=[Z_t]+d\beta_t+\partial_t\bar\partial_t R_t,\qquad t\in B\setminus\{0\}.\tag{7.115}$$

Conclusion 7.2.5. *If X_t is a $\partial\bar\partial$-manifold for all $t\in B\setminus\{0\}$, the γ_t-volumes of any C^∞ family of relative $(n-p)$-cycles $(Z_t)_{t\in B\setminus\{0\}}$ can be expressed as*

$$v_{\gamma_t}(Z_t):=\int\limits_X[Z_t]\wedge\gamma_t^{n-p}=\int\limits_X\alpha\wedge\gamma_t^{n-p}-\int\limits_X d\beta_t\wedge\gamma_t^{n-p}-\int\limits_X\partial_t\bar\partial_t R_t\wedge\gamma_t^{n-p},\qquad t\in B\setminus\{0\},\tag{7.116}$$

for any family of Hermitian metrics $(\gamma_t)_{t\in B}$ on the fibres $(X_t)_{t\in B}$, where α is a fixed real $(2p)$-form in the De Rham class that is common to all $[Z_t]$, $(\beta_t)_{t\in B\setminus\{0\}}$ are given by formula (7.114) by adding their components inductively defined in formulae (7.108), (7.112) and (7.113), while $(R_t)_{t\in B\setminus\{0\}}$ are given by (7.115).

Recall that what is at stake is ensuring that $v_{\gamma_t}(Z_t)$ is uniformly bounded as $t\in B\setminus\{0\}$ approaches $0\in B$. If γ_t is chosen to vary in a C^∞ way with $t\in B$ (up to $t=0$), the first term in the right-hand side of (7.116) stays bounded when t varies in a relatively compact neighbourhood $U\Subset\Delta$ of $0\in B$, since α is independent of t. The other two terms are problematic as both β_t and R_t are only defined off $t=0\in B$.

The first observation is that, when the cycles Z_t are *divisors* (i.e. $p=1$), the third term in the right-hand side of (7.116) can be easily handled. The reason is that, thanks to Proposition 4.1.13, the Hermitian metrics γ_t of the fibres X_t can be chosen to be *Gauduchon metrics*, i.e. such that $\partial_t\bar\partial_t\gamma_t^{n-1}=0$ for all $t\in B$. With this special choice for $(\gamma_t)_{t\in B}$, Stokes's Theorem gives:

$$\int\limits_X \partial_t\bar\partial_t R_t\wedge\gamma_t^{n-1} = -\int\limits_X R_t\wedge\partial_t\bar\partial_t\gamma_t^{n-1}=0, \qquad t\in B\setminus\{0\},$$

so this term vanishes in the case of divisors. However, achieving uniform boundedness for this term in the case of higher codimensional cycles (i.e. for $p\geq 2$) is a major challenge.

As for uniformly bounding the term depending on β_t in the right-hand side of (7.116), the difficulty stems from the possible jump of the Hodge numbers $h^{p,\,q}(t):=\dim_{\mathbb{C}}H^{p,\,q}(X_t,\,\mathbb{C})$ at $t=0$. The family of strongly elliptic operators $(\Delta''_t)_{t\in B}$ defined in J_t-bidegree $(p,\,q)$ varies in a C^∞ way with t, while a classical Kodaira-Spencer Theorem D in §.2.5 ensures that the corresponding family of Green operators $(\Delta_t^{''-1})_{t\in B}$ varies in a C^∞ with t if the dimension (as a $\mathbb{C}$-vector space) of the kernel $\ker\Delta''_t$ is independent of $t\in B$. Since $\ker\Delta''_t$ is isomorphic to the Dolbeault cohomology space $H^{p,\,q}(X_t,\,\mathbb{C})$ by the Hodge Isomorphism Theorem, we have differentiability of the families of operators $(\Delta_t^{''-1})_{t\in B}$ (and hence of the families of forms $(\beta_t^{l,\,2p-l-1})_{t\in B}$, $l=0,1,\dots,p-1$, thanks to the formulae (7.108) and (7.112)) if the Hodge numbers $h^{l,\,2p-l-1}(t)$, $l=0,1,\dots,p-1$, of the fibres do not jump at $t=0\in B$. This condition is fulfilled, for instance, under the hypothesis of Conjecture 7.2.3 since the *class* $\mathcal{C}$ assumption on the fibres ensures the degeneracy at $E_1^\bullet$ of the Frölicher spectral sequence of each fibre which, in turn, implies the local constancy of the Hodge numbers of the fibres (cf. Theorem 2.6.3). Thus, the term depending on β_t in the expression (7.116) for $v_{\gamma_t}(Z_t)$ is uniformly bounded when t varies in a relatively compact neighbourhood $U\Subset\Delta$ of $0\in B$ under the hypothesis of Conjecture 7.2.3. However, controlling this term in the more general situation of Conjecture 7.2.4 poses a major challenge as the Hodge numbers might *a priori* jump at $t=0$ if the *class* $\mathcal{C}$ assumption skips X_0 (unless they can be shown not to do so, which seems to be a daunting task).

A by-product of these considerations is that the divisor case of Conjecture 7.2.3 holds true.

We now choose the metrics $(\gamma_t)_{t\in B}$ to be *Gauduchon* and assume that $p=1$. It remains to control the term depending on β_t in (7.116). (We stress that the control of this term in the case $p\geq 2$ falls completely outside the method of the present proof and is widely open.)

As $p=1$, formula (7.108) defining $\beta_t^{0,\,1}$ reads:

$$\beta_t^{0,\,1}=\Delta_t^{''-1}\bar\partial_t^\star\alpha_t^{0,\,2}, \qquad t\in B\setminus\{0\}, \tag{7.117}$$

while $\beta_t=\overline{\beta_t^{0,\,1}}+\beta_t^{0,\,1}$ (cf. (7.113) and (7.114)) is now a 1-form. Thus, only the $(1,\,1)$-component of $d\beta_t$ has a non-trivial contribution to $v_{\gamma_t}(Z_t)$ and we get

$$\int_X d\beta_t \wedge \gamma_t^{n-1} = \int_X (\partial_t \beta_t^{0,1} + \bar\partial_t \beta_t^{1,0}) \wedge \gamma_t^{n-1},$$

where we have set $\beta_t^{1,0} := \overline{\beta_t^{0,1}}$. As $\partial_t \beta_t^{0,1}$ and $\bar\partial_t \beta_t^{1,0}$ are conjugate to each other, it suffices to uniformly bound

$$I_t := \int_X \partial_t \beta_t^{0,1} \wedge \gamma_t^{n-1}, \qquad t \in B \setminus \{0\}. \tag{7.118}$$

The difficulty is that $\beta_t^{0,1}$ (hence also $\partial_t \beta_t^{0,1}$) might *explode* as $t \in B \setminus \{0\}$ approaches $0 \in B$, if $h^{0,1}(t)$ *jumps* at $t = 0$. However, $\bar\partial_t \beta_t^{0,1} = \alpha_t^{0,2}$ (see equation (7.107) with $p = 1$) and thus $\bar\partial_t \beta_t^{0,1}$ extends in a C^∞ way to $t = 0$ since the $(0, 2)$-component $\alpha_t^{0,2}$ of the fixed form α w.r.t. to the holomorphic family of complex structures $(J_t)_{t\in B}$ does. Hence the idea of trying to substitute $\bar\partial_t \beta_t^{0,1}$ for $\partial_t \beta_t^{0,1}$ in (7.118) appears as natural. Stokes' theorem gives

$$I_t = \int_X \beta_t^{0,1} \wedge \partial_t \gamma_t^{n-1}, \qquad t \in B \setminus \{0\}. \tag{7.119}$$

Since the metric γ_t is *Gauduchon*, $d(\partial_t \gamma_t^{n-1}) = 0$ for every $t \in B$, so the $\partial\bar\partial$-assumption on every X_t with $t \neq 0$ implies that the d-closed form $\partial_t \gamma_t^{n-1}$ of pure type $(n, n-1)$, which is obviously ∂_t-exact, must also be $\bar\partial_t$-exact for every $t \neq 0$. However, it is not clear *a priori* whether $\partial_0 \gamma_0^{n-1}$ is $\bar\partial_0$-exact since X_0 is not supposed to be a $\partial\bar\partial$-manifold.

It is at this point that we need Theorem 7.1.1. Under our assumptions, it ensures the existence of a *strongly Gauduchon (sG) metric* γ_0 on X_0. Moreover, the sG property of compact complex manifolds is *deformation open* by Theorem 4.2.4, so we can deform γ_0 to a C^∞ family $(\gamma_t)_{t\in B}$ of *strongly Gauduchon (sG) metrics* on the fibres $(X_t)_{t\in B}$. Finally, recall that the proof of Theorem 4.2.4 gives a real d-closed C^∞ form Ω of degree $2n-2$ on X (the C^∞ manifold underlying the fibres X_t) such that its component of J_0-type $(n-1, n-1)$ is positive definite (i.e. $\Omega_0^{n-1,n-1} > 0$). Thus, if X_0 carries a *strongly Gauduchon metric* γ_0, the components $\Omega_t^{n-1,n-1}$ of J_t-type $(n-1, n-1)$ of Ω vary in a C^∞ way with $t \in B$ and, therefore, the strict positivity condition is preserved in a small neighbourhood of $0 \in B$ (and thus on the whole B if B is shrunk sufficiently about 0):

$$\Omega_t^{n-1,n-1} > 0, \qquad t \in B.$$

The induced sG metric γ_t on X_t satisfies $\gamma_t^{n-1} = \Omega_t^{n-1,n-1}$ for $t \in B$. Moreover, since the form Ω is *real*, the closedness condition $d\Omega = 0$ is equivalent to

$$\partial_t \Omega_t^{n-1,n-1} = -\bar\partial_t \Omega_t^{n,n-2}, \qquad t \in B.$$

Thus, the $\bar\partial_t$-potentials $\Omega_t^{n,n-2}$ of $\partial_t \Omega_t^{n-1,n-1}$ also vary in a C^∞ way with $t \in B$ since they are components of pure J_t-type $(n, n-2)$ of the fixed form Ω.

Put $\zeta_t = -\Omega_t^{n,n-2}$ for $t \in B$. Then, (7.119) reads:

$$I_t = \int_X \beta_t^{0,1} \wedge \partial_t \gamma_t^{n-1} = \int_X \beta_t^{0,1} \wedge \bar\partial_t \zeta_t^{n,n-2} = \int_X \bar\partial_t \beta_t^{0,1} \wedge \zeta_t^{n,n-2} = \int_X \alpha_t^{0,2} \wedge \zeta_t^{n,n-2}, \qquad t \in B \setminus \{0\},$$

where the third identity follows from Stokes's theorem. As both families of forms $(\alpha_t^{0,2})_{t\in B}$ and $(\zeta_t^{n,n-2})_{t\in B}$ vary in a C^∞ way with t (up to $t = 0$), I_t is bounded independently of $t \in B \setminus \{0\}$

after possibly shrinking B about 0. Hence, the volume $v_{\gamma_t}(Z_t)$ is bounded independently of t when $t \in B \setminus \{0\}$ approaches $0 \in B$ (see (7.116)).

To show properness over B of an arbitrary irreducible component $S \subset \mathcal{C}^{n-1}(\mathcal{X}/B)$, one has to show that for every compact subset $K \subset B$, $\mu_{n-1}^{-1}(K) \cap S$ is a compact subset of $\mathcal{C}^{n-1}(\mathcal{X}/B)$. If $(Z_s)_{s \in S}$ is the analytic family of divisors associated with S (such that $Z_s \subset X_{\mu_{n-1}(s)}$, $s \in S$), this amounts to proving that the volumes

$$v_{\gamma_s}(Z_s) = \int\limits_X [Z_s] \wedge \gamma_s^{n-1}$$

are uniformly bounded when s ranges over $\mu_{n-1}^{-1}(K) \cap S$. Here we have denoted for convenience $\gamma_s = \gamma_{\mu_{n-1}(s)}$. As mentioned in §.7.2.1, the absolute Barlet space $\mathcal{C}^{n-1}(X_t)$ of divisors of every fibre X_t is known to have compact irreducible components. Thus, $v_{\gamma_s}(Z_s)$ stays uniformly bounded when Z_s varies across any irreducible component of any given fibre. It then suffices to show uniform boundedness of the volumes in the *horizontal directions*, i.e. when $Z_t \subset X_t$ varies in a differentiable family $(Z_t)_{t \in B \setminus \{0\}}$ with $t \in B \setminus \{0\}$ approaching $0 \in B$. This has been done above. The proof of Theorem 7.2.2 is complete. $\square$

Recall that we have already proved the implications:

$$\text{Theorem 7.2.2} \implies \text{Theorem 7.2.1} \implies \text{Theorem 7.0.4}.$$

Together with the proof of Theorem 7.2.2 given in this §.7.2.2, these implications complete the first proof of Theorem 7.0.4.

(II) Second proof of Theorem 7.0.4

We will present it in two stages over sections 7.3 and 7.4.

7.3 The Frölicher approximating vector bundle (FAVB)

This section, taken from [Pop19], is the analogue in this more conceptual approach to Theorem 7.0.4 of §.7.1. We prove the following slightly weaker version of Theorem 7.1.1 that will turn out to be equally effective for the proof of Theorem 7.0.4.

Theorem 7.3.1. *([Pop19, Theorem 1.4 and Theorem 3.3]) Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex n-dimensional manifolds over an open ball $B \subset \mathbb{C}^N$ about the origin. Suppose that the fibre $X_t := \pi^{-1}(t)$ is a* $\partial\bar\partial$**-manifold** *for all $t \in B \setminus \{0\}$.*

Then, the fibre $X_0 := \pi^{-1}(0)$ is an E_r**-sG manifold**, *where r is the smallest positive integer such that the Frölicher spectral sequence of X_0 degenerates at E_r.*

Furthermore, X_0 is even an E_r**-sGG manifold**.

Recall that *E_r-sG metrics* and *E_r-sG manifolds* were introduced in Definition 4.4.1 and discussed in §.4.4. In particular, recall that the *E_r-sG* property becomes weaker and weaker as $r \in \mathbb{N}^\star$ increases, it coincides with the *strongly Gauduchon (sG)* property when $r = 1$ and only the cases $r = 1, 2, 3$ correspond to new properties (i.e. any E_r-sG metric with some $r \geq 4$ is also E_3-sG).

The proof of Theorem 7.3.1 presented in this section relies heavily on the *adiabatic limit* theory for complex structures introduced and discussed in §.3.5. In §.7.3.1, we construct pseudo-differential operators $(\widetilde\Delta_h)_{h\in\mathbb{C}}$, resp. $(\widetilde\Delta_h^{(r)})_{h\in\mathbb{C}}$ with $r \in \mathbb{N}$ and $r \geq 3$, as deformations of the pseudo-differential Laplacians $\widetilde\Delta$, resp. $\widetilde\Delta^{(r)}$, introduced in Definition 3.1.2, resp. (iii) of Proposition 3.2.6, such that $\widetilde\Delta_0 = \widetilde\Delta$ and $\widetilde\Delta_0^{(r)} = \widetilde\Delta^{(r)}$. The operators $(\widetilde\Delta_h)_{h\in\mathbb{C}}$, resp. $(\widetilde\Delta_h^{(r)})_{h\in\mathbb{C}}$, will then be used to display the second page, resp. the r-th page with $r \geq 3$, of the Frölicher spectral sequence of X as the limit, when $\mathbb{C} \ni h \to 0$, of the d_h-cohomology when $E_2(X) = E_\infty(X)$, resp. when $E_r(X) = E_\infty(X)$. This limiting construction, carried out in §.7.3.2, will produce what we call the *Frölicher approximating vector bundle (FAVB)* of X (in its absolute version), resp. of a family $(X_t)_{t\in B}$ of manifolds (in its relative verion). The proof of Theorem 7.3.1 will then follow easily (cf. §.7.3.3) from these constructions.

7.3.1 h-theory for the Frölicher spectral sequence

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.

Recall that $(\Delta_h)_{h\in\mathbb{C}}$, introduced in Definition 3.5.6, is a C^∞ family of elliptic differential operators such that $\Delta_0 = \Delta''$. So, the Δ_h's can be regarded as a deformation (allowing for more flexibility) of the standard $\bar\partial$-Laplacian Δ''. The kernel of Δ'' is classically isomorphic to the Dolbeault cohomology of X (thus, to the first page of the Frölicher spectral sequence), cf. e.g. Corollary 1.1.7.

(A) Second page: the pseudo-differential Laplacians $\widetilde\Delta_h$

We will now introduce and study a similar deformation of the pseudo-differential Laplacian

$$\widetilde\Delta = \partial p''\partial^\star + \partial^\star p''\partial + \Delta'' : \mathbb{C}^\infty_{p,q}(X, \mathbb{C}) \longrightarrow \mathbb{C}^\infty_{p,q}(X, \mathbb{C}), \qquad p, q = 0, \dots, n,$$

introduced in Definition 3.1.2 and proved in §.3.1 to define a Hodge theory for the second page of the Frölicher spectral sequence, namely a Hodge isomorphism (cf. Theorem 3.1.4):

$$\mathcal{H}^{p,\,q}_{\widetilde{\Delta}}(X,\,\mathbb{C}) := \ker(\widetilde{\Delta} : \mathbb{C}^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow \mathbb{C}^\infty_{p,\,q}(X,\,\mathbb{C})) \simeq E^{p,\,q}_2(X)$$

in every bidegree $(p,\,q)$. Note that $\widetilde{\Delta} = (\partial p'')(\partial p'')^\star + (p''\partial)^\star(p''\partial) + \Delta''$, so we will approximate $\partial p''$ and $p''\partial$ by adding to each a small h-multiple of its conjugate, while still approximating Δ'' by Δ_h.

Definition 7.3.2. *Let $(X,\,\omega)$ be a compact complex Hermitian manifold with $\dim_{\mathbb{C}} X = n$. For every $h \in \mathbb{C}$ and every $k = 0, \dots, 2n$, we let*

$$\widetilde{\Delta}_h = (\partial p'' + h\,\bar\partial p')(\partial p'' + h\,\bar\partial p')^\star + (p''\partial + h\,p'\bar\partial)^\star(p''\partial + h\,p'\bar\partial) + \Delta_h : \mathbb{C}^\infty_k(X,\,\mathbb{C}) \longrightarrow \mathbb{C}^\infty_k(X,\,\mathbb{C}),$$

where $p' = p'_\omega : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow \ker(\Delta' : \mathbb{C}^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow \mathbb{C}^\infty_{p,\,q}(X,\,\mathbb{C})) := \mathcal{H}^{p,\,q}_{\Delta'}(X,\,\mathbb{C})$ and $p'' = p''_\omega : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow \ker(\Delta'' : \mathbb{C}^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow \mathbb{C}^\infty_{p,\,q}(X,\,\mathbb{C})) := \mathcal{H}^{p,\,q}_{\Delta''}(X,\,\mathbb{C})$ are the orthogonal projections onto the Δ'-, resp. Δ'-harmonic spaces of any fixed bidegree $(p,\,q)$. These projections are then extended by linearity to

$$p' = p'_\omega : C^\infty_k(X,\,\mathbb{C}) \longrightarrow \mathcal{H}^k_{\Delta'}(X,\,\mathbb{C}), \qquad p'' = p''_\omega : C^\infty_k(X,\,\mathbb{C}) \longrightarrow \mathcal{H}^k_{\Delta''}(X,\,\mathbb{C}),$$

where $\mathcal{H}^k_{\Delta'}(X,\,\mathbb{C}) := \oplus_{p+q=k}\mathcal{H}^{p,\,q}_{\Delta'}(X,\,\mathbb{C})$ and $\mathcal{H}^k_{\Delta''}(X,\,\mathbb{C}) := \oplus_{p+q=k}\mathcal{H}^{p,\,q}_{\Delta''}(X,\,\mathbb{C})$.

For every $h \in \mathbb{C}$, $\widetilde{\Delta}_h$ is a non-negative, self-adjoint pseudo-differential operator and $\widetilde{\Delta}_0 = \widetilde{\Delta}$. Further properties include the following.

Lemma 7.3.3. *For every $h \in \mathbb{C}\setminus\{0\}$, $\widetilde{\Delta}_h$ is an elliptic pseudo-differential operator whose kernel is*

$$\begin{aligned}\ker\widetilde{\Delta}_h &= \ker(\partial p'' + h\,\bar\partial p')^\star \cap \ker(p''\partial + h\,p'\bar\partial) \cap \ker d_h \cap \ker d_h^\star \\ &= \ker d_h \cap \ker d_h^\star = \ker\Delta_h, \qquad k = 0, \dots, 2n.\end{aligned} \tag{7.120}$$

Hence, the 3-space orthogonal decompositions induced by $\widetilde{\Delta}_h$ and Δ_h coincide when $h \in \mathbb{C}\setminus\{0\}$:

$$C^\infty_k(X,\,\mathbb{C}) = \ker\widetilde{\Delta}_h \oplus Im\,d_h \oplus Im\,d_h^\star, \qquad k = 0, \dots, 2n, \tag{7.121}$$

where $\ker d_h = \ker\widetilde{\Delta}_h \oplus Im\,d_h$, $\ker d_h^\star = \ker\widetilde{\Delta}_h \oplus Im\,d_h^\star$ and $Im\,\widetilde{\Delta}_h = Im\,d_h \oplus Im\,d_h^\star$.

Consequently, we have the Hodge isomorphism:

$$\mathcal{H}^k_{\widetilde{\Delta}_h}(X,\,\mathbb{C}) = \mathcal{H}^k_{\Delta_h}(X,\,\mathbb{C}) \simeq H^k_{d_h}(X,\,\mathbb{C}), \qquad k = 0, \dots, 2n, \quad h \in \mathbb{C}\setminus\{0\}. \tag{7.122}$$

Moreover, the decomposition (7.121) is stable under $\widetilde{\Delta}_h$, namely

$$\widetilde{\Delta}_h(Im\,d_h) \subset Im\,d_h \quad \text{and} \quad \widetilde{\Delta}_h(Im\,d_h^\star) \subset Im\,d_h^\star. \tag{7.123}$$

Proof. The first identity in (7.120) follows immediately from the fact that $\widetilde{\Delta}_h$ is a sum of non-negative operators of the shape $A^\star A$ and $\ker(A^\star A) = \ker A$ for every A, since $\langle\langle A^\star Au,\,u\rangle\rangle = ||Au||^2$.

To prove the second identity in (7.120), we will prove the inclusions $\ker d_h \subset \ker(p''\partial + h\,p'\bar\partial)$ and $\ker d_h^\star \subset \ker(\partial p'' + h\,\bar\partial p')^\star$.

Let $u = \sum_{r+s=k} u^{r,\,s}$ be a smooth k-form such that $d_h u = 0$. This amounts to $h\partial u^{r,\,s} + \bar\partial u^{r+1,\,s-1} = 0$ whenever $r + s = k$. Applying p' and respectively p'', we get

$$p'\bar\partial u^{r+1,\,s-1} = 0 \quad \text{and} \quad p''\partial u^{r,\,s} = 0, \quad \text{whenever } r+s=k,$$

since $h \neq 0$, while $p'\partial = 0$ and $p''\bar\partial = 0$. Hence,

$$(p''\partial + h\,p'\bar\partial)\,u = \sum_{r+s=k}(p''\partial u^{r,\,s} + h\,p'\bar\partial u^{r+1,\,s-1}) = 0.$$

This proves the inclusion $\ker d_h \subset \ker(p''\partial + h\,p'\bar\partial)$.

The ellipticity of the (pseudo)-differential operators Δ_h and $\widetilde\Delta_h$, combined with the compactness of the manifold X, implies that the images of d_h and $\partial p'' + h\,\bar\partial p'$ are *closed* in $C^\infty_k(X,\,\mathbb{C})$. Hence, these images coincide with the orthogonal complements of the kernels of the adjoint operators $d_h^\star$ and $(\partial p'' + h\,\bar\partial p')^\star$. Therefore, proving the inclusion $\ker d_h^\star \subset \ker(\partial p'' + h\,\bar\partial p')^\star$ is equivalent to proving the inclusion $\operatorname{Im}(\partial p'' + h\,\bar\partial p') \subset \operatorname{Im} d_h$. (Actually, the closedness of these images is not needed here, we would have taken closures otherwise.)

Let $u = \partial p''v + h\,\bar\partial p'v$ be a smooth k-form lying in the image of $\partial p'' + h\,\bar\partial p'$. Since $\partial p' = 0$ and $\bar\partial p'' = 0$, while $h\neq 0$, we get

$$u = (h\partial)\,(\frac{1}{h}\,p''v + h\,p'v) + \bar\partial\,(\frac{1}{h}\,p''v + h\,p'v) = d_h\,(\frac{1}{h}\,p''v + h\,p'v) \in \operatorname{Im} d_h.$$

This completes the proof of (7.120).

Since Δ_h commutes with both d_h and $d_h^\star$, to prove (7.123) it suffices to prove the stability of $\operatorname{Im} d_h$ and $\operatorname{Im} d_h^\star$ under $\widetilde\Delta_h - \Delta_h$. Now, since $(p''\partial + hp'\bar\partial)\,d_h = 0$ (immediate verification), we get

$$(\widetilde\Delta_h - \Delta_h)\,d_h = (\partial p'' + h\bar\partial p')(\partial p'' + h\bar\partial p')^\star(h\partial + \bar\partial).$$

Since $\operatorname{Im}(\partial p'' + h\bar\partial p') \subset \operatorname{Im} d_h$ (as seen above), we get $(\widetilde\Delta_h - \Delta_h)(\operatorname{Im} d_h) \subset \operatorname{Im} d_h$. Similarly, an immediate verification shows that $(\partial p'' + h\bar\partial p')^\star d_h^\star = 0$. Consequently,

$$(\widetilde\Delta_h - \Delta_h)\,d_h^\star = (p''\partial + hp'\bar\partial)^\star(p''\partial + hp'\bar\partial)d_h^\star.$$

Meanwhile, $\operatorname{Im}(p''\partial + hp'\bar\partial)^\star \subset \operatorname{Im} d_h^\star$ (since this is equivalent to the inclusion $\ker d_h \subset \ker(p''\partial + h\,p'\bar\partial)$ that was proved above). Therefore, $(\widetilde\Delta_h - \Delta_h)(\operatorname{Im} d_h^\star) \subset \operatorname{Im} d_h^\star$. The proof of (7.123) is complete.

The remaining statements follow from the standard elliptic theory as in §.3.5. $\square$

Conclusion 7.3.4. *Let $(X,\,\omega)$ be a compact complex Hermitian manifold with $\dim_{\mathbb{C}} X = n$. For every degree $k\in\{0,\dots,2n\}$, we have C^∞ families of elliptic differential operators $(\Delta_h)_{h\in\mathbb{C}}$ and, respectively, elliptic pseudo-differential operators $(\widetilde\Delta_h)_{h\in\mathbb{C}}$ from $C^\infty_k(X,\,\mathbb{C})$ to $C^\infty_k(X,\,\mathbb{C})$ such that*

(i) $\Delta_0 = \Delta''$ *and* $\widetilde\Delta_0 = \widetilde\Delta$*;*

(ii) $\mathcal{H}^k_{\Delta_h}(X,\,\mathbb{C}) = \mathcal{H}^k_{\widetilde\Delta_h}(X,\,\mathbb{C}) \simeq H^k_{d_h}(X,\,\mathbb{C})$ *for all* $h\in\mathbb{C}\setminus\{0\}$*;*

(iii) $\mathcal{H}^k_{\Delta_0}(X,\,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,\,q}(X,\,\mathbb{C})$ *and* $\mathcal{H}^k_{\widetilde\Delta_0}(X,\,\mathbb{C}) \simeq \bigoplus_{p+q=k} E_2^{p,\,q}(X)$*.*

Proof. Only the latter part of (iii) still needs a proof. Since $\widetilde\Delta$ preserves the pure type of forms and since the kernel of $\widetilde\Delta : C^\infty_{p,\,q}(X,\,\mathbb{C}) \longrightarrow C^\infty_{p,\,q}(X,\,\mathbb{C})$ is isomorphic to $E_2^{p,\,q}(X,\,\mathbb{C})$ for every bidegree $(p,\,q)$ (cf. Theorem 3.1.4), the isomorphism follows. $\square$

(B) Page $r\geq 3$: the pseudo-differential Laplacians $\widetilde\Delta^{(r)}_h$

Besides the case of E_2 treated in §.7.3.1 **(A)**, only the case of E_3 will be needed for the proof of Theorem 7.0.4. However, we will treat the general case of E_r for the sake of completeness.

With the construction and the notation of §.3.2.2 (mainly those of Proposition 3.2.6) and of §.3.2.3 in place, we now introduce, for every $r\in\mathbb{N}^\star$, a smooth family $(\widetilde\Delta^{(r+1)}_h)_{h\in\mathbb{C}}$ of pseudo-differential operators whose member for $h=0$ is the pseudo-differential Laplacian $\widetilde{\Delta}^{(r+1)}$ constructed in (iii) of Proposition 3.2.6. When $r=1$, this will be the smooth family $(\widetilde{\Delta}_h)_{h\in\mathbb{C}}$ constructed in Conclusion 7.3.4 as a deformation of the pseudo-differential Laplacian $\widetilde{\Delta}^{(2)}=\widetilde{\Delta}$. Following the model of Definition 7.3.2, we will deform each factor in the above definition of $\widetilde{\Delta}^{(r+1)}$ by adding to it a small h-multiple of its conjugate.

Definition 7.3.5. *Let $(X,\,\omega)$ be a compact complex Hermitian manifold with $\dim_{\mathbb{C}}X=n$. For every $h\in\mathbb{C}$ and every $k=0,\dots,2n$, we define the pseudo-differential operator $\widetilde{\Delta}_h^{(r+1)}:C_k^\infty(X,\,\mathbb{C})\longrightarrow C_k^\infty(X,\,\mathbb{C})$ by induction on $r\geq 2$ as follows:*

$$\begin{aligned}\widetilde{\Delta}_h^{(r+1)} &= \left(\partial D_{r-1}p_r + h\,\bar\partial\overline{D_{r-1}}\bar{p}_r\right)\left(\partial D_{r-1}p_r + h\,\bar\partial\overline{D_{r-1}}\bar{p}_r\right)^\star\\ &+ \left(p_r\partial D_{r-1} + h\,\bar{p}_r\bar\partial\overline{D_{r-1}}\right)^\star\left(p_r\partial D_{r-1} + h\,\bar{p}_r\bar\partial\overline{D_{r-1}}\right)\\ &+ \widetilde{\Delta}_h^{(r)},\end{aligned}$$

where $\widetilde{\Delta}_h^{(r)}:C_k^\infty(X,\,\mathbb{C})\longrightarrow C_k^\infty(X,\,\mathbb{C})$ has been defined at the previous induction step and $\widetilde{\Delta}_h^{(2)}:=\widetilde{\Delta}_h$ was defined in Definition 7.3.2. For every bidegree $(p,\,q)$, by $\bar{p}_r:C_{p,q}^\infty(X,\,\mathbb{C})\longrightarrow\ker(\overline{\widetilde{\Delta}^{(r)}}:C_{p,q}^\infty(X,\,\mathbb{C})\longrightarrow C_{p,q}^\infty(X,\,\mathbb{C}))$ we mean the orthogonal projection onto the kernel of the conjugate of $\widetilde{\Delta}^{(r)}$ acting in bidegree $(p,\,q)$. Both the projections p_r and $\bar{p}_r$ are then extended by linearity to the whole space $C_k^\infty(X,\,\mathbb{C})$.

As in the case of $\widetilde{\Delta}_h=\widetilde{\Delta}_h^{(2)}$ (cf. Lemma 7.3.3), we need to prove that $\widetilde{\Delta}_h^{(r+1)}$ has the same kernel as Δ_h for every $r\geq 2$. A priori, the kernel of $\widetilde{\Delta}_h^{(r+1)}$ might be smaller than that of Δ_h.

Lemma 7.3.6. *For every $h\in\mathbb{C}\setminus\{0\}$, the following identities of kernels hold:*

$$\ker\Delta_h=\ker\widetilde{\Delta}_h^{(2)}=\dots=\ker\widetilde{\Delta}_h^{(r)}=\ker\widetilde{\Delta}_h^{(r+1)}=\dots$$

in every degree $k=0,\dots,2n$.

Proof. Fix any k. We will prove by induction on $r\geq 1$ that $\ker\widetilde{\Delta}_h^{(r+1)}=\ker\Delta_h$ in degree k. The case $r=1$ was proved in Lemma 7.3.3. Since each operator $\widetilde{\Delta}_h^{(r+1)}$ is a sum of non-negative self-adjoint operators of the shape $AA^\star$ and since $\ker(AA^\star)=\ker A^\star$, we have:

$$\ker\widetilde{\Delta}_h^{(r+1)}=\ker\left(\partial D_{r-1}p_r + h\,\bar\partial\overline{D_{r-1}}\bar{p}_r\right)^\star\cap\ker\left(p_r\partial D_{r-1} + h\,\bar{p}_r\bar\partial\overline{D_{r-1}}\right)\cap\ker\widetilde{\Delta}_h^{(r)}.$$

In particular, $\ker\widetilde{\Delta}_h^{(r+1)}\subset\ker\widetilde{\Delta}_h^{(r)}\subset\dots\subset\ker\widetilde{\Delta}_h^{(2)}\subset\ker\Delta_h$ for every r and $\ker\widetilde{\Delta}_h^{(2)}=\ker\Delta_h$ thanks to Lemma 7.3.3.

Suppose, as the induction hypothesis, that $\ker\widetilde{\Delta}_h^{(r)}=\ker\Delta_h$ for some $r\geq 2$. Since $\ker\Delta_h=\ker d_h\cap\ker d_h^\star$, to prove that $\ker\widetilde{\Delta}_h^{(r+1)}=\ker\Delta_h$, it suffices to prove the inclusions

$$\ker(h\partial+\bar\partial)\subset\ker\left(p_r\partial D_{r-1} + h\,\bar{p}_r\bar\partial\overline{D_{r-1}}\right)\ \text{ and }\ \ker(h\partial^\star+\bar\partial^\star)\subset\ker\left(\partial D_{r-1}p_r + h\,\bar\partial\overline{D_{r-1}}\bar{p}_r\right)^\star\tag{7.124}$$

• To prove the first inclusion of (7.124), let $u=\sum_{l+s=k}u^{l,\,s}\in\ker(h\partial+\bar\partial)$. This amounts to $h\partial u^{l,\,s}+\bar\partial u^{l+1,\,s-1}=0$ for all l,s such that $l+s=k$. For any fixed $r\geq 1$, applying p_r and $\bar{p}_r$ to this identity and using the fact that $h\neq 0$, we get

$$p_r\partial u^{l,\,s}=0\quad\text{ and }\quad\bar{p}_r\bar\partial u^{l+1,\,s-1}=0\quad\text{ for all } l,s \text{ such that } l+s=k,\tag{7.125}$$

since $p_r\bar\partial = 0$ and $\bar{p}_r\partial = 0$. The last two identities follow from the fact that $\mathrm{Im}\,\bar\partial$ (resp. $\mathrm{Im}\,\partial$) is orthogonal to $\ker\Delta''$ (resp. $\ker\Delta'$), hence also to its subspace $\mathcal{H}_r^{p,\,q}$ (resp. $\overline{\mathcal{H}_r^{p,\,q}}$) onto which p_r (resp. $\bar{p}_r$) projects orthogonally.

Meanwhile, for such a u, we have:

$$\bigg(p_r\partial D_{r-1} + h\,\bar{p}_r\bar\partial\overline{D_{r-1}}\bigg)\,u$$

$$= \sum_{l+s=k}\bigg(p_r\partial D_{r-2}\,(\widetilde\Delta^{(r-1)})^{-1}\bar\partial^\star(\partial u^{l,\,s}) + h\,\bar{p}_r\bar\partial\,\overline{D_{r-2}}(\overline{\widetilde\Delta^{(r-1)}})^{-1}\partial^\star(\bar\partial u^{l,\,s})\bigg)$$

$$= \sum_{l+s=k}\bigg(-\frac{1}{h}\,p_r\partial D_{r-2}\,(\widetilde\Delta^{(r-1)})^{-1}\bar\partial^\star(\bar\partial u^{l+1,\,s-1}) - h^2\,\bar{p}_r\bar\partial\,\overline{D_{r-2}}(\overline{\widetilde\Delta^{(r-1)}})^{-1}\partial^\star(\partial u^{l-1,\,s+1})\bigg),$$

where the last line followed from the properties of the forms $u^{l,\,s}$: $\partial u^{l,\,s} = -\frac{1}{h}\,\bar\partial u^{l+1,\,s-1}$ and $\bar\partial u^{l,\,s} = -h\,\partial u^{l-1,\,s+1}$.

Now, the orthogonal decomposition $C^\infty_{l+1,\,s-1}(X,\,\mathbb{C}) = \ker\Delta''\oplus\mathrm{Im}\,\bar\partial\oplus\mathrm{Im}\,\bar\partial^\star$ induces a splitting $u^{l+1,\,s-1} = \alpha^{l+1,\,s-1} + \bar\partial\xi^{l+1,\,s-2} + \bar\partial^\star\eta^{l+1,\,s}$ with $\alpha^{l+1,\,s-1}\in\ker\Delta''$. Similarly, the orthogonal decomposition $C^\infty_{l-1,\,s+1}(X,\,\mathbb{C}) = \ker\Delta'\oplus\mathrm{Im}\,\partial\oplus\mathrm{Im}\,\partial^\star$ induces a splitting $u^{l-1,\,s+1} = \beta^{l-1,\,s+1} + \partial\zeta^{l-2,\,s+1} + \partial^\star\rho^{l,\,s+1}$ with $\beta^{l-1,\,s+1}\in\ker\Delta'$. Therefore, in the last sum over $l+s=k$, we can re-write the following quantity as follows:

$$(\widetilde\Delta^{(r-1)})^{-1}\bar\partial^\star(\bar\partial u^{l+1,\,s-1}) = (\widetilde\Delta^{(r-1)})^{-1}(\bar\partial^\star\bar\partial)(\bar\partial^\star\eta^{l+1,\,s}) = (\widetilde\Delta^{(r-1)})^{-1}\Delta''(\bar\partial^\star\eta^{l+1,\,s})$$

and this quantity equals $\bar\partial^\star\eta^{l+1,\,s}$ when $r=2$ since $\widetilde\Delta^{(1)} = \Delta''$. Similarly,

$$(\overline{\widetilde\Delta^{(r-1)}})^{-1}\partial^\star(\partial u^{l-1,\,s+1}) = (\overline{\widetilde\Delta^{(r-1)}})^{-1}(\partial^\star\partial)(\partial^\star\rho^{l,\,s+1}) = (\overline{\widetilde\Delta^{(r-1)}})^{-1}\Delta'(\partial^\star\rho^{l,\,s+1})$$

and this quantity equals $\partial^\star\rho^{l,\,s+1}$ when $r=2$ since $\overline{\widetilde\Delta^{(1)}} = \Delta'$.

Suppose that $r=2$. We get

$$\bigg(p_r\partial D_{r-1} + h\,\bar{p}_r\bar\partial\overline{D_{r-1}}\bigg)\,u = \sum_{l+s=k}\bigg(-\frac{1}{h}\,p_2\partial\bar\partial^\star\eta^{l+1,\,s} - h^2\,\bar{p}_2\bar\partial\partial^\star\rho^{l,\,s+1}\bigg)$$

$$= \sum_{l+s=k}\bigg(-\frac{1}{h}\,p_2\partial\,(\alpha^{l+1,\,s-1} + \bar\partial\xi^{l+1,\,s-2} + \bar\partial^\star\eta^{l+1,\,s}) - h^2\,\bar{p}_2\bar\partial\,(\beta^{l-1,\,s+1} + \partial\zeta^{l-2,\,s+1} + \partial^\star\rho^{l,\,s+1})\bigg)$$

$$= \sum_{l+s=k}\bigg(-\frac{1}{h}\,p_2\partial u^{l+1,\,s-1} - h^2\,\bar{p}_2\bar\partial u^{l-1,\,s+1}\bigg) = 0$$

where the last identity follows from (7.125), while the identity on the second row follows from $p_2\partial\bar\partial\xi^{l+1,\,s-2} = -(p_2\bar\partial)\,\partial\xi^{l+1,\,s-2} = 0$ (since $p_2\bar\partial = 0$ as already explained), from $\bar{p}_2\bar\partial\partial\zeta^{l-2,\,s+1} = -(\bar{p}_2\partial)\,\bar\partial\zeta^{l-2,\,s+1} = 0$ (since $\bar{p}_2\partial = 0$ as already explained) and from $p_2\partial\alpha^{l+1,\,s-1} = 0$ and $\bar{p}_2\bar\partial\beta^{l-1,\,s+1} = 0$.

Let us explain the identity $p_2\partial\alpha^{l+1,\,s-1} = 0$. (To get $\bar{p}_2\bar\partial\,\beta^{l-1,\,s+1} = 0$, it will suffice to conjugate all the operators involved.) Since $\alpha^{l+1,\,s-1}\in\ker\Delta''$, we have $\alpha^{l+1,\,s-1} = p_1\alpha^{l+1,\,s-1}$, so $p_2\partial\,\alpha^{l+1,\,s-1} = p_2\partial p_1\alpha^{l+1,\,s-1}$. Now, the following identity of operators holds in every bidegree:

$$p_2\partial p_1 = 0. \tag{7.126}$$

This is because $\mathrm{Im}\,(\partial p_1) \subset \mathrm{Im}\,(\partial_{|\ker\bar\partial}) \subset \mathrm{Im}\,\bar\partial + \mathrm{Im}\,(\partial_{|\ker\bar\partial})$ and $\ker\widetilde\Delta^{(2)}$ is orthogonal to $(\mathrm{Im}\,\bar\partial + \mathrm{Im}\,(\partial_{|\ker\bar\partial}))$ (as can be checked at once, see also [Pop16, the orthogonal 3-space decomposition (26) of Lemma 3.3]). Since p_2 is the orthogonal projection onto $\ker\widetilde\Delta^{(2)}$, it must vanish on any subspace that is orthogonal to $\ker\widetilde\Delta^{(2)}$. In particular, p_2 vanishes on $\mathrm{Im}\,(\partial p_1)$, which proves (7.126).

Thus, the first inclusion of (7.124) is proved in the case when $r=2$. In fact, more has been proved when $r=2$, namely that $\ker(h\partial+\bar\partial) \subset \ker(p_2\partial D_1)\cap\ker(\bar p_2\bar\partial\bar D_1)$ (and even that for every $u\in\ker(h\partial+\bar\partial)$, every $u^{l,\,s}\in\ker(p_2\partial D_1)\cap\ker(\bar p_2\bar\partial\bar D_1)$). The following stronger form of the second inclusion of (7.124) can be proved in a similar fashion when $r=2$: $\ker(h\partial^\star+\bar\partial^\star)\subset\ker(\partial D_1p_2)^\star\cap\ker(\overline{\partial D_1}\bar p_2)^\star$.

• We will now prove by induction on $r\geq 3$ the analogous stronger forms of the inclusions of (7.124). Suppose we have already proved the inclusions

$$\ker(h\partial+\bar\partial)\subset\ker(p_j\partial D_{j-1})\cap\ker(\bar p_j\bar\partial\bar D_{j-1}),\quad \ker(h\partial^\star+\bar\partial^\star)\subset\ker(\partial D_{j-1}p_j)^\star\cap\ker(\bar\partial\bar D_{j-1}\bar p_j)^\star \tag{7.127}$$

for all $j=1,\dots,r-1$ (and even their stronger versions according to which for every $u\in\ker(h\partial+\bar\partial)$, every $u^{l,\,s}\in\ker(p_2\partial D_1)\cap\ker(\bar p_2\bar\partial\bar D_1)$ and the analogous statement for the other inclusion) and let us prove the inclusion $\ker(h\partial+\bar\partial)\subset\ker(p_r\partial D_{r-1})\cap\ker(\bar p_r\bar\partial\bar D_{r-1})$. Its counterpart $\ker(h\partial^\star+\bar\partial^\star)\subset\ker(\partial D_{r-1}p_r)^\star\cap\ker(\bar\partial\bar D_{r-1}\bar p_r)^\star$ can be proved in a similar way.

Given $u=\sum_{l+s=k}u^{l,\,s}\in\ker(h\partial+\bar\partial)$, we have seen that

$$(p_r\partial D_{r-1})\,u = -\frac{1}{h}\sum_{l+s=k}p_r\partial D_{r-2}(\widetilde\Delta^{(r-1)})^{-1}\bar\partial^\star(\bar\partial u^{l+1,\,s-1}).$$

Now, according to the orthogonal 3-space decomposition (3.27) with $r+1$ replaced by $r-1$, every form $u^{l+1,\,s-1}$ splits uniquely as

$$u^{l+1,\,s-1}=\alpha^{l+1,\,s-1}_{(r-1)}+A^{l+1,\,s-1}_{(r-1)}+B^{l+1,\,s-1}_{(r-1)},$$

where $\alpha^{l+1,\,s-1}_{(r-1)}\in\ker\widetilde\Delta^{(r-1)}=\ker\bar\partial\cap\cdots\cap\ker(p_{r-2}\partial D_{r-3})\cap\ker\bar\partial^\star\cap\cdots\cap\ker(\partial D_{r-3}p_{r-2})^\star$, $A^{l+1,\,s-1}_{(r-1)}\in\mathrm{Im}\,\bar\partial+\mathrm{Im}\,(\partial p_1)+\cdots+\mathrm{Im}\,(\partial D_{r-3}p_{r-2})\subset\ker\bar\partial\cap\ker(p_1\partial)\cap\cdots\cap\ker(p_{r-2}\partial D_{r-3})$ and $B^{l+1,\,s-1}_{(r-1)}\in\mathrm{Im}\,\bar\partial^\star+\cdots+\mathrm{Im}\,(p_{r-2}\partial D_{r-3})^\star\subset\ker\bar\partial^\star\cap\ker(\partial p_1)^\star\cap\cdots\cap\ker(\partial D_{r-3}p_{r-2})^\star$.

Therefore, since $\bar\partial u^{l+1,\,s-1}=\bar\partial B^{l+1,\,s-1}_{(r-1)}$ and $\bar\partial^\star B^{l+1,\,s-1}_{(r-1)}=0$, we get

$$(\widetilde\Delta^{(r-1)})^{-1}\bar\partial^\star(\bar\partial u^{l+1,\,s-1})=(\widetilde\Delta^{(r-1)})^{-1}\Delta''B^{l+1,\,s-1}_{(r-1)}.$$

We claim that $\Delta''B^{l+1,\,s-1}_{(r-1)}=\widetilde\Delta^{(r-1)}B^{l+1,\,s-1}_{(r-1)}$. Proving this claim amounts to proving that

$$B^{l+1,\,s-1}_{(r-1)}\in\bigg(\ker(p_1\partial)\cap\cdots\cap\ker(p_{r-2}\partial D_{r-3})\bigg)\cap\bigg(\ker(\partial p_1)^\star\cap\cdots\cap\ker(\partial D_{r-3}p_{r-2})^\star\bigg).$$

We already know that $B^{l+1,\,s-1}_{(r-1)}$ lies in the latter big paranthesis. To see that it also lies in the former, we recall that $B^{l+1,\,s-1}_{(r-1)}=u^{l+1,\,s-1}-(\alpha^{l+1,\,s-1}_{(r-1)}+A^{l+1,\,s-1}_{(r-1)})$ and that $\alpha^{l+1,\,s-1}_{(r-1)}+A^{l+1,\,s-1}_{(r-1)}\in\ker\bar\partial\cap\ker(p_1\partial)\cap\cdots\cap\ker(p_{r-2}\partial D_{r-3})$, while $u^{l+1,\,s-1}\in\ker(p_1\partial)\cap\cdots\cap\ker(p_{r-1}\partial D_{r-2})$ by the induction hypothesis (see the first inclusion in (7.127) for $j=1,\dots,r-1$). Thus, the claim is proved and we get

$$(\widetilde\Delta^{(r-1)})^{-1}\bar\partial^\star(\bar\partial u^{l+1,\,s-1})=(\widetilde\Delta^{(r-1)})^{-1}\widetilde\Delta^{(r-1)}B^{l+1,\,s-1}_{(r-1)}=B^{l+1,\,s-1}_{(r-1)},$$

where for the last identity we also used the fact that $B^{l+1,\,s-1}_{(r-1)}$ lies in a subspace that is orthogonal to $\ker\widetilde\Delta^{(r-1)}$. Consequently, we get

$$(p_r\partial D_{r-1})\,u=-\frac{1}{h}\sum_{l+s=k}(p_r\partial D_{r-2})\,B^{l+1,\,s-1}_{(r-1)}. \tag{7.128}$$

The next observation is that, for every $r\geq 2$ and in every bidegree, the following identity holds:

$$p_r\partial D_{r-2}p_{r-1}=0. \tag{7.129}$$

Indeed, in the orthogonal 3-space decomposition (3.27) with $r+1$ replaced by r, $\mathrm{Im}\,(\partial D_{r-2}p_{r-1})$ is a subspace of $\mathrm{Im}\,\bar\partial+\cdots+\mathrm{Im}\,(\partial D_{r-2}p_{r-1})$ which is orthogonal on $\ker\widetilde\Delta^{(r)}$. Since p_r is the orthogonal projection onto $\ker\widetilde\Delta^{(r)}$, the restriction of p_r to $\mathrm{Im}\,(\partial D_{r-2}p_{r-1})$ must vanish, hence (7.129).

In our case, $\alpha^{l+1,\,s-1}_{(r-1)}\in\ker\widetilde\Delta^{(r-1)}$, so $\alpha^{l+1,\,s-1}_{(r-1)}=p_{r-1}\alpha^{l+1,\,s-1}_{(r-1)}$, hence using (7.129) we get:

$$(p_r\partial D_{r-2})\,\alpha^{l+1,\,s-1}_{(r-1)}=(p_r\partial D_{r-2}p_{r-1})\,\alpha^{l+1,\,s-1}_{(r-1)}=0. \tag{7.130}$$

The next observation is that

$$(p_r\partial D_{r-2})\,A^{l+1,\,s-1}_{(r-1)}=0. \tag{7.131}$$

To see this, recall that $A^{l+1,\,s-1}_{(r-1)}$ is of the shape $A^{l+1,\,s-1}_{(r-1)}=\bar\partial a+\partial b$. Since D_{r-2} is a composition of operators ending with ∂, we get $D_{r-2}A^{l+1,\,s-1}_{(r-1)}=D_{r-2}\bar\partial a$. On the other hand, if $u^{l+1,\,s-1}=\alpha^{l+1,\,s-1}_{(r)}+A^{l+1,\,s-1}_{(r)}+B^{l+1,\,s-1}_{(r)}$ is the splitting of u w.r.t. the orthogonal 3-space decomposition (3.27) with $r+1$ replaced by r, we do have $(p_{r-1}\partial D_{r-2})\,A^{l+1,\,s-1}_{(r)}=0$, which amounts to $(p_{r-1}\partial D_{r-2})\,\bar\partial a=0$. Then also $(p_r\partial D_{r-2})\,\bar\partial a=0$, hence $(p_r\partial D_{r-2})\,A^{l+1,\,s-1}_{(r-1)}=0$, proving (7.131).

Putting together (7.128), (7.130) and (7.131), we get

$$(p_r\partial D_{r-1})\,u=-\frac{1}{h}\sum_{l+s=k}(p_r\partial D_{r-2})\,(\alpha^{l+1,\,s-1}_{(r-1)}+A^{l+1,\,s-1}_{(r-1)}+B^{l+1,\,s-1}_{(r-1)})=-\frac{1}{h}\,(p_r\partial D_{r-2})\,u=0,$$

where the last identity followed from the induction hypothesis $(p_{r-1}\partial D_{r-2})\,u=0$ (see the first inclusion in (7.127) for $j=r-1$).

We have thus proved the inclusion $\ker(h\partial+\bar\partial)\subset\ker(p_r\partial D_{r-1})$. The inclusion $\ker(h\partial+\bar\partial)\subset\ker(\bar p_r\bar\partial\bar D_{r-1})$ can be proved by conjugating the above arguments as we did in the case $r=2$. $\square$

Summing up, as in the case of $\widetilde\Delta_h=\widetilde\Delta_h^{(2)}$ described in Conclusion 7.3.4, we get an analogous family of pseudo-differential operators $(\widetilde\Delta_h^{(r)})_{h\in\mathbb{C}}$ for every integer $r\geq 2$ (and the already discussed family of differential operators $(\Delta_h)_{h\in\mathbb{C}}$ for $r=1$). The kernel of $\widetilde\Delta_h^{(r)}:C^\infty_k(X,\,\mathbb{C})\to C^\infty_k(X,\,\mathbb{C})$ will be denoted by $\mathcal{H}^k_{\widetilde\Delta_h^{(r)}}(X,\,\mathbb{C})$ and the analogous notation is used for Δ_h.

Conclusion 7.3.7. *Let $(X,\,\omega)$ be a compact complex Hermitian manifold with $\dim_{\mathbb{C}}X=n$. For every integer $r\geq 2$ and every degree $k\in\{0,\dots,2n\}$, we have C^∞ families of elliptic differential operators $(\Delta_h)_{h\in\mathbb{C}}$ (independent of r) and, respectively, elliptic pseudo-differential operators $(\widetilde\Delta_h^{(r)})_{h\in\mathbb{C}}$ from $C^\infty_k(X,\,\mathbb{C})$ to $C^\infty_k(X,\,\mathbb{C})$ such that*

(i) $\Delta_0=\Delta''$ and $\widetilde\Delta_0^{(r)}=\widetilde\Delta^{(r)}$, where $\widetilde\Delta^{(r)}$ was defined in (iii) of Proposition 3.2.6 for an arbitrary $r+1$;

(ii) $\mathcal{H}^k_{\Delta_h}(X,\,\mathbb{C})=\mathcal{H}^k_{\widetilde\Delta_h^{(r)}}(X,\,\mathbb{C})\simeq H^k_{d_h}(X,\,\mathbb{C})$ for all $h\in\mathbb{C}\setminus\{0\}$;

(iii) $\mathcal{H}^k_{\Delta_0}(X,\,\mathbb{C})\simeq\bigoplus_{p+q=k}H^{p,\,q}(X,\,\mathbb{C})$ and $\mathcal{H}^k_{\widetilde\Delta_0^{(r)}}(X,\,\mathbb{C})\simeq\bigoplus_{p+q=k}E_r^{p,\,q}(X)$.

7.3.2 Construction of the FAVB

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Recall the following map, introduced in §.3.5.1:

$$\theta_h : \Lambda^k T^\star X \longrightarrow \Lambda^k T^\star X, \qquad \theta_h\bigg(\sum_{p+q=k} u^{p,\,q}\bigg) = \sum_{p+q=k} h^p\, u^{p,\,q},$$

defined for all $h \in \mathbb{C}$ and $k \in \{0, 1, \dots, 2n\}$. When $h = 0$, $\theta_0(\sum_{p+q=k} u^{p,\,q}) = u^{0,\,k}$. As a preliminary to our construction, we notice that this projection onto the $(0,\,k)$-component at the level of forms induces the analogous projection in cohomology, that will still be denoted by θ_0, in the context of the splitting $H^k_{DR}(X,\,\mathbb{C}) \simeq \oplus_{p+q=k} E_\infty^{p,\,q}(X)$ provided by the Frölicher spectral sequence of X.

Lemma 7.3.8. *For every $k \in \{0, \dots, n\}$, the canonical linear map:*

$$\theta_0 : H^k_{DR}(X,\,\mathbb{C}) \longrightarrow E_\infty^{0,\,k}(X), \qquad \{\alpha\}_{DR} \longmapsto \{\alpha^{0,\,k}\}_{E_\infty} = \{\theta_0\alpha\}_{E_\infty}, \tag{7.132}$$

is **well defined** *and* **surjective**.

Proof. Let $r \in \mathbb{N}^\star$ be the smallest positive integer l such that the Frölicher spectral sequence of X degenerates at E_l. In particular, $E_\infty^{0,\,k}(X) = E_r^{0,\,k}(X)$.

To show well-definedness, we have to show two things, namely that

(a) $\alpha^{0,\,k} = \theta_0\alpha$ is E_r-closed for every d-closed k-form α. (This will show that $\alpha^{0,\,k} = \theta_0\alpha$ represents an E_r-cohomology class, or equivalently an E_∞-cohomology class.)

(b) for any De Rham cohomologous d-closed k-forms α and β, their $(0,\,k)$-components $\alpha^{0,\,k}$ and $\beta^{0,\,k}$ are E_r-cohomologous. (This will show that the E_∞-cohomology class of $\alpha^{0,\,k} = \theta_0\alpha$ does not depend on the choice of representative of the De Rham class $\{\alpha\}_{DR}$.)

To prove (a), let $\alpha \in C^\infty_k(X,\,\mathbb{C})$ be d-closed. The condition $d\alpha = 0$ is equivalent to the following tower of $(k+2)$ equations:

$$\begin{aligned}
\partial\alpha^{k,\,0} &= 0\\
\partial\alpha^{k-1,\,1} &= -\bar\partial\alpha^{k,\,0}\\
&\vdots\\
\partial\alpha^{0,\,k} &= -\bar\partial\alpha^{1,\,k-1}\\
\bar\partial\alpha^{0,\,k} &= 0.
\end{aligned} \tag{7.133}$$

When read from bottom to top, this tower of equations expresses the fact that $\alpha^{0,\,k}$ is E_l-closed for every $l \geq k+2$. (Note that $\partial\alpha^{k,\,0}$ is of type $(k+1,\,0)$, so it vanishes if and only if it is $\bar\partial$-exact.)

Now, if $k+2 \geq r$, any E_{k+2}-closed form is also E_r-closed. So, $\alpha^{0,\,k}$ is E_r-closed in this case. If $k+2 < r$, we have already noticed above that $\alpha^{0,\,k}$ is E_r-closed. Thus, $\alpha^{0,\,k}$ is always E_r-closed.

To prove (b), let $\alpha, \beta \in C^\infty_k(X,\,\mathbb{C})$ such that $d\alpha = d\beta = 0$ and $\alpha = \beta + d\gamma$ for some $\gamma \in C^\infty_{k-1}(X,\,\mathbb{C})$. The last identity implies that $\alpha^{0,\,k} - \beta^{0,\,k} = \bar\partial\gamma^{0,\,k-1}$. Thus, being $\bar\partial$-exact (equivalently, E_1-exact), $\alpha^{0,\,k} - \beta^{0,\,k}$ is also E_l-exact for every $l \geq 1$, hence E_r-exact, i.e. E_∞-exact. Therefore, $\{\alpha^{0,\,k}\}_{E_\infty} = \{\beta^{0,\,k}\}_{E_\infty}$.

To show surjectivity, let $\{\alpha^{0,\,k}\}_{E_r} \in E_r^{0,\,k}(X)$. Pick an arbitrary representative $\alpha^{0,\,k} \in C^\infty_{0,\,k}(X,\,\mathbb{C})$ of this class. It is necessarily E_r-closed. This means that, if $r \geq k+2$, there exist smooth pure-type forms $\alpha^{1,\,k-1}, \alpha^{2,\,k-2}, \dots, \alpha^{k-1,\,1}, \alpha^{k,\,0}$ of the shown types that, together with $\alpha^{0,\,k}$, satisfy the tower

(7.133) of $(k+2)$ equations. This expresses the fact that the smooth k-form $\alpha := \alpha^{k,0} + \cdots + \alpha^{0,k}$ is d-closed. It is obvious, by construction, that $\theta_0(\{\alpha\}_{DR}) = \{\alpha^{0,k}\}_{E_\infty}$.

If $r \leq k+1$, then $E_r^{0,k}(X) = E_\infty^{0,k}(X) = E_{k+2}^{0,k}(X)$ and the E_r-closed forms coincide with the E_{k+2}-closed forms. Hence, we still get forms $\alpha^{l,k-l}$ as above satisfying the tower of equations (7.133) and the conclusion is the same. □

(I) The FAVB in the absolute case

As a first application of the pseudo-differential operators $\widetilde{\Delta}_h$, we obtain a holomorphic vector bundle over $\mathbb{C}$ whose fibre above 0 is defined by the page in the Frölicher spectral sequence of X on which degeneration occurs.

Corollary and Definition 7.3.9. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Let $r \in \mathbb{N}^\star$ be the smallest positive integer such that the Frölicher spectral sequence of X degenerates at E_r.*

For every $k \in \{0, \dots, 2n\}$, there exists a holomorphic vector bundle $\mathcal{A}^k \longrightarrow \mathbb{C}$, of rank equal to the k-th Betti number b_k of X, whose fibres are

$$\mathcal{A}_h^k = H_{d_h}^k(X, \mathbb{C}) \quad \text{if } h \in \mathbb{C} \setminus \{0\}, \quad \text{and} \quad \mathcal{A}_0^k = \bigoplus_{p+q=k} E_r^{p,q}(X) \quad \text{if } h = 0,$$

and whose restriction to $\mathbb{C} \setminus \{0\}$ is isomorphic to the constant vector bundle $\mathcal{H}^k_{|\mathbb{C}^\star} \longrightarrow \mathbb{C} \setminus \{0\}$ of fibre $H_{DR}^k(X, \mathbb{C})$ under the holomorphic vector bundle isomorphism $\theta = (\theta_h)_{h \in \mathbb{C}^\star} : \mathcal{H}^k_{|\mathbb{C}^\star} \longrightarrow \mathcal{A}^k_{|\mathbb{C}^\star}$.

The vector bundle $\mathcal{A}^k \longrightarrow \mathbb{C}$ will be called the **Frölicher approximating vector bundle (FAVB)** *of X in degree k.*

Proof. Recall that $\dim_{\mathbb{C}} H_{d_h}^k(X, \mathbb{C}) = b_k$ for every $h \neq 0$. Fix any Hermitian metric ω on X.

If $r = 1$, the dimension of $\oplus_{p+q=k} E_1^{p,q}(X, \mathbb{C})$ equals b_k and the fibre $\mathcal{A}_0^k$ is isomorphic to the kernel of $\Delta'' = \Delta_0 : C_k^\infty(X, \mathbb{C}) \longrightarrow C_k^\infty(X, \mathbb{C})$. Thus, the C^∞ family $(\Delta_h)_{h \in \mathbb{C}}$ of elliptic differential operators has the property that the dimension of the kernel of $\Delta_h : C_k^\infty(X, \mathbb{C}) \longrightarrow C_k^\infty(X, \mathbb{C})$ is independent of $h \in \mathbb{C}$. The classical Kodaira-Spencer Theorem C of §.2.5.1 ensures that the harmonic spaces $\mathcal{H}^k_{\Delta_h}(X, \mathbb{C})$ depend in a C^∞ way on $h \in \mathbb{C}$. Therefore, they form a C^∞ vector bundle over $\mathbb{C}$, as do the vector spaces $\mathcal{A}_h^k$ to which they are isomorphic.

If $r = 2$, the dimension of $\oplus_{p+q=k} E_2^{p,q}(X, \mathbb{C})$ equals b_k and the fibre $\mathcal{A}_0^k$ is isomorphic to the kernel of $\widetilde{\Delta} = \widetilde{\Delta}_0 : C_k^\infty(X, \mathbb{C}) \longrightarrow C_k^\infty(X, \mathbb{C})$ by Theorem 3.1.4. The classical Kodaira-Spencer Theorem C of §.2.5.1 still applies to the C^∞ family $(\widetilde{\Delta}_h)_{h \in \mathbb{C}}$ of elliptic pseudo-differential operators (cf. argument in [Mas18] for the case $h = 0$), whose kernels have dimension independent of $h \in \mathbb{C}$ (and equal to b_k, see Conclusion 7.3.4), to ensure that the harmonic spaces $\mathcal{H}^k_{\widetilde{\Delta}_h}(X, \mathbb{C})$ depend in a C^∞ way on $h \in \mathbb{C}$. As above, we infer that the vector spaces $\mathcal{A}_h^k$, to which the harmonic spaces $\mathcal{H}^k_{\widetilde{\Delta}_h}(X, \mathbb{C})$ are isomorphic for all $h \in \mathbb{C}$ (cf. Conclusion 7.3.4), form a C^∞ vector bundle over $\mathbb{C}$.

If $r \geq 3$, the dimension of $\oplus_{p+q=k} E_r^{p,q}(X, \mathbb{C})$ equals b_k and the fibre $\mathcal{A}_0^k$ is isomorphic to the kernel of $\widetilde{\Delta}^{(r)} = \widetilde{\Delta}_0^{(r)} : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C})$ (cf. Conclusion 7.3.7). The classical Kodaira-Spencer Theorem C of §.2.5.1 still applies to the C^∞ family $(\widetilde{\Delta}_h^{(r)})_{h \in \mathbb{C}}$ of elliptic pseudo-differential operators (cf. argument in [Mas18] for the case of $\widetilde{\Delta}$) whose kernels have dimension independent of $h \in \mathbb{C}$ (and equal to b_k) to ensure that the harmonic spaces $\mathcal{H}^k_{\widetilde{\Delta}_h^{(r)}}(X, \mathbb{C})$ depend in a C^∞ way on $h \in \mathbb{C}$. We infer as above that the vector spaces $\mathcal{A}_h^k$, to which the harmonic spaces $\mathcal{H}^k_{\widetilde{\Delta}_h^{(r)}}(X, \mathbb{C})$ are isomorphic for all $h \in \mathbb{C}$ (cf. Conclusion 7.3.7), form a C^∞ vector bundle over $\mathbb{C}$.

Meanwhile, we know from (ii) of Lemma 3.5.5 that for every $h \neq 0$, the linear map $\theta_h : H^k_{DR}(X,\,\mathbb{C}) \longrightarrow H^k_{d_h}(X,\,\mathbb{C})$ defined by $\theta_h(\{u\}_{DR}) = \{\theta_h u\}_{d_h}$ is an isomorphism of $\mathbb{C}$-vector spaces. Since θ_h depends *holomorphically* on h and the space $H^k_{DR}(X,\,\mathbb{C})$ is independent of h, we infer that the $\mathbb{C}$-vector spaces $H^k_{d_h}(X,\,\mathbb{C})$ form a *holomorphic* vector bundle over $\mathbb{C}\setminus\{0\}$. However, we know from the above argument that this holomorphic vector bundle extends in a C^∞ way across 0 to the whole of $\mathbb{C}$. This extension must then be holomorphic. □

(II) The FAVB in the relative case

We will now define the Frölicher approximating vector bundles of a holomorphic family $(X_t)_{t\in B}$ of compact complex n-dimensional manifolds induced by a proper holomorphic submersion $\pi : \mathcal{X} \longrightarrow B$ whose base $B \subset \mathbb{C}^N$ is an open ball about the origin in some complex Euclidean vector space.

By the Ehresmann's classical Theorem 2.1.1, the differential structure of the fibres X_t is independent of $t \in B$, hence so is the Poincaré differential d, which splits differently as $d = \partial_t + \bar\partial_t$ as the complex structure of X_t varies. In particular, the differential operators d_h depend on t (except when $h = 1$), so we put

$$d_{h,t} := h\partial_t + \bar\partial_t : C^\infty_k(X,\,\mathbb{C}) \longrightarrow C^\infty_{k+1}(X,\,\mathbb{C}), \qquad h\in\mathbb{C},\;\; t\in B,\;\; k\in\{0,\dots,2n\},$$

where X is the C^∞ manifold underlying the fibres X_t. Likewise, the pointwise linear maps θ_h (which are isomorphisms when $h \neq 0$) depend on t (because the splitting of k-forms into pure-type-forms depends on the complex structure of X_t), so we put

$$\theta_{h,t} : \Lambda^k T^\star X \longrightarrow \Lambda^k T^\star X, \qquad u = \sum_{p+q=k} u_t^{p,\,q} \mapsto \theta_{h,t}u := \sum_{p+q=k} h^p\, u_t^{p,\,q}.$$

When $h \neq 0$, this induces an *isomorphism* in cohomology:

$$\theta_{h,t} : H^k_{DR}(X,\,\mathbb{C}) \longrightarrow H^k_{d_{h,t}}(X_t,\,\mathbb{C}), \qquad \theta_{h,t}(\{u\}_{DR}) = \{\theta_{h,t}u\}_{d_{h,t}}, \tag{7.134}$$

for every $t \in B$, since $\theta_{h,t}d = d_{h,t}\theta_{h,t}$. When $h = 0$, we saw in Lemma 7.3.8 that $\theta_{0,t}$ induces a *surjective* linear map:

$$\theta_{0,t} : H^k_{DR}(X,\,\mathbb{C}) \longrightarrow E_\infty^{0,\,k}(X_t), \qquad \theta_{0,t}(\{u\}_{DR}) = \{u_t^{0,\,k}\}_{E_\infty}, \tag{7.135}$$

for every $t \in B$, where $u_t^{0,\,k}$ is the component of type $(0,\,k)$ of u w.r.t. the complex structure of X_t.

For every k, let $\mathcal{H}^k \longrightarrow B$ be the *constant vector bundle* of rank $b_k = b_k(X)$ (the k^{th} Betti number of X, or equivalently of any X_t) whose fibre is the k^{th} De Rham cohomology group $H^k(X,\,\mathbb{C})$ of X (= of any X_t). Thus, $\mathcal{H}^k_t = H^k_{DR}(X_t,\,\mathbb{C}) = H^k_{DR}(X,\,\mathbb{C})$ for every $t\in B$. Let $\widetilde\nabla$ be the *Gauss-Manin* connection on $\mathcal{H}^k$. Recall that this is the trivial connection, given in the local trivialisations of $\mathcal{H}^k$ by the usual differentiation d (i.e. $\widetilde\nabla(\sum_j f_j\otimes e_j) = \sum_j(df_j)\otimes e_j$ for any local frame $\{e_j\}$ of $\mathcal{H}^k$ and any locally defined functions f_j) thanks to the transition matrices of $\mathcal{H}^k$ having *constant entries.*

Recall that the degeneration at E_1 of the Frölicher spectral sequence is a *deformation open* property of compact complex manifolds. Thus, if $E_1(X_0) = E_\infty(X_0)$, then $E_1(X_t) = E_\infty(X_t)$ for every $t\in B$, after possibly shrinking B about 0. (This follows at once from the upper-semicontinuity of the Hodge numbers $h^{p,\,q}(t)$ and from the invariance of the Betti numbers b_k of the fibres X_t.) However, when $r \geq 2$, the degeneration at E_r of the Frölicher spectral sequence is not deformation open, so we will have to assume it on all the fibres X_t for the sake of convenience.

Corollary and Definition 7.3.10. *Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex n-dimensional manifolds over an open ball $B \subset \mathbb{C}^N$ about the origin. Suppose that for an $r \in \mathbb{N}^\star$, the Frölicher spectral sequence of X_t degenerates (at least) at E_r for all $t \in B$ and that r is the smallest positive integer with this property.*

For every $k \in \{0, \dots, 2n\}$, there exists a holomorphic vector bundle $\mathcal{A}^k \longrightarrow \mathbb{C} \times B$, of rank equal to the k-th Betti number b_k of X (= of any fibre X_t), whose fibres are

$$\mathcal{A}^k_{h,\,t} = H^k_{d_{h,t}}(X_t,\,\mathbb{C}) \quad \textit{for } (h,\,t) \in \mathbb{C}^\star \times B, \quad \textit{and} \quad \mathcal{A}^k_{0,\,t} = \bigoplus_{p+q=k} E_r^{p,\,q}(X_t) \quad \textit{for } (0,\,t) \in \{0\} \times B,$$

and whose restriction to $\mathbb{C}^\star \times B$ is isomorphic to the constant vector bundle $\mathcal{H}^k_{|\mathbb{C}^\star \times B} \longrightarrow \mathbb{C}^\star \times B$ of fibre $H^k_{DR}(X,\,\mathbb{C})$ under the holomorphic vector bundle isomorphism $\theta = (\theta_{h,t})_{(h,\,t)\in\mathbb{C}^\star\times B} : \mathcal{H}^k_{|\mathbb{C}^\star\times B} \longrightarrow \mathcal{A}^k_{|\mathbb{C}^\star\times B}$.

The vector bundle $\mathcal{A}^k \longrightarrow \mathbb{C} \times B$ is called the **Frölicher approximating vector bundle (FAVB)** *of the family $(X_t)_{t\in B}$ in degree k.*

Proof. We know that $\dim_{\mathbb{C}} H^k_{d_{h,t}}(X_t,\,\mathbb{C}) = b_k$ for all $h \neq 0$ and $t \in B$. Moreover, thanks to the E_r-degeneration assumption on every fibre X_t, $\dim_{\mathbb{C}} \oplus_{p+q=k} E_r^{p,\,q}(X_t,\,\mathbb{C}) = b_k$ for all $t \in B$. Thus, $\dim_{\mathbb{C}} \mathcal{A}^k_{h,\,t} = b_k$ for all $(h,\,t) \in \mathbb{C} \times B$.

Now, fix an arbitrary C^∞ family $(\omega_t)_{t\in B}$ of Hermitian metrics on the fibres $(X_t)_{t\in B}$ and consider the C^∞ family $(\Delta_{h,\,t})_{(h,\,t)\in\mathbb{C}^\star\times B}$ of elliptic differential operators defined in every degree k by analogy with the absolute case as

$$\Delta_{h,\,t} = d_{h,\,t}d^\star_{h,\,t} + d^\star_{h,\,t}d_{h,\,t} : C^\infty_k(X,\,\mathbb{C}) \longrightarrow C^\infty_k(X,\,\mathbb{C}),$$

where the formal adjoint $d^\star_{h,\,t}$ is computed w.r.t. the metric ω_t. The kernels $\ker \Delta_{h,\,t}$ are isomorphic to the vector spaces $\mathcal{A}^k_{h,\,t}$, hence they have a dimension independent of $(h,\,t) \in \mathbb{C}^\star \times B$ (and equal to b_k). This implies, thanks to the classical Kodaira-Spencer Theorem C of §.2.5.1, that $\mathcal{A}^k \longrightarrow \mathbb{C}^\star \times B$ is a C^∞ complex vector bundle of rank b_k. This vector bundle is even holomorphic since, as pointed out in the statement, the C^∞ vector bundle isomorphism $\theta = (\theta_{h,t})_{(h,\,t)\in\mathbb{C}^\star\times B} : \mathcal{H}^k \longrightarrow \mathcal{A}^k$, viewed as a section of $\mathrm{End}\,(\mathcal{H}^k,\,\mathcal{A}^k)$, depends in a holomorphic way on $(h,\,t) \in \mathbb{C}^\star \times B$. Note that no assumption on the spectral sequence is necessary to get this conclusion on $\mathbb{C}^\star \times B$.

On the other hand, for every fixed $t \in B$, we know from the absolute case of Corollary and Definition 7.3.9 that $\mathbb{C} \ni h \mapsto \mathcal{A}^k_{h,\,t}$ is a *holomorphic* vector bundle (of rank b_k) over $\mathbb{C}$.

We conclude that near the points of the hypersurface $\{0\} \times B \subset \mathbb{C} \times B$, the entries of the transition matrices of the vector bundle $\mathcal{A}^k \longrightarrow \mathbb{C}^\star \times B$ are functions $g(h,\,t)$ on open subsets $U \setminus (\{0\} \times B) \subset \mathbb{C}^\star \times B$ (where U is an open subset of $\mathbb{C} \times B$) with the following two properties:

-the function $(h,\,t) \mapsto g(h,\,t)$ is holomorphic in the complement of the hypersurface $U \cap (\{0\} \times B)$ in U;

-for every $t \in B$, the holomorphic function $0 \neq h \mapsto g(h,\,t)$ extends holomorphically across 0.

Therefore, the resulting functions $g(h,\,t)$, defined for all $(h,\,t) \in U \subset \mathbb{C} \times B$, must be holomorphic on the whole of U, proving that $\mathbb{C} \times B \ni (h,\,t) \mapsto \mathcal{A}^k_{h,\,t}$ is a holomorphic vector bundle over $\mathbb{C} \times B$. $\square$

7.3.3 Proof of Theorem 7.3.1.

Let γ_0 be an arbitrary Gauduchon metric on X_0. It is known that, after possibly shrinking B about 0, γ_0 can be extended to a C^∞ family $(\gamma_t)_{t\in\,B}$ of C^∞ 2-forms on X (= the C^∞ manifold underlying the complex manifolds X_t) such that γ_t is a Gauduchon metric on X_t for every $t\in B$ (see, e.g., [Pop13, section 3]). Let n be the complex dimension of the fibres X_t.

The Gauduchon property of the γ_t's implies that $d_{h,\,t}(\partial_t\gamma_t^{n-1})=0$ for all $(h,\,t)\in\mathbb{C}^\star\times B$ and that $\partial_t\gamma_t^{n-1}$ is $E_r(X_t)$-closed for all $t\in B$. Thus, the following object is well defined:

$$\sigma(h,\,t):=\begin{cases}\{\partial_t\gamma_t^{n-1}\}_{d_{h,\,t}}\in H^{2n-1}_{d_{h,\,t}}(X_t,\,\mathbb{C})=\mathcal{A}^{2n-1}_{h,\,t}, & \text{if} \quad (h,\,t)\in\mathbb{C}^\star\times B,\\ \{\partial_t\gamma_t^{n-1}\}_{E_r(X_t)}\in\bigoplus\limits_{p+q=2n-1}E_r^{p,\,q}(X_t)=\mathcal{A}^{2n-1}_{0,\,t}, & \text{if} \quad (h,\,t)=(0,\,t)\in\{0\}\times B,\end{cases}$$

where $\mathcal{A}^{2n-1}\longrightarrow\mathbb{C}\times B$ is the Frölicher approximating vector bundle of the family $(X_t)_{t\in B}$ in degree $2n-1$ defined in Corollary and Definition 7.3.10. Note that the $\partial\bar\partial$-assumption on the fibres X_t with $t\neq 0$ implies that the Frölicher spectral sequence of each of these fibres degenerates at E_1, hence also at any E_r with $r\geq 1$. Thus, the assumption of Corollary and Definition 7.3.10 is satisfied and that result ensures that $\mathcal{A}^{2n-1}\longrightarrow\mathbb{C}\times B$ is a holomorphic vector bundle of rank $b_{2n-1}=b_1$ (= the $(2n-1)$-st, respectively the first Betti numbers of X, that are equal by Poincaré duality).

This last fact, in turn, implies that σ is a global C^∞ section of $\mathcal{A}^{2n-1}$ on $\mathbb{C}\times B$. Indeed, ∂_t varies holomorphically with $t\in B$, γ_t^{n-1} varies in a C^∞ way with $t\in B$, while the vector space $\mathcal{A}^{2n-1}_{h,\,t}$ varies holomorphically with $(h,\,t)\in\mathbb{C}\times B$.

Meanwhile, the $\partial\bar\partial$-assumption on every X_t with $t\in B^\star$ implies that the d-closed ∂_t-exact $(n,\,n-1)$-form $\partial_t\gamma_t^{n-1}$ is $(\partial_t\bar\partial_t)$-exact, hence also $d_{h,\,t}$-exact for every $h\in\mathbb{C}$. (Indeed, if $\partial_t\gamma_t^{n-1}=\partial_t\bar\partial_t u_t$, then $\partial_t\gamma_t^{n-1}=d_{h,\,t}(-\partial_t u_t)$.) This translates to $\sigma(h,\,t)=\{\partial_t\gamma_t^{n-1}\}_{d_{h,\,t}}=0\in\mathcal{A}^{2n-1}_{h,\,t}$ for all $(h,\,t)\in\mathbb{C}^\star\times B^\star$. (We even have $\sigma(h,\,t)=0$ for all $(h,\,t)\in\mathbb{C}\times B^\star$.)

Thus, the restriction of σ to $\mathbb{C}^\star\times B^\star$ is identically zero. Then, by continuity, σ must be identically zero on $\mathbb{C}\times B$. In particular,

$$\sigma(0,\,t)=\{\partial_t\gamma_t^{n-1}\}_{E_r(X_t)}=0\in\mathcal{A}^{2n-1}_{0,\,t}\quad\text{for all } t\in B,$$

which means precisely that $\partial_t\gamma_t^{n-1}$ is $E_r(X_t)$-exact for every $t\in B$. In other words, γ_t is an E_r-sG metric on X_t for every $t\in B$, including $t=0$. In particular, X_0 is an E_r-sG manifold and even an E_r-sGG manifold since the Gauduchon metric γ_0 was chosen arbitrarily on X_0 in the first place. $\square$

7.4 Uniform control of volumes of divisors and of masses of currents using E_r-sG metrics

This section is the analogue in this more conceptual approach to Theorem 7.0.4 of §.7.2.

7.4.1 Deformation limits of real $(1,\,1)$-classes

We now derive a key application of the FAVB construction that will be used later on.

By $H^{p,\,q}_{DR}(X,\,\mathbb{C})$ we will mean the space of De Rham cohomology classes of degree $p+q$ that can be represented by a (d-closed) pure-type $(p,\,q)$-form. These classes will be said to be of *type* $(p,\,q)$.

The next statement will play a key role despite its simplicity. It gives a criterion for a *real* De Rham 2-class to be of *type* $(1,\,1)$ on a possibly *non-$\partial\bar\partial$-manifold* that is analogous to the familiar criterion on $\partial\bar\partial$-manifolds requiring the vanishing of the projection onto $H^{0,\,2}(X,\,\mathbb{C})$ in the canonical Hodge decomposition $H^2_{DR}(X,\,\mathbb{C})\simeq H^{2,\,0}(X,\,\mathbb{C})\oplus H^{1,\,1}(X,\,\mathbb{C})\oplus H^{0,\,2}(X,\,\mathbb{C})$. On an arbitrary X, there is no Hodge decomposition, but its role is played in a certain sense by the non-canonical isomorphism $H^2_{DR}(X,\,\mathbb{C})\simeq E_\infty^{2,\,0}(X)\oplus E_\infty^{1,\,1}(X)\oplus E_\infty^{0,\,2}(X)$, as the following result shows.

Lemma 7.4.1. *Let $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$ be a* **real** *class. The following equivalence holds:*

$$\{\alpha\}_{DR} \in H^{1,1}_{DR}(X, \mathbb{C}) \iff \theta_0(\{\alpha\}_{DR}) = 0 \in E^{0,2}_\infty(X). \tag{7.136}$$

Proof. Recall that $\theta_0(\{\alpha\}_{DR}) = \{\alpha^{0,2}\}_{E_\infty}$ by Lemma 7.3.8.

" $\implies$ " If $\{\alpha\}_{DR} \in H^{1,1}_{DR}(X, \mathbb{C})$, there exists a d-closed form $u^{1,1} \in C^\infty_{1,1}(X, \mathbb{C})$ such that $\{\alpha\}_{DR} = \{u^{1,1}\}_{DR}$. Then, $\theta_0 u^{1,1} = 0$, hence $\theta_0(\{\alpha\}_{DR}) = \{\theta_0 u^{1,1}\}_{DR} = 0 \in E^{0,2}_\infty(X)$.

"$\Longleftarrow$" Since the class $\{\alpha\}_{DR}$ is *real*, it can be represented by a *real* form $\alpha = \alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}$. The condition $\alpha = \overline{\alpha}$ translates to $\alpha^{1,1} = \overline{\alpha^{1,1}}$ and $\alpha^{2,0} = \overline{\alpha^{0,2}}$, while the condition $d\alpha = 0$ for the real form α translates to either of the following two equivalent conditions being satisfied:

$$(\partial\alpha^{2,0} = 0 \text{ and } \partial\alpha^{1,1} + \bar\partial\alpha^{2,0} = 0) \iff (\bar\partial\alpha^{0,2} = 0 \text{ and } \partial\alpha^{0,2} + \bar\partial\alpha^{1,1} = 0). \tag{7.137}$$

On the other hand, $\theta_0\alpha = \alpha^{0,2}$, so the hypothesis $\theta_0(\{\alpha\}_{DR}) = 0$ amounts to $\{\alpha^{0,2}\}_{E_\infty} = 0$. This is equivalent to $\alpha^{0,2}$ being E_r-exact, where r is the smallest positive integer l such that the Frölicher spectral sequence of X degenerates at E_l. However, for bidegree reasons, a $(0, q)$-form is E_r-exact if and only if it is $\bar\partial$-exact. (See characterisation of E_r-exactness in (ii) of Proposition 3.2.4. In an arbitrary bidegree, $\bar\partial$-exactness, which coincides with E_1-exactness, is a stronger property than E_r-exactness when $r \geq 2$.) Thus, our assumption $\theta_0(\{\alpha\}_{DR}) = 0$ translates to the existence of a form $u^{0,1} \in C^\infty_{0,1}(X, \mathbb{C})$ such that

$$\alpha^{0,2} = \bar\partial u^{0,1}.$$

Conjugating the above identity, we get $\alpha^{2,0} = \partial u^{1,0}$, where we put $u^{1,0} := \overline{u^{0,1}}$. This yields:

$$\alpha^{2,0} + \alpha^{0,2} = du - (\bar\partial u^{1,0} + \partial u^{0,1}), \qquad \text{where } u := u^{1,0} + u^{0,1},$$

hence finally

$$\alpha - du = \alpha^{1,1} - (\bar\partial u^{1,0} + \partial u^{0,1}).$$

This shows that $\alpha - du$ is a representative of bidegree $(1, 1)$ of the De Rham cohomology class $\{\alpha\}_{DR}$, proving that $\{\alpha\}_{DR} \in H^{1,1}_{DR}(X, \mathbb{C})$. □

We can now prove the following

Theorem 7.4.2. *Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds over an open ball $B \subset \mathbb{C}^N$ about the origin. Suppose that the fibre $X_t := \pi^{-1}(t)$ is a* **$\partial\bar\partial$-manifold** *for all $t \in B \setminus \{0\}$. Let $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$ be a* **real** *class.*

If $\{\alpha\}_{DR} \in H^{1,1}_{DR}(X_t, \mathbb{C})$ for every $t \in B \setminus \{0\}$, then $\{\alpha\}_{DR} \in H^{1,1}_{DR}(X_0, \mathbb{C})$.

Proof. Let $\theta : \mathcal{H}^2 \longrightarrow \mathcal{A}^2$ be the vector bundle morphism from the constant bundle of fibre $H^2_{DR}(X, \mathbb{C})$ on $\mathbb{C} \times B$ to the *Frölicher approximating vector bundle* $\mathcal{A}^2 \longrightarrow \mathbb{C} \times B$ of the family $(X_t)_{t\in B}$ in degree 2 defined by the family of linear maps:

$$\theta_{h,t} : H^2_{DR}(X, \mathbb{C}) \longrightarrow \mathcal{A}^2_{h,t}, \qquad (h, t) \in \mathbb{C} \times B.$$

(See (7.134) and (7.135).)

By Lemma 7.4.1, the hypothesis $\{\alpha\}_{DR} \in H^{1,1}_{DR}(X_t, \mathbb{C})$ for every $t \in B \setminus \{0\}$ translates to

$$\theta_{0,t}(\{\alpha\}_{DR}) = 0 \in \mathcal{A}^2_{0,t}, \qquad t \in B \setminus \{0\}.$$

Since $\theta_{0,0}(\{\alpha\}_{DR}) = \lim_{t\to 0} \theta_{0,t}(\{\alpha\}_{DR})$, we get

$$\theta_{0,0}(\{\alpha\}_{DR}) = 0 \in \mathcal{A}^2_{0,0} = E^{2,0}_\infty(X_0) \oplus E^{1,1}_\infty(X_0) \oplus E^{0,2}_\infty(X_0).$$

We know from Lemma 7.3.8 that $\theta_{0,0}(\{\alpha\}_{DR}) \in E^{0,2}_\infty(X_0)$, so $\theta_{0,0}(\{\alpha\}_{DR}) = 0 \in E^{0,2}_\infty(X_0)$.

By Lemma 7.4.1, this is equivalent to $\{\alpha\}_{DR} \in H^{1,1}_{DR}(X_0, \mathbb{C})$ and we are done. □

7.4.2 Deformation limits of Moishezon manifolds

Besides Theorem 7.3.1, the second main ingredient in the second proof of Theorem 7.0.4 is the following

Theorem 7.4.3. *Let $\pi : \mathcal{X} \longrightarrow B$ be a holomorphic family of compact complex n-dimensional manifolds over an open ball $B \subset \mathbb{C}^N$ about the origin such that the fibre $X_t := \pi^{-1}(t)$ is a $\partial\bar\partial$-***manifold** *for all $t \in B \setminus \{0\}$. Let X be the C^∞ manifold that underlies the fibres $(X_t)_{t\in B}$ and let J_t be the complex structure of X_t.*

Suppose there exists a C^∞ family $(\widetilde\omega_t)_{t\in B}$ of d-closed, smooth, real 2-forms on X such that, for every $t \in B$, the J_t-pure-type components of $\widetilde\omega_t$ are d-closed. Fix an integer $r \geq 1$ and suppose there exists a C^∞ family $(\gamma_t)_{t\in B}$ of E_r-sG metrics on the fibres $(X_t)_{t\in B}$ with potentials depending in a C^∞ way on t.

(i) If, for every $t \in B^\star$, there exists a Kähler metric ω_t on X_t that is De Rham-cohomologous to $\widetilde\omega_t$, then there exists a constant $C > 0$ independent of $t \in B^\star$ such that the γ_t-masses of the metrics ω_t are uniformly bounded above by C:

$$0 \leq M_{\gamma_t}(\omega_t) := \int\limits_X \omega_t \wedge \gamma_t^{n-1} < C < +\infty, \qquad t \in B^\star.$$

In particular, there exists a sequence of points $t_j \in B^\star$ converging to $0 \in B$ and a d-closed positive J_0-$(1,\,1)$-current T on X_0 such that ω_{t_j} converges in the weak topology of currents to T as $j \to +\infty$.

(ii) If, for every $t \in B^\star$, there exists an effective analytic $(n-1)$-cycle $Z_t = \sum_l n_l(t)\, Z_l(t)$ on X_t (i.e. a finite linear combination with integer coefficients $n_l(t) \in \mathbb{N}^\star$ of irreducible analytic subsets $Z_l(t) \subset X_t$ of codimension 1) that is De Rham-cohomologous to $\widetilde\omega_t$, then there exists a constant $C > 0$ independent of $t \in B^\star$ such that the γ_t-volumes of the cycles Z_t are uniformly bounded above by C:

$$0 \leq v_{\gamma_t}(Z_t) := \int\limits_X [Z_t] \wedge \gamma_t^{n-1} < C < +\infty, \qquad t \in B^\star.$$

Proof. We will prove (ii). The proof of (i) is very similar and we will indicate the minor differences after the proof of (ii). The method is almost the same as the one used to prove Theorem 7.2.2.

Since the positive $(1,\,1)$-current $[Z_t] = \sum_l n_l(t)\,[Z_l(t)]$ (a linear combination of the currents $[Z_l(t)]$ of integration on the hypersurfaces Z_t) on X_t is De Rham cohomologous to $\widetilde\omega_t$ for every $t \in B^\star$, there exists a *real* current β'_t of degree 1 on X such that

$$\widetilde\omega_t = [Z_t] + d\beta'_t, \qquad t \in B^\star. \tag{7.138}$$

This implies that

$$\bar\partial_t \beta_t^{\prime\,0,\,1} = \widetilde\omega_t^{0,\,2}, \qquad t \in B^\star. \tag{7.139}$$

In particular, $\widetilde\omega_t^{0,\,2}$ is $\bar\partial_t$-exact for every $t \in B^\star$, so it can be regarded as the right-hand side term of equation (7.139) whose unknown is $\beta_t^{\prime\,0,\,1}$.

For every $t \in B^\star$, let $\beta_t^{0,\,1}$ be the minimal $L^2_{\gamma_t}$-norm solution of equation (7.139). Thus, $\beta_t^{0,\,1}$ is the C^∞ J_t-type $(0,\,1)$-form given by the Neumann formula

$$\beta_t^{0,\,1} = \Delta_t^{\prime\prime\,-1}\bar\partial_t^\star \widetilde\omega_t^{0,\,2}, \qquad t \in B^\star, \tag{7.140}$$

where $\Delta_t''^{-1}$ is the Green operator of the $\bar\partial$-Laplacian $\Delta_t'' := \bar\partial_t\bar\partial_t^\star + \bar\partial_t^\star\bar\partial_t$ induced by the metric γ_t on the forms of X_t. The difficulty we are faced with is that the family of operators $(\Delta_t''^{-1})_{t\in B^\star}$, hence also the family of forms $(\beta_t^{0,1})_{t\in B^\star}$, need not extend in a continuous way to $t=0$ if the Hodge number $h^{0,1}(t)$ of X_t jumps at $t=0$ (i.e. if $h^{0,1}(0) > h^{0,1}(t)$ for $t\in B^\star$ close to 0).

As in the proof of Theorem 7.2.2, the way around this goes through the use of special metrics on the fibres X_t. Set

$$\beta_t^{1,0} := \overline{\beta_t^{0,1}} \quad \text{and} \quad \beta_t := \beta_t^{1,0} + \beta_t^{0,1}, \quad t\in B^\star.$$

Since $\widetilde\omega_t$ is real, this and equation (7.139) satisfied by $\beta_t^{0,1}$ imply that $\widetilde\omega_t - [Z_t] - d\beta_t$ is a J_t-type $(1,1)$-current. Since this current is d-exact (it equals $d(\beta_t' - \beta_t)$) and since every fibre X_t with $t\in B^\star$ is supposed to be a $\partial\bar\partial$-manifold, we infer that the current $\widetilde\omega_t - [Z_t] - d\beta_t$ is $\partial_t\bar\partial_t$-exact. (Indeed, the $\partial\bar\partial$-property can be equivalently expressed in terms of smooth forms or currents since it is equivalent to the canonical maps between the Bott-Chern and Aeppli cohomologies being isomorphic and both these cohomologies can be defined using either smooth forms or currents.) Hence, there exists a family of distributions $(R_t)_{t\in B^\star}$ on $(X_t)_{t\in B^\star}$ such that

$$\widetilde\omega_t = [Z_t] + d\beta_t + \partial_t\bar\partial_t R_t \quad \text{on } X_t \quad \text{for all } t\in B^\star. \tag{7.141}$$

Consequently, for the γ_t-volume of the divisor Z_t we get:

$$v_{\gamma_t}(Z_t) := \int\limits_X [Z_t]\wedge\gamma_t^{n-1} = \int\limits_X \widetilde\omega_t\wedge\gamma_t^{n-1} - \int\limits_X d\beta_t\wedge\gamma_t^{n-1}, \quad t\in B^\star, \tag{7.142}$$

since $\int_X \partial_t\bar\partial_t R_t\wedge\gamma_t^{n-1} = 0$ thanks to the Gauduchon property of γ_t and to integration by parts. Now, the families of forms $(\widetilde\omega_t)_{t\in B}$ and $(\gamma_t^{n-1})_{t\in B}$ depend in a C^∞ way on t up to $t=0$, so the quantity $\int_X \widetilde\omega_t\wedge\gamma_t^{n-1}$ is bounded as $t\in B^\star$ converges to $0\in B$. Thus, we are left with proving the boundedness of the quantity $\int_X d\beta_t\wedge\gamma_t^{n-1} = \int_X \partial_t\beta_t^{0,1}\wedge\gamma_t^{n-1} + \int_X \bar\partial_t\beta_t^{1,0}\wedge\gamma_t^{n-1}$ whose two terms are conjugated to each other. Consequently, it suffices to prove the boundedness of the quantity

$$I_t := \int\limits_X \partial_t\beta_t^{0,1}\wedge\gamma_t^{n-1} = \int\limits_X \beta_t^{0,1}\wedge\partial_t\gamma_t^{n-1}, \quad t\in B^\star,$$

as t approaches $0\in B$.

So far, the proof has been identical to the one in [Pop10]. The assumption made on the C^∞ family $(\gamma_t)_{t\in B}$ of E_r-sG metrics implies the existence of C^∞ families of J_t-type $(n, n-2)$-forms $(\Gamma_t^{n,n-2})_{t\in B}$ and of J_t-type $(n-1, n-1)$-forms $(\zeta_{r-2,t})_{t\in B}$ such that

$$\partial_t\gamma_t^{n-1} = \bar\partial_t\Gamma_t^{n,n-2} + \partial_t\zeta_{r-2,t}, \quad t\in B, \tag{7.143}$$

and

$$\begin{aligned}\bar\partial_t\zeta_{r-2,t} &= \partial_t v_{r-3,t}^{(r-2)}\\ \bar\partial_t v_{r-3,t}^{(r-2)} &= 0.\end{aligned} \tag{7.144}$$

(We have already noticed that, for bidegree reasons, tower (1.29) reduces to its first two rows when we start off in bidegree $(n, n-1)$.)

On the other hand, $\bar\partial_t(\partial_t\beta_t^{0,1}) = -\partial_t(\bar\partial_t\beta_t^{0,1}) = -\partial_t\widetilde\omega_t^{0,2} = 0$, the last identity being a consequence of the d-closedness assumption made on the J_t-pure-type components of $\widetilde\omega_t$. The $\partial\bar\partial$-assumption on

X_t for every $t\in B^\star$ implies that the J_t-type $(1,\,1)$-form $\partial_t\beta_t^{0,\,1}$ is $\bar\partial_t$-exact (since it is already d-closed and ∂_t-exact), so there exist J_t-type $(1,\,0)$-forms $(u_t)_{t\in B^\star}$ such that

$$\partial_t\beta_t^{0,\,1}=\bar\partial_t u_t,\qquad t\in B^\star. \tag{7.145}$$

This, in turn, implies that the J_t-type $(2,\,0)$-form $\partial_t u_t$ is $\bar\partial_t$-closed, hence d-closed. The $\partial\bar\partial$-assumption on X_t for every $t\in B^\star$ implies that $\partial_t u_t$ is $\bar\partial_t$-exact, hence zero, for bidegree reasons. Thus

$$\partial_t u_t=0,\qquad t\in B^\star. \tag{7.146}$$

Putting (7.143), (7.144), (7.145) and (7.146) together and integrating by parts several times, we get:

$$\begin{aligned} I_t &= \int\limits_X\bar\partial_t\beta_t^{0,\,1}\wedge\Gamma_t^{n,\,n-2}+\int\limits_X\partial_t\beta_t^{0,\,1}\wedge\zeta_{r-2,\,t}=\int\limits_X\widetilde\omega_t^{0,\,2}\wedge\Gamma_t^{n,\,n-2}+\int\limits_X\bar\partial_t u_t\wedge\zeta_{r-2,\,t}\\ &= \int\limits_X\widetilde\omega_t^{0,\,2}\wedge\Gamma_t^{n,\,n-2}+\int\limits_X u_t\wedge\bar\partial_t\zeta_{r-2,\,t}=\int\limits_X\widetilde\omega_t^{0,\,2}\wedge\Gamma_t^{n,\,n-2}+\int\limits_X u_t\wedge\partial_t v_{r-3,\,t}^{(r-2)}\\ &= \int\limits_X\widetilde\omega_t^{0,\,2}\wedge\Gamma_t^{n,\,n-2}+\int\limits_X\partial_t u_t\wedge v_{r-3,\,t}^{(r-2)}=\int\limits_X\widetilde\omega_t^{0,\,2}\wedge\Gamma_t^{n,\,n-2},\qquad t\in B^\star.\end{aligned}$$

Since the families of forms $(\Gamma_t^{n,\,n-2})_{t\in B}$ and $(\widetilde\omega_t^{0,\,2})_{t\in B}$ vary in a C^∞ way with t up to $t=0\in B$, we infer that the quantities $(I_t)_{t\in B^\star}$ are bounded as $t\in B^\star$ converges to $0\in B$. This completes the proof of (ii).

The proof of (i) is identical to that of (ii), except for the fact that $[Z_t]$ has to be replaced by ω_t in (7.138), (7.141) and (7.142), while β'_t and R_t are smooth. $\square$

Proof of Theorem 7.2.1 as a consequence of Theorems 7.3.1, 7.4.3 and 7.4.2

By Theorem 7.3.1, X_0 is an E_r-sG manifold, where $r\in\mathbb{N}^\star$ is the smallest positive integer such that $E_r(X_0)=E_\infty(X_0)$. Therefore, thanks to Lemma 4.4.2, after possibly shrinking B about 0, there exists a C^∞ family $(\gamma_t)_{t\in B}$ of E_r-sG metrics on the fibres $(X_t)_{t\in B}$ whose potentials depend in a C^∞ way on $t\in B$.

Let $(Z_t)_{t\in B^\star}$ be a C^∞ family of effective analytic divisors such that $Z_t\subset X_t$ for all $t\in B^\star$. The De Rham cohomology class $\{[Z_t]\}_{DR}\in H^2(X,\,\mathbb{R})$ of the current $[Z_t]$ of integration over $Z_t=\sum_l n_l(t)\,Z_l(t)$ (where $n_l(t)\in\mathbb{N}^\star$ and the $Z_l(t)$'s are irreducible analytic hypersurfaces of X_t) is integral. Therefore, the continuous, integral-class-valued map

$$B^\star\ni t\mapsto\{[Z_t]\}_{DR}\in H^2(X,\,\mathbb{Z})$$

must be constant, equal to an integral De Rham 2-class that we denote by $\{\alpha\}$. Moreover, the current $[Z_t]$ is of bidegree $(1,\,1)$ for J_t, so $\{\alpha\}_{DR}\in H^{1,\,1}_{DR}(X_t,\,\mathbb{C})$ for every $t\in B^\star$. By Theorem 7.4.2, $\{\alpha\}_{DR}\in H^{1,\,1}_{DR}(X_0,\,\mathbb{C})$.

Therefore, there exists a C^∞ family $(\widetilde\omega_t)_{t\in B}$ of d-closed, smooth, real 2-forms on X lying in the De Rham class $\{\alpha\}$ such that, for every $t\in B$, the J_t-pure-type components of $\widetilde\omega_t$ are d-closed. (Actually, Theorem 7.4.2 implies more, but this suffices for our purposes.) In particular, for every $t\in B^\star$, the current $[Z_t]$ is De Rham-cohomologous to $\widetilde\omega_t$.

Thus, all the hypotheses of Theorem 7.4.3 are satisfied. From (ii) of that theorem we get that the γ_t-volumes $(v_{\gamma_t}(Z_t))_{t\in B^\star}$ of the divisors Z_t are uniformly bounded. This implies, thanks to Lieberman's strengthened form ([Lie78, Theorem 1.1]) of Bishop's Theorem [Bis64], that a limiting effective divisor $Z_0 \subset X_0$ for the family of relative effective divisors $(Z_t)_{t\in B^\star}$ exists. Since this family has been chosen arbitrarily, it follows that X_0 has at least as many divisors as the nearby fibres X_t with $t \neq 0$ and t close to 0. Meanwhile, we know (see, e.g., [CP94, Remark 2.22]) that the algebraic dimension of any compact complex manifold X is the maximal number of effective prime divisors meeting transversally at a generic point of X. It follows that the algebraic dimension of X_0 is $\geq$ the algebraic dimension of the generic fibre X_t with $t \in B^\star$ close to 0. $\square$

Recall that in §.7.2 we proved the implication:

$$\text{Theorem 7.2.1} \implies \text{Theorem 7.0.4}.$$

Together with the second proof of Theorem 7.2.1 given in this §.7.4.2, this implication completes the second proof of Theorem 7.0.4.

Chapter 8

Appendix: Nilmanifolds and Solvmanifolds

For the convenience of the reader, we collect here some well-known facts about Lie groups, Lie algebras, nilmanifolds and solvmanifolds that are a rich source of examples of various classes of manifolds and have been used throughout the book. Many statements are given without proofs, but the references for those proofs are indicated.

8.1 Nilpotent Lie algebras (NLA's)

Most of this section is taken from Salamon [Sal01].

Let $(\mathfrak{g}, [\cdot, \cdot])$ be a real **Lie algebra** with $\dim_{\mathbb{R}}\mathfrak{g} = 2n$, where the alternating $\mathbb{R}$-bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying the *Jacobi identity*:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \qquad x, y, z \in \mathfrak{g},$$

is its Lie bracket. The standard example is the Lie algebra $\mathfrak{g}$ of a real **Lie group** G, in which case $\mathfrak{g}$ consists of the *left-invariant vector fields* on G. Let $(\mathfrak{g}^\star, d)$ be the **dual** of $(\mathfrak{g}, [\cdot, \cdot])$. So, one has a complex:

$$0 \longrightarrow \mathfrak{g}^\star \xrightarrow{d} \Lambda^2\mathfrak{g}^\star \xrightarrow{d} \Lambda^3\mathfrak{g}^\star \xrightarrow{d} \dots \xrightarrow{d} \Lambda^{2n}\mathfrak{g}^\star \longrightarrow 0$$

obtained by extending the linear map $d : \mathfrak{g}^\star \longrightarrow \Lambda^2\mathfrak{g}^\star$ which is the dual of the Lie bracket. The vanishing of the composition $\mathfrak{g}^\star \longrightarrow \Lambda^3\mathfrak{g}^\star$ corresponds to the *Jacobi identity* satisfied by $[\cdot, \cdot]$. When $\mathfrak{g}$ is the Lie algebra of a Lie group G, $\mathfrak{g}^\star$ consists of the *left-invariant* 1-*forms* on G.

We denote by

$$b_k := \dim \frac{\ker(d : \Lambda^k\mathfrak{g}^\star \longrightarrow \Lambda^{k+1}\mathfrak{g}^\star)}{\mathrm{Im}\,(d : \Lambda^{k-1}\mathfrak{g}^\star \longrightarrow \Lambda^k\mathfrak{g}^\star)}, \qquad k = 1, \dots, 2n. \tag{8.1}$$

Moreover, we have *isomorphisms*:

$$\frac{\ker(d : \Lambda^k\mathfrak{g}^\star \longrightarrow \Lambda^{k+1}\mathfrak{g}^\star)}{\mathrm{Im}\,(d : \Lambda^{k-1}\mathfrak{g}^\star \longrightarrow \Lambda^k\mathfrak{g}^\star)} \simeq H^k(\mathfrak{g}), \qquad k = 1, \dots, 2n,$$

where $H^k(\mathfrak{g})$ is the Lie algebra *cohomology group* of degree k.

The **descending central series** of a Lie algebra $\mathfrak{g}$ is the chain of ideals:

$$\mathfrak{g}^0 := \mathfrak{g} \supset \mathfrak{g}^1 := [\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{g}^2 := [[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] \supset \cdots \supset \mathfrak{g}^p := [\mathfrak{g}^{p-1}, \mathfrak{g}] \supset \dots.$$

In other words, we define inductively: $\mathfrak{g}^0 := \mathfrak{g}$ and $\mathfrak{g}^i := [\mathfrak{g}^{i-1}, \mathfrak{g}]$ for all $i \geq 1$.

Definition 8.1.1. *A Lie algebra $\mathfrak{g}$ is said to be* s**-step nilpotent** *if $\mathfrak{g}^s = 0$ and $\mathfrak{g}^{s-1} \neq 0$.*

A Lie algebra $\mathfrak{g}$ is said to be a **nilpotent Lie algebra (NLA)** *if $\mathfrak{g}$ is s-step nilpotent for some $s \in \mathbb{N}^\star$.*

The nilpotency condition can be expressed in terms of differential forms as follows. Define vector subspaces $V_i \subset \mathfrak{g}^\star$ inductively as:

$$V_0 := \{0\} \quad \text{and} \quad V_i := \{\sigma \in \mathfrak{g}^\star \mid d\sigma \in \Lambda^2 V_{i-1}\}, \qquad i \geq 1.$$

Observation 8.1.2. *$V_1 = \{\sigma \in \mathfrak{g}^\star \mid d\sigma = 0\} = \ker(d : \mathfrak{g}^\star \to \Lambda^2\mathfrak{g}^\star) \simeq H^1(\mathfrak{g})$, so $\dim V_1 = b_1$.*

Proof. It is obvious from the definitions and what has been said above. □

Notation 8.1.3. *The* **annihilator** *of an ideal $\mathfrak{h}$ of $\mathfrak{g}$ is denoted by $(\mathfrak{h})^o$.*

Lemma 8.1.4. *([Sal01, Lemma 1.1]) For every $i \geq 0$, V_i is the* **annihilator** *of $\mathfrak{g}^i$, namely $V_i = (\mathfrak{g}^i)^o$.*

Proof. We proceed by induction on $i \geq 0$. Since the annihilator of $\mathfrak{g}^0 = \mathfrak{g}$ is $\{0\} = V_0 \subset \mathfrak{g}^\star$, the statement is true for $i = 0$.

Suppose that, for a given $i \geq 0$, V_i is the annihilator of $\mathfrak{g}^i$. Fix an arbitrary $\sigma \in \mathfrak{g}^\star$. The following equivalences hold:

$$\begin{aligned} \sigma \in V_{i+1} \quad &\overset{(a)}{\Longleftrightarrow} \quad d\sigma \in \Lambda^2 V_i \overset{(b)}{\Longleftrightarrow} d\sigma(X, Y) = 0 \quad \forall\, X \in \mathfrak{g},\ \forall\, Y \in \mathfrak{g}^i \\ &\overset{(c)}{\Longleftrightarrow} \quad \sigma([X, Y]) = 0 \quad \forall\, X \in \mathfrak{g},\ \forall\, Y \in \mathfrak{g}^i \overset{(d)}{\Longleftrightarrow} \sigma \text{ annihilates } \mathfrak{g}^{i+1}, \end{aligned}$$

where (a) is the definition of V_{i+1}; (b) expresses the fact that V_i is the annihilator of $\mathfrak{g}^i$; (c) follows from the fact that $d\sigma(X, Y) = -\sigma([X, Y])$ for any $\sigma \in \mathfrak{g}^\star$ and any $X, Y \in \mathfrak{g}$ (as a special case of Cartan's formula $d\sigma(X, Y) = X.\sigma(Y) - Y.\sigma(X) - \sigma([X, Y])$ on a manifold in which $\sigma(Y)$ and $\sigma(X)$ are constant, hence $X.\sigma(Y)$ and $Y.\sigma(X)$ vanish); (d) follows from the definition of $\mathfrak{g}^{i+1}$.

We conclude that $V_{i+1} = (\mathfrak{g}^{i+1})^o$. □

Corollary 8.1.5. *The subspaces V_i form an* **ascending sequence***:*

$$\{0\} = V_0 \subset V_1 = \ker(d : \mathfrak{g}^\star \to \Lambda^2\mathfrak{g}^\star) = (\mathfrak{g}^1)^o \subset V_2 = (\mathfrak{g}^2)^o \subset \cdots \subset V_p = (\mathfrak{g}^p)^o \subset \cdots \subset \mathfrak{g}^\star.$$

In particular:

(a) the Lie algebra $\mathfrak{g}$ is s**-step nilpotent** *if and only if $V_s = \mathfrak{g}^\star$ and $V_{s-1} \subsetneq \mathfrak{g}^\star$;*

(b) A real Lie group G of dimension m is **nilpotent** *if and only if there exists a basis $\{e^1, \ldots, e^m\}$ of left-invariant 1-forms on G (i.e. a basis of $\mathfrak{g}^\star$) such that*

$$de^i \in \Lambda^2\langle e^1, \ldots, e^{i-1}\rangle, \qquad 1 \leq i \leq m, \tag{8.2}$$

with the right-hand side interpreted as zero when $i = 1$.

(c) $H^1(\mathfrak{g}) \simeq V_1 = ([\mathfrak{g}, \mathfrak{g}])^o$. Hence, $H^1(\mathfrak{g})$ is isomorphic to (the dual of) the quotient $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

Characterisation (b) of nilpotency in Corollary 8.1.5 leads to the following notation for NLA's. To make a choice, we give it in real dimension 6.

Notation 8.1.6. *Let $\mathfrak{g}$ be an NLA of real dimension 6. We write, for example, $\mathfrak{g} = (0, 0, 0, 0, 12, 34)$ to signify that $\mathfrak{g}^\star$ has a basis $\{e^1, \ldots, e^6\}$ such that*

$$de^1 = de^2 = de^3 = de^4 = 0, \;\; de^5 = e^1 \wedge e^2, \;\; de^6 = e^3 \wedge e^4. \tag{8.3}$$

The well-known **Cartan formula**:

$$d\sigma(X,\,Y) = X.\sigma(Y) - Y.\sigma(X) - \sigma([X,\,Y]) \tag{8.4}$$

that holds for any smooth 1-form σ and any vector fields X, Y on any manifold, implies that description (8.3) of the 6-dimensional NLA $\mathfrak{g} = (0,\,0,\,0,\,0,\,12,\,34)$ in terms of 1-forms is equivalent to the following description in terms of vector fields: $\mathfrak{g}$ has a basis $\{e_1, \dots, e_6\}$ (dual to the basis $\{e^1, \dots, e^6\}$ of $\mathfrak{g}^\star$) such that

$$[e_1,\,e_2] = -e_5,\;\; [e_3,\,e_4] = -e_6 \;\text{ and }\; [e_i,\,e_j] = 0 \;\text{ in all the other cases.} \tag{8.5}$$

Finally, let us mention that the nilpotency of a Lie group implies the following interpretation of the numbers b_k defined in (8.1).

Theorem 8.1.7. *(***Nomizu***) Let G be a* **nilpotent** *Lie group. For any* **discrete co-compact subgroup** *$\Gamma \subset G$, b_k equals the k-***th Betti number** *of the quotient manifold G/Γ.*

Proof. See [Nom54]. □

8.2 Left-invariant complex structures on Lie groups

Most of this section is taken from Salamon [Sal01], Cordero-Fernandez-Gray-Ugarte [CFGU00] and Ceballos-Otal-Ugarte-Villacampa [COUV14].

Definition 8.2.1. *A* **left-invariant almost complex structure** *on a Lie group G is a complex structure on its Lie algebra $\mathfrak{g} = T_eG$, namely an $\mathbb{R}$-linear map $J : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $J^2 = -Id_{\mathfrak{g}}$.*

Given a left-invariant almost complex structure J on a Lie group G, one defines the space of **left-invariant** $(1,\,0)$**-forms** on G as the subspace

$$\Lambda^{1,\,0} := \{\alpha - iJ\alpha \;\mid\; \alpha \in \mathfrak{g}^\star\} \subset \mathfrak{g}^\star_{\mathbb{C}} \tag{8.6}$$

of the complexification $\mathfrak{g}^\star_{\mathbb{C}}$ of $\mathfrak{g}^\star$. One then defines the space of **left-invariant** $(0,\,1)$**-forms** on G by

$$\Lambda^{0,\,1} := \overline{\Lambda^{1,\,0}}$$

and, more generally, the spaces $\Lambda^{p,\,q}$ of **left-invariant** $(p,\,q)$**-forms** on G as subspaces of $\Lambda^{p+q}\mathfrak{g}^\star_{\mathbb{C}}$.

Theorem and Definition 8.2.2. *Let J be a left-invariant almost complex structure on a Lie group G.*

(i) The following equivalences hold:

$$\begin{aligned} J \textit{ is } \textbf{integrable} \;\; &\overset{(a)}{\Longleftrightarrow} \;\; [JX,\,JY] = [X,\,Y] + J\,[JX,\,Y] + J\,[X,\,JY], \qquad \forall X, Y \in \mathfrak{g} \\ &\overset{(b)}{\Longleftrightarrow} \;\; d(\Lambda^{1,\,0}) \subset \Lambda^{2,\,0} \oplus \Lambda^{1,\,1}. \end{aligned}$$

In this case, we say that J is a left-invariant **complex structure** *on G.*

(ii) One defines the space $\mathfrak{g}^{0,\,1}$ of **left-invariant vector fields of type** $(0,\,1)$ *as the annihilator of $\Lambda^{1,\,0}$:*

$$\mathfrak{g}^{0,\,1} := (\Lambda^{1,\,0})^o = (\Lambda^{0,\,1})^\star.$$

The space $\mathfrak{g}^{1,0}$ *of* **left-invariant vector fields of type** $(1, 0)$ *is then defined by conjugating* $\mathfrak{g}^{0,1}$*:*

$$\mathfrak{g}^{1,0} := \overline{\mathfrak{g}^{0,1}}.$$

One has:

$$\mathfrak{g}^{1,0} := \{X - iJX \mid X \in \mathfrak{g}\} \subset \mathfrak{g}_{\mathbb{C}}, \tag{8.7}$$

where $\mathfrak{g}_{\mathbb{C}}$ *is the complexification of* $\mathfrak{g}$*.*

(iii) If J *is* **integrable**, $\mathfrak{g}^{0,1}$ *has the structure of a* **complex** *Lie algebra.*

Equivalence (a) in (i) of Theorem and Definition 8.2.2 is the definition of integrability, while equivalence (b) expresses the fact that J is integrable if and only if the operator d splits as $d = \partial + \bar\partial$, with ∂ of type $(1, 0)$ and $\bar\partial$ of type $(0, 1)$. Meanwhile, (8.7) follows from (8.6) and from (ii).

(I) Two opposite types of extreme left-invariant complex structures on a Lie group

(1) The first of these is described in the following

Theorem and Definition 8.2.3. *Let* J *be a left-invariant complex structure on a Lie group* G*.*
The following equivalences hold:

$$\begin{aligned} d(\Lambda^{1,0}) \subset \Lambda^{2,0} \quad &\iff \quad J[X, Y] = [JX, Y] \quad \forall X, Y \in \mathfrak{g} \\ &\iff \quad \mathfrak{g} \text{ is the real Lie algebra underlying } \mathfrak{g}^{0,1}. \end{aligned}$$

A left-invariant complex structure J *that satisfies the above equivalent conditions is said to be* **complex parallelisable**. *In this case,* $\mathfrak{g}$ *is a* **complex Lie algebra**.

Note that the condition $J[X, Y] = [JX, Y]$ for all $X, Y \in \mathfrak{g}$ (which is equivalent to $J[X, Y] = [X, JY]$ for all $X, Y \in \mathfrak{g}$ thanks to $[\cdot\, ,\, \cdot]$ being alternating) expresses the $\mathbb{C}$-bilinearity of the Lie bracket $[\cdot\, ,\, \cdot]$. Also note that, when $\mathfrak{g}$ is a *complex* Lie algebra, G is a *complex* Lie group, so the group operation is *holomorphic*. Thus, any left-invariant form in $\Lambda^{1,0}$ is *holomorphic*. In particular, we have the implications:

$$u \in \Lambda^{1,0} \implies \bar\partial u = 0 \implies du = \partial u \in \Lambda^{2,0}.$$

Proof of Theorem and Definition 8.2.3. It remains to prove the first equivalence in the statement.
For any $\sigma \in \Lambda^{1,0}$, the following equivalence holds:

$$d\sigma \in \Lambda^{2,0} \iff (d\sigma)(Z, \overline{W}) = 0 \quad \forall Z, W \in \mathfrak{g}^{1,0}.$$

On the other hand, from Cartan's formula (8.4) we get

$$(d\sigma)(Z, \overline{W}) = -\sigma([Z, \overline{W}]) \quad \forall Z, W \in \mathfrak{g}^{1,0},$$

since $\sigma(\overline{W})$ and $\sigma(Z)$ are constant, so their derivatives (including $Z.\sigma(\overline{W})$ and $\overline{W}.\sigma(Z)$) vanish.
We conclude that

$$d(\Lambda^{1,0}) \subset \Lambda^{2,0} \iff [Z, \overline{W}] = 0 \quad \forall Z, W \in \mathfrak{g}^{1,0}.$$

Now, writing $Z = X - iJX$ and $W = Y - iJY$ with $X, Y \in \mathfrak{g}$ (see (8.7)), the above equivalence translates to the first equivalence below:

$$\begin{aligned} d(\Lambda^{1,0}) \subset \Lambda^{2,0} \quad & \iff \quad [X - iJX,\, Y + iJY] = 0 \quad \forall X, Y \in \mathfrak{g} \\ & \iff \quad ([X,\, Y] + [JX,\, JY]) + i\,([X,\, JY] - [JX,\, Y]) = 0 \quad \forall X, Y \in \mathfrak{g} \\ & \iff \quad [X,\, Y] = -[JX,\, JY] \quad \forall X, Y \in \mathfrak{g} \\ (& \overset{(a)}{\iff} \quad [X,\, JY] = [JX,\, Y] \quad \forall X, Y \in \mathfrak{g}) \\ & \overset{(b)}{\iff} \quad -[X,\, Y] = [X,\, Y] + J[JX,\, Y] + J[X,\, JY] \quad \forall X, Y \in \mathfrak{g} \\ & \overset{(c)}{\iff} \quad -2[X,\, Y] = 2J[JX,\, Y] \quad \forall X, Y \in \mathfrak{g} \\ & \overset{(d)}{\iff} \quad J[X,\, Y] = [JX,\, Y] \quad \forall X, Y \in \mathfrak{g}, \end{aligned}$$

where (a) follows by replacing Y with JY in the previous equivalence and using $J^2 = -1$, (b) follows from the integrability condition (a) of (i) of Theorem and Definition 8.2.2, (c) follows from (b) and (a), while (d) follows by applying the isomorphism J to (c).

The first equivalence in the statement is proved. $\square$

Finally, let us mention the following addition to Wang's Theorem 4.5.30. It relates the notion of *complex parallelisable* for compact complex manifolds (in the sense of Definition 4.5.29) and for left-invariant complex structures on a Lie group (in the sense of Theorem and Definition 8.2.3).

Theorem 8.2.4. *([Wan54]) Let $X = G/\Gamma$ be a compact complex manifold defined as the quotient of a simply connected, connected Lie group G by a discrete subgroup $\Gamma \subset G$.*

Then, $X = G/\Gamma$ is **complex parallelisable** *if and only if the complex structure of X is induced by a* **complex parallelisable** *left-invariant complex structure on G.*

Proof. See [Wan54]. The proof is actually implicit in that of Theorem and Definition 8.2.3). $\square$

(2) The other type of extreme left-invariant complex structure on a Lie group is described in the following

Theorem and Definition 8.2.5. *Let J be a left-invariant complex structure on a Lie group G.*

The following equivalence holds:

$$\begin{aligned} d(\Lambda^{1,0}) \subset \Lambda^{1,1} \quad & \iff \quad [JX,\, JY] = [X,\, Y], \quad \forall X, Y \in \mathfrak{g} \\ & \iff \quad \mathfrak{g}^{0,1} \text{ is an abelian Lie algebra.} \end{aligned}$$

A left-invariant complex structure J that satisfies the above equivalent conditions is said to be **abelian**.

Proof. • We start by proving the first equivalence. For any $\sigma \in \Lambda^{1,0}$, the following equivalence holds:

$$d\sigma \in \Lambda^{1,1} \iff (d\sigma)(Z,\, W) = 0 \quad \forall Z, W \in \mathfrak{g}^{1,0}.$$

On the other hand, from Cartan's formula (8.4) we get

$$(d\sigma)(Z,\, W) = -\sigma([Z,\, W]) \quad \forall Z, W \in \mathfrak{g}^{1,0},$$

since $\sigma(W)$ and $\sigma(Z)$ are constant, so their derivatives (including $Z.\sigma(W)$ and $W.\sigma(Z)$) vanish.

We conclude that

$$d(\Lambda^{1,0}) \subset \Lambda^{1,1} \iff [Z, W] = 0 \quad \forall Z, W \in \mathfrak{g}^{1,0}.$$

Now, writing $Z = X - iJX$ and $W = Y - iJY$ with $X, Y \in \mathfrak{g}$ (see (8.7)), the above equivalence translates to the first equivalence below:

$$\begin{aligned} d(\Lambda^{1,0}) \subset \Lambda^{1,1} \quad &\iff \quad [X - iJX,\, Y - iJY] = 0 \quad \forall X, Y \in \mathfrak{g} \\ &\iff \quad ([X,\, Y] - [JX,\, JY]) - i\,([X,\, JY] + [JX,\, Y]) = 0 \quad \forall X, Y \in \mathfrak{g} \\ &\iff \quad [X,\, Y] = [JX,\, JY] \quad \forall X, Y \in \mathfrak{g} \\ (&\overset{(a)}{\iff} \quad [X,\, JY] = -[JX,\, Y] \quad \forall X, Y \in \mathfrak{g}), \end{aligned}$$

where (a) follows by replacing Y with JY in the previous equivalence and using $J^2 = -1$.

The first equivalence in the statement is proved.

- We now prove the second equivalence. We have the following equivalences:

$$\begin{aligned} \mathfrak{g}^{0,1} \text{ is abelian} \quad &\iff \quad \mathfrak{g}^{1,0} \text{ is abelian} \overset{(a)}{\iff} [X - iJX,\, Y - iJY] = 0 \text{ for all } X, Y \in \mathfrak{g} \\ &\iff \quad ([X,\, Y] - [JX,\, JY]) - i\,([X,\, JY] + [JX,\, Y]) = 0 \text{ for all } X, Y \in \mathfrak{g} \\ &\iff \quad [X,\, Y] = [JX,\, JY] \text{ and } [X,\, JY] = -[JX,\, Y] \text{ for all } X, Y \in \mathfrak{g} \\ &\overset{(b)}{\iff} \quad [X,\, Y] = [JX,\, JY] \text{ for all } X, Y \in \mathfrak{g}, \end{aligned}$$

where (a) follows from (8.7) and (b) follows from the obvious equivalence:

$$[X,\, Y] = [JX,\, JY] \text{ for all } X, Y \in \mathfrak{g} \iff [X,\, JY] = -[JX,\, Y] \text{ for all } X, Y \in \mathfrak{g}.$$

□

(II) Existence and classification of complex structures on NLA's

We will now use the following notation. If $\omega^1, \dots, \omega^i \in \mathfrak{g}^\star_{\mathbb{C}}$, the **ideal** $\mathcal{I}(\omega^1, \dots, \omega^i)$ in $\Lambda^\bullet \mathfrak{g}^\star_{\mathbb{C}}$ generated by $\omega^1, \dots, \omega^i$ consists of the 2-forms of the shape

$$\sum_{\substack{1 \le j \le i \\ k}} A_{jk}\, \omega^j \wedge \alpha_k, \tag{8.8}$$

with arbitrary 1-forms $\alpha_k \in \mathfrak{g}^\star_{\mathbb{C}}$ and constants $A_{jk} \in \mathbb{C}$. Thus, $\mathcal{I}(\omega^1, \dots, \omega^i) = \langle \omega^1, \dots, \omega^i \rangle \wedge \mathfrak{g}^\star_{\mathbb{C}}$.

Theorem 8.2.6. *([Sal01, Theorem 1.3]) A* **nilpotent** *Lie group G admits a* **left-invariant complex structure** *if and only if $\mathfrak{g}^\star_{\mathbb{C}}$ has a basis $\{\omega^1, \dots, \omega^n, \overline{\omega}^1, \dots, \overline{\omega}^n\}$ such that*

$$d\omega^{i+1} \in \mathcal{I}(\omega^1, \dots, \omega^i), \qquad i = 0, 1, \dots, n-1, \tag{8.9}$$

where $\mathcal{I}(\omega^1, \dots, \omega^i)$ is the ideal in $\Lambda^\bullet \mathfrak{g}^\star_{\mathbb{C}}$ generated by $\omega^1, \dots, \omega^i$ and is interpreted as zero when $i = 0$.

In this case, $\Lambda^{1,0}$ is the span of $\omega^1, \dots, \omega^n$.

In particular, when G admits a left-invariant complex structure, we have:

(a) $d\omega^1 = 0$, so there exists a non-zero d**-closed** $(1, 0)$**-form** *$\omega^1 \in \mathfrak{g}^\star_{\mathbb{C}}$;*

(b) the $(n, 0)$-form $\omega^1 \wedge ... \wedge \omega^n$ is d**-closed**.

The last statement in (b) follows at once from (8.9) since the latter implies that $(d\omega^{i+1}) \wedge \omega^1 \wedge \ldots \wedge \omega^i = 0$ for every $i = 0, 1, \ldots, n-1$.

Proof of Theorem 8.2.6. "$\Longleftarrow$" Any basis $\{\omega^1, \ldots, \omega^n, \overline{\omega}^1, \ldots, \overline{\omega}^n\}$ of $\mathfrak{g}^\star_{\mathbb{C}}$ determines an almost-complex structure J on G by decreeing $\Lambda^{1,0}$ to be the span of $\omega^1, \ldots, \omega^n$. The integrability of J then follows from condition (8.9).

"$\Longrightarrow$" Suppose that G admits a left-invariant complex structure J. Then, (i) of Theorem and Definition 8.2.2 gives:

$$d(\Lambda^{1,0}) \subset \Lambda^{2,0} \oplus \Lambda^{1,1} \tag{8.10}$$

On the other hand, let

$$V_i^{1,0} := (V_i)_{\mathbb{C}} \cap \Lambda^{1,0}, \qquad 0 \leq i \leq s,$$

where, for each i, $(V_i)_{\mathbb{C}}$ is the complexification of the vector subspace $V_i \subset \mathfrak{g}^\star$ of Corollary 8.1.5.

We construct the ω^i's inductively by successively extending a basis of each $V_j^{1,0}$ to one of $V_{j+1}^{1,0}$. Suppose we have constructed $\omega^1, \ldots, \omega^i$ linearly independent such that

$$d\omega^l \in \mathcal{I}(\omega^1, \ldots, \omega^{l-1}), \qquad l = 1, \ldots, i.$$

Let j be the least positive integer (depending on i) such that $\omega^1, \ldots, \omega^i \in V_j^{1,0}$. We may assume that $\{\omega^1, \ldots, \omega^i\}$ is a basis of $V_j^{1,0}$. Then[1], $(V_j)_{\mathbb{C}}$ is generated by $\omega^1, \ldots, \omega^i, \overline{\omega}^1, \ldots, \overline{\omega}^i$ and, possibly, for some positive integer p, by p other linearly independent 1-forms of the shape:

$$\sigma^1 + \rho^1, \ldots, \sigma^p + \rho^p,$$

where the σ^l's are of type (1, 0) and the ρ^l's are of type (0, 1).

Then, $\Lambda^2(V_j)_{\mathbb{C}}$ is generated by the wedge products of two elements of the above basis, namely by the 2-forms of the following kinds:

(1) $\omega^k \wedge \omega^l, \qquad k, l \in \{1, \ldots, i\};$

(2) $\omega^k \wedge \overline{\omega}^l, \qquad k, l \in \{1, \ldots, i\};$

(3) $\omega^k \wedge (\sigma^r + \rho^r), \qquad k \in \{1, \ldots, i\},\ r \in \{1, \ldots, p\};$

(4) $\overline{\omega}^k \wedge \overline{\omega}^l, \qquad k, l \in \{1, \ldots, i\};$

(5) $\overline{\omega}^k \wedge (\sigma^r + \rho^r), \qquad k \in \{1, \ldots, i\},\ r \in \{1, \ldots, p\};$

(6) $(\sigma^r + \rho^r) \wedge (\sigma^s + \rho^s), \qquad r, s \in \{1, \ldots, p\}.$

Let $\omega^{i+1} \in V_{j+1}^{1,0}$ such that $\{\omega^1, \ldots, \omega^i, \omega^{i+1}\}$ is linearly independent. Since J is integrable, (8.10) implies that $d\omega^{i+1}$ is a linear combination of forms of the above types (1), (2) and (3) only. This amounts to $d\omega^{i+1} \in \langle \omega^1, \ldots, \omega^i \rangle \wedge \mathfrak{g}^\star_{\mathbb{C}} = \mathcal{I}(\omega^1, \ldots, \omega^i)$. □

[1]The remaining part of the argument starting here was pointed out to the author by L. Ugarte as an addition to the proof given in [Sal01, Theorem 1.3].

Definition 8.2.7. *([CFGU00]) Let J be a complex structure on a nilpotent Lie algebra (NLA) $\mathfrak{g}$ of real dimension $2n$. One says that J is* **nilpotent** *if $\Lambda^{1,0}$ has a $\mathbb{C}$-basis $\{\omega^1, \ldots, \omega^n\}$ such that*

$$d\omega^{i+1} \in \Lambda^2\langle\omega^1, \ldots, \omega^i, \overline{\omega}^1, \ldots, \overline{\omega}^i\rangle, \qquad i = 0, 1, \ldots, n-1, \tag{8.11}$$

where the right-hand side is interpreted as zero when $i = 0$.

Explicitly, this means that $d\omega^1 = 0$ and

$$d\omega^{i+1} = \sum_{j<k\leq i} A_{ijk}\, \omega^j \wedge \omega^k + \sum_{j,\, k\leq i} B_{ijk}\, \omega^j \wedge \overline{\omega}^k \qquad i = 0, 1, \ldots, n-1, \tag{8.12}$$

where $A_{ijk}, B_{ijk} \in \mathbb{C}$.

One has the obvious

Observation 8.2.8. *Let J be a complex structure on a nilpotent Lie algebra (NLA) $\mathfrak{g}$ of real dimension $2n$. If J is either complex parallelisable or abelian, then J is nilpotent. Specifically:*

(i) J is **complex parallelisable** *if and only if $\Lambda^{1,0}$ has a $\mathbb{C}$-basis $\{\omega^1, \ldots, \omega^n\}$ such that identities (8.12) are satisfied with $B_{ijk} = 0$ for all $1 \leq j, k \leq i \leq n$;*

(ii) J is **abelian** *if and only if $\Lambda^{1,0}$ has a $\mathbb{C}$-basis $\{\omega^1, \ldots, \omega^n\}$ such that identities (8.12) are satisfied with $A_{ijk} = 0$ for all $1 \leq j < k \leq i \leq n$.*

Using Notation 8.1.6, we now cite the following result of Salamon's (also reproduced as Theorem 2.1. in [COUV]) classifying 6-dimensional NLA's in terms of the types of complex structures they admit.

Theorem 8.2.9. *([Uga07, Theorem 8]) Let $\mathfrak{g}$ be an NLA of real dimension 6.*

(I) There exists a **complex structure** *on $\mathfrak{g}$ if and only if $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:*

$\mathfrak{h}_1 = (0,\ 0,\ 0,\ 0,\ 0,\ 0)$,
$\mathfrak{h}_2 = (0,\ 0,\ 0,\ 0,\ 12,\ 34)$,
$\mathfrak{h}_3 = (0,\ 0,\ 0,\ 0,\ 0,\ 12+34)$,
$\mathfrak{h}_4 = (0,\ 0,\ 0,\ 0,\ 12,\ 14+23)$,
$\mathfrak{h}_5 = (0,\ 0,\ 0,\ 0,\ 13+42,\ 14+23)$,
$\mathfrak{h}_6 = (0,\ 0,\ 0,\ 0,\ 12,\ 13)$,
$\mathfrak{h}_7 = (0,\ 0,\ 0,\ 12,\ 13,\ 23)$,
$\mathfrak{h}_8 = (0,\ 0,\ 0,\ 0,\ 0,\ 12)$,
$\mathfrak{h}_9 = (0,\ 0,\ 0,\ 0,\ 12,\ 14+25)$,
$\mathfrak{h}_{10} = (0,\ 0,\ 0,\ 12,\ 13,\ 14)$,
$\mathfrak{h}_{11} = (0,\ 0,\ 0,\ 12,\ 13,\ 14+23)$,
$\mathfrak{h}_{12} = (0,\ 0,\ 0,\ 12,\ 13,\ 24)$,
$\mathfrak{h}_{13} = (0,\ 0,\ 0,\ 12,\ 13+14,\ 24)$,
$\mathfrak{h}_{14} = (0,\ 0,\ 0,\ 12,\ 14,\ 13+42)$,
$\mathfrak{h}_{15} = (0,\ 0,\ 0,\ 12,\ 13+42,\ 14+23)$,
$\mathfrak{h}_{16} = (0,\ 0,\ 0,\ 12,\ 14,\ 24)$,
$\mathfrak{h}_{19}^- = (0,\ 0,\ 0,\ 12,\ 23,\ 14-35)$,
$\mathfrak{h}_{26}^+ = (0,\ 0,\ 12,\ 13,\ 23,\ 14+25)$.

(II) From the point of view of **nilpotency***, the complex structures on the above NLA's satisfy the following dichotomy.*

(a) Any complex structure on $\mathfrak{h}_{19}^-$ and on $\mathfrak{h}_{26}^+$ is **non-nilpotent***.*

(b) For every $1 \leq k \leq 16$, any complex structure on $\mathfrak{h}_k$ is **nilpotent***.*

(III) From the point of view of **abelianity***, the nilpotent complex structures on the above NLA's are of the following types.*

(i) Any complex structure on $\mathfrak{h}_1$, $\mathfrak{h}_3$, $\mathfrak{h}_8$ and $\mathfrak{h}_9$ is **abelian***.*

(ii) There exist both **abelian** *and* **non-abelian nilpotent** *complex structures on* $\mathfrak{h}_2$, $\mathfrak{h}_4$, $\mathfrak{h}_5$ *and* $\mathfrak{h}_{15}$.

(iii) Any complex structure on $\mathfrak{h}_6$, $\mathfrak{h}_7$, $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$, $\mathfrak{h}_{12}$, $\mathfrak{h}_{13}$, $\mathfrak{h}_{14}$ *and* $\mathfrak{h}_{16}$ *is* **non-abelian nilpotent**.

Of particular interest to us is the following application of the above classification of complex structures on NLA's, given by Ceballos, Otal, Ugarte and Villacampa, showing, in particular, that deformation limits of **balanced** manifolds need not even be **strongly Gauduchon**, let alone balanced.

Theorem 8.2.10. *([COUV16, Theorem 5.9.]) Let* $B = \{t \in \mathbb{C} \mid |t| < 1\}$ *be the open unit disc in the complex plane. There exists a holomorphic family* $(X, J_t)_{t\in B}$ *of compact complex manifolds such that* (X, J_t) *is* **balanced** *for every* $t \in B \setminus \{0\}$, *but* (X, J_0) *does not admit any* **strongly Gauduchon metric**.

In particular, the **balanced** *and* **sG** *properties of compact complex manifolds are* **not closed** *under holomorphic deformations of complex structures.*

Sketch of proof (according to [COUV16]). Let $X = G/\Gamma$ be a **nilmanifold** (i.e. G is a simply connected **nilpotent** real Lie group and $\Gamma \subset G$ is a discrete co-compact subgroup) whose underlying Lie algebra is $\mathfrak{h}_4 = (0, 0, 0, 0, 12, 14 + 23)$ (i.e. $\mathfrak{h}_4$ is the Lie algebra of G). Let J_0 be the **abelian** complex structure on $\mathfrak{h}_4$.

There exists a basis $\{\eta^1, \eta^2, \eta^3\}$ of $(1, 0)$-forms for J_0 (i.e. a $\mathbb{C}$-basis of $\Lambda^{1,0}_{J_0}$) satisfying the structure equations:

$$d\eta^1 = d\eta^2 = 0 \quad \text{and} \quad d\eta^3 = \frac{i}{2}\,\eta^1\wedge\bar\eta^1 + \frac{1}{2}\,\eta^1\wedge\bar\eta^2 + \frac{1}{2}\,\eta^2\wedge\bar\eta^1.$$

Thanks to results of Maclaughlin-Pedersen-Poon-Salamon (2006), the Kuranishi family of J_0 consists entirely of invariant complex structures that can be completely described in terms of the invariant forms η^1, η^2, η^3, $\bar\eta^1$, $\bar\eta^2$, $\bar\eta^3$ as follows. Any complex structure J_Φ on X sufficiently close to J_0 has a basis $\{\mu^1_\Phi, \mu^2_\Phi, \mu^3_\Phi\}$ of $(1, 0)$-forms such that:

$$\begin{aligned}\mu^1_\Phi &= \eta^1 + \Phi^1_1\,\bar\eta^1 + \Phi^1_2\,\bar\eta^2\\ \mu^2_\Phi &= \eta^2 + \Phi^2_1\,\bar\eta^1 + \Phi^2_2\,\bar\eta^2\\ \mu^3_\Phi &= \eta^3 + \Phi^3_3\,\bar\eta^3,\end{aligned}\tag{8.13}$$

where the coefficients $\Phi^j_k \in \mathbb{R}$ are sufficiently small and satisfy the condition:

$$i(1+\Phi^3_3)\,\Phi^1_2 = (1-\Phi^3_3)(\Phi^1_1 - \Phi^2_2).\tag{8.14}$$

Moreover, the deformed complex structure remains abelian if and only if $\Phi^1_2 = 0$ and $\Phi^1_1 = \Phi^2_2$.

To exhibit a particular holomorphic family of deformations of J_0 that are **not abelian** but have **balanced metrics**, one considers, for every $t \in B \setminus \{0\}$ sufficiently close to 0, the following linearly independent family of complex-valued 1-forms on X:

$$\mu^1_t := \eta^1 + t\,\bar\eta^1 - it\,\bar\eta^2, \qquad \mu^2_t := \eta^2, \qquad \mu^3_t := \eta^3.$$

This choice of 1-forms corresponds to the choice of coefficients $\Phi^1_1 = t$, $\Phi^1_2 = -it$, $\Phi^2_1 = \Phi^2_2 = \Phi^3_3 = 0$ in (8.13). One immediately checks that these coefficients satisfy condition (8.14). The linearly independent family $\{\mu^1_t, \mu^2_t, \mu^3_t\}$ of 1-forms defines an invariant complex structure J_t on X

by decreeing that the forms $\mu_t^1,\,\mu_t^2,\,\mu_t^3$ are of type $(1,\,0)$ for J_t (i.e. by decreeing $\Lambda^{1,0}_{J_t}$ to be the $\mathbb{C}$-span of $\mu_t^1,\,\mu_t^2,\,\mu_t^3$).

Now, a straightforward computation shows that the structure equations for J_t in this basis are:

$$d\mu_t^1 = d\mu_t^2 = 0, \quad 2(1-|t|^2)\,d\mu_t^3 = 2\bar{t}\,\mu_t^{12} + i\,\mu_t^{1\bar{1}} + \mu_t^{1\bar{2}} + \mu_t^{2\bar{1}} - i|t|^2\,\mu_t^{2\bar{2}}, \qquad t\in B. \tag{8.15}$$

(Here, as elsewhere, μ_t^{jk} stands for $\mu_t^j\wedge\mu_t^k$ and $\mu_t^{j\bar{k}}$ stands for $\mu_t^j\wedge\bar{\mu}_t^k$.)

By certain results from [COUV16], the authors conclude that the complex manifold $(X,\,J_0)$ is **not sG** because, on the one hand, J_0 being abelian implies the equivalence: $(X,\,J_0)$ is sG $\iff$ $(X,\,J_0)$ is balanced, while on the other hand, $(X,\,J_0)$ is seen to not be balanced by an explicit computation.

Meanwhile, for every $t\in B\setminus\{0\}$, the complex structure J_t is nilpotent but not abelian. One can normalise the coefficient of μ_t^{12} in (8.15) by substituting $(1-|t|^2)/\bar{t}\,\mu_t^3$ for μ_t^3. Then, one goes on to replace the $(1,\,0)$-basis $\{\mu_t^1,\,\mu_t^2,\,\mu_t^3\}$ with the $(1,\,0)$-basis

$$\{\tau_t^1 := \mu_t^1 - i\mu_t^2,\ \tau_t^2 := -2\bar{t}i\,\mu_t^2,\ \tau_t^3 := -2\bar{t}i\,\mu_t^3\}.$$

The structure equations for J_t in this new basis are:

$$d\tau_t^1 = d\tau_t^2 = 0, \quad d\tau_t^3 = \tau_t^{12} + \tau_t^{1\bar{1}} - \frac{1}{t}\,\tau_t^{1\bar{2}} + \frac{1-|t|^2}{4|t|^2}\,\tau_t^{2\bar{2}}, \qquad t\in B. \tag{8.16}$$

Finally, the proof of another result from [COUV16] shows that, for any $t\in B\setminus\{0\}$, $(X,\,J_t)$ admits balanced metrics if and only if

$$\frac{1}{|t|^2}\left(\frac{1}{|t|^2} - 4\,\frac{1-|t|^2}{4|t|^2}\right) = \frac{1}{|t|^2} > 0.$$

The conclusion is that $(X,\,J_t)$ is a **balanced manifold** for every $t\in\mathbb{C}$ such that $0<|t|<1$. $\square$

Salamon's Theorem 8.2.6 is also used in the proof of the following result of Fino, Parton and Salamon which shows, in particular, that on a 6-dimensional nilmanifold equipped with an invariant complex structure, the SKT condition depends only on the underlying Lie algebra and the complex structure . This is in stark contrast with the sG property studied in [COUV16].

Theorem 8.2.11. *([FPS02, Theorem 1.2]) Let $X=G/\Gamma$ be a* **nilmanifold** *with $\dim_{\mathbb{R}}X=6$ and let J be a* **left-invariant complex structure** *on X. Then, either every invariant Hermitian metric ω on $(X,\,J)$ is* **SKT** *or none is.*

Moreover, $(X,\,J)$ is an **SKT manifold** *if and only if J has a basis $\{\omega^1,\,\omega^2,\,\omega^3\}$ of $(1,\,0)$-forms such that*

$$\begin{aligned} d\omega^1 &= 0\\ d\omega^2 &= 0\\ d\omega^3 &= A\,\omega^{\bar{1}2} + B\,\omega^{\bar{2}2} + C\,\omega^{1\bar{1}} + D\,\omega^{1\bar{2}} + E\,\omega^{12}, \end{aligned} \tag{8.17}$$

where the constants $A,B,C,D,E\in\mathbb{C}$ satisfy the condition

$$|A|^2 + |D|^2 + |E|^2 + 2\,Re\,(\bar{B}\,C) = 0. \tag{8.18}$$

Proof. See [FPS02, §.1 and §.2]. $\square$

The following result of Ugarte's is a refinement of Theorem 8.2.11.

Theorem 8.2.12. *([Uga07, Theorem 19]) In the setting of Theorem 8.2.11, (X, J) is an* **SKT manifold** *if and only if J has a basis $\{\omega^1, \omega^2, \omega^3\}$ of $(1, 0)$-forms such that*

$$\begin{aligned} d\omega^1 &= 0 \\ d\omega^2 &= 0 \\ d\omega^3 &= \rho\,\omega^{12} + \omega^{1\bar{1}} + B\,\omega^{\bar{1}2} + D\,\omega^{2\bar{2}}, \end{aligned} \tag{8.19}$$

where the constants $\rho, B, D \in \mathbb{C}$ satisfy the conditions:

$$\rho \in \{0,\, 1\} \quad \text{and} \quad \rho + |B|^2 = 2\,Re\,(D). \tag{8.20}$$

Proof. See [Uga07, Theorem 19]. □

Note that the third equation in (8.17) implies that

$$d\omega^3 \in \Lambda^2\langle\omega^1,\, \overline{\omega}^1,\, \omega^2,\, \overline{\omega}^2\rangle,$$

so the complex structure J is **nilpotent** in the sense of Definition 8.2.7.

8.3 Adjoint representations and solvable Lie algebras

The material in this section is classical and was gleaned from various well-known sources, such as Serre's classical book [Ser64]. It is included here to provide the reader with a quick rundown on some basic facts. Most of the proofs will be skipped and the reader referred to standard sources.

8.3.1 The adjoint representation of a Lie group

The basic notion here is the following

Definition 8.3.1. *Let G be a Lie group.*

(i) For every $a \in G$, the **inner automorphism** *induced by a is the map:*

$$Int\,a : G \longrightarrow G, \qquad x \mapsto axa^{-1}.$$

(ii) The **adjoint representation** *of G is the linear representation $Ad : G \longrightarrow End\,(\mathfrak{g})$ of G on its Lie algebra $\mathfrak{g}$ defined as:*

$$G \ni a \mapsto Ad_a := d_e(Int\,a) \in End\,(\mathfrak{g}),$$

where $d_e(Int\,a)$ is the differential at the identity element e of G of $Int\,a$.

Note that $(\mathrm{Int}\,a)(e) = e$, so the differential $d_e(\mathrm{Int}\,a) : T_eG \longrightarrow T_eG$ lies in $\mathrm{End}\,(\mathfrak{g})$ when $\mathfrak{g}$ is identified with the tangent space T_eG to G at e. Note also that

$$\mathrm{Int}\,a = R_{a^{-1}} \circ L_a, \tag{8.21}$$

where $L_g : G \longrightarrow G$, resp. $R_g : G \longrightarrow G$, is the left, resp. right, translation by a given element $g \in G$: $L_g(x) = gx$, resp. $R_g(x) = xg$, for every $x \in G$.

Some basic properties of the adjoint representation of a Lie group are given in the following

Proposition 8.3.2. *Let G be a Lie group.*

(i) If G is a group of **isomorphisms** *of a vector space V, namely $G \subset GL(V)$, then for every $a \in G$, the map $Ad_a : \mathfrak{g} \longrightarrow \mathfrak{g}$ is given by*

$$Ad_a(X) = aXa^{-1}, \qquad X \in T_eG = \mathfrak{g} \subset End(V).$$

(ii) If $Z(G)$ is the centre of G, one has the inclusion:

$$Z(G) \subset \ker(Ad).$$

(iii) If G is connected and the ground field has characteristic zero, then

$$Z(G) = \ker(Ad).$$

Proof. It follows at once from the definitions and from the obvious equivalence:

$$a \in Z(G) \iff \operatorname{Int} a = \operatorname{Id}_G.$$

□

Recall that, when V is finite dimensional, End (V) is the Lie algebra of $GL(V)$ and its Lie bracket is given by $[T, S] := T \circ S - S \circ T$ for all $S, T \in \operatorname{End}(V)$.

8.3.2 The adjoint representation of a Lie algebra

The basic notion here is the following

Definition 8.3.3. *The* **adjoint representation** *of a Lie algebra $\mathfrak{g}$ is the linear representation $ad : \mathfrak{g} \longrightarrow End(\mathfrak{g})$ of $\mathfrak{g}$ on the module $\mathfrak{g}$ defined as:*

$$\mathfrak{g} \ni x \longmapsto ad_x := [x, \cdot] : \mathfrak{g} \longrightarrow \mathfrak{g},$$

where $[\cdot\ ,\ \cdot]$ is the Lie bracket of $\mathfrak{g}$.

Recall that a **derivation** on a Lie algebra $(\mathfrak{g}, [\cdot\ ,\ \cdot])$ is a linear map $D : \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying the **Leibniz rule**:

$$D([x, y]) = [Dx, y] + [x, Dy], \qquad x, y \in \mathfrak{g}.$$

The set Der $(\mathfrak{g})$ of all derivations on $\mathfrak{g}$ is a Lie algebra with the Lie bracket:

$$[D, D'] := DD' - D'D, \qquad D, D' \in \operatorname{Der}(\mathfrak{g}).$$

Some basic properties of the adjoint representation of a Lie group are given in the following

Proposition 8.3.4. *Let $\mathfrak{g}$ be a Lie algebra.*

(i) If V is a finite dimensional vector space and $\mathfrak{g} = End(V)$ is the Lie algebra of $GL(V)$, then

$$\ker(ad) = Z(\mathfrak{g})$$

(the centre of $\mathfrak{g}$).

(ii) For every $x \in \mathfrak{g}$, $ad_x : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a **derivation** *on $\mathfrak{g}$ (called the* **inner derivation** *induced by x).*

(iii) The map

$$\mathfrak{g} \ni x \overset{ad}{\longmapsto} ad_x \in Der(\mathfrak{g})$$

is a **Lie algebra homomorphism**, *namely*

$$ad_{[x,y]} = [ad_x, ad_y], \qquad x, y \in \mathfrak{g}.$$

Proof. (i) is obvious, while both (ii) and (iii) are equivalent to the Jacobi identity satisfied by the Lie bracket $[\cdot\,,\,\cdot]$. $\square$

Definition 8.3.5. *Let $\mathfrak{g}$ be a Lie algebra. The image $ad(\mathfrak{g})$ of the map $\mathfrak{g} \ni x \overset{ad}{\longmapsto} ad_x \in Der(\mathfrak{g})$ is called the* **adjoint linear Lie algebra** *associated with $\mathfrak{g}$.*

We list two basic properties of the object defined above.

Proposition 8.3.6. *Let $\mathfrak{g}$ be a Lie algebra.*

(i) $ad(\mathfrak{g}) \subset Der(\mathfrak{g})$ is an **ideal**.

(ii) The quotient $Der(\mathfrak{g})/ad(\mathfrak{g})$ is the cohomology space $H^1(\mathfrak{g})$ of degree 1 of $\mathfrak{g}$. In particular, if the ground field is of characteristic zero, one has the equivalence:

$$ad(\mathfrak{g}) = Der(\mathfrak{g}) \iff \mathfrak{g} \text{ is } \textbf{semi-simple}.$$

Finally, the following result relates the adjoint representation of a Lie group G to the one of its Lie algebra $\mathfrak{g}$ viewed as the Lie algebra of **left-invariant** vector fields on G.

Proposition 8.3.7. *Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. Then, ad is the differential of Ad at the identity element e of G:*

$$ad = d_e(Ad) : \mathfrak{g} \longrightarrow End(\mathfrak{g}). \tag{8.22}$$

Moreover, $Der(\mathfrak{g})$ is the Lie algebra of the automorphism group $Aut(\mathfrak{g})$ of $\mathfrak{g}$.

Proof. Fix $a \in G$. From Definition 8.3.1, we get: $Ad_a = d_e(R_{a^{-1}} \circ L_a) = d_a(R_{a^{-1}}) \circ d_e(L_a)$. Hence, for every $Y \in \mathfrak{g}$ (i.e. for every left-invariant vector field Y on G), we get:

$$Ad_a(Y) = d_a(R_{a^{-1}})(Y_a) \in \mathfrak{g}, \tag{8.23}$$

where we have used the identity $d_e(L_a)(Y) = Y_a$ that follows from Y being left-invariant.

On the other hand, for any vector fields X, Y on G, their Lie bracket is given by the following standard formula:

$$[X,\, Y] = \lim_{t \to 0} \frac{1}{t} \Big(Y - (d\varphi_t)(Y) \Big) = \lim_{t \to 0} \frac{1}{t} \Big((d\varphi_{-t})(Y) - Y \Big), \tag{8.24}$$

where $\varphi_t : G \to G$ is the flow generated by X for $t \in (-\varepsilon,\, \varepsilon)$ and some small $\varepsilon > 0$.

Now, it turns out that $\varphi_t(a) = a\,\varphi_t(e) = R_{\varphi_t(e)}(a)$ for all $a \in G$ because both $\varphi_t(a)$ and $a\,\varphi_t(e)$ satisfy the same ODE defining the flow of X. This means that

$$\varphi_t = R_{\varphi_t(e)}, \qquad t \in (-\varepsilon,\, \varepsilon). \tag{8.25}$$

If $\varphi_t(e) = a$, then $a^{-1} = \varphi_{-t}(e)$, so (8.23) and (8.25) yield:

$$Ad_a(Y) = d_a(\varphi_{-t})(Y_a), \qquad \text{or equivalently} \qquad Ad_{\varphi_t(e)}(Y) = d_{\varphi_t(e)}(\varphi_{-t})(Y).$$

Together with (8.24), this gives:

$$[X,\, Y] = \lim_{t \to 0} \frac{1}{t} \Big(Ad_{\varphi_t(e)}(Y) - Y \Big) = d_e(Ad)(X)(Y),$$

for any left-invariant vector fields X, Y on G (i.e. $X, Y \in \mathfrak{g}$). $\square$

8.3.3 Construction of Lie algebras from known ones

The following reminder of classical constructions is taken from [Ser64, I(vi)].

Proposition and Definition 8.3.8. *(a) If $\mathfrak{g}$ is a Lie algebra and $J \subset \mathfrak{g}$ is an ideal, then the quotient $\mathfrak{g}/J$ is a Lie algebra.*

(b) If $(\mathfrak{g}_i)_{i\in I}$ is a family of Lie algebras, then $\Pi_{i\in I}\mathfrak{g}_i$ is a Lie algebra.

(c) Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{a} \subset \mathfrak{g}$ an ideal and $\mathfrak{b} \subset \mathfrak{g}$ a subalgebra.
We say that $\mathfrak{g}$ is a **semidirect product** *of $\mathfrak{b}$ by $\mathfrak{a}$ if the natural map*

$$\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{a}$$

induces an **isomorphism** $\mathfrak{b} \xrightarrow{\simeq} \mathfrak{g}/\mathfrak{a}$.

In other words, (c) requires the subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ to be realised as a quotient algebra of $\mathfrak{g}$. As for (a), for all $x, y \in \mathfrak{g}$ and every $a \in J$, we have: $[x + a,\, y] = [x,\, y] + [a,\, y]$ and $[a,\, y] \in J$, so $[x + a,\, y] = [x,\, y]$ modulo J. Thus, we may define the Lie bracket on $\mathfrak{g}/J$ as

$$[\hat{x},\, \hat{y}] := [x,\, y], \qquad \hat{x}, \hat{y} \in \mathfrak{g}/J.$$

8.3.4 Solvable Lie algebras

Most of this subsection is again taken from [Ser64], especially from chapter V, §.2-§.5.
Let $\mathfrak{g}$ be a Lie algebra.

Definition 8.3.9. *The* **derived series** *of $\mathfrak{g}$ is the inductively defined chain of ideals:*

$$\mathfrak{g}^{(0)} := \mathfrak{g} \supset \mathfrak{g}^{(1)} := [\mathfrak{g},\, \mathfrak{g}] \supset \mathfrak{g}^{(2)} := [\mathfrak{g}^{(1)},\, \mathfrak{g}^{(1)}] = [[\mathfrak{g},\, \mathfrak{g}],\, [\mathfrak{g},\, \mathfrak{g}]] \supset \cdots \supset \mathfrak{g}^{(n)} := [\mathfrak{g}^{(n-1)},\, \mathfrak{g}^{(n-1)}] \supset \dots$$

Theorem and Definition 8.3.10. *The following conditions are equivalent.*

(i) There exists $n \in \mathbb{N}^\star$ such that $\mathfrak{g}^{(n)} = \{0\}$.

(In other words, the derived series of $\mathfrak{g}$ terminates in the zero subalgebra.)

(ii) There exist $n \in \mathbb{N}^\star$ and a sequence of ideals

$$\mathfrak{g} = \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_n = \{0\}$$

such that $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ is **abelian** *(equivalently, $[\mathfrak{a}_i,\, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$) for every i.*

(In other words, $\mathfrak{g}$ is a successive extension of abelian Lie algebras.)

If $\mathfrak{g}$ satisfies either of the equivalent conditions (i) and (ii), $\mathfrak{g}$ is said to be a **solvable Lie algebra**.

The standard example of a **nilpotent**, resp. **solvable**, Lie algebra is the following.

Example 8.3.11. *Let V be a finite-dimensional k-vector space, where k is a field, and let $F = (V_i)_i$ be a* **flag** *in V, namely a sequence of vector subspaces:*

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_i \subset V_{i+1} \subset \cdots \subset V_n = V$$

such that $\dim V_i = i$ for every i.

(i) Let $\mathfrak{u}(F) := \{u \in End(V) \mid u(V_i) \subset V_{i-1} \quad \text{for all} \quad i \geq 1\} \subset End(V)$.
Then, $\mathfrak{u}(F)$ *is a* **nilpotent Lie subalgebra** *of* $End(V)$ *under the bracket* $[S, T] = S \circ T - T \circ S$.

(ii) Let $\mathfrak{b}(F) := \{u \in End(V) \mid u(V_i) \subset V_i \quad \text{for all} \quad i \geq 1\} \subset End(V)$.
Then, $\mathfrak{b}(F)$ *is a* **solvable Lie subalgebra** *of* $End(V)$ *under the bracket* $[S, T] = S \circ T - T \circ S$.

Proof. (i) We see that $\mathfrak{u}(F)$ is the space of endomorphisms of V that take each V_i into itself and factor to the *zero* endomorphisms $V_i/V_{i-1} \longrightarrow V_i/V_{i-1}$ of the successive quotient spaces. It is obvious that $\mathfrak{u}(F)$ is a Lie subalgebra of $\mathrm{End}\,(V)$.

Now, let $\{v_1, \dots, v_n\}$ be a basis for V adapted to F in the sense that

$$V_i = kv_1 + \cdots + kv_i, \qquad i = 1, \dots, n.$$

The Lie algebra $\mathfrak{u}(F)$ consists of those endomorphisms of V whose matrix w.r.t. such a basis is *strictly superdiagonal*, namely it has zeros on and below the main diagonal.

For every k, define

$$\mathfrak{u}_k(F) := \{u \in \mathrm{End}\,(V) \mid u(V_i) \subset V_{i-k} \quad \text{for all} \quad i \geq k\} \subset \mathrm{End}\,(V).$$

We have

$$\cdots \subset \mathfrak{u}_{k+1}(F) \subset \mathfrak{u}_k(F) \subset \cdots \subset \mathfrak{u}_1(F) = \mathfrak{u}(F)$$

and $\mathfrak{u}_k(F) = 0$ for k large enough. Furthermore, $\mathfrak{u}(F)\,\mathfrak{u}_k(F) \subset \mathfrak{u}_{k+1}(F)$ and $\mathfrak{u}_k(F)\,\mathfrak{u}(F) \subset \mathfrak{u}_{k+1}(F)$, hence $[\mathfrak{u}(F), \mathfrak{u}_k(F)] \subset \mathfrak{u}_{k+1}(F)$ for every k. This implies that every $\mathfrak{u}_k(F)$ is an *ideal* of $\mathfrak{u}(F)$.

Since the nilpotency of a Lie algebra $\mathfrak{g}$ is equivalent, by a standard general result, to the existence of a chain of *ideals*:

$$\mathfrak{g} = \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_n = \{0\}$$

such that $[\mathfrak{g}, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$ for every i, we conclude from the above observations that the Lie algebra $\mathfrak{u}(F)$ is nilpotent.

(ii) In a basis $\{v_1, \dots, v_n\}$ for V adapted to F, the Lie algebra $\mathfrak{b}(F)$ consists of those endomorphisms of V whose matrix w.r.t. such a basis is *upper triangular*, namely it has zeros below the main diagonal.

One easily checks that $\mathfrak{u}(F) \subset \mathfrak{b}(F)$, that $\mathfrak{u}(F)$ is an *ideal* of $\mathfrak{b}(F)$ and that the quotient Lie algebra $\mathfrak{b}(F)/\mathfrak{u}(F)$ is *abelian*. Thus, for some large enough n, the sequence of ideals

$$\mathfrak{b}(F) \supset \mathfrak{u}(F) = \mathfrak{u}_1(F) \supset \cdots \supset \mathfrak{u}_k(F) \supset \mathfrak{u}_{k+1}(F) \supset \cdots \supset \mathfrak{u}_n(F) = \{0\}$$

has the properties in (ii) of Theorem and Definition 8.3.10 because $[\mathfrak{u}_k(F), \mathfrak{u}_k(F)] \subset [\mathfrak{u}(F), \mathfrak{u}_k(F)] \subset \mathfrak{u}_{k+1}(F)$ for every k.

Consequently, $\mathfrak{b}(F)$ is solvable by (ii) of Theorem and Definition 8.3.10. □

An immediate observation is that the *derived series* and the *descending central series* of a Lie algebra $\mathfrak{g}$ compare as follows:

$$\mathfrak{g}^{(0)} = \mathfrak{g}^0 = \mathfrak{g}, \;\; \mathfrak{g}^{(1)} = \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{(k)} \subset \mathfrak{g}^k, \quad k \geq 2.$$

Hence, we get

Corollary 8.3.12. *Every* **nilpotent** *Lie algebra is* **solvable**.

As a kind of weak converse, we have the following

Theorem 8.3.13. *Let $\mathfrak{g}$ be a* **finite-dimensional** *Lie algebra over a field of* **characteristic zero***. The following statements are equivalent.*

(i) $\mathfrak{g}$ *is* **solvable***.*

(ii) The adjoint representation ad $: \mathfrak{g} \longrightarrow End(\mathfrak{g})$ *of* $\mathfrak{g}$ *is* **solvable***.*

(iii) $[\mathfrak{g}, \mathfrak{g}]$ *is* **nilpotent***.*

Note that the implication (iii) $\Longrightarrow$ (i) is always (i.e. the finite-dimensionality and the characteristic zero assumptions are not needed) trivially true. The implication (i) $\Longrightarrow$ (iii) follows from the main theorem on solvable Lie algebras that we now state.

Theorem 8.3.14. *(***Lie's theorem on solvable Lie algebras***) Let $\mathfrak{g}$ be a* **solvable** *Lie algebra over an algebraically closed field k of characteristic 0. Let $\rho : \mathfrak{g} \longrightarrow End(V)$ be a linear representation of $\mathfrak{g}$ on a vector space V.*

Then, there exists a flag $F = (V_i)_i$ in V such that $\rho(\mathfrak{g}) \subset \mathfrak{b}(F)$.

In other words, there exists a flag in V that is preserved by every element of $\mathfrak{g}$ under the given linear representation ρ. Let us make this last piece of terminology more precise.

Definition 8.3.15. *Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra over a field k. A* **$\mathfrak{g}$-module** *is a k-vector space V equipped with a k-bilinear map*

$$\mathfrak{g} \times V \longrightarrow V, \qquad (x, v) \longmapsto xv,$$

that satisfies the condition:

$$[x, y]\, v = x(yv) - y(xv), \qquad x, y \in \mathfrak{g}, \;\; v \in V. \tag{8.26}$$

The corresponding Lie homomorphism:

$$\rho : \mathfrak{g} \longrightarrow End(V), \qquad x \longmapsto \rho(x) := (V \ni v \mapsto xv \in V),$$

is called a **linear representation** *of $\mathfrak{g}$ on V.*

As usual, the Lie algebra structure of $\mathrm{End}(V)$ is defined by the Lie bracket $[S, T] := S \circ T - T \circ S$ for all $S, T \in \mathrm{End}(V)$. Note that $\rho : (\mathfrak{g}, [\cdot, \cdot]) \longrightarrow (\mathrm{End}(V), [\cdot, \cdot])$ being a *Lie algebra homomorphism*, namely the condition

$$\rho([x, y]) = [\rho(x), \rho(y)], \qquad x, y \in \mathfrak{g},$$

is equivalent to property (8.26).

Observation 8.3.16. *Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra over a field k. Then:*

(i) $\mathfrak{g}$ is a $\mathfrak{g}$-module with the k-bilinear map:

$$\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \qquad (x, v) \longmapsto [x, v].$$

(ii) the adjoint representation ad $: \mathfrak{g} \longrightarrow End(\mathfrak{g})$ *is a linear representation of $\mathfrak{g}$ on itself.*

Proof. (i) We need to check that condition (8.26) is satisfied in this case, namely that

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]], \qquad x, y, z \in \mathfrak{g}.$$

This is precisely the Jacobi identity satisfied by the Lie bracket of $\mathfrak{g}$.

(ii) is an immediate consequence of (i) and of Definition 8.3.15. $\square$

Corollary 8.3.17. *Let $\mathfrak{g}$ be a* **solvable** *Lie algebra over an algebraically closed field k of characteristic 0. Then, there exists a* **flag of ideals** *in $\mathfrak{g}$.*

Proof. Lie's Theorem 8.3.14 applied to the adjoint representation $ad : \mathfrak{g} \longrightarrow \mathrm{End}\,(\mathfrak{g})$ yields a flag $F = (V_i)_i$ in $\mathfrak{g}$ such that $x(V_i) \subset V_i$ for all i and all $x \in \mathfrak{g}$. This means precisely that each V_i is an ideal of $\mathfrak{g}$. □

The next corollary of Lie's Theorem 8.3.14 proves implication (i) $\Longrightarrow$ (iii) in Theorem 8.3.13.

Corollary 8.3.18. *Let $\mathfrak{g}$ be a* **solvable** *Lie algebra over a field k of characteristic 0. Then, the Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is* **nilpotent***.*

Proof. The statement being *linear*, we may suppose that k is *algebraically closed*. Otherwise, we consider an extension field k' of k and we put $\mathfrak{g}' := \mathfrak{g} \otimes_k k'$. Then, $\mathfrak{g}$ is solvable, resp. nilpotent, if and only if $\mathfrak{g}'$ is solvable, resp. nilpotent. Moreover, $[\mathfrak{g}, \mathfrak{g}]' = [\mathfrak{g}', \mathfrak{g}']$.

By Corollary 8.3.17, there exists a flag of ideals in $\mathfrak{g}$:

$$\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_n = \{0\}.$$

Fix an arbitrary $x \in [\mathfrak{g}, \mathfrak{g}]$. Since $x \in \mathfrak{g}$, $ad_x(\mathfrak{g}_i) \subset \mathfrak{g}_i$ for every i because $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$. But more is true: $ad_x(\mathfrak{g}_i) \subset \mathfrak{g}_{i+1}$ for every i because the induced map $ad_x : \mathfrak{g}_i/\mathfrak{g}_{i+1} \longrightarrow \mathfrak{g}_i/\mathfrak{g}_{i+1}$ on the quotient vanishes identically as a result of $\mathrm{End}\,(\mathfrak{g}_i/\mathfrak{g}_{i+1}) \simeq k$ being commutative and of the special form $[x, \cdot]$ of the map ad_x. This means that ad_x is nilpotent on $\mathfrak{g}$ and all the more so on $[\mathfrak{g}, \mathfrak{g}]$.

We have thus shown that ad_x is nilpotent for each $x \in [\mathfrak{g}, \mathfrak{g}]$. By a well-known theorem (see [Ser64, V., Theorem 3.1]), this is equivalent to $[\mathfrak{g}, \mathfrak{g}]$ being nilpotent. □

We end this brief review of general basic notions and results with the following link between the notions of solvability and nilpotency for Lie algebras, on the one hand, and for Lie groups, on the other hand.

Theorem 8.3.19. *Let $\mathfrak{g}$ be a Lie algebra over a field k of* **characteristic** *0. Suppose that $\mathfrak{g}$ is the Lie algebra of a* **connected** *Lie group G. The following equivalences hold:*

$\mathfrak{g}$ is **solvable** *(resp.* **nilpotent***) as a Lie algebra $\Longleftrightarrow$ G is* **solvable** *(resp.* **nilpotent***) in the group theoretic sense.*

Recall that a group G is called:

(a) **solvable** if there exist **normal subgroups**

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

such that the quotients G_j/G_{j-1} are all **abelian**;

(b) **nilpotent** if there exist **normal subgroups**

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

such that $G_j/G_{j-1} \subset Z(G/G_{j-1})$ for every j, where $Z(G/G_{j-1})$ stands for the centre of G/G_{j-1}.

8.3.5 Classification of complex parallelisable 3-dimensional solvmanifolds

This subsection is taken from Nakamura's work [Nak75]. The starting point is the following

Proposition 8.3.20. *([Nak75, Proposition 1.4.]) Any simply connected, connected,* **solvable complex** *Lie group G is biholomorphically equivalent to $\mathbb{C}^n$, where $n = dim_{\mathbb{C}}G$.*

Sketch of proof. We proceed by induction on $n \geq 1$. The case $n = 1$ is easy. Suppose we have proved the result for $n-1$. When $n = \dim_{\mathbb{C}}G \geq 2$, there exists a connected normal Lie subgroup $N \subset G$ such that $\dim_{\mathbb{C}}N = 1$. Thus, $(G,\, \pi,\, G/N)$ is a holomorphic fibre bundle with fibre N. We easily find that N and G/N are simply connected, connected and solvable, so by the induction hypothesis G/N, resp. N, is biholomorphically equivalent to $\mathbb{C}^{n-1}$, resp. $\mathbb{C}$. It then follows from Oka's principle that G is biholomorphically equivalent to $\mathbb{C}^n$. $\square$

Nakamura goes on to deduce the following classification of complex parallelisable solvmanifolds of complex dimension 3. They are all of the shape $X = G/\Gamma$, where G is a **solvable** simply connected, connected **complex** Lie group with $\dim_{\mathbb{C}}G = 3$ and Γ is a discrete co-compact subgroup of G. By Proposition 8.3.20, G is biholomorphically equivalent to $\mathbb{C}^3$. By Lie's Theorem 8.3.14 applied to the (necessarily solvable) Lie algebra $\mathfrak{g}$ of G, there exists a $\mathbb{C}$-basis $\{\varphi_1,\, \varphi_2,\, \varphi_3\}$ of $\Lambda^{1,0}\mathfrak{g}^\star$ such that

$$d\varphi_\nu = \xi_\nu \wedge \varphi_\nu + \eta_\nu, \qquad \nu = 1, 2, 3, \tag{8.27}$$

where $\xi_\nu \in \{\varphi_1, \dots, \varphi_{\nu-1}\}$ and $\eta_\nu \in \{\varphi_j \wedge \varphi_k \mid 1 \leq j < k \leq 3\}$ for each ν.

Theorem 8.3.21. *Let $X = G/\Gamma$ be a compact* **complex paralellisable solvmanifold** *with $dim_{\mathbb{C}}X = 3$. Let $\{\varphi_1,\, \varphi_2,\, \varphi_3\}$ be a $\mathbb{C}$-basis of $H^{1,0}(X,\, \mathbb{C})$ that satisfies (8.27).*

Then, the manifolds X are classified into the following 3 classes:

III-(1): $\quad d\varphi_1 = d\varphi_2 = d\varphi_3 = 0;$

III-(2): $\quad d\varphi_1 = d\varphi_2 = 0, \quad d\varphi_3 = -\varphi_1 \wedge \varphi_2;$

III-(3): $\quad d\varphi_1 = 0, \quad d\varphi_2 = \varphi_1 \wedge \varphi_2, \quad d\varphi_3 = -\varphi_1 \wedge \varphi_3.$

By duality, properties III-(1), III-(2), III-(3) are equivalent to the following properties satisfied by the dual basis $\{\theta_1,\, \theta_2,\, \theta_3\}$ of $(1,\, 0)$-vector fields on X:

III-(1'): $\quad [\theta_\lambda,\, \theta_\nu] = 0, \qquad \lambda, \nu \in \{1,\, 2,\, 3\};$

III-(2'): $\quad [\theta_1,\, \theta_2] = -[\theta_2,\, \theta_1] = \theta_3, \quad [\theta_\lambda,\, \theta_\nu] = 0$ *otherwise;*

III-(3'): $\quad [\theta_1,\, \theta_2] = -[\theta_2,\, \theta_1] = -\theta_2, \quad [\theta_1,\, \theta_3] = -[\theta_3,\, \theta_1] = \theta_3, \quad [\theta_2,\, \theta_3] = 0.$

In the case III-(1), G is abelian and X is a **complex torus**. In the case III-(2), G is a **nilpotent** complex Lie group (as will be seen below) and X is of the type of the Iwasawa manifold (cf. §1.3.3). In the case III-(3), G is a **solvable non-nilpotent** complex Lie group (as will be seen below). In all three cases, $\mathbb{C}^3$ is the universal covering space of the manifold X, as follows from Proposition 8.3.20.

• Determination of the solvable Lie group structures on $\mathbb{C}^3$

Let 0 be the origin of $\mathbb{C}^3$.

Case III-(2). For each $\nu \in \{1, 2\}$, let

$$\Phi_\nu(z) := \int_0^z \varphi_\nu, \qquad z \in \mathbb{C}^3.$$

Since the 1-forms φ_1 and φ_2 are d-closed on $G = \mathbb{C}^3$, $\Phi_1,\, \Phi_2 : \mathbb{C}^3 \longrightarrow \mathbb{C}$ are *single-valued* holomorphic functions and we have

$$\varphi_\nu = d\Phi_\nu \quad \text{on } \mathbb{C}^3 \quad \text{for} \quad \nu = 1, 2.$$

In particular, we get:

$$d\varphi_3 = -\varphi_1 \wedge \varphi_2 = -d\Phi_1 \wedge d\Phi_2 = -d(\Phi_1 d\Phi_2) \iff d(\varphi_3 + \Phi_1 d\Phi_2) = 0.$$

Thus, the 1-form $\varphi_3 + \Phi_1 d\Phi_2$ is d-closed on $\mathbb{C}^3$, so

$$\Phi_3(z) := \int_0^z (\varphi_3 + \Phi_1 d\Phi_2), \qquad z \in \mathbb{C}^3,$$

defines a *single-valued* holomorphic function $\Phi_3 : \mathbb{C}^3 \longrightarrow \mathbb{C}$ and we have

$$\varphi_3 = d\Phi_3 - \Phi_1 d\Phi_2 \quad \text{on } \mathbb{C}^3.$$

Now, fix an arbitrary $g \in \Gamma$ and consider its orbit $\{z' = gz \mid z \in \mathbb{C}^3\}$. Since φ_1 and φ_2 are Γ-invariant, $\varphi_1(gz) = \varphi_1(z)$ and $\varphi_2(gz) = \varphi_2(z)$ for every $z \in \mathbb{C}^3$, or equivalently $d\Phi_1(gz) = d\Phi_1(z)$ and $d\Phi_2(gz) = d\Phi_2(z)$ for every $z \in \mathbb{C}^3$. Hence, there exist constants $\omega_1(g), \omega_2(g) \in \mathbb{C}$ depending only on g such that

$$\Phi_1(gz) = \Phi_1(z) + \omega_1(g) \quad \text{and} \quad \Phi_2(gz) = \Phi_2(z) + \omega_2(g) \quad \text{for all} \quad z \in \mathbb{C}^3. \tag{8.28}$$

On the other hand, we have

$$\varphi_3(gz) = d\Phi_3(gz) - \Phi_1(gz) d\Phi_2(gz) = d\Phi_3(gz) - (\Phi_1(z) + \omega_1(g))\, d\Phi_2(z).$$

Since $\varphi_3(gz) = \varphi_3(z)$ (because φ_3 is Γ-invariant), we get:

$$d\Big(\Phi_3(gz) - \omega_1(g)\,\Phi_2(z)\Big) = \varphi_3(z) + \Phi_1(z) d\Phi_2(z) = d\Phi_3(z) \iff d\Big(\Phi_3(gz) - \Phi_3(z) - \omega_1(g)\,\Phi_2(z)\Big) = 0.$$

Hence, there exists a constant $\omega_3(g) \in \mathbb{C}$ depending only on g such that

$$\Phi_3(gz) = \Phi_3(z) + \omega_1(g)\,\Phi_2(z) + \omega_3(g) \quad \text{for all} \quad z \in \mathbb{C}^3. \tag{8.29}$$

Consequently, (8.28) and (8.29) leads one to define the following **multiplication on** $\mathbb{C}^3$:

$$(\zeta_1,\, \zeta_2,\, \zeta_3) \star (z_1,\, z_2,\, z_3) := (\zeta_1 + z_1,\, \zeta_2 + z_2,\, \zeta_3 + \zeta_1 z_2 + z_3). \tag{8.30}$$

This multiplication coincides with the matrix multiplication for upper triangular matrices:

$$\begin{pmatrix} 1 & \zeta_1 & \zeta_3 \\ 0 & 1 & \zeta_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_1 + \zeta_1 & z_3 + \zeta_1 z_2 + \zeta_3 \\ 0 & 1 & z_2 + \zeta_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and makes $\mathbb{C}^3$ into a **nilpotent complex Lie group** with Lie algebra of type III-(2'). The forms $\varphi_1,\, \varphi_2,\, \varphi_3$ on $G \simeq (\mathbb{C}^3,\, \star)$ pass to the quotient and define forms denoted by the same symbols on the **nilmanifold** $X = G/\Gamma$.

Case III-(3). Let

$$\Phi_1(z) := \int\limits_0^z \varphi_1, \qquad z \in \mathbb{C}^3.$$

Since the 1-form φ_1 is d-closed on $G = \mathbb{C}^3$, $\Phi_1 : \mathbb{C}^3 \longrightarrow \mathbb{C}$ is a *single-valued* holomorphic function on $\mathbb{C}^3$ and

$$\varphi_1 = d\Phi_1 \quad \text{on } \mathbb{C}^3.$$

Meanwhile, we have:

$$d\varphi_2 = \varphi_1 \wedge \varphi_2 \iff e^{-\Phi_1} d\varphi_2 - e^{-\Phi_1} d\Phi_1 \wedge \varphi_2 = 0 \iff d(e^{-\Phi_1}\varphi_2) = 0.$$

Thus, the 1-form $e^{-\Phi_1}\varphi_2$ is d-closed on $G = \mathbb{C}^3$, so

$$\Phi_2(z) := \int\limits_0^z e^{-\Phi_1}\varphi_2, \qquad z \in \mathbb{C}^3,$$

defines a *single-valued* holomorphic function $\Phi_2 : \mathbb{C}^3 \longrightarrow \mathbb{C}$ and

$$\varphi_2 = e^{\Phi_1} d\Phi_2 \quad \text{on } \mathbb{C}^3.$$

Similarly,

$$d\varphi_3 = -\varphi_1 \wedge \varphi_3 \iff e^{\Phi_1} d\varphi_3 - e^{\Phi_1} d\Phi_1 \wedge \varphi_3 = 0 \iff d(e^{\Phi_1}\varphi_3) = 0.$$

Thus, the 1-form $e^{\Phi_1}\varphi_3$ is d-closed on $G = \mathbb{C}^3$, so

$$\Phi_3(z) := \int\limits_0^z e^{\Phi_1}\varphi_3, \qquad z \in \mathbb{C}^3,$$

defines a *single-valued* holomorphic function $\Phi_3 : \mathbb{C}^3 \longrightarrow \mathbb{C}$ and

$$\varphi_3 = e^{-\Phi_1} d\Phi_3 \quad \text{on } \mathbb{C}^3.$$

By the arguments used in the case III-(2), for every $g \in \Gamma$, we get the existence of constants $\omega_1(g), \omega_2(g), \omega_3(g) \in \mathbb{C}$ depending only on g such that

$$\begin{array}{rcl}\Phi_1(gz) & = & \Phi_1(z) + \omega_1(g)\\ \Phi_2(gz) & = & e^{-\omega_1(g)}\Phi_2(z) + \omega_2(g)\\ \Phi_3(gz) & = & e^{\omega_1(g)}\Phi_3(z) + \omega_3(g)\end{array}$$

for all $z \in \mathbb{C}^3$. This leads one to define the following **multiplication on** $\mathbb{C}^3$:

$$(\zeta_1,\, \zeta_2,\, \zeta_3) \star (z_1,\, z_2,\, z_3) := (\zeta_1 + z_1,\, \zeta_2 + e^{-\zeta_1} z_2,\, \zeta_3 + e^{\zeta_1} z_3). \tag{8.31}$$

This multiplication makes $\mathbb{C}^3$ into a **solvable non-nilpotent complex Lie group** with Lie algebra of type III-(3'). The forms $\varphi_1,\, \varphi_2,\, \varphi_3$ on $G \simeq (\mathbb{C}^3,\, \star)$ pass to the quotient and define forms denoted by the same symbols on the **solvmanifold** $X = G/\Gamma$.

• Examples of complex parallelisable manifolds as above

As already mentioned, in the case III-(2), $G = (\mathbb{C}^3, \star)$ is the **Heisenberg group** and $X = G/\Gamma$ is the **Iwasawa manifold** discussed in §.1.3.3.

In the case III-(3), Nakamura constructs an example that we now briefly describe.

Nakamura's example. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a, b, c, d \in \mathbb{Z}$, A is invertible and the entries of A^{-1} are all integers. Suppose that trace$\,(A) \geq 3$. Let α be an eigenvalue of A.

The elliptic curve $E = \mathbb{C}/\Lambda$ and the group H of analytic automorphisms of $\mathbb{C} \times E \times E$ are defined in the same way as in example (a) using this new choice of α, except that we now put

$$\sigma_2 : (z_1,\, z_2,\, z_3) \longmapsto (z_1 + \log\alpha,\, az_2 + bz_3,\, cz_2 + dz_3).$$

One checks that the action of H on $\mathbb{C} \times E \times E$ is **properly discontinuous** and **fixed-point free**. The quotient manifold

$$X := \mathbb{C} \times E \times E/H$$

is a **complex parallelisable solvmanifold** of **type III-(3)** with $h^{0,\,1}_{\bar\partial}(X) = 3$. $\square$

We get the following addition to Nakamura's Classification Theorem 8.3.21. By T^k we mean a complex torus of complex dimension k.

Observation 8.3.22. *([Nak75, Theorem 1]) The manifolds in class III-(1) are 3-dimensional* **complex tori** T^3.

The manifolds in class III-(2) are **nilmanifolds** *that arise as T^1-bundles over some T^2.*

The manifolds in class III-(3) are **solvmanifolds** *that arise as T^2-bundles over some T^1.*

• Further description of solvmanifolds of type III-(3)

Recall that in this case, we are given a compact complex parallelisable solvmanifold $X = \mathbb{C}^3/\Gamma$ for which we have constructed holomorphic functions $(\Phi_1,\, \Phi_2,\, \Phi_3)$ on $\mathbb{C}^3$ which can be chosen as coordinates, henceforth denoted by $(z_1,\, z_2,\, z_3)$, on $\mathbb{C}^3$. The action of Γ on $\mathbb{C}^3$ is described in (8.31) and there exists a basis of holomorphic 1-forms $\varphi_1, \varphi_2, \varphi_3$ on X obtained from the following holomorphic 1-forms on $\mathbb{C}^3$ by passing to the quotient:

$$\varphi_1 = dz_1, \qquad \varphi_2 = e^{z_1}dz_2, \qquad \varphi_3 = e^{-z_1}dz_3.$$

By duality and Cartan's formula, the dual basis of $(1,\, 0)$-vector fields is

$$\theta_1 = \frac{\partial}{\partial z_1}, \qquad \theta_2 = e^{-z_1}\frac{\partial}{\partial z_2}, \qquad \theta_2 = e^{z_1}\frac{\partial}{\partial z_3}.$$

Since $d\varphi_1 = 0$, $\partial\varphi_1 = 0$, hence $\bar\partial\overline{\varphi}_1 = 0$, so $[\overline{\varphi}_1]_{\bar\partial} \in H^{0,\,1}_{\bar\partial}(X,\, \mathbb{C})$.

On the other hand,

$$\partial\varphi_2 = \varphi_1 \wedge \varphi_2 \neq 0 \qquad \text{and} \qquad \partial\varphi_3 = -\varphi_1 \wedge \varphi_3 \neq 0,$$

so $\overline{\varphi}_2$ and $\overline{\varphi}_3$ are not $\bar\partial$-closed and, therefore, do not represent cohomology classes in $H^{0,\,1}_{\bar\partial}(X,\, \mathbb{C})$. However, they can be modified to $(0,\, 1)$-forms $\varphi_2^\star$ and $\varphi_3^\star$ that are $\bar\partial$-closed and induce cohomology classes in $H^{0,\,1}_{\bar\partial}(X,\, \mathbb{C})$ that, together with $[\overline{\varphi}_1]_{\bar\partial}$, form a basis. It is roughly for this reason that one either has $h^{0,\,1}_{\bar\partial}(X) = 1$ or $h^{0,\,1}_{\bar\partial}(X) = 3$ (as in Nakamura's above example).

Case III-(3)(a). This is the case when $h^{0,\,1}_{\bar\partial}(X) = 1$ and

$$H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C}) = \mathbb{C}\left\langle[\overline{\varphi}_1]_{\bar\partial}\right\rangle.$$

Since $T^{1,\,0}X$ is trivial, this implies that

$$H^{0,\,1}_{\bar\partial}(X,\,T^{1,\,0}X) = \mathbb{C}\left\langle[\theta_i\,\overline{\varphi}_1]_{\bar\partial}\ \mid\ i = 1, 2, 3\right\rangle.$$

In particular, $\dim_{\mathbb{C}} H^{0,\,1}_{\bar\partial}(X,\,T^{1,\,0}X) = 3$.

Case III-(3)(b). This is the case when $h^{0,\,1}_{\bar\partial}(X) = 3$ and

$$H^{0,\,1}_{\bar\partial}(X,\,\mathbb{C}) = \mathbb{C}\left\langle[\overline{\varphi}_1]_{\bar\partial},\,[\varphi^\star_2]_{\bar\partial},\,[\varphi^\star_3]_{\bar\partial}\right\rangle,$$

where the $\bar\partial$-closed $(0,\,1)$-forms $\varphi^\star_2$ and $\varphi^\star_3$ are defined as:

$$\varphi^\star_2 := e^{z_1-\bar z_1}\,\overline{\varphi}_2 = e^{z_1}d\bar z_2 \quad \text{and} \quad \varphi^\star_3 := e^{-z_1+\bar z_1}\,\overline{\varphi}_3 = e^{-z_1}d\bar z_3.$$

Since $T^{1,\,0}X$ is trivial, this implies that

$$H^{0,\,1}_{\bar\partial}(X,\,T^{1,\,0}X) = \mathbb{C}\left\langle[\theta_i\,\varphi^\star_\lambda]_{\bar\partial}\ \mid\ i = 1, 2, 3;\ \lambda = 1, 2, 3\right\rangle,$$

where we put $\varphi^\star_1 := \overline{\varphi}_1$. In particular, $\dim_{\mathbb{C}} H^{0,\,1}_{\bar\partial}(X,\,T^{1,\,0}X) = 9$.

Chapter 9

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