Introduction to several complex variables

Nivoche Stéphanie

Some basic aspects of the theory of holomorphic (complex analytic) functions, are essentially the same in all dimensions, such as Cauchy's formula and inequalities. However the multidimensional theory reveals striking new phenomena (for power series expansions, integral representations, partial differential equations, extension theorems and geometry, for example). In the first two chapters of this course, we introduce this theory and sketch some of these phenomena.
Then in a third chapter, we introduce the notion of entire curve, i.e. a non constant holomorphic map $f : \mathbb{C} \to X$ into a complex n-dimensional manifold X. We study very famous Liouville's Theorem (Cauchy 1844 ?, Liouville 1847. If X is a bounded open subset $\Omega \subset \mathbb{C}^n$, then there are no entire curves $f : \mathbb{C} \to \Omega$) and Little Picard's Theorem (1879) about entire curves whose target is of dimension 1 ($X = \mathbb{C} \setminus \{0, 1\}$ has no entire curve). This last one is more than 140 years old and it has a lot of connections with other areas of mathematics.
Finally, we introduce the notion of complex hyperbolicity, which is a notion in complex geometry in connection with value distribution of entire curves in a complex manifold. The Brody hyperbolicity (a complex manifold X is Brody hyperbolic if it does not contain the image of any non-trivial holomorphic curve) and the Kobayashi hyperbolicity (X is Kobayashi hyperbolic if the Kobayashi metric (analog in higher dimensions of the hyperbolic metric in dimension 1) on X is non-degenerate) are defined.
We conclude with two results: any Kobayashi hyperbolic manifold is Brody hyperbolic as well. For a compact manifold, the two notions are equivalent (Brody, 1978). For non-compact manifolds, there are examples of Brody hyperbolic manifolds which are not Kobayashi hyperbolic.

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