Introduction to Complex Differential Geometry

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References:

- *Complex Analytic and Differential Geometry*, Jean Pierre Demailly.
- *Lectures on Kähler geometry*, Andrei Moroianu.
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- *Introduction to smooth manifolds*, John M. Lee.

Class 1: Real and Complex Vector Bundles

Let M be a C^∞ differentiable manifold of dimension m and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ be the scalar field.

Definition

A (real, complex) vector bundle of rank r over M is a C^∞ manifold E together with

(i) *a C^∞ map $\pi : E \longrightarrow M$ called the projection,*

(ii) *a $\mathbb{K}$-vector space structure of dimension r on each fiber $E_x = \pi^{-1}(x)$*

such that there exists an open covering $\{V_\alpha\}_{\alpha \in I}$ of M and diffeomorphisms

$$\theta_\alpha : \pi^{-1}(V_\alpha) \longrightarrow V_\alpha \times \mathbb{K}^r,$$

satisfying:

- $p_1 \circ \theta_\alpha = \pi$
- *for all $x \in V_\alpha$ of M, the map $E_x \to \{x\} \times \mathbb{K}^r \to \mathbb{K}^r$ is a $\mathbb{K}$-linear isomorphism.*

θ_α is called a local trivialization.

In the above conditions, for each $\alpha, \beta \in I$, the map

$$\theta_{\alpha\beta} = \theta_\alpha \circ \theta_\beta^{-1} : (V_\alpha \cap V_\beta) \times \mathbb{K}^r \longrightarrow (V_\alpha \cap V_\beta) \times \mathbb{K}^r$$

verifies $\theta_{\alpha\beta}(\{x\} \times \mathbb{K}^r) = \{x\} \times \mathbb{K}^r$ and is a $\mathbb{K}$-isomorphism.
We thus define

$$\theta_{\alpha\beta}(x, \xi) = (x, g_{\alpha\beta}(x) \cdot \xi), \quad (x, \xi) \in (V_\alpha \cap V_\beta) \times \mathbb{K}^r$$

where $g_{\alpha\beta} : V_\alpha \cap V_\beta \to \mathrm{GL}(r, \mathbb{K})$ is C^∞.
On triple intersections of the form $V_\alpha \cap V_\beta \cap V_\gamma$, they satisfy the cocycle relation

$$(1.1) \quad g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{on } V_\alpha \cap V_\beta \cap V_\gamma.$$

The collection $(g_{\alpha\beta})$ is called a *system of transition matrices*.

Example. The product manifold $E = M \times \mathbb{K}^r$ is a vector bundle over M, and is called the *trivial vector bundle* of rank r over M.

Example. The *tangent bundle* TM of a differentiable manifold is a real vector bundle over M.
The transition matrices are given by $g_{\alpha\beta} = d\tau_{\alpha\beta}$ where $\tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta^{-1}$.

Example. Similarly, the cotangent bundle T^*M of TM and the p-th exterior power $\Lambda^p T^*M$ (bundle of differential forms of degree p on M) are real vector bundles over M.

The complex projective space $\mathbb{CP}^n$ is the set of complex lines on $\mathbb{C}^{n+1}$:

$$\mathbb{CP}^n = \mathbb{C}^{n+1} - \{0\}/\sim,$$

where, for $z, w \in \mathbb{C}^{n+1}$, $z \sim w \Leftrightarrow z = \lambda w$ for some $\lambda \in \mathbb{C}^*$.
Local charts for $\mathbb{CP}^n$: for $i = 0, \ldots, n$, set

$$U_i = \{[z] : z_i \neq 0\}, \quad \varphi_i : U_i \to \mathbb{C}^n, \varphi_i([z]) = (\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots \frac{z_n}{z_i}).$$

The tautological line bundle L over $\mathbb{CP}^n$ is defined as follows:

$$L = \{([z], w) \mid [z] \in \mathbb{CP}^n, w \in [z]\}$$

Consider the natural map

$$\pi : L \longrightarrow \mathbb{C}P^n, \quad \pi([z], w) = [z],$$

The fiber over the line $[z] \in \mathbb{C}P^n$ consists of all pairs $([z], w)$ with $w \in []$.
Since the fibers are $\mathbb{C}$-lines, we can introduce a natural $\mathbb{C}$-vector space structure on them.

$$([z], w) + ([z], u) := ([z], w + u)$$
$$\kappa \cdot ([z], w) := ([z], \kappa \cdot w).$$

This turns the fibers of $\pi^{-1}([z])$ into $\mathbb{C}$-vector spaces of dimension one.

For a non-zero vector $z \in \mathbb{C}^{n+1}$, we denote by $[z]$ the $\mathbb{C}$-line spanned by z. For $1 \leq i \leq n+1$, let $U_i = \{[z] \in \mathbb{C}P^n \mid x_i \neq 0\}$ and consider the bijections

$$\Phi_i : U_i \times \mathbb{C} \longrightarrow \tilde{\mathbb{C}}^{n+1}|_{U_i}, \quad \Phi_i([z], \lambda z) = ([z], \lambda z_i)\,.$$

It is easy to see that $\tilde{\mathbb{C}}^{n+1}$ has a unique structure as a smooth manifold such that the canonical map $\pi : L \longrightarrow \mathbb{C}P^n$ is a complex line bundle for which the bijections Φ_i are local trivializations, see Lemma 1.3.4.

Let M be a smooth manifold, and suppose that for each $x \in M$ we are given a $\mathbb{K}$-vector space E_x of some fixed dimension r. Let $E = \bigsqcup_{x \in M} E_x$, and let $\pi : E \longrightarrow M$ be the map that takes each element of E_x to the point x.

Suppose furthermore that we are given the following data:

(I) an open cover $\{U_\alpha\}_{\alpha \in I}$ of M

(II) for each $\alpha \in I$, a bijective map $\theta_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^r$ whose restriction to each E_x is a vector space isomorphism from E_x to $\{x\} \times \mathbb{K}^r \cong \mathbb{K}^r$

(III) for each $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(r, \mathbb{K})$ such that the map $\theta_\alpha \circ \theta_\beta^{-1}$ from $(U_\alpha \cap U_\beta) \times \mathbb{K}^r$ to itself has the form

$$\theta_\alpha \circ \theta_\beta^{-1}(x, \xi) = (x, g_{\alpha\beta}(x)\xi).$$

Then E has a unique topology and smooth structure making it into a smooth manifold and a rank-r vector bundle over M.

Let E, F be $\mathbb{K}$-vector bundles of rank r_1, r_2 over M. We can construct new $\mathbb{K}$ vector bundles using vector space operations.

For example $E^*, E \oplus F, \mathrm{Hom}(E, F)$ are defined by

$$(E^*)_x = (E_x)^*, \quad (E \oplus F)_x = E_x \oplus F_x, \quad \mathrm{Hom}(E, F)_x = \mathrm{Hom}(E_x, F_x).$$

Let $\{V_\alpha\}$ be a local trivialization for both bundles E and F. If $(g_{\alpha\beta})$ and $(\gamma_{\alpha\beta})$ are the transition matrices of E and F, then for example $E \otimes F, E \oplus F, E^*, Hom(E, F)$ are the bundles defined by the transition matrices

$$g_{\alpha\beta} \otimes \gamma_{\alpha\beta}, \quad g_{\alpha\beta} \oplus \gamma_{\alpha\beta}, \quad (g_{\alpha\beta}^{-1})^T, \quad (g_{\alpha\beta}^{-1})^T \otimes \gamma_{\alpha\beta}.$$

Let M be a differentiable manifold and E a vector bundle over M.

Example The bundle of k-forms on M with values on E is $\Lambda^k T^*M \otimes_{\mathbb{R}} E$. The fiber at x consists of

$$\Lambda^k T_x^* M \otimes E_x = \{\sigma : T_x M \times \ldots \times T_x M \to E_x \mid \sigma \text{ is } k-\text{linear and alternating}\}.$$

Example The complexified tangent bundle $TM^{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$ has fiber $T_x M \otimes_{\mathbb{R}} \mathbb{C}$.

Let $M, \tilde{M}$ be C^∞ manifolds and $\psi : \tilde{M} \longrightarrow M$ a smooth map. If E is a vector bundle on M, one can define in a natural way a C^∞ vector bundle $\bar{\pi} : \bar{E} \longrightarrow \tilde{M}$ and a C^∞ linear morphism $\Psi : \bar{E} \longrightarrow E$ such that the diagram

$$\begin{array}{ccc} \bar{E} & \stackrel{\Psi}{\longrightarrow} & E \\ \downarrow \bar{\pi} & & \downarrow \pi \\ \tilde{M} & \stackrel{\psi}{\longrightarrow} & M \end{array}$$

commutes and such that $\Psi : \bar{E}_{\tilde{x}} \longrightarrow E_{\psi(\tilde{x})}$ is an isomorphism for every $\tilde{x} \in \tilde{M}$. The bundle $\bar{E}$ can be defined by

$$\bar{E} = \{(\tilde{x}, \xi) \in \tilde{M} \times E : \psi(\tilde{x}) = \pi(\xi)\}$$

and the maps $\bar{\pi}$ and Ψ are then the restrictions to $\bar{E}$ of the projections of $\tilde{M} \times E$ onto $\tilde{M}$ and E, respectively.

Definition

Let $U \subset M$ be an open subset of M. A smooth section of E on U is a differentiable function $s : U \to E$ such that $\pi \circ s = id_U$ (i.e. $s(x) \in E_x$ for all $x \in U$). We denote $C^\infty(U, E)$ the set of all smooth sections on U.

If $s_1, s_2 \in C^\infty(U, E)$ and $f \in C^\infty(M, \mathbb{K})$, then $s_1 + s_2$ and fs_1 are sections on U as well.

Example: The Zero Section

The zero section is the map $s : M \longrightarrow E$ defined by $s(x) = 0_x$ for all $x \in M$, where 0_x denotes the zero element in the fiber E_x. This is always a C^∞ section of E.

Example: Vector Fields

Vector fields on M are exactly the C^∞ sections of the tangent bundle TM.

Example: Differential Forms

Differential p-forms on M are the C^∞ sections of the bundle $\Lambda^p T^*M$, where $\Lambda^p T^*M$ is the p-th exterior power of the cotangent bundle.

(On the blackboard)

Summary:

On a local chart $\theta : \pi^{-1}(U) \to U \times \mathbb{K}^r$, take $\{\epsilon_\lambda\}_{\lambda=1}^r$ a basis of $\mathbb{K}^r$ and set $e_\lambda(x) := \theta^{-1}(x, \epsilon_\lambda)$, which are sections of E. Given any section $s : U \to E$, we have

$$s = \sum_\lambda \sigma_\lambda e_\lambda,$$

where $\sigma_\lambda \in C^\infty(U)$.

Example: Forms with Values in a Vector Bundle
Sections of $\Lambda^k T^*M \otimes E$ are C^∞ maps $\omega : M \longrightarrow \Lambda^k T^*M \otimes E$ that are k-forms on M with values in the vector bundle E.

These generalize differential forms by allowing coefficients in the fibers of E rather than just scalar values.

For $k \geq 1$, we denote $C_k^\infty(M, E) := C^\infty(M, \Lambda^k T^*M \otimes E)$ the sections of this bundle, and $C_0^\infty(M, E) := C^\infty(M, E)$.
With this notation, $C_k^\infty(M, \mathbb{K})$ is the space of $\mathbb{K}$-valued k-forms on M.

- Write sections in coordinates
- Transition functions
- Local frames and form of sections of $\Lambda^k T^*M \otimes E$

(On the blackboard)

Summary:

On a local chart $\theta : \pi^{-1}(U) \to U \times \mathbb{K}^r$, take e_λ as before. Given $s \in C_k^\infty(M, E)$ we have

$$s = \sum_\lambda \sigma_\lambda \otimes e_\lambda,$$

where $\sigma_\lambda \in C_k^\infty(U, \mathbb{K})$ are usual k-forms on $U \subset M$.
If $\bar{\theta} : \pi^{-1}(\bar{U}) \to \bar{U} \times \mathbb{K}^r$ is another trivialization and $s = \sum_\lambda \bar{\sigma}_\lambda \otimes \bar{e}_\lambda$, then

$$\bar{\sigma}(x) = g(x)\sigma(x), \quad \forall x \in U \cap \bar{U},$$

where $g : U \cap \bar{U} \to \mathrm{GL}(r, \mathbb{K})$ are the transition functions verifying $\bar{\theta} \circ \theta^{-1}(x, \xi) = (x, g(x)\xi)$.

Class 2: Linear Connections and Curvature

Definition

A (linear) connection D on the bundle E is a linear differential operator of order 1 acting on $C_q^\infty(M,E)$, for all $q \geq 0$, and satisfying the following properties:

1. $D : C_q^\infty(M,E) \longrightarrow C_{q+1}^\infty(M,E)$,
2. $D(f \wedge s) = df \wedge s + (-1)^p f \wedge Ds$

for any $f \in C_p^\infty(M,\mathbb{K})$ and $s \in C_q^\infty(M,E)$, where df stands for the usual exterior derivative of f.

In the particular case where $f \in C^\infty(M, \mathbb{K})$ is a function (i.e., a 0-form) and $s \in C^\infty(M, E)$ is a section of E (i.e., a 0-form with values in E), the Leibniz rule simplifies to:

$$D(f \cdot s) = df \otimes s + f \cdot Ds$$

This shows that D behaves like a derivation with respect to scalar multiplication.

Connection on a Trivial Bundle
Consider the trivial bundle $E = M \times V$ over M, where V is a fixed vector space of dimension r. Let $A \in C_1^\infty(M, End(\mathbb{K}^r))$.

We can then define a connection on E as follows.

Let $\sigma \in C_q^\infty(M, E)$ and write $s = \sum \sigma_\lambda \otimes e_\lambda$, where e_λ is a global frame and σ_λ are q-forms on M.

We thus set $Ds = d\sigma + A \wedge \sigma$. One can check that this is in fact a connection.

Locally, any connection is trivial. That is, with respect to a local trivialization $\theta_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{K}^r$, a connection D can be described by a matrix of 1-forms

$$A \in \Omega^1(U_\alpha, End(\mathbb{K}^r))$$

where the action of D on sections in coordinates is determined by:

$$Ds|_{U_\alpha} = d\sigma + A \wedge \sigma$$

Under a change of local trivialization corresponding to the transition function $g_{\alpha\beta}$, the matrices of 1-forms transform as:

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$$

where A_α, A_β denote, respectively the matrix of 1-forms corresponding to the trivializations θ_α, θ_β.

The difference of two connections D and D' on the same bundle E is given by a global 1-form:

$$D' - D = Q \in C_1^\infty(M, \mathrm{End}(E))$$

where Q is a section of the endomorphism bundle $\mathrm{End}(E) = \mathrm{Hom}(E, E)$. Note that Q is a matrix of 1-forms, globally defined on M, contrary to the connection 1-forms A that depend on a trivialization.

This means the space of all connections on E is an affine space modeled on $C^\infty(M, \mathrm{End}(E))$.

Given a connection R^D on E, the *curvature* is defined as:

$$R = D \circ D : C^\infty(M, E) \longrightarrow C^\infty(M, \Lambda^2 T^* M \otimes E)$$

Locally, with respect to a trivialization θ, the curvature is given by a matrix of 2-forms:

$$R^D = dA + A \wedge A \in C_2^\infty(U, End(\mathbb{K}^r))$$

where A is the matrix of 1-forms corresponding to θ.

These local matrices glue together to form a global 2-form R^D on M with values in $\mathrm{End}(E)$.

For a line bundle L over M, the curvature is a global (usual) 2-form on M:

$$R \in C_2^\infty(M, \mathbb{K}) = \Omega^2(M)$$

An important property is that the curvature of any connection on a line bundle is always *closed*:

$$dR = 0$$

This means the curvature defines a cohomology class in de Rham cohomology $H^2_{\mathrm{dR}}(M)$, which is an invariant of the line bundle independent of the choice of connection.

Let D be a conection on E and $\xi \in C^{\infty}(M, TM)$ a vector field on M.

Definition

Given a section $s \in C^{\infty}(M, E)$, the covariant derivative of s in the direction of ξ is the section $\nabla_{\xi} s \in C^{\infty}(M, E)$ given by

$$(\nabla_{\xi} s)(x) = Ds(x)(\xi_x).$$

For any ξ fixed, this gives a linear operator ξ_D satisfying the Leibniz rule:

$$\nabla_{\xi}(f\, s) = \xi(f)\, s + f\, \nabla_{\xi} s, \qquad f \in C^{\infty}(M, \mathbb{K}),\ s \in C^{\infty}(M, E).$$

Proposition

For any vector fields ξ, η on M and any section $s \in C^\infty(M, E)$, the commutator of covariant derivatives is given by:

$$\nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s = \nabla_{[\xi,\eta]} s + R^D(\xi, \eta) s$$

This relation shows that the failure of covariant derivatives to commute is precisely measured by the curvature of the connection.

Proof. (On the blackboard)

Class 3: Hermitian Vector Bundles and Complex Manifolds

Let $E \to M$ be a complex rank k bundle over some differentiable manifold M.

Definition

A Hermitian structure H on E is a smooth field of Hermitian products on the fibers of E, that is, for every $x \in M$, $H : E_x \times E_x \longrightarrow \mathbb{C}$ satisfies:

- *$H(u, v)$ is $\mathbb{C}$-linear in u for every $v \in E_x$.*
- *$H(u, v) = \overline{H(v, u)}$ for all $u, v \in E_x$.*
- *$H(u, u) > 0$ for all $u \neq 0$.*
- *$H(u, v)$ is a smooth function on M for every smooth sections u, v.*

From the above conditions:

- H is $\mathbb{C}$ anti-linear in the second variable.
- The third condition shows that H is non-degenerate so we get a bundle isomorphism $H : E \longrightarrow E^*$, $H(\xi) = H(\cdot, \xi)$.

Every rank k complex vector bundle E admits Hermitian structures.
To see this, just take a trivialization (U_i, θ_i) of E and a partition of the unity f_i subordinate to the open cover $\{U_i\}$ of M.

For every $x \in U_i$, let $(H_i)_x$ denote the pull-back of the Hermitian metric on $\mathbb{C}^k$ by the $\mathbb{C}$-linear map $\theta_i|_{E_x}$. Then

$$H = \sum_i f_i H_i$$

is a well-defined Hermitian structure on E.

Assume H is a Hermitian structure on E, a complex bundle. Let θ be a trivialization $\theta : \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^r$ and let $e_1, \ldots, e_r$ be the local frame induced by θ.

We define the matrix of the Hermitian form with respect to this local frame as

$$h_{ij}(x) := H(e_i(x), e_j(x))$$

where $h_{ij} \in C^\infty(U, \mathbb{C})$. This gives a Hermitian matrix $h = (h_{ij})$ of smooth functions on U.

We can extend the Hermitian metric to sections:

$$C_p^\infty(M,E) \times C_q^\infty(M,E) \longrightarrow C_{p+q}^\infty(M,\mathbb{C})$$

$$(s,t) \mapsto H(s,t)$$

such that for $s = \sum_\lambda \sigma_\lambda \otimes e_\lambda$ and $t = \sum_\mu \tau_\mu \otimes e_\mu$, we set

$$H(s,t) = \sum_{\lambda,\mu} \sigma_\lambda \wedge \overline{\tau_\mu} H(e_\lambda, e_\mu).$$

Note: If $s, t \in C_0^\infty(M, E)$, then $s = \sum_\lambda \sigma_\lambda e_\lambda$ and $t = \sum_\mu \tau_\mu e_\mu$.

Then $H(s, t) \in C_0^\infty(M, E)$ and its exterior derivative satisfies:

$$dH(s,t)(\xi) = d(H(s,t))(\xi) = \xi(H(s,t))$$

for every vector field ξ on M.

Note: If $u \in C_1^\infty(M, E)$, $t \in C_0^\infty(M, E)$ then $H(u, t) \in C_1^\infty(M, E)$.
Moreover, for every $\xi \in \mathcal{X}(M)$,

$$H(u,t)(\xi) = H(u(\xi), t)$$

Definition

A linear connection D is called Hermitian if for all $s \in C^\infty_p(M, E)$, $t \in C^\infty_p(M, E)$, we have

$$dH(s,t) = H(Ds,t) + (-1)^p H(s, Dt).$$

Given the equalities above, if H is Hermitian and $s, t \in C^\infty_0(M, E)$, then

$$\xi H(s,t) = H(Ds(\xi), t) + H(s, Dt(\xi)) = H(\nabla_\xi s, t) + H(s, \nabla_\xi t), \quad \forall \xi \in \mathcal{X}(M).$$

Remark: Given an Hermitian vector bundle, there always exists an hermitian connection. In general the hermitian structure is not unique.

Now let's assume $E \to M$ is a Hermitian bundle and D is a Hermitian connection. Let θ be a trivialization, and assume e_λ is an orthonormal basis.

Then $De_\lambda = \sum_\mu a_{\lambda\mu} e_\mu$ where $a_{\lambda\mu} \in C_1^\infty(M, \mathbb{C})$.

We have $dH(e_\lambda, e_\mu) = 0$ since $H(e_\lambda, e_\mu) = \delta_{\lambda\mu}$. Therefore:

$$0 = a_{\lambda\mu} + \overline{a_{\mu\lambda}}$$

Thus $A = -A^*$, so $(iA)^* = iA$.

For the curvature, we have:

$$R_{\lambda\mu} = da_{\lambda\mu} + \sum_{\kappa} a_{\lambda\kappa} \wedge a_{\kappa\mu}$$

Taking conjugates:

$$\overline{R_{\lambda\mu}} = d\overline{a_{\lambda\mu}} + \sum_{\kappa} \overline{a_{\lambda\kappa}} \wedge \overline{a_{\kappa\mu}} = -da_{\mu\lambda} + \sum_{\kappa} a_{\kappa\lambda} \wedge a_{\mu\kappa} = -R_{\mu\lambda}$$

Thus $(R_{\lambda\mu})$ is a skew-Hermitian matrix. Hence $(iR^D)^* = iR^D$, which implies that iR^D has real-valued 2-forms on its diagonal. Define:

$$\mathrm{tr}(iR^D) := \sum_{\lambda} \omega_{\lambda\lambda}, \quad \text{where} \quad \omega_{\lambda\lambda} = (R^D)_{\lambda\lambda}$$

Since R^D is a globally defined 2-form with values in E, $\mathrm{tr}(iR^D)$ is a real-valued 2-form defined on M.

We claim that $d\mathrm{tr}(R^D) = 0$.
Note that

$$\sum_{\lambda,\mu=1}^{r} a_{\lambda\mu} \wedge a_{\mu\lambda} = -\sum_{\lambda,\mu=1}^{r} a_{\mu\lambda} \wedge a_{\mu\lambda} = -\sum_{\lambda,\mu=1}^{r} a_{\lambda\mu} \wedge a_{\mu\lambda}$$

where first we used the skew-symmetry of the wedge product first and later a change of summation index.
Using this, one can prove that

$$\mathrm{tr}(R^D) = d\left(\sum_{\lambda} a_{\lambda\lambda}\right) \implies \text{closed 1-form}$$

Therefore:

1. $\mathrm{tr}(iR^D)$ is closed and real.
2. $\mathrm{tr}(iR^D)$ defines a cohomology class in $H^2_{\mathrm{dR}}(M, \mathbb{R})$.
3. This class does not depend on the Hermitian connection (on the blackboard: use the formula for difference of connections)

- If V is a real vector space and $J \in End(V)$ such that $J^2 = -1$ then V has a structure of complex vector space. We shall denote V^J this complex vector space.

- Identify $\mathbb{C}$ with $\mathbb{R}^2$ and using the differentiable structure as real manifold, we give to $T_{(x,y)}\mathbb{R}^2 = T_z\mathbb{C}$ a complex structure:

$$J : T_{(x,y)}\mathbb{R}^2 \to T_{(x,y)}\mathbb{R}^2, \quad J(v)(f) = v(f \circ m),\ \forall f \in C^\infty(\mathbb{R}^2)$$

  where $m : \mathbb{R}^2 \to \mathbb{R}^2$, $m(x,y) = (-y,x)$, (i.e. $m(z) = iz$). One can easily check that $J^2 = -I$.

  This is the canonical complex structure of $T_{(x,y)}\mathbb{R}^2$.

  One can extend this procedure to $\mathbb{R}^{2n} = \mathbb{C}^n$ using the multiplication by i on each variable and obtain a canonical complex structure J_n on each tangent of $\mathbb{R}^{2n} = \mathbb{C}^n$.

- Let $U \subset \mathbb{C}$ be an open set and $F : U \to \mathbb{C}$ be a smooth function, which we view as a real function $F(x, y) = (f(x, y), g(x, y))$, if $z = x + iy$.

Definition

We say that F is holomorphic if $\partial_x f = \partial_y g$ and $\partial_y f = -\partial_x g$.

This condition is equivalent to

$$J \circ dF = dF \circ J,$$

that is, $dF_{(x,y)}$ is a $\mathbb{C}$-linear map of $(T_{(x,y)}\mathbb{R}^2)^J$ for all $(x, y) \in \mathbb{R}^2$.

Similarly, one can consider holomorphic functions $F : U \subset \mathbb{C}^n \to \mathbb{C}$, and being holomorphic is now equivalent to dF commuting with the canonical complex structure J_n.

Definition

A complex manifold of complex dimension m is a topological space M with an open covering $\mathcal{U}$ such that for every point $x \in M$ there exists $U \in \mathcal{U}$ containing x and a homeomorphism $\varphi_U : U \longrightarrow \tilde{U} \subset \mathbb{C}^m$, such that for every intersecting $U, V \in \mathcal{U}$, the map between open sets of $\mathbb{C}^m$

$$\varphi_{UV} := \varphi_U \circ \varphi_V^{-1}$$

is holomorphic. A pair (U, φ_U) is called a (holomorphic) chart and the collection of all charts is called a holomorphic structure.

Example: Complex projective space $\mathbb{CP}^n$.

Definition

A function $F : M \to \mathbb{C}$ is holomorphic if $F \circ \varphi_U^{-1}$ is holomorphic for every holomorphic chart.

Clearly, any complex manifold is a real C^∞ manifold. We can define an almost complex structure on M using the canonical complex structure J_n of $\mathbb{C}^n$:

For $p \in M$, $J_p : T_pM \longrightarrow T_pM$ is defined as follows: let (U, φ) be a holomorphic chart at p, then

$$J_p := d(\varphi_U^{-1}) \circ J_n \circ d\varphi_U$$

The definition is independent of the local chart.

Clearly, $J_p^2 = -I$.

By definition, if $\varphi_U = (x_1, \ldots, x_n, y_1, \ldots, y_n)$, then $J(\partial/\partial x_i) = \partial/\partial y_i$.

In addition, $(d\varphi_U)_p : T_pM \longrightarrow T_{\varphi(p)}\mathbb{C}^m$ is a $\mathbb{R}$-linear isomorphism and $d(\varphi_U) \circ J_p = J_n \circ d\varphi_U$.

Note: J is a $(1,1)$ tensor on M, i.e. $J \in C^\infty(M, End(TM))$.

Definition

A section $J \in C^\infty(M, End(TM))$ *on a differential manifold* M *(i.e. a* $(1,1)$*-tensor on* M*) satisfying* $J^2 = -\mathrm{Id}$ *is called an almost complex structure. The pair* (M, J) *is then referred to as an almost complex manifold.*

The previous slide shows taht every complex manifold carries an almost complex structure.

The converse holds only under some extra (integrability) assumption that we shall see next.

Let (M, J) be an almost complex manifold.

Consider $TM^{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$, that is, $T_pM^{\mathbb{C}} = T_pM \otimes_{\mathbb{R}} \mathbb{C}$, which is a complex vector space.

$T_pM \otimes_{\mathbb{R}} \mathbb{C}$ is a complex vector space; $i(v \otimes z) = v \otimes iz$ and $T_pM \subset T_pM^{\mathbb{C}}$ by $v \mapsto v \otimes 1$.

Any $\mathbb{R}$-linear map $S : T_pM \to T_pM$ is extended to $T_pM^{\mathbb{C}}$ as $S(v \otimes z) = S(v) \otimes z$, which is $\mathbb{C}$-linear.

In particular we extend J to $T_pM^{\mathbb{C}}$. This extension (which we also denote by J) also verifies $J^2 = -I$ on $T_pM^{\mathbb{C}}$, so its eigenvalues are $\pm i$. We denote $T_p^{1,0}M$ and $T_p^{0,1}M$ the eigenbundles corresponding to these eigenvalues.

PROPOSITION

$$T_p^{1,0} = \{x - iJx : x \in TM\}, \quad T_p^{0,1} = \{x + iJx : x \in TM\}$$

Theorem (Newlander-Nirenberg)

Let (M, J) be an almost complex manifold. The almost complex structure J is induced by a structure of complex manifold on M if and only if $T^{0,1}$ is integrable.

Proof: Assume M is a complex manifold and let (U, φ) be a holomorphic chart. We have $\varphi = (z_1, \ldots, z_n) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$. We denote:

$$\frac{\partial}{\partial z_\alpha} := \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}\right)$$

$$\frac{\partial}{\partial \bar{z}_\alpha} := \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}\right)$$

Recall that $J\left(\frac{\partial}{\partial x_\alpha}\right) = \frac{\partial}{\partial y_\alpha}$, so $\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\alpha}$ are local sections of $T^{1,0}M$ and $T^{0,1}M$, respectively. They actually form a local basis of them.
Given $X, Y \in T^{0,1}M$, $X = \sum_\alpha a_\alpha \frac{\partial}{\partial \bar{z}_\alpha}$ with $Y = \sum_\mu b_\mu \frac{\partial}{\partial \bar{z}_\mu}$:

$$[X, Y] = \sum_{\alpha,\mu} a_\alpha \frac{\partial b_\mu}{\partial \bar{z}_\alpha}\frac{\partial}{\partial \bar{z}_\mu} - \sum_{\alpha,\mu} b_\mu \frac{\partial a_\alpha}{\partial \bar{z}_\mu}\frac{\partial}{\partial \bar{z}_\alpha} \in T^{0,1}M.$$

(complexified Lie bracket).
The converse holds $\rightsquigarrow$ see Demailly or Kobayashi-Nomizu.

We continue with (M, J) an almost complex manifold.

Set $\Lambda^k M_{\mathbb{C}} = \Lambda^k M \otimes \mathbb{C}$, whose sections are k-forms on M with values on $\mathbb{C}$.

J extends to T_pM^* by setting $J(\alpha) = \alpha \circ J$ for all $\alpha \in M^*$. Later, we extend it to $\Lambda^1_{\mathbb{C}} M = T_pM^* \otimes \mathbb{C}$ as explained before. One can easily check that, again, $J^2 = -1$ on $\Lambda^1_{\mathbb{C}} M = T_pM^* \otimes \mathbb{C}$ so it provides a decomposition on eigenbundles:

$$\Lambda^{1,0}M \quad = \{\xi \in \Lambda^1_{\mathbb{C}} M / J\xi = i\xi\}, \; \Lambda^{0,1}M \quad = \{\xi \in \Lambda^1_{\mathbb{C}} M / J\xi = -\xi\}.$$

Similarly as before, one has

Lemma

$$\begin{aligned} \Lambda^{1,0}M &= \{\xi \in \Lambda^1_{\mathbb{C}} M / \xi(z) = 0 \; \forall z \in T^{0,1}M\} \\ \Lambda^{0,1}M &= \{\xi \in \Lambda^1_{\mathbb{C}} M / \xi(z) = 0 \; \forall z \in T^{1,0}M\} \end{aligned}$$

Note that $\omega \in \Lambda^{k,0}M \Leftrightarrow X \lrcorner \omega = 0 \quad \forall X \in T^{0,1}M$.

Suppose that J is complex (that is, J comes from a complex manifold structure on M). Let (U, φ) be a holomorphic chart, with coordinates $z_\alpha = x_\alpha + iy_\alpha$.
Extending the exterior derivative to $\Lambda^k M^{\mathbb{C}} := \Lambda^k M \otimes \mathbb{C}$ we get

$$dz_\alpha = dx_\alpha + idy_\alpha, \quad d\bar{z}_\alpha = dx_\alpha - idy_\alpha.$$

Using that

$$J\left(\frac{\partial}{\partial x_\alpha}\right) = \frac{\partial}{\partial y_\alpha},$$

we have $Jdx_\alpha = -dy_\alpha$, $Jdy_\alpha = -dx_\alpha$ and thus

$$\begin{aligned} \Lambda^{1,0} M &= \text{span}\{dz_\alpha = dx_\alpha + idy_\alpha : \alpha = 1, \dots, n\}, \\ \Lambda^{0,1} M &= \text{span}\{d\bar{z}_\alpha = dx_\alpha - idy_\alpha : \alpha = 1, \dots, n\} \end{aligned}$$

Notation: $\Lambda^{k,0} M$ (resp. $\Lambda^{0,k} M$) denotes the k-th exterior power of $\Lambda^{1,0} M$ (resp. $\Lambda^{0,1} M$). Furthermore, $\Lambda^{p,q} M := \Lambda^{p,0} M \oplus \Lambda^{0,q} M$ and thus

$$\Lambda^k M^{\mathbb{C}} := \bigoplus_{p+q=k} \Lambda^{p,q} M$$

To every almost complex structure J one can associate a $(2,1)$-tensor N^J called the *Nijenhuis tensor*, satisfying:

$$N^J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY], \quad \forall X,Y \in \mathcal{X}(M)$$

Proposition

Let J be an almost complex structure on M^{2m}. The following statements are equivalent:

(a) *J is a complex structure.*

(b) *$T^{0,1}M$ is integrable.*

(c) *$dC^\infty(\Lambda^{1,0}M) \subset C^\infty(\Lambda^{2,0}M \oplus \Lambda^{1,1}M)$.*

(d) *$dC^\infty(\Lambda^{p,q}M) \subset C^\infty(\Lambda^{p+1,q}M \oplus \Lambda^{p,q+1}M)$ for all $0 \leq p,q \leq m$.*

(e) *$N^J = 0$.*

Proof. (On the blackboard)

Proposition

Let M be a complex manifold. Let $f : M \longrightarrow \mathbb{C}$ be a smooth complex-valued function on M. The following assertions are equivalent:

(1) *f is holomorphic.*

(2) *$Z(f) = 0$ for all $Z \in T^{0,1}M$.*

(3) *df is a form of type $(1, 0)$ (i.e. $df \in \Lambda^{1,0}M$).*

Proof. (On the blackboard)

Recall that on a complex manifold:

$$dC^\infty(M,\Lambda^{p,q}M)\subset C^\infty(M,\Lambda^{p+1,q}M\oplus\Lambda^{p,q+1}M)$$

We can therefore define

$$\begin{aligned}\partial &: C^\infty(M,\Lambda^{p,q}M)\longrightarrow C^\infty(\Lambda^{p+1,q}M)\\ \bar\partial &: C^\infty(M,\Lambda^{p,q}M)\longrightarrow C^\infty(\Lambda^{p,q+1}M)\end{aligned}$$

by composing d with projections so that $d=\partial+\bar\partial$.

Note that

$$\begin{aligned}\partial^2 &: \Lambda^{p,q} \to \Lambda^{p+1,q}\to\Lambda^{p+2,q}\\ \bar\partial^2 &: \Lambda^{p,q} \to \Lambda^{p,q+1}\to\Lambda^{p,q+2}\\ \bar\partial\partial &: \Lambda^{p,q} \to \Lambda^{p+1,q}\to\Lambda^{p+1,q+1}\\ \partial\bar\partial &: \Lambda^{p,q} \to \Lambda^{p,q+1}\to\Lambda^{p+1,q+1}\end{aligned}$$

Lemma

The following identities hold:

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0$$

Corolary

$f : M \to \mathbb{C}$ *is holomorphic if and only if* $\bar{\partial} f = 0$.

The operator $\bar{\partial}$ is called the Dolbeault operator.

Class 4: Holomorphic bundles and Chern connection

Definition

A complex vector bundle $\pi : E \longrightarrow M$ over a complex manifold M is said to be holomorphic if E is a complex manifold, the projection map π is holomorphic and there exists a covering $(V_\alpha)_{\alpha \in I}$ of X and a family of holomorphic trivializations $\theta_\alpha : \pi^{-1}(V_\alpha) \longrightarrow V_\alpha \times \mathbb{C}^r$.

Not every complex vector bundle over a complex manifold is holomorphic.

A complex vector bundle may admit different holomorphic structures.

For a holomorphic vector bundle the transition functions between holomorphic charts $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{C})$ are holomorphic.

Examples The tangent bundle of a complex manifold is holomorphic.

Examples If M is a complex manifold, $\Lambda^{p,0}M$ is holomorphic. Indeed, Let (U, φ) be a chart on M, with $\varphi = (z_1, \dots, z_n)$.
Then $dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ is a $(p, 0)$-form and thus we can define $\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^r$ with

$$\alpha \mapsto (\pi(\alpha), (\alpha(dz_{i_1}, \dots, dz_{i_p})))$$

If (V, ψ) is another local chart with $\psi = (w_1, \dots, w_n)$, then the change of coordinates is holomorphic.

In particular, if $\dim_{\mathbb{R}} M = 2n$, then $\Lambda^{n,0}M$ is a holomorphic line bundle, that is, whose fibers have dimension 1.

Note: $\Lambda^{0,p}$ is not in general holomorphic.

Let E be a holomorphic bundle over a complex manifold. We let $\Lambda^{p,q}E = \Lambda^{p,q}M \otimes E$ be the E-valued (p,q)-forms.

Let e_λ be a local holomorphic frame. Given $\sigma \in C^\infty(M, \Lambda^{p,q}E)$ we have

$$\sigma = \sum_\lambda \sigma_\lambda \otimes e_\lambda, \quad \text{where } \sigma_\lambda \in \Lambda^{p,q}M$$

We define $\bar\partial : C^\infty(\Lambda^{p,q}E) \longrightarrow C^\infty(\Lambda^{p,q+1}E)$ as follows:

$$\bar\partial\sigma = \sum_\lambda \bar\partial\sigma_\lambda \otimes e_\lambda$$

By construction, $\bar\partial^2 = 0$.

Note: If $\sigma = \sum_\lambda \tau_\lambda \otimes f_\lambda$ for some other holomorphic trivialization, then $\tau_\lambda = \sum_\lambda g_{\lambda\mu}\sigma_\mu$ where $g_{\lambda\mu}$ are holomorphic.
Therefore $\bar\partial\tau_\lambda = \sum_\lambda g_{\lambda\mu}\bar\partial\sigma_\mu$, so $\bar\partial$ does not depend on the trivialization.

Theorem

Let E be a complex vector bundle over a complex manifold. Then E is holomorphic if and only if there exists an operator

$$\bar{\partial} : C^{\infty}(\Lambda^{p,q}E) \longrightarrow C^{\infty}(\Lambda^{p,q+1}E)$$

satisfying the Leibniz rule and such that $\bar{\partial}^2 = 0$.

We saw that:

$$\left.\begin{array}{r} M \text{ complex} \\ E \text{ holom.} \end{array}\right\} \quad \Rightarrow \quad \exists \bar{\partial} : C^\infty(\Lambda^{p,q}E) \to C^\infty(\Lambda^{p,q+1}E)$$

We will see:

$$\left.\begin{array}{r} M \text{ complex} \\ E \text{ complex v.bdl.} \\ D \text{ connection} \end{array}\right\} \quad \Rightarrow \quad \exists D^{0,1} : C^\infty(\Lambda^{p,q}E) \to C_\infty(\Lambda^{p,q+1}E)$$

Definition

Let M be a complex vector bundle, E an holomorphic bundle and H an hermitian structure on E. An hermitian connection D is called a *Chern connection if $D^{0,1} = \bar{\partial}$.*

Remark: The curvature of a Chern connection has zero $(0, 2)$-part.

Theorem

For every Hermitian structure H on a holomorphic vector bundle E, there exists a unique hermitian connection D such that $D^{0,1} = \bar{\partial}$.

Proof. Assume that such D exists and let ∇ denote its covariant derivative.

- Note that if E is holomorphic then E^* is also holomorphic. Moreover, since D on E satisfies $D^{0,1} = \bar{\partial}$, this equation also holds for the induced connection on E^*.
- D induces a connection in E^*, which we also denote by D. Indeed, it is the connection induced by the covariant derivative

$$(\nabla_X \sigma^*)(\sigma) := X(\sigma^*(\sigma)) - \sigma^*(\nabla_X \sigma),$$

for every $X \in \mathcal{X}(M)$, $\sigma^* \in C^\infty(M, E^*)$, $\sigma \in C^\infty(M, E)$

- D also induces a connection in $End(E)$ (which we also denote by D) by declaring its covariant derivative as:

$$(\nabla_X Q)(\sigma) := \nabla_X(Q(\sigma)) - Q(\nabla_X \sigma),$$

for every $X \in \mathcal{X}(M),\ Q \in C^\infty(M, Hom(E)),\ \sigma \in C^\infty(M, E)$
In particular, if $Q = H$, the formulas of two induced connections above give

$$\begin{aligned}(\nabla_X H)(\sigma)(\tau) &= \nabla_X(H(\sigma))(\tau) - H(\nabla_X \sigma)(\tau) \\ &= X(H(\sigma)(\tau)) - H(\tau)(\nabla_X \sigma) - H(\nabla_X \sigma)(\tau) \\ &= X(H(\tau, \sigma)) - H(\nabla_X \sigma, \tau) - H(\tau, \nabla_X \sigma) = 0\end{aligned}$$

and this vanishes because D is an hermitian connection, for any $\tau, \sigma \in C^\infty(M, E)$. Therefore

$$\nabla_X(H(\sigma)) = H(\nabla_X \sigma),\ \forall X \in TM, \sigma \in C^\infty(M, E). \tag{1}$$

- Applying (1) to $Z = X + iY \in TM^{\mathbb{C}}$ we get,

$$\begin{array}{rcl} \nabla_Z(H(\sigma)) & = & \nabla_X(H(\sigma)) + i\nabla_Z(H(\sigma)) \\ H(\nabla_Z\sigma) & = & H(\cdot, \nabla_X\sigma + i\nabla_Y\sigma = H(\nabla_X\sigma) - iH(\nabla_Y\sigma) \end{array}$$

Therefore,

$$\nabla_Z(H(\sigma)) = H(\nabla_{\bar{Z}}\sigma), \ \forall X \in TM, \sigma \in C^\infty(M, E). \tag{2}$$

- For $Z \in T^{0,1}$, we get

$$\nabla_Z(H(\sigma)) = D(H(\sigma))(Z) = D^{0,1}H(\sigma).$$

Moreover, since $\bar{Z} \in T^{1,0}$, we get

$$H(\nabla_{\bar{Z}}\sigma) = H((D\sigma)(\bar{Z})) = H(D^{1,0}\sigma(\bar{Z}))$$

Using this two equalities in (2), we get

$$D^{0,1}H(\sigma) = H(D^{1,0}\sigma(\bar{Z})) \tag{3}$$

Note that on the left hand side $D^{0,1}$ is the connection on E^* and on the right hand side, $D^{1,0}$ is on E. By hypothesis, $D^{0,1} = \bar{\partial}$ (on E^*) and therefore we can re-write the previous equality as
$D^{1,0} = H^{-1}\bar{\partial}H$.

Finally, we can conclude that if a D as in the statement exists, then it must satisfy

$$D = D^{1,0} + D^{0,1} = H^{-1}\bar{\partial}H + \bar{\partial}. \tag{4}$$

Therefore, given a holomorphic vector bundle E and a hermitian structure H, D is determined by E and H by the formula above, so it must be unique.

In addition, given E and H, we can define D on sections of E as in (4) and extend it using Leibniz rule and prove that this is a connection that satisfies the required conditions.

Proposition (Kobayashi-Nomizu)

To every complex vector bundle E over a smooth manifold M one can associate a cohomology class $c_1(E) \in H^2(M, \mathbb{Z})$ called the first Chern class of E satisfying the following axioms:

- **(naturality)** *For every smooth map $f : M \longrightarrow N$ and complex vector bundle E over N, one has $f^*(c_1(E)) = c_1(f^*E)$.*
- **(Whitney sum formula)** *For every bundles E, F over M one has $c_1(E \oplus F) = c_1(E) + c_1(F)$.*
- **(normalization)** *The first Chern class of the tautological line bundle of $\mathbb{CP}^1$ is equal to -1 in $H^2(\mathbb{CP}^1, \mathbb{Z}) \simeq \mathbb{Z}$, which means that the integral over $\mathbb{CP}^1$ of any representative of this class equals -1.*

Theorem

Let D be a connection on a complex bundle E over M. The real cohomology class

$$c_1(D) := \left[\frac{i}{2\pi} tr(R^D) \right]$$

is equal to the image of $c_1(E)$ in $H^2_{dR}(M, \mathbb{R})$.

Note that a procedure on the proof generalizes to any line bundle.

Let E be a holomorphic line bundle with an Hermitian metric over a complex manifold. Given a nowhere vanishing holomorphic section σ, we set

$$u = H(\sigma, \sigma) = e^{-\varphi},$$

for some $\varphi : M \to \mathbb{R}$.

Then, for the Chern connection, a procedure similar as above gives

$$R^D = \bar{\partial}\partial \log u = \partial\bar{\partial}\varphi.$$

The function φ is called a local weight of the bundle.

Definition

A Hermitian metric on an almost complex manifold (M, J) *is a Riemannian metric* h *such that:*

$$h(X, Y) = h(JX, JY), \quad \forall X, Y \in TM$$

The fundamental form of a Hermitian metric is defined by:

$$\Omega(X, Y) := h(JX, Y)$$

We say that (M, h, J) is an Hermitian manifold if J is an almost complex structure on M and h is Hermitian on (M, J).

The extension (also denoted by h) of the Hermitian metric to $TM^{\mathbb{C}}$ by $\mathbb{C}$-linearity satisfies:

$$\begin{cases} h(\bar{Z}, \bar{W}) = \overline{h(Z, W)}, & \forall Z, W \in TM^{\mathbb{C}} \\ h(Z, Z) > 0 & \forall Z \in TM^{\mathbb{C}} - \{0\} \\ h(Z, W) = 0, & \forall Z, W \in T^{1,0}M \text{ and } \forall Z, W \in T^{0,1}M \end{cases}$$

Conversely, each symmetric tensor on $TM^{\mathbb{C}}$ with these properties defines a Hermitian metric by restriction to TM.

The tangent bundle of an almost complex manifold (M, J) is a complex vector bundle. If h is a Hermitian metric on M, then:

$$H(X, Y) := h(X, Y) - ih(JX, Y) = (h - i\Omega)(X, Y)$$

defines a Hermitian structure on the complex vector bundle (TM, J).

Conversely, any Hermitian structure H on TM defines a Hermitian metric h on M by:

$$h := \mathrm{Re}(H)$$

Every almost complex manifold admits Hermitian metrics.
Construction: Simply choose an arbitrary Riemannian metric g and define:

$$h(X, Y) := g(X, Y) + g(JX, JY)$$

This is automatically a Hermitian metric with respect to the almost complex structure J.

Definition

A Hermitian metric h on an almost complex manifold (M, J) is called a Kähler metric if J is a complex structure and the fundamental form Ω is closed, that is,

$$h \text{ is Kähler } \iff N^J = 0 \text{ and } d\Omega = 0.$$

Lemma

Let (M, h, J) be an Hermitian manifold and let ∇ denote the Levi-Civita connection. Then

$$N^J = 0 \Leftrightarrow (\nabla_{JX} J)Y = J(\nabla_X J)Y, \quad \forall X, Y \in TM \ (\star)$$

Recall that

$$(\nabla_Y J)X = \nabla_Y(JX) - J(\nabla_Y X)$$

- Let $X, Y \in T_pM$ and denote also by X, Y local vector fields around p extending these vectors and such that that $(\nabla X)_p = 0 = \nabla Y)_p$. This implies that $\nabla_W X = \nabla_W Y = 0$ for all W. In particular,

$$\nabla_X Y = \nabla_Y X = 0, \quad (\nabla_{JX} J)(Y) = \nabla_{JX} JY,$$

$$\nabla_Y J)X = \nabla_Y(JX), \quad (\nabla_X J)Y = \nabla_X(JY).$$

- using that $[X, Y] = \nabla_X Y - \nabla_Y X$ and the above we get

$$\begin{aligned} [X, Y]_p &= 0 \\ [JX, Y]_p &= -(\nabla_Y J)X \\ [X, JY]_p &= (\nabla_X J)Y \\ [JX, JY]_p &= (\nabla_{JX} J)Y - (\nabla_{JY} J)X \end{aligned}$$

Therefore

$$\begin{aligned} N^J(X, Y) &= J(\nabla_X J)Y - J(\nabla_Y J)X - (\nabla_{JX} J)Y + (\nabla_{JY} J)X \\ &= \{J(\nabla_X J)Y - (\nabla_{JX} J)Y\} + \{(\nabla_{JY} J)X - J(\nabla_Y J)X\} \end{aligned}$$

Clearly, if $(\star)$ holds, $N^J = 0$.

For the converse, we define

$$A(X, Y, Z) = h(J(\nabla_X J)Y - (\nabla_{JX} J)Y, Z).$$

Excercise: A is skew-symmetric in the last two variables.

If $N^J = 0$, the computation above gives $A(X, Y, Z) = A(Y, X, Z)$. So applying these properties in different orders, we get

$$A(X, Y, Z) = -A(X, Z, Y) = -A(Z, X, Y)$$

but also

$$A(X, Y, Z) = A(Y, X, Z) = -A(Y, Z, X) = -A(Z, Y, X) = A(Z, X, Y)$$

so $A = 0$ and the lemma follows.

Theorem

A hermitian metric h on (M, J) almost complex is Kähler if and only if $\nabla J = 0$. Equivalently, if and only if $\nabla_X(JY) = J(\nabla_X Y)$ for all $X, Y \in TM$.

Proof. If $\nabla J = 0$, then J is integrable by the previous lemma. Also, since $\Omega = h(J\cdot, \cdot)$, one can prove that $\nabla\Omega = 0$ and therefore $d\Omega = 0$.

Conversely, suppose that h is Kähler and denote by B the tensor:

$$B(X, Y, Z) := h((\nabla_X J)Y, Z) = (\nabla_X \Omega)(Y, Z).$$

Since J and $\nabla_X J$ anti-commute we have:

$$B(X, Y, JZ) = B(X, JY, Z)$$

In addition, from the previous lemma,

$$B(X, Y, JZ) + B(JX, Y, Z) = 0$$

Combining these two relations also yields:

$$B(X, JY, Z) + B(JX, Y, Z) = 0$$

We now use that $d\Omega$ is the skew-symmetrization of $\nabla\Omega$ and apply $d\Omega = 0$ twice, first on X, Y, JZ, then on X, JY, Z and get:

$$B(X, Y, JZ) + B(Y, JZ, X) + B(JZ, X, Y) = 0$$

$$B(X, JY, Z) + B(JY, Z, X) + B(Z, X, JY) = 0$$

Adding these two relations and using the previous properties of B yields $2B(X, Y, JZ) = 0$, that is, J is parallel. $\square$