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A Complex Analytic Approach to Differential and Algebraic Geometry

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Introduction to Several Complex Variables

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Chapter 1

Elementary properties of holomorphic functions in several complex variables

1.1 Preliminaries

Let u be a complex valued function in $\mathcal{C}^1(\Omega)$, where Ω is an open set in $\mathbb{R}^{2n}$. We shall denote the real coordinates by x_j, $1 \leq j \leq n$ and y_j, $1 \leq j \leq n$, and the complex coordinates by $z_j = x_j + iy_j$, $j = 1, \dots, n$. With this $\mathbb{R}$-linear isomorphism, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ is the cartesian product of n copies of $\mathbb{C}$, which carries the structure of an n-dimensional complex vector space. The standard hermitian inner product on $\mathbb{C}^n$ is defined by $(z, z') = \sum_{j=1}^n z_j \overline{z'_j}$, z, $z' \in \mathbb{C}^n$. The associated norm $\mid z \mid= (z, z)^{1/2}$ induces the euclidian metric in the usual way: for z, $z' \in \mathbb{C}^n$, $\text{dist}(z, z') =\mid z - z' \mid$.

We can express du as a linear combination of the differentials dz_j and $d\bar{z}_j$,

$$du = \sum_1^n \frac{\partial u}{\partial z_j} dz_j + \sum_1^n \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j,$$

where we have used the notation $dz_j = dx_j + idy_j$ and $d\bar{z}_j = dx_j - idy_j$ with

$$\frac{\partial u}{\partial z_j} = \frac{1}{2}(\frac{\partial u}{\partial x_j} - i\frac{\partial u}{\partial y_j}), \qquad \frac{\partial u}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial u}{\partial x_j} + i\frac{\partial u}{\partial y_j}).$$

We write du in the form $du = \partial u + \bar{\partial} u$ with

$$\partial u = \sum_1^n \frac{\partial u}{\partial z_j} dz_j, \qquad \bar{\partial} u = \sum_1^n \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j.$$

Differential forms which are linear combinations of the differentials dz_j are said to be of type $(1, 0)$, and those which are linear combinations of $d\bar{z}_j$ are said to be of type $(0, 1)$. Thus ∂u (resp. $\bar{\partial} u$) is the component of du of type $(1, 0)$ (resp. $(0, 1)$).

We extend the definition of the ∂ and $\bar\partial$ operators to arbitrary differential forms. A differential form f is said to be of type (p,q) if it can be written in the form

$$f = \sum_{|I|=p} \sum_{|J|=q} f_{I,J} dz^I \wedge d\bar z^J,$$

where $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are multi-indices, that is, sequences of indices between 1 and n. Coefficients $f_{I,J}$ are distributions in open sets in $\mathbb{C}^n$ (a distribution on an open set U is a linear functional on $\mathcal{D}^0(U) = \mathcal{C}_c^\infty(U)$ that is continuous when $\mathcal{D}^0(U)$ is given a topology called the canonical LF topology (a locally convex inductive limit of a countable inductive system of Fréchet spaces (generalizations of Banach spaces)). This leads to the space of (all) distributions on U, usually denoted by $\mathcal{D}'(U) = (\mathcal{D}^0(U))'$, that is the continuous dual space of $\mathcal{D}^0(U)$). We have here used the notation

$$dz^I \wedge d\bar z^J = dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar z_{j_1} \wedge \ldots \wedge d\bar z_{j_q}.$$

Every differential form can be written in one and only one way as a sum of forms of type (p,q), where $0 \le p, q \le n$. And every differential form of type (p,q) can be written in one and only one way as a sum $\sum'$ of forms where $|I| = p$ and $|J| = q$ are strictly increasing.

If f is of type (p,q) and in $\mathcal{C}^1$, the exterior differential

$$df = \sum df_{I,J} dz^I \wedge d\bar z^J$$

can be written $df = \partial f + \bar\partial f$, where

$$\partial f = \sum_{I,J} \partial f_{I,J} \wedge dz^I \wedge d\bar z^J, \qquad \bar\partial f = \sum_{I,J} \bar\partial f_{I,J} \wedge dz^I \wedge d\bar z^J$$

are of type $(p+1,q)$ and $(p,q+1)$, respectively. If f is in $\mathcal{C}^2$, since $0 = d^2 f = \partial^2 f + (\partial\bar\partial + \bar\partial\partial)f + \bar\partial^2 f$ and all terms are of different types, we obtain

$$\partial^2 = 0, \qquad \partial\bar\partial + \bar\partial\partial = 0 \qquad \bar\partial^2 = 0.$$

Hence the equation

$$\bar\partial u = f,$$

where f is of type $(p,q+1)$, cannot have a solution u unless

$$\bar\partial f = 0.$$

This shows that it is natural to study the $\bar\partial$ operator for any forms of type (p,q) and not only for functions u.

1.2 Holomorphic functions

1.2.1 Definition and first properties

If Ω is an open set in $\mathbb{C}^n = \mathbb{R}^{2n}$ and u is in $\mathcal{C}^1(\Omega)$, $du(a)$ is the differential of u in $a \in \Omega$ and it is the unique $\mathbb{R}$-linear map : $\mathbb{R}^{2n} \to \mathbb{R}^2$ such that $u(z+a) = u(a) + du(a)(z) + o(|z|)$ when z tends to $0 \in \mathbb{C}^n$. This $\mathbb{R}$-linear map can be uniquely written as the sum of a $\mathbb{C}$-linear map, $\partial u(a)(z) = \sum_j \frac{\partial u}{\partial z_j}(a) z_j$, and the conjugate of an other $\mathbb{C}$-linear map, $\sum_j \frac{\partial \bar{u}}{\partial z_j}(a) z_j$.

Definition 1.2.1 *A function $u \in \mathcal{C}^1(\Omega)$ is said to be holomorphic in Ω if du is of type $(1,0)$, that is, if $\bar{\partial} u(a) = 0$ in Ω (the Cauchy-Riemann equations) or*

$$\frac{\partial u}{\partial \bar{z}_j}(a) = 0, \ \textit{for any } a \in \Omega \textit{ and for any } 1 \leq j \leq n.$$

Or equivalently, it meens that $du(a)$ is a $\mathbb{C}$-linear map for any $a \in \Omega$.

The set of all holomorphic functions in Ω is denoted by $\mathcal{O}(\Omega)$.

Proposition 1.2.2 *1) $\mathcal{O}(\Omega)$ is a $\mathbb{C}$-algebra for addition, product of holomorphic functions and product with constants in $\mathbb{C}$.*
2) If $u \in \mathcal{O}(\Omega)$ and if $u(z) \neq 0$, $\forall z \in \Omega$, then $1/u \in \mathcal{O}(\Omega)$.
3) Suppose that Ω is connected and that $u \in \mathcal{O}(\Omega)$. If u is real valued or if $\mid u \mid$ is constant, then u is constant in Ω.

Proof. 1) The differential operators ∂ and $\bar{\partial}$ are obviously linear and satisfy the product rule : if u and $v \in \mathcal{C}^1(\Omega)$, $\partial(uv) = v\partial u + u\partial v$ and $\bar{\partial}(uv) = v\bar{\partial}u + u\bar{\partial}v$.
2) If $u \in \mathcal{C}^1(\Omega)$ and if $u(z) \neq 0$, $\forall z \in \Omega$, $1/u \in \mathcal{C}^1(\Omega)$ and $\bar{\partial}(1/u) = -u^{-2}\bar{\partial}u$.
3) If u is real valued, then $\partial u/\partial x_j$ and $\partial u/\partial y_j$ are real valued for any $1 \leq j \leq n$. The Cauchy-Riemann equations give : $\frac{\partial u}{\partial x_j} = -i\frac{\partial u}{\partial y_j}$, then $\frac{\partial u}{\partial x_j} = \frac{\partial u}{\partial y_j} = 0$, $1 \leq j \leq n$ and u is constant in Ω.
If $\mid u \mid = \varrho > 0$ is constant, we have locally $u(z) = \varrho e^{i\theta(z)}$ with $\bar{\partial} u = \varrho e^{i\theta(z)} i \bar{\partial}\theta(z) = 0$, then θ is holomorphic and real valued in Ω, then θ is constant in Ω. □

Remark 1.2.3 *If a function u is holomorphic in Ω, then u is an holomorphic function of each z_j when the other variables are kept fixed. The reverse is true and it is Hartog's Theorem (1906) (we will see it later).*

1.2.2 Holomorphic maps

Now let u be an holomorphic function in Ω (open set in $\mathbb{C}^n$) with values in $\mathbb{C}^m$, that is

$$u = (u_1, \ldots, u_m),$$

where each component u_j is holomorphic in Ω. We say that u is an holomorphic map.
If $u = (u_1, \ldots, u_m)$ is an holomorphic map in Ω, the matrix

$$J_u(a) = \left(\frac{\partial u_j}{\partial z_k}(a) \right)_{1 \le j \le m,\, 1 \le k \le n}$$

is called the jacobian matrix of u in a. The differential of u in a, $du(a)$, is a $\mathbb{C}$-linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$, such that the matrix with respect to the canonical basis of $\mathbb{C}^n$ to $\mathbb{C}^m$ is $J_u(a)$. We have $u(a+z) = u(a) + J_u(a)z + o(|z|)$ when z tends to O.
If u is an holomorphic map in Ω with values in $\mathbb{C}^m$ and v is an holomorphic function in an open set ω such that $\mathbb{C}^m \supset \omega \supset u(\Omega)$, then $v \circ u$ is an holomorphic function: for any $1 \le k \le n$, we have

$$\frac{\partial}{\partial \bar{z}_k}(v(u(z))) = \sum_{l=1}^{m} \frac{\partial v}{\partial w_l}(u(z)).\frac{\partial u_l}{\partial \bar{z}_k}(z) + \sum_{l=1}^{m} \frac{\partial v}{\partial \bar{w}_l}(u(z)).\frac{\partial \bar{u}_l}{\partial \bar{z}_k}(z)$$

and $J_{v \circ u}(a) = J_v(u(a))J_u(a)$, since $d(v \circ u)(a) = dv(u(a)) \circ du(a)$.

The implicit function theorem extends to holomorphic functions.

Theorem 1.2.4 *Let $f_j(w,z)$, $j = 1, \ldots, m$, be holomorphic functions of $(w,z) = (w_1, \ldots, w_m, z_1, \ldots, z_n)$ in a neighborhood of a point (w^0, z^0) in $\mathbb{C}^m \times \mathbb{C}^n$, and assume that $f_j(w^0, z^0) = 0, j = 1, \ldots, m$ and that*

$$det \left(\frac{\partial f_j}{\partial w_k} \right)_{j,k=1}^{m} \neq 0 \qquad at\ (w^0, z^0).$$

Then the equations $f_j(w,z) = 0$, $j = 1, \ldots, m$, have a uniquely determined holomorphic solution $w(z)$ in a neighborhood of a point z^0 in $\mathbb{C}^n$, such that $w(z^0) = w^0$.

Proof. We have

$$0 \neq \left| \det\left(\frac{\partial f_j}{\partial w_k}\right)_{j,k=1}^{m} \right|^2 = \det \begin{pmatrix} \left(\frac{\partial f_j}{\partial w_k}\right) & 0 \\ 0 & \left(\frac{\partial \bar{f}_j}{\partial \bar{w}_k}\right) \end{pmatrix} = \det \frac{D(f_1, \ldots, f_m, \bar{f}_1, \ldots, \bar{f}_m)}{D(w_1, \ldots, w_m, \bar{w}_1, \ldots, \bar{w}_m)}.$$

We write $f_j = u_j + iv_j$ where u_j and v_j are real valued for $1 \le j \le m$. We write $w_k = x_k + iy_k$. Then

$$\frac{D(u_1, v_1, \ldots, u_m, v_m)}{D(x_1, y_1, \ldots, x_m, y_m)} \neq 0.$$

We apply the usual implicit real valued function theorem, and we obtain $w = w(z)$.
Functions w_k are holomorphic since $\sum_1^m \frac{\partial f_j}{\partial w_k} dw_k + \sum_1^n \frac{\partial f_j}{\partial z_k} dz_k = df_j = 0$; and we can solve this system of equations for dw_k and we find that dw_k is a linear combination of $dz_1, \ldots, dz_n$. □

Exercise 1.2.5 *Express the previous determinant* $\dfrac{D(f_1,\ldots,f_m,\bar f_1,\ldots,\bar f_m)}{D(w_1,\ldots,w_m,\bar w_1,\ldots,\bar w_m)}$ *in terms of* $\dfrac{D(u_1,v_1,\ldots,u_m,v_m)}{D(x_1,y_1,\ldots,x_m,y_m)}$. *We could remark for instance that*

$$\frac{1}{2}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial v_1}{\partial x_1} \\ \frac{\partial u_1}{\partial y_1} & \frac{\partial v_1}{\partial y_1} \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_1}{\partial y_1} \end{pmatrix} = \frac{\partial f_1}{\partial z_1}$$

and that the product of the following three matrices gives

$$\begin{pmatrix} 1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \end{pmatrix}\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial v_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial v_2}{\partial x_1} \\ \frac{\partial u_1}{\partial y_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial u_2}{\partial y_1} & \frac{\partial v_2}{\partial y_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial v_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial v_2}{\partial x_2} \\ \frac{\partial u_1}{\partial y_2} & \frac{\partial v_1}{\partial y_2} & \frac{\partial u_2}{\partial y_2} & \frac{\partial v_2}{\partial y_2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \\ i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 \\ 0 & i & 0 & -i \end{pmatrix} = 2\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_2}{\partial z_1} & 0 & 0 \\ \frac{\partial f_1}{\partial z_2} & \frac{\partial f_2}{\partial z_2} & 0 & 0 \\ 0 & 0 & \overline{\frac{\partial f_1}{\partial z_1}} & \overline{\frac{\partial f_2}{\partial z_1}} \\ 0 & 0 & \overline{\frac{\partial f_1}{\partial z_2}} & \overline{\frac{\partial f_2}{\partial z_2}} \end{pmatrix}$$

which is exactly $2\dfrac{D(f_1,f_2,\bar f_1,\bar f_2)}{D(z_1,z_2,\bar z_1,\bar z_2)}$.

Corollary 1.2.6 Inverse function theorem. *An holomorphic map of an open set in* $\mathbb{C}^n$ *into itself has locally an holomorphic inverse where the Jacobian does not vanish.*

Proof. We apply the previous theorem to $f(w) - z$ with $m = n$. □

Exercise 1.2.7 *If* u *is a holomorphic map of* $\Omega \subset \mathbb{C}^n$ *into* $\mathbb{C}^m$ *and if* $f = \sum f_{I,J}(w)dw^I \wedge d\bar w^J$ *is a form defined in an open neighborhood of the range of* u, *we can define the pullback of* f *relative to* u: *the form* u^*f *in* Ω *by*

$$u^*f = \sum f_{I,J}(u(z))du^I \wedge d\bar u^J,$$

where du_k *and* $d\bar u_k$ *for* $k = 1,\ldots,m$ *are differential forms in* Ω.
Prove that these differential forms are of type $(1,0)$ *and* $(0,1)$, *respectively, since* u_k *is an holomorphic function, for* $k = 1,\ldots,m$.
Hence u^*f *is of type* (p,q) *if* f *is of type* (p,q).
Prove that $d(u^*f) = u^*(df)$ *and deduce that* $\partial(u^*f) = u^*(\partial f)$ *and* $\bar\partial(u^*f) = u^*(\bar\partial f)$.

1.3 Cauchy's integral formula

Till the beginning of the 1930's, the only multidimensional integral formula was the Cauchy formula for a polydomain $D = \prod_1^n D_j$ in $\mathbb{C}^n$, where each D_j is a bounded domain in $\mathbb{C}$ with rectifiable boundary. A set $D \subset \mathbb{C}^n$ is called a polydisc if there are discs $D_1,\ldots,D_n$ in $\mathbb{C}$ such that

$$D = \prod_1^n D_j = \{z, z_j \in D_j, j = 1,\ldots,n\}.$$

The set $\prod_1^n \partial D_j$ is called the distinguished boundary, or Shilov's boundary, of D and we denote it by $\partial_0 D \subset \partial D$, but it is not equal to ∂D.

Theorem 1.3.1 *Let D be an open polydisc and let u be a continuous function in $\bar{D}$ which is (in D) an holomorphic function of each z_j when the other variables are kept fixed. Then, for any $z \in D$, we have*

$$u(z) = \frac{1}{(2i\pi)^n} \int_{\partial_0 D} \frac{u(\zeta_1, \ldots, \zeta_n)}{\prod_{j=1}^n (\zeta_j - z_j)} d\zeta_1 \ldots d\zeta_n.$$

Hence $u \in \mathcal{C}^\infty(D)$ and u is in fact holomorphic in D.

Proof. Fix $1 \le j \le n$. Let $z_k \in \bar{D}_k$, for any $k \ne j$ be fixed. If $((\zeta_1^p, \ldots, \zeta_{j-1}^p, \zeta_{j+1}^p, \ldots, \zeta_n^p))_p$ is a sequence in $\prod_{k \ne j} D_k$ which converges to $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$, then the sequence of holomorphic function (of one variable) in D_j, $(u(\zeta_1^p, \ldots, \zeta_{j-1}^p, z_j, \zeta_{j+1}^p, \ldots, \zeta_n^p))_p$ converges uniformly in any compact sets in D_j to $u(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n)$, which is an holomorphic function in D_j. For $n = 1$, we have

$$u(z_1) = \frac{1}{2i\pi} \int_{\partial D_1} \frac{u(\zeta_1)}{\zeta_1 - z_1} d\zeta_1.$$

For any $n \in \mathbb{N}^*$, $(z_1, \ldots, z_{n-1}) \in \bar{D}_1 \times \ldots \bar{D}_{n-1}$, the function $z_n \mapsto u(z_1, \ldots, z_n)$ is holomorphic in D_n and we have

$$u(z_1, \ldots, z_n) = \frac{1}{2i\pi} \int_{\partial D_n} \frac{u(z_1, \ldots, z_{n-1}, \zeta_n)}{\zeta_n - z_n} d\zeta_n.$$

Then $u(z_1, \ldots, z_{n-1}, \zeta_n)$ is holomorphic in $z_{n-1} \in D_{n-1}$ when $(z_1, \ldots, z_{n-2}, \zeta_n) \in \bar{D}_1 \times \ldots \bar{D}_{n-2} \times \bar{D}_n$. According to the previous formula for $n = 1$, we obtain

$$u(z_1, \ldots, z_{n-1}, \zeta_n) = \frac{1}{2i\pi} \int_{\partial D_{n-1}} \frac{u(z_1, \ldots, z_{n-2}, \zeta_{n-1}, \zeta_n)}{\zeta_{n-1} - z_{n-1}} d\zeta_{n-1}.$$

By induction on n and according to Fubini's theorem, we obtain the Cauchy formula.
The partial derivatives for any order under $\int$ in this formula are possible, then $u \in \mathcal{C}^\infty(D)$ and in particular $\bar{\partial} u = 0$. □

Corollary 1.3.2 *If Ω is an open set in $\mathbb{C}^n$ and $u \in \mathcal{O}(\Omega)$, it follows that $u \in \mathcal{C}^\infty(\Omega)$ and that all derivatives of u are also holomorphic in Ω.*

Proof. Let $z_0 \in \Omega$ and D be an open polydisc centered in z_0 such that $\bar{D} \subset \Omega$. Then according to theorem 1.3.1, u is $\mathcal{C}^\infty$ in D, and in Ω. The partial derivatives of u in $z_1, \ldots, z_n$ are obtained by derivation under the integral sign in Cauchy's formula and are holomorphic in D, since for any $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$, $\prod_{j=1}^n (\zeta_j - z_j)^{\nu_j}$ is holomorphic in D, with respect to z. □

Corollary 1.3.3 *(Osgood) If a function u, continuous in an open set Ω in $\mathbb{C}^n$, is an holomorphic function of each z_j when the other variables are kept fixed, then u is holomorphic in Ω.*

In fact the hypothesis of continuity is not necessary.

Theorem 1.3.4 Hartog's Theorem (1906). *If u is a complex valued function defined in the open set $\Omega \subset \mathbb{C}^n$ and u is holomorphic in each variable z_j when the other variables are given arbitrary fixed values, then u is holomorphic in Ω.*

Proof. See for instance [Ho1] Theor 2.2.8. $\square$

A corresponding result would be false for functions of real variables. Indeed, let study $f_1 : \mathbb{R}^2 \to \mathbb{R}$ defined by $f_1(O) = 0$ and $f_1(x,y) = \frac{xy}{x^4+y^4}$ if $(x,y) \neq (0,0)$. f_1 is infinitely differentiable with respect to x (or y) when y (or x) is kept fixed, but in spite of that f_1 is not bounded around origin.
Let study the function f_2 such that $f_2(x,y) = xy/(x^2+y^2)$, $f_2(0,0) = 0$. f_2 is infinitely differentiable with respect to x (or y) when y (or x) is kept fixed, but in spite of that f_2 is bounded and not even continuous at the origin (use polar coordinates).

1.4 Applications of the Cauchy's integral formula

Cauchy formula allows one to prove the fundamental properties of holomorphic functions of several variables, for example, the local representation of holomorphic functions by power series, the property of uniqueness of analytic continuation, etc...
We can also immediately obtain bounds for the derivatives of u. In doing so we shall call an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers a multi-order and write $\partial^\alpha = \frac{\partial}{\partial z_1}^{\alpha_1} \cdots \frac{\partial}{\partial z_n}^{\alpha_n}$.
The operator $\bar{\partial}^\alpha$ is defined similarly and we write $\alpha! = \alpha_1! \ldots \alpha_n!$ and $\mid \alpha \mid = \alpha_1 + \ldots + \alpha_n$.
Since the partial derivatives of any order under $\int$ in Cauchy's integral formula are possible, we obtain:

Proposition 1.4.1 *Let Ω be an open set in $\mathbb{C}^n$. For any $u \in \mathcal{O}(\Omega)$ and any $z \in \Omega$, we have:*

$$\partial^\alpha u(z) = \frac{\alpha!}{(2i\pi)^n} \int_{\partial_0 D} \frac{u(\zeta_1, \ldots, \zeta_n)}{\prod_{k=1}^n (\zeta_k - z_k)^{\alpha_k+1}} d\zeta_1 \ldots d\zeta_n,$$

where D is an open polydisc containing z and relatively compact in Ω.

Theorem 1.4.2 (Cauchy's inequalities)
1) For every compact set $K \subset \Omega$ (open set in $\mathbb{C}^n$) and every open neighborhood ω of K, there are constants C_α for all multi-orders α such that

$$\sup_K \mid \partial^\alpha u \mid \leq C_\alpha \mid\mid u \mid\mid_{L^1(\omega)}, \qquad u \in \mathcal{O}(\Omega).$$

1') For every compact set $K \subset \Omega$ (open set in $\mathbb{C}^n$) and every bounded open neighborhood ω of K, for any $p \geq 1$, there are constants $C_{\alpha,p}$ for all multi-orders α such that

$$\sup_K \mid \partial^\alpha u \mid \leq C_{\alpha,p} \mid\mid u \mid\mid_{L^p(\omega)}, \qquad u \in \mathcal{O}(\Omega).$$

2) If u is holomorphic in the polydisc $D = \{z : | z_j | < r_j, 1 \leq j \leq n\}$ and $| u | \leq M$ in D, it follows that

$$| \partial^\alpha u(0) | \leq M\alpha!/r^\alpha.$$

Proof. 1) There exist $r'_j > r_j > 0$ such that any closed polydisc centered in $z \in K$ with multi-radius $(r'_1, \ldots, r'_n)$ is contained in ω. Indeed, it is sufficient to choose $0 < r_j < r'_j < \inf\{\sup_j | z_j - w_j | : z \in K, w \in \partial\omega\}$.
Fix $z \in K$ and r_j for any $1 \leq j \leq n$ chosen as above. We can apply Proposition 1.4.1 to the polydisc D centered in z and with multi-radius $(r_1, \ldots, r_n)$

$$\partial^\alpha u(z) = \frac{\alpha!}{(2i\pi)^n} \int_{\partial_0 D} \frac{u(\zeta_1, \ldots, \zeta_n)}{\prod_{k=1}^n (\zeta_k - z_k)^{\alpha_k+1}} d\zeta_1 \ldots d\zeta_n,$$

where for $1 \leq j \leq n$, $\zeta_j = z_j + r_j e^{i\theta_j}$. Then

$$|\partial^\alpha u(z)| \leq \frac{\alpha!}{r^{\alpha+\mathbf{1}}(2\pi)^n} \int_{[2\pi]^n} |u(\zeta_1(\theta), \ldots, \zeta_n(\theta))| r_1 \ldots r_n d\theta_1 \ldots d\theta_n.$$

$$|\partial^\alpha u(z)| \int_{[0,r'_1]\times\ldots\times[0,r'_n]} r^{\alpha+\mathbf{1}} dr_1 \ldots dr_n = |\partial^\alpha u(z)| \frac{r'^{\alpha+\mathbf{2}}}{(\alpha_1+2)\ldots(\alpha_n+2)}$$

$$\leq \frac{\alpha!}{(2\pi)^n} \int_{[0,r'_1]\times\ldots\times[0,r'_n]} \int_{[2\pi]^n} |u(\zeta_1(\theta), \ldots, \zeta_n(\theta))| r_1 \ldots r_n d\theta_1 \ldots d\theta_n dr_1 \ldots dr_n$$

$$= \frac{\alpha!}{(2\pi)^n} \int_{D(z,r')} |u(\zeta)| dV(\zeta).$$

To conclude,

$$|\partial^\alpha u(z)| \leq \frac{\alpha!\,(\alpha_1+2)\ldots(\alpha_n+2)}{(2\pi)^n\, r'^{\alpha+\mathbf{2}}} \int_{D(z,r')} |u(\zeta)| dV(\zeta)$$

and

$$\sup_K | \partial^\alpha u | \leq \frac{\alpha!\,(\alpha_1+2)\ldots(\alpha_n+2)}{(2\pi)^n\, r'^{\alpha+\mathbf{2}}} || u ||_{L^1(\omega)}.$$

1') If in addition ω is bounded, then for any $p > 1$, we have (according to Hölder's inequality)

$$|| u ||_{L^1(\omega)} \leq V(\omega)^{1-1/p} || u ||_{L^p(\omega)}.$$

2) It is a direct consequence of proposition 1.4.1. □

We endow $\mathcal{O}(\Omega)$ with the topology of uniform convergence on compact sets in Ω. Let $(K_j)_{j\geq 1}$ be an exhaustive sequence of compact sets in Ω : $\Omega = \cup_j K_j$ and $K_j \subset \text{Int}(K_{j+1})$. For example $K_j = \{z \in \Omega : d(z, \partial\Omega) \geq 1/j,\ |z| \leq j\}$.
If f and $g \in \mathcal{C}^0(\Omega)$, we note $\delta(f,g) = \sum_{j=1}^\infty \frac{1}{2^j} \inf\{1, ||f-g||_{K_j}\}$.

Exercise 1.4.3 *Prove that*
1) δ *is a metric which defines the previous topology.*
2) $\mathcal{C}^0(\Omega, \mathbb{C})$ *endowed with this topology, is a complete metric space (and then it is a Fréchet space).*

Then the following corollary proves that $\mathcal{O}(\Omega)$ is closed in $\mathcal{C}^0(\Omega, \mathbb{C})$ and the Cauchy inequalities show that all derivations ∂^α are continuous operators on $\mathcal{O}(\Omega)$ to itself.

Corollary 1.4.4 *If* $u_k \in \mathcal{O}(\Omega)$ *and* $u_k \to u$ *when* $k \to \infty$, *uniformly on compact subsets of* Ω, *it follows that* $u \in \mathcal{O}(\Omega)$.

Proof. Application of theorem 1.4.2 to $u_n - u_m$ shows that ∂u_n converges uniformly on compact subsets of Ω (Cauchy's sequence). Since $\bar{\partial} u_n = 0$, it follows that $\partial u_n / \partial x_j$ converges uniformly on compact sets in Ω. Hence $u \in \mathcal{C}^1(\Omega)$ and $\bar{\partial} u = 0$. □

Definition 1.4.5 *A subset* S *in* $\mathcal{O}(\Omega)$ *is bounded iff for any compact set* $K \subset \Omega$ *we have*

$$\sup_{f \in S} ||f||_K < +\infty.$$

The following theorem essentially says that a subset S in $\mathcal{O}(\Omega)$ is compact iff it is bounded and closed in $\mathcal{O}(\Omega)$.

Corollary 1.4.6 (Montel or Stieltjes-Vitali) *If* $u_k \in \mathcal{O}(\Omega)$ *and the sequence* $(\mid u_k \mid)_k$ *is uniformly bounded on every compact subset of* Ω, *there is a subsequence* $(u_{k_j})_j$ *converging uniformly on every compact subset of* Ω *to a limit* $u \in \mathcal{O}(\Omega)$.

Proof. First let recall **Ascoli's theorem.** *Let* K *be a compact set and* (E, d) *a metric space. The space* $C(K, E)$ *of continuous functions in* K *valued in* E, *with the topology induced by uniform distance, is a metric space. A part* A *in* $C(K, E)$ *is relatively compact iff the two following conditions are satisfied:*
- A is equicontinuous, i.e for any $x \in K$, *we have:* $\forall \epsilon > 0$, $\exists V \in \mathcal{V}(x)$, $\forall f \in A$, $\forall y \in V$, $d(f(x), f(y)) < \epsilon$;
- for all $x \in K$, *the set* $A(x) = \{f(x) : f \in A\}$ *is relatively compact.*

According to theorem 1.4.2, we obtain that there are uniform bounds for the first-order derivatives of u_n on any compact set. Hence this sequence is equicontinuous on any compact subset in Ω.
Let $(K_\nu)_\nu$ be an increasing sequence of compact subsets in Ω such that $\cup_\nu K_\nu = \Omega$. First in K_1, we can apply Ascoli's theorem and we obtain that there exists a subsequence of $(u_k)_k$ which converges uniformly in K_1 to a function u^1. Denote by $u_{1,1}$ a term of this subsequence such that $|| u_{1,1} - u^1 ||_{K_1} \leq 1$. In K_2, we apply again Ascoli's theorem and we obtain that there exists a subsequence of the previous one which converges uniformly in K_2 to a function u^2. It is clear that $u^2 = u^1$ in K_1. Denote by $u_{2,2}$ a term of this subsequence such that $|| u_{2,2} - u^2 ||_{K_2} \leq 1/2$. We continue like that for any k, and we finally obtain that the subsequence $(u_{k,k})_k$ of $(u_k)_k$ converges uniformly in any compact subset of Ω to a function u defined by $u = u^\nu$ on any compact set K_ν. Finally we conclude with corollary 1.4.4. □

Theorem 1.4.7 *If u is holomorphic in the polydisc $D = \{z :| z_j |< r_j, 1 \leq j \leq n\}$, we have*

$$u(z) = \sum_{\alpha} \frac{\partial^\alpha u(0)}{\alpha!} z^\alpha, \qquad z \in D,$$

with normal convergence on any compact polydiscs $\bar{D}(O, r')$, where $r' < r$.

Proof. The power series expansions

$$(\zeta_j - z_j)^{-1} = \frac{1}{\zeta_j} \sum_{\alpha_j} \frac{z_j^{\alpha_j}}{\zeta_j^{\alpha_j}}, \ \forall\, 1 \leq j \leq n$$

and

$$\prod_{j=1}^{n} (\zeta_j - z_j)^{-1} = \sum_{\alpha} \frac{z^\alpha}{\zeta^{\alpha+\mathbf{1}}}$$

converges normally when $(\zeta, z) \in \partial_0 D(O, r") \times D(O, r')$, where $r' < r'' < r$. Hence we can multiply by $u(\zeta_1, \ldots, \zeta_n)$ and integrate term by term in Cauchy's formula (Theorem 1.3.1) since u is continuous in $\bar{D}(O, r")$. According to proposition 1.4.1,

$$\frac{\partial^\alpha u(0)}{\alpha!} = \frac{1}{(2i\pi)^n} \int_{\partial_0 D(O, r")} \frac{u(\zeta_1, \ldots, \zeta_n)}{\prod_{k=1}^{n} \zeta_k^{\alpha_k+1}} d\zeta_1 \ldots d\zeta_n.$$

If we commute $\int$ and $\sum$, we obtain the theorem, with normal convergence in $\bar{D}(O, r')$, according to Cauchy's inequalities. □

Remark 1.4.8 *Conversely, any series in z in $D = \{z :| z_j |< r_j, 1 \leq j \leq n\}$, which converges normally on any compact polydiscs $\bar{D}(O, r')$ where $r' < r$, is an holomorphic function in D. Indeed, in this case a differentiation under the sign $\sum$ is valid.*

Theorem 1.4.9 *Let Ω be an open set in $\mathbb{C}^n$. $f \in \mathcal{O}(\Omega)$, i.e. f is holomorphic in Ω, iff for any $z \in \Omega$, f has a power series expension in a neighborhood of z, i.e. f is analytic in Ω.*

In the following we will use indiscriminately the two words holomorphic and analytic.
A domain $\Omega \subseteq \mathbb{C}^n$ is, by definition, an open and connected set.

Theorem 1.4.10 Uniqueness principle for holomorphic functions. *Let f and g be two holomorphic functions in a domain Ω in $\mathbb{C}^n$; then if there exists a non empty open set ω in Ω such that $f = g$ on ω, we have $f = g$ in Ω.*

Lemma 1.4.11 *Let Ω be a domain in $\mathbb{C}^n$ and $u \in \mathcal{O}(\Omega)$ such that there exists $z_0 \in \Omega$ with $\partial^\alpha u(z_0) = 0$, for any $\alpha \in \mathbb{N}^n$. Then $u \equiv 0$ in Ω.*

Proof of Lemma 1.4.11. Let $E = \{z \in \Omega : \partial^\alpha u(z) = 0, \text{ for any } \alpha \in \mathbb{N}^n\}$. E is closed in Ω, since it is an intersection of closed sets in Ω. If $w \in E$, there exists an open polydisc D, centered in w, with closure contained in Ω; according to Theorem 1.4.7, u is equal to 0 in D. And E is open in Ω. $z_0 \in E$, then $E \neq \emptyset$. Since Ω is connected, we have $E = \Omega$. □

Proof of Theorem 1.4.10. We apply the previous lemma to $u = f - g$. □

Exercise 1.4.12 *Let*

$$\Omega_1^\epsilon = \{z \in \mathbb{C}^2 :\mid z_1 \mid< 1+\epsilon,\ 1-\epsilon <\mid z_2 \mid< 1+\epsilon\}$$

and

$$\Omega_2^\epsilon = \{z \in \mathbb{C}^2 :\mid z_1 - 1 \mid< \epsilon,\ \mid z_2 \mid< 1+\epsilon\}.$$

1) *Prove that* $\Omega_1^\epsilon \cup \Omega_2^\epsilon$ *is a domain, contained in an* ϵ*-neighbourhood of the boundary of the bidisc*

$$\Omega = \{z \in \mathbb{C}^2 :\mid z_1 \mid< 1, \mid z_2 \mid< 1\}.$$

2) *Prove that any function* f*, holomorphic in* $\Omega_1^\epsilon \cup \Omega_2^\epsilon$*, has a holomorphic continuation to the bidisc*

$$\Omega^\epsilon = \{z \in \mathbb{C}^2 :\mid z_1 \mid< 1+\epsilon, \mid z_2 \mid< 1+\epsilon\}.$$

Indeed, for any z *such that* $|z_1| < 1+\epsilon$ *and* $|z_2| < 1+\epsilon$*, let choose any* ϵ' *such that* $0 < \epsilon' < \epsilon$ *and* $|z_2| < 1+\epsilon'$ *and we could consider the function* F *defined by*

$$F(z_1, z_2) = \frac{1}{2i\pi} \int_{|\zeta_2|=1+\epsilon'} \frac{f(z_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2.$$

Why this formula doesn't depend on ϵ' *chosen such that* $|z_2| < 1+\epsilon' < 1+\epsilon$ *?*
Prove that F *is holomorphic in* Ω^ϵ.
Prove that F *is agrees with* f *in* Ω_2^ϵ *and prove finally that* F *is agrees with* f *in* Ω_1^ϵ.

Theorem 1.4.13 *Let* Ω *be a domain in* $\mathbb{C}^n$ *and* $f \in \mathcal{O}(\Omega)$. *If* f *is not constant then the mapping* $f : \Omega \to \mathbb{C}$ *is open, i.e. the set* $f(\Omega)$ *is open in* $\mathbb{C}$.

Exercise 1.4.14 *Prove this theorem.*

Theorem 1.4.15 (Maximum Principle) *Let* Ω *be a domain in* $\mathbb{C}^n$. *If* $f \in \mathcal{O}(\Omega)$ *and if there exists* $w \in \Omega$ *such that* $\mid f(z) \mid\leq\mid f(w) \mid$ *for any* z *in a neighbourhood of* w*, then* $f(z) = f(w)$ *for any* $z \in \Omega$.

Proof. Let $D = D(w, r)$ be an open polydisc such that $\bar{D} \subset \Omega$. Then

$$V(D) \mid f(w) \mid\leq \int_D \mid f(\zeta) \mid dV(\zeta),$$

where $V(D)$ is the volum of D and dV is the volum form in $\mathbb{C}^n$. Indeed, Cauchy's integral formula gives for $n = 1$, $f(w) = \frac{1}{2\pi}\int_0^{2\pi} f(w + \varrho e^{i\theta})d\theta$ and $\pi r_1^2 f(w) = 2\pi f(w)\int_0^{r_1} \varrho d\varrho = \int_{D(w,r_1)} f(\zeta)dV(\zeta)$. Then $V(D)f(w) = \int_D f(\zeta)dV(\zeta)$. And we deduce the same formula for an arbitrary n with Fubini's formula.
Now let $D = D(w, r)$ be an open polydisc centered in w, such that for any $z \in D$, we have $\mid f(w) \mid - \mid f(z) \mid \geq 0$. Then

$$0 \leq \int_D (\mid f(w) \mid - \mid f(\zeta) \mid)dV(\zeta) = V(D) \mid f(w) \mid - \int_D \mid f(\zeta) \mid dV(\zeta) \leq 0,$$

according to the previous inequality. Thus $\mid f(w) \mid - \mid f(z) \mid = 0$ for any $z \in D$. According to proposition 1.2.2, f is constant in D : $f(z) = f(w)$, and according to theorem 1.4.10, f is constant in Ω. □

Another version of Maximum Principle.

Theorem 1.4.16 *Let Ω be a bounded domain in $\mathbb{C}^n$ and $f \in \mathcal{O}(\Omega)\cap\mathcal{C}(\overline{\Omega})$. If f is not constant in Ω, for any $z \in \Omega$*

$$\mid f(z) \mid < \sup_{w\in\partial\Omega} \mid f(w) \mid .$$

Exercise 1.4.17 *Prove this theorem.*

Theorem 1.4.18 (Schwarz's Lemma) *Let f be an holomorphic function in a neighbourhood of $\bar{D}(O, r) = (\bar{\Delta}(0, r))^n$, with order k in O and such that $\mid f(z) \mid \leq M$ in $\bar{D}(O, r)$. Then, we have*

$$\mid f(z) \mid \leq M\left(\frac{max_j \mid z_j \mid}{r}\right)^k, \text{ in } \bar{D}(O, r).$$

Proof. $f(z) = p_k(z) + \ldots$, where $p_k \not\equiv 0$ is an homogeneous polynomial. Let $z \in D^*(O, r)$ be fixed and let $t \in \mathbb{C}$, $\mid t \mid \leq r$. Let denote g by $g(t) = t^{-k} f(tz/\max_j \mid z_j \mid)$. Then Taylor expension of g is

$$g(t) = p_k(z/\max_j \mid z_j \mid) + p_{k+1}(z/\max_j \mid z_j \mid)t + \ldots$$

and g is an holomorphic function of one complex variable in $\Delta(0, r)$. By hypothesis we have $\mid f(tz/\max_j \mid z_j \mid) \mid \leq M$, then $\mid g(t) \mid \leq Mr^{-k}$ when $\mid t \mid = r$. According to maximum principle in $\mathbb{C}$, we have $\mid g(t) \mid \leq Mr^{-k}$ when $\mid t \mid \leq r$; in particular for $t = \max_j \mid z_j \mid$, we have $(\max_j \mid z_j \mid)^{-k} \mid f(z) \mid = \mid g(\max_j \mid z_j \mid) \mid \leq Mr^{-k}$. □

Exercise 1.4.19 *It follows easily from Theorem 1.4.18 that every bounded holomorphic function on $\mathbb{C}^n$ is constant (Liouville's theorem), and more generally, every holomorphic function f on $\mathbb{C}^n$ such that $\mid f(z) \mid \leq A(1+ \mid z \mid)^B$ with suitable constants A, $B \geq 0$ is in fact a polynomial of degree $\leq B$.*

Using the classical Cauchy formula, Hartogs (1906) showed that in $\mathbb{C}^n$, for $n > 1$, there is a domain D such that each function holomorphic on D necessarily has a holomorphic continuation to some larger domain $\Omega \supset D$.
Poincaré (1907) using the expansion of a function on a sphere by spherical harmonics, showed that each function, holomorphic in a neighbourhood of the boundary of a ball in $\mathbb{C}^2$, extends holomorphically to the interior of this ball.

1.5 Poisson formula and Jensen inequality

Theorem 1.5.1 *Let f be a holomorphic function in a neighbourhood of $\overline{D(O,\varrho)}$, where $D(O,\varrho) = D(0,\varrho_1) \times \ldots \times D(0,\varrho_n)$, in $\mathbb{C}^n$. Then for any $z \in D(O,\varrho)$, we have*

$$\Re f(z) = \left(\frac{1}{2\pi}\right)^n \int_{\zeta \in \prod_j \partial D_j} \prod_{j=1}^{n} \frac{\varrho_j^2 - \mid z_j \mid^2}{\mid \zeta_j - z_j \mid^2} \Re f(\zeta_1, \ldots, \zeta_n) d(arg\zeta_1) \wedge \ldots \wedge d(arg\zeta_n), \qquad (P)$$

$$\ln \mid f(z) \mid \leq \left(\frac{1}{2\pi}\right)^n \int_{\zeta \in \prod_j \partial D_j} \prod_{j=1}^{n} \frac{\varrho_j^2 - \mid z_j \mid^2}{\mid \zeta_j - z_j \mid^2} \ln \mid f(\zeta_1, \ldots, \zeta_n) \mid d(arg\zeta_1) \wedge \ldots \wedge d(arg\zeta_n). \qquad (J)$$

Proof. We will first prove the formula and the inequality when $n = 1$.
In this case introduce $\{a_1, \ldots, a_p\}$ the zero set of f (repeated with multiplicity) in $\overline{D(O,\varrho)}$ and the holomorphic function F defined by $F(z) = f(z) \prod_{j=1}^{p} \frac{\varrho^2 - \bar{a}_j z}{\varrho(z - a_j)}$ in the same open set as f. Remark that F has no zero in $\overline{D(O,\varrho)}$, $\mid F(z) \mid = \mid f(z) \mid$ in $\partial D(O,\varrho)$ and $\mid F(z) \mid \geq \mid f(z) \mid$ in $D(O,\varrho)$. And we conclude by using Poisson formula with harmonic functions $\Re f$ and $\ln \mid F \mid$.
In the case $n > 1$, we will apply the previous formula and inequality with respect to each variable z_j of $z = (z_1, \ldots, z_n)$. □

Corollary 1.5.2 *Let f be a holomorphic function in a neighbourhood of $\overline{D(O,\varrho)}$, where $D(O,\varrho) = D(0,\varrho_1) \times \ldots \times D(0,\varrho_n)$, in $\mathbb{C}^n$. Then, if $f(O) \neq 0$, $\ln \mid f \mid$ is integrable in $\bar{D}$ and we have*

$$\ln \mid f(O) \mid \leq \frac{1}{V(D)} \int_{\bar{D}} \ln \mid f \mid dv.$$

Proof. If we apply Jensen inequality to $\ln|f|$ in any polydisc $D(O,r) = D(0,r_1) \times \ldots \times D(0,r_n)$ (where $r_j \leq \varrho_j$ for any j) then

$$\ln \mid f(O) \mid \leq \left(\frac{1}{2\pi}\right)^n \int_{\zeta \in \prod_j \partial D(0,r_j)} \ln \mid f(\zeta_1, \ldots, \zeta_n) \mid d(arg\zeta_1) \wedge \ldots \wedge d(arg\zeta_n).$$

We multiply each side of the previous inequality by $\prod_{j=1}^{n} r_j$ and we integrate with respect to $r_1, \ldots, r_n$.
Without any restriction, we can suppose that $||f||_{\overline{D(O,\varrho)}} \leq 1$.
To conclude, by using Fubini theorem, it is sufficient to prove that $\ln \mid f \mid$ is integrable with

respect to dv. For any $m \geq 1$, let f_m be a function defined by $f_m = \max\{-m, \ln \mid f \mid\}$. $-m \leq f_m \leq 0$ and f_m is mesurable and integrable with respect to dv.
$\ln \mid f \mid \leq f_m$ and $V(D) \ln \mid f(O) \mid \leq \int_{\prod_j [0,\varrho_j]} \int_{\prod_j \partial D(0,r_j)} f_m \prod_{j=1}^n r_j d\theta_1 \ldots d\theta_n dr_1 \ldots dr_n = \int_{\bar{D}} f_m dv$. The sequence (f_m) is decreasing and pointwise converging to $\ln \mid f \mid$. According to monotonic convergence theorem, we conclude that $\ln \mid f \mid$ is integrable with respect to dv and we obtain finally the required inequality. $\square$

Corollary 1.5.3 *Let Ω be a domain in $\mathbb{C}^n$ and f be an holomorphic function in Ω. We suppose that $f \not\equiv 0$ in Ω. Then the set $Z(f) := \{z \in \Omega : f(z) = 0\}$ has a $2n$-dimensional Lebesgue measure equal to 0.*

Proof. First remark that the interior of $Z(f)$ is empty. Then $\Omega \setminus Z(f)$ is dense in Ω and there exists a sequence of polydiscs $D(z_\nu, \varrho_\nu) \subset \Omega$ such that $f(z_\nu) \neq 0$ and $\cup_\nu \overline{D(z_\nu, \varrho_\nu)} = \Omega$. Indeed, since $\Omega \setminus Z(f) = \cup_{z \notin Z(f)} B(z, \mathrm{dist}(z, Z(f) \cup \partial\Omega))$, we have $(\mathbb{Q}^n + i\mathbb{Q}^n) \cap (\Omega \setminus Z(f))$ is dense in Ω. Let denote $(z_\nu)_{\nu \in \mathbb{N}} = (\mathbb{Q}^n + i\mathbb{Q}^n) \cap (\Omega \setminus Z(f))$. Let denote also $(r_{\mu,\nu})_{\mu \in \mathbb{N}} = \mathbb{Q}^+ \cap [0, dist(z_\nu, \partial\Omega))$. Then $\cup_{(\nu,\mu) \in \mathbb{N}^2} \overline{D(z_\nu, r_{\mu,\nu})} = \Omega$.
Then we can apply Corollary 1.5.2 to each $f(z_\nu)$ and deduce for any ν, that $Z(f) \cap \overline{D(z_\nu, \varrho_\nu)}$ has a $2n$-dimensional Lebesgue measure equal to 0. Then we conclude. $\square$

Chapter 2

Introduction to the $\bar{\partial}$-problem, extension theorems, applications and different notions of convexity

2.1 Cauchy Formula in One Variable

We start by recalling a few elementary facts in one complex variable theory. Let $\Omega \subset \mathbb{C}$ be an open set and let $z = x + iy$ be the complex variable, where x, $y \in \mathbb{R}$. If f is a function of class $\mathcal{C}^1$ on Ω, we have

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

with the usual notations $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.
The function f is holomorphic on Ω iff df is $\mathbb{C}$-linear, that is, $\partial f/\partial \bar{z} = 0$.

Theorem 2.1.1 Cauchy-Green-Pompéiu formula (1904) *Let $K \subset \mathbb{C}$ be a compact set with piecewise $\mathcal{C}^1$ boundary ∂K. Then for every $f \in \mathcal{C}^1(K, \mathbb{C})$*

$$f(w) = \frac{1}{2i\pi} \int_{\partial K} \frac{f(z)}{z - w} dz + \frac{1}{2i\pi} \int_K \frac{1}{z - w} \frac{\partial f}{\partial \bar{z}}(z) dz \wedge d\bar{z}, \;\; \forall\, w \in K^\circ,$$

where $d\lambda(z) = \frac{i}{2} dz \wedge d\bar{z} = dx \wedge dy$ is the Lebesgue measure on $\mathbb{C}$.

Proof. Assume for simplicity that $w = 0$. As the function $z \mapsto 1/z$ is locally integrable at $z = 0$, we get

$$\int_K \frac{1}{\pi z} \frac{\partial f}{\partial \bar{z}} d\lambda(z) = \lim_{\epsilon \to 0} \int_{K \setminus D(0,\epsilon)} \frac{1}{\pi z} \frac{\partial f}{\partial \bar{z}} \frac{i}{2} dz \wedge d\bar{z} =$$

$$\lim_{\epsilon \to 0} \int_{K \setminus D(0,\epsilon)} d[\frac{1}{2i\pi} f(z) \frac{dz}{z}] = \frac{1}{2i\pi} \int_{\partial K} f(z) \frac{dz}{z} - \lim_{\epsilon \to 0} \frac{1}{2i\pi} \int_{\partial D(0,\epsilon)} f(z) \frac{dz}{z}$$

by Stokes' formula. The last integral is equal to $\frac{1}{2\pi}\int_0^{2\pi} f(\epsilon e^{i\theta})d\theta$ and converges to $f(0)$ as ϵ tends to 0. □

When f is holomorphic on Ω, we get the usual Cauchy formula

$$f(w) = \frac{1}{2\pi}\int_{\partial K} \frac{f(z)}{z-w}dz, \;\; w \in K^\circ.$$

The Cauchy and Cauchy-Green-Pompéiu formulae are fundamental technical tools in the theory of functions of one complex variable. Examples of profound applications of these formulas are given in the works of Carleson and Vitushkin. In the first of these, the famous "Corona" problem for the disc in $\mathbb{C}$ is solved. In the second is solved the problem, going back to Weierstrass and Runge, on the uniform approximation by holomorphic functions on compact sets in $\mathbb{C}$.
Many other basic properties of holomorphic functions can be derived from theses formulas: power and Laurent series expansions, Cauchy residue formula, ...
Another interesting consequence is:

Corollary 2.1.2 *The L^1_{loc} function $E(z) = 1/\pi z$ is a fundamental solution of the operator $\partial/\partial\bar{z}$ on $\mathbb{C}$, i.e. $\partial E/\partial\bar{z} = \delta_0$ (Dirac measure at 0). As a consequence, if v is a distribution with compact support in $\mathbb{C}$, then the convolution $u = (1/\pi z) * v$ is a solution of the equation $\partial u/\partial\bar{z} = v$.*

Proof. Apply the previous Cauchy-Green-Pompéiu formula with $w = 0$, $f \in \mathcal{D}(\mathbb{C})$ and $K \supset \text{Supp } f$, so that $f = 0$ on the boundary ∂K. Then $< \delta_0, f >= f(0) =< 1/\pi z, -\partial f/\partial\bar{z} >$ $=< \partial/\partial\bar{z}(1/\pi z), f >$.
u is a solution of the equation $\partial u/\partial\bar{z} = v$ iff for any $f \in \mathcal{D}(\mathbb{C})$, we have

$$< v, f >=< u, -\partial f/\partial\bar{z} > .$$

And we have $< E * v, -\dfrac{\partial f}{\partial\bar{z}} >=< \dfrac{\partial(E*v)}{\partial\bar{z}}, f >=< v, f >$, because
$\dfrac{\partial(E*v)}{\partial\bar{z}}(z) = \dfrac{\partial}{\partial\bar{z}}(\int_{\mathbb{C}} E(z-w)v(w)d\lambda(w)) = \int_{\mathbb{C}} \dfrac{\partial E}{\partial\bar{z}}(z-w)v(w)d\lambda(w) = v(z).$
To summarise, in the sense of distributions we have $\dfrac{\partial}{\partial\bar{z}}(E*v) = \dfrac{\partial E}{\partial\bar{z}} * v = \delta_0 * v = v.$ □

Remark 2.1.3 *If u is a solution, $u + h$ is an other solution for any $h \in \mathcal{O}(\mathbb{C})$.*
It should be observed that the previous formula cannot be used to solve the equation $\partial u/\partial\bar{z} = v$ when Supp v is not compact.
If Supp v is compact, a solution u with compact support need not always exist.
If v is a distribution with compact support in $\mathbb{C}$ (as in the previous corollary) such that there exists a solution u with compact support, then $< v, z^n >= 0$ for all integers $n \geq 0$.
It is sufficient to choose $f \in \mathcal{D}(\mathbb{C})$ such that $f(z) = z^n$ in $D(0,R)$, and Supp $u \cup$ Supp $v \subset D(0,R)$.

Conversely, when the necessary condition $< v, z^n >= 0$ is satisfied and Supp v is contained in the disk $|z| < R$, then the canonical solution $u = (1/\pi z) * v$ has compact support.
Indeed, this is easily seen by means of the power series expansion

$$(z - w)^{-1} = \sum w^n z^{-n-1},$$

where $|w| < R < |z|$. If $R < |z|$, we have $E * v(z) = \int_{\mathbb{C}} E(z - w)v(w)d\lambda(w)$

$$= \frac{1}{\pi} \sum_n z^{-n-1} \int_{\mathbb{C}} w^n v(w) d\lambda(w) = \frac{1}{\pi} \sum_n z^{-n-1} \int_{D(0,R)} w^n v(w) d\lambda(w) = 0.$$

2.2 The inhomogeneous Cauchy-Riemann equations in $\mathbb{C}^n$, when $n \geq 2$ and some extension theorems

We first consider the equation

$$\bar{\partial} u = f$$

where f is a given form of type $(0, 1)$ with compact support ($f(z) = \sum_j f_j(z) d\bar{z}_j$), and the unknown u is a function. $\bar{\partial} f = 0$ is a necessary condition for the existence of a solution. We want to solve the differential equations

$$\partial u / \partial \overline{z_j} = f_j,\ j = 1, \ldots, n \tag{2.1}$$

when the compatibility conditions

$$\partial f_j / \partial \bar{z}_k - \partial f_k / \partial \bar{z}_j = 0,\ j, k = 1, \ldots, n \tag{2.2}$$

are fulfilled.
$\mathcal{C}_0^k(\mathbb{C}^n)$ is the subset of $\mathcal{C}^k(\mathbb{C}^n)$ containing functions with compact support.

Theorem 2.2.1 *Let $f_j \in \mathcal{C}_0^k(\mathbb{C}^n)$, $j = 1, \ldots, n$ where $k > 0$, and assume that (2.2) is fulfilled ($n > 1$). Then there is a function $u \in \mathcal{C}_0^k(\mathbb{C}^n)$ satisfying (2.1).*

Remark 2.2.2 *Note that this theorem is false when $n = 1$, for $f_1 \in \mathcal{C}_0^\infty$ with Lebesgue integral different from 0.*
Let suppose that there exists $u_1 \in \mathcal{C}_0^\infty(\Delta)$ such that $\frac{\partial u_1}{\partial \bar{z}} = f_1$ on Δ (an open disc containing the supports of f_1 and u_1). We can apply theorem 2.1.1 to u_1. Then $u_1(z) = \frac{1}{2i\pi} \int_\Delta \frac{f_1(w)}{w-z} dw d\bar{w}$. In addition, $\int_\Delta f_1 dz d\bar{z} = \int_{\mathbb{C}} f_1 dz d\bar{z} = \int_{\mathbb{C}} \frac{\partial u_1}{\partial \bar{z}} dz d\bar{z}$

$$= \lim_{R \to \infty} \int_{D(O,R)} \frac{\partial u_1}{\partial \bar{z}} dz d\bar{z} = - \lim_{R \to \infty} \int_{D(O,R)} d(u_1(z) dz) = 0,$$

if u_1 has a compact support, according to Stokes' formula. There is a contradiction.

Proof of Theorem 2.2.1 We set for any $z \in \mathbb{C}^n$

$$u(z) = \frac{1}{2i\pi} \int_{\mathbb{C}} (\tau - z_1)^{-1} f_1(\tau, z_2, \ldots, z_n) d\tau \wedge d\bar{\tau}$$

$$= \frac{1}{2i\pi} \int_{\mathbb{C}} \tau^{-1} f_1(z_1 + \tau, z_2, \ldots, z_n) d\tau \wedge d\bar{\tau}.$$

The second form of the definition shows that $u \in \mathcal{C}^k(\mathbb{C}^n)$, and it is clear that $u(z) = 0$ if $\mid z_2 \mid + \ldots + \mid z_n \mid$ is large enough.
Let $z' = (z_2, \ldots, z_n)$ be fixed. $u(z) = (E_{z_1} * f_{1,z'})(z_1)$, where $f_{1,z'}(z_1) = f_1(z)$.
Denote by $K_{z'}$ the compact support in $\mathbb{C}$ of $\tau \mapsto f_1(\tau, z')$. Choose $K \subset \mathbb{C}$ a compact set with piecewise $\mathcal{C}^1$ boundary ∂K, such that K contains z_1 and $K_{z'}$ is included in the interior of K. Then we verify that $\displaystyle\int_{\partial K} \frac{u(\lambda, z')}{\lambda - z_1} d\lambda = 0$.
Indeed, this last integral is equal to $\displaystyle\frac{1}{2i\pi} \int_{\tau \in \mathbb{C}} \int_{\lambda \in \partial K} \frac{d\lambda}{(\lambda - z_1)(\tau - \lambda)} f_1(\tau, z') d\tau d\bar{\tau}$, where $\displaystyle\int_{\lambda \in \partial K} \frac{d\lambda}{(\lambda - z_1)(\tau - \lambda)} = 0$, according to residus theorem, in the two cases $z_1 = \tau$ and $z_1 \neq \tau$.
Then from Theorem 2.1.1 or Corollary 2.1.2, we deduce that $\partial u / \partial \bar{z}_1 = f_1$.
If $k > 1$, by differentiating under the sign of integration and using the fact that $\partial f_1 / \partial \bar{z}_k = \partial f_k / \partial \bar{z}_1$, we obtain

$$\partial u / \partial \bar{z}_k = \frac{1}{2i\pi} \int_{\mathbb{C}} (\tau - z_1)^{-1} \frac{\partial f_k(\tau, z_2, \ldots, z_n)}{\partial \bar{\tau}} d\tau \wedge d\bar{\tau} = f_k(z),$$

where the last equality follows from theorem 2.1.1. Hence u satisfies all the equations (2.1), which means in particular that u is holomorphic outside the compact set supp $f_1 \cup \ldots \cup$supp f_n. Since $u(z) = 0$ if $\mid z_2 \mid + \ldots + \mid z_n \mid$ is large enough, from the uniqueness of analytic continuation, we conclude that $u(z) = 0$ in the unbounded component of $\mathbb{C}^n \setminus$ supp $f_1 \cup \ldots \cup$ supp f_n, i.e., u has compact support. □

Theorem 2.2.3 (Hartogs) *Let Ω be an open set in $\mathbb{C}^n$, $n > 1$, and let K be a compact subset of Ω such that $\Omega \setminus K$ is connected. For every $f \in \mathcal{O}(\Omega \setminus K)$ one can find an unique $F \in \mathcal{O}(\Omega)$ so that $f = F$ in $\Omega \setminus K$.*

This is a striking contrast with the situation in the case of one complex variable. Indeed, the previous result is false in one variable. It is sufficient to think about an holomorphic function with a singularity. And more generally, we have the following result.

Theorem 2.2.4 *Let Ω be an open set in $\mathbb{C}$. Then there exists a holomorphic function in Ω that does not extend to any open set containing Ω.*

Proof. See for instance [Ho1] p14-15, with the use of Weierstrass theorem.

Proof of Theorem 2.2.3. Let $\varphi \in \mathcal{C}_0^\infty(\Omega)$ be equal to 1 in a neighborhood of K. Set $f_0 = (1-\varphi)f$, defined as 0 in a neighborhood of K. Then $f_0 \in \mathcal{C}^\infty(\Omega)$, and we want to find $g \in \mathcal{C}^\infty(\mathbb{C}^n)$ so that

$$F = f_0 - g$$

has the required properties. The function F will be analytic in Ω iff

$$\bar\partial g = \bar\partial f_0 = -f\bar\partial\varphi := \psi,$$

where ψ, defined as 0 in a neighborhood of K and outside Ω ($spt(\psi) \subset \Omega \setminus K$), is a $(0,1)$-form with components in $\mathcal{C}_0^\infty(\mathbb{C}^n)$. In addition ψ satisfies the compatibility conditions (2.2). Hence the previous equation has a solution g, according to theorem 2.2.1, which vanishes in the unbounded component of the complement of the support of φ. The boundary of the support of g belongs to Ω, so there exists a non-empty open set ω in $\Omega \setminus (K \cup \text{supp}\ \varphi)$ where $g = 0$, $F = f_0$ and $\varphi = 0$. Hence the analytic function F in Ω which we have defined, coincides with f on $\omega \subset \Omega \setminus K$, which is connected. Then $f = F$ in $\Omega \setminus K$.
Let remark that Ω is necessary connected and consequently, F is unique. □

A refined version of the Hartogs extension theorem 2.2.3, due to Bochner, shows that f need only be given as a $\mathcal{C}^1$ function on $\partial\Omega$, satisfying the tangencial Cauchy-Riemann equations (a so-called CR-function). Then f extends as a holomorphic function $F \in \mathcal{O}(\Omega) \cap \mathcal{C}^0(\bar\Omega)$, provided that $\partial\Omega$ is connected. (see for instance Theorem 2.3.2', [Ho1] p.31.).

Corollary 2.2.5 *Let $n \geq 2$, and let $D \subset \mathbb{C}^n$ be a domain. If $f \in \mathcal{O}(D)$, $f \not\equiv 0$ and $N(f) = \{z \in D : f(z) = 0\}$, then*
(i) $D \setminus N(f)$ is connected,
(ii) $N(f)$ is not compact.

Proof. $D \setminus N(f)$ is open (because $N(f)$ is closed in D) and it is dense in D (because $N(f)$ has an empty interior).
(i) We only have to prove that for every open ball $B \subset D$ the set $B \setminus N(f)$ is connected.
Indeed, let suppose that it is the case and that $D \setminus N(f) = U_1 \cup U_2$, where U_j are non empty disjoint open sets. We know that $D = \overline{D \setminus N(f)} = \bar U_1 \cup \bar U_2$. D is connected, then $\bar U_1 \cap \bar U_2 \neq \emptyset$. Let $a \in \bar U_1 \cap \bar U_2$. Let B be a ball centered in a such that $B \setminus N(f)$ is a domain. We have $B \setminus N(f) = (B \cap U_1) \cup (B \cap U_2)$. Consequently, one $B \cap U_j$ is empty, which is impossible since B is a neighborhood of $a \in \bar U_j$. Then we will deduce that $D \setminus N(f)$ is connected.
To do this we consider two arbitrary points z, $w \in B \setminus N(f)$ and show that z and w belong to the same component of $B \setminus N(f)$. Let X be a complex line which contains z and w. Then the restriction of f to $X \cap B$ can be considered as a holomorphic function of one complex variable. Since f is not identically equal to zero on $X \cap B$, $X \cap B \cap N(f)$ is discrete in $X \cap B$. Since $X \cap B$ is a disc and therefore connected, this implies that $(X \cap B) \setminus N(f)$ is connected. Consequently, there exists a continuous path between z and w in $(X \cap B) \setminus N(f)$ and finally z and w belong to the same component of $B \setminus N(f)$.
(ii) Assume that $N(f)$ is compact. Since, by part (i), $D \setminus N(f)$ is connected, then it follows from

Hartog's theorem that $1/f$ can be continued holomorphically to $N(f)$. This is a contradiction, because $f = 0$ on $N(f)$. $\square$

We can generalize this result to the case of a finite number of holomorphic functions: we replace $N(f)$ by an analytic set A (for a definition of analytic sets and some properties, see [Gu], [Na]).

To solve the inhomogeneous Cauchy-Riemann equations locally without the hypothesis on compact support is slightly more complicated than the previous proof. For example, one cannot, in general, solve these equations with regularity $\mathcal{C}^\infty$ for any open set: this is linked to the notion of pseudo-convexity.
Here is the following result in polydiscs, which also implies local solvability.

Theorem 2.2.6 *Let D be an open set in $\mathbb{C}^n$ and let f be a $\mathcal{C}^\infty(D)$ $(0,1)$-form such that the compatibility condition $\bar\partial f = 0$ is fulfilled in D. Then, for every open polydisc P relatively compact in D, there exists $u \in \mathcal{C}^\infty(P)$ satisfying $\bar\partial u = f$ in P.*

Proof. We shall prove inductively that the theorem is true if $f_{m+1} = \ldots = f_n = 0$.
This is trivial if $m = 0$.
Assume that it has already been proved for $m-1$. Let $D' = \prod_{j=1}^n D'_j$ and D'' be open polydiscs such that $P \Subset D'' \Subset D' \Subset D$ and choose $\chi \in \mathcal{C}_0^\infty(D'_m)$ such that $\chi = 1$ in D''_m. Define for $z \in D'$

$$v(z) := -\frac{1}{2i\pi} \int_{\zeta \in D'_m} \frac{\chi(\zeta) f_m(z_1, \ldots, z_{m-1}, \zeta, z_{m+1}, \ldots, z_n)}{z_m - \zeta} d\zeta \wedge d\bar\zeta.$$

Since $\chi \in \mathcal{C}_0^\infty(D'_m)$, after the change of variables $\zeta - z_m \to \zeta$, we see that $v \in \mathcal{C}^\infty(D')$. Since $\chi = 1$ in D''_m, it follows from Corollary 2.1.2 that

$$\frac{\partial v}{\partial \bar{z_m}} = f_m, \text{ in } D''.$$

Since $\frac{\partial f_m}{\partial \bar z_j} = \frac{\partial f_j}{\partial \bar{z_m}} = 0$ for $j = m+1, \ldots, n$, we obtain by differentiation under the sign of integration that

$$\frac{\partial v}{\partial \bar z_j} = 0 = f_j \text{ in } D'' \text{ for } j = m+1, \ldots, n.$$

Then the functions $\tilde{f_j} := f_j - \frac{\partial v}{\partial \bar z_j}$, $j = 1, \ldots, n$, fulfil the compatibility conditions (2.2) and it follows from above that $\tilde{f_m} = \ldots = \tilde{f_n} = 0$ in D''. Therefore, by the inductive hypothesis, we can find $w \in \mathcal{C}^\infty(P)$ such that $\frac{\partial w}{\partial \bar z_j} = \tilde{f_j}$ $\forall j = 1, ..., n$ in P and $u := v + w$ is the required solution of $\bar\partial u = f$ in P. $\square$

See [Ho1] p32-33, for a result about (p,q)-forms with $q \geq 1$.

As we have seen with Hartogs and Bochner's theorems, for some open sets in $\mathbb{C}^n$, any holomorphic function automatically extends to a strictly larger open set. This phenomenon, absent in dimension 1 shows that any open of $\mathbb{C}^n$ is not a "natural" open set of definition of holomorphic functions. The "natural" open sets are those for which there are holomorphic functions that do not extend to a larger open set; they are called "domains of holomorphy".

A last extension theorem when $n \geq 2$.
We usually call the following open set Ω the Hartogs pot, according to its form in the space $\mathbb{R}^3$ of the points $(z_1, \mid z_2 \mid)$. This theorem says that the domain of definition of f can be extended to the filled pot.

Theorem 2.2.7 *Let consider the following domain $\Omega := \Omega_1 \cup \Omega_2$ in $\mathbb{C}^n$:*

$$\{(z_1, z_2) \in \mathbb{C}^2 :\mid z_1 \mid< r_1, \mid z_2 \mid< \epsilon\} \cup \{(z_1, z_2) \in \mathbb{C}^2 : r_1 - \epsilon <\mid z_1 \mid< r_1, \mid z_2 \mid< r_2\},$$

where $0 < \epsilon < r_1$.
Let f be a continuous function in Ω, holomorphic in z_1 in Ω_1 and in z_2 in Ω_2. Then f has an unique holomorphic extension in all the polydisc $D(O, r)$.

Proof. Let fix δ_1 and δ_2 such that $r_1 - \epsilon < \delta_1 < r_1$ and $\epsilon < \delta_2 < r_2$.
When $\mid z_2 \mid< \epsilon$ is fixed, the function $f(z_1, z_2)$ is holomorphic in the open disc $\mid z_1 \mid< r_1$. According to the Cauchy's formula in z_1, when $\mid z_1 \mid< \delta_1$,

$$f(z) = \frac{1}{2i\pi} \int_{|\zeta_1|=\delta_1} (\zeta_1 - z_1)^{-1} f(\zeta_1, z_2) d\zeta_1.$$

When $r_1 - \epsilon <\mid \zeta_1 \mid< r_1$, the function $f(\zeta_1, z_2)$ is holomorphic in z_2 in the disc $\mid z_2 \mid< r_2$. According to Cauchy's formula in z_2, when $\mid z_2 \mid< \delta_2$

$$f(\zeta_1, z_2) = \frac{1}{2i\pi} \int_{|\zeta_2|=\delta_2} (\zeta_2 - z_2)^{-1} f(\zeta_1, \zeta_2) d\zeta_2.$$

Otherwise, the function $F_{(\delta_1,\delta_2)}$ defined by

$$\frac{1}{(2i\pi)^2} \int_{|\zeta_1|=\delta_1} \int_{|\zeta_2|=\delta_2} (\zeta_1 - z_1)^{-1} (\zeta_2 - z_2)^{-1} f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2$$

is an holomorphic function in $D(O, \delta)$. According to the two previous equalities, $F_{(\delta_1,\delta_2)} = f$ in the open set $\mid z_1 \mid< \delta_1, \mid z_2 \mid< \epsilon$. According to the identity theorem, we deduce that $F_{(\delta_1,\delta_2)} = f$ in all $\Omega \cap D(O, \delta)$.
If $r_1 - \epsilon < \delta_1 < \delta_1' < r_1$ and $\epsilon < \delta_2 < \delta_2' < r_2$, $F_{(\delta_1,\delta_2)} = F_{(\delta_1',\delta_2')}$ in $D(O, \delta)$. □

2.3 Domains of holomorphy, holomorphic convexity and pseudoconvexity in $\mathbb{C}^n$, $n \geq 2$

The phenomena of Hartogs described in Theorem 2.2.3, leds Hartogs to the following definition: a domain of holomorphy is an open subset D in $\mathbb{C}^n$ such that there is no part of ∂D across which all functions $f \in \mathcal{O}(D)$ can be extended. More precisely:

Definition 2.3.1 *Let $D \subset \mathbb{C}^n$ be an open subset. D is said to be a domain of holomorphy if for every connected open set $U \subset \mathbb{C}^n$ which meets ∂D and every connected component V of $U \cap D$ there exists $f \in \mathcal{O}(D)$ such that $f_{|V}$ has no holomorphic extension to U.*

Example 2.3.2 1*) Every open subset $D \subset \mathbb{C}$ is a domain of holomorphy (for any $z_0 \in \partial D$, $f(z) = (z - z_0)^{-1}$ cannot be extended at z_0).*
2*) $\mathbb{C}^n$ is a domain of holomorphy.*
3*) In $\mathbb{C}^n$ every convex open subset is a domain of holomorphy: if $\Re\langle z - z_0, \xi_0\rangle = 0$ is a supporting hyperplane of ∂D at z_0, the function $f(z) = (\langle z - z_0, \xi_0\rangle)^{-1}$ is holomorphic on D but cannot be extended at z_0.*
4*) Hartogs figure. Assume that $n \geq 2$. Let $\omega \subset \mathbb{C}^{n-1}$ be a connected open set and $\omega' \subsetneq \omega$ an open connected subset. Consider the open sets in $\mathbb{C}^n$:*

$$\begin{aligned} &\Omega = ((D(R) \setminus \overline{D(r)}) \times \omega) \cup (D(R) \times \omega') && \textit{Hartogs figure,} \\ &\tilde{\Omega} = D(R) \times \omega && \textit{filled Hartogs figure.} \end{aligned}$$

where $0 \leq r < R$ and $D(r) \subset \mathbb{C}$ denotes the open disk centered at 0 with radius r.
Then every function $f \in \mathcal{O}(\Omega)$ can be extended to $\tilde{\Omega} = D(R) \times \omega$ by means of the Cauchy formula:

$$\tilde{f}(z_1, z') = \frac{1}{2i\pi} \int_{|\zeta_1| = \varrho} \frac{f(\zeta_1, z')}{\zeta_1 - z_1} d\zeta_1, \; z \in \tilde{\Omega}, \quad max\{| \, z_1 \, |, r\} < \varrho < R.$$

Remark first that this last definition doesn't depend on the choice of $max\{| \, z_1 \, |, r\} < \varrho < R$. $\tilde{f} \in \mathcal{O}(D(R) \times \omega)$ and $\tilde{f} = f$ on $D(R) \times \omega'$, so we must have $\tilde{f} = f$ on Ω since Ω is connected (prove it). It follows that Ω is not a domain of holomorphy. □

Let us quote one interesting last extension theorem.

Theorem 2.3.3 (Riemann's extension theorem) *Let D be an open set in $\mathbb{C}^n$ and S a closed submanifold of codimension ≥ 2. Then every $f \in \mathcal{O}(\Omega \setminus S)$ extends holomorphically to Ω.*

According to **Hartogs theorem**, we see that if K is a compact subset in a domain Ω in $\mathbb{C}^n$ ($n \geq 2$) such that $\Omega \setminus K$ is connected, then any holomorphic function in $\Omega \setminus K$, can be extended holomorphically in all Ω. Consequently, $\Omega \setminus K$ is not a domain of holomorphy.

We first introduce the notion of holomorphic hull of a compact set K in an open set D. This can be seen somehow as the complex analogue of the notion of (affine) convex hull for a compact set in a real vector space. It is shown that domains of holomorphy in $\mathbb{C}^n$ are characterized by a property of holomorphic convexity. And finally, we will see that holomorphic convexity is equivalent to another notion of convexity: pseudoconvexity, another complex analogue of the geometric notion of convexity.

Definition 2.3.4 *Let D be an open set in $\mathbb{C}^n$ and let K be a compact subset in D. Then the holomorphic hull of K in D is defined to be*

$$\hat{K} = \hat{K}_D = \hat{K}_{\mathcal{O}(D)} = \{z \in D :\mid f(z) \mid \leq \sup_K \mid f \mid, \forall f \in \mathcal{O}(D)\}.$$

Remark 2.3.5 *However, when Ω is arbitrary, $\hat{K}_{\mathcal{O}(\Omega)}$ is not always compact in Ω. For instance, in the case when $\Omega = \mathbb{C}^n \setminus \{O\}$, $n \geq 2$, then $\mathcal{O}(\Omega) = \mathcal{O}(\mathbb{C}^n)$ and the holomorphic hull of $K = S(0,1)$ is the non compact set $\hat{K} = \bar{B}(O,1) \setminus \{O\}$.*
In general, $\hat{K}_{\mathcal{O}(\Omega)} \subset \hat{K}_{aff}$ and they are not equal.

Definition 2.3.6 *An open set Ω in $\mathbb{C}^n$ is said to be holomorphically convex if the holomorphic hull $\hat{K}_{\mathcal{O}(\Omega)}$ of every compact set $K \subset \Omega$ is compact.*

Example 2.3.7 *Let $\Omega = \{z \in \mathbb{C}^n : \frac{1}{2} <\mid z \mid< 2\}$ and $K = S(O,1)$ be a compact subset in Ω.*
If $n = 1$, we can prove (by using holomorphic functions $1/z$ and z) that $\hat{K}_{\mathcal{O}(\Omega)} = K$.
If $n \geq 2$, we can prove that $\bar{B}(O,1) \cap \Omega = \hat{K}_{\mathcal{O}(\Omega)}$.
Indeed, the Hartogs phenomena implies that any holomorphic function f in Ω can be extended holomorphically in all $B(O,2)$. Denote by $\tilde{f}$ its extension. According to the maximum principle, $\mid \tilde{f}(z) \mid=\mid f(z) \mid \leq \sup_K \mid f \mid$ for any $\frac{1}{2} <\mid z \mid< 1$. Then $\bar{B}(O,1) \cap \Omega \subset \hat{K}_{\mathcal{O}(\Omega)}$. In addition, $\hat{K}_{\mathcal{O}(\Omega)} \subset \hat{K}_{aff} = \bar{B}(O,1) \cap \Omega$.
Consequently $\hat{K}_{\mathcal{O}(\Omega)}$ is not relatively compact in Ω and Ω is not holomorphically convex.

Then we have the following theorem which characterizes domains of holomorphy in terms of holomorphic convexity, obtained by H. Cartan and P. Thullen (1932).

Theorem 2.3.8 *Let Ω be an open subset of $\mathbb{C}^n$. The following properties are equivalent:*
a) Ω is a domain of holomorphy;
b) Ω is holomorphically convex;
c) For every countable subset $\{w_j\}_{j\in\mathbb{N}} \subset \Omega$ without accumulation points in Ω and every sequence of complex numbers (a_j), there exists an interpolation function $F \in \mathcal{O}(\Omega)$ such that $F(w_j) = a_j$.
d) There exists a function $F \in \mathcal{O}(\Omega)$ which is unbounded on any neighborhood of any point of $\partial\Omega$.

Example 2.3.9 *Let Ω be a domain in $\mathbb{C}^n$ and $(f_j)_{1\leq j\leq N}$ be a family of analytic functions in Ω.*

$$P = \{z \in \Omega; \mid f_j(z) \mid< 1, 1 \leq j \leq N\}$$

is called an analytic polyhedron in Ω.
Since the polydisc $P_N(O,1)$ *is convex and then it is a domain of holomorphy in* $\mathbb{C}^N$, P *is also a domain of holomorphy in* $\mathbb{C}^n$ *if* Ω *is a domain of holomorphy in* $\mathbb{C}^n$ *or if* $P \subset\subset \Omega$.

We now finish this chapter by seeing that a holomorphically convex open set in $\mathbb{C}^n$ must satisfy some more geometric convexity condition, known as pseudoconvexity, which is most easily described in terms of the existence of plurisubharmonic exhaustion functions.

Plurisubharmonic functions are the several variables complex counterparts of subharmonic functions in $\mathbb{C}$. These objects are relatively soft in comparison to holomorphic functions which are rigid objects.

Definition 2.3.10 *A function* $u : \Omega \to [-\infty, +\infty)$ *defined on an open subset* $\Omega \subset \mathbb{C}^n$ *is said to be plurisubharmonic (psh) if*
a) u is upper semicontinuous (usc);
b) for every complex line $L \subset \mathbb{C}^n$, $u\,|_{\Omega\cap L}$ *is subharmonic as a function of one complex variable, in the open set* $\Omega \cap L$ *in* $\mathbb{C}$.
The set of plurisubharmonic functions on Ω *is denoted by* $Psh(\Omega)$.

Definition 2.3.11 *A function* $\Psi : X \to [-\infty, +\infty[$ *on a topological space* X *is said to be an exhaustion if all sublevel sets* $X_c := \{z \in X; \Psi(z) < c\}$, $c \in \mathbb{R}$, *are relatively compact in* X. *Equivalently,* Ψ *is an exhaustion if and only if* Ψ *tends to* $+\infty$ *relatively to the filter of complements* $X \setminus K$ *of compact subsets of* X *(for any* $M \in \mathbb{R}$, *there exists a compact set* $K \subset X$, *such that* $u \geq M$ *in* $X \setminus K$*).*

A function Ψ on an open set $\Omega \subset \mathbb{R}^n$ is thus an exhaustion if and only if $\Psi(x) \to +\infty$ as $x \to \partial\Omega$ or $|x| \to \infty$. Since plurisubharmonic functions appear as the natural generalization of convex functions in complex analysis, we are led to the following definition.

Definition 2.3.12 *Let* Ω *be an open set in* $\mathbb{C}^n$. *Then* Ω *is said to be*
a) weakly pseudoconvex if there exists a smooth plurisubharmonic exhaustion function $\Psi \in Psh(\Omega) \cap \mathcal{C}^\infty(\Omega)$;
b) strongly pseudoconvex if there exists a smooth strictly plurisubharmonic exhaustion function $\Psi \in Psh(\Omega) \cap \mathcal{C}^\infty(\Omega)$, *i.e.* $H\Psi$ *is positive definite at every point.*

Theorem 2.3.13 *Every holomorphically convex open set* Ω *is weakly pseudoconvex.*

Theorem 2.3.14 *Let* $\Omega \subset \mathbb{C}^n$ *be an open subset. The following properties are equivalent:*
a) Ω *is strongly pseudoconvex;*
b) Ω *is weakly pseudoconvex;*
c) Ω *has a plurisubharmonic exhaustion function* Ψ.
d) $-\log d(z, \complement\Omega)$ *is plurisubharmonic and continuous on* Ω.

Definition 2.3.15 *Let $\Omega \subset \mathbb{C}^n$ be an open subset. If one of the previous properties holds, Ω is said to be a pseudoconvex open set.*

Definition 2.3.16 *Let D be an open set in $\mathbb{C}^n$ and let K be a compact subset of D. Then the plurisubharmonic convex-hull of K in D is defined to be*

$$\hat{K}_{Psh(D)} = \{z \in D : u(z) \leq \sup_K u, \, \forall u \in Psh(D)\}.$$

Since $f \in \mathcal{O}(D)$ implies that $\mid f \mid \in PSH(D)$, it is clear that $\hat{K}_{Psh(D)} \subset \hat{K}_{\mathcal{O}(D)}$.

Using the classical Cauchy formula as in examples of this chapter 2, Hartogs showed that for any domain of holomorphy in $\mathbb{C}^n$, the following *Continuity Principle* holds (1909).

Theorem 2.3.17 Hartogs, 1909, Continuity Principle *Let Ω be a domain of holomorphy. If (Δ_ν) is an arbitrary sequence of analytic discs whose closures are contained in Ω and such that $\lim_{\nu \to \infty} \Delta_\nu = \Delta_0$, $b\Delta_0 \subset \Omega$, where Δ_0 is an analytic disc, then $\Delta_0 \subset \Omega$.*

Here, an analytic disc δ means a non constant holomorphic mapping $\varphi : \Delta \to \mathbb{C}^n$, where Δ is the unit disc in $\mathbb{C}$. $\delta := \varphi(\Delta)$. If φ is continuous up to $\bar{\Delta}$, we will say that $\varphi(\bar{\Delta}) = \overline{\varphi(\Delta)}$ (according to the continuity of φ and the compacity of $\bar{\Delta}$) is a closed analytic disc and $\varphi(\partial\Delta)$ is its boundary $b\delta$.

Following Hartogs and Levi a domain in $\mathbb{C}^n$ is called *pseudoconvex* if the continuity principle is valid for it.

Theorem 2.3.18 *Let $\Omega \subset \mathbb{C}^n$ be an open set. The following properties are equivalent:*
a) Ω is pseudoconvex;
b) If K is a compact subset in Ω then $\hat{K}_{Psh(D)} \Subset \Omega$;
c) The Continuity Principle is satisfies : Let $(\delta_\alpha)_{\alpha \in A}$ be a family of closed analytic discs in Ω. If $\cup_{\alpha \in A} b\delta_\alpha \Subset \Omega$ then $\cup_{\alpha \in A} \delta_\alpha \Subset \Omega$.

A consequence of this theorem is the following one

Theorem 2.3.19 *Any domain of holomorphy (or holomorphically convex) is pseudoconvex.*

E. Levi (1911) formulated a natural problem: *any pseudoconvex domain is a domain of holomorphy.* This problem turned out to be exceedingly difficult and was solved by Oka only in 1942.

Theorem 2.3.20 (Oka 1942) *Any pseudoconvex open set in $\mathbb{C}^n$ is a domain of holomorphy.*

Chapter 3

Liouville's theorem, Ahlfors' Lemma, Picard's theorems

3.1 Theorems

In $\mathbb{C}$ the complex plane, $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ is the open disc with radius $r > 0$ and $\mathbb{D}_r^* := \mathbb{D}_r \backslash \{0\}$. Also $\mathbb{D} := \mathbb{D}_1$ and $\mathbb{D}^* := \mathbb{D}_1^*$.
$\mathcal{O}(\Omega)$ is the family of holomorphic functions on the domain Ω,
$\mathcal{O}(\Omega_1, \Omega_2)$ is the family of holomorphic mappings from a domain Ω_1 to domain Ω_2.

Theorem 3.1.1 (Schwarz's lemma) *Assume $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ and $f(0) = 0$.*
(a) Then $|f(z)| \leq |z|$ for every $z \in \mathbb{D}$ and $|f'(0)| \leq 1$.
(b) If $|f'(0)| = 1$, or if $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}^*$, then there exists $a \in \partial\mathbb{D}$ such that $f(z) = az$.

Proof. We just apply the maximum principle on an auxiliary function $f(z)/z$, which is holomorphic on $\mathbb{D}$, since $f(0) = 0$. □

Schwarz's lemma could be very easily reformulated for discs of arbitrary radii.
Supposing that $f \in \mathcal{O}(\mathbb{D}_{r_1}, \mathbb{D}_{r_2})$, where $r_1, r_2 > 0$ and $f(0) = 0$. Then the mapping $F(z) := r_2^{-1} f(r_1 z)$ meets the conditions of the lemma, so we get $|F(z)| \leq |z|$ for $z \in \mathbb{D}$. Then $|f(z)| \leq (r_2/r_1)|z|$ for $z \in \mathbb{D}_{r_1}$. Let f be the entire function, i.e. holomorphic on $\mathbb{C}$ such that $f(\mathbb{C}) \subset \mathbb{D}_{r_2}$ for a fixed $r_2 > 0$. The radius r_1 can be arbitrary large, so we get $f(z) \equiv 0$.

Theorem 3.1.2 (Liouville) *Every bounded entire function is constant.*

The connection between Schwarz's lemma and Liouville's theorem is a wonderful and simple example of Bloch's principle: there is nothing in the infinite which was not first in the finite. In consequence, for a global result like Liouville's, there must be a more powerful local result, such as Schwarz's result: Bloch's principle.
Let $\mathbb{D}_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$ be an open disc with radius $r > 0$ and centre $z_0 \in \mathbb{C}$.
$\mathcal{A}(\Omega)$ is the set of all continuous functions on $\bar{\Omega}$ which are holomorphic on Ω.

Theorem 3.1.3 (Bloch's principle) *There is a universal constant $B > 0$ with the property that for every value of $0 < R < B$, every function $f \in \mathcal{A}(\mathbb{D})$ with $|f'(0)| = 1$ maps a domain $\Omega \subset \mathbb{D}$ biholomorphically onto $\mathbb{D}_R(z_0)$ for some $z_0 \in f(\mathbb{D})$.*

We name the discs from the theorem simple ("schlicht") discs. Bloch's theorem is interesting because it guarantees the existence of simple discs with a fixed radius in the image of "quite a large family" of holomorphic functions on the unit disc.
In accordance with his principle, Bloch derived the following global result from his "local" theorem.

Theorem 3.1.4 (The Little Picard Theorem) *Any entire function whose range omits at least two distinct values is a constant.*

The above theorem is a remarkable generalization of Liouville's theorem.
It is simple to find entire functions whose range is the entire $\mathbb{C}$; nonconstant polynomials, for instance.
The exponential function is an example of an entire function whose range omits only one value, namely zero.
But there does not exist a nonconstant entire function whose range omits 0 and 1. The latter statement is actually equivalent to the Little Picard theorem since $(b-a)z+a$ is a biholomorphic mapping between $\mathbb{C}\backslash\{0,1\}$ and $\mathbb{C}\backslash\{a,b\}$, where $a \neq b$. Theorem 3.1.4 was proved first by C.E. Picard, by using arguments based on the modular function (a covering map from the upper halfplane $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$ onto $\mathbb{C}\backslash\{0,1\}$. See for instance [Ahl79, 7.3.4]).

Theorem 3.1.5 (The Big Picard Theorem) *In the neighborhood of an isolated essential singularity a holomorphic function takes every value in $\mathbb{C}$ infinitely often with no more than one exception.*

Similarly to the relation between Liouville's theorem and the Little Picard theorem, there is a weaker and more accessible theorem in the case of the Big Picard theorem.
We know that a holomorphic function on $\Omega\backslash\{a\}$ has in a one and only one type of isolated singularity: removable singularity, pole or essential singularity. In the latter case, the limit $\lim\limits_{z\to a}|f(z)|$ does not exist and this happens if and only if the image of the neighborhood of the point a is dense in $\mathbb{C}$.
Theorem 3.1.5 can be reformulated as a meromorphic extension: if a holomorphic function in the neighborhood of an isolated singularity omits two distinct values, then the singularity is removable or it is a pole. In this case, the function becomes meromorphic.

3.2 The Poincaré metric on a disc

In this section we introduce the Poincaré metric on a disc and we compute the corresponding distance in order to obtain the Schwarz-Pick lemma.

$\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. The **Poincaré metric on** $\mathbb{D}_r$ is

$$d\rho_r^2 := \frac{4r^2 \ \mathrm{d}z \wedge \ \mathrm{d}\bar{z}}{\left(r^2 - |z|^2\right)^2} \tag{1}$$

This is a form of a Hermitian pseudometric, which is on domain $\Omega \subseteq \mathbb{C}$ defined by

$$ds_\Omega^2 := 2\lambda(z) \ \mathrm{d}z \wedge \ \mathrm{d}\bar{z} \tag{2}$$

where $\lambda(z) \in \mathcal{C}^2\left(\Omega, \mathbb{R}_0^+\right)$is twice real-differentiable function with $\lambda(z) = \overline{\lambda(z)}$ and $Z(\lambda) := \{z \in \Omega : \lambda(z) = 0\}$ is a discrete set. If $Z(\lambda) = \varnothing$, then ds_Ω^2 is said to be a Hermitian metric.
We can observe that (1) is really a Hermitian metric. For the sake of simplicity, let us write $d\rho^2 := d\rho_1^2$.

Let Ω_1 and Ω_2 be two domains on $\mathbb{C}$ and $f \in \mathcal{O}(\Omega_1, \Omega_2)$. The pullback of arbitrary pseudometric $ds_{\Omega_2}^2$ is defined by

$$f^*\left(ds_{\Omega_2}^2\right) := 2\lambda(f(z))\left|f'(z)\right|^2 \ \mathrm{d}z \wedge \ \mathrm{d}\bar{z},$$

which is a pseudometric on Ω_1. For the **Möbius transformation**

$$\varphi_a(z) := \frac{z-a}{1-\bar{a}z},$$

where $a \in \mathbb{D}$, $\varphi_a \in \mathrm{Aut}(\mathbb{D})$, $\varphi_a^{-1} = \varphi_{-a}$ and $\varphi_a^*\left(d\rho^2\right) = d\rho^2$. $\mathrm{Aut}(\mathbb{D})$ is the family of holomorphic automorphisms of the unit disc. Therefore Möbius transformations are isometries for the Poincaré metric. We have (according to Schwarz's lemma [BG91, Examples 2.3.12])

$$\mathrm{Aut}(\mathbb{D}) = \left\{e^{\mathrm{i}a}\varphi_b(z) : a \in \mathbb{R}, b \in \mathbb{D}\right\}.$$

A pseudodistance (differs from a distance in metric spaces only in that the distance between two different points might be zero) can always be assigned to a **Hermitian pseudometric**. The process is the following.

Let $\Omega \subseteq \mathbb{C}$ be an arbitrary domain and $x, y \in \Omega$ arbitrary points. The mapping $\gamma : [0,1] \to \Omega$ is called $\mathcal{C}^n$-path from x to y for $n \geq 0$ if γ is n-times differentiable mapping and $\gamma(0) = x, \gamma(1) = y$. In the case $n = 0$ we speak about $\mathcal{C}$-paths and γ is a continuous mapping. The concatenation of $\mathcal{C}^n$-paths γ_1 from x to y and γ_2 from y to z is $\mathcal{C}$-path

$$\left(\gamma_1 * \gamma_2\right)(t) := \begin{cases} \gamma_1(2t),\ t \in [0, 1/2] \\ \gamma_2(2t-1),\ t \in [1/2, 1] \end{cases}$$

from x to z. Piecewise $\mathcal{C}^n$-path γ from x to y is $\gamma := \gamma_1 * \cdots * \gamma_k$ where $\gamma_1, \ldots, \gamma_k$ are $\mathcal{C}^n$-paths and $\gamma(0) = x, \gamma(1) = y$.
Assume that domain Ω is equipped with a Hermitian pseudometric ds_Ω^2. Let $\gamma : [0,1] \to \Omega$ be a piecewise $\mathcal{C}^1$-path from x to y. The length of γ is defined by

$$L_{ds_\Omega^2}(\gamma) := \int_0^1 \sqrt{2(\lambda \circ \gamma)}|\dot{\gamma}|\mathrm{d}t$$

The **pseudodistance** between the points x and y is defined by

$$d_\Omega(x,y) := \inf L_{ds^2_\Omega}(\gamma),$$

where the infimum goes through all piecewise $\mathcal{C}^1$-paths γ from x to y.

Let domains $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ be equipped with pseudometrics $ds^2_{\Omega_1} = 2\lambda_1(z)|\mathrm{d}z|^2$ and $ds^2_{\Omega_2} = 2\lambda_2(z)|\mathrm{d}z|^2$. Let there be points $x, y \in \Omega_1$, $f \in \mathcal{O}(\Omega_1, \Omega_2)$ and a piecewise $\mathcal{C}^1$-path $\gamma : [0,1] \to \Omega_1$ from x to y. Assume that $f^*\left(ds^2_{\Omega_2}\right) \leq ds^2_{\Omega_1}$. Then $f(\gamma)$ is a piecewise $\mathcal{C}^1$-path from $f(x)$ to $f(y)$. We have

$$d_{\Omega_2}(f(x), f(y)) \leq \int_0^1 \sqrt{2\left(\lambda_2 \circ f \circ \gamma\right)}\, |f'(\gamma)|\, |\dot{\gamma}| \mathrm{d}t \leq \int_0^1 \sqrt{2\left(\lambda_1 \circ \gamma\right)} |\dot{\gamma}| \mathrm{d}t \tag{3}$$

Because this is valid for every such path, it follows

$$d_{\Omega_2}(f(x), f(y)) \leq d_{\Omega_1}(x,y). \tag{4}$$

If $f \in \mathcal{O}(\Omega_1, \Omega_2)$ is a biholomorphic mapping and f is an isometry for pseudometrics i.e. $f^*\left(ds^2_{\Omega_2}\right) = ds^2_{\Omega_1}$, then we can, with similar inequality as (3), but on inverse mapping f^{-1}, obtain $d_{\Omega_2}(f(x), f(y)) = d_{\Omega_1}(x,y)$. In this case f is also an isometry for the induced pseudodistances.

To the Poincaré metric on a disc we can explicitly write down the distance function between arbitrary points $p, q \in \mathbb{D}$. We denote it with $\rho(p,q)$ and we call it the **Poincaré distance**.

Proposition 3.2.1 *For arbitrary points $p, q \in \mathbb{D}$, the Poincaré distance is*

$$\rho(p,q) = \ln \frac{|1 - \bar{p}q| + |p - q|}{|1 - \bar{p}q| - |p - q|} = \ln \frac{1 + |\varphi_p(q)|}{1 - |\varphi_p(q)|} = 2 \operatorname{artanh} |\varphi_p(q)| \,. \tag{5}$$

The area hyperbolic tangent $\operatorname{artanh}(x) := \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), x \in (-1,1)$, is an increasing function, is equal to zero at $x = 0$. Also $\lim\limits_{x \to -1} \operatorname{artanh}(x) = -\infty$ and $\lim\limits_{x \to 1} \operatorname{artanh}(x) = +\infty$.

Proof. The second and third equalities are clear from the definitions.
Let's prove the first equality. Since rotations and Möbius transformations are isometries for the Poincaré metric, it is sufficient to show for every $a \in [0,1)$ that

$$\rho(0,a) = \ln \frac{1+a}{1-a} \tag{6}$$

Because

$$\rho\left(0, \left|\frac{p-q}{1-\bar{p}q}\right|\right) = \rho\left(0, \frac{q-p}{1-\bar{p}q}\right) = \rho\left(0, \varphi_p(q)\right) = \rho\left(\varphi_{-p}(0), q\right) = \rho\left(p, q\right),$$

equation (5) follows from (6). Let $\gamma(t) := x(t) + \mathrm{i}y(t)$ be a piecewise $\mathcal{C}^1$-path from 0 to a and $\bar{\gamma}(t) := at$. Then

$$L_{d\rho^2}(\gamma) = \int_0^1 \frac{2\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}}{1 - x^2(t) - y^2(t)}\mathrm{d}t \geq \int_0^1 \frac{2\dot{x}(t)\mathrm{d}t}{1 - x^2(t)}$$
$$= \ln \frac{1 + x(t)}{1 - x(t)}\bigg|_{t=0}^{t=1} = \ln \frac{1+a}{1-a} = L_{d\rho^2}(\bar{\gamma})$$

The inequality above becomes equality if and only if $y \equiv 0$. And we verify that $L_{d\rho^2}(\bar{\gamma}) = \rho(0, a)$. □

The above proof makes it evident that the shortest path in the Poincaré metric from 0 to $a \in [0, 1)$ is a chord between those points. We call **the shortest path in arbitrary metric a geodesic**. For a general domain and metric on it, the geodesic does not always exist; think about a nonconvex domain, equipped with the Euclidean metric. If it exists, it may not be the only one.
Let's now connect Schwarz's lemma and the Poincaré metric to obtain the Schwarz-Pick lemma.

Theorem 3.2.2 (The Schwarz-Pick Lemma) *Assume* $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$.
(a) Then

$$\rho(f(p), f(q)) \leq \rho(p, q), \;\; \forall\, p,\, q \in \mathbb{D} \tag{7}$$

and

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \;\; \forall\, z \in \mathbb{D}. \tag{8}$$

(b) If $p \neq q$ *exist such that the equality in (7) is valid or if* z_0 *exists such that the equality in (8) is valid, then* $f \in \mathrm{Aut}(\mathbb{D})$.

Proof. Choose arbitrary $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ and arbitrary points $p, q \in \mathbb{D}$. Define $F(z) := \left(\varphi_{f(p)} \circ f \circ \varphi_{-p}\right)(z)$. Then $F \in \mathcal{O}(\mathbb{D}, \mathbb{D}), F(0) = 0$ ($\varphi_{-p}(0) = p$ and $\varphi_{f(p)}(f(p)) = 0$) and

$$F'(0) = \frac{f'(p)\,(1 - |p|^2)}{1 - |f(p)|^2}.$$

According to Schwarz's lemma we have $|F(z)| \leq |z|$ in $\mathbb{D}$, then $\left|\varphi_{f(p)}(f(z))\right| \leq |\varphi_p(z)|$. This is equivalent to (7), since artanh is an increasing function. According again to Schwarz's lemma we have $|F'(0)| \leq 1$ and we obtain (8).
If $f(0) = 0$, then with (7) we get $\rho(0, f(z)) \leq \rho(0, z)$, which is equivalent to $|f(z)| \leq |z|$ and with (8) we get $|f'(0)| \leq 1$. Therefore part (a) of Schwarz's lemma is equivalent to part (a) of the Schwarz-Pick lemma.
To have $p \neq q$ such that the equality in (7) is valid ($|\varphi_{f(p)}(f(q))| = |\varphi_p(q)|$), is equivalent to have $w \neq 0$ such that $|F(w)| = |w|$ ($q = \varphi_{-p}(w)$). To have z_0 such that the equality in (8) is valid is equivalent to have $|F'(0)| = 1$. In these two situations $f \in \mathrm{Aut}(\mathbb{D})$. Consequently parts (b) of both lemmas are equivalent. □

3.3 Inner distances

What makes the Poincaré distance exceptional ?
In the Poincaré distance the boundary is infinitely far away from every point.
It can also be observed that closed balls in the Poincaré metric are compact. From the explicit expression for ρ we can prove that every Cauchy sequence with respect to ρ is convergent in $\mathbb{D}$. We say that $(\mathbb{D}, \rho)$ **is a complete metric space**. And there is a connection among the infiniteness of the boundary, the compactness of closed balls and the completeness of the metric space.

Let $\gamma : [0,1] \to \Omega$ be a piecewise $\mathcal{C}^1$-path from x to y and $\delta := \{0 = t_0 < t_1 < \cdots < t_k = 1\}$ partition of $[0,1]$ on k pieces. Length of γ in space (Ω, d_Ω) is defined by

$$L_{d_\Omega}(\gamma) := \sup_\delta \sum_{n=1}^{k} d_\Omega\left(\gamma\left(t_{n-1}\right), \gamma\left(t_n\right)\right)$$

We call $d^i_\Omega(x,y) := \inf L_{d_\Omega}(\gamma)$, where the infimum goes through all piecewise $\mathcal{C}^1$-paths γ from x to y, **inner pseudodistance**.
It is not difficult to prove that this is indeed a pseudodistance.
Because it is always $d_\Omega(x,y) \leq L_{d_\Omega}(\gamma)$, it follows that $d_\Omega(x,y) \leq d^i_\Omega(x,y)$.
If the opposite inequality is valid, then we say that d_Ω is **inner**. In that case we have $d_\Omega = d^i_\Omega$.

Proposition 3.3.1 *Let d_Ω be a pseudodistance, generated with (2). Then d_Ω is inner.*

Proof. Let $\gamma : [0,1] \to \Omega$ be a piecewise $\mathcal{C}^1$-path from p to q. Because we have

$$\sum_{n=1}^{k} d_\Omega\left(\gamma\left(t_{n-1}\right), \gamma\left(t_n\right)\right) \leq \sum_{n=1}^{k} \int_{t_{n-1}}^{t_n} \sqrt{2(\lambda \circ \gamma)}|\dot{\gamma}|\mathrm{d}t = L_{ds^2_\Omega}(\gamma)$$

for every partition δ, it follows $L_{d_\Omega}(\gamma) \leq L_{ds^2_\Omega}(\gamma)$. Therefore $d^i_\Omega(p,q) \leq d_\Omega(p,q)$. □

The balls $B_{d_\Omega}(x,r) := \{y \in \Omega : d_\Omega(x,y) < r\}$ and $\bar{B}_{d_\Omega}(x,r) := \{y \in \Omega : d_\Omega(x,y) \leq r\}$.

Proposition 3.3.2 *Assume that d_Ω is a continuous inner distance. Then d_Ω is equivalent to the Euclidean topology on Ω.*

Proof. Let $x \in \Omega$ be arbitrary point. $d_\Omega : \{x\} \times \Omega \to [0, \infty)$ is a continuous function. The set $[0,r) \subset [0,\infty)$ is open. Because $B_{d_\Omega}(x,r) = \left(\mathrm{pr}_2 \circ d^{-1}_\Omega\right)([0,r))$ where pr_2 is a projection to the second component, every ball $B_{d_\Omega}(x,r)$ is open in the Euclidean topology.
Conversely, we are going to prove that every open set in Ω for the Euclidean topology is open for d_Ω. Let $U \subset \Omega$ be an arbitrary neighborhood of a point $x \in \Omega$. We must show that there exists $r > 0$ such that $B_{d_\Omega}(x,r) \subset U$. Choose a relatively compact neighborhood $U' \subset U$ of x. Define

$$r := d_\Omega\left(x, \partial U'\right) = \inf_{y \in \partial U'} d_\Omega(x,y).$$

This infimum is a minimum because d_Ω is continuous with respect to the Euclidean topology. Because d_Ω is an inner distance, for every point $y \in B_{d_\Omega}(x,r)$, there exists a piecewise $\mathcal{C}^1$-path γ_y from x to y such that $L_{d_\Omega}(\gamma_y) < r$. This means that $\mathrm{spt}\gamma_y \subset B_{d_\Omega}(x,r)$. Hence $B_{d_\Omega}(x,r) \subset \overline{U'}$, because contrary, for $y \in B_{d_\Omega}(x,r)\backslash\overline{U'}$ there will be $x' \in \partial U' \cap \mathrm{spt}\gamma_y$ such that $r > L_{d_\Omega}(\gamma_y) \geq r + d_\Omega(x', y)$. As this is impossible, the proposition is thus proved. $\square$

Remember that a complete metric space (X, d_Ω) means that every Cauchy sequence converges in d_Ω. If there is a continuous inner distance, then compactness of closed balls characterizes completeness of a metric space.

Theorem 3.3.3 (Hopf-Rinow) *Assume that d_Ω is a continuous inner distance. Then (Ω, d_Ω) is a complete metric space if and only if every closed ball $\bar{B}_{d_\Omega}(x,r)$ is compact.*

Proof. The easy part of the proof is the implication from compactness of closed balls to completeness of the space and is valid without the assumption of innerness. Let every closed d_Ω-ball be compact. Because in a metric space every Cauchy sequence has one accumulation point at most and in a compact space every sequence has one accumulation point at least, it follows that (Ω, d_Ω) is complete.
Let (Ω, d_Ω) be a complete space. Fix $x_0 \in \Omega$. Then $r > 0$ exists (small enought) such that $B_{d_\Omega}(x_0, r)$ is relatively compact. We want to prove that this is true for all $r > 0$. Assuming the contrary, set

$$r_0 := \sup\left\{r : \bar{B}_{d_\Omega}(x_0, r) \text{ is compact}\right\} < \infty.$$

Then the set $\bar{B}_{d_\Omega}(x_0, r_0 - \varepsilon)$ is compact for all $\varepsilon > 0$. Therefore a sequence $\{y_i\}_{i=1}^n \subset \bar{B}_{d_\Omega}(x_0, r_0 - \varepsilon)$ exists such that

$$\bar{B}_{d_\Omega}(x_0, r_0 - \varepsilon) \subset \bigcup_{i=1}^n B_{d_\Omega}(y_i, \varepsilon)$$

For any $\varepsilon' \geq 0$, we will demonstrate that $\{B_{d_\Omega}(y_i, \varepsilon' + 2\varepsilon)\}_{i=1}^n$ is an open cover of $B_{d_\Omega}(x_0, r_0 + \varepsilon')$. Let us take arbitrary

$$x \in B_{d_\Omega}(x_0, r_0 + \varepsilon') \backslash \bar{B}_{d_\Omega}(x_0, r_0 - \varepsilon)$$

By innerness a piecewise $\mathcal{C}^1$-path γ exists from x_0 to x such that $L_{d_\Omega}(\gamma) < r_0 + \varepsilon'$. Then $t_0 \in (0,1)$ and $y_j \in \{y_i\}_{i=1}^n$ exist such that $\gamma(t_0) \in \partial B_{d_\Omega}(x_0, r_0 - \varepsilon)$ and $\gamma(t_0) \in B_{d_\Omega}(y_j, \varepsilon)$. Then we have

$$\begin{aligned} L_{d_\Omega}\left(\gamma|_{[t_0,1]}\right) &= L_{d_\Omega}(\gamma) - L_{d_\Omega}\left(\gamma|_{[0,t_0]}\right) \\ &< r_0 + \varepsilon' - (r_0 - \varepsilon) = \varepsilon' + \varepsilon \end{aligned}$$

This means that $d_\Omega(\gamma(t_0), x) < \varepsilon' + \varepsilon$ and $d_\Omega(x, y_j) \leq d_\Omega(x, \gamma(t_0)) + d_\Omega(\gamma(t_0), y_j) < \varepsilon' + 2\varepsilon$. It follows

$$B_{d_\Omega}(x_0, r_0 + \varepsilon') \subset \bigcup_{i=1}^n B_{d_\Omega}(y_i, \varepsilon' + 2\varepsilon)$$

For $\varepsilon = \varepsilon' = r_0/6$, we have $B_{d_\Omega}(x_0, r_0 + r_0/6) \subset \bigcup_{i=1}^n B_{d_\Omega}(y_i, r_0/6 + 2r_0/6)$.
Then there exists $n_1 \in \mathbb{N}, 1 \leq n_1 \leq n$ such that $\bar{B}_{d_\Omega}(y_{n_1}, r_0/2)$ is not compact with $y_{n_1} \in \bar{B}_{d_\Omega}(x_0, 5r_0/6)$. Set

$$0 < r_1 := \sup\left\{r : \bar{B}_{d_\Omega}(y_{n_1}, r) \text{ is compact}\right\} \leq r_0/2.$$

We inductively continue this process as above. We obtain a sequence of points $y_{n_k} \in B_{d_\Omega}\left(y_{n_{k-1}}, 5.2^{-k} r_0/3\right)$, where $\bar{B}_{d_\Omega}\left(y_{n_k}, r_0 2^{-k}\right)$ is not compact for every $k \in \mathbb{N}$. The sequence (y_{n_k}) doesn't converge. Indeed if $w = \lim y_{n_k}$ then for k large enough, $\bar{B}_{d_\Omega}\left(y_{n_k}, r_0 2^{-k}\right) \subset \bar{B}_{d_\Omega}\left(w, r_0 2^{-k+1}\right)$, which is not compact. This is impossible for any k large enough. But the nonconvergent sequence (y_{n_k}) is Cauchy (for any $k' \geq k$, $d(y_{n_{k'}}, y_{n_k}) \leq \frac{10r_0}{3}\frac{1}{2^k}$), which is in contradiction with the assumption of the completeness of (Ω, d_Ω). The theorem is therefore proved. □

Let $\Omega \subseteq \mathbb{C}$ be a domain and $x \in \Omega$ an arbitrary point. The mapping $\gamma : [0,1) \to \Omega$ is a piecewise $\mathcal{C}^1$-path from x to $y \in \partial\Omega \cup \{\infty\}$ if for every $t_0 \in (0,1)$, the mapping $\gamma|_{[0,t_0]}$ is a $\mathcal{C}^1$-path, $\gamma(0) = x$ and $\lim_{t\to 1}\gamma(t) = y$.

Definition 3.3.4 *A domain Ω is b-complete with respect to the distance d_Ω if for arbitrary points $x \in \Omega, y \in \partial\Omega \cup \{\infty\}$ and for an arbitrary piecewise $\mathcal{C}^1$-path γ from x to y, it follows that $\lim_{t\to 1} L_{d_\Omega}\left(\gamma|_{[0,t]}\right) = \infty$.*

Letter "b" stands for "boundary". Intuitively speaking, (Ω, d_Ω) is b-complete if and only if the boundary is "infinitely far away" from every inner point.

Corollary 3.3.5 *Assume that d_Ω is a continuous inner distance. Then (Ω, d_Ω) is a complete metric space if and only if (Ω, d_Ω) is b-complete.*

Proof. Assume that Ω is not b-complete. Then there exists a piecewise $\mathcal{C}^1$-path $\gamma : [0,1) \to \Omega$ from $x \in \Omega$ to $y \in \partial\Omega \cup \{\infty\}$ such that $\lim_{t\to 1} L_{d_\Omega}\left(\gamma|_{[0,t]}\right) = r$ for some $r > 0$. For every sequence $(y_n) \subset \gamma([0,1))$, where $y_n \to y$, it follows that $d_\Omega(x, y_n) \leq r$ for every $n \in \mathbb{N}$. Because the closed ball $\bar{B}_{d_\Omega}(x, r)$ is not compact, Theorem 3.3.3 guarantees that (Ω, d_Ω) is not complete.
Now let assume that Ω is not a complete metric space. Then there exists a Cauchy sequence $(x_i) \subset \Omega \subset \mathbb{C}$ with the limit $x \in \partial\Omega$.
Let fix arbitrary $\varepsilon \in (0,1)$. Since the sequence is Cauchy, then there exists a subsequence $\{k_i\} \subset \mathbb{N}$ such that $d_\Omega\left(x_{k_i}, x_{k_{i+1}}\right) < \varepsilon^i$. Because d_Ω is an inner distance, there exist piecewise $\mathcal{C}^1$-paths γ_i with $\gamma_i(0) = x_{k_i}$ and $\gamma_i(1) = x_{k_{i+1}}$ such that

$$d_\Omega\left(x_{k_i}, x_{k_{i+1}}\right) \leq L_{d_\Omega}(\gamma_i) \leq d_\Omega\left(x_{k_i}, x_{k_{i+1}}\right) + \varepsilon^i < 2\varepsilon^i$$

Define a piecewise $\mathcal{C}^1$-path $\gamma : [0,1) \to \Omega$ from x_{k_1} to x with $\gamma(t) := \gamma_i\left(2^i(t-1)+2\right)$ for $t \in [1-2^{1-i}, 1-2^{-i}]$. Take arbitrary $t_0 \in (0,1)$. Then there exists $j \in \mathbb{N}$ such that $t_0 \in$

$[1-2^{1-j}, 1-2^{-j}]$. Therefore

$$L_{d_\Omega}\left(\gamma|_{[0,t_0]}\right) = \sum_{i=1}^{j-1} L_{d_\Omega}(\gamma_i) + L_{d_\Omega}\left(\gamma|_{[1-2^{1-j},t_0]}\right) < 2\left(\varepsilon + \varepsilon^2 + \cdots + \varepsilon^j\right) < \frac{2\varepsilon}{1-\varepsilon}$$

Since $\lim_{t\to 1} L_{d_\Omega}\left(\gamma|_{[0,t]}\right) < \infty$, the domain Ω is not b-complete. □

3.4 Ahlfors' generalization of the Schwarz-Pick lemma

Ahlfors' generalization is based on curvature.

Definition 3.4.1 Gaussian curvature $K_{ds^2_\Omega}$ *of a pseudometric (2) is defined by*

$$K_{ds^2_\Omega}(z) := -\frac{1}{\lambda}\frac{\partial^2}{\partial z \partial \bar{z}} \log \lambda(z) \tag{9}$$

for $z \in \Omega \backslash Z(\lambda)$ *and* $-\infty$ *for the rest of the points.*

We can compute: $K_{d\rho_r^2} \equiv -1$ for the Poincaré metric (1) on $\mathbb{D}_r$.
This curvature is indeed connected to Gaussian curvature of Riemannian metric on surfaces in real differential geometry.
An important property of Gaussian curvature is invariance on the pullback. For an arbitrary $f \in \mathcal{O}(\Omega_1, \Omega_2)$ there is

$$K_{ds^2_{\Omega_2}}(f(z)) = K_{ds^2_{\Omega_1}}(z) \tag{10}$$

where $ds^2_{\Omega_2}$ is an arbitrary Hermitian pseudometric on Ω_2 and $ds^2_{\Omega_1} := f^*\left(ds^2_{\Omega_2}\right)$. This can be easily seen from (9), using the chain rule and $\partial f / \partial \bar{z} \equiv 0$ since f is holomorphic.

We will need later to have weaker assumptions for the function λ. Assume that λ is only continuous function and then $ds^2_\Omega = 2\lambda dz \wedge d\bar{z}$ is a continuous Hermitian metric.

Definition 3.4.2 *A pseudometric* $ds^2_{supp} = 2\lambda_{supp}(z) dz \wedge d\bar{z}$ *is supporting pseudometric for* ds^2 *at* $z_0 \in \Omega$ *if there is a neighborhood* $U \ni z_0$ *in* Ω *such that* $\lambda_{supp} \in \mathcal{C}^2\left(U, \mathbb{R}_0^+\right)$ *and* $\lambda_{supp}|_U \leq \lambda|_U$ *with equality at* z_0.

We do not need a supporting pseudometric, defined on the whole domain Ω. When a supporting pseudometric exists for a continuous pseudometric, this is defined as local existence, which can change from point to point.

Theorem 3.4.3 (Ahlfors' lemma) *Let* Ω *be a domain with a continuous Hermitian pseudometric* ds^2_Ω, *for which a supporting pseudometric* ds^2_{supp} *exists. Assume that* $K_{ds^2_{supp}}\Big|_\Omega \leq L$ *for some* $L < 0$. *Then for every* $f \in \mathcal{O}(\mathbb{D}, \Omega)$ *we have*

$$f^*\left(ds^2_\Omega\right) \leq |L|^{-1} d\rho^2 \tag{11}$$

where $d\rho^2$ *is the Poincaré metric (1) in* $\mathbb{D}$.

Proof. There is a continuous Hermitian pseudometric $ds^2_\Omega = 2\lambda dz \wedge d\bar{z}$ on Ω. Define

$$ds^2 := |L| f^*\left(ds^2_\Omega\right) = 2|L|\lambda(f) \cdot |f'|^2 \, dz \wedge d\bar{z}$$

Then ds^2 is a continuous Hermitian pseudometric on $\mathbb{D}$. Define $\lambda_1 := |L|\lambda(f).|f'|^2$. The equation (11) is equivalent to $ds^2 \leq d\rho^2$.
For every $r \in \mathbb{R}^+$, define $\mu_r(z) := 2r^2\left(r^2 - |z|^2\right)^{-2}$ on $\mathbb{D}_r$. $d\varrho_r^2 = 2\mu_r dz \wedge d\bar{z}$ and $d\rho^2 = 2\mu_1(z) dz \wedge d\bar{z}$.
Define the function $u_r(z) := \lambda_1(z)\mu_r^{-1}(z)$. Hence $ds^2 = u_r d\rho_r^2$. If we show that $u_1 \leq 1$ on $\mathbb{D}$, then $ds^2 \leq d\rho^2$.
If we show that for any $r < 1$: $u_r(z) \leq 1$ for every $z \in \mathbb{D}_r$, then $u_1 \leq 1$ on $\mathbb{D}$, because when $z_0 \in \mathbb{D}$ is fixed and $r \to 1$ and it follows $u_r(z_0) \to u_1(z_0)$.
Let fix an arbitrary $r \in (0,1)$.
Since λ_1 is bounded on $\mathbb{D}_r$, when $|z| \to r$ follows that $u_r(z) \to 0$. Function u_r is continuous, hence there exists $z_0 \in \overline{\mathbb{D}_r}$ such that $\max_{\overline{\mathbb{D}_r}} u_r = u_r(z_0)$.
Let there be a supporting pseudometric ds^2_{supp} for ds^2_Ω at $f(z_0) \in \Omega$. Then we define

$$ds'^2_{\text{supp}} := |L| f^*\left(ds^2_{\text{supp}}\right).$$

It is a supporting pseudometric for ds^2 at z_0, whose curvature is -1 at most, according to (9) and (10): $K_{ds'^2_{supp}} = \frac{1}{|L|} K_{f^* ds^2_{supp}} = \frac{1}{|L|} K_{ds^2_{supp}}(f) \leq \frac{L}{|L|} = -1$.
Then there exist a neighborhood $U \ni z_0$ and $\lambda'_{\text{supp}}(z) \in \mathcal{C}^2\left(U, \mathbb{R}_0^+\right)$ such that $\lambda'_{\text{supp}}\big|_U \leq \lambda_1|_U$ with equality in z_0. Define function

$$v_r(z) := \frac{\lambda'_{\text{supp}}(z)}{\mu_r(z)} = \frac{\lambda'_{\text{supp}}(z)}{\lambda_1(z)} u_r(z).$$

Hence $\max_{z \in U} v_r = u_r(z_0)$.
Although what follows is related to the theory of real functions, it is a crucial element of the proof. Let us have $u \in \mathcal{C}^2(\Omega, \mathbb{R}^+)$, where $\Omega \subset \mathbb{C}$ is a domain. Assume that a function u reaches its maximum at $(x_0, y_0) \in \Omega$. Because this point is singular, it follows that $\partial u/\partial x\,(x_0, y_0) = \partial u/\partial y\,(x_0, y_0) = 0$. But the point is a maximum, so $\partial^2 u/\partial x^2\,(x_0, y_0) \leq 0$ and $\partial^2 u/\partial y^2\,(x_0, y_0) \leq 0$. A computation shows that

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log u = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log u = \frac{u\left(u_{xx} + u_{yy}\right) - \left(u_x^2 + u_y^2\right)}{4u^2}.$$

Then we have $\left.\dfrac{\partial^2 \log u}{\partial z \partial \bar{z}}\right|_{z = x_0 + iy_0} \leq 0$.
Remember that the maximum of function $v_r|_U$ is reached at point z_0. Hence

$$\begin{aligned} 0 \geq \left.\frac{\partial^2 \log v_r|_U}{\partial z \partial \bar{z}}\right|_{z_0} &= \left.\frac{\partial^2 \log \lambda'_{\text{supp}}}{\partial z \partial \bar{z}}\right|_{z_0} - \left.\frac{\partial^2 \log \mu_r}{\partial z \partial \bar{z}}\right|_{z_0} \\ &= -\lambda'_{\text{supp}}(z_0) K_{ds'^2_{\text{supp}}}(z_0) - \mu_r(z_0) \\ &= \mu_r(z_0)\left(-v_r(z_0) K_{ds'^2_{\text{supp}}}(z_0) - 1\right) \\ &\geq \mu_r(z_0)\left(v_r(z_0) - 1\right) \end{aligned}$$

because $K_{ds'^2_{\text{supp}}}(z_0) \leq -1$. We get $v_r(z_0) \leq 1$ and $u_r(z_0) \leq 1$. Since z_0 is the maximum of u_r, it follows $u_r(z) \leq 1$ on $\mathbb{D}_r$. □

The original version of Ahlfors' lemma is for Riemann surfaces (one dimensional complex manifolds), so the proof is essentially the same as one above.
Assume that $L = -1$ in Ahlfors' lemma. Then we have $f^*(ds^2_\Omega) \leq d\rho^2$. Therefore we can use inequality (4) and get

$$d_\Omega(f(p), f(q)) \leq \rho(p, q) \tag{13}$$

In the case of domain $(\mathbb{D}, d\rho^2)$, we get (7) of the Schwarz-Pick lemma.

3.5 Applications and Proofs

In this section we prove the theorems mentioned in the introduction.
Firstly, we will prove Bloch's theorem and a familiar theorem due to Landau, which drops out the assumption about simple discs. Bloch's and Landau's theorems are examples of applications of Ahlfors' lemma.
Next, a Hermitian metric is constructed on domain $\mathbb{C}\backslash\{0,1\}$, which satisfies the assumptions of Ahlfors' lemma. From that point, we are able to provide a proof of the Little Picard theorem. Then Schottky theorem permits us to prove the Big Picard theorem.

3.5.1 The Bloch Theorem

$\mathcal{B} := \{f \in \mathcal{A}(\mathbb{D}) : |f'(0)| = 1\}$ where $\mathcal{A}(\mathbb{D}) = \mathcal{C}(\overline{\mathbb{D}}) \cap \mathcal{O}(\mathbb{D})$.
Remember that Bloch's theorem (Theorem 3.1.3) guarantees the existence of a constant $B > 0$ such that $\mathrm{B}(f) \geq B$ for every $f \in \mathcal{B}$, where $\mathrm{B}(f)$ be a supremum of all radii of simple discs in $f(\mathbb{D})$ (discs in $f(\mathbb{D})$ which are biholomorphic with f to an open subset in $\mathbb{D}$).

Proof of Bloch's theorem. By $S := \{z \in \mathbb{D} : f'(z) = 0\}$ we denote the set of singular points. It is a discrete set of points. According to the open mapping theorem, $\Omega := f(\mathbb{D})$ is a domain and $f(\overline{\mathbb{D}}) \subseteq \bar{\Omega} \subset \mathbb{C}$. For every point $w \in \Omega$ there is a number $\tilde{\rho}(w)$ such that $\mathbb{D}_{\tilde{\rho}(w)}(w)$ is the largest simple disc. Therefore $\mathrm{B}(f) = \sup_{w\in\Omega} \tilde{\rho}(w)$ and $\mathrm{B}(f) < \infty$ (otherwise we would have $\Omega = \mathbb{C}$. This is impossible since f is continuous on $\overline{\mathbb{D}}$ and Ω is bounded). On $\mathbb{D}$ we define a metric

$$\lambda(z) := \frac{A^2 |f'(z)|^2}{\tilde{\rho}(f(z)) (A^2 - \tilde{\rho}(f(z)))^2} \tag{14}$$

where A is a constant, which satisfies $A^2 > \mathrm{B}(f)$. Since $\tilde{\rho}$ is a continuous function and $\tilde{\rho}(f(z)) = 0$ if and only if $z \in S$, then (14) is a continuous Hermitian metric at nonsingular points.
We must care only at singular points. Take arbitrary $z_0 \in S$. We know that there is a small neighborhood $U \ni z_0$ in $\mathbb{D}, n \geq 2$ and biholomorphic function φ on U ($\varphi(z_0) = 0$ and $\varphi'(z_0) \neq 0$) such that $f(z) = f(z_0) + \varphi^n(z)$ on U ($f(z) = f(z_0) + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n g(z)$ with $g(z_0) = 1$. We

can choose U small enought such that $|g(z)-1|<1$ on U. There exists a holomorphic nth root on $\mathbb{D}_1(1))$ and $\tilde{\rho}(f(z)) = |f(z) - f(z_0)|$ on U. Therefore the equation (14) can be rewritten as

$$\lambda(z) = \frac{A^2 n^2 |\varphi(z)|^{n-2} |\varphi'(z)|^2}{(A^2 - |\varphi(z)|^n)^2}$$

for $z \in U$. Therefore λ is a Hermitian pseudometric in the neighborhood U of the singular point z_0, with constant curvature -1.

If we want to use Ahlfors' lemma, we need to construct a supporting pseudometric for (14) in a neighborhood of any nonsingular point $z_0 \in \mathbb{D}\backslash S$. There exists $s_0 \in \overline{\mathbb{D}}$ such that the boundary of $\mathbb{D}_{\tilde{\varrho}(f(z_0))}(f(z_0))$ contains the point $f(s_0)$ and in the neighborhood U of z_0, we have $\tilde{\varrho}(f(z)) \leq |f(z) - f(s_0)|$. On U define a Hermitian metric

$$\lambda_{\text{supp}}(z) := \frac{A^2 |f'(z)|^2}{|f(z) - f(s_0)|.(A^2 - |f(z) - f(s_0)|)^2}. \tag{15}$$

Let choose A such that $x \mapsto x(A^2 - x)^2$ is an increasing function on $[0,\ \mathrm{B}(f)]$. A quick calculation shows that this function is increasing on $[0, A^2/3]$. Thereforewe choose $A^2 > 3\ \mathrm{B}(f)$. Then the inequality $\lambda_{\text{supp}}(z) \leq \lambda(z)$ is satisfied on U. Since $\lambda_{\text{supp}}(z_0) = \lambda(z_0)$, metric (15), which has constant curvature -1 , is supporting for (14) at z_0.

Let $f(0) = z_0$. By assumption $|f'(0)| = 1$, the upper bounds combined with Ahlfors' lemma give

$$3\ \mathrm{B}(f) < A^2 \leq 4\tilde{\rho}(z_0)\left(A^2 - \tilde{\rho}(z_0)\right)^2 \leq 4\ \mathrm{B}(f)\left(A^2 - \mathrm{B}(f)\right)^2.$$

Pushing A^2 toward $3\ \mathrm{B}(f)$, we get $\mathrm{B}(f) \geq \sqrt{3}/4$. Hence $B \geq \sqrt{3}/4$. □

Edmund G. H. Landau dropped the assumption about simple discs in Bloch's theorem.

Theorem 3.5.1 *Assume $f \in \mathcal{A}(\mathbb{D})$ and $|f'(0)| = 1$. Then there exists a universal constant $L > 0$ such that there exists a disc with radius $R \geq L$ in the image $f(\mathbb{D})$.*

Proof. Proving this theorem is very similar to proving Bloch's theorem. Let there be a real and positive function $\tilde{\rho}(z)$ such that $\mathbb{D}_{\tilde{\rho}(z)}(z)$ is the largest disc in $\Omega := f(\mathbb{D})$. Define $\mathrm{L}(f) := \sup_{z\in\Omega} \tilde{\rho}(z)$. Since we are not dealing with singular points, we take metrics

$$\lambda(z) := \frac{1}{2}\left(\tilde{\rho}(z)\ln\frac{C}{\tilde{\rho}(z)}\right)^{-2} \text{ and } \lambda_{\text{supp}}(z) := \frac{1}{2}\left(|z-s_0|\ln\frac{C}{|z-s_0|}\right)^{-2}$$

on Ω. The metric $\lambda_{\text{supp}}(z)$ is defined on a neighborhood U of a point $z_0 \in \Omega$ and $s_0 \in \partial\mathbb{D}_{\tilde{\rho}(z_0)}(z_0) \cap \partial\Omega$ where it has constant curvature -1 . We have $\tilde{\varrho}(z) \leq |z - s_0|$ in U. Therefore λ_{supp} will be supporting for λ at z_0 if the inequality $\lambda_{\text{supp}}(z) \leq \lambda(z)$ is satisfied on U. This will be true if $x\ln(Cx^{-1})$ is an increasing function on $[0,\ \mathrm{L}(f)]$. This function is increasing for $e.x < C$, therefore the metric is supporting if $e.\mathrm{L} < C$.

Assume $f(0) = z_0$. We have $|f'(0)| = 1$. According to Ahlfors' lemma it follows

$$1 \leq \left(2\tilde{\rho}(z_0)\ln\frac{C}{\tilde{\rho}(z_0)}\right)^2 \leq \left(2\ \mathrm{L}(f)\ln\frac{C}{\mathrm{L}(f)}\right)^2$$

Pushing C toward $e.\mathrm{L}(f)$, we get $\mathrm{L}(f) \geq 1/2$ and hence $L \geq 1/2$. □

3.5.2 The Little Picard Theorem

The Little Picard theorem (Theorem 3.1.4) deals with domain $\mathbb{C}\backslash\{0,1\}$. Therefore, we are going to construct a Hermitian metric with curvature, bounded with negative constant and use Ahlfors' lemma.
On looking at the definition of $\lambda_{\mathbb{C}^{**}}$, one sees that the first factor is singular at 0 and the second is singular at 1. Let us concentrate on the first of these. Since the expression defining curvature is rotationally invariant, it is plausible that the metric we define would also be rotationally invariant about its singularities. Thus it should be a function of $|z|$. Hence one would like to choose exponents α, β so that $(1+|z|^\alpha)^\beta$ defines a metric of negative curvature. However, a calculation reveals that the α, β which are suitable for z large are not suitable for z small and vice-versa. This explains why the expression has powers both of $|z|$ (for behavior near 0) and of $(1+|z|)$ (for behavior near ∞). A similar discussion applies to the factors $|z-1|^\alpha$.

Proof of the Little Picard theorem. Introduce $\mathbb{C}^{**} := \mathbb{C}\backslash\{0,1\}$. Define

$$\lambda_{\mathbb{C}^{**}}(z) := \frac{(1+|z|^{1/3})^{1/2}(1+|z-1|^{1/3})^{1/2}}{C|z|^{5/3}|z-1|^{5/3}} \tag{16}$$

with a constant $C>0$. $\lambda_{\mathbb{C}^{**}}$ is positive and smooth on $\mathbb{C}^{**}$. The metric $ds^2_{\mathbb{C}^{**}} := 2\lambda_{\mathbb{C}^{**}}(z)dz\wedge d\bar{z}$ is a Hermitian metric on $\mathbb{C}^{**}$ and for C sufficiently large its curvature $K(z) := K_{ds^2_{\mathbb{C}^{**}}}(z) < -1$. Let proceed to calculate its curvature. First notice that, away from the origin, $\frac{\partial^2(\ln(|z|^{5/3}))}{\partial z\partial\bar{z}} = 0$, since $z\mapsto \ln|z|$ is harmonic in $\mathbb{C}^*$. The situation is the same around 1. Thus in $\mathbb{C}^{**}$ we have

$$\begin{aligned}
K(z) &= -\frac{1}{\lambda_{\mathbb{C}^{**}}}\frac{\partial^2}{\partial z\partial\bar{z}}\ln\left[\frac{(1+|z|^{1/3})^{1/2}(1+|z-1|^{1/3})^{1/2}}{C|z|^{5/3}|z-1|^{5/3}}\right]\\
&= -\frac{1}{\lambda_{\mathbb{C}^{**}}}\frac{\partial^2}{\partial z\partial\bar{z}}\ln\left[(1+|z|^{1/3})^{1/2}(1+|z-1|^{1/3})^{1/2}\right]\\
&= -\frac{1}{72\lambda_{\mathbb{C}^{**}}}\left[\frac{1}{|z|^{5/3}(1+|z|^{1/3})^2}+\frac{1}{|z-1|^{5/3}(1+|z-1|^{1/3})^2}\right]\\
&= -\frac{C}{72}\left[\frac{|z-1|^{5/3}}{(1+|z|^{1/3})^{5/2}(1+|z-1|^{1/3})^{1/2}}+\frac{|z|^{5/3}}{(1+|z|^{1/3})^{1/2}(1+|z-1|^{1/3})^{5/2}}\right]
\end{aligned}$$

It can be derived from the expression above that

$$\lim_{z\to 0}K(z) = \lim_{z\to 1}K(z) = -\frac{C}{72\sqrt{2}} \text{ and } \lim_{|z|\to\infty}K(z) = -\infty.$$

It follows immediately that K is bounded from above by -1 for C large enought.

Let $f\in\mathcal{O}(\mathbb{D}_r,\mathbb{C}^{**})$, with $r>0$. Then $g(z) := f(rz)\in\mathcal{O}(\mathbb{D},\mathbb{C}^{**})$. According to Ahlfors' lemma it follows that $\lambda_{\mathbb{C}^{**}}(g(0))|g'(0)|^2 \le 2$ and

$$(r\,|f'(0)|)^2 \le \frac{2}{\lambda_{\mathbb{C}^{**}}(f(0))} \tag{17}$$

We can now prove the Little Picard theorem. Assume that f is an entire function such that $f(\mathbb{C}) \subseteq \mathbb{C}^{**}$. Choose an arbitrary point $z_0 \in \mathbb{C}$ and introduce a function $g(z) := f(z + z_0)$. Choose an increasing and unbounded sequence (r_n) of positive real numbers. Let $g_n := g|_{\mathbb{D}_{r_n}}$. By equation (17) for every $n \in \mathbb{N}$ it follows

$$|f'(z_0)|^2 = |g_n'(0)|^2 \leq \frac{2}{r_n^2 \lambda_{\mathbb{C}^{**}}(g_n(0))} = \frac{2}{r_n^2 \lambda_{\mathbb{C}^{**}}(f(z_0))} \xrightarrow{n \to \infty} 0$$

since $g_n(0) = f(z_0)$ and $g_n'(0) = f'(z_0)$. Hence $f'(z_0) = 0$. Because z_0 was an arbitrary point, it follows $f' \equiv 0$ on $\mathbb{C}$. This means that f is a constant function. □

By using inequality (17) we are able to provide a very easy proof of the following Landau theorem.

Theorem 3.5.2 *Assume that $f \in \mathcal{O}(\mathbb{D}_r, \mathbb{C}^{**})$ for some $r > 0$ and $f'(0) \neq 0$. Then there is a constant $C > 0$, depending only on $f(0)$ and $f'(0)$ such that $r \leq C$.*

Proof. Inequality (17) suggests that a good choice for a constant is

$$C = \frac{2}{|f'(0)|\, \lambda_{\mathbb{C}^{**}}^{1/2}(f(0))}$$

which only depends on $f(0)$ and $f'(0)$. □

Assume that $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ is a power series expansion of f at 0 . Then $f(0) = a_0$ and $f'(0) = a_1$. Theorem 3.5.2 has the following equivalent form: if f omits 0 and 1 and $a_1 \neq 0$, then a constant $C(a_0, a_1) > 0$ exists such that the convergence radius of f is not greater than $C(a_0, a_1)$.

In the same vein, H. Schottky studied the size of an image of a disc under a holomorphic mapping, which omits two distinct points on $\mathbb{C}$. The theorem is a bridge between the Little and Big Picard theorems.

Theorem 3.5.3 (Schottky's Theorem) *Let R and C be positive real numbers. Assume that we have $f \in \mathcal{O}(\mathbb{D}_R, \mathbb{C}^{**})$ such that $|f(0)| < C$. Then for every $r \in (0, R)$, there exits a constant M, depending only on R, r and C such that $|f(z)| \leq M$ for $|z| \leq r$.*

3.5.3 The Big Picard Theorem

While studying the properties of $\mathcal{O}(\Omega)$, we should introduce the concept of normal families: a family $\mathcal{F} \subset \mathcal{O}(\Omega)$ is normal if every sequence in $\mathcal{F}$ has a convergent subsequence in $\mathcal{O}(\Omega)$, where convergence is uniformly on compact sets. For $\mathcal{F}$ we say that it is bounded on Ω if for every compact set $K \subset \Omega$ a constant $C(K)$ exists such that

$$\sup_{f \in \mathcal{F}} \left(\sup_{z \in K} |f(z)| \right) \leq C(K). \tag{18}$$

We denote $\|f\|_K := \sup_{z\in K}|f(z)|$.

Theorem 3.5.4 (Montel) *A family $\mathcal{F} \subset \mathcal{O}(\Omega)$ is bounded on Ω if and only if it is normal.*

It is useful to expand the definition of normality in the direction that allows uniform convergence on compact sets to ∞. A closed family $\mathcal{F} \subset \mathcal{O}(\Omega_1, \Omega_2)$ is normal if every sequence in $\mathcal{F}$ has convergent subsequence or has a compactly divergent subsequence. A sequence $(g_n) \subset \mathcal{O}(\Omega_1, \Omega_2)$ is compactly divergent if for arbitrary compact sets $K \subset \Omega_1$ and $L \subset \Omega_2$, there exists an integer $N \in \mathbb{N}$ such that $g_n(K) \cap L = \emptyset$ for all $n > N$.

For the proof of the next theorem we need the classical result by A. Hurwitz [Ahl79, p. 178]: Assume that $(f_n) \subset \mathcal{O}(\Omega)$ is a convergent sequence with the limit $f \in \mathcal{O}(\Omega)$. If there exists $a \in \mathbb{C}$ such that $a \notin f_n(\Omega)$ for every $n \in \mathbb{N}$, then $a \notin f(\Omega)$ or $f \equiv a$.

Theorem 3.5.5 (The Normality Theorem) *Let there be $a, b \in \mathbb{C}, a \neq b$. Then any family $\mathcal{F} \subseteq \mathcal{O}(\Omega, \mathbb{C}\backslash\{a,b\})$ is normal for every domain $\Omega \subset \mathbb{C}$.*

Proof. Since $z \mapsto (b-a)z + a$ is a biholomorphisme mapping between $\mathbb{C}\backslash\{0,1\})$ and $\mathbb{C}\backslash\{a,b\})$, we can suppose that $\mathcal{F} \subset \mathcal{O}(\Omega, \mathbb{C}\backslash\{0,1\})$ and $(f_n) \subset \mathcal{F}$ is an arbitrary sequence. It is enough to show that for every point $x \in \Omega$ there is a neighborhood $U \subset \Omega$ such that the family $\{\, f_n|_U : n \in \mathbb{N}\}$ is normal.
Choose a fixed but arbitrary point $x \in \Omega$. Choose $R > 0$ such that $\mathbb{D}_R(x) \subset \Omega$. If the sequence $(f_n(x))$ is bounded (by C), then according to Schottky's theorem, for any $0 < r < R$, there exists $M > 0$ such that $f_n\left(\overline{\mathbb{D}_r}(x)\right) \subset \mathbb{D}_M$: $\sup_{f\in\{f_n\}} \|f\|_{\overline{\mathbb{D}_r}(x)} \leq M$. According to Montel's theorem, there exists a subsequence $(f_{n(j)})_j \subset (f_n)$ such that $(f_{n(j)})_j$ uniformly converges to $f \in \mathcal{O}(\mathbb{D}_r(x))$ on compact sets in $\mathbb{D}_r(x)$. If $f(\mathbb{D}_r(x)) \subset \mathbb{C}\backslash\{0,1\}$, the goal has been achieved. If it isn't the case, according to Hurwitz' Theorem, since the sequence $(f_{n(j)}) \subset \mathcal{O}(\Omega, \mathbb{C}\backslash\{0,1\})$, this implies that $f \equiv 0$ or 1. Therefore, the sequence is compactly divergent.
If the sequence $(f_n(x))$ is unbounded, there exists a subsequence $(|f_{n(j)}(x)|)_j$ which goes to ∞. Then we can do the same thing as before with $g_n = 1/f_n$ and we obtain the same conclusion.
□

The Normality theorem is used to prove the Big Picard theorem (Theorem 3.1.5).

Proof of Theorem 3.1.5. Assume that $f \in \mathcal{O}(\mathbb{D}^*_\epsilon)$, where 0 is an essential singularity and there are two values $a \neq b$ such that both equations $f(z) = a$ and $f(z) = b$ have only finitely many solutions in $\mathbb{D}^*_\epsilon$. Hence there is a $\epsilon \geq \delta > 0$ such that there are no solution in $\mathbb{D}^*_\delta$. By a simple change of variable we can assume $\delta = 2$. Consider now the domain $D = \{z \in \mathbb{C} : 1/2 < |z| < 2\}$. Define the family $\mathcal{F} := (f_n|_D)$, where $f_n(z) := f(2^{-n}z)$. f_n is defined in $\{z \in \mathbb{C} : 0 < |z| < 2^{n+1}\}$. Since

$$\mathcal{F} \subset \mathcal{O}(D, \mathbb{C}\backslash\{a,b\}),$$

the Normality theorem guarantees that $\mathcal{F}$ is a normal family. Therefore there is a subsequence $(f_{n(j)})$ such that $f_{n(j)} \to g \in \mathcal{O}(D)$ (normally in D). Either $g \equiv \infty$ or g is holomorphic in D.

If $g \not\equiv \infty$, then we can assume that the sequence $(f_{n(j)})$ is uniformly bounded on the unit circle $\partial\mathbb{D}$, which is a compact subset of D. Then there exists $M > 0$ such that $|f(z)| \leq M$ and $|f_{n(j)}(z)| \leq M$, for any j and any $z \in \partial\mathbb{D}$. Then we have $|f(z)| \leq M$ for $|z| = 2^{-n(j)}$. By the maximum principle, whenever $2^{-n(j)} \leq |z| \leq 1$, $|f(z)| \leq M$. Hence $|f(z)| \leq M$ in $\mathbb{D}^*$, since $2^{-n(j)} \to 0$. This means that the singularity is removable. This is in contradiction with the assumption of an essential singularity.
If $g \equiv \infty$, then $(f_{n(j)}) \to \infty$ uniformly in $\partial\mathbb{D}$ and we can assume that $(1/f_{n(j)})$ is bounded in $\partial\mathbb{D}$. Then this implies that 0 is a removable singularity for $1/f$, which implies that f has a pole at 0. It is also a contradiction. □

Chapter 4

Entire curve and a glimpse of hyperbolic complex manifolds

We briefly describe main properties of Kobayashi hyperbolic complex manifolds and Brody hyperbolic complex manifolds. We are especially interested on those properties which are in direct connection with Picard's theorems.

4.1 Kobayashi hyperbolicity

We begin with the notion of invariant pseudodistances. These are pseudodistances which can be constructed on the category of complex manifolds and they become isometries for biholomorphic mappings. In 1967, S.Kobayashi constructed one of those pseudodistances.
For every $x, y \in M$ and every $f \in \mathcal{O}(M, N)$, where M and N are (connected) complex manifolds, the Kobayashi pseudodistance d_M^K has the following properties

$$d_N^K(f(x), f(y)) \leq d_M^K(x, y) \tag{1}$$

$$d_{\mathbb{D}}^K(x, y) = \rho(x, y) \tag{2}$$

Thus d_M^K is an invariant pseudodistance, which coincides with the Poincaré distance on a disc. Explicit construction is carried out by the so-called chain of holomorphic discs. A chain of holomorphic discs from p to q (points in M) is the following data

$$\alpha : \begin{cases} p = p_0, p_1, \ldots, p_n = q \in M \\ a_1, a_2, \ldots, a_n \in \mathbb{D} \\ f_1, f_2, \ldots, f_n \in \mathcal{O}(\mathbb{D}, M) \end{cases}$$

where $f_k(0) = p_{k-1}$ and $f_k(a_k) = p_k$ for all $k \in \{1, \ldots, n\}$. The lenght of α is $l(\alpha) := \sum_{k=1}^{n} \rho(0, a_k)$ and **Kobayashi pseudodistance** is then defined as

$$d_M^K(p, q) := \inf_{\alpha} \left\{ \sum_{k=1}^{n} \rho(0, a_k) \right\}$$

It is a pseudo-distance: $d_M^K(p,q) = d_M^K(q,p)$ and $d_M^K(p,q) \leq d_M^K(p,w) + d_M^K(w,q)$, for all p, q and $w \in M$.
We have the distance decreasing property (1) because any chain of holomorphic discs joining x to y can be composed with f to induce a chain of holomorphic discs joining $f(x)$ and $f(y)$. The result just follows by taking infimums.
And we have (2), according to Schwarz Lemma:
$\rho(0, \varphi_p(q)) = \rho(p,q) \leq \sum_{k=1}^n \rho(p_{k-1}, p_k) = \sum_{k=1}^n \rho(f_k(0), f_k(a_k)) \leq \sum_{k=1}^n \rho\,(0, a_k)$.
The Kobayashi pseudo-distance is also inner.

Generally speaking, d_M^K is not a distance.
We have $d_{\mathbb{C}}^K \equiv 0$. We say that $d_{\mathbb{C}}^K$ is degenerate. To see this, take a holomorphic mapping $f(z) := p + \varepsilon^{-1}(q-p)z$ from $\mathbb{D}$ into $\mathbb{C}$, where $p, q \in \mathbb{C}$ are arbitrary points and $\varepsilon > 0$ is an arbitrary small number. Then $f(0) = p$ and $f(\varepsilon) = q$. From (1) we get $d_{\mathbb{C}}^K(p,q) \leq \varrho(0,\epsilon) = \ln \frac{1+\epsilon}{1-\epsilon} \lesssim 2\varepsilon$.
Similarly, the Kobayashi pseudo-distance of $\mathbb{C}^*$ is also degenerate because we have the holomorphic mapping $\exp : \mathbb{C} \to \mathbb{C}^*$ which is surjective.

This distance decreasing property characterizes the Kobayashi pseudo-distance.

Proposition 4.1.1 *Let $\delta : M \times M \to \mathbb{R}_0^+$ be a pseudo-distance on M such that for any holomorphic map $f \in \mathcal{O}(\mathbb{D}, M)$, and for any z, $w \in \mathbb{D}$ one has*

$$\delta(f(z), f(w)) \leq \varrho(z,w)$$

then $\delta(x,y) \leq d_M^K(x,y)$, $\forall\, x$, $y \in M$.

Proof. Let x, $y \in M$ and consider a chain α of holomorphic discs given by the data

$$\alpha : \begin{cases} x = x_0, x_1, \ldots, x_n = y \in M \\ a_1, a_2, \ldots, a_n \in \mathbb{D} \\ f_1, f_2, \ldots, f_n \in \mathcal{O}(\mathbb{D}, M) \end{cases}$$

By hypothesis and the triangular inequality, one has

$$\delta(x,y) \leq \sum_{i=1}^n \delta(x_{i-1}, x_i) = \sum_{i=1}^n \delta(f_i(0), f_i(a_i)) \leq \sum_{i=1}^n \varrho(0, a_i) = l(\alpha).$$

By passing to the infimum, one obtains $\delta(x,y) \leq d_M^K(x,y)$. □

We call pseudodistances, which satisfy the properties (1) and (2), contractible pseudodistances. Then the Kobayashi pseudodistance is the largest among contractible pseudodistances.

Definition 4.1.2 *A complex manifold M is* **Kobayashi hyperbolic** *if the Kobayashi pseudo-distance d_M^K becomes a distance and is* **complete Kobayashi hyperbolic** *if (M, d_M^K) is a complete metric space.*

Definition 4.1.3 Property (Kobayashi, Royden) *Let X be a complex manifold. $x \in X$ and $\xi \in T_{X,x}$ a tangent vector at X in x. The Kobayashi-Royden infinitesinal pseudo-metric on X is defined by*

$$F_X(\xi) = \inf\{||u||_\varrho : \exists f : \mathbb{D} \to X,\ f(0) = x,\ u \in T_{\mathbb{D},0},\ df(u) = \xi\}$$

$$= \inf\{\lambda > 0 : \exists f : \mathbb{D} \to X,\ f(0) = x,\ \lambda f'(0) = \xi\}.$$

The pseudo-distance of Kobayashi d_X^K is the associated integrated pseudo-distance of this pseudo-norm: for any x, $y \in X$,

$$d_X^K(x, y) = \inf_\gamma \int_0^1 F_X(\gamma'(t))dt$$

where the infimum is taken over all piecewise smooth curves $\gamma : [0, 1] \to X$ joining x to y.

Holomorphic maps $f : X \to Y$ are distance decreasing with respect to it: for any $x \in X$ and $\xi \in T_{X,x}$, one has

$$F_Y(f_*\xi) \leq F_X(\xi).$$

For instance: $F_\mathbb{D}(\xi) = ||\xi||_\varrho$ and $F_\mathbb{C} \equiv 0$.

Kobayashi hyperbolic manifolds have several important properties, including the fact that direct product of (complete) Kobayashi hyperbolic manifolds is (complete) Kobayashi hyperbolic.
If M and N are two complex manifolds. For any z_1, $z_2 \in M$ and any w_1, $w_2 \in N$, we have $d_{M\times N}^K((z_1, w_1), (z_2, w_2)) = \max\{d_M^K(z_1, z_2), d_N^K(w_1, w_2)\}$.
And $d_M^K : M \times M \to \mathbb{R}_0^+$ is a continuous map, where M is endowed with the Euclidean topology and $\mathbb{R}_0^+$ with the standard topology.
Any submanifold of a Kobayashi hyperbolic manifold, is Kobayashi hyperbolic.
With this pseudo-distance, we can see (1), as a generalization of the Schwarz'Lemma in any dimension.

$\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{P}^1$ are not Kobayashi hyperbolic. Generally, a Riemann surface is hyperbolic if and only if it is uniformazed by the unit disc (its universal covering is $\mathbb{D}$).
Ahlfors' lemma implies that every planar domain (or more generally every Riemann surface), which carries a complete Hermitian metric (ds^2) of curvature not greater than -1 (then $f^*(ds^2) \leq d\varrho^2$), is complete Kobayashi hyperbolic:
we have $d(p, q) \leq \sum_k d(p_{k-1}, p_k) = \sum_k d(f_k(0), f_k(a_k)) \leq \sum_k \varrho(0, a_k)$; and passing to the infimum over all chain of holomorphic discs we obtain that: $d(p, q) \leq d^K(p, q)$.

Therefore, the domain $\mathbb{C}\backslash\{0, 1\}$ is complete Kobayashi hyperbolic. From this it is easy to see that every domain $\Omega \subseteq \mathbb{C}\backslash\{a, b\}, a \neq b$, is complete Kobayashi hyperbolic; observe that every Cauchy sequence in Ω is also Cauchy in $\mathbb{C}\backslash\{a, b\}, a \neq b$.

In higher dimensions, any bounded domain in $\mathbb{C}^n$ is Kobayashi hyperbolic. There exist unbounded domains in $\mathbb{C}^n$ which are not Kobayashi hyperbolic and there exist bounded domains in $\mathbb{C}^n$ which are not complete Kobayashi hyperbolic.
1) The Kobayashi distance on $\mathbb{D}\times\mathbb{D}$ is given by $d^K_{\mathbb{D}^2}((z_1,w_1),(z_2,w_2)) = \max\{\varrho(z_1,z_2),\varrho(w_1,w_2)\}$.
2) The Kobayashi distance on $\mathbb{D}\times\mathbb{C}$ is given by $d^K((z_1,w_1),(z_2,w_2)) = \varrho(z_1,z_2)$.
3) An example of Eisenman and Taylor. Consider the following open set U in $\mathbb{C}^2$

$$U = \{(z,w)\in\mathbb{C}^2 : |z|<1, |zw|<1\}\setminus\{(0,w) : |w|\geq 1\}.$$

U is unbounded and is not Kobayashi hyperbolic.
For any (z_1,w_1), $(z_2,w_2)\in U$, we have

$$d^K_U((z_1,w_1),(z_2,w_2)) = 0, \text{ if } z_1=z_2=0$$

$$d^K_U((z_1,w_1),(z_2,w_2)) = d^K_{\mathbb{D}^2}((z_1,z_1w_1),(z_2,z_2w_2)) \text{ otherwise.}$$

Indeed, for any $|w_0|<1$, let $f_1(z)=(z,0)$, $f_2(z)=(\frac{1}{n}+\frac{z}{2},w_0)$ defined in $\mathbb{D}$. $f_1(0)=(0,0)$, $f_1(\frac{1}{n})=(\frac{1}{n},w_0)=f_2(0)$ and $f_2(\frac{-2}{n})=(0,w_0)$. Then when n goes to ∞, $d^K_U((0,0),(0,w_0))$ goes to 0.
4) The punctured bidisc $\mathbb{D}_2\backslash\{(0,0)\}$ is a simple example of bounded domain not complete Kobayashi hyperbolic.
Denote punctured bidisc by X. Because X is bounded, it is Kobayashi hyperbolic.
Define the following sequences $a_n:=(0,\alpha_n)\,,b_n:=(\alpha_n,0)$ and $c_{n,m}:=(\alpha_n,\alpha_m)$ where $\{\alpha_n\}\subset\mathbb{D}$ is a sequence with property $\rho(0,\alpha_n)=2^{-n}$. Introducing domains $X_1:=\mathbb{D}\times\mathbb{D}^*\subset X$ and $X_2:=\mathbb{D}^*\times\mathbb{D}\subset X$ yields

$$d^K_X(a_n,b_n)\leq d^K_X(a_n,c_{n,n})+d^K_X(b_n,c_{n,n})\leq d^K_{X_1}(a_n,c_{n,n})+d^K_{X_2}(b_n,c_{n,n})$$

$$d^K_X(b_n,a_{n+1})\leq d^K_X(b_n,c_{n,n+1})+d^K_X(a_{n+1},c_{n,n+1})\leq d^K_{X_2}(b_n,c_{n,n+1})+d^K_{X_1}(a_{n+1},c_{n,n+1})\,.$$

Define $f_n\in\mathcal{O}(\mathbb{D},X_1)$ with $f_n(z):=(z,\alpha_n)$ and $g_n\in\mathcal{O}(\mathbb{D},X_2)$ with $g_n(z):=(\alpha_n,z)$.
Then $d^K_{X_1}(a_n,c_{n,n})=d^K_{X_1}(f_n(0),f_n(\alpha_n))\leq\rho(0,\alpha_n)$ and
$d^K_{X_1}(a_{n+1},c_{n,n+1})=d^K_{X_1}(f_{n+1}(0),f_{n+1}(\alpha_n))\leq\rho(0,\alpha_n)$.
Equivalently $d^K_{X_2}(b_n,c_{n,n})\leq\rho(0,\alpha_n)$ and $d^K_{X_2}(b_n,c_{n,n+1})\leq\rho(0,\alpha_n)$.
Thus $d^K_X(a_n,b_n)\leq 2^{1-n}$, $d^K_X(b_n,a_{n+1})\leq 2^{1-n}$ and $d^K_X(a_n,a_{n+1})\leq 2^{2-n}$. Therefore (a_n) is a Cauchy sequence which converges to $(0,0)\notin X$. □

Definition 4.1.4 *Let X be a complex manifold. Let h be a metric on T_X, namely a smooth map $h : T_X\times T_X\to\mathbb{C}$ such that for any $x\in X$ the induced map $h_x : T_{X,x}\times T_{X,x}\to\mathbb{C}$ is a hermitian metric (we identy it to a hermitian $(1,1)$-form).*
Set $||.||_h : T_X\to\mathbb{R}^+_0$ the associated lenght function $||\xi||_h=\sqrt{h(\xi,\xi)}$, for all $\xi\in T_X$.
The **holomorphic sectional curvature** *of h is the following function defined by*

$$HSC_h(x,[\xi]) = \sup K_{f^*h}(0)$$

for all $x\in X$ and $\xi\in T_{X,x}\backslash\{O\}$, where the supremum is taken over all analytic disc $f:\mathbb{D}\to X$ such that $f(0)=x$ and $\xi\in\mathbb{C}f'(0)$. K_{f^*h} is the Gaussian curvature of the pseudo-metric f^*h.

If X is a complex curve then $HSC_h = K_h$ (see chapter 3).

Theorem 4.1.5 *Let X be a complex manifold endowed with a metric h such that $HSC_h \leq -\gamma$ for some $\gamma > 0$. Then X is Kobayashi hyperbolic.*

Proof. After normalization, one can suppose that $HSC_h \leq -1$ (just we replace h by $\gamma.h$). Let δ be the distance induced by h

$$\delta(x,y) = \inf_{\varphi} \int_0^1 ||\varphi'(t)||_h dt$$

with $\varphi : [0,1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$.
Let $f : \mathbb{D} \to X$ be a holomorphic map. The Ahlfors' Lemma implies $f^*h \leq d\varrho^2$, where $d\varrho^2$ is the Poincaré metric. $f^*||.||_h \leq ||.||_\varrho$. Integrating yields, for the associated integrated pseudodistances, we have

$$\delta(f(z), f(w)) \leq \varrho(z,w), \text{ for any } z,\, w \in \mathbb{D}.$$

By the distance decreasing characterization of the Kobayashi pseudo-distance, one finds for any $x,\, y \in X$

$$\delta(x,y) \leq d_X^K(x,y).$$

Hence d_X^K is a distance. □

4.2 Brody hyperbolicity

Definition 4.2.1 *Let X be a complex manifold. An entire curve in X is a non-constant holomorphic map $f : \mathbb{C} \to X$.*

Hyperbolicity is closely related to the Little Picard theorem. If we assume $f \in \mathcal{O}(\mathbb{C}, X)$, then we get $d_X^K(f(z), f(w)) = 0$ by (1), since d_X^K is contractible and $d_{\mathbb{C}}^K \equiv 0$. This shows that every holomorphic map from $\mathbb{C}$ to a Kobayashi hyperbolic manifold is constant.
For instance, if X is a bounded open subset in $\mathbb{C}^n$, then there are no entire curves $f : \mathbb{C} \to X$, according to Liouville's theorem.

Proposition 4.2.2 *If X is Kobayashi hyperbolic, then X doesn't contain any entire curve.*

This justifies the following definition

Definition 4.2.3 *A complex manifold X is said to be* **Brody hyperbolic** *if X doesn't contain any entire curve.*

Then we have proved that

$$\text{Kobayashi hyperbolicity} \Rightarrow \text{Brody hyperbolicity}.$$

The converse doesn't always hold. A counter example is given by the Eisenmann-Taylor example: the unbounded open set U in $\mathbb{C}^2$

$$U = \{(z, w) \in \mathbb{C}^2 : |z| < 1, |zw| < 1\} \setminus \{(0, w) : |w| \geq 1\}.$$

We have already seen that U is not Kobayashi hyperbolic. We can prove that U is Brody hyperbolic.
Indeed, let π_1 be the first projection $U \to \mathbb{D}$, $(z, w) \mapsto z$. π_1 is holomorphic with fibers which are discs (If $z = 0$ then $|w| < 1$. If $z \neq 0$, $|z| < 1$ then $|w| < 1/|z|$).
Let $f = (f_1, f_2)$ be an entire curve in U. Then according to Liouville's Theorem, $f_1 = \pi_1 \circ f : \mathbb{C} \to \mathbb{D}$ is a constant c_1 ($|c_1| < 1$). If $c_1 = 0$, then $f_2 : \mathbb{C} \to \mathbb{D}$ and f_2 is constant too. If $c_1 \neq 0$, then $f_2 : \mathbb{C} \to \mathbb{D}_{1/|c_1|}$ and f_2 is constant too.

According to Little Picard Theorem, we have that $\mathbb{C}\backslash\{a, b\}$ (where $a \neq b$) is also Brody hyperbolic.

The converse of the previous proposition is true on compact complex manifolds in view of a fundamental theorem of R.Brody from 1978.

Theorem 4.2.4 *A compact complex manifold is Kobayashi hyperbolic if and only if it is Brody hyperbolic.*

Lemma 4.2.5 (Brody's reparametrization lemma) *Let X be a complex manifold endowed with a hermitian metric h. Let $f : \mathbb{D} \to X$ be an analytic disc in X. For any $0 \leq r < 1$ there exists $R \geq r||f'(0)||_h$ and a biholomorphic map $\varphi : \mathbb{D}_R \to \mathbb{D}_r$ such that*
(i) $||(f \circ \varphi)'(0)||_h = 1$
(ii) For all $t \in \mathbb{D}_R$, $||(f \circ \varphi)'(t)||_h \leq \dfrac{1}{1 - |t/R|^2}$.

Proof. Let $z_0 \in \mathbb{D}$ be such that $(1 - |z|^2)||f'(rz)||_h$ has a maximum at z_0, i.e. the norm of the differential of the map $f_r : z \mapsto f(rz)$ (which is holomorphic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$) with respect to the Poincaré metric ϱ and the metric h is maximal .
Let $\varphi(t) = r\frac{t+Rz_0}{R+\bar{z}_0 t} = rg_{z_0}(t/R)$, where $g_{z_0}(t) = \frac{t+z_0}{1+\bar{z}_0 t}$ is a Möbius transformation and an isometry for the Poincaré metric ϱ. $\varphi(0) = rz_0$ and $\varphi'(0) = r\frac{1-|z_0|^2}{R}$. We have

$$||(f \circ \varphi)'(0)||_h = |\varphi'(0)|.||f'(rz_0)||_h = (1 - |z_0|^2)\frac{r}{R}||f'(rz_0)||_h.$$

If we choose $R = r(1 - |z_0|^2)||f'(rz_0)||_h \geq r||f'(0)||_h$, we obtain (i).
$\left(1 - |t/R|^2\right) ||(f \circ \varphi)'(t)||_h$ is the norm of the differential of $f \circ \varphi$ at t, with respect to the

Poincaré metric ϱ and the metric h.
Since g_{z_0} is an isometry for ϱ, there exists a constant $C > 0$ such that $C.\left(1 - |t/R|^2\right) ||(f \circ \varphi)'(t)||_h$ is the norm of the differential of the map f_r at $g_{z_0}(t/R)$, with respect to the Poincaré metric and the metric h.
Since this last one is maximal for $t = 0$, $\left(1 - |t/R|^2\right) ||(f \circ \varphi)'(t)||_h$ is also maximal for $t = 0$. □

Lemma 4.2.6 *Let X be a compact complex manifold (with a hermitian metric h and the associated distance d with respect to this metric). Then X is Kobayashi hyperbolic if and only if* $\sup |f'(0)|_h < \infty$*, where the supremum is taken over any $f \in \mathcal{O}(\mathbb{D}, X)$.*

Proof. $\sup |f'(0)|_h = \sup |f_*(v)|_\varrho$, where $v \in T_\mathbb{D}$ with $|v|_\varrho = 1$.
If there exists $0 < c < \infty$ such that $\sup |f'(0)|_h = c$. Then $d_h(x, y) \leq c d_X^K(x, y)$ for any x, $y \in X$. Then X is Kobayashi hyperbolic.
Conversely, if this supremum is infinite, there is a sequence of mappings $f_n \in \mathcal{O}(\mathbb{D}, X)$ with $|f_n'(0)|_h \to \infty$. By compactness, there exists a subsequence (we call it again (f_n)) such that $(f_n(0))$ converges to a point $p \in X$. Let U be a little coordinate neighborhood of p. According to the Cauchy estimates, for any m large enough there exists n_m (the sequence $(n_m) \to \infty$) such that $f_{n_m}(\mathbb{D}_{1/m}) \cap \partial U \neq \emptyset$. There exists a sequence $(x_m) \subset \partial U$ such that each $x_m \in f_{n_m}(\mathbb{D}_{1/m})$, and therefore

$$d_X^K(f_{n_m}(0), x_m) \leq \varrho(0, 1/m) \to 0, \text{ as } m \to \infty.$$

By the continuity of d_X^K and the compactness of ∂U, there exists $x \in \partial U$ with $d_X^K(p, x) = 0$. Then X is not Kobayashi hyperbolic. □

Proof of the Theorem. Let suppose that X is a compact complex manifold not Kobayashi hyperbolic. Then there exists a sequence $(f_n) \subset \mathcal{O}(\mathbb{D}, X)$ such that $(||f_n'(0)||_h)$ is unbounded. According to Brody's reparametrization lemma, we obtain a sequence (g_n) such that $g_n : \mathbb{D}(R_n) \to X$, $R_n \geq \frac{1}{2}||f_n'(0)||_h$, $||g_n'(0)||_h = 1$ and $||g_n'(z)||_h \leq \frac{1}{1-|\frac{z}{R_n}|^2}$ for any $z \in \mathbb{D}(R_n)$.
The sequence $(R_n) \to \infty$. We can assume that it increases.
For any N, the family

$$\{g_n|_{\mathbb{D}(R_N)} : \mathbb{D}(R_N) \to X, \, n \geq N\}$$

is equicontinuous on $\mathbb{D}(R_N)$ with respect to the Poincaré distance and the distance on X induced by h. Therefore, this family is also equicontinuous on $\mathbb{D}(R_N)$ with respect to the Euclidean metric on $\mathbb{D}(R_N)$ and the distance on X induced by h.
According to Ascoli's theorem, there exists a subsequence (with a diagonal extraction. We call it again (g_n)) which converges uniformly on any compact subset in $\mathbb{C}$ to a map $g : \mathbb{C} \to X$ such that $||g'(z)||_h \leq ||g'(0)||_h = 1$ for any $z \in \mathbb{C}$. In particular g is not constant and X is not Brody hyperbolic. □