The wave front set and intersections of currents

CIMPA School: A Complex Analytic Approach to Differential and Algebraic Geometry

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Contents

1 Introduction

We explain an intuitive condition that ensures that a well-behaved product of two currents can be defined: Namely, if at any point $x \in X$, two currents S, T are 'singular along different directions', they can be multiplied.

Why would one want to multiply currents?

One motivation comes from intersection theory: If δ_Z, δ_W are currents associated to submanifolds $Z, W \subseteq X$, (Definition recalled below), the current $\delta_Z \wedge \delta_W$ should be related to the intersection of Z and W. E.g. one can compute de Rham cohomology using currents instead of forms and δ_Z, δ_W are closed and define canonical representatives de Rham cohomology classes $[Z] = [\delta_Z]$, $[W] = [\delta_W]$ and the current $\delta_Z \wedge \delta_W$ should represent the product of these classes. If we were only interested in the cohomology classes, we could use form representatives as well, but having a canonical representatives gives additional information, e.g. on the exact location and local properties of the intersection and not just its homology class. Moreover, if instead of the product in de Rham cohomology one wants to compute 'higher' products, e.g. Massey products or linking numbers, one is naturally lead to consider products of non-closed currents.[1]

[1] For three de Rham cohomology classes $[\alpha], [\beta], [\gamma] \in H_{dR}(X)$ with the additional condition that there exist x, y s.t. $\alpha \wedge \beta = dx$ and $\beta \wedge \gamma = dy$, their Massey product is defined as $\langle [\alpha], [\beta], [\gamma] \rangle \in H_{dR}(X)/([\alpha], [\gamma])$. It depends only on the input classes and not on the choices of $\alpha, \beta, \gamma, x, y$. If Z, W are submanifolds (or more generally cycles) of an n-dimensional comacpct manifold of dimensions e, f with $e+f = n-1$, which are null-homologous, δ_Z, δ_W the respective fundamental currents and $\delta_Z = dT$,

2 Forms and currents

Let X denote a smooth manifold of dimension n. Given the context of this school, we will mostly be interested in X actually underlying a complex manifold (in which case $n = 2m$ is even). In that case, one can replace the words 'grading' and 'cochain complex' by 'bigrading' and 'double complex' everywhere. That also mean we will always assume our manifolds to be oriented and maps to be orientation preserving (as complex manifolds and holomorphic maps are).

We denote by $\mathcal{A}(X)$ the vector space of smooth $\mathbb{C}$-valued differential forms[2]. It carries the following extra structure:

1. A grading $\mathcal{A}(X) = \bigoplus \mathcal{A}^k(X)$ by degree of the forms, i.e.

$$\mathcal{A}^k(X) = \{\omega \in \mathcal{A}^k(X) \mid \omega \overset{loc}{=} \sum_{I=(i_1,\dots i_k)} a_I dx_{i_1} \wedge \dots \wedge dx_{i_k}\}$$

2. A product $\mathcal{A}(X) \times \mathcal{A}(X) \to \mathcal{A}(X)$ which respects the degree, i.e. if $\alpha \in \mathcal{A}^k(X)$ and $\beta \in \mathcal{A}^l(X)$, then $\alpha \wedge \beta \in \mathcal{A}^{k+l}$. This product is associative and graded commutative, i.e. $\alpha \wedge \beta = (-1)^{|\alpha||\beta|}\beta \wedge \alpha$.

3. The exterior derivative,

$$d : \mathcal{A}(X) \to \mathcal{A}(X)$$

an operator of degree 1. It squares to zero $d^2 = 0$ and satisfies the Leibniz-rule $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta$.

I.e. $\mathcal{A}^k(X)$ is simultaneously a cochain complex and a graded-commutative algebra, and the two structures are compatible (via the Leibniz rule). We call such a structure a graded-commutative, differential graded algebra, or cdga for short.

The forms with compact support $\mathcal{A}_c(X) \subseteq \mathcal{A}(X)$ form a sub-cdga, i.e. they are a graded subspace preserved by d and $\wedge$.

Moreover, we can consider $\mathcal{A}(X)$ not just as an abstract vector space, but actually as a topological vector space. To this end, consider the following semi-norms on this space: For any chart (U, φ) of X, compact subset $K \subseteq U$ and non-negative integer p, for $\alpha \in \mathcal{A}^k(X)$, we write $\varphi^*\alpha = \sum_{|I|=k} a_I(x)dx_I$ and set

$$|\alpha|_{\varphi,K,p} := \sup_{x\in K} \max_{\substack{|I|=k \\ |\alpha|=p}} |\partial^\alpha a_I|$$

The collections of all these seminorms (for varying φ, K, p) defines a Fréchet space topology on the spaces $\mathcal{A}^k(X)$. For any compact subset $L \subseteq X$ we moreover equip the space of smooth k-forms with support in L, denoted $\mathcal{A}^k_L(X)$ with the induced topology and we equip the space of compactly supported forms

$$\mathcal{A}^k_c(X) = \bigcup_{L\subseteq X \text{ compact}} \mathcal{A}^k_L(X)$$

where T is a current, one may want to define its linking number as $\langle Z, W\rangle := \int_W T = \delta_W \wedge T(1)$ and so one needs the product to be well-defined.

[2]As an overall convention, unless explicitly stated otherwise, all our functions, forms, and currents will be $\mathbb{C}$-valued and we will stop mentioning this explicitly. If we ever want to consider the subspace of $\mathbb{R}$-valued forms, we will indicate this by a subscript, e.g. $\mathcal{A}(X)_\mathbb{R} \subseteq \mathcal{A}(X)$

with the finest locally convex topology s.t. for any compact set $L \subseteq X$ the inclusion $\mathcal{A}^k_L(X) \subseteq \mathcal{A}^k_c(X)$ is continuous.

Remark 2.0.1. One might whether we are actually simply describing the subspace topology on $\mathcal{A}_c(X) \subseteq \mathcal{A}(X)$. In fact, the topology described above is strictly finer. For concreteness, consider $X = \mathbb{R}$. For a sequence of functions converging to zero, it must eventually have support concentrated in one fixed compact set $K \subseteq X$, on which it (and all its derivatives) need to converge uniformly to zero. On the other hand, in the subspace topology, a sequence of functions whose support moves ever closer 'to infinity' (i.e. eventually lies outside every compact interval) converges to zero.

Definition 2.0.2. Let X be a manifold.

1. The space of currents $\mathcal{M}(X)$ is the topological dual of $\mathcal{A}_c(X)$.

   Concretely, a linear map $T : \mathcal{A}_c(X) \to \mathbb{C}$ is in $\mathcal{M}(X)$ if and only if $T|_{\mathcal{A}_L(X)}$ is continuous for every $L \subseteq X$, if and only if for every chart $\varphi : U \to \mathbb{R}^n$, compact set $K \subseteq U$ and $p \in \mathbb{N}$, there exist a $C > 0$ s.t. $|T(\alpha)| \leq C|\alpha|_{\varphi,K,p}$.

2. The space of currents with compact support $\mathcal{M}_c(X)$, is the topological dual of $\mathcal{A}(X)$.

Remark 2.0.3 (Locality: Currents form a sheaf)**.** Given an open immersion $j : U \hookrightarrow X$ (e.g. the inclusion of an open subset), there is an inclusion $j_* : \mathcal{A}_c(U) \subseteq \mathcal{A}_c(X)$ via extending by zero. Dually, this induces a restriction map: For $\mathcal{M}(X) \to \mathcal{M}(U), T \mapsto T|_U =: j^*T$. If there is an open cover $X = \bigcup U_i$ and currents $T_i \in \mathcal{M}(U_i)$ with $T_i|_{U_i \cap U_j} = T_j|_{U_i \cap U_j}$ for all i, j then there exists a unique current $T \in \mathcal{M}(U)$ with $T|_{U_i} = T_i$.

Remark 2.0.4 (Currents with compact support are currents whose support is compact)**.** The subset $\mathcal{A}_c(X) \subseteq \mathcal{A}(X)$ is dense (exercise), and so the restriction induces an injective map $\mathcal{M}_c \subseteq \mathcal{M}$. The support of a current $T \in \mathcal{A}(X)$ is then the set

$$\operatorname{supp} T = \{x \in X \mid \text{there exists no open nbhd } x \in U \subseteq X \text{ s.t. } T|_U = 0\}.\}$$

One may check that the image of $\mathcal{M}_c \to \mathcal{M}$ consists indeed of those currents for which the support is compact (exercise).

Remark 2.0.5 (Currents form a complex)**.** The currents are again a graded vector space $\mathcal{M}(X) = \bigoplus \mathcal{M}^k(X)$ with $\mathcal{M}^k = \mathrm{Hom}_{cont}(\mathcal{A}^{n-k}_c(X), \mathbb{C})$. They are also a cochain complex, where $dT(\alpha) := (-1)^{k+1}T(d\alpha)$ for $T \in \mathcal{M}^k$ and $\alpha \in \mathcal{A}^{n-k-1}_c$. The same holds for $\mathcal{M}_c$ and the inclusion $\mathcal{M}_c \hookrightarrow \mathcal{M}$ is a map of complexes. Sometimes, one also considers a 'lower grading' defined by $\mathcal{M}_k = \mathcal{M}^{n-k} = Hom(\mathcal{A}^k, \mathbb{C})$, but for the most part, the upper grading will be more convenient for us, as it is well-attuned to comparing forms and currents (see next point).

Example 2.0.6 (Current associated to a differential form)**.** For any $\alpha \in \mathcal{A}^k(X)$, there is a current T_α defined by $T_\alpha(\beta) = \int_X \alpha \wedge \beta$. This defines an injective map of cochain complexes $(\mathcal{A}^\bullet(X), d) \to (\mathcal{M}^\bullet(X), d)$, $\alpha \mapsto T_\alpha$. If a current is of the form T_α it is said to be smooth. Sometimes, one abuses notation and simply writes α for the current T_α. One has $\operatorname{supp}(T_\alpha) = \operatorname{supp}(\alpha)$. In particular, if $\alpha \in \mathcal{A}_c(X)$, then $T_\alpha \in \mathcal{M}_c(X)$.

Example 2.0.7 (Current associated to a submanifold)**.** If $Z \subseteq X$ is a submanifold of codimension p, then it has a fundamental current $\delta_Z \in \mathcal{M}^p(X)$, defined by $\delta_Z(\beta) = \int_Z \beta$. By Stokes' theorem, this is closed, i.e. it satisfies $d\delta_Z = 0$ (exercise). If X is actually a complex manifold and $Z \subseteq X$ is a complex submanifold of complex codimension p, then δ_Z is a current of bidegree (p,p) and one can define it more generally for any analytic subset $Z \subseteq X$, i.e. a subset locally defined by the vanishing of holomorphic functions, by defining δ_Z via the integral over the non-singular part $\delta_Z(\beta) = \int_{Z^{reg}} \beta$. It is non-obvious (but true) that this is indeed a convergent integral.

Remark 2.0.8 (Wedge product: Module structure)**.** The there is a pairing $\wedge : \mathcal{M}(X) \wedge \mathcal{A}(X) \to \mathcal{M}(X)$ defined by $(T \wedge \alpha)(\beta) = T(\alpha \wedge \beta)$ for $T \in \mathcal{M}(X)$, $\alpha \in \mathcal{A}(X)$, $\beta \in \mathcal{A}_c(X)$. This pairing respects the grading and satisfies the Leibniz rule $d(T \wedge \alpha) = dT \wedge \alpha + (-1)^{|T|} T \wedge d\alpha$). I.e., it turns $\mathcal{M}(X)$ into a differential graded $\mathcal{A}(X)$-module. Analogously, $\mathcal{M}_c(X)$ is a differential graded $\mathcal{A}_c(X)$-module.

Example 2.0.9. The previous three points fit together by the formula $T_\alpha = \delta_X \wedge \alpha$.

Remark 2.0.10 (Proper pushforward of currents)**.** For any smooth map $f : X \to Y$, there is a map of cdga's $f^* : \mathcal{A}(Y) \to \mathcal{A}(X)$. If f is in addition proper,[3] then f^* restricts to a map $\mathcal{A}_c(Y) \to \mathcal{A}_c(X)$, which one checks to be continuous (exercise). Hence, we obtain a dual map $f_* : \mathcal{M}(X) \to \mathcal{M}(Y)$, concretely given by $(f_*T)(\alpha) := T(f^*\alpha)$. One checks that this is a map of complexes, up to a degree shift, i.e. $f_* dT = df_*T$ but $f_*(\mathcal{M}^k(X)) = f_*(\mathcal{M}^{k+\dim_Y - \dim X}(X))$. Moreover, one has the so-called projection formula: for any $T \in \mathcal{M}(X)$ and $\alpha \in \mathcal{A}(Y)$, one has

$$f_*(T \wedge f^*\alpha) = f_*T \wedge \alpha$$

Algebraically, this formula says f_* is a map of $\mathcal{A}(Y)$-modules. If $f : X \to Y$ is a diffeomorphism, then $f_*T = (f^{-1})^*T$ for $T \in \mathcal{M}(X)$.

While we can multiply a current with a form, in general we cannot multiply two currents in a natural way which behaves reasonable with respect to degree and exterior derivative.[4] However, sometimes we can, and finding a good criterion when will be the topic of this mini-course.

Example 2.0.11. If $Y, Z \subseteq X$ are two submanifolds of codimension p, q, which intersect transversely, so that the intersection $Y \cap Z \subseteq X$ is a submanifold of codimension $p+q$, then it seems reasonable to define $\delta_Y \wedge \delta_Z := \delta_{Y \cap Z} \in \mathcal{M}^{p+q}(X)$. However, already when we want to check compatiblity with the $\mathcal{A}(X)$-module structure we need to explain more general products since we want that for any $\alpha \in \mathcal{A}(X)$, $(\delta_Y \wedge \delta_Z) \wedge \alpha = \delta_Y \wedge (\delta_Z \wedge \alpha)$.[5]

In a first approximation, we might restrict to intersections where the singularities of the currents never meet: Recall that a current is smooth on U if $T|_U = T_\alpha$ for some $\alpha \in \mathcal{A}(U)$.

[3] i.e. pre-images of compact sets are compact. E.g., any smooth map between compact manifolds is proper.

[4] In fancy terms: It is (provably) impossible to extend the product in such a way that $\mathcal{M}$ defines a functor $\{manifolds\} \to \{cdga's\}$

[5] In this case the product on the right can again be explained in an ad hoc way but we now proceed with the general study.

Definition 2.0.12. The singular support of a current $T \in \mathcal{M}(X)$ is the set

$$\operatorname{sing\,supp}(T) := \{x \in X \mid \text{ there is no open nbhd } x \in U \subseteq X \text{ s.t. } T|_U \text{ is smooth}\}$$

Lemma 2.0.13 (Properties of the singular support)**.**

1. For a submanifold $Z \subsetneq X$, $\operatorname{sing\,supp} \delta_Z = Z$.[6]
2. For $\alpha \in \mathcal{A}(X)$ and $T \in \mathcal{M}(X)$, $\operatorname{sing\,supp}(T \wedge \alpha) \subseteq \operatorname{sing\,supp}(T)$.
3. For any diffeomorphism $f : X \to Y$ and $T \in \mathcal{M}(X)$, there is an equality $f(\operatorname{sing\,supp}(T)) = \operatorname{sing\,supp}(f_* T)$.

Proposition 2.0.14. If $T \in \mathcal{M}^k$, $T \in \mathcal{M}^l$ with $\operatorname{sing\,supp} T \cap \operatorname{sing\,supp} S = \emptyset$, there exists a current $T \wedge S \in \mathcal{M}^{l+k}(X)$ which satisfies

1. If $T = T_\alpha$, $S = T_\beta$ for $\alpha, \beta \in \mathcal{A}(X)$, then $T_\alpha \wedge T_\beta = T_{\alpha \wedge \beta}$.
2. $d(S \wedge T) = dS \wedge T + (-1)^{|S|} S \wedge T$.
3. If $S, T, R \in \mathcal{M}^k$ with pairwise disjoint singular support, then $(T \wedge S) \wedge R = T \wedge (R \wedge S)$.

Proof. Let $X = \bigcup U_i$ be a locally cover such that on each open set either S or T is smooth. Then for any test for $\alpha \in \mathcal{A}_c(X)$. Then on each U_i we can define $(S \wedge T)_{U_i} := S|_{U_i} \wedge T|_{U_i}$ and this determines $S \wedge T$ uniquely by the glueing property. One checks that this is independent from the choice of covering and decomposition $\alpha = \sum \alpha_i$. All claimed properties follow from the definition and the corresponding properties for the product of forms and currents. ☐

However, in many geometric situations, this is not satisfying. In fact, this does not even cover the case of two submanifolds intersection transversely (but nontrivially)! The solution will be to remember not only the singularities of the currents, but also the 'directions' in which these occur. To make this more precise, we need to talk about the Fourier transform.

3 Reminder on the Fourier transform

A good reference for this material, which I mostly follow, is [3, Ch. 7].

3.1 The Fourier transform for functions

Definition 3.1.1. For a function $f \in L^1(\mathbb{R}^n)$, the Fourier transform is the bounded continuous function defined by sending $\xi \in \mathbb{R}^n$ to

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-\langle x, \xi \rangle} f(x) dx$$

Note that the Fourier transform takes in integrable functions and produces a bounded (but not necessarily integrable function). Restricted to the following space, one obtains the same source and target:

[6] On the other hand, for $Z = X$, $\delta_X = T_1$ is smooth.

Definition 3.1.2. The space of Schwartz functions $\mathcal{S} \subseteq C^\infty(\mathbb{R}^n)$ is given by those functions for which all derivatives decay faster than any polynomial:

$$f \in \mathcal{S} \overset{Def}{\iff} \sup_{x\in\mathbb{R}^n} |x^I \partial^J f(x)| < \infty \text{ for all multi-indices } I, J.$$

The space $\mathcal{S}$ is a Fréchet space with topology induced by the family of semi-norms defined by

$$f \mapsto |f|_{I,J} =: \sup_x |x^I \partial^J f(x)|$$

Exercise 3.1.3. $\mathcal{S} \subseteq L^1(\mathbb{R}^n)$ and for every $f \in \mathcal{S}$, and multi-indices I, J, also $x^I \partial^J f(x) \in \mathcal{S}$. For $f, g \in \mathcal{S}$, also $f \cdot g \in \mathcal{S}$.

Proposition 3.1.4. The Fourier transformation induces a continuous isomorphism

$$\mathcal{F} : \mathcal{S} \to \mathcal{S}$$

The inverse is given by

$$\mathcal{F}^{-1}(f)(x) := \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{i\langle \xi, x\rangle} f(\xi) d\xi.$$

Even though it now takes functions in $\mathcal{S}$ to functions in $\mathcal{S}$, we will follow tradition to denote the variable for functions in the "left" $\mathcal{S}$ by x and in the "right" $\mathcal{S}$ by ξ. We further denote by ∂_i the partial derivative in the i-th coordinate direction and set $D_j = -i\partial_j$. Then we have

Proposition 3.1.5 (Properties of the Fourier transform)**.** The Fourier transform...

1. ... interchanges differentiation and multiplication. For all $f \in \mathcal{S}$,

$$\mathcal{F}(D_j f) = \xi_j \cdot \mathcal{F}(f) \qquad \text{and} \qquad \mathcal{F}(x_j \cdot f) = -D_j \mathcal{F}(f)$$

2. ... interchanges multiplication and convolution: For $f, g \in \mathcal{S}$,

$$\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g) \qquad \text{and} \qquad \mathcal{F}(f \cdot g) = \mathcal{F}(f) * \mathcal{F}(g),$$

where we recall $(f * g) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$.

3. ... intertwines complex conjugation with additive inversion of the argument: For $f \in \mathcal{S}$,

$$\mathcal{F}(\bar{f})(-\xi) = \overline{\mathcal{F}(f)(\xi)}$$

In particular, f is real and even if and only if $\mathcal{F}(f)$ is real and even.

4. ... is compatible with the L^2-pairing: For all $f, g \in \mathcal{S}$,

$$\int_{\mathbb{R}^n} \mathcal{F}f \cdot g \, dx = \int_{\mathbb{R}^n} f \cdot \mathcal{F}g \, dx \qquad \text{and} \qquad \int_{\mathbb{R}^n} f \cdot \bar{g} \, dx = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \mathcal{F}f \cdot \overline{\mathcal{F}(g)} \, dx$$

3.2 Fourier transform for distributions

Given an open set $U \subseteq \mathbb{R}^n$, denote by $C^\infty(U)$, resp. $C_c^\infty(U)$ the smooth functions with (with compact support). There is a sequence of inclusions

$$\mathcal{C}_c^\infty(\mathbb{R}^n) \subset \mathcal{S} \subset \mathcal{C}^\infty(\mathbb{R}^n)$$

We have defined topologies on the middle space in the previous subsection and on the outer spaces in the first section. (Note that $\mathcal{C}_c^\infty(\mathbb{R}^n) = \mathcal{A}_c^0(\mathbb{R}^n)$ and $\mathcal{C}^\infty(\mathbb{R}^n) = \mathcal{A}^0(\mathbb{R}^n)$). In each case, the subset is dense in the ambient space. Passing to the topological dual spaces, we obtain the the spaces of distributions, resp. tempered distributions, resp. distributions with compact supports:

$$\mathcal{D}'(\mathbb{R}^n) \supset \mathcal{S}'(\mathbb{R}^n) \supset \mathcal{E}'(\mathbb{R}^n)$$

Remark 3.2.1. We follow traditional notation here, which arises from the fact that Schwartz wrote $\mathcal{D}(X) = C_0^\infty(X)$ and $\mathcal{E}(X) = C^\infty(X)$ and then $(\ )'$ for dual spaces. I personally find that unfortunate since I constantly have to suppress the urge of calling the space of **d**istributions $\mathcal{D}$ instead of $\mathcal{D}'$. Anyways. In our previous notation, we have the admittedly not that economical expressions $\mathcal{D}'(\mathbb{R}^n) = \mathcal{M}_0(\mathbb{R}^n)$ and $\mathcal{E}' = (\mathcal{M}_c)_0(\mathbb{R}^n)$.

One may now extend the Fourier transform to these spaces of distributions by duality, i.e. for a distribution $T \in \mathcal{D}'$ and a function f, one defines $\mathcal{F}(T)$ via

$$\mathcal{F}(T)(f) := T(\mathcal{F}(f))$$

A key to the definition of the wave-front set is the following property:

Proposition 3.2.2. Let $T \in \mathcal{E}'(\mathbb{R}^n)$. Then, $\mathcal{F}(T)$ is (the image of) a smooth function, given explicitly by

$$\mathcal{F}(T)(\xi) = T(e_\xi) \qquad \text{where } e_\xi(x) = e^{-i\langle x,\xi\rangle}$$

Morover, $T \in C_0^\infty(\mathbb{R}^n)$ if and only if there are constants C_N such that

$$|\mathcal{F}(T)(\xi)| \leq C_N(1+|\xi|)^{-N} \qquad \text{for all } N \in \mathbb{N}, \xi \in \mathbb{R}^n \tag{1}$$

Proof. We omit the proof of the first part.[7] The second follows since it says that $\mathcal{F}(T)$ is a Schwartz function, so since the Fourier transform is invertible on Schwartz functions, also T itself is a Schwartz function, hence smooth (and it was compactly supported to begin with). □

4 The wave front set

Recall that a subset of a real vector space is called conical if it is closed under multiplication by positive real numbers. For a vector bundle $V \to X$ of rank r, we denote $V^* = V \backslash s(X) \to X$ the $\mathbb{C}^r - \{0\}$-bundle arising by removing the image of the zero section $s : X \to V$.

[7] essentially, you write out both sides and note that they differ up to the order of applying current and integral, but these can be exchanged (which needs additional justification).

Definition 4.0.1.

1. A smooth function $f = f(\xi) \in C^\infty(\mathbb{R}^n)$ is said to have rapid decay in direction ξ_0 if there is a conic open neighborhood $V \subseteq \mathbb{R}^n \backslash \{0\}$ and constants $C_N > 0$ for all N such that

$$|f(\xi)| \leq C_N(1 + |\xi|)^{-N} \quad \text{for all } N \in \mathbb{N}, \xi \in V$$

2. Given a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$, the wave-front set $WF_0(T) \subseteq \mathbb{R}^n$ is given by all directions in which the Fourier transform of suffienciently small compact cutoffs of T decays rapidly, i.e.

$$WF_0(T) := \bigcap_{\substack{\psi:\mathbb{R}^n \to \mathbb{R} \\ 0 \in \operatorname{supp}\psi \text{ compact} \\ \psi \equiv 1 \text{ near } 0}} \{\xi \in \mathbb{R}^n \backslash \{0\} \mid \mathcal{F}(\psi \cdot T) \text{ has rapid decay in direction } \xi\}$$

3. For a distribution $T \in \mathcal{D}'(X)$, the wave-front set $WF(T) \subseteq T^\vee(X)^*$ is the subset of the (punctured) cotangent bundle defined by $WF = \bigcup WF_x(T)$, where $W_x(T)$ corresponds to the aforementioned set $WF_0(T)$ for some (equivalently: any) chart centered at x, i.e. given a chart $x \in U \xrightarrow{\varphi} \mathbb{R}^n$ with $\varphi(x) = 0$ and a cutoff-function $\psi : U \to \mathbb{R}$ with $\operatorname{supp}\psi \subseteq U$, we have

$$WF_x(T) := \varphi^* WF_0(\varphi_*(\psi T)).$$

4. For an arbitrary current $T \in \mathcal{M}(X)$, one defines, for any $U \subseteq X$ s.t. $TX|_U$ is trivial, $WF(T)|_U$, to be the union of the wave-front sets of the coefficients of T in some trivialization. Then $WF(T) = \bigcup_i WF(T)|_U$ for some (any) open cover $X = \bigcup U_i$ consisting of open sets over which the tangent bundle is trivial.

For a vector bundle $V \to X$ we denote $V^* = V \backslash s(X) \to X$ the $\mathbb{C}^r - \{0\}$-bundle arising by removing the image of the zero section $s : X \to V$. We now get an upgrade of our statements for singular support:

Proposition 4.0.2 (Properties of the Wave-front set)**.**

1. If $\pi : T^\vee X \to X$ is the projection, $\pi WF(T) = \operatorname{sing\,supp}(T)$. The preimage $\pi(x) \cap WF(T) \subseteq T_x^\vee X$ is an open conical subset.

2. For any submanifold $Z \subseteq X$, the wave-front set equals the (punctured) normal bundle $WF(\delta_Z) = \mathcal{N}^*_{Z|X}$.

3. For $\alpha \in \mathcal{A}(X)$ and $T \in \mathcal{M}(X)$, $WF(T \wedge \alpha) \subseteq WF(\alpha)$.

4. For a diffeomorphism $f : X \to Y$ and $T \in \mathcal{M}(X)$, there is an equality $f(WF(T)) = WF(f_*T)$.

5. If P is a linear partial differential operator with smooth coefficients and $T \in \mathcal{M}(X)$, then $WF(PT) \subseteq WF(T)$. If P is elliptic, the equality $WF(PT) = WF(T)$ holds.

For simplicity, we now pass to the projectivized wave front set defined as the image of $WF(T)$ under the projection $T^\vee X^* \to \mathbb{P}(T^\vee X)$, where $\mathbb{P}$ denotes the (real) projectivization. We write $\mathbb{P}WF(T) \subseteq \mathbb{P}(T^\vee X)$. I.e., we only remember the collection of (unsigned) directions determined by $WF(T)$.

Theorem 4.0.3. If $S \in \mathcal{M}^k$, $T \in \mathcal{M}^l$ with $\mathbb{P}WF(T) \cap \mathbb{P}WF(S) = \emptyset$, there exists a current $T \wedge S \in \mathcal{M}^{l+k}(X)$ which satisfies

1. $\mathbb{P}WF(T \wedge S) \subseteq \mathbb{P}(WF(T) + WF(S))$.
2. If $T = T_\alpha$, $S = T_\beta$ for $\alpha, \beta \in \mathcal{A}(X)$, then $T_\alpha \wedge T_\beta = T_{\alpha\wedge\beta}$.
3. $d(S \wedge T) = dS \wedge T + (-1)^{|S|} S \wedge T$.
4. If for $S, T, R \in \mathcal{M}^k$ the following products have disjoint projective wave front sets, then $(T \wedge S) \wedge R = T \wedge (R \wedge S)$.

Sketh of proof. Let us only explain how to define the product of two distributions $S, T \in \mathcal{D}'(\mathbb{R}^n)$ with disjoint projective wave front sets: First, again using a partition of unity function, we may write $S = \sum S_i$, $T = \sum T_i$ as locally finite sums of compactly supported distributions and so without loss of generality $S, T \in \mathcal{E}'(\mathbb{R}^n)$. Then, since the Fourier transform interchanges multiplication and convolution, a natural attempt is to define

$$T \cdot S := \mathcal{F}^{-1}(\mathcal{F}(T) * \mathcal{F}(W))$$

but the Fourier transforms $\mathcal{F}(T)$ and $\mathcal{F}(S)$ are smooth functions and so for the right hand side to be defined, we want that the integral

$$\mathcal{F}(T) * \mathcal{F}(W)(\theta) = (2\pi)^{-2n} \int \mathcal{F}(T)(\xi)\mathcal{F}(S)(\theta - \xi) d\xi$$

converges for all θ, at least after potentially replacing T, S by their multiplication with cutoff functions with smaller support. This is ensured since in (a neighborhood of) every direction, at least one of the two factors in the integrand decreases rapidly enough. $\square$

References

[1] Christian Brouder, N. V. D., and Hélein, F. A smooth introduction to the wave front set. *J. Phys. A: Math. Theor. 47*, 44.

[2] Demailly, J.-P. *Complex analytic and differential geometry.* 2012. Online book draft.

[3] Hörmander, L. *The analysis of linear partial differential operators. I: Distribution theory and Fourier analysis.*, reprint of the 2nd edition 1990 ed. Class. Math. Berlin: Springer, 2003.