Exercises for the training session

CIMPA School: A Complex Analytic Approach to Differential and Algebraic Geometry

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This is a collection of exercises related to the introductory courses in complex and Kähler geometry. The selection is somewhat biased by my own preferences and certainly has some blank spots with respect to the material covered. Also, it is possible that some exercises use notation or even concepts which have not, or not fully, been introduced, but which should easily be found online - and of course in case of doubt you can always ask. I encourage you to try to do as many of these problems as possible; it will be beneficial for your understanding of the material even if you don't succeed in finding the solution. Also, it is a good additional training to present your solutions (or attempts) in class.

Exercise 1. Show that $\mathbb{CP}^n = \mathbb{C}^{n+1}\backslash\{0\}/\mathbb{C}^*$ is a compact complex manifold. Show that there is a diffeomorphism $\mathbb{CP}^1 \cong S^2$.

Exercise 2. Formulate and prove the following theorems for holomorphic functions $f : U \to \mathbb{C}$ with $U \subseteq \mathbb{C}^n$:

1. Open mapping theorem
2. Maximum principle
3. Liouville's theorem

Exercise 3 (Projective hypersurfaces)**.**

1. Remind yourself of the definition of a smooth submanifold and of the theorem of the regular value for smooth manifolds. Formulate the analogous definition and theorem for complex manifolds and convince yourself that it still holds.
2. Let $P \in \mathbb{C}[X_0, ..., X_n]_k$ be a homogenous polynomial of degree k. Denote by P_i the i-th dehomogenization, i.e. the polynomial in n variables, where we insert 1 for X_i. Show that $0 \in \mathbb{C}$ is a regular value of the map $\mathbb{C}^{n+1}\backslash\{0\}, z \mapsto P(z)$ if and only if 0 is a regular value for each P_i.

3. Assuming P as before and 0 is a regular value for the induced map $\mathbb{C}^{n+1}\backslash\{0\} \to \{0\}$, deduce that the projective vanishing set

$$V_{\mathbb{CP}^n}(P) = \{[z_0 : ... : z_n] \mid P(z_0 : ... : z_n) = 0\}$$

is a complex submanifold of $\mathbb{CP}^n$ and that the projection $V(P) \cap \mathbb{C}^{n+1}\backslash\{0\} \to V_{\mathbb{CP}^n}(P)$ is a holomorphic fibre bundle with fibre $\mathbb{C}^\times$.

Exercise 4 (Not every submanifold a complete intersection)**.**
Show that the *Veronese embedding*

$$\begin{aligned} \nu : \mathbb{CP}^1 &\longrightarrow \mathbb{CP}^3 \\ [z_0 : z_1] &\longmapsto [z_0^3 : z_0^2 z_1 : z_0 z_2^2 : z_1^3] \end{aligned}$$

is indeed an embedding. Find 2 or 3 homogenous polynomials in four variables such that the image of ν is described as their (projective, as above) vanishing locus. Show that 0 is not a regular value for the map $\mathbb{C}^4\backslash\{0\} \to \mathbb{C}^k$ ($k = 2$ or $k = 3$) arising from your polynomials.

Exercise 5. Let (X, J) be an almost complex manifold. Show there is at most one complex structure on X inducing J.

Exercise 6 (Many complex structures on the real 2-torus)**.**

1. Let $\Lambda, \Lambda' \subseteq \mathbb{C}$ be lattices. Use covering space theory to show that any biholomorphism

$$\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$$

has the form $f([z]) = [a \cdot z + b]$ for some $a \in \mathbb{C}^\times$ $b \in \mathbb{C}$ s.t. $a \cdot \Lambda \subseteq \Lambda'$.

2. We say an elliptic curve is any manifold of the form $\mathbb{C}/\Lambda$ for $\Lambda \subseteq \mathbb{C}$ a lattice.
Consider the action of $GL_2(\mathbb{Z})$ on $\mathbb{CP}^1$ induced by the action of $\mathbb{C}^2\backslash\{0\}$. Show there is an isomorphism of sets

$$(\mathbb{CP}^1 \setminus \mathbb{RR}^1)/GL_2(\mathbb{Z}) \cong \{\text{biholomorphism classes of elliptic curves}\}.$$

Exercise 7. A Hopf surface is a complex manifold of the form

$$H_\lambda = (\mathbb{C}^2\backslash\{0\})/_{z\sim\lambda z}$$

for some $\lambda \in \mathbb{C}^\times\backslash S^1$.

1. Show that every Hopf surface has the structure of a fibre bundle over $\mathbb{CP}^1$ with fibres elliptic curves.

2. Show that H_λ is diffeomorphic to $S^1 \times S^3$.

Exercise 8. Let X, Y be complex manifolds and $\varphi : X \to Y$ a smooth map. Show that the following assertions are equivalent:

1. φ is holomorphic.

2. $\varphi_* = D(\varphi) : T_{\mathbb{C}}X \to T_{\mathbb{C}}Y$ respects the splitting into i and $-i$ eigenspaces, i.e. $\varphi_*(T^{1,0}X) \subseteq T^{1,0}Y$.

3. $\varphi^*\mathcal{A}(Y) \to \mathcal{A}(X)$ is a map of double complexes, i.e. it respects the bigrading and satisfies $\partial\varphi^* = \varphi^*\partial$ and $\bar{\partial}\varphi^* = \varphi^*\bar{\partial}$

4. For any holomorphic function f on an open subset $U \subseteq Y$, $f \circ \varphi$ is again holomorphic.

Exercise 9 (general Cauchy formula)**.** Let $f : \mathbb{C} \supseteq \overline{B_\varepsilon(0)} \to \mathbb{C}$ be a continuously differentiable function. Show that for any $z \in B_\varepsilon(0)$, we have the formula

$$f(z) = \frac{1}{2\pi i}\left[\int_{\partial\overline{B_\varepsilon(0)}} \frac{f(\xi)}{\xi - z} d\xi + \int_{B_\varepsilon(0)} \frac{\partial f(\xi)}{\partial\bar{\xi}} \frac{d\xi d\bar{\xi}}{\xi - z}\right]$$

Deduce Cauchy's integral formula for holomorphic functions as a special case.

Exercise 10. Compute the cohomologies H_{dR}, H_{BC}, H_A, H_{∂_1} and H_{∂_2} for the following four bi-complexes:

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & \mathbb{C} \end{array} \qquad \begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\ & & \uparrow & & \\ & & 0 & & \end{array} \qquad \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ & & \uparrow \\ & & \mathbb{C} \end{array} \qquad \begin{array}{ccccc} \mathbb{C} & \longrightarrow & \mathbb{C} & & \\ & & \uparrow & & \\ & & \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

where apart from the second picture, all drawn arrows are isomorphisms (say, $\pm\mathrm{id}$) and all other maps are zero. Can you describe a general pattern for arbitrarily long 'zigzags' (generalizing the last cases)?

Exercise 11. Let $B \subseteq \mathbb{C}^n$ be a polydisc. Let $(\Omega(X), \partial)$ be the holomorphic de Rham complex. Deduce from the $\bar{\partial}$-Poincaré Lemma that for all k, there is an isomorphism

$$H^k(\Omega(B), \partial) \cong H^k_{dR}(B)$$

Is this true for an arbitrary complex manifold X?

Exercise 12. Let X be a complex manifold. Show that there is a group isomorphism

$$\check{H}^1(X, \mathcal{O}_X^\times) \cong Pic(X)$$

where $Pic(X)$ denotes the group of holomorphic line bundles and the right hand side denotes Cech cohomology with coefficients in the sheaf of invertible holomorphic functions.

The proof should give analogous results for the sheaf of smooth functions and complex vector bundles or of locally constant functions and flat vector bundles (i.e. with locally constant transition functions)

Exercise 13 (Iwasawa manifold)**.** Let $H \subseteq GL_3(\mathbb{C})$ be the subgroup of matrices of the form

$$H = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{C}) \right\}$$

The group $H_{\mathbb{Z}} = H \cap GL_3(\mathbb{Z}[i])$ acts freely and properly discontinuously on H and the quotient manifold $X := H/H_{\mathbb{Z}}$ is called the Iwasawa manifold. Show that

1. X admits the structure of a holomorphic fibre bundle with base $(\mathbb{C}/\mathbb{Z}[i])^2$ and fibre $\mathbb{C}/\mathbb{Z}[i]$.
2. X is holomorphically parallelizable, i.e. $T^{1,0}X$ is, as a holomorphic vector bundle, isomorphic to the trivial rank three bundle

   (Hint: Look for sections on H which are invariant under the action of H on itself).
3. Does H admit a Kähler metric?

Exercise 14. Let X be a complex manifold of dimension 1. Establish a $1:1$ correspondence between meromorphic functions on X and holomorphic maps $X \to \mathbb{CP}^1$ which are not constant equal to ∞.

Exercise 15. Show that a holomorphic line bundle L on a compact connected complex manifold X is trivial if and only if both L and L^{-1} admit a nontrivial global section.

Exercise 16. Let $Z \subseteq X$ be a hypersurface and $I_Z \subseteq \mathcal{O}_X$ the sheaf of ideals of functions vanishing on X. Show that there is an isomorphism $I_Z \cong \mathcal{O}(-Z)$.

Exercise 17. Define meromorphic sections of line bundles. Show that on any connected complex manifold X, the image of the map $Div(X) \to Pic(X)$ consists of isomorphism classes of those line bundles which admit a nontrivial meromorphic section.

Exercise 18. Show that the Fermat hypersurfaces, defined as the vanishing loci

$$V(\sum_{i=0}^{n} z_i^d) \subseteq \mathbb{CP}^N$$

are complex manifolds. Show that for $d = n + 1$ they are Calabi-Yau manifolds.

Exercise 19. Show that $H^0(\mathbb{CP}^n, \Omega^q) = 0$ for all $q > 0$.

Exercise 20. Show that for any short exact sequence of holomorphic vector bundles

$$0 \to L \to E \to F \to 0$$

where L is a line bundle, there are induced short exact sequences of the form

$$0 \to L \otimes \Lambda^{i-1} F \to \Lambda^i E \to \Lambda^i F \to 0$$

Exercise 21 (Segre embedding)**.** Consider $X = \mathbb{CP}^n \times \mathbb{CP}^m$ and the line bundle $L = p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(1)$ on X (where p_i denote the projection maps). Describe the map associated with the linear system $H^0(X, L)$ explicitly and show it is an embedding.

Exercise 22. The surface given as the projectivized vector bundle $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(n))$ is called the Hirzebruch surface. Show that Σ_n is biholomorphic to the hypersurface $V(x_0^n y_1 - x_1^n y_2) \subseteq \mathbb{CP}^1 \times \mathbb{CP}^2$, where $[x_0 : x_1]$ and $[y_0 : y_1 : y_2]$ are the homogeneous coordinates on $\mathbb{CP}^1$, resp. $\mathbb{CP}^2$.

Exercise 23. Describe tangent, cotangent and canonical bundle of $\mathbb{CP}^n$ and $\mathbb{CP}^n \times \mathbb{CP}^m$.

Exercise 24. Use Serre duality and Kodaira vanishing theorem to compute $H^d(\mathbb{CP}^n, \mathcal{O}(k))$ for all d, n, k.

Exercise 25. Let $D = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$. Show:

1. The form $\omega := -\frac{i}{2} \partial\bar{\partial} \log(1 - \|z\|^2)$ is (the $(1,1)$-form of) a Kähler metric on D.
2. Identify D with the set of points $[z] \in \mathbb{CP}^n$ s.t. $\langle z, z, \rangle > 0$, where $\langle\ ,\ \rangle$ denotes (non-positive) the hermitian inner product given by the diagonal matrix with Eigenvalues $(1, -1, ..., -1)$. Show that ω is invariant under the corresponding action of $SU(1, n)$.

Exercise 26. Let (X, J) be an almost complex manifold. Construct bijections between the sets consisting of the objects in (1), (2), (3) below:

1. Hermitian metrics h on TX.
2. Riemannian metrics g on TX s.t. $g(J\_, J\_) = g(\_, \_)$
3. positive real $(1, 1)$-forms ω

Hint: You can reduce this to a calculation at a single fibre of TX, and hence to linear algebra on a complex vector space.

Exercise 27 (Products and the Kähler property)**.** Show that a product $X \times Y$ of complex manifolds is Kähler if and only if both factors are Kähler.

Exercise 28 (Conformal deformations of Kähler metrics are not Kähler)**.** Show that for a Kähler metric h on a connected complex manifold X of dimension $\geq$, and a smooth function $f : X \to \mathbb{R}_{>0}$, the form $f \cdot \omega$ defines a Kähler metric if and only if f is constant.

In the next two exericses, you may want to use the Weil identity on hermitian manifolds:

$$*L^j\alpha = (-1)^{\binom{k+1}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j} I(\alpha) \qquad \forall \alpha \in P^k.$$

Exercise 29. Let $(V, \langle\ ,\ \rangle, I)$ be a Euclidean vector space of real dimension $2n$ with compatible complex structure. Let $k \leq n$ and

$$\begin{aligned} Q : \Lambda^k V_{\mathbb{C}}^\vee \times \Lambda^k V_{\mathbb{C}}^\vee &\longrightarrow \mathbb{C} \\ (\alpha, \beta) &\longmapsto (-1)^{\binom{k}{2}} \alpha \wedge \beta \wedge \omega^{n-k}. \end{aligned}$$

Show

1. $Q(\Lambda^{p,q} V_{\mathbb{C}}^\vee, \Lambda^{r,s} V_{\mathbb{C}}^\vee) = 0$ unless $(p, q) = (s, r)$.
2. $i^{p-q} Q(\alpha, \bar{\alpha}) = (n - (p + q))\langle \alpha, \alpha \rangle_{\mathbb{C}} > 0$ if $0 \neq \alpha \in P^{p,q}$ primitive.

Exercise 30.

1. With notation as above, show that the Lefschetz and type decompositions

$$\Lambda^k V^\vee = \bigoplus_{j \geq 0} L^j P^{k-2j} \qquad L^j P^{k-2j} = \bigoplus_{p+q=k-2j} L^j P^{p,q}$$

are orthogonal with respect to the L^2-pairing.

2. Show that the intersection pairing on a compact Kähler surface has index $(2h^{2,0}+1, h^{1,1}(X)-1)$ and that restricted to $H^{1,1}(X)$ it has index $(1, h^{1,1}(X)-1)$. Harder: Can you compute the signature intersection pairing on compact Kähler manifolds of even complex dimension in general in terms of the Hodge numbers?

3. Deduce that $\mathbb{CP}^2\#\mathbb{CP}^2$ does not admit the structure of a Kähler manifold.

Exercise 31. Let $(A, \partial, \bar{\partial})$ be a double complex of complex vector spaces. Show that the following assertions are equivalent:

1. The map $H^{p,q}_{BC}(A) \to H^{p,q}_{A}(A)$ is an isomorphism for all $p, q \in \mathbb{Z}$.

2. A is isomorphic to a direct sum of dots and squares, i.e. to double complexes isomorphic to one of the following

$$\begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\ & & \uparrow & & \\ & & 0 & & \end{array}, \qquad \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

Exercise 32. For any complex manifold X, let

$$Pic^0(X) := \ker(Pic(X) \xrightarrow{c_1} H^2(X;\mathbb{Z}))$$

be the group of complex structures on the trivial line bundle.

1. Show that if X is compact Kähler, $Pic^0(X)$ is a complex torus of dimension $b_1(X)$.

2. Find an example of a compact non-Kähler complex manifold, where $Pic^0(X)$ is not a torus.

Exercise 33. Give three examples of compact Kähler manifolds and three examples of compact manifolds which do not admit a Kähler structure.

Exercise 34. Let X be a compact complex manifold and $C_X \subseteq H^2(X;\mathbb{R})$ the set of all classes which can be represented by the $(1,1)$-form associated to a Kähler metric. Show that C_X is an open convex cone in $H^{1,1}(X;\mathbb{R})$.

Exercise 35. Show that the product of two projective manifolds is again projective, using the Kodaira embedding theorem.

Exercise 36. For Riemann surfaces, give a direct proof of the Kodaira vanishing and embedding theorem, using the notion of degree of a divisor deg : $\sum a_x \mapsto \sum a_x x$ and that there is an equality $\deg(D) = \int_X c_1(\mathcal{O}(D))$. First show that on a compact connected Riemann surface, a line bundle $\mathcal{O}(D)$ is positive if and only if $\deg(D) > 0$.

Exercise 37. Let X be a compact Riemann surface. Show that $c_1(K_X) = -\chi(X) = 2g(X) - 2$. Deduce the degree-genus formula: If $X \subseteq \mathbb{CP}^2$ is a smooth hypersurface of degree d, then $g(X) = \frac{(d-1)(d-2)}{2}$.

Some of these problems are taken from [1] and perhaps others (subconsciously) from other texts.

References

[1] HUYBRECHTS, D. *Complex geometry.* Universitext. Springer-Verlag, Berlin, 2005. An introduction.