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Métriques hermitiennes spéciales sur les variétés complexes compactes lisses

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To my dear mother

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Abstract

In this thesis we are interested in investigating the existence of special metrics on compact complex non-Kähler manifolds and their deformations. In particular we focus on Hermitian symplectic and lcK metrics. This thesis consists mainly of two parts. Recently, Dinew and Popovici have introduced and studied an energy functional F acting on the metrics in the Aeppli cohomology class of a Hermitian-symplectic metric and showed that in dimension 3 its critical points (if any) are Kähler.

In the first part of this we further investigate the critical points of this functional in higher dimensions and under holomorphic deformations. We first prove that being a critical point for F is a closed property under holomorphic deformations. We then show that the existence of a Kähler metric ω_k in the Aeppli cohomology class is an open property under holomorphic deformations. Furthermore, we consider the case when the (2, 0)-torsion form $\rho_{\omega}^{2,\,0}$ of ω is ∂-exact and prove that this property is closed under holomorphic deformations. Finally, we give an explicit formula for the differential of F when the (2, 0)-torsion form $\rho_{\omega}^{2,\,0}$ is ∂-exact.

In the second part, we employ the variational method in [12] and propose an approach to study the existence problem of locally conformally Kähler metrics on compact complex manifolds by introducing and studying a functional, L, which changes according to whether the complex dimension of the manifold is 2 or higher. We show that in dimension 3 the critical points of L are exactly the set of lcK metrics. We then introduce a normalised functional with respect to another Hermitian metric ρ. Finally we prove that the set of critical points of L and the normilised functional are related.

Keywords

Functional approach, Hermitian-symplectic metrics, Holomorphic family of compact complex manifolds, Kähler metrics, Locally confromally Kähler metrics.

Résumé

Dans cette thèse, nous nous intéressons à l'étude de l'existence de métriques spéciales sur les variétés compactes complexes non-Kähleriennes et leurs déformations. En particulier, nous nous concentrons sur les métriques symplectique hermitienne et lcK. Cette thèse se compose principalement de deux parties. Récemment, Dinew et Popovici ont introduit et étudié une fonctionnelle énergétique F agissant sur les métriques de la classe de cohomologie d'Aeppli d'une métrique hermitienne-symplectique et ont montré qu'en dimension 3 ses points critiques (le cas échéant) sont de Kähler.

Dans une première partie, nous approfondissons les points critiques de cette fonctionnelle en dimension supérieure et sous déformations holomorphes. Nous montrons d'abord qu'être un point critique pour F est une propriété fermée par déformation holomorphe. Nous montrons ensuite que l'existence d'une métrique de Kähler ω_k dans la classe de cohomologie d'Aeppli est une propriété ouverte sous déformations holomorphes. De plus, nous considérons le cas où la (2, 0)-forme de torsion $\rho_\omega^{2,0}$ de ω est ∂-exacte et prouvons que cette propriété est fermée sous les déformations holomorphes. Enfin, nous donnons une formule explicite pour la différentielle de F lorsque la (2, 0)-forme de torsion $\rho_\omega^{2,0}$ est ∂-exacte

Dans la deuxième partie, nous utilisons la méthode variationnelle dans [12] et proposons une approche pour étudier le problème d'existence de métriques localement conformément Kähleriennes sur des variétés complexes compactes en introduisant et en étudiant une fonctionnelle, L, qui change selon que la dimension complexe de la variété est 2 ou plus. Nous montrons qu'en dimension 3 les points critiques de L sont exactement l'ensemble des métriques lcK. Nous introduisons ensuite une fonctionnelle normalisée par rapport à une autre métrique Hermitienne ρ. Enfin nous prouvons que l'ensemble des points critiques de L et la fonctionnelle normalisée sont liés.

Mots-clés

Approche fonctionnelle, Métriques hermitiennes-symplectiques, Famille holomorphe de variétés complexes compactes, Métriques de Kähler, Métriques localement conformément Kähleriennes.

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0.1 Introduction Générale

Le but de cette section est de présenter l'idée derrière cette thèse, qui se compose de deux parties principales comme suit :

1) L'étude des propriétés de déformation des variétés complexes compactes.
2) L'existence de métriques spéciales sur les variétés complexes compactes.

Nous expliquons brièvement les deux dans la suite. Pour plus de détails, le lecteur est renvoyé aux sections 2.1 et 3.1. Tout d'abord, rappelons que par **métrique hermitienne** sur une variété complexe compacte X, nous entendons une C^∞ (1, 1)-forme ω réelle définie positive. Ce qui signifie que pour toute coordonnée locale $(z_1, \ldots, z_n)$ sur X, ω peut être représenté par :

$$\omega(x) = \sum_{i,j} \omega_{i\bar{j}}(x) dz_i \wedge \bar{z}_j,$$

telle que la matrice $(\omega_{i\bar{j}})_{i,j}$ est définie positive (de manière équivalente, ses valeurs propres sont toutes positives) pour tout $x \in X$.

Propriétés de déformation des variétés complexes compactes

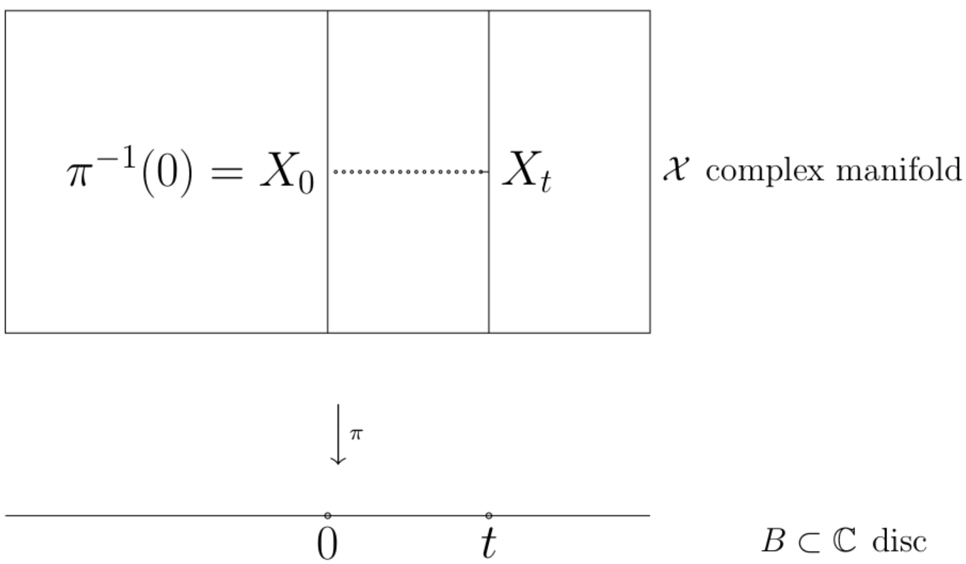
L'étude des déformations des structures complexes est une autre méthode pour étudier la classification des variétés complexes compactes. La théorie de la déformation est un outil fondamental en la géométrie complexe, car elle nous permet d'étudier le comportement de structures complexes sous de petites perturbations. En particulier, la théorie de la déformation fournit un moyen d'étudier l'espace des modules de structures complexes sur une variété fixée. Dans cette section, nous nous concentrons principalement sur le point de vue de Kodaira-Spencer.

Cette méthode a été employée par diverses personnes. Par exemple, K. Kodaira et D.C. Spencer ont étudié en profondeur les propriétés de déformation des variétés compactes complexes de Kähler dans leurs célèbres articles [22], [23] and [23]. Ici, une déformation est vue comme une variété complexe fibrée sur un espace de base, les fibres étant des déformations d'une fibre distinguée isomorphe à une variété complexe donnée.

Pour mieux comprendre, considérons les variétés complexes $\mathcal{X}$ et B avec une submersion propre $\pi : \mathcal{X} \to B$. Cela signifie que, pour tout $t \in B$, la fibre $\pi^{-1}(t) = X_t$ est une variété complexe compacte avec une structure complexe J_t et une métrique hermitienne ω_t dépendant de J_t. De plus, pour tout t_1, $t_2 \in B$ tel que $t_1 \neq t_2$, nous avons $\dim_{\mathbb{C}} X_{t_1} = \dim_{\mathbb{C}} X_{t_2}$.

Ainsi, faire varier la structure complexe J_t revient à faire varier t. Par conséquent, $\mathcal{X}$ est considérée comme une **famille** de variétés complexes compactes $(X_t)_{t\in B}$ telles que chacune de ces variétés porte une structure complexe J_t, qui varie régulièrement avec t.

Supposons maintenant que B soit une boule unité autour de l'origine dans $\mathbb{C}^m$ pour un certain $m \in \mathbb{N}$. Comme B est une variété contractible, alors toutes les fibres X_t C^∞ sont difféomorphes à la fibre centrale X_0 grâce au théorème d'Ehresmann, Théorème 1.6.1 (cf. aussi à [43], Théorème 9.3). On peut donc reconnaître $\mathcal{X}$ comme X_0 avec une famille C^∞ de structures complexes $(J_t)_{t\in B}$, en particulier avec une famille $\mathbb{C}^\infty$ de métriques hermitiennes $(\omega_t)_{t\in B}$. La figure suivante montre un résumé de à la.



Considérons une famille holomorphe de variétés complexes compactes $(X_t)_{t\in B}$, et soit P une propriété qui dépend de structures complexes. Alors, il y a au moins deux questions spécifiques dans la théorie de la déformation comme suit :

Question 0.1.1 *1) Si la fibre centrale X_0 vérifie la propriété P, est-ce que toutes les fibres voisines X_t vérifient également la propriété P (propriété d'ouverture) ?*

2) Si pour t suffisamment proche de 0, toutes les fibres voisines X_t vérifient la propriété P, la fibre centrale X_0 vérifie-t-elle également la propriété P (propriété de fermeture) ?

L'un des résultats fondamentaux de la théorie de la déformation est l'ouverture de la déformation de la propriété de Kähler, que Kodaira et Spencer ont prouvé dans leur célèbre article (cf. [24], également le Théorème 1.6.8). Ils ont montré que si $(X_t)_{t\in B}$ est une famille C^∞ de variétés complexes compactes et que la fibre centrale X_0 satisfait la propriété de Kähler (i.e. X étant Kähleriennes), alors toutes les fibres voisines X_t satisfont également la propriété de Kähler. L'idée de la preuve consiste en trois étapes principales (pour les détails, le lecteur se réfèrera la section 1.6).

Étape 1 Supposons que Δ_{BC} est l'opérateur elliptique du Laplacien de Bott-Chern (cf. Définition 1.1.9-(i)). Si la fibre centrale X_0 est Kähleriennes alors, pour t suffisamment proche de 0, $\dim\ker \Delta_{BC}(t)$ est constant. Donc la famille $(\Delta_{BC}(t))$ est une famille C^∞ d'opérateurs elliptiques, où $\Delta_{BC}(t) : C^\infty_{\bullet,\bullet}(X_t, \mathbb{C}) \to C^\infty_{\bullet,\bullet}(X_t, \mathbb{C})$.

Étape 2 Une métrique hermitienne ω est Kähleriennes si et seulement si $\Delta_{BC}(\omega) = 0$.

Étape 3 Supposons que $(\omega_t)_{t\in B}$ est une famille C^∞ de métriques hermitiennes. Cela signifie que pour tout $t \in B$, ω_t est une métrique hermitienne sur X_t et $(\omega_t)_{t\in B}$ est une famille C^∞ de $(1,1)$-formes. Si ω_0 est une métrique de Kähler sur X_0, alors pour t suffisamment proche de 0, $\tilde{\omega} = \frac{1}{2}(P_t(\omega_t) + \overline{P_t(\omega_t)})$ st une métrique de Kähler sur X_t, où les $P_t : C^\infty_{1,1}(X, C) \to \ker \Delta_{BC}(t)$ sont projections $L^2_{\omega_t}$-orthogonales.

En utilisant des arguments similaires mais sur un opérateur elliptique différent, on peut prouver que si la fibre centrale X_0 admet une métrique de Gauduchon ω, i.e. telle que $\partial\bar{\partial}\omega^{n-1} = 0$, alors toutes les fibres voisines admettent des métriques de Gauduchon. Dans le tableau 1.6, nous présentons quelques exemples de différents types de métriques et leurs propriétés sous des déformations de structures complexes.

On peut même se poser une question plus générale, à savoir comment les diverses propriétés des variétés complexes compactes, qui dépendent de la structure complexe, varient sous les déformations de la structure complexe. Par exemple, dans [36], D. Popovici a prouvé que si la fibre centrale X_0 admet une métrique **fortement Gauduchon** alors toutes les fibres voisines X_t admettent également. Rappelons qu'une métrique hermitienne ω est appelée fortement Gauduchon si $\partial\omega^{n-1}$ est $\bar{\partial}$-exact.

Dans la section 2.3, nous présenterons un résultat sur la déformation d'une métrique de Kähler dans la classe de cohomologie d'Aeppli (cf. Définition 1.1.8) d'un certain type de métrique appelé **hermitienne-symplectique** (cf. Définition 1.1.6-(5)). L'idée de la preuve est inspirée du théorème fondamental de Kodaira-Spencer (cf. Théorème 1.6.3). En particulier, nous prouvons

Theorem 0.1.2 *(chapitre 2, Théorème 2.3.1) Supposons que B est une boule ouverte dans $\mathbb{C}^m$ contenant l'origine et $(X_t)_{t\in B}$ est une famille holomorphe de variétés complexes compactes de dimension complexe n vérifiant les conditions suivantes :*

1) pour tout $t \in B$, X_t est muni d'une métrique hermitienne-symplectique ω_t et la famille $(\omega_t)_{t\in B}$ est une C^∞-famille de $(1,1)$-formes,

2) pour $t = 0$, ω_0 est une métrique de Kähler sur X_0.

Alors après éventuellement rétrécissement de B autour de 0, il existe une famille de $(1,1)$-formes $(\tilde{\omega}_t)_{t\in B}$ telle que

a) $\tilde{\omega}_t \in \{\omega_t\}_A$, où $\{\omega_t\}_A$ est la classe de cohomologie d'Aeppli ω_t,

b) $\tilde{\omega}_t$ est une métrique de Kähler sur X_t pour tout $t \in B$,

c) $\tilde{\omega}_0 = \omega_0$,

d) $(\tilde{\omega}_t)_{t\in B}$ est une famille C^∞ de métriques.

Selon Kodaira et Spencer, la propriété de Kähler est une propriété d'ouverture de la déformation (Théorème 1.6.8, chapitre 1. Le théorème précédent prouve que l'existence d'une métrique de Kähler dans une classe de cohomologie Aeppli donnée d'une métrique hermitienne-symplectique est une propriété ouverte. Ce résultat renforce l'ouverture classique de la propriété de Kähler.

Les résultat suivant traite des points critiques de la fonctionnelle énergétique de Dinew-Popovici. Dans [12], les auteurs ont introduit une fonctionnelle énergétique F sur une variété hermitienne-symplectique complexe compacte X. Le but était de montrer que les points critiques de F sont des métriques de Kähler. Lorsque $\dim_{\mathbb{C}} X = 3$, ils ont montré que les points critiques de F sont exactement les métriques de Kähler.

En dimension supérieure, la formule de $d_\omega F$, la première variation de F à ω, est compliquée, et doit donc être simplifiée. La proposition suivante donne une formule simplifiée pour $d_\omega F$, lorsque $\rho_\omega^{2,0}$ est ∂-exacte.

Proposition 0.1.3 *(chapitre 2, Proposition 2.4.1) Supposons que (X, ω_0) est une variété hermitienne-symplectique complexe compacte de dimension n. Fixons un $\omega \in \mathcal{S}_{\{\omega_0\}}$. Si $\rho_\omega^{2,0} = \partial\xi$, pour un certain $(1,0)$-forme ξ, alors la différentielle en ω de la fonctionnelle d'énergie de Dinew-Popovici F, définie dans l'équation (2.2), évaluée $\gamma = \bar\partial\xi + \partial\bar\xi$ est*

$$d_\omega F(\gamma) = 2\|\rho_\omega^{2,0}\|^2 + 2Re\int_X \bar\partial\xi \wedge \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge \omega_{n-3}.$$

Existence de métriques spéciales sur les variétés complexes compactes

Alors que les variétés admettant des métriques de Kähler ont été intensivement étudiées comme l'un des problèmes centraux de la géométrie complexe, la théorie des variétés avec des structures de type Kähleriennes (SKT, lcK, etc.) a également été développée par de nombreuses personnes au cours des 15 dernières années, par exemple, le lecteur est renvoyé à [6] et [26].

Une variété complexe compacte X peut admettre des métriques de type Kähler telles que équilibrée ($d\omega^{n-1} = 0$) , SKT ($\bar\partial\partial\omega = 0$) etc. (cf. Definition 1.1.6). Parmi ces métriques, nous nous intéressons plus particulièrement aux métriques **hermitiennes-symplectiques** et **localement conformément Kähleriennes**. En particulier, dans la section 2.4 nous étudions principalement la question suivante qui a été posée récemment par Dinew et Popovici dans [12] comme un renforcement de la question posée par Streets et Tian (cf. Question 2.1.1).

Question 0.1.4 *([12]) Supposons que (X, ω) est une variété hermitienne-symplectique compacte, quand existe-t-il une métrique de Kähler dans la classe de cohomologie d'Aeppli d'un ω hermitien-symplectique ?*

Rappelons qu'une métrique hermitienne ω sur X est une hermitienne-symplectique nommée (H-s) s'il existe une 2-forme réelle d-fermée Ω et une $(2,0)$-forme $\rho^{2,0}$ telle que

$$\Omega = \rho^{2,0} + \omega + \overline{\rho^{2,0}}.$$

Il existe plusieurs raisons d'étudier les métriques symplectiques hermitiennes, dont deux sont mentionnées ci-dessous.

Premièrement, on sait qu'il existe des variétés SKT n'admettant aucune métrique Kählerienne. Bien que toute métrique hermitienne-symplectique soit une métrique SKT, la question de savoir s'il existe une variété hermitienne-symplectique qui n'admet pas de métrique Kählerienne est toujours ouverte. Cette question a été posée par J. Streets et G. Tian (cf. Question 2.1.1).

Une autre raison qui peut être mentionnée pour appuyer une réponse négative à la question de Streets-Tian est que les métriques hermitiennes-symplectiques ont les mêmes propriétés de déformation que les métriques Kählerienne (cf. Tableau 1.6). En effet, tout comme la propriété de Kähler, la propriété hermitienne-symplectique se propage de la fibre centrale à toutes les fibres voisines. De plus, d'après la proposition 3.3 de [8], il existe une famille de variétés complexes compactes telles que toutes les fibres, sauf une, sont des variétés hermitiennes-symplectiques.

Pour répondre à la question 0.1.4, nous utilisons la méthode qui a été décrite dans [12]. Nous montrons que l'existence d'une métrique Kählerienne dans la classe de cohomologie d'Aeppli d'une métrique hermitienne-symplectique ω est liée à la ∂-exactitude de la $(2,\,0)$-forme de torsion de ω définie par Dinew-Popoivici dans [12] (pour plus de détails, le lecteur est renvoyé au chapitre 2). En particulier, nous prouvons

Corollary 0.1.5 *(chapitre 2, Corollary 2.4.2) Supposons que (X, ω_0) est une variété hermitienne-symplectique complexe compacte de dimension n. Fixons un $\omega \in \mathcal{S}_{\{\omega_0\}}$, si*

(i) ω est un point critique pour F, et

(ii) la $(2,\,0)$-forme de torsion $\rho_\omega^{2,\,0} = \partial\xi$ est telle que $\bar{\partial}\xi$ soit faiblement semi-positive,,

alors ω est une métrique Kählerienne sur X.

Un autre type de variétés proche des variétés Kählerienne que nous étudions au chapitre 3 sont les variétés **localement conformément Kähleriennes** (lcK). Une variété est localement conformément Kähleriennes si elle admet une métrique localement conformément Kähleriennes. Rappelons qu'une métrique hermitienne ω est appelée localement conformément Kähleriennes si $d\omega = \theta \wedge \omega$, où θ est une 1-forme d-fermée et est appelée la **forme de Lee** de ω.

Quelques exemples de variétés de localement conformément Kähleriennes qui ne sont pas Kählerienne sont les variétés de Hopf [28], et certaines variétés de Oeljeklaus-Toma [27]. De plus, contrairement à la propriété de Kähler, la propriété d'être localement conformément Kähleriennes ne se propage pas à partir de la fibre centrale par [4]. Dans le chapitre 3 de cette thèse, nous abordons la question suivante :

Question 0.1.6 *Supposons que X est une variété complexe compacte, quand existe-t-il une métrique localement conformément Kähleriennes sur X ?*

Malgré toutes les différences entre les métriques localement conformément Kähleriennes et Kählerienn, la raison qui nous pousse à considérer la question ci-dessus est la complémentarité de la structure localement conformément Kähleriennes avec les structures équilibrée et SKT dans le sens suivant :

Proposition 0.1.7 *(1) Si une métrique hermitienne ω est simultanément balanced et localement conformément Kähleriennes alors ω est Kähler (pour la preuve cf. Observation 3.6.3).*

(2) Si une métrique hermitienne ω est simultanément SKT et localement conformément Kähleriennes alors ω est Kähler (pour la preuve cf. Remarque 1.1.7).

Pour tenter de répondre à la question 0.1.6, nous utiliserons une approche variationnelle. En d'autres termes, nous allons définir une fonctionnelle différentiable non négative L sur l'ensemble de toutes les métriques hermitiennes sur X, telle que L ait la propriété suivante :

L'ensemble des zéro de L est l'ensemble de toutes les métriques localement conformément Kähleriennes sur X.

Le but est de montrer que l'ensemble des points critiques de L est l'ensemble des métriques lcK sur X. On considère deux cas, $\dim_{\mathbb{C}} X = 2$ et $\dim_{\mathbb{C}} X \geqslant 3$. Dans le cas où $\dim_{\mathbb{C}} X \geqslant 3$ après calcul de la première variation de L, nous prouvons que l'ensemble des points critiques de L est l'ensemble de toutes les métriques localement conformément Kähleriennes sur X. Cependant, ce n'est pas le cas pour $\dim_{\mathbb{C}} X = 2$, nous dérivons donc une forme simplifiée de la première variation de la fonctionnelle L dans ce cas.

Rappelons que pour chaque métrique hermitienne ω nous avons la **décomposition de Lefschetz** suivante

$$d\omega = (d\omega)_{prim} \wedge \theta \wedge \omega, \tag{1}$$

où $(d\omega)_{prim}$ est une 3-forme ω-primitive. En de basent sur la dimension complexe de la variété compacte complexe X, nous étudions soit la propriété de d**-fermeture**, soit la propriété **d'existence**. Cela signifie que lorsque dim $\dim_{\mathbb{C}} X = 2$, a cause de degré l'équation (1) se réduit à

$$d\omega = \theta \wedge \omega,$$

mais dans ce cas, θ **n'est pas nécessairement** d-fermée. Notre question se réduit donc à la d-fermeture de la forme de Lee de ω. Dans ce cas, comme mentionné précédemment, la dérivée première de la fonctionnelle L nécessite d'être simplifiée. Dans le théorème suivant, nous avons dérivé une forme simplifiée de la première variation de la fonctionnelle L.

Theorem 0.1.8 *(chapitre 3, Théorème 3.4.4) Soit S une surface complexe compacte sur laquelle une métrique hermitienne ω a été fixée.*

(i) La différentielle en $\omega \in \mathcal{H}_S$ de la fonctionnelle $L : \mathcal{H}_S \longrightarrow [0, +\infty)$ évaluée en toute forme

$\gamma \in C^\infty_{1,1}(S, \mathbb{R})$ *est donnée par l'une des trois formules suivantes :*

$$\begin{aligned}
(d_\omega L)(\gamma) &= -2\,Re\int\limits_S \Lambda_\omega(\gamma)\,\partial\theta^{1,0}_\omega \wedge \bar\partial\theta^{0,1}_\omega - 2\,Re\int\limits_S \partial\theta^{1,0}_\omega \wedge \bar\partial\Lambda_\omega(\gamma)\wedge\theta^{0,1}_\omega + 2\,Re\int\limits_S \partial\theta^{1,0}_\omega \wedge \bar\partial\Lambda_\omega(\bar\partial\gamma)\\
&\quad -2\,Re\int\limits_S i\partial\theta^{1,0}_\omega \wedge \bar\partial(\xi_{\theta^{0,1}_\omega} \lrcorner \gamma)\\
&= -2\,Re\int\limits_S \Lambda_\omega(\gamma)\,|\partial\theta^{1,0}_\omega|^2_\omega\, dV_\omega - 2\,Re\int\limits_S \partial\theta^{1,0}_\omega \wedge \bar\partial\Lambda_\omega(\gamma)\wedge\theta^{0,1}_\omega - 2\,Re\, i\langle\langle\partial\bar\partial\theta^{1,0}_\omega,\, \partial\gamma\rangle\rangle_\omega\\
&\quad -2\,Re\int\limits_S i\partial\theta^{1,0}_\omega \wedge \bar\partial(\xi_{\theta^{0,1}_\omega} \lrcorner \gamma)\\
&= -2\,Re\int\limits_S \partial\theta^{1,0}_\omega \wedge \bar\partial\Lambda_\omega(\gamma\wedge\theta^{0,1}_\omega) - 2\,Re\, i\langle\langle\partial\bar\partial\theta^{1,0}_\omega,\, \partial\gamma\rangle\rangle_\omega,
\end{aligned}$$

où $\star = \star_\omega$ *est l'opérateur étoile de Hodge défini par la métrique* ω *et* $\xi_{\theta^{0,1}_\omega}$ *est le champ vectoriel de type* $(1,\,0)$ *défini par l'exigence* $\xi_{\theta^{0,1}_\omega} \lrcorner \omega = i\theta^{0,1}_\omega$.

(ii) *En particulier, pour tout* $\omega \in \mathcal{H}_S$*, donné, si on choisit* $\gamma = \partial\theta^{0,1}_\omega + \bar\partial\theta^{1,0}_\omega$*, nous avons*

$$(d_\omega L)(\gamma) = -2\,Re\int\limits_S i\partial\theta^{1,0}_\omega \wedge \bar\partial\Big(\xi_{\theta^{0,1}_\omega} \lrcorner \gamma\Big) = -2\,Re\int\limits_S \partial\theta^{1,0}_\omega \wedge \bar\partial\Lambda_\omega(\gamma\wedge\theta^{0,1}_\omega).$$

D'autre part, lorsque $\dim_{\mathbb{C}} X \geqslant 3$, si $(d\omega)_{prim}$ dans l'équation (1) s'annule, alors la forme de Lee θ est toujours **d-fermée**. Cependant, en général, $(d\omega)_{prim}$ **ne s'annule pas** nécessairement. Ainsi, lorsque $\dim_{\mathbb{C}} X \geqslant 3$ nous avons affaire à **l'existence** d'une forme de Lee θ satisfaisant la condition lcK (i.e. sans $(d\omega)_{prim}$ sur r.h.s de l'équation (1)). Finalement, nous avons affaire à la **(non-)annulation** de $(d\omega)_{prim}$.

Pour tenter d'étudier le problème d'existence, nous avons introduit une fonctionnelle L telle que l'ensemble des zéros de L soit l'ensemble des métriques lcK sur X. La proposition suivante nous calcule la première variation de L.

Theorem 0.1.9 *(chapitre 3, Théorème 3.5.1) Pour toute métrique hermitienne* ω *et toute* $(1,\,1)$*-forme réelle* γ*, nous avons :*

$$\begin{aligned}
(d_\omega L)(\gamma) &= \int_X i(\bar\partial\omega)_{prim} \wedge \overline{(\bar\partial\omega)_{prim}} \wedge \gamma \wedge \omega_{n-4}\\
&\quad +2Re\,\langle\langle(\bar\partial\omega)_{prim},\, (\bar\partial\gamma)_{prim}\rangle\rangle_\omega - 2Re\,\langle\langle\theta^{0,1}_\omega \wedge \gamma,\, (\bar\partial\omega)_{prim}\rangle\rangle_\omega.
\end{aligned}$$

Rappelons que nous nous intéressons à l'ensemble des points critiques de L. Nous remarquons maintenant qu'un choix convenable de γ dans le résultat précédent conduit à une description explicite de cet ensemble. Puisque l'équation ci-dessus est valable pour toutes les $(1,\,1)$-formes réelles γ, le choix $\gamma = \omega$ est licite, comme tout autre choix. On obtient ce qui suit

Corollary 0.1.10 *(chapitre 3, Corollary 3.5.3) Soit X une variété complexe compacte avec $dim_{\mathbb{C}}X = n \geq 3$et soit L la fonctionnelle définie en 3.3.1-(ii). Pour toute métrique hermitienne ω sur X, nous avons :*

$$(d_\omega L)(\omega) = (n-1)\,\|(\bar\partial\omega)_{prim}\|^2_\omega = (n-1)\,L(\omega).$$

Une conséquence immédiate du corollaire ci-dessus est :

*ω est un point **critique** de L si et seulement si ω est **lcK**.*

Chapitre 1

Preliminaries

This section will recall essential definitions and results in Hermitian geometry. Our primary references for this section are [10] and [43]. In this chapter, X is a compact complex manifold of complex dimension n and $B \in \mathbb{C}^m$ is an open ball around the origin, where $n,\, m \in \mathbb{N}$.

1.1 Hermitian Geometry

A differential form u of degree $(p,\, q)$, or briefly a $(p,\, q)$-form over X, is a map u on X with values $u(x) \in \Lambda^{p,\,q} T^\star_{X,x}$. In a coordinate open set $\Omega \subset X$, a differential $(p,\, q)$-form can be written as :

$$u(x) = \sum_{|I|=p,\,|J|=q} u_{I\bar{J}}(x)\, dz_I \wedge d\bar{z}_J,$$

where $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are a multi-indices with integer components, $i_1 < \ldots < i_p$, $j_1 < \ldots < j_q$, $dz_I := dz_{i_1} \wedge \ldots \wedge dz_{i_p}$ and $d\bar{z}_J = d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$. The notation $|I|$ stands for the number of components of I, and is read *length* of I. For all integers $p,\, q = 0, 1, \ldots, n$ and $s \in \mathbb{N} \cup \{\infty\}$, we denote by $C^s_{p,\,q}(X,\, \mathbb{C})$ the space of differential $(p,\, q)$-forms of class C^s, i.e. with C^s coefficients $u_{I\bar{J}}$.

Definition 1.1.1 *A **Hermitian** metric ω is a positive definite real C^∞ $(1,\, 1)$-form. Meaning that for any local coordinate $(z_1, \ldots, z_n)$ on X, ω can be represented as*

$$\omega(x) = \sum_{i,\,j} \omega_{i\bar{j}}(x) dz_i \wedge \bar{z}_j,$$

such that the matrix $(\omega_{i\bar{j}}(x))_{i,\,j}$ is positive definite (equivalently, its eigenvalues are all positive) for all $x \in X$.

So, the Hermitian metric ω can induce a metric on $C^\infty_{p,\,q}(X,\, \mathbb{C})$. We are now able to define the L^2 inner product on $C^\infty_{p,\,q}(X,\, \mathbb{C})$ induced by a Hermitian metric ω on X as follows :

$$\langle\langle u,\, v\rangle\rangle = \langle\langle u,\, v\rangle\rangle_\omega = \int_X \langle u(x),\, v(x)\rangle_\omega dV_\omega(x),$$

where $dV_\omega := \frac{\omega^n}{n!}$, is the volume form induced by ω.

Definition 1.1.2 ***(Hodge Star Operator)*** *The Hodge-Poincaré-De Rham operator $\star_\omega$ is the collection of linear maps defined by*

$$\star_\omega : \Lambda^p T_X^\star \to \Lambda^{2n-p} T_X^\star, \qquad u \wedge \star\, v = \langle u, v\rangle\, dV, \qquad \forall u, v \in \Lambda^p T_X^\star.$$

The existence and uniqueness of this operator is easily seen by using the duality pairing

$$\begin{aligned} &\Lambda^p T_X^\star \times \Lambda^{m-p} T_X^\star \longrightarrow \mathbb{R} \\ &(u, v) \longmapsto u \wedge v/dV = \sum \varepsilon(I, \complement I)\, u_I v_{\complement I}, \end{aligned}$$

where $u = \sum_{|I|=p} u_I\, \xi_I^\star$, $v = \sum_{|J|=m-p} v_J\, \xi_J^\star$, $\complement I$ stands for the (ordered) complementary multi-index of I and $\varepsilon(I, \complement I)$ for the signature of the permutation $(1, 2, \ldots, m) \mapsto (I, \complement I)$. From this, we find

$$\star\, v = \sum_{|I|=p} \varepsilon(I, \complement I) v_I\, \xi_{\complement I}^\star.$$

The Hodge $\star_\omega$ operator can be extended to $\mathbb{C}$-valued forms by the formula :

$$u \wedge \star_\omega \overline{v} = \langle u,\, v\rangle dV.$$

It follows that $\star_\omega$ is a $\mathbb{C}$-linear isometry

$$\star_\omega : \Lambda^{p,\,q} T_X^\star \to \Lambda^{n-q,\,n-p} T_X^\star,$$

and it satisfies

$$\star_\omega \star_\omega \alpha = (-1)^{(p+q)(n-1)} \alpha,$$

where α is a $(p,\, q)$-form. When there is no ambiguity we omit the sub-index ω in $\star_\omega$.

Once a Hermitian metric has been fixed on X, one defines **formal adjoints** for $d^\star = d^\star_\omega : C_k^\infty(X, \mathbb{C}) \to C_{k-1}^\infty(X, \mathbb{C})$, $\partial^\star = \partial^\star_\omega : C_{p,\,q}^\infty(X, \mathbb{C}) \to C_{p-1,\,q}^\infty(X, \mathbb{C})$ and $\bar\partial^\star = \bar\partial^\star_\omega : C_{p,\,q}^\infty(X, \mathbb{C}) \to C_{p,\,q-1}^\infty(X, \mathbb{C})$ with respect to ω of the operators d, ∂ and $\bar\partial$, by requiring the identities

$$\langle\langle du,\, v\rangle\rangle_\omega = \langle\langle u,\, d^\star v\rangle\rangle_\omega \quad \langle\langle \partial u,\, v\rangle\rangle_\omega = \langle\langle u,\, \partial^\star v\rangle\rangle_\omega \quad \langle\langle \bar\partial u,\, v\rangle\rangle_\omega = \langle\langle u,\, \bar\partial^\star v\rangle\rangle_\omega,$$

where $C_k^\infty(X, \mathbb{C}) = \bigoplus_{p+q=k} C_{p,\,q}^\infty(X, \mathbb{C})$. By the above equations we get :

$$d^\star = -\star d \star \quad \partial^\star = -\star \bar\partial \star \quad \bar\partial^\star = -\star \partial \star. \tag{1.1}$$

Another important operator is the **Lefschetz operator** $L_\omega : C_{p,\,q}^\infty(X,\, \mathbb{C}) \to C_{p+1,\,q+1}^\infty(X,\, \mathbb{C})$ of type $(1,\, 1)$ defined by :

$$L_\omega(\alpha) = \omega \wedge \alpha,$$

where α is a $(p,\, q)$-form. Its formal adjoint, the **trace operator**, is denoted by $\Lambda_\omega = \star^{-1} L_\omega \star$. This means that for any $(p,\, q)$-form α and any $(p+1,\, q+1)$-form β we have :

$$\langle\langle L_\omega(\alpha),\, \beta\rangle\rangle_\omega = \langle\langle \alpha,\, \Lambda_\omega(\beta)\rangle\rangle_\omega.$$

A $(p,\, q)$-form α is called ω**-primitive** if $\Lambda_\omega(\alpha) = 0$. It is obvious that any form of bidegree $(p,\, 0)$ or $(0,\, q)$ is ω-primitive. In the following, we recall two lemmas that play a critical role in chapters 2 and 3.

Lemma 1.1.3 *([10], p. 301) Suppose that $L^r_\omega = \cdot \wedge \omega^r$, then for any $(p,\, q)$-form u we have :*

$$[L^r_\omega,\, \Lambda_\omega](u) = r(k-n+r-1)L^{r-1}_\omega(u),$$

where $[L^r_\omega,\, \Lambda_\omega] = L^r_\omega\Lambda_\omega - \Lambda_\omega L^r_\omega$ and $k = p+q$.

Lemma 1.1.4 *([13], Lemma 3.3) Let η be a $(1,\, 1)$-form on X. The following formula holds for operators acting on differential forms of any bidegree on X :*

$$\star_\omega(\eta \wedge \cdot) = (\bar\eta \wedge \cdot)^\star \star_\omega .$$

In particular we have

$$\star_\omega L_\omega = \Lambda_\omega \star_\omega .$$

Proof. Let $(p,\, q)$ be an arbitrary bidegree and let $u \in \Lambda^{p+1,\, q+1}T^\star X$, $v \in \Lambda^{p,\, q}T^\star X$ be arbitrary forms of the indicated bidegrees.

The definition of the Hodge star operator $\star_\omega$ yields the first and the third equalities below :

$$u \wedge \star_\omega\overline{(\eta \wedge v)} \;\; = \;\; \langle u,\, \eta\wedge v\rangle_\omega\, dV_\omega = \langle(\eta\wedge\cdot)^\star_\omega u,\, v\rangle_\omega\, dV_\omega = (\eta\wedge\cdot)^\star_\omega u \wedge \star_\omega \bar v. \tag{1.2}$$

The formula to prove being pointwise, we fix an arbitrary point $x \in X$ and we choose local holomorphic coordinates $(z_1, \dots, z_n)$ about x such that ω is given by the identity matrix at x in these coordinates. In particular, the adjoints of the multiplication operators $dz_j \wedge \cdot$ and $d\bar z_j \wedge \cdot$ w.r.t. the pointwise inner product induced by ω at x are given by the contractions with the corresponding tangent vectors at x :

$$(dz_j \wedge \cdot)^\star_\omega = \frac{\partial}{\partial z_j} \lrcorner \cdot, \qquad (d\bar z_j \wedge \cdot)^\star_\omega = \frac{\partial}{\partial \bar z_j} \lrcorner \cdot, \tag{1.3}$$

for every j. Let

$$\eta = \sum_{j,\, k=1}^n \eta_{j\bar k}\, i dz_j \wedge d\bar z_k$$

be the local expression of η.

The last term in (1.2) reads :

$$(\eta\wedge\cdot)^\star_\omega u \wedge \star_\omega \bar v \;\; = \;\; -\sum_{j,\, k=1}^n \bar\eta_{j\bar k}\, i\left(\frac{\partial}{\partial \bar z_k}\lrcorner\frac{\partial}{\partial z_j}\lrcorner u\right) \wedge \star_\omega \bar v. \tag{1.4}$$

Meanwhile, we have :

$$\begin{aligned}
\left(\frac{\partial}{\partial \bar z_k}\lrcorner\frac{\partial}{\partial z_j}\lrcorner u\right) \wedge \star_\omega \bar v \;\; &= \;\; \frac{\partial}{\partial \bar z_k}\lrcorner\left[\left(\frac{\partial}{\partial z_j}\lrcorner u\right)\wedge \star_\omega \bar v\right] - (-1)^{p+q+1}\left(\frac{\partial}{\partial z_j}\lrcorner u\right)\wedge\left(\frac{\partial}{\partial \bar z_k}\lrcorner \star_\omega\, \bar v\right) \\
&= \;\; (-1)^{p+q}\left(\frac{\partial}{\partial z_j}\lrcorner u\right)\wedge\left(\frac{\partial}{\partial \bar z_k}\lrcorner \star_\omega\, \bar v\right) \\
&= \;\; (-1)^{p+q}\,\frac{\partial}{\partial z_j}\lrcorner\left[u\wedge\left(\frac{\partial}{\partial \bar z_k}\lrcorner \star_\omega\, \bar v\right)\right] - u\wedge\left(\frac{\partial}{\partial z_j}\lrcorner\frac{\partial}{\partial \bar z_k}\lrcorner \star_\omega\, \bar v\right) \\
&= \;\; -u\wedge\left(\frac{\partial}{\partial z_j}\lrcorner\frac{\partial}{\partial \bar z_k}\lrcorner \star_\omega\, \bar v\right),
\end{aligned}$$

where the first term on the r.h.s. of each of the first and third lines above vanishes for bidegree reasons. Indeed, $((\partial/\partial z_j)\lrcorner u)\wedge\star_\omega\bar{v}$ is of bidegree $(n,\,n+1)$ and $u\wedge((\partial/\partial\bar{z}_k)\lrcorner\star_\omega\bar{v})$ is of bidegree $(n+1,\,n)$ since u is of bidegree $(p+1,\,q+1)$ and $\star_\omega\bar{v}$ is of bidegree $(n-p,\,n-q)$. Using again (1.3), this translates to

$$\left(\frac{\partial}{\partial\bar{z}_k}\lrcorner\frac{\partial}{\partial z_j}\lrcorner u\right)\wedge\star_\omega\bar{v}=-u\wedge(d\bar{z}_k\wedge dz_j\wedge\cdot)^\star_\omega\,(\star_\omega\bar{v}). \tag{1.5}$$

Putting (1.4) and (1.5) together, we get :

$$\begin{aligned}(\eta\wedge\cdot)^\star_\omega u\wedge\star_\omega\bar{v} &= \sum_{j,k=1}^n\bar{\eta}_{j\bar{k}}\,i\,u\wedge(d\bar{z}_k\wedge dz_j\wedge\cdot)^\star_\omega\,(\star_\omega\bar{v})=-u\wedge\left(\sum_{j,\,k=1}^n\eta_{j\bar{k}}\,i\,d\bar{z}_k\wedge dz_j\wedge\cdot\right)^\star_\omega(\star_\omega\bar{v})\\ &= u\wedge(\eta\wedge\cdot)^\star_\omega\,(\star_\omega\bar{v}).\end{aligned} \tag{1.6}$$

Finally, putting (1.2) and (1.6) together, we get :

$$u\wedge\star_\omega\overline{(\eta\wedge v)}=u\wedge\overline{(\bar{\eta}\wedge\cdot)^\star_\omega\,(\star_\omega v)}$$

for all forms $u\in\Lambda^{p+1,\,q+1}T^\star X$, $v\in\Lambda^{p,\,q}T^\star X$. This proves the Lemma. □

For any $(1,\,1)$-form $\rho\geq 0$, we will also use the following notation :

$$\rho_k:=\frac{\rho^k}{k!},\qquad 1\leq k\leq n.$$

When $\rho=\omega$ is C^∞ and positive definite (i.e. ω is a Hermitian metric on X), it can immediately be checked that

$$d\omega_k=\omega_{k-1}\wedge d\omega\quad\text{ and }\quad\star_\omega\,\omega_k=\omega_{n-k} \tag{1.7}$$

for all $1\leq k\leq n$, where $\star=\star_\omega$ is the Hodge star operator induced by ω.

The following proposition plays an important role in our discussion later.

Proposition 1.1.5 *([43], Proposition 6.29) If $u\in C^\infty_{p,\,q}(X,\,\mathbb{C})$ is ω-primitive then*

$$\star\,u=(-1)^{\frac{(p+q)^2+p+q}{2}}\,i^{p-q}\omega_{n-q-p}\wedge u. \tag{1.8}$$

In the following definition, we mention different types of metrics that will be used frequently in this thesis.

Definition 1.1.6 *Suppose that ω is a Hermitian metric on X,*

(1) ω is said to be ***Kähler*** *if $d\omega=0$. X is called a Kähler manifold if it admits a Kähler metric.*

(2) ω is said to be ***balanced*** *if $d\omega^{n-1}=0$. X is called a balanced manifold if it admits a balanced metric.*

*(3) ω is said to be **SKT or pluriclosed** if $\partial\bar{\partial}\omega = 0$. X is called an SKT manifold if it admits an SKT metric.*

*(4) ω is said to be **locally conformally Kähler (lcK)** if $d\omega = \theta \wedge \omega$, where θ is a d-closed 1-form on X. X is called an lcK manifold if it admits an lcK metric. The 1-form θ is uniquely determined, is real and is called the **Lee form** of ω.*

*(5) ω is said to be **Hermitian-symplectic (H-s)** if there exists d-closed 2-form Ω and $(2,\,0)$-form $\rho^{2,\,0}$ such that,*

$$\Omega = \rho^{2,\,0} + \omega + \rho^{0,\,2}, \tag{1.9}$$

where $\rho^{0,\,2} = \overline{\rho^{2,\,0}}$. X is called an H-s manifold if it admits an H-s metric.

*6) ω is said to be **Gauduchon** if $\bar{\partial}\partial\omega^{n-1} = 0$. X is called a Gauduchon manifold if it admits a Gauduchon metric.*

Remark 1.1.7

1) We have the following implications among the above metrics :

$$\begin{tikzcd}
\textit{balanced} + \textit{SKT} \arrow[dr, "(a)"] & \textit{SKT} + \textit{lcK} \arrow[d, "(b)"] & \textit{Balanced} + \textit{lcK} \arrow[dl, "(c)"'] \\
\textit{balanced} \arrow[d, "(d)"] & \textit{Kähler} \arrow[l, "(e)"] \arrow[r, "(f)"] \arrow[dl, "(g)"] \arrow[d, "(h)"] \arrow[dr, "(i)"] & \textit{lcK} \\
\textit{Gauduchon} & \textit{SKT} & \textit{H-s} \arrow[l, "(k)"]
\end{tikzcd}$$

All statements are easy to compute except (a), (b) and (c). We only prove (a) and (b) whilst the proof of (c) is given in Observation 3.6.3.

***Proof of (a).**([12], Proposition 2.6, also c.f. [2]) The SKT assumption on ω translates to any of the following equivalent properties :*

$$\partial\bar{\partial}\omega = 0 \Longleftrightarrow \partial\omega \in \ker\bar{\partial} \Longleftrightarrow \star(\partial\omega) \in \ker\partial^\star, \tag{1.10}$$

where the last equivalence follows from the standard formula $\partial^\star = -\star\bar{\partial}\star$ involving the Hodge-star isomorphism $\star = \star_\omega : \Lambda^{p,\,q}T^\star X \to \Lambda^{n-q,\,n-p}T^\star X$ defined by ω for arbitrary $p, q = 0, \dots, n$.

Meanwhile, the balanced assumption on ω translates to any of the following equivalent properties :

$$d\omega^{n-1} = 0 \Longleftrightarrow \partial\omega^{n-1} = 0 \Longleftrightarrow \omega^{n-2}\wedge\partial\omega = 0 \Longleftrightarrow \partial\omega \text{ is primitive.}$$

Moreover, since $\partial\omega$ is primitive when ω is balanced, the general formula (1.8) yields :

$$\star(\partial\omega) = i\,\frac{\omega^{n-3}}{(n-3)!}\wedge\partial\omega = \frac{i}{(n-2)!}\,\partial\omega^{n-2} \in Im\,\partial. \tag{1.11}$$

Thus, if ω is both SKT and balanced, we get from (1.10) and (1.11) that

$$\star(\partial\omega) \in \ker\partial^\star \cap Im\,\partial = \{0\},$$

where the last identity follows from the subspaces $\ker \partial^\star$ *and* $Im\,\partial$ *of* $C^\infty_{n-1,\,n-2}(X,\,\mathbb{C})$ *being* L^2_ω*-orthogonal. We infer that* $\partial\omega = 0$*, i.e.* ω *is Kähler.* □

Proof of (b). *Consider the pure-type decomposition of the Lee form* θ *into its pure-type components as follows :*

$$\theta = \theta^{1,\,0} + \theta^{0,\,1}.$$

Since θ *is real, then* $\theta^{0,\,1} = \overline{\theta^{1,\,0}}$*, therefore to prove that* ω *is Kähler it suffices to prove that* $\theta^{1,\,0} = 0$ *or equivalently* $\|\theta^{1,\,0}\|^2_\omega = 0$*. By (1.8) we get*

$$\star_\omega\overline{\theta^{1,\,0}} = \star\overline{\theta^{1,\,0}} = \star\theta^{0,\,1} = i\theta^{0,\,1} \wedge \omega_{n-1}. \tag{1.12}$$

On the other hand

$$\bar\partial\partial(\omega_{n-1}) = \bar\partial(\partial\omega \wedge \omega_{n-2}) = \bar\partial\omega \wedge \partial\omega \wedge \omega_{n-3},$$

where the last equality comes from the fact that ω *is an SKT metric, i.e.* $\bar\partial\partial\omega = 0$*. But*

$$\bar\partial\omega \wedge \partial\omega \wedge \omega_{n-3} = -(n-1)(n-2)\theta^{1,\,0} \wedge \theta^{0,\,1} \wedge \omega_{n-1} \overset{(1.12)}{=} (n-1)(n-2)i\theta^{0,\,1} \wedge \star\overline{\theta^{1,\,0}}$$

Since X *is a compact complex without a boundary,* $\int_X \bar\partial\partial\omega_{n-1} = 0$*, this implies that* $(n-1)(n-2)\|\theta^{1,\,0}\|^2_\omega = 0$*. This prove that* ω *is Kähler.* □

(2) *Suppose that* ω *is an H-s metric. We mention four equations easily implied by equation (1.9). Since* $d\Omega = 0$ *we have :*

   (i) $\partial\omega = -\bar\partial\rho^{2,\,0}$ *and* $\bar\partial\omega = -\partial\rho^{0,\,2}$.

   (ii) $\partial\rho^{2,\,0} = 0$ *and* $\bar\partial\rho^{0,\,2} = 0$.

   (iii) ω *is Kähler if and only if* $\rho^{2,\,0} = 0$.

   (iv) $\partial\bar\partial\omega = 0$.

(3) *Suppose that* ω *is balanced then by formulae (1.7) and (1.1) we get*

$$0 \;=\; d\omega^{n-1} \Longleftrightarrow 0 = d \star \omega \Longleftrightarrow 0 = \star d \star \omega \Longleftrightarrow 0 = d^\star\omega.$$

In other words ω *is* ***co-closed****.*

(4) *Every compact complex manifold carries a Gauduchon metric. Indeed by [16], every conformal class of any Hermitian metric* ω *on* X *contains a* ***unique*** *(up to multiplications by positive constants)* ***Gauduchon metric****.*

In order to define suitable cohomology groups for Hermitian-symplectic and SKT metrics, we recall the definitions of the Bott-Chern cohomology and the Aeppli cohomology groups along with other cohomology groups definitions.

Definition 1.1.8 *For every* $p, q \in \{0, \dots n\}$ *one defines :*

(i) the De Rham cohomology group of degree K of X as

$$H^k_{DR}(X,\,\mathbb{C}) = \frac{\ker d}{Imd},$$

(ii) the Bott-Chern cohomology group of bidegree (or type) $(p,\,q)$ of X as

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) = \frac{\ker\partial\cap\ker\bar\partial}{\mathrm{Im}(\partial\bar\partial)},$$

(iii) the Aeppli cohomology group of bidegree (or type) $(p,\,q)$ of X as

$$H^{p,\,q}_{A}(X,\,\mathbb{C}) = \frac{\ker(\partial\bar\partial)}{\mathrm{Im}\,\partial+\mathrm{Im}\,\bar\partial},$$

(iv) the Dolbeault cohomology group of bidegree (or type) $(p,\,q)$ of X as

$$H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) = \frac{\ker\bar\partial}{Im\bar\partial},$$

(v) the conjugate Dolbeault cohomology group of bidegree (or type) $(p,\,q)$ of X as

$$H^{p,\,q}_{\partial}(X,\,\mathbb{C}) = \frac{\ker\partial}{Im\partial},$$

where all the kernels and images are considered as $\mathbb{C}$-vector subspaces of $C^\infty_{p,\,q}(X,\,\mathbb{C})$ according to the case.

From Definition 1.1.6 one can see if ω is a Hermitian-symplectic metric then the Aeppli cohomology class of ω, which will be denoted by $\{\omega\}_A$, is well-defined. Moreover if ω is chosen to be an SKT metric then the Bott-Chern cohomology class of $\partial\omega$, which will be denoted by $\{\partial\omega\}_{BC}$ is also well-defined.

To fix the notation suppose that α is a k-form, then we denote by

$$\{\alpha\}_\sharp$$

the cohomology class of α in cohomology group $H_\sharp(X,\,\mathbb{C})$, where $\sharp\in\{A,\,BC,\,\partial,\,\bar\partial,\,DR\}$.

In the following definition, we recall formal definitions of five elliptic self-adjoint operators and mention the Hodge decompositions for $C^\infty_{p,\,q}(X,\,\mathbb{C})$ of these operators.

Definition 1.1.9 *Fix $p,\,q\in\{0,\,\dots\,n\}$ then*

*(i) The **Bott-Chern Laplacian operator** $\Delta^{p,\,q}_{BC}: C^\infty_{p,\,q}(X,\,\mathbb{C})\to C^\infty_{p,\,q}(X,\,\mathbb{C})$ is defined as follows*

$$\Delta^{p,\,q}_{BC} := \partial^\star\partial+\bar\partial^\star\bar\partial+(\partial\bar\partial)^\star(\partial\bar\partial)+(\partial\bar\partial)(\partial\bar\partial)^\star+(\partial^\star\bar\partial)^\star(\partial^\star\bar\partial)+(\partial^\star\bar\partial)(\partial^\star\bar\partial)^\star.$$

(ii) *The* ***Aeppli Laplacian operator*** $\Delta_A^{p,\,q} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C})$ *is defined as follows*

$$\Delta_A^{p,\,q} := \partial\partial^\star + \bar\partial\bar\partial^\star + (\partial\bar\partial)^\star(\partial\bar\partial) + (\partial\bar\partial)(\partial\bar\partial)^\star + (\partial\bar\partial^\star)(\partial\bar\partial^\star)^\star + (\partial\bar\partial^\star)^\star(\partial\bar\partial^\star),$$

(iii) *The* ***Dolbeault Laplacian operator*** $\Delta_{\bar\partial}^{p,\,q} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C})$ *is defined as follows*

$$\Delta_{\bar\partial}^{p,\,q} := \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial.$$

(iv) *The* ***conjugate Dolbeault Laplacian*** *operator* $\Delta_{\partial}^{p,\,q} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C})$ *is defined as follows*

$$\Delta_{\partial}^{p,\,q} := \partial\partial^\star + \partial^\star\partial.$$

(v) *The* ***De Rham Laplacian operator*** $\Delta : C^\infty_k(X,\,\mathbb{C}) \to C^\infty_k(X,\,\mathbb{C})$ *is defined as follows*

$$\Delta := dd^\star + d^\star d.$$

It is worth noting that that by definition, the Bott-Chern Laplacian operator is a real self-adjoint operator, whereas the Dolbeault operator is not.

For each of the above operators, we have the following L^2_ω**-orthogonal two-space decompositions** for $C^\infty_{p,\,q}(X,\,\mathbb{C})$ as follows :

$$C^\infty_{p,\,q}(X,\,\mathbb{C}) = \ker\Delta_{\mathrm{BC}}^{p,\,q} \oplus \mathrm{Im}\,\Delta_{\mathrm{BC}}^{p,\,q},$$

$$C^\infty_{p,\,q}(X,\,\mathbb{C}) = \ker\Delta_{\mathrm{A}}^{p,\,q} \oplus \mathrm{Im}\,\Delta_{\mathrm{A}}^{p,\,q},$$

$$C^\infty_{p,\,q}(X,\,\mathbb{C}) = \ker\Delta_{\bar\partial}^{p,\,q} \oplus \mathrm{Im}\,\Delta_{\bar\partial}^{p,\,q},$$

$$C^\infty_{p,\,q}(X,\,\mathbb{C}) = \ker\Delta_{\partial}^{p,\,q} \oplus \mathrm{Im}\,\Delta_{\partial}^{p,\,q}.$$

$$C^\infty_k(X,\,\mathbb{C}) = \ker\Delta \oplus \mathrm{Im}\,\Delta,$$

From the above equations we get :

$$\begin{cases} \ker\partial \cap \ker\bar\partial = \ker\Delta_{BC} \oplus \mathrm{Im}\partial\bar\partial \\ \ker\Delta_{BC} = \ker\partial \cap \ker\bar\partial \cap \ker(\partial\bar\partial)^\star \end{cases}. \tag{1.13}$$

$$\begin{cases} \ker(\partial\bar\partial) = \ker\Delta_A \oplus (\mathrm{Im}\partial + \mathrm{Im}\bar\partial) \\ \ker\Delta_A = \ker\partial^\star \cap \ker\bar\partial^\star \cap \ker(\partial\bar\partial) \end{cases}. \tag{1.14}$$

$$\begin{cases} \ker\bar\partial = \ker\Delta_{\bar\partial} \oplus \mathrm{Im}\bar\partial \\ \ker\Delta_{\bar\partial} = \ker\bar\partial \cap \ker\bar\partial^\star \end{cases}.$$

$$\begin{cases} \ker \partial = \ker \Delta_\partial \oplus \mathrm{Im}\partial \\ \ker \Delta_\partial = \ker \partial \cap \ker \partial^\star \end{cases} .$$

$$\begin{cases} \ker d = \ker \Delta \oplus \mathrm{Im} d \\ \ker \Delta = \ker d \cap \ker d^\star \end{cases} .$$

Which yield the following **Hodge isomorphisms**

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\,\mathbb{C}),$$

where $\mathcal{H}^{p,\,q}_{\Delta_{BC}}(X,\,\mathbb{C}) = \ker(\Delta_{BC} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C}))$ is the **Bott-Chern harmonic space.**

$$H^{p,\,q}_{A}(X,\,\mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta_{A}}(X,\,\mathbb{C}),$$

where $\mathcal{H}^{p,\,q}_{\Delta_{A}}(X,\,\mathbb{C}) = \ker(\Delta_{A} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C}))$ is the **Aeppli harmonic space.**

$$H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta_{\bar\partial}}(X,\,\mathbb{C})$$

where $\mathcal{H}^{p,\,q}_{\Delta_{\bar\partial}}(X,\,\mathbb{C}) = \ker(\Delta_{\bar\partial} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C}))$ is the **Dolbeault harmonic space.**

$$H^{p,\,q}_{\partial}(X,\,\mathbb{C}) \simeq \mathcal{H}^{p,\,q}_{\Delta_{\partial}}(X,\,\mathbb{C})$$

where $\mathcal{H}^{p,\,q}_{\Delta_{\partial}}(X,\,\mathbb{C}) = \ker(\Delta_{\partial} : C^\infty_{p,\,q}(X,\,\mathbb{C}) \to C^\infty_{p,\,q}(X,\,\mathbb{C}))$ is the **conjugate Dolbeault harmonic space.**

$$H^{k}_{DR}(X,\,\mathbb{C}) \simeq \mathcal{H}^{k}_{\Delta}(X,\,\mathbb{C})$$

where $\mathcal{H}^{k}_{\Delta}(X,\,\mathbb{C}) = \ker(\Delta : C^\infty_{k}(X,\,\mathbb{C}) \to C^\infty_{k}(X,\,\mathbb{C}))$ is the **De Rham harmonic space**.

In the following we give a **Frölicher-type** inequality for the Bott-Chern and Aeppli cohomologies. As a matter of notation, for every $p, q \in \{0, \cdots, n\}$, for every $k \in \mathbb{N}$ and for $\sharp \in \left\{\overline{\partial},\, \partial,\, BC,\, A\right\}$, we will denote

$$h^{p,q}_\sharp \;:=\; \dim_{\mathbb{C}} H^{p,q}_\sharp(X,\,\mathbb{C}) \qquad \text{and} \qquad h^k_\sharp \;:=\; \sum_{p+q=k} h^{p,q}_\sharp \;,$$

and we will denote the Betti numbers by

$$b_k \;:=\; \dim_{\mathbb{C}} H^k_{DR}(X,\,\mathbb{C}) \;.$$

Theorem 1.1.10 *([3], Theorem A) Let X be a compact complex manifold of complex dimension n. Then, for every $p, q \in \{0, \cdots, n\}$, the following inequality holds :*

$$h^{p,q}_{BC} + h^{p,q}_{A} \;\geq\; h^{p,q}_{\bar\partial} + h^{p,q}_{\partial} \;. \tag{1.15}$$

In particular, for every $k \in \mathbb{N}$, the following inequality holds :

$$h^{k}_{BC} + h^{k}_{A} \;\geq\; 2\, b_k \;, \tag{1.16}$$

where $h^k_{BC} := \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{BC}(X,\,\mathbb{C})$ and $h^k_{A} := \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{A}(X,\,\mathbb{C})$.

Proof. First of all, we need to recall two exact sequences from [44]. Following J. Varouchas, one defines the finite-dimensional bi-graded vector spaces :

$$A^{\bullet,\bullet} := \frac{\operatorname{im}\overline{\partial}\cap\operatorname{im}\partial}{\operatorname{im}\partial\overline{\partial}}\,, \quad B^{\bullet,\bullet} := \frac{\ker\overline{\partial}\cap\operatorname{im}\partial}{\operatorname{im}\partial\overline{\partial}}\,, \quad C^{\bullet,\bullet} := \frac{\ker\partial\overline{\partial}}{\ker\overline{\partial}+\operatorname{im}\partial}$$

and

$$D^{\bullet,\bullet} := \frac{\operatorname{im}\overline{\partial}\cap\ker\partial}{\operatorname{im}\partial\overline{\partial}}\,, \quad E^{\bullet,\bullet} := \frac{\ker\partial\overline{\partial}}{\ker\partial+\operatorname{im}\overline{\partial}}\,, \quad F^{\bullet,\bullet} := \frac{\ker\partial\overline{\partial}}{\ker\overline{\partial}+\ker\partial}\,.$$

For every $p, q \in \mathbb{N}$ and $k \in \mathbb{N}$, we will denote

$$a^{p,q} := \dim_{\mathbb{C}} A^{p,q}\,, \quad \ldots\,, \quad f^{p,q} := \dim_{\mathbb{C}} F^{p,q}$$

and

$$a^{k} := \sum_{p+q=k} a^{p,q}\,, \quad \ldots\,, \quad f^{k} := \sum_{p+q=k} f^{p,q}\,.$$

One has the following exact sequences, see [44, §3.1] :

$$0 \to A^{\bullet,\bullet} \to B^{\bullet,\bullet} \to H^{\bullet,\bullet}_{\overline{\partial}}(X) \to H^{\bullet,\bullet}_{A}(X,\,\mathbb{C}) \to C^{\bullet,\bullet} \to 0 \tag{1.17}$$

and

$$0 \to D^{\bullet,\bullet} \to H^{\bullet,\bullet}_{BC}(X,\,\mathbb{C}) \to H^{\bullet,\bullet}_{\overline{\partial}}(X,\,\mathbb{C}) \to E^{\bullet,\bullet} \to F^{\bullet,\bullet} \to 0\,. \tag{1.18}$$

Note also (see [44, §3.1]) that the conjugation and the maps $\overline{\partial}\colon C^{\bullet,\bullet} \xrightarrow{\simeq} D^{\bullet,\bullet+1}$ and $\partial\colon E^{\bullet,\bullet} \xrightarrow{\simeq} B^{\bullet+1,\bullet}$ induce, for every $p, q \in \mathbb{N}$, the equalities

$$a^{p,q} = a^{q,p}\,, \qquad f^{p,q} = f^{q,p}\,, \qquad d^{p,q} = b^{q,p}\,, \qquad e^{p,q} = c^{q,p} \tag{1.19}$$

and

$$c^{p,q} = d^{p,q+1}\,, \qquad e^{p,q} = b^{p+1,q}\,,$$

from which one gets, for every $k \in \mathbb{N}$, the equalities

$$d^{k} = b^{k}\,, \qquad e^{k} = c^{k} \qquad \text{and} \qquad c^{k} = d^{k+1}\,, \qquad e^{k} = b^{k+1}\,.$$

Fix $p, q \in \{1, \cdots, n\}$, using the symmetries $h^{p,q}_{A} = h^{q,p}_{A}$ and $h^{p,q}_{\bar{\partial}} = h^{q,p}_{\partial}$, the exact sequences (1.17), (1.18) and the equalities (1.19), we have

$$\begin{aligned} h^{p,q}_{BC} + h^{p,q}_{A} &= h^{p,q}_{BC} + h^{q,p}_{A} \\ &= h^{p,q}_{\bar{\partial}} + h^{q,p}_{\bar{\partial}} + f^{p,q} + a^{q,p} + d^{p,q} - b^{q,p} - e^{p,q} + c^{q,p} \\ &= h^{p,q}_{\bar{\partial}} + h^{p,q}_{\partial} + f^{p,q} + a^{p,q} \\ &\geq h^{p,q}_{\bar{\partial}} + h^{p,q}_{\partial}\,, \end{aligned}$$

which proves (1.15).
Now, fix $k \in \mathbb{N}$; summing over $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $p + q = k$, we get

$$\begin{aligned} h^k_{BC} + h^k_A &= \sum_{p+q=k} (h^{p,q}_{BC} + h^{p,q}_A) \\ &\geq \sum_{p+q=k} \left(h^{p,q}_{\bar\partial} + h^{p,q}_{\partial}\right) \; = \; h^k_{\bar\partial} + h^k_{\partial} \\ &\geq 2\, b_k \,, \end{aligned}$$

from which we get 1.16. □

Moreover, there is a **canonical** isomorphism between the Aeppli cohomology group and dual of the Bott-Chern cohomology group of the complementary bidegree.

Theorem 1.1.11 *For all $p,\, q \in \{0, \cdots ,\; n\}$ the bilinear pairing*

$$H^{p,\,q}_{BC}(X,\, \mathbb{C}) \times H^{n-p,\,n-q}_A(X,\, \mathbb{C}), \qquad (\{\alpha\}_{BC},\, \{\beta\}_A) = \int_X \alpha \wedge \beta,$$

*is **well-defined** (it does not depend on the representative of $\{\alpha\}_{BC}$ or $\{\beta\}_A$), and* **non-degenerate.**

Proof. Let $\alpha + \partial\bar\partial u$ be another representative in the class $\{\alpha\}_{BC}$. Then

$$\int_X (\alpha + \partial\bar\partial u) \wedge \beta = \int_X \alpha \wedge \beta + \int_X \partial\bar\partial u \wedge \beta = \int_X \alpha \wedge \beta \pm \int_X u \wedge \partial\bar\partial\beta = \int_X \alpha \wedge \beta,$$

where the last equality is deduced by the fact that β represents an Aeppli class. Similarly, let $\beta + \partial\psi + \bar\partial\zeta$ be another representative in the class $\{\alpha\}_A$. Then

$$\int_X \alpha \wedge (\beta + \partial\psi + \bar\partial\zeta) = \pm \int_X \partial\alpha \wedge \psi \pm \int_X \bar\partial\alpha \wedge \zeta + \int_X \alpha \wedge \beta,$$

where the last equality is deduced by the fact that α represents a Bott-Chern class. We conclude that the bilinear map in the statement is well defined.

To prove non-degeneracy, we fix an arbitrary Hermitian metric ω on X. We know that if $\{\alpha\}_{BC} \in H^{p,\,q}_{BC}(X,\, \mathbb{C})$ is a non-zero class, then $\{\star\bar\alpha\}_{BC} \in H^{n-p,\,n-q}_A(X,\, \mathbb{C})$. Indeed by equations (1.1), (1.13) and (1.14), we get the following equivalences for every form α

$$\begin{aligned} \alpha \in \mathcal{H}^{p,\,q}_{\Delta_{BC}} &\iff \partial\alpha = 0,\, \bar\partial\alpha = 0,\, (\partial\bar\partial)^\star\alpha = 0 \\ &\iff \bar\partial^\star(\star\alpha) = 0,\, \partial^\star(\star\alpha) = 0,\, (\partial\bar\partial)(\star\alpha) = 0 \iff \star\alpha \in \mathcal{H}^{n-q,\,n-p}_{\Delta_A}. \end{aligned}$$

Therefore by the Hodge isomorphism we get what we claimed. By the above argument we have :

$$(\{\alpha\}_{BC},\, \{\star\bar\alpha\}_A) = \int_X \alpha \wedge \star\bar\alpha = \|\alpha\|^2_\omega \neq 0.$$

Similarly if $\{\beta\}_A \in H^{p,\,q}_A(X,\, \mathbb{C})$ is a non-zero class, by the latest argument we have $\{\star\overline{\beta}\}_{BC} \in H^{n-p,\,n-q}_{BC}(X,\, \mathbb{C})$. Therefore

$$(\{\star\bar\beta\}_{BC},\, \{\beta\}_A) = \int_X \star\bar\beta \wedge \beta = \|\beta\|^2_\omega \neq 0.$$

□

Note that the above statement depends only on the complex structure of the manifold, no metric is involved. This is why we called the isomorphism in Theorem 1.1.11 the **canonical** Isomorphism.

1.2 Frölicher Spectral Sequence (FSS)

The Frölicher spectral sequence (FSS) of X is an object that relates the complex structure of X to its differential structure at the cohomological level. In this section we will briefly describe the more recent point of view on the FSS. This point of view is more concrete and it is introduced by Cordero, Fernández, Gray and Ugarte in [7].

Definition 1.2.1 *(i) Fix $r \geq 1$. A form $\alpha \in C^\infty_{p,\,q}(X,\,\mathbb{C})$ is* E_r**-closed** *(i.e. α represents an E_r-cohomology class) if and only if there exist forms $u_l \in C^\infty_{p+l,\,q-l}(X,\,\mathbb{C})$ with $l \in \{1,\dots,r-1\}$ satisfying the following r equations :*

$$\begin{aligned}\bar\partial\alpha &= 0\\ \partial\alpha &= \bar\partial u_1\\ \partial u_1 &= \bar\partial u_2\\ &\vdots\\ \partial u_{r-2} &= \bar\partial u_{r-1}.\end{aligned}$$

*(When $r=1$, the above equations reduce to $\bar\partial\alpha = 0$.) An $(r-1)$-tuple $(u_1,\cdots,u_{r-1})$ of forms with the above property is called a **system of $\bar\partial$-potentials** for $\partial\alpha$.*

*(i') If we only have $\bar\partial\alpha = 0$, we say that α is E_1**-closed** or $\bar\partial$**-closed**.*

(i") We set $\mathcal{X}_r^{p,\,q} := \{\alpha \in C^\infty_{p,\,q}(X,\,\mathbb{C})\,|\,\alpha \text{ is } E_r-\text{closed}\}$.

(ii) Fix $r \geq 1$. A form $\alpha \in C^\infty_{p,\,q}(X,\,\mathbb{C})$ is E_r**-exact** *(i.e. α represents the zero E_r-cohomology class) if and only if there exist forms $\zeta_{r-2} \in C^\infty_{p-1,\,q}(X,\,\mathbb{C})$ and $\xi_0 \in C^\infty_{p,\,q-1}(X,\,\mathbb{C})$ such that*

$$\alpha = \partial\zeta_{r-2} + \bar\partial\xi_0,$$

with ξ_0 arbitrary and ζ_{r-2} satisfying the following additional condition (which is empty when $r=1$, denoting $\zeta_{-1}=0$, and reduces to requiring that $\zeta_{r-2}=\zeta_0$ be $\bar\partial$-closed when $r=2$.)

There exist C^∞ forms $v_0^{(r-2)}, v_1^{(r-2)},\dots,v_{r-3}^{(r-2)}$ satisfying the following $(r-1)$ equations :

$$\begin{aligned}\bar\partial\zeta_{r-2} &= \partial v_{r-3}^{(r-2)}\\ \bar\partial v_{r-3}^{(r-2)} &= \partial v_{r-4}^{(r-2)}\\ &\vdots\\ \bar\partial v_1^{(r-2)} &= \partial v_0^{(r-2)}\\ \bar\partial v_0^{(r-2)} &= 0,\end{aligned} \tag{1.20}$$

with the convention that any form $v_l^{(r-2)}$ with $l<0$ vanishes.

(Note that, thanks to (i), equations (1.20), when read from bottom to top, express precisely the condition that the form $v_0^{(r-2)} \in C^\infty_{p-r+1,\,q+r-2}(X,\,\mathbb{C})$ be E_{r-1}-closed. Moreover, the form $\partial\zeta_{r-2}$ featuring on the r.h.s. of the above expression for α represents the E_{r-1}-class $(-1)^r d_{r-1}(\{v_0^{(r-2)}\}_{E_{r-1}})$.)

*(i') If $\alpha \in Im\bar\partial$, we say that α is E_1**-exact** or $\bar\partial$**-exact**.*

(ii”) We set $\mathcal{Y}_r^{p,q} := \{\alpha \in C^\infty_{p,q}(X, \mathbb{C}) \,|\, \alpha$ *is* $E_r -$ *exact*$\}$.

Note the obvious inclusions :

$$\cdots \mathcal{Y}_{r-1}^{p,q} \subset \mathcal{Y}_r^{p,q} \subset \mathcal{Y}_{r+1}^{p,q} \subset \cdots \subset \mathcal{X}_{r+1}^{p,q} \subset \mathcal{X}_r^{p,q} \subset \mathcal{X}_{r-1}^{p,q} \subset \cdots .$$

For any $r \in \mathbb{N}^\star$ and any $p, q \in \{0, \cdots, n\}$ put

$$E_r^{p,q}(X) = \frac{\mathcal{X}_r^{p,q}}{\mathcal{Y}_r^{p,q}},$$

and every class in $E_r^{p,q}(X)$ with representative α is shown as $\{\alpha\}_{E_r}$ when there is no confusion on bidegree. It is obvious that

$$\cdots \subset \overbrace{\mathcal{Y}_{r-1}^{p,q} \subset \overbrace{\mathcal{Y}_r^{p,q} \subset \overbrace{\mathcal{Y}_{r+1}^{p,q} \subset \cdots \subset \mathcal{X}_{r+1}^{p,q}}^{E_{r+1}^{p,q}(X)} \subset \mathcal{X}_r^{p,q}}^{E_r^{p,q}(X)} \subset \mathcal{X}_{r-1}^{p,q}}^{E_{r-1}^{p,q}(X)} \subset \cdots .$$

Therefore it is evident that for any $r \in \mathbb{N}$ and any $p, q \in \{0, \cdots, n\}$ we have the following **complex** of $\mathbb{C}$-vector spaces

$$\cdots \xrightarrow{d_r} E_r^{p,q}(X) \xrightarrow{d_r} E_r^{p+r,\, q-r+1}(X) \xrightarrow{d_r} \cdots ,$$

where the differentials d_r are given by

$$d_r : E_r^{p,q}(X) \to E_r^{p+r,\, q-r+1}(X)$$
$$d_r\big(\{\alpha\}_{E_r}\big) = (-1)^{r-1}\{\partial u_{r-1}\}_{E_r},$$

for any choice of E_r-closed representatives $\alpha \in C^\infty_{p,q}(X, \mathbb{C})$ and any choice of $\bar\partial$-potentials $u_1, \cdots, u_{r-1}$ of $\partial\alpha$ as in (i) of Definition 1.2.1.

Definition 1.2.2 *(i) The sequence of complexes*

$$(E_r^{p,q}(X),\, d_r)$$

is called the ***Frölicher spectral sequence (FSS)*** *of the compact complex manifold* X*.*

(ii) For every $r \in \mathbb{N}$ *the family of complexes* $(E_r^{\bullet,\bullet}(X), d_r)$ *is called the* ***r-th page*** *of the Frölicher spectral sequence.*

(iii) The Frölicher spectral sequence of X *is said to* ***degenerate at*** E_r *, or* ***at the*** r***-th page****, if* $E_r^{p,q}(X) = E_{r+1}^{p,q}(X) = E_{r+2}^{p,q}(X) = \cdots$ *for all* $p, q \in \{0, \cdots, n\}$*. In this case, we write* $E_r^{p,q}(X) = E_\infty^{p,q}(X)$*.*

Remark 1.2.3 *The above definition (Definition 1.2.2) has at least two consequences which we mention in the following*

(1) The zero-th page of the FSS is the Dolbeault complex of X and the 1st page of the FSS is the Dolbeault cohomology groups of X. Therefore the maps d_1 are defined by ∂ in the Dolbeault cohomology :

$$E_1^{p,\,q}(X) = H_{\bar\partial}^{p,\,q}(X,\,\mathbb{C}) \xrightarrow{d_1} E_1^{p,\,q}(X) = H_{\bar\partial}^{p+1,\,q}(X,\,\mathbb{C}).$$

(2) For every r, the $(r+1)$-th page is the cohomology of the previous page r-th page.

In the following, we recall a proposition that helps us to prove that the Iwasawa manifold (cf. Definition 1.6.9) is not a Kähler manifold.

Proposition 1.2.4 *Suppose that the Frölicher spectral sequence of X degenerates at E_1. Then for every $p \in \{1, \cdots, n\}$, every holomorphic p-form on X is d-**closed**. By a holomorphic p-form on X we mean a $(p,\,0)$-form α such that $\bar\partial\alpha = 0$.*

Proof. Let α be a holomorhic p-form. Since $\bar\partial\alpha = 0$, $\{\alpha\}_{\bar\partial}$ is well-defined and $\{\alpha\}_{\bar\partial} \in H_{\bar\partial}^{p,\,0}(X,\,\mathbb{C}) = E_1^{p,\,0}(X)$. On the other hand $E_1(X) = E_\infty(X)$ means that all the maps d_1 vanish identically. Hence $d_1(\{\alpha\}_{\bar\partial}) = \{\partial\alpha\}_{\bar\partial} = 0$, which means that $(p+1,\,0)$-form $\partial\alpha$ is $\bar\partial$-exact. Hence $\partial\alpha = \bar\partial\beta$ for some $(p+1,\,-1)$-form β. For bidegree reasons $\beta = 0$ hence $\partial\alpha = 0$, so $d\alpha = 0$. $\square$

Proposition 1.2.5 *Let $r \in \mathbb{N}^\star$. The Frölicher spectral sequence of X degenerates at E_r if and only if there exists a **non-necessarily canonical isomorphism***

$$H_{DR}^k(X,\,\mathbb{C}) \simeq \bigoplus_{p+q=k} E_\infty^{p,\,q}(X),$$

for every $k \in \{0, \cdots, 2n\}$. In particular we have

$$b_k = \bigoplus_{p+q=k} e_r^{p,\,q},$$

*where $b_k = \dim H_{DR}^k(X,\,\mathbb{C})$ are the **Betti numbers** and $e_r^{p,\,q} = \dim E_r^{p,\,q}(X) = \dim E_\infty^{p,\,q}(X)$.*

1.3 $\partial\bar\partial$-Manifolds

In this section we mention the definition of $\partial\bar\partial$-manifolds and recall the $\partial\bar\partial$-Lemma which plays an essential role in the sequel and prove that every Kähler manifold is a $\partial\bar\partial$-manifold. Moreover, we recall equivalences between the $\partial\bar\partial$-property and the canonical isomorphisms between the various cohomologies.

Definition 1.3.1 *Compact complex manifold X is called a $\partial\bar\partial$-**manifold** if for any d-closed pure-type $(p,\,q)$-form u, the following exactness properties are equivalent :*

$$u \textit{ is } d\textit{-exact} \Leftrightarrow u \textit{ is } \partial\textit{-exact} \Leftrightarrow u \textit{ is } \bar\partial\textit{-exact} \Leftrightarrow u \textit{ is } \partial\bar\partial\textit{-exact.}$$

If X is a $\partial\bar\partial$-manifold then X satisfies some properties in cohomological level. For example if X is a $\partial\bar\partial$-manifold then there exists a canonical isomorphisms between the Bott-Chern and the Aeppli cohomology groups mentioned in Definition 1.1.8.

Theorem 1.3.2 *The following statements are equivalent*

(i) X *is a* $\partial\bar{\partial}$*-manifold.*

*(ii) The **canonical** map*

$$\begin{aligned} &H^{p,\,q}_{BC}(X,\,\mathbb{C}) \to H^{p,\,q}_{A}(X,\,\mathbb{C}) \\ &\{\alpha\}_{BC} \to \{\alpha\}_{A} \qquad \alpha \in C^{\infty}_{p,\,q}(X,\,\mathbb{C}) \end{aligned}$$

*is **bijective**.*

Sketch of proof. First note that the canonical map being injective is clearly equivalent to property

$$\ker\partial \cap \ker\bar{\partial} \cap (\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}) = \mathrm{Im}(\partial\bar{\partial}). \tag{1.21}$$

Indeed suppose that the above property satisfies for every $(p,\,q)$-forms. Let α be a fixed closed $(p,\,q)$-form. then for some forms β and γ, we have

$$\{\alpha\}_A = 0 \Longleftrightarrow \bar{\partial}\partial\alpha = 0 \,\text{and}\, \alpha = \partial\beta + \bar{\partial}\gamma \Longrightarrow \partial\alpha = \bar{\partial}\alpha = 0 \,\text{and}\, \alpha \in \mathrm{Im}(\partial\bar{\partial}) \Longrightarrow \{\alpha\}_{BC} = 0.$$

This implies the injectivity. Now suppose that the map $\{\alpha\}_{BC} \to \{\alpha\}_A$ is injective then,

$$\alpha \in \ker\partial \cap \ker\bar{\partial} \cap (\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}) \Longrightarrow \{\alpha\}_A = 0 \overset{(1.21)}{\Longrightarrow} \{\alpha\}_{BC} = 0 \Longrightarrow \alpha \in \mathrm{Im}(\partial\bar{\partial}).$$

The other direction in obvious. Meanwhile the canonical map being surjective is clearly equivalent to property

$$\mathrm{Im}\partial + \mathrm{Im}\bar{\partial} + (\ker\partial \cap \ker\bar{\partial}) = \ker\partial\bar{\partial}, \tag{1.22}$$

Indeed, let the above property is true for all $(p,\,q)$-forms, and let $\{\alpha\}_A \in H^{p,\,q}_A(X,\,\mathbb{C})$ has been chosen. Then $\partial\bar{\partial}\alpha = 0$, so by (1.22) there exist three forms β, γ and ζ such that

$$\begin{aligned} &\alpha = \partial\beta + \bar{\partial}\gamma + \zeta \\ &\partial\zeta = \bar{\partial}\zeta = 0 \end{aligned}$$

So, the class $\{\zeta\}_{BC}$ is well-defined and its image under the canonical map is coincided with $\{\alpha\}_A$. Suppose that the canonical map $\{\alpha\}_{BC} \to \{\alpha\}_A$ is surjective. Let $\alpha \in \ker\partial\bar{\partial}$ has been chosen, this means that $\{\alpha\}_A$ is well-defined, by surjectivity there exists a $(p,\,q)$-form β such that the class $\{\beta\}_{BC}$ is well-defined and the image $\{\beta\}_{BC}$ under the canonical map is $\{\alpha\}_A$, in other words $\{\alpha-\beta\}_A = 0$. This implies that there are two forms ζ and γ such that

$$\alpha = \beta + \bar{\partial}\gamma + \partial\zeta,$$

which spells that $\alpha \in \mathrm{Im}\partial + \mathrm{Im}\bar{\partial} + (\ker\partial \cap \ker\bar{\partial})$. The other direction is obvious. Finally it is clear that, X being a $\partial\bar{\partial}$-manifold is equivalent to the simultaneous occurrence of (1.21) and (1.22). □

Therefore if X is a $\partial\bar{\partial}$-manifold, by the above theorem and Theorem 1.1.10 we get

$$\begin{cases} h^{p,\,q}_{BC} = h^{p,\,q}_A,\; h^k_{BC} = h^k_A \qquad \forall p, q \in \{0, \cdots, n\}\; k \in \{0, \cdots, 2n\}. \\ h^k_{BC} \geqslant b_k \qquad \forall k \in \{0, \cdots, 2n\}. \end{cases}$$

In the following we prove another cohomological property of a $\partial\bar{\partial}$-manifold called the **Hodge decomposition**.

Theorem 1.3.3 *A compact complex n-dimensional manifold X is $\partial\bar\partial$-manifold if and only if the identity* ***induces an isomorphism*** *between $\bigoplus_{p+q=k} H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ and $H^k_{DR}(X,\,\mathbb{C})$ for every $k \in \{0, \cdots, 2n\}$, in the following sense :*

(a) for every bidegree $(p,\,q)$ with $p+q=k$, every Dolbeault cohomology class $\{\alpha^{p,\,q}\}_{\bar\partial} \in H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C})$ contains a ***d-closed representative*** *$\alpha^{p,\,q}$.*

(b) the linear map

$$\bigoplus_{p+q=k} H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) \ni \sum_{p+q=k}\{\alpha^{p,\,q}\}_{\bar\partial} \mapsto \{\sum_{p+q=k}\alpha^{p,\,q}\}_{DR} \in H^k_{DR}(X,\,\mathbb{C}) \tag{1.23}$$

is ***well-defined*** *by means of d-closed representatives (meaning that it does not depend on the d-closed representative $\alpha^{p,\,q}$ of the Dolbeault class $\{\alpha^{p,\,q}\}_{\overline\partial}$) and* ***bijective****.*

The above latter property of manifolds has a name :

Definition 1.3.4 *If the identity induces an isomorphism $\bigoplus_{p+q=k} H^{p,\,q}_{\bar\partial}(X,\,\mathbb{C}) \simeq H^k_{DR}(X,\,\mathbb{C})$ in the sense of Theorem 1.3.3 for every $k \in \{0, \cdots, 2n\}$, we say that the manifold X has the* ***Hodge Decomposition*** *property.*

Proof of Theorem 1.3.3. Suppose that X is a $\partial\bar\partial$-manifold.

" $\implies$" Suppose that X is a $\partial\bar\partial$-manifold.

(a) Let $\{\alpha^{p,\,q}\}_{\overline\partial} \in H^{p,\,q}_{\overline\partial}(X,\,\mathbb{C})$ be an arbitrary class and let $\alpha^{p,\,q}$, be an arbitrary representative of it, then we have $\overline\partial\alpha^{p,\,q} = 0$. Since the $(p+1,\,q)$-form $\partial\alpha^{p,\,q}$ is d-closed and ∂-exact, by $\partial\bar\partial$-assumption on X, $\partial\alpha^{p,\,q}$ is $\partial\bar\partial$-exact, so $\partial\alpha^{p,\,q} = \partial\overline\partial\beta$. Now the $(p,\,q)$-form $\alpha^{p,\,q} - \overline\partial\beta$ is the d-closed representative $\alpha^{p,\,q}$.

(b) To prove well-definedness, fix a bidegree $(p,\,q)$ and let $\alpha^{p,\,q}_1$ and $\alpha^{p,\,q}_2$ be d-closed representatives of the same Dolbeault cohomology class. This means that $\alpha^{p,\,q}_1 - \alpha^{p,\,q}_2$ is $\overline\partial$-exact and d-closed. Since it is pure type by the $\partial\overline\partial$-assumption on X it must be d-exact. Thus $\{\alpha^{p,\,q}_1\}_{DR} = \{\alpha^{p,\,q}_2\}_{DR}$.

We will now prove that, for every bidegree $(p,\,q)$ with $p+q=k$, the identity induces an *injection*

$$\begin{aligned} H^{p,\,q}_{\overline\partial}(X,\,\mathbb{C}) &\hookrightarrow H^k_{DR}(X,\,\mathbb{C}) \\ \{\alpha^{p,\,q}\}_{\overline\partial} &\mapsto \{\alpha^{p,\,q}\}_{DR}, \end{aligned}$$

by means of d-closed representatives $\alpha^{p,\,q}$ of their respective Dolbeault cohomology classes. Let $\{\alpha^{p,\,q}\}_{DR} = 0$, this means that $\alpha^{p,\,q}$ is d-exact, since it is d-closed and of pure-type, by $\partial\overline\partial$-assumption on X it must be $\overline\partial$-exact. Thus $\{\alpha^{p,\,q}\}_{\overline\partial} = 0$, this proves the injectivity of the above map.

Now we prove that for any fixed k and any distinct bidegrees $(p,\,q) \neq (r,\,s)$ with $p+q=r+s=k$ the images of $H^{p,\,q}_{\overline\partial}(X,\,\mathbb{C})$ and $H^{r,\,s}_{\overline\partial}(X,\,\mathbb{C})$ in $H^k_{DR}(X,\,\mathbb{C})$ only meet at 0. Suppose to the contrary that there exist d-closed and non-$\overline\partial$-exact forms $\alpha^{p,\,q}$ and $\alpha^{r,\,s}$ of the shown bidegree such that $\{\alpha^{p,\,q}\}_{DR} = \{\alpha^{r,\,s}\}_{DR}$. Then $\alpha^{p,\,q} - \alpha^{r,\,s} = d\beta$ for some form β, so

$$\alpha^{p,\,q} = \partial\beta^{p-1,\,q} + \overline{\partial}\beta^{p,\,q-1} \text{ and } \alpha^{r,\,s} = \partial\beta^{r-1,\,s} + \overline{\partial}\beta^{r,\,s-1}.$$

Then $\alpha^{p,\,q} - \overline{\partial}\beta^{p,\,q-1} = \partial\beta^{p-1,\,q}$. Meanwhile the pure-type form $\partial\beta^{p-1,\,q}$ is d-closed and ∂-exact, by the $\partial\overline{\partial}$-assumption on X it must be $\overline{\partial}$-exact. This means that $\alpha^{p,\,q}$ is $\overline{\partial}$-exact which is a contradiction. So we conclude that the map (1.23) is injective.

On the other hand we know that for every compact complex manifold X (whether it is $\partial\overline{\partial}$ or not) we have

$$b_k \leqslant \sum_{p+q=k} h^{p,\,q}_{\overline{\partial}}.$$

this means that the map (1.23) is surjective.

" $\Longleftarrow$ " Suppose that X has the Hodge decomposition property.

Fix a d-closed $(p,\,q)$-form α and put $p+q=k$.

- Let us first prove the equivalence

$$\alpha \in \mathrm{Im}\overline{\partial} \Longleftrightarrow \alpha \in \mathrm{Im}d.$$

Suppose α is $\bar{\partial}$-exact, since it is of pure-type and d-closed, it is $\bar{\partial}$-closed. So $\{\alpha\}_{\overline{\partial}} = 0$, and this implies that $\{\alpha\}_{DR} = 0$ (because the image of 0 under a linear map is 0). If α is d-exact, then $\{\alpha\}_{DR} = 0$. Since the identity induces a linear injection we have $\{\alpha\}_{\overline{\partial}} = 0$, meaning that α is $\overline{\partial}$-exact. This proves the equivalence. Since the above equivalence has been proved in every bidegree, by conjugation we also get the equivalence

$$\alpha \in \mathrm{Im}\partial \Longleftrightarrow \alpha \in \mathrm{Im}d.$$

- For the proof of the following equivalence

$$\alpha \in \mathrm{Im}(\partial\overline{\partial}) \Longleftrightarrow \alpha \in \mathrm{Im}\overline{\partial},$$

we refer the reader to Proposition (5.17) of [11].

□

One of the immediate consequences of Theorem 1.3.3 is that when X is a $\partial\bar{\partial}$-manifold the Frölicher spectral sequence of X degenerates at E_1 by Proposition 1.2.5.

Remark 1.3.5 *Suppose that the Frölicher spectral sequence of X degenerates at E_1, then we get the isomorphism*

$$H^k_{DR}(X,\,\mathbb{C}) \simeq \bigoplus_{p+q=k} E^{p,\,q}_{\infty}(X),$$

by Proposition 1.2.5. But the main difference between the above isomorphism and the isomorphism in Theorem 1.3.3 is that the first one in ***not canonical*** *in* ***general*** *but the later one is* ***canonical****. Therefore the $\partial\bar{\partial}$-assumption on X satisfies $E_1(X) = E_\infty(X)$.*

Moreover if X is a $\partial\overline{\partial}$-manifold, the canonical map

$$\begin{aligned}&H^{p,\,q}_{BC}(X,\,\mathbb{C}) \to H^{p,\,q}_{\overline{\partial}}(X,\,\mathbb{C})\\ &\{\alpha\}_{BC} \to \{\alpha\}_{\overline{\partial}} \qquad\quad \alpha\in C^{\infty}_{p,\,q}(X,\,\mathbb{C})\end{aligned}$$

is **injective**. Indeed if α is a fixed $(p,\,q)$-form, $\{\alpha\}_{\overline{\partial}}=0$ implies that $\{\alpha\}_A=0$. Hence by Theorem 1.3.2 we get $\{\alpha\}_{BC}=0$.

In other word in the level of the Hodge numbers we have $h^k_{BC}\leqslant h^k_{\overline{\partial}}$, for all $k\in\{0,\cdots,2n\}$. Therefore by Theorem 1.1.10 we get

$$\text{If } X \text{ is a } \boldsymbol{\partial\bar{\partial}}\textbf{-manifold} \implies h^k_{BC}=h^k_A=h^k_{\overline{\partial}}=b_k.$$

Recall that in Theorem 1.1.10 we saw that for every $k\in\mathbb{N}$, the following inequality holds :

$$h^k_{BC}+h^k_A \;\geq\; 2\,b_k\;. \tag{1.24}$$

In the following theorem we show that when X is a $\partial\bar{\partial}$-manifold we get the equality in (1.24).

Theorem 1.3.6 *The equality*

$$h^k_{BC}+h^k_A \;=\; 2\,b_k$$

in (1.24) holds for every $k\in\mathbb{N}$ if and only if X satisfies the $\partial\overline{\partial}$-Lemma.

Proof. Obviously, if X satisfies the $\partial\overline{\partial}$-Lemma, then, for every $k\in\mathbb{N}$, one has

$$h^k_{BC} \;=\; h^k_A \;=\; h^k_{\overline{\partial}} \;=\; b_k$$

and hence, in particular,

$$h^k_{BC}+h^k_A \;=\; 2\,b_k\;.$$

We split the proof of the converse into the following claims.

`Claim 1.` *If $h^k_{BC}+h^k_A\;=2\,b_k$ holds for every $k\in\mathbb{N}$, then $E_1\simeq E_\infty$ and $a^k\;=0=\;f^k$ for every $k\in\mathbb{N}$.*
Since, for every $k\in\mathbb{N}$, we have

$$2\,b_k \;=\; h^k_{BC}+h^k_A \;=\; 2\,h^k_{\overline{\partial}}+a^k+f^k \;\geq\; 2\,b_k\;,$$

then $h^k_{\overline{\partial}}=b_k$ and $a^k=0=f^k$ for every $k\in\mathbb{N}$.

`Claim 2.` *Fix $k\in\mathbb{N}$. If $a^{k+1}:=\sum_{p+q=k+1}\dim_{\mathbb{C}}A^{p,q}=0$, then the natural map*

$$\bigoplus_{p+q=k} H^{p,q}_{BC}(X,\,\mathbb{C})\to H^k_{dR}(X,\,\mathbb{C})$$

is surjective.
Let $\mathfrak{a} = [\alpha] \in H^k_{dR}(X, \mathbb{C})$. We have to prove that $\mathfrak{a}$ admits a representative whose pure-type components are d-closed. Consider the pure-type decomposition of α :

$$\alpha \;=:\; \sum_{j=0}^{k} (-1)^j \; \alpha^{k-j,j} \;,$$

where $\alpha^{k-j,j} \in \wedge^{k-j,j} X$. Since $\mathrm{d}\,\alpha = 0$, we get that

$$\partial\alpha^{k,0} = 0 \;, \qquad \overline{\partial}\alpha^{k-j,j} - \partial\alpha^{k-j-1,j+1} = 0 \text{ for } j \in \{0, \ldots, k-1\} \;, \qquad \overline{\partial}\alpha^{0.k} = 0 \;;$$

by the hypothesis $a^{k+1} = 0$, for every $j \in \{0, \ldots, k-1\}$, we get that,

$$\overline{\partial}\alpha^{k-j,j} \;=\; \partial\alpha^{k-j-1,j+1} \;\in\; \left(\operatorname{im}\overline{\partial} \cap \operatorname{im}\partial\right) \cap \wedge^{k-j,j+1} X \;=\; \operatorname{im}\partial\overline{\partial} \cap \wedge^{k-j,j+1} X$$

and hence there exists $\eta^{k-j-1,j} \in \wedge^{k-j-1,j} X$ such that

$$\overline{\partial}\alpha^{k-j,j} \;=\; \partial\overline{\partial}\eta^{k-j-1,j} \;=\; \partial\alpha^{k-j-1,j+1} \;.$$

Define

$$\eta \;:=\; \sum_{j=0}^{k-1} (-1)^j \; \eta^{k-j-1,j} \;\in\; \wedge^{k-1}(X, \mathbb{C}) \;.$$

The claim follows noting that

$$\begin{aligned}
\mathfrak{a} \;&=\; [\alpha] \;=\; [\alpha + \mathrm{d}\,\eta] \\
&=\; \left[\left(\alpha^{k,0} + \partial\eta^{k-1,0}\right) + \sum_{j=1}^{k-1} (-1)^j \; \left(\alpha^{k-j,j} + \partial\eta^{k-j-1,j} - \overline{\partial}\eta^{k-j,j-1}\right)\right. \\
&\qquad \left. + \,(-1)^k \; \left(\alpha^{0,k} - \overline{\partial}\eta^{0,k-1}\right)\right] \\
&=\; \left[\alpha^{k,0} + \partial\eta^{k-1,0}\right] + \sum_{j=1}^{k-1} (-1)^j \; \left[\alpha^{k-j,j} + \partial\eta^{k-j-1,j} - \overline{\partial}\eta^{k-j,j-1}\right] \\
&\qquad + (-1)^k \; \left[\alpha^{0,k} - \overline{\partial}\eta^{0,k-1}\right] \;,
\end{aligned}$$

that is, each of the pure-type components of $\alpha + \mathrm{d}\,\eta$ is both ∂-closed and $\overline{\partial}$-closed.

`Claim 3.` *If $h^k_{BC} \geq b_k$ and $h^k_{BC} + h^k_A = 2\, b_k$ for every $k \in \mathbb{N}$, then $h^k_{BC} = b_k$ for every $k \in \mathbb{N}$.*
If n is the complex dimension of X, then, for every $k \in \mathbb{N}$, we have

$$b_k \;\leq\; h^k_{BC} \;=\; h^{2n-k}_A \;=\; 2\, b_{2n-k} - h^{2n-k}_{BC} \;\leq\; b_{2n-k} \;=\; b_k$$

and hence $h^k_{BC} = b_k$ for every $k \in \mathbb{N}$.

Now, by `Claim 1`, we get that $a^k = 0$ for each $k \in \mathbb{N}$; hence, by `Claim 2`, for every $k \in \mathbb{N}$ the map

$$\bigoplus_{p+q=k} H^{p,q}_{BC}(X) \to H^k_{dR}(X, \mathbb{C})$$

is surjective and hence, in particular, $h^k_{BC} \geq b_k$. By `Claim 3` we get therefore that $h^k_{BC} = b_k$ for every $k \in \mathbb{N}$. Hence, the canonical map $H^{\bullet,\bullet}_{BC}(X) \to H^\bullet_{dR}(X, \mathbb{C})$ is actually an isomorphism, which is equivalent to say that X satisfies the $\partial\overline{\partial}$-Lemma. □

In the following we prove the fundamental fact that underlies the theory $\partial\bar{\partial}$-manifolds.

Theorem 1.3.7 *($\partial\overline{\partial}$-lemma) Every compact Kähler manifold X is a $\partial\overline{\partial}$-manifold.*

Sketch of proof. Suppose that (X, ω) is a compact Hermitian manifold and fix a (p, q)-form $u \in C^\infty_{p,\,q}(X, \mathbb{C})$. The following implications always hold for every compact Hermitian manifold.

$$u \in \mathrm{Im}(\bar{\partial}\partial) \Longrightarrow u \in \mathrm{Im}\partial \quad \text{and} \quad u \in \mathrm{Im}\bar{\partial} \quad \text{and} \quad u \in \mathrm{Im}d,$$

for the last note that $u = \partial\overline{\partial}v$, then $u = d(\overline{\partial}v)$. Moreover, the following equivalences always hold for every compact Hermitian manifold.

$$\begin{aligned} u \in \mathrm{Im}\partial &\Longrightarrow u \quad \perp \quad \mathcal{H}^{p,\,q}_{\Delta_\partial}(X, \mathbb{C}) \\ u \in \mathrm{Im}\bar{\partial} &\Longrightarrow u \quad \perp \quad \mathcal{H}^{p,\,q}_{\Delta_{\bar{\partial}}}(X, \mathbb{C}) \\ u \in \mathrm{Im}d &\Longrightarrow u \quad \perp \quad \mathcal{H}^{k}_{\Delta}(X, \mathbb{C}) \end{aligned}$$

Now suppose that ω is Kähler, by Corollary 6.5 of [10] we have

$$\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta. \tag{1.25}$$

So the above equivalences satisfy the following equivalences

$$u \in \mathrm{Im}\partial \Longleftrightarrow u \in \mathrm{Im}\bar{\partial} \Longleftrightarrow u \in \mathrm{Im}d,$$

so it remains to prove the implication :

$$u \in \mathrm{Im}\partial \Longleftrightarrow u \in \mathrm{Im}(\bar{\partial}\partial). \tag{1.26}$$

To prove this implication suppose that $u = \bar{\partial}v$ for some $(p,\, q-1)$ form v. by L^2_ω-orthogonal 3-space decomposition

$$C^\infty_{p,\,q-1} = \ker \Delta_\partial \oplus \mathrm{Im}\partial \oplus \mathrm{Im}\partial^\star,$$

v splits uniquely as $v = w + \partial\alpha + \partial^\star\beta$, with $w \in \ker \Delta_\partial$. Therefore by (1.25) we get :

$$u = \bar{\partial}v = \bar{\partial}w + \bar{\partial}\partial\alpha + \bar{\partial}\partial^\star\beta = \bar{\partial}\partial\alpha + \bar{\partial}\partial^\star\beta = \bar{\partial}\partial\alpha - \partial^\star\bar{\partial}\beta.$$

where the last equality comes from $[\bar{\partial},\, \bar{\partial}^\star] = 0$ when ω is Kähler. Therefore we have :

$$\mathrm{Im}\bar{\partial} \ni u - \bar{\partial}\partial\alpha = -\partial^\star\bar{\partial}\beta \in \mathrm{Im}\bar{\partial}^\star.$$

Which means that, $u - \bar{\partial}\partial\alpha = 0$. □

Up to now we have investigated some cohomological properties of $\partial\overline{\partial}$-manifolds. To give a brief statement of main results of this section we can mention to the following diagram

$$
\begin{tikzcd}
 & X \text{ is Kähler} \arrow[d, Rightarrow] & \\
H^{p,q}_{BC}(X,\mathbb{C}) \overset{\text{canonical}}{\simeq} H^{p,q}_{A}(X,\mathbb{C}) \arrow[r, Leftrightarrow] & X \text{ is } \partial\bar{\partial} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(X,\mathbb{C}) \simeq H^k_{DR}(X,\mathbb{C}) \\
 & E_1(X) = E_\infty(X) &
\end{tikzcd}
$$

In [7], the author constructed a 3-dimensional nilmanifold which is not a complex tori, such that its Frölicher spectral sequence degenerates at E_1. Since nilmanifolds does not satisfy $\partial\bar{\partial}$-lemma unless they are complex tori then In the above diagram the converse direction of the vertical implications fails to be true in general.

1.4 Positivity Notion for Differential Forms

Let V be a complex vector space of dimension n and $(z_1, \dots, z_n)$ be a coordinate on V. We denote the corresponding basis of V by $(\partial/\partial z_1, \dots, \partial/\partial z_n)$ and its dual basis in V^* by $(dz_1, \dots, dz_n)$. Consider the exterior algebra

$$\Lambda V^*_{\mathbb{C}} = \bigoplus \Lambda^{p,q} V^*, \quad \Lambda^{p,q} V^* = \Lambda^p V^* \otimes \Lambda^q \overline{V^*}.$$

Since V is a complex vector space, it has a canonical orientation, given by the (n, n)-form

$$\tau(z) = idz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge idz_n \wedge d\bar{z}_n = 2^n dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

where $z_j = x_j + iy_j$. In fact, if $(w_1, \dots, w_n)$ are the other coordinates, we find

$$dw_1 \wedge \cdots \wedge dw_n = \det(\partial w_j/\partial z_k) dz_1 \wedge \cdots \wedge dz_n,$$

$$\tau(w) = |\det(\partial w_j/\partial z_k)|^2 \tau(z).$$

So one can define the notion of positivity independent of local coordinates.

Definition 1.4.1 *(1) A (q, q)-form $v \in \Lambda^{q,q} V^*$ is said to be* ***strongly semi-positive*** *(resp.* ***strongly strictly positive****) if v is a convex combination*

$$v = \sum \gamma_s i\alpha_{s,1} \wedge \bar{\alpha}_{s,1} \wedge \cdots \wedge i\alpha_{s,q} \wedge \bar{\alpha}_{s,q}$$

where $\alpha_{j,s} \in V^*$ and $\gamma_s \geq 0$ (resp. $\gamma_s > 0$).

(2) A (p, p)-form $u \in \Lambda^{p,p} V^*$ is said to be ***weakly semi-positive***[1] *(resp.* ***weakly strictly positive****) if for all $\alpha_j \in V^*$, $1 \leq j \leq q = n - p$, then*

$$u \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_q \wedge \bar{\alpha}_q \geq 0 \quad (\textit{resp.} \quad u \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_q \wedge \bar{\alpha}_q > 0 \qquad \forall \alpha_j \neq 0)$$

1. In the original definition in [10], the author use semi-positive. We added the have word " weakly " to distinguish better from the " strongly positive " notion.

Remark 1.4.2 *Locally any Hermitian metric ω is a strongly strictly positive $(1,\, 1)$-form, this means that at any point $x \in X$, we can choose a local holomorphic coordinate $(z_1, \ldots, z_n)$ and an open set $x \in U \subset X$, such that ω has the following representation*

$$\omega = \sum idz_i \wedge d\bar{z}_i.$$

Fortunately, the concepts of weakly semi-positive (resp. weakly strictly positive) and strongly semi-positive (resp. strongly strictly positive) coincide in bidegree $(1, 1)$ and $(n-1, n-1)$.

Proposition 1.4.3 *([10], Chapter III, Proposition 1.11) If $u_1, \ldots, u_s$ are strongly semi-positive (resp. strongly strictly positive) forms, then $u_1 \wedge \cdots \wedge u_s$ is also strongly semi-positive (resp. strongly strictly positive) form.*

1.5 Currents

Definition 1.5.1 (1) *Suppose that K is a compact subset of X. For any $s \in \mathbb{N} \cup \{\infty\}$ and any $0 \leq p,\, q \leq n$, let ${}^s\mathcal{D}^{p,\,q}(K)$ be the set of all $u \in C^s_{p,\,q}(X,\, \mathbb{C})$ with support contained in K, and ${}^s\mathcal{D}^{p,\,q}(X) := \bigcup_K {}^s\mathcal{D}^{p,\,q}(K)$, where the union is taken over all compact subsets of X.*

(2) *A* ***Current*** *T of degree $(n-p,\, n-q)$ and of order s is a linear form on ${}^s\mathcal{D}^{p,\,q}(X)$ such that the restriction of T to any subspace ${}^s\mathcal{D}^{p,\,q}(K)$ is continuous.*

Note that if X is a compact manifold then ${}^s\mathcal{D}^{p,\,q}(X) = C^s_{p,\,q}(X,\, \mathbb{C})$. In the sequel, we let $\langle T,\, u\rangle$ be the pairing between a current T and a test form $u \in {}^s\mathcal{D}^{p,\,q}(X)$

Example 1.5.2 *([10], Chapter I, Example 2.5) If u is a differential $(p,\, q)$-form on X, we can associate to u the current of dimension $(n-p,\, n-q)$:*

$$\langle T_u, v\rangle = \int_X T \wedge v = \int_X u \wedge v, \qquad v \in {}^s\mathcal{D}^{n-p,\, n-q}(X).$$

T_u is of degree $(p,\, q)$ and of order 0. The correspondence $u \longmapsto T_u$ is injective.

Definition 1.5.3 *([10], Chapter III, Definition 1.13) A current T of degree $(n-p,\, n-p)$ is said to be positive (resp. strongly positive) if for all test forms $u \in {}^s\mathcal{D}^{p,\,q}(X)$ that are strongly positive (resp. weakly positive) at each point we have, $\langle T, u\rangle \geq 0$.*

1.6 Holomorphic deformations of Complex Structures

Let $\mathcal{X}$ and B be two complex manifolds and $\varphi : \mathcal{X} \to B$ be a holomorphic map. Also let $X_t = \varphi^{-1}(t)$ denote the fiber of φ above the point $t \in B$. We say that $\mathcal{X} \xrightarrow{\varphi} B$ is a **holomorphic family of compact complex manifolds** if φ is a proper holomorphic submersion. Thus for every $t \in B$, X_t is a compact complex manifold.

Theorem 1.6.1 *([43], Theorem 9.3) Ehresmann Theorem*
Suppose that $\mathcal{X} \overset{\varphi}{\to} B$ is a holomorphic family of compact complex manifolds. If B is a contractible manifold equipped with a base point 0, then there exists a diffeomorphism

$$T : \mathcal{X} \cong X_0 \times B.$$

Given such a diffeomorphism, we can assume that for every $t \in B$, X_t is diffeomorphic to X_0. In other words, the differential structure of X_0 does not depend on t. In general since T is not a holomorphic map, the complex structure of X_t depends on t. However, the existence of such T enables us to consider the complex structure on X_t as a complex structure on X_0 varying with t. From now on B is an open disc around the origin in $\mathbb{C}^m$, for some $m \in \mathbb{N}$ and we denote $(X_t)_{t\in B}$ instead of $\mathcal{X} \overset{\varphi}{\to} B$ for referring to a holomorphic family of compact complex manifolds.

The following statement contains the definition of C^∞ family of complex vector bundles, sections, linear operators and metrics. First we fix notation for the next definition.

Suppose Y is a complex bundle over X. By $\Gamma(Y) = C^\infty(X, Y)$ we mean the $\mathbb{C}$-vector space of global C^∞ sections of Y.

Definition 1.6.2 *(i) Suppose $(Y_t)_{t\in B}$, $Y_t \longrightarrow X$, is a family of complex bundles over X. We say that the family $(Y_t)_{t\in B}$ is a C^∞* ***family of*** *C^∞* ***complex vector bundles*** *over X if there is a complex bundle $\pi : \mathcal{Y} \longrightarrow X \times B$ such that*

$$Y_t = \pi^{-1}(X \times \{t\}) = \mathcal{Y}|_{X\times\{t\}}, \qquad t \in B.$$

(ii) Suppose $(Y_t)_{t\in B}$ is a C^∞ family of C^∞ complex vector bundles over X.

(1) For every $t \in B$, let $\gamma_t \in \Gamma(Y_t)$. We say that the family $(\gamma_t)_{t\in B}$ is a C^∞ ***family of sections*** *if there is a $\hat{\gamma} \in \Gamma(\mathcal{Y}) = C^\infty(X \times B, \mathcal{Y})$ such that*

$$\gamma_t = \hat{\gamma}|_{X\times\{t\}}.$$

(2) For every $t \in B$, suppose $L_t : \Gamma(Y_t) \to \Gamma(Y_t)$ is a self-adjoint elliptic operator of even order s on $\Gamma(Y_t)$, we say that the family $(L_t)_{t\in B}$ is a C^∞ ***family of linear operators*** *if for every C^∞ family of sections $(\gamma_t)_{t\in B}$, $(L_t\gamma_t)_{t\in B}$ is a C^∞ family of sections.*

In (i), we implicitly assumed that on every t there exists a Hermitian metric h_t on Y_t, i.e., the $(h_t)_{t\in B}$ is a family of positive definite inner products on $(Y_t)_{t\in B}$ such that for every $t \in B$, h_t varies C^∞ with $y \in Y_t$.

(iii) We say that the family $(h_t)_{t\in B}$ is a C^∞ ***family of metrics*** *if there exists a Hermitian metric* $\mathbf{h}$ *on $\mathcal{Y}$ such that*

$$\mathbf{h}|_{X\times\{t\}} = h_t.$$

We recall the Green operator of a self-adjoint elliptic operator. For every fixed $p, q \in \{1, \ldots, n\}$ suppose E is a self-adjoint elliptic operator on $C^\infty_{p,q}(X, \mathbb{C})$. Since X is a compact manifold, $\ker E$ is a finite-dimensional complex vector space. We denote by $P_E : C^{\infty}_{p,q}(X, \mathbb{C}) \to \ker E$ the L^2_ω orthogonal projection. One can define the **Green operator** of E, $E^{-1} : C^{\infty}_{p,q}(X, \mathbb{C}) \longrightarrow \operatorname{Im} E$, such that

$$E^{-1}E(\gamma) = EE^{-1}(\gamma) = \gamma - P_E(\gamma), \qquad \gamma \in C^{\infty}_{p,q}(X, \mathbb{C}).$$

If we restrict E to $\operatorname{Im} E$ then E is a bijection and so $E^{-1} : \operatorname{Im} E \to \operatorname{Im} E$ is the inverse of this restriction. In particular one can define, P_{BC}, $P_{\bar\partial}$, $\Delta^{-1}_{\mathrm{BC}}$, and $\Delta^{-1}_{\bar\partial}$.

The following theorem gives us a criteria to determine whether these families are C^∞ families of linear operators. This is the main key to proving Theorem 2.3.1.

Theorem 1.6.3 *([24]) Kodaira-Spencer fundamental theorem.*
Suppose that $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds.

(i) If the dim $\ker \Delta_{BC,t} : C^{\infty}_{p,q}(X_t, \mathbb{C}) \to C^{\infty}_{p,q}(X_t, \mathbb{C})$ (resp. dim $\ker \Delta_{\bar\partial,t}$) is independent of $t \in B$, then the family $(P_{BC,t})_{t\in B}$ (resp. $(P_{\bar\partial,t})_{t\in B}$) is a C^∞ family of linear operators.

(ii) If the dim $\ker \Delta_{BC,t} : C^{\infty}_{p,q}(X_t, \mathbb{C}) \to C^{\infty}_{p,q}(X_t, \mathbb{C})$ (resp. dim $\ker \Delta_{\bar\partial,t}$) is independent of $t \in B$, then the family $(\Delta^{-1}_{BC,t})_{t\in B}$ (resp. $(\Delta^{-1}_{\bar\partial,t})_{t\in B}$) is a C^∞ family of linear operators.

Two important and commonly used concepts in deformation theory are openness and closedness properties. The following terminology was introduced by D. Popovici in [35].

Definition 1.6.4 *([35]) Suppose $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds.*

*(i) A given property (P) of a compact complex manifold is said to be **open** under holomorphic deformations if*

$$X_0 \ \textit{ has property } \ (P) \Rightarrow X_t \ \textit{ has property } \ (P), \ \textit{ for } \ t \in B \ \textit{ sufficiently close to } 0.$$

*(ii) A given property (P) of a compact complex manifold is said to be **closed** under holomorphic deformations if*

$$X_t \ \textit{ has property } \ (P) \ \textit{ for } \ t \in B \setminus \{0\} \ \textit{ sufficiently close to } 0 \Rightarrow X_0 \ \textit{ has property } \ (P).$$

The following table shows openness and closedness properties for some specific classes of metrics mentioned in Definition 1.1.1.

Metrics	**Open**	**closed**
Kähler	✓	×
SKT	×	×
Balanced	×	×
H-s	✓	×
lcK	×	×
$\partial\bar\partial$	✓	×

In the remaining of this section we show that the $\partial\bar\partial$ and the Kähler properties are open properties and give two counter-examples to describe why SKT and balanced properties are not open under holomorphic deformations. Moreover in Theorem 4 of [4] the author proved that the lcK property is not stable under holomorphic deformation. First we mention the **upper-semicontinuity** of the **Hodge numbers** under deformations which plays an important role to prove Theorem 1.6.6.

Theorem 1.6.5 *Let $(X_t)_{t\in B}$ be a holomorphic family of compact complex manifolds and fix an arbitrary bidegree (p, q). Then the following maps are* ***upper-semicontinuous****.*

$$\begin{aligned} B \ni t &\to h^{p,\,q}(t) := \dim H^{p,\,q}_{\bar\partial}(X_t,\,\mathbb{C}) \\ B \ni t &\to h^{p,\,q}_{BC}(t) := \dim H^{p,\,q}_{BC}(X_t,\,\mathbb{C}) \\ B \ni t &\to h^{p,\,q}_{A}(t) := \dim H^{p,\,q}_{A}(X_t,\,\mathbb{C}) \end{aligned}$$

One main consequence of Theorem 1.6.5 is the **deformation openness** of the $\partial\bar\partial$-property of compact complex manifolds. This fact was first proved by Wu in [45] and was reproved by Angella and Tomassini in [3].

Theorem 1.6.6 *($\partial\bar\partial$ openness) Suppose that $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds. If the central fiber X_0 satisfies the $\partial\bar\partial$-lemma then for all t sufficiently close to 0, X_t satisfies the $\partial\bar\partial$-lemma.*

Sketch of proof. By D. Angella and A. Tomassini in [3] (also cf. Theorem 1.3.6), the $\partial\bar\partial$-assumption on X_0 is equivalent with

$$\sum_{p+q=k} (h^{p,\,q}_{BC}(0) + h^{p,\,q}_{A}(0)) = 2b_k,$$

where b_k is the k-th Betti number of X (equation (1.16)). By Theorem 1.6.5 we have :

$$h^{p,\,q}_{BC}(t) \leq h^{p,\,q}_{BC}(0) \ \text{ and } \ h^{p,\,q}_{A}(t) \leq h^{p,\,q}_{A}(0).$$

Moreover by Theorem 1.1.10 we always have :

$$\sum_{p+q=k} (h^{p,\,q}_{BC}(t) + h^{p,\,q}_{A}(t)) \geq 2b_k,$$

for every $t \in B$ and $k \in \{0,\,\cdots,2n\}$. Putting together all the above pieces of information, we get :

$$2b_k \leq \sum_{p+q=k} (h^{p,\,q}_{BC}(t) + h^{p,\,q}_{A}(t)) \leq \sum_{p+q=k} (h^{p,\,q}_{BC}(0) + h^{p,\,q}_{A}(0)) = 2b_k, \tag{1.27}$$

which is exactly what we claimed by Theorem 1.3.6. $\square$

Now we are ready to prove one of the key theorems in deformation theory which is Theorem 1.6.8 proved by Kodaira and Spencer. Let us first mention a lemma that plays a central role to prove Theorem 1.6.8.

Lemma 1.6.7 *Let ω be a Hermitian metric on X. Then*

$$\omega \text{ is Kähler} \iff \Delta_{BC}(\omega) = 0,$$

where $\Delta_{BC} : C^\infty_{1,\,1}(X,\,\mathbb{C}) \to C^\infty_{1,\,1}(X,\,\mathbb{C})$ is the Bott-Chern Laplacian induced by ω.

Proof. By (1.13) we know that $\ker \Delta_{BC} = \ker \partial \cap \ker \bar{\partial} \cap \ker(\partial\bar{\partial})^{\star}$, so the implication "$\Longleftarrow$" is obvious.

To prove the other direction of implication, suppose that ω is a Kähler metric. Then by equations (1.1) and (1.7) we get :

$$(\partial\bar{\partial})^{\star}\omega = 0 \Longleftrightarrow \star(\partial\bar{\partial})(\star\omega) = 0 \Longleftrightarrow \partial\bar{\partial}\omega_{n-1} = 0,$$

where in the last equality we have used the fact that ω is a Kähler metric and $\star$ is an isomorphism. □

Theorem 1.6.8 *([24], Theorem 15) Suppose $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds. If X_0 is a Kähler manifold, then for all t close enough to 0, X_t is again a Kähler manifold.*

Sketch of proof. Suppose that ω_0 is a Kähler metric on X. For every $t \in B$ let ω_t be the J_t-type of the 2-form ω_0, therefore family $(\omega_t)_{t\in B}$ is a C^∞ family of forms. In addition since ω is positive definite for t sufficiently close to 0, ω_t is also positive definite. So, $(\omega_t)_{t\in B}$ is a C^∞ family of Hermitian metrics for t sufficiently close to 0. Let us therefore consider the $L^2_{\omega_t}$-orthogonal projectors :

$$P_t : C^\infty_{1,1}(X, \mathbb{C}) \to \ker H^{1,1}_{BC}(X_t, \mathbb{C}), \quad t \in B.$$

The crucial piece of information that we need is $h^{p,q}_{BC}(t) = h^{p,q}_{BC}(0)$, for sufficiently small t (which is provided by (1.27)). By Theorem 1.6.3, the $(P_t)_{t\in B}$ is a C^∞ family of linear operators. Now define

$$\widetilde{\omega_t} = \frac{1}{2}(P_t\omega_t + \overline{P_t(\omega_t)}).$$

The family $(\widetilde{\omega_t})_{t\in B}$ is a C^∞ family of Kähler metrics on the respective fibers X_t, whose member for $t = 0$ is the originally given Kähler metric ω_0 on X_0. □

Note that, the proof of Theorem 1.6.8 not only shows that the Kähler property is an open property but also proves that any Kähler metric ω_0 on X can be deformed to a C^∞ family of Kähler metrics ω_t on the nearby fibers X_t, which is a stronger statement.

In the following we point out two counterFexamples that show, SKT and Balanced properties are not open under holomorphic deformations. The reader can refer to [1], [31], [8] and [15] for more details.

Balanced Counter-Example

Definition 1.6.9 *The **Iwasawa manifold** $X = G/\Gamma$, denoted sometimes by $I^{(3)}$, is the compact complex manifold of complex dimension 3 defined as the quotient of the complex Heisenberg group*

$$G := \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} ; z_1, z_2, z_3 \in \mathbb{C} \right\}$$

by its discrete subgroup $\Gamma \subset G$ of matrices with entries $z_1, z_2, z_3 \in Z[i]$.

Hence, X is a compact complex parallelisable (meaning that its holomorphic tangent bundle is trivial) nilmanifold of complex dimension 3. One finds that

$$\begin{cases} \varphi_1 := dz_1 \\ \varphi_2 := dz_2 \\ \varphi_3 := dz_3 - z_1 dz_2 \end{cases}$$

are G-left-invariant holomorphic 1-forms on G so they descend to a holomorphic 1-forms on X (denoted by the same symbols) and that the structure equations with respect to these 1-forms are

$$\begin{cases} d\varphi_1 := 0 \\ d\varphi_2 := 0 \\ d\varphi_3 := -\varphi_1 \wedge \varphi_2 \end{cases} .$$

Since φ_3 is not d-closed we get the following by Proposition 1.2.4

Proposition 1.6.10 *The Frölicher spectral sequence of the Iwasawa manifold* ***does not degenerate at*** E_1*. In particular, the Iwasawa manifold is* ***not a*** $\partial\bar{\partial}$***-manifold****, hence* ***not a Kähler manifold****.*

Proof. Since there is a non zero holomorphic 1-form φ_3 on X such that $d\varphi_3 \neq 0$, by Proposition 1.2.4, $E_1(X) \neq E_\infty(X)$. On the other hand we know that the Frölicher spectral sequence of any $\partial\bar{\partial}$-manifold degenerates at E_1, therefore the Iwasawa manifold is not a $\partial\bar{\partial}$-manifold. Finally by Theorem 1.3.7 it is not a Kähler manifold. $\square$.

Let φ_1, φ_2, φ_3 be the holomorhphic 1-forms on X. They are linearly independent at every point of X. Since φ_1 and φ_2, are d-closed while φ_3 is not d-closed the $\mathbb{C}$-vector space $H^{0,\,1}_{\bar{\partial}}(X,\,\mathbb{C})$ has complex dimension 2 and is spanned by the Dolbeault cohomology classes $\{\bar{\varphi}_1\}$ and $\{\bar{\varphi}_2\}$.
Let θ_1, θ_2, θ_3, $\in H^0(X, \Omega^1_X)$ be the holomorphic vector fields dual to φ_1, φ_2, φ_3. They are given by

$$\begin{cases} \theta_1 = \frac{\partial}{\partial z_1} \\ \theta_2 = \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_3} \\ \theta_3 = \frac{\partial}{\partial z_3} \end{cases} .$$

and satisfy the relations

$$[\theta_1,\,\theta_2] = -[\theta_2,\,\theta_1] = \theta_3, \qquad [\theta_2,\,\theta_3] = [\theta_1,\,\theta_3] = 0.$$

In particular, we get :

$$[\theta_i\bar{\varphi}_\lambda,\,\theta_k\bar{\varphi}_\nu] = [\theta_i,\,\theta_k]\bar{\varphi}_\lambda \wedge \bar{\varphi}_\nu, \qquad i,\,k,\,\lambda,\,\nu = 1, 2, 3.$$

Since the holomorphic tangent bundle $T^{1,\,0}X$ is trivial and spanned by θ_1, θ_2, θ_3 the cohomology group $H^{0,\,1}(X,\,T^{1,\,0}X)$ of $T^{1,\,0}X$-valued $(0,\,1)$-forms on X is a $\mathbb{C}$-vector space of dimension 6 spanned by the classes of $\theta_i\bar{\varphi}_\lambda$:

$$H^{0,\,1}(X,\,T^{1,\,0}X) = \bigoplus_{1\leqslant i\leqslant 3,\,1\leqslant\lambda\leqslant 2} \mathbb{C}\{\theta_i\bar{\varphi}_\lambda\}, \qquad \dim H^{0,\,1}(X,\,T^{1,\,0}X) = 6.$$

Consequently the Kuranishi family of X can be described by 6 parameters $t = (t_{i\lambda})_{1\leqslant i\leqslant 3,\, 1\leqslant \lambda\leqslant 2}$. By the **Kuranishi family** of X we mean that there is a holomorphic family of compact complex manifolds $(X_t)_{t\in B}$ whose central fiber X_0 is the given X and whose base B is an open ball centered at the origin in $H^{0,1}(X, T^{1,0}X)$. This family of small deformations of X is called the Kuranishi family of X. The Kuranishi family is a general object that exists for every compact complex manifold X (see [25]).

In the 6-parameter Kuranishi family $(X_t)_{t\in B}$, $t = (t_{i\lambda})_{1\leqslant i\leqslant 3,\, 1\leqslant \lambda\leqslant 2}$ of the Iwasawa manifold $X_0 = X = G/\Gamma$, Alessandrini and Bassanelli [1] single out the direction corresponding to parameters t such that

$$t_{12} \neq 0, \quad t_{ij} = 0, \qquad \text{for all} \ \ (i,\, j) \neq (1,\, 2).$$

The following proposition is a counter-example that shows the balanced property is **not open under holomorphic deformation**.

Proposition 1.6.11 *(Alessandrini-Bassanelli, [1], p 1062) Let $(X_t)_{t\in B}$ be the 6-parameter Kuranishi family of the Iwasawa manifold $X_0 = G/\Gamma$, $t = (t_{i\lambda})_{1\leqslant i\leqslant 3,\, 1\leqslant \lambda\leqslant 2}$.*
*Then , for parameters such that $t_{12} \neq 0$, $t_{ij} = 0$, for all $(i,\, j) \neq (1,\, 2)$, X_t is **not balanced** for any $t := t_{12} \neq 0$ satisfying $|t_{12}| < \epsilon$ if $\epsilon > 0$ is small enough.*

SKT Counter Example

Consider the Heisenberg group H

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; x,\, y,\, z \in \mathbb{R} \right\} \tag{1.28}$$

The following set is a basis of left-invariant 1-forms on H

$$\{\alpha_1 = dx,\, \alpha_2 = dy,\, \alpha_3 = xdy - dz\},$$

and the structure equations are given by :

$$\begin{cases} d\alpha_1 = 0 \\ d\alpha_2 = 0 \\ d\alpha_3 = \alpha_1 \wedge \alpha_2 \end{cases}.$$

Let us consider the lattice Γ given by the matrices in (1.28) with $(x,\, y,\, z)$-entries lying in $\mathbb{Z}$. From now on, we denote by $N = H/\Gamma$ the 3-dimensional nilmanifold and we will refer to N as the **Heisenberg nilmanifold.** Let us take another copy of N with basis of 1-forms $\beta_1,\, \beta_2,\, \beta_3$ satisfying the equations

$$\begin{cases} d\beta_1 = 0 \\ d\beta_2 = 0 \\ d\beta_3 = \beta_1 \wedge \beta_2 \end{cases}.$$

We consider the invariant almost-complex structure J_0 on $N \times N$ defined by

$$J_0(\alpha_1) = -\alpha_2 \quad J_0(\beta_1) = -\beta_2 \quad J_0(\alpha_3) = -\beta_3.$$

Indeed, in terms of the basis of 1-forms $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ we have :

$$e_1 = \alpha_1; \quad e_2 = \alpha_2; \quad e_3 = \beta_1; \quad e_4 = \beta_2; \quad e_5 = \alpha_3; \quad e_6 = \beta_3,$$

the structure equations are

$$de_1 = de_2 = de_3 = de_4 = 0 \quad de_5 = \alpha_1 \wedge \alpha_2 \quad de_6 = \beta_1 \wedge \beta_2.$$

By the above equations we can define a basis $\{\omega_{0,1}, \omega_{0,2}, \omega_{0,3}\}$ of invariant complex 1-forms of bidegree (1, 0) with respect to J_0 as follows :

$$\begin{cases} \omega_{0,1} = e_1 - iJ_0(e_1) = e_1 + ie_2 \\ \omega_{0,2} = e_3 - iJ_0(e_3) = e_3 + ie_4 \\ \omega_{0,3} = 2e_6 - 2iJ_0(e_6) = 2e_6 - 2ie_5 \end{cases}.$$

By a straightforward computation we get :

$$d\omega_{0,1} = d\omega_{0,2} = 0 \quad d\omega_{0,3} = \omega_{0,1} \wedge \bar{\omega}_{0,1} + i\omega_{0,2} \wedge \bar{\omega}_{0,2}.$$

Now we consider the small deformation J_t on $N \times N$ given by

$$t\frac{\partial}{\partial z_2} \otimes \bar{\omega}_{0,1} + it\frac{\partial}{\partial z_1} \otimes \bar{\omega}_{0,2} \in H^{0,1}(X_0, T^{1,0}X_0),$$

where X_0 denotes the complex manifold $(N \times N, J_0)$. This deformation is defined for any $t \in \mathbb{C}$, i.e. we can take $B = \mathbb{C}$. The analytic family of compact complex manifolds. $(N \times N, J_t)_{t \in B}$ has a complex basis $\{\omega_{t,1}, \omega_{t,2}, \omega_{t,3}\}$ of type (1, 0) with respect to J_t given by

$$J_t : \omega_{t,1} = \omega_{0,1} + it\bar{\omega}_{0,2}, \quad \omega_{t,2} = \omega_{0,2} + t\bar{\omega}_{0,1}, \quad \omega_{t,3} = \omega_{0,3}.$$

The complex basis satisfies the following equation for any $t \in \mathbb{C}$:

$$d\omega_{t,1} = d\omega_{t,2} = 0, \quad d\omega_{t,3} = \frac{2i\bar{t}}{1+|t|^4}\omega_{t,1} \wedge \omega_{t,2} + \frac{1-i|t|^2}{1+|t|^4}\omega_{t,1} \wedge \bar{\omega}_{t,1} + \frac{i-|t|^2}{1+|t|^4}\omega_{t,2} \wedge \bar{\omega}_{t,2}.$$

Now consider the following **Hermitian** metric F_t on $(N \times N, J_t)$ for every $t \in \mathbb{C}$

$$F_t = \frac{i}{2}(\omega_{t,1} \wedge \bar{\omega}_{t,1} + \omega_{t,2} \wedge \bar{\omega}_{t,2} + \omega_{t,3} \wedge \bar{\omega}_{t,3}).$$

A direct calculation shows that the metric F_t is **SKT** if and only if $t = 0$. Since by [15] on a 6-dimensional SKT nilmanifold all the invariant Hermitian metrics are SKT, then one gets that $((N \times N, J_t)_{t \in \mathbb{C}})$ is a holomorphic family of 6-dimensional nilmanifolds such that X_0 is SKT but $X_t = (N \times N, J_t)$ does not admit any SKT metric for $t \neq 0$.

Deformation Openness of Gauduchon Metric

In this section we show that if $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds then any Gauduchon metric ω_0 on X_0 deforms C^∞ to Gauduchon metrics ω_t on the nearby fibers X_t. Let X be a compact complex manifold and ω be a Hermitian metric X define the following the Laplace-type operator :

$$P_\omega := i\Lambda_\omega\bar{\partial}\partial : C^\infty(X,\,\mathbb{C}) \to C^\infty(X,\,\mathbb{C})$$

Its adjoint is the operator $P^\star_\omega : C^\infty(X,\,\mathbb{C}) \to C^\infty(X,\,\mathbb{C})$ and is given by :

$$P^\star_\omega(f) = i\star_\omega \bar{\partial}\partial(f\omega_{n-1}) : C^\infty(X,\,\mathbb{C}) \to C^\infty(X,\,\mathbb{C})$$

It is proved in [16] that the operators P_ω and $P^\star_\omega$ are elliptic operators and of vanishing index. Moreover, $\ker P_\omega = \mathbb{C}$. Hence by ellipticity and vanishing index we get $\dim P^\star_\omega = 1$. On the other hand if $f \in \ker P^\star_\omega$ then either $f < 0$ or $f > 0$. To make a choice, suppose that φ is a generator of $\ker\dim P^\star_\omega$ such that $\varphi > 0$, hence $\varphi^{\frac{1}{n-1}}\omega$ is a **Gauduchon metric** on X. Note that the dimension of $\ker P^\star_\omega$ does not depend on ω. Meaning that if ρ is another Hermitian metric on X then $\dim\ker P^\star_\rho = 1$.

Now suppose that $((X_t),\,\gamma_t)_{t\in B}$ is a C^∞ family of compact complex Hermitian manifolds. The family $(P^\star_{\gamma_t})_{t\in B}$ is a C^∞ family of elliptic differential operators on the fibers X_t with kernels of equal dimensions $(=1)$. By theorem 1.6.3, the kernels define C^∞ vector bundles $B \ni t \to \ker P^\star_{\gamma_t}$. Suppose that $0 < f_0 \in \ker P^\star_{\gamma_0}|_{C^\infty(X,\mathbb{C})}$, extend f_t to the local C^∞ section $B \ni t \to f_t$ of the C^∞ real vector bundle $B \ni t \to \ker P^\star_{\gamma_t}|_{C^\infty(X,\mathbb{C})}$. Since $f_0 > 0$ and by the continuous dependence of f_t on t, for t sufficiently close to 0 we get $f_t > 0$. Thus, we get a C^∞ family $\omega_t = f_t^{\frac{1}{n-1}}\gamma_t$ of Gauduchon metrics on fibers X_t.

Note that in the above argument we proved a slightly stronger statement, in the sense that we need not assume γ_0 to be Gauduchon from the start. In other word, we have proved the following Proposition.

Proposition 1.6.12 *(Proposition 2.1, [38]) Let $(X_t)_{t\in B}$, be a holomorphic family of compact complex manifolds. After possibly shrinking B about 0, there exists a C^∞ family of 2-forms $(\omega_t)_{t\in B}$ such that ω_t is a Gauduchon metric on X_t, for every $t \in B$.*

Chapitre 2

Properties of Critical Points of the Dinew-Popovici Energy Functional

2.1 Introduction

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ and ω a Hermitian metric on X. By Theorem 1.6.8, the property of being a Kähler manifold is open under holomorphic deformations. However it is proved by H. Hironaka in [17] and [18] that the Kähler property is not closed under holomorphic deformations.

The class of Kähler metrics is not the only class that is open under holomorphic deformations. In [37], Popovici showed that the strongly Gauduchon property is open under holomorphic deformations as well, but it is not a closed property by Proposition 3.4 in [31]. The notion of a strongly Gauduchon manifold was introduced by Popovici in [36]. Recall that ω is called strongly Gauduchon if $\partial\omega^{n-1}$ is $\bar{\partial}$-exact and, we say that X is said to be strongly Gauduchon manifold if there exists a strongly Gauduchon metric on X.

However, the openness property for an arbitrary class of metrics does not always hold. As a famous example, consider a holomorphic family of compact complex manifolds $(X_t)_{t\in B}$, and suppose ω_0 is a balanced metric on X_0.
In [1], it is shown that the balanced property is not open under holomorphic deformations. Alessandrini and Bassanelli pointed out the counterexample of the Iwasawa manifold endowed with the holomorphically parallelizable complex structure.

The closedness property for balanced metrics does not hold either. In [8], M. Ceballos, A. Otal, L. Ugarte and R. Villacampa constructed a holomorphic family of compact complex manifolds $(X_t)_{t\in B}$ such that for $t \in B \setminus \{0\}$, X_t is a balanced manifold, but the central fiber X_0 does not admit any strongly Gauduchon metric (so it does not admit any balanced metric).

Another example is that of class SKT manifolds. The behavior of this class of manifolds under holomorphic deformations is the same as the class of balanced manifolds. This means the SKT property is neither closed nor open under holomorphic deformations. See for example Proposition 3.1

for openness and Proposition 3.4 for closedness in [31] (see also [15] for closedness).

Another class of manifolds that has drawn a lot of attention is the one of $\partial\bar{\partial}$-manifolds because they satisfy the Hodge decomposition and the Hodge symmetry. In [45] C.C. Wu proved that the $\partial\bar{\partial}$-property is open under holomorphic deformations.
In fact, if one considers a holomorphic family of compact complex manifolds $(X_t)_{t\in B}$ and supposes that the central fiber X_0 is a $\partial\bar{\partial}$-manifold then both the SKT and the balanced properties become open under holomorphic deformations.

In a more general setting, we do not consider our manifolds to be $\partial\bar{\partial}$-manifolds. The main class of metrics that we discuss in this chapter is that of Hermitian-symplectic metrics. In dimension 2 any Hermitian-symplectic manifold is Kähler (see [41]) but in higher dimensions, the following question is still open.

Question 2.1.1 *([41], Question 1.7]) Do there exist non-Kähler Hermitian-symplectic complex manifolds X with $dim_{\mathbb{C}}X \geqslant 3$?*

In [46], S. Yang proved that the property of having a Hermitian-symplectic metric is open under holomorphic deformations (see also [6]). But it is not a closed property under holomorphic deformations by Theorem 3.8 of [31].
In Definition 1.1.6 (5) Ω is not of type $(1,1)$ and $\rho^{2,0}$ is not unique. One can find a unique $(2,\,0)$-form such that has the minimal L^2_ω-norm among such all forms, which we call the $(2,0)$-**torsion form** of ω and it is denoted by $\rho^{2,0}_\omega$.

The main discussion of this chapter is based on [12], where Dinew and Popovici introduced the **Dinew-Popovici energy functional**. When the dimension of X is 3, the critical points for F are exactly the Kähler metrics in the Aeppli cohomology class of ω_0. In Theorem 2.3.1 we show that this property is open under holomorphic deformations in any dimension.In higher dimension, $\dim_{\mathbb{C}}X > 3$ the following question is still open

Question 2.1.2 *When $dim_{\mathbb{C}}X > 3$, are the critical points of the Dinew-Popovici energy functional $F : \mathcal{S}_{\{\omega_0\}} \to [0,\,+\infty)$ exactly the Kähler metrics in the Aeppli cohomology class of ω_0 ?*

We give a partial answer to this question in Proposition 2.4.1 and Corollary 2.4.2. Moreover, in Proposition 2.3.8, we prove that the property of being a critical point for F is closed under holomorphic deformations.
In section 2.2 we first give the definitions and tools to state the main results and in section 2.3 we state our new results and prove them.

2.2 Preliminaries

Throughout this section, X is a compact complex manifold of dimension n equipped with a Hermitian metric ω. This means that ω is a C^∞ positive definite $(1,\,1)$-form on X. Recall that (cf. 1.1.6, (5)) an H-s metric ω, is a positive definite $(1\,1)$-form such that

$$\Omega = \rho^{2,\,0} + \omega + \rho^{0,\,2},$$

Ω is a d-closed 2-form, where $\rho^{2,\,0}$ is $(2,\,0)$-form and $\rho^{0,\,2} = \overline{\rho^{2,\,0}}$. We call Ω, a **completion** of ω.

Lemma 2.2.1 *([12], Lemma and Definition 3.1) For every Hermitian-symplectic metric ω on X, there exists a unique smooth $(2,\,0)$-form $\rho_\omega^{2,\,0}$ on X such that*

$$(i)\ \partial\rho_\omega^{2,\,0} = 0 \quad \text{and} \quad (ii)\ \bar\partial\rho_\omega^{2,\,0} = -\partial\omega \quad \text{and} \quad (iii)\ \rho_\omega^{2,\,0} \in Im\,\partial_\omega^\star + Im\,\bar\partial_\omega^\star.$$

Moreover, property (iii) ensures that $\rho_\omega^{2,\,0}$ has **minimal L^2_ω norm** *among all the $(2,\,0)$-forms satisfying properties (i) and (ii).*

We call $\rho_\omega^{2,\,0}$ the **$(2,\,0)$-torsion form** *and its conjugate $\rho_\omega^{0,\,2}$ the* **$(0,\,2)$-torsion form** *of the Hermitian-symplectic metric ω. One has the explicit* **Neumann-type formula** *:*

$$\rho_\omega^{2,\,0} = -\Delta_{BC}^{-1}[\bar\partial^\star\partial\omega + \bar\partial^\star\partial\partial^\star\partial\omega], \tag{2.1}$$

where Δ_{BC}^{-1} is the Green operator of the Bott-Chern Laplacian Δ_{BC} induced by ω, while $\partial^\star = \partial_\omega^\star$ and $\bar\partial^\star = \bar\partial_\omega^\star$ are the formal adjoints of ∂, resp. $\bar\partial$, w.r.t. the L^2 inner product defined by ω.

Notice that in (2.1) by Δ_{BC}^{-1} we mean $(\Delta_{BC}^{2,0})^{-1} : C^\infty_{2,\,0}(X,\,\mathbb{C}) \longrightarrow \mathrm{Im}\,\Delta_{BC}$. Moreover, since Δ_{BC} is a real operator (so Δ_{BC}^{-1} is also real) we thus have

$$\rho_\omega^{0,\,2} = -\Delta_{BC}^{-1}[\partial^\star\bar\partial\omega + \partial^\star\bar\partial\bar\partial^\star\bar\partial\omega].$$

Let ω_0 be a fixed Hermitian-symplectic metric on X. In the following we introduce an energy functional, defined by S. Dinew and D. Popovici, which plays a central role is this chapter. But first let us define $\mathcal{S}_{\{\omega_0\}}$ as follows :

$$\mathcal{S}_{\{\omega_0\}} := \{\omega_0 + \partial\bar u_0 + \bar\partial u_0 \ \mid\ u_0 \in C^\infty_{1,\,0}(X,\,\mathbb{C}) \ \text{ such that } \ \omega_0 + \partial\overline{u_0} + \bar\partial u_0 > 0\}.$$

The set $\mathcal{S}_{\{\omega\}}$ is an *open convex subset* of the real affine space $\{\omega\}_A \cap C^\infty_{1,\,1}(X,\,\mathbb{R}) = \{\omega + \partial\bar u + \bar\partial u \ \mid\ u \in C^\infty_{1,\,0}(X,\,\mathbb{C})\}$.

Definition 2.2.2 *([12], Definition 3.3) The definition of Dinew-Popovici energy functional F is given by :*

$$F : \mathcal{S}_{\{\omega_0\}} \to [0,\,+\infty), \qquad F(\omega) = \int\limits_X |\rho_\omega^{2,\,0}|^2_\omega\, dV_\omega = ||\rho_\omega^{2,\,0}||^2_\omega, \tag{2.2}$$

where $\omega \in \mathcal{S}_{\{\omega_0\}}$ and $\rho_\omega^{2,\,0}$ is the $(2,0)$-torsion form of ω, while $|\ \ |_\omega$ is the pointwise norm and $||\ \ ||_\omega$ is the L^2 norm induced by ω.

From Definition 2.2.2, it is clear that the zero set of F is the set of Kähler metrics in $\mathcal{S}_{\{\omega_0\}}$. We are interested in the set of critical points of F. For this purpose we will compute the first variation of F long the path $\omega + t\gamma$, where $\gamma = \partial\bar u + \bar\partial u \in C^\infty_{1,\,1}(X,\,\mathbb{R})$ is a fixed real $(1,\,1)$-form chosen to be Aeppli cohomologous to zero.

Proposition 2.2.3 *([12], Proposition 3.5) The differential at ω of F is given by the formula :*

$$(d_\omega F)(\gamma) = \frac{d}{dt}_{|t=0} \widetilde{F}_t(\omega) = -2\,Re\,\langle\langle u,\, \bar\partial^\star\omega\rangle\rangle_\omega + 2\,Re\int\limits_X u\wedge\rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\bar\partial\bigg(\frac{\omega^{n-3}}{(n-3)!}\bigg)$$

$$= -\langle\langle\gamma\,,\omega\rangle\rangle + 2\,Re\int\limits_X u\wedge\rho_\omega^{2,\,0}\wedge\overline{\rho_\omega^{2,\,0}}\wedge\bar\partial\bigg(\frac{\omega^{n-3}}{(n-3)!}\bigg), \qquad (2.3)$$

for every $(1,\,1)$*-form* $\gamma = \partial\bar u + \bar\partial u$.

It is obvious that every Kähler metric ω is a critical point for F. Now suppose that ω is a critical point for F and $\dim_{\mathbb{C}} X = 3$, $\bar\partial\omega^{n-3} = 0$, so (2.3) reduces to $(d_\omega F)(\gamma) = -2\,\mathrm{Re}\,\langle\langle u,\, \bar\partial^\star\omega\rangle\rangle_\omega$. This amounts to $\mathrm{Re}\,\langle\langle u,\, \bar\partial^\star\omega\rangle\rangle_\omega = 0$ for every $(1,\,0)$-form u. Thus, by taking $u = \bar\partial^\star\omega$ we get $\bar\partial^\star\omega = 0$. This is equivalent to ω being balanced. However, ω is already SKT since it is Hermitian-symplectic, so ω must be Kähler by Remark 1.1.7.

Now consider a holomorphic family of compact complex manifolds $(X_t)_{t\in B}$ and $(\omega_t)_{t\in B}$ is a C^∞ family of Hermitian-symplectic metrics on $(X_t)_{t\in B}$. This means that for every $t\in B$, ω_t is a Hermitian-symplectic metric on X_t. Therefore like (2.2) one can define a **family of Dinew-Popovici energy functionals** $(F_t)_{t\in B}$, $F_t : \mathcal{S}_{\{\omega_t\}} \to [0,\,\infty)$, as follows

$$F_t : \mathcal{S}_{\{\omega_t\}} \to [0,\,+\infty), \qquad F_t(\bar\omega_t) = \int\limits_{X_t} |\rho_{\bar\omega_t}^{2,\,0}|_{\bar\omega_t}^2\, dV_{\bar\omega_t} = ||\rho_{\bar\omega_t}^{2,\,0}||_{\bar\omega_t}^2,$$

where like the Equation (2.2) $\bar\omega_t \in \mathcal{S}_{\{\omega_t\}}$ and $\rho_{\bar\omega_t}^{2,\,0}$ is the $(2,\,0)$-torsion form of $\bar\omega_t$, while $|\;\;|_{\bar\omega_t}$ is the pointwise norm and $||\;\;||_{\bar\omega_t}$ is the L^2 norm induced by $\bar\omega_t$.
Henceforth if we fix any Hermitian symplectic $\bar\omega_t \in \mathcal{S}_{\{\omega_t\}}$ then for every $t\in B$ one can define the differential at $\bar\omega_t$ of F_t exactly like Proposition 2.2.3.

2.3 Deformation properties of the Dinew-Popovic energy functional and H-S metrics

This section is devoted to our new results based on [12]. We give a proof for Theorem 2.3.1. This theorem shows that if a compact complex manifold X admits a Hermitian-symplectic metric ω_0, then the existence of a Kähler metric $\tilde\omega_0$ in the Aeppli cohomology class of ω_0 is an open property under holomorphic deformations.

Theorem 2.3.1 *Suppose B is an open ball in $\mathbb{C}^m$ containing the origin and $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds of complex dimension n satisfying the following conditions :*

1) for every $t\in B$, X_t is equipped with a Hermitian-symplectic metric ω_t and the family $(\omega_t)_{t\in B}$ is a C^∞-family of $(1,\,1)$-forms,

2) for $t=0$, ω_0 is a Kähler metric on X_0.

Then after possibly shrinking B about 0, there exists a family of $(1,\,1)$-forms $(\tilde\omega_t)_{t\in B}$ such that *a)* $\tilde{\omega}_t \in \{\omega_t\}_A$, *where* $\{\omega_t\}_A$ *is the Aeppli cohomology class of* ω_t*,*

b) $\tilde{\omega}_t$ *is a Kähler metric on* X_t *for every* $t \in B$*,*

c) $\tilde{\omega}_0 = \omega_0$*,*

d) $(\tilde{\omega}_t)_{t\in B}$ *is a* C^∞ *family of metrics.*

By Theorem 1.6.8, the open property for Kähler metrics is known. But the way that we constructed the C^∞ family of Kähler metrics $(\tilde{\omega}_t)_{t\in B}$ is different. The new result of Theorem 2.3.1 is that we have constructed a Kähler metric in a specific Aeppli cohomology class.
Before we present the proof of Theorem 2.3.1, we mention three theorems which play a crucial role in our proof.

Theorem 2.3.2 *([34], Theorem 4.1) Fix a compact Hermitian manifold* (X, ω)*. For any* C^∞ (p, q)*-form* $v \in Im(\partial\bar{\partial})$*, the (unique) minimal* L^2*-norm solution of the equation*

$$\partial\bar{\partial}u = v$$

is given by the formula

$$u = (\partial\bar{\partial})^\star \Delta_{BC}^{-1} v,$$

where Δ_{BC}^{-1} *is the Green operator of the Bott-Chern Laplacian* Δ_{BC} *induced by* ω*.*

Theorem 2.3.3 *([45], Theorem 5.12) Let* $(X_t)_{t\in B}$ *be a holomorphic family of compact complex manifolds of complex dimension* n*. If the central fiber* X_0 *is a* $\partial\bar{\partial}$*-manifold, then after possibly shrinking* B *about* 0*,* X_t *is a* $\partial\bar{\partial}$*-manifold for all* $t \in B$*.*

Theorem 2.3.4 *([11], Section 6) Every compact Kähler manifold is a* $\partial\bar{\partial}$*-manifold.*

Proof. Since ω_0 is a Kähler metric on X_0 by Theorem 2.3.4, X_0 is a $\partial\bar{\partial}$-manifold, therefore by Theorem 2.3.3 after possibly shrinking B about 0 one can assume that X_t is a $\partial\bar{\partial}$-manifold for every $t \in B$. Let us fix a $t \in B$, ω_t is a Hermitian-symplectic metric on X_t then by Remark 1.1.7 (2) in Section 1.1 one implies that $\partial_t\omega_t$ is d-closed and ∂_t-exact. Since X_t is a $\partial\bar{\partial}$-manifold, $\partial_t\omega_t$ is $\partial_t\bar{\partial}_t$-exact. So the following equation

$$-\partial_t\bar{\partial}_t u_t = \partial_t\omega_t. \tag{2.4}$$

has at least one solution, u_t, for $t \in B$. By Theorem 2.3.2 we are able to choose the minimal L^2-norm solution with respect to ω_t among all such u_t. The minimal $L^2_{\omega_t}$-norm solution of equation (2.4) is given by

$$u_t^{min} = -(\partial_t\bar{\partial}_t)^\star \Delta_{BC,t}^{-1}(\partial_t\omega_t), \tag{2.5}$$

where $\Delta_{BC,t}^{-1}$ is the Green operator of the Bott-Chern Laplacian $\Delta_{BC,t}$ induced by ω_t, mentioned in Section 1.6. Now we define,

$$\tilde{\omega}_t = \omega_t + \partial_t\overline{u_t^{min}} + \bar{\partial}_t u_t^{min},$$

for all $t \in B$.
By the construction of $\tilde{\omega}_t$, one can see that

$$\partial_t\bar{\partial}_t\tilde{\omega}_t = \partial_t\bar{\partial}_t(\omega_t + \partial_t\overline{u_t^{min}} + \bar{\partial}_t u_t^{min}) = \partial_t\bar{\partial}_t\omega_t = 0.$$

Therefore $\{\tilde{\omega}_t\}_A$ is well-defined and by the definition of the Aeppli cohomology group, adding $\partial_t\overline{u_t^{min}}$ and $\bar{\partial}_t u_t^{min}$ to ω_t does not change the Aeppli cohomology class of ω_t. Hence $\tilde{\omega}_t \in \{\omega_t\}_A$, this proves (a).
Also for every $t \in B$, $\tilde{\omega}_t$ is d-closed because

$$d\tilde{\omega}_t = d(\omega_t + \partial_t\overline{u_t^{min}} + \bar{\partial}_t u_t^{min}) = \partial_t\omega_t + \bar{\partial}_t\omega_t + \bar{\partial}_t\partial_t\overline{u_t^{min}} + \partial_t\bar{\partial}_t u_t^{min}. \tag{2.6}$$

Equation (2.5) implies that $\partial\bar{\partial}_t u_t^{min} = -\partial_t\omega_t$, put this in the equation (2.6) one can see that $\tilde{\omega}_t$ is d-closed. On the other hand, the strict positivity of ω_0 implies strict positivity of $\tilde{\omega}_t$ for all $t \in B$ sufficiently close to 0, henceforth $\tilde{\omega}_t$ is a strictly positive d-closed $(1,1)$-form on X_t, this proves (b). So we have a family of Kähler metrics $(\tilde{\omega}_t)_{t\in B}$ on $(X_t)_{t\in B}$. At $t = 0$, there are two Kähler metrics on X_0. One of them is ω_0, which is given by assumption (2) and the other one is $\tilde{\omega}_0$ by our construction. Since ω_0 is a Kähler metric on X_0, $\partial_0\omega_0 = 0$. Hence

$$u_0^{min} = -(\partial_0\bar{\partial}_0)^\star \Delta_{BC,0}^{-1}(\partial_0\omega_0) = 0.$$

So,

$$\tilde{\omega}_0 = \omega_0 + \partial_0\overline{u_0^{min}} + \bar{\partial}_0 u_0^{min} = \omega_0.$$

This means that these two metrics coincide on X_0 which proves (c).
For every $t \in B$ we denote by $h_{\mathrm{BC},t}(X_t)$ the dimension of $\ker\Delta_{\mathrm{BC},t}$ ($\Delta_{\mathrm{BC},t} : C^\infty_{2,1}(X_t, \mathbb{C}) \to C^\infty_{2,1}(X_t, \mathbb{C})$). Since $(X_t)_{t\in B}$, after possibly shrinking B about 0, is a holomorphic family of compact complex $\partial\bar{\partial}$-manifolds, by Theorem 5.12 in [45], $h_{\mathrm{BC},t}(X_t) = h_{\mathrm{BC},0}(X_0)$ for every $t \in B$. By Theorem 1.6.3 (ii) the family of linear operators $(\Delta_{BC,t}^{-1})_{t\in B}$ acting on $(2,1)$-forms is a C^∞ family of linear operators, therefore the family $(u_t^{min})_{t\in B}$ is a C^∞ family of $(1,0)$-forms, and since $(\omega_t)_{t\in B}$ is a C^∞ family of metrics one can say $(\tilde{\omega}_t)$ is a C^∞ family of metrics, this proves (d). $\square$
We saw that in dimension 3, Kähler metrics are the critical points for the Dinew-Popovici energy functional F. So as a consequence of Theorem 2.3.1 one can get the following corollary.

Corollary 2.3.5 *Suppose $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds of dimension 3, $(\omega_t)_{t\in B}$ is a C^∞ family of Hermitian-symplectic metrics on $(X_t)_{t\in B}$, and $(F_t)_{t\in B}$ is a family of Dinew-Popovici energy functionals mentioned in section 2.2. If for $t = 0$, ω_0 is a critical point of F_0, then after possibly shrinking B about 0 there exists a C^∞ family of Kähler metrics $(\tilde{\omega}_t)_{t\in B}$ such that for every $t \in B$, $\tilde{\omega}_t \in \mathcal{S}_{\{\omega_t\}}$ and $\tilde{\omega}_t$ is a critical point of F_t and $\tilde{\omega}_0 = \omega_0$.*

Proof. The existence of a C^∞ family of Kähler metrics $(\tilde{\omega}_t)_{t\in B}$ such that for every $t \in B$ each $\tilde{\omega}_t \in \mathcal{S}_{\{\omega_t\}}$ and $\tilde{\omega}_0 = \omega_0$ come directly from Theorem 2.3.1 and since the dimension of each X_t is 3, $\tilde{\omega}_t$ being a Kähler for each $t \in B$ implies that $\tilde{\omega}_t$ is a critical point of F_t . $\square$

By Corollary 4.2 of [12], in dimension 3 if ω is a Hermitian-symplectic metric and the given Aeppli class $\{\omega\}_A$ contains a Kähler metric ω_k, then its $(0,2)$-torsion form $\rho_\omega^{0,2}$ is $\bar{\partial}$-exact. Therefore by Theorem 2.3.1 if the given Aeppli class $\{\omega\}_A$ contains a Kähler metric ω_k, then the $\bar{\partial}$-exactness for $\rho_\omega^{0,2}$ is an open property under holomorphic deformations.
So it is natural to investigate the openness and the closedness properties of the $(0,2)$-torsion form $\rho_\omega^{0,2}$ in higher dimensions.
In the following proposition we show that for a Hermitian-symplectic metric ω, the $\bar{\partial}$-exactness

for the (0, 2)-torsion form $\rho_\omega^{0,2}$ is a closed property under small holomorphic deformations in any dimension.
First, we fix some notations for next proposition. For every $t \in B$ let $h_{BC,t} = \dim\ker \Delta^{0,2}_{BC,t}$ and $h_{\bar\partial,t} = \dim\ker \Delta^{0,2}_{\bar\partial,t}$

Proposition 2.3.6 *Suppose that $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds, $(\omega_t)_{t\in B}$ is a C^∞ family of Hermitian-symplectic metrics on $(X_t)_{t\in B}$. If*

(1) for every $t \in B$ sufficiently close to 0, $h_{BC,t} = h_{BC,0}$,

(2) for every $t \in B$ sufficiently close to 0, $h_{\bar\partial,t} = h_{\bar\partial,0}$,

(3) for every $t \in B \setminus \{0\}$ and sufficiently close to 0, the (0, 2)-torsion form $\rho_{\omega_t}^{0,2}$ is $\bar\partial_t$-exact.

Then

(a) the family $(\rho_{\omega_t}^{0,2})_{t\in B}$ is a C^∞ family of (0, 2)-forms,

(b) for $t = 0$, the (0, 2)-torsion form $\rho_{\omega_0}^{0,2}$ is $\bar\partial_0$-exact.

Before giving the proof of Proposition 2.3.6, we recall the following lemma which will be used in the proof.

Lemma 2.3.7 *([38], p 11-12) Let (X,ω) be an n-dimensional compact Hermitian manifold. For every $\rho \in \bar\partial(C^\infty_{0,2}(X,\mathbb{C})$, the minimal L^2-norm solution of the equation*

$$\bar\partial\varphi = \rho$$

is given by the following Neumann formula

$$\varphi = \bar\partial^\star(\Delta_{\bar\partial})^{-1}\rho, \tag{2.7}$$

where $(\Delta_{\bar\partial})^{-1}$ is the Green operator of the $\bar\partial$-Laplacian $\Delta_{\bar\partial}$ induced by ω.

Proof of Proposition 2.3.6. From equation (2.1) one sees that the $\rho_{\omega_t}^{2,0}$ has the following form

$$\rho_{\omega_t}^{2,0} = -\Delta^{-1}_{BC,t}[\bar\partial_t^\star\partial_t\omega_t + \bar\partial_t^\star\partial_t\partial_t^\star\partial_t\omega_t],$$

for all $t \in B$. By conjugating the above equation one can see that

$$\rho_{\omega_t}^{0,2} = -\Delta^{-1}_{BC,t}[\partial_t^\star\bar\partial_t\omega_t + \partial_t^\star\bar\partial_t\bar\partial_t^\star\bar\partial_t\omega_t], \tag{2.8}$$

for all $t \in B$. Note that in (2.8) we used the fact that $\Delta_{BC} = \overline{\Delta_{BC}}$. Since $h_{BC,t} = h_{BC,0}$ for t sufficiently close to the origin, by Theorem 1.6.3.(ii), the family $(\Delta^{-1}_{BC,t})_{t\in B}$ of linear operators, acting on (0, 2)-form, is a C^∞ family of linear operators. This means that the family $(\rho_{\omega_t}^{0,2})_{t\in B}$ is a C^∞ family of (0, 2)-forms. In particular $\rho_{\omega_t}^{0,2} \to \rho_{\omega_0}^{0,2}$, when $t \longrightarrow 0$. This proves (a).
By assumption (3) for every $t \in B\backslash\{0\}$, $\rho_{\omega_t}^{0,2}$ is $\bar\partial_t$-exact. So after possibly shrinking B about the origin the following equation

$$\rho_{\omega_t}^{0,2} = \bar\partial_t\beta_t \tag{2.9}$$

has at least one solution β_t in $C^\infty_{0,1}(X_t,\mathbb{C})$ for every $t \in B\backslash\{0\}$. By Lemma 2.3.7, we are able to choose the unique solution among such β_t with the minimal L^2-norm induced by ω_t. Hence by equation (2.7), the minimal L^2-norm solution of equation (2.9) has the following form

$$\beta_t^{\min} = \bar\partial_t^\star(\Delta_{\bar\partial,t})^{-1}\rho_{\omega_t}^{0,2} \overset{(1)}{=} (\Delta_{\bar\partial,t})^{-1}\bar\partial_t^\star\rho_{\omega_t}^{0,2}. \tag{2.10}$$

Where (I) is implied as follows

$$\bar\partial^\star\Delta_{\bar\partial}=\bar\partial^\star(\bar\partial\bar\partial^\star+\bar\partial^\star\bar\partial)=\bar\partial^\star\bar\partial\bar\partial^\star=\bar\partial^\star\bar\partial\bar\partial^\star+\bar\partial\bar\partial^\star\bar\partial^\star=(\bar\partial\bar\partial^\star+\bar\partial^\star\bar\partial)\bar\partial^\star=\Delta_{\bar\partial}\bar\partial^\star.$$

By assumption (2) after possibly shrinking B about the origin, $h_{\bar\partial,t}=h_{\bar\partial,0}$ for every $t\in B$. Therefore by Theorem 1.6.3 (ii) the family $(\Delta^{-1}_{\bar\partial,t})_{t\in B}$ is a C^∞ family of linear operators acting on (0, 1)-forms. On the other hand from (a) one can imply that the family $(\rho^{0,2}_{\omega_t})_{t\in B}$ is a C^∞ family of (0, 2)-forms. Hence there exists a $\beta_0=(\Delta_{\bar\partial,0})^{-1}\bar\partial_0^\star\rho^{0,2}_{\omega_0}\in C^\infty_{0,1}(X_0,\mathbb{C})$ such that the family $(\beta_t^{\min})_{t\in B}$ is a C^∞ family of (0, 1)-forms. In other words

$$\lim_{t\to 0}\beta_t^{\min}=\beta_0. \tag{2.11}$$

From equations (2.10) and (2.11) we get

$$\bar\partial_0\beta_0=\bar\partial_0\lim_{t\to 0}\beta_t^{\min}\overset{(\mathrm{I})}{=}\lim_{t\to 0}\bar\partial_t\beta_t^{\min}\overset{(\mathrm{II})}{=}\lim_{t\to 0}\bar\partial_t\bar\partial_t^\star(\Delta_{\bar\partial,t})^{-1}\rho^{0,2}_{\omega_t}\overset{(\mathrm{III})}{=}\lim_{t\to 0}\rho^{0,2}_{\omega_t}.$$

In the above equation, (I) comes from the fact that the family $(\bar\partial_t)_{\ t\in B}$ is a C^∞ family of smooth linear operators so it commutes with lim, (II) comes from the definition of $\beta_t^{\min}$ in equation (2.10) and finally, we have (III) because

$$\begin{aligned}\rho^{0,2}_{\omega_t}=\Delta_{\bar\partial,t}(\Delta_{\bar\partial,t})^{-1}\rho^{0,2}_{\omega_t}&=(\bar\partial_t^\star\bar\partial_t+\bar\partial_t\bar\partial_t^\star)(\Delta_{\bar\partial,t})^{-1}\rho^{0,2}_{\omega_t}\\&=\bar\partial_t\bar\partial_t^\star(\Delta_{\bar\partial,t})^{-1}\rho^{0,2}_{\omega_t}+\bar\partial_t^\star\bar\partial_t(\Delta_{\bar\partial,t})^{-1}\rho^{0,2}_{\omega_t},\end{aligned} \tag{2.12}$$

first note that

$$\bar\partial^\star\bar\partial\Delta_{\bar\partial}=\bar\partial^\star\bar\partial(\bar\partial^\star\bar\partial+\bar\partial\bar\partial^\star)=\bar\partial^\star\bar\partial\bar\partial^\star\bar\partial=\bar\partial^\star\bar\partial\bar\partial^\star\bar\partial+\bar\partial\bar\partial^\star\bar\partial^\star\bar\partial=(\bar\partial^\star\bar\partial+\bar\partial\bar\partial^\star)\bar\partial^\star\bar\partial=\Delta_{\bar\partial}\bar\partial^\star\bar\partial,$$

so $(\Delta_{\bar\partial,t})^{-1}$ commutes with $\bar\partial^\star\bar\partial$, hence in equation (2.12) one gets

$$\bar\partial_t^\star\bar\partial_t(\Delta_{\bar\partial,t})^{-1}\rho^{0,2}_{\omega_t}=(\Delta_{\bar\partial,t})^{-1}\bar\partial_t^\star\bar\partial_t\rho^{0,2}_{\omega_t}$$

and since $\bar\partial_t\rho^{0,2}_{\omega_t}=0$, $(\Delta_{\bar\partial,t})^{-1}\bar\partial_t^\star\bar\partial_t\rho^{0,2}_{\omega_t}$ vanishes, so

$$\bar\partial_t\bar\partial_t^\star(\Delta_{\bar\partial,t})^{-1}\rho^{0,2}_{\omega_t}+\bar\partial_t^\star\bar\partial_t(\Delta_{\bar\partial,t})^{-1}\rho^{0,2}_{\omega_t}=\bar\partial_t\bar\partial_t^\star(\Delta_{\bar\partial,t})^{-1}\rho^{0,2}_{\omega_t}.$$

From (a) one can see that the family $(\rho^{0,2}_{\omega_t})_{t\in B}$ is a C^∞ family of (0, 2)-forms. Which means that

$$\lim_{t\to 0}\rho^{0,2}_{\omega_t}=\rho^{0,2}_{\omega_0}.$$

This proves (b). □

In Proposition 2.3.6, not only did we prove that the $\bar\partial$-exactness for the family $\rho^{0,2}_{\omega_t}$ is a closed property under holomorphic deformations but also we showed that the family $(\beta_t^{\min})_{t\in B}$ of minimal $L^2_{\omega_t}$ solutions is a C^∞ family of (1, 0)-forms and the existence of a minimal $L^2_{\omega_t}$ solution is closed property under holomorphic deformations.

From now on we focus on the Dinew-Popovici energy functional F defined in section 2.1 and its critical points. In the following we give a proof of Proposition 2.3.8. We show that for a fix Hermitian-symplectic metric ω being a critical point for Dinew-Popovici energy functional F is a closed property under holomorphic deformations.

Proposition 2.3.8 *Suppose $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds, $(\omega_t)_{t\in B}$ is a C^∞ family of Hermitian-symplectic metrics on $(X_t)_{t\in B}$ and $(F_t)_{t\in B}$ is the associated family of Dinew-Popovici energy functionals $F_t : \mathcal{S}_{\{\omega_t\}} \to [0, \infty)$ (cf. section 2.2). If after possibly shrinking B about 0,*

(1) for every $t \in B \setminus \{0\}$, ω_t is a critical point in F_t,

(2) for every $t \in B$, $h_{BC,t} = h_{BC,0}$, where $h_{BC,t}$ is the dimension of $H^{0,2}_{BC}(X_t, \mathbb{C})$,

Then ω_0 is a critical point for F_0.

Proof From Proposition 2.2.3 for every $t \in B$ and for every $(1,1)$-form $\gamma_t = \bar\partial_t u_t + \partial_t \bar u_t$, one gets

$$(d_{\omega_t}F_t)(\gamma_t) = -2\,\mathrm{Re}\,\langle\langle u_t,\, \bar\partial_t^\star\omega_t\rangle\rangle_{\omega_t} + 2\,\mathrm{Re}\int\limits_{X_t} u_t \wedge \rho^{2,0}_{\omega_t} \wedge \overline{\rho^{2,0}_{\omega_t}} \wedge \bar\partial_t \frac{\omega_t^{n-3}}{(n-3)!}.$$

Since ω_t is a critical point of F_t for $t \in B\backslash\{0\}$, $(d_{\omega_t}F_t)(\gamma_t) = 0$ for every $\gamma_t \in C^\infty_{1,1}(X_t, \mathbb{C})$. Also by assumption (2) and Proposition 2.3.6 the family $(\rho^{0,2}_{\omega_t})_{t\in B}$ is a C^∞ family of $(0, 2)$-forms. It is obvious that the smooth J_t-$(1,0)$ forms u_t on X_t determine $d_{\omega_t}F_t$. Define for every $t \in B$,

$$T_t : C^\infty_{1,0}(X_t, \mathbb{C}) \longrightarrow \mathbb{R} \qquad\qquad T_t(u_t) = G_t(u_t) + H_t(u_t),$$

where

$$G_t(u_t) = -2\,\mathrm{Re}\,\langle\langle u_t,\, \bar\partial_t^\star\omega_t\rangle\rangle_{\omega_t}$$

and

$$H_t(u_t) = 2\,\mathrm{Re}\int\limits_{X_t} u_t \wedge \rho^{2,0}_{\omega_t} \wedge \overline{\rho^{2,0}_{\omega_t}} \wedge \bar\partial_t \frac{\omega_t^{n-3}}{(n-3)!}.$$

In order to prove that ω_0 is a critical point for F_0, it is sufficient to show that the family $(T_t)_{t\in B}$ is a C^∞ family of linear operators. Therefore it is sufficient to consider a C^∞ family of $(1, 0)$-forms $(u_t)_{t\in B}$ and show that $T_t(u_t) = 0$ for all $t \in B$.
Now suppose that the C^∞ family of $(1, 0)$-forms $(u_t)_{t\in B}$ is given and B is sufficiently shrunk about the origin. For every $t \in B$,

$$\bar\partial_t^\star : C^\infty_{1,1}(X_t, \mathbb{C}) \longrightarrow C^\infty_{1,0}(X_t, \mathbb{C}),$$

is a smooth linear operator and the family $(\bar\partial_t^\star)_{t\in B}$ is a C^∞ family of linear operators. Also for every $t \in B$, the map

$$\langle\langle\ \ ,\, \omega_t\rangle\rangle_{\omega_t} : C^\infty_{1,1}(X_t, \mathbb{C}) \longrightarrow \mathbb{C}, \qquad \langle\langle\ \ ,\, \omega_t\rangle\rangle_{\omega_t}(\alpha) = \langle\langle \alpha\,,\, \omega_t\rangle\rangle_{\omega_t}$$

is a smooth linear map and the family $(\langle\langle\ \ ,\, \omega_t\rangle\rangle_{\omega_t})_{t\in B}$ is a C^∞ family of linear operators. So for every $t \in B$, G_t is a smooth linear operator and the family $(G_t)_{t\in B}$ is a C^∞ family of linear operators. In other words

$$\lim_{t\to 0} G_t(u_t) = G_0(u_0) = -2\,\mathrm{Re}\,\langle\langle u_0,\, \bar\partial_0^\star\omega_0\rangle\rangle_{\omega_0}. \tag{2.13}$$

We show that the family $(H_t)_{t\in B}$ is a C^∞ family of linear operators. First it is obvious that for every $t \in B$, the map

$$\bar\partial_t : C^\infty_{n-3,\,n-3}(X_t, \mathbb{C}) \longrightarrow C^\infty_{n-3,\,n-2}(X_t, \mathbb{C}),$$

is a smooth linear operator and the family $(\bar{\partial}_t)_{t\in B}$ is a C^∞ family of linear operators. On the other hand, the family $(\omega_t)_{t\in B}$ is a C^∞ family of metrics, henceforth the family $(\frac{\bar{\partial}\omega_t^{n-3}}{(n-3)!})_{t\in B}$ is a C^∞ family of $(n-3,\, n-2)$-forms. Also, assumption (2) allows us to employ Proposition 2.3.6 and say that both families $(\rho_{\omega_t}^{0,\,2})_{t\in B}$ and $(\overline{\rho_{\omega_t}^{0,\,2}})_{t\in B}$ are C^∞ family of (2, 0)-forms and (0, 2)-forms respectively. Therefore for every $t \in B$ the map H_t is a smooth real-valued linear map and the family $(H_t)_{t\in B}$ is a C^∞ family of linear operators. In other words,

$$\lim_{t\to 0} H_t(u_t) = H_0(u_0) = \operatorname{Re} \int_{X_0} u_0 \wedge \rho_{\omega_0}^{2,\,0} \wedge \overline{\rho_{\omega_0}^{2,\,0}} \wedge \bar{\partial}_0 \frac{\omega_0^{n-3}}{(n-3)!}. \tag{2.14}$$

The smoothness of T_t for every $t \in B$ is implied by the smoothness of G_t and H_t, and by equations (2.13) and (2.14) one can get

$$T_0(u_0) = \lim_{t\in B} T_t(u_t) = \lim_{t\in B} G_t(u_t) + \lim_{t\in B} H_t(u_t) = 0.$$

Which means that ω_0 is a critical point of F_0. $\square$

2.4 Critical points of the Dinev-Popovici energy functional when $\dim_{\mathbb{C}} X > 3$

In section 2.2 we saw that in dimension 3 the explicit formula for differential of the Dinew-Popovici energy functional F at ω is simpler than in higher dimensions. In the next result of this section we give a proof to Proposition 2.4.1, where we compute the differential of F at ω, when ω is a fixed Hermitian-symplectic metric on compact complex manifold X of dimension n and the (2, 0)-torsion form $\rho_\omega^{2,0}$ is ∂-exact.

Proposition 2.4.1 *Suppose that (X, ω_0) is a compact complex Hermitian-symplectic manifold of dimension n. Fix an $\omega \in \mathcal{S}_{\{\omega_0\}}$. If $\rho_\omega^{2,\,0} = \partial\xi$, for some $(1,0)$-form ξ, then the differential at ω of the Dinew-Popovici energy functional F defined in equation (2.2) evaluated on $\gamma = \bar{\partial}\xi + \partial\bar{\xi}$ is*

$$d_\omega F(\gamma) = 2\|\rho_\omega^{2,0}\|^2 + 2Re \int_X \bar{\partial}\xi \wedge \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,\,0}} \wedge \omega_{n-3}. \tag{2.15}$$

Proof. First note that since $\omega \in \mathcal{S}_{\{\omega_0\}}$, it is a Hermitian-symplectic metric on X so the (2, 0)-torsion form $\rho_\omega^{2,\,0}$ satisfies, $\bar{\partial}\omega = -\partial\overline{\rho_\omega^{2,\,0}}$ and $\partial\omega = -\bar{\partial}\rho_\omega^{2,\,0}$ and $\bar{\partial}\overline{\rho_\omega^{2,\,0}} = 0$. On the other hand, since X is a compact complex manifold it has no boundary so for every $(n-1,\, n)$-form α and every $(n,\, n-1)$-form β

$$\int_X \partial\alpha = 0 \quad \text{and} \quad \int_X \bar{\partial}\beta = 0$$

by the Stokes' theorem. From (2.3), one can observe that when $\gamma = \bar{\partial}\xi + \partial\bar{\xi}$ the differential at ω of F evaluated on γ is

$$(d_\omega F)(\gamma) = -2\operatorname{Re}\langle\langle \xi,\, \bar{\partial}^\star\omega\rangle\rangle_\omega + 2\operatorname{Re} \int_X \xi \wedge \rho_\omega^{2,\,0} \wedge \overline{\rho_\omega^{2,\,0}} \wedge \bar{\partial}\omega_{n-3}.$$

First we compute $\langle\langle \xi,\, \bar{\partial}^\star\omega\rangle\rangle_\omega$. By the definition of the L^2_ω inner product ($u\wedge\star\bar{v} = \langle u,\, v\rangle_\omega dV_\omega$), we have

$$\langle\langle \xi,\, \bar{\partial}^\star\omega\rangle\rangle_\omega = \int_X \langle \xi,\, \bar{\partial}^\star\omega\rangle_\omega dV_\omega = \int_X \xi\wedge\star\partial^\star\omega. \tag{2.16}$$

By standard computation for the Hodge star operator $\star^2 = -id$ on odd-degree forms and by equations (1.1) and (1.7), one gets

$$\star\,\partial^\star\omega = -\star\star\bar{\partial}\star\omega = \bar{\partial}\omega_{n-1}, \tag{2.17}$$

hence by equations (2.16) and (2.17),

$$\langle\langle \xi,\, \bar{\partial}^\star\omega\rangle\rangle_\omega = \int_X \xi\wedge\bar{\partial}\omega_{n-1}. \tag{2.18}$$

Now equation (2.18) allows us to compute $\langle\langle \xi,\, \bar{\partial}^\star\omega\rangle\rangle_\omega$. We get

$$\langle\langle \xi,\, \bar{\partial}^\star\omega\rangle\rangle_\omega = \int_X \xi\wedge\bar{\partial}\omega_{n-1} = \int_X \xi\wedge\bar{\partial}\omega\wedge\omega_{n-2} = -\int_X \xi\wedge\partial\overline{\rho^{2,\,0}_\omega}\wedge\omega_{n-2}. \tag{2.19}$$

By the Stokes' theorem $0 = \int_X \partial(\xi\wedge\overline{\rho^{2,\,0}_\omega}\wedge\omega_{n-2})$, so

$$0 = \int_X \partial\xi\wedge\overline{\rho^{2,\,0}_\omega}\wedge\omega_{n-2} - \int_X \xi\wedge\partial\overline{\rho^{2,\,0}_\omega}\wedge\omega_{n-2} - \int_X \xi\wedge\overline{\rho^{2,\,0}_\omega}\wedge\partial\omega_{n-2}.$$

Therefore,

$$-\int_X \xi\wedge\partial\overline{\rho^{2,\,0}_\omega}\wedge\omega_{n-2} = \int_X \xi\wedge\overline{\rho^{2,\,0}_\omega}\wedge\partial\omega_{n-2} - \int_X \partial\xi\wedge\overline{\rho^{2,\,0}_\omega}\wedge\omega_{n-2}.$$

By assumption $\rho^{2,0}_\omega = \partial\xi$ so,

$$\int_X \partial\xi\wedge\overline{\rho^{2,\,0}_\omega}\wedge\omega_{n-2} = \int_X \rho^{2,\,0}_\omega\wedge\overline{\rho^{2,\,0}_\omega}\wedge\omega_{n-2}.$$

Since $\rho^{2,0}_\omega$ is a primitive form of bidegree $(2,\,0)$, we can apply (1.8) and we get :

$$\int_X \rho^{2,\,0}_\omega\wedge\overline{\rho^{2,\,0}_\omega}\wedge\omega_{n-2} = \int_X \rho^{2,\,0}_\omega\wedge\star\overline{\rho^{2,\,0}_\omega} = \langle\langle \rho^{2,\,0}_\omega,\, \rho^{2,\,0}_\omega\rangle\rangle_\omega = \|\rho^{2,0}_\omega\|^2_\omega.$$

Hence

$$-\int_X \xi\wedge\partial\overline{\rho^{2,\,0}_\omega}\wedge\omega_{n-2} = \int_X \xi\wedge\overline{\rho^{2,\,0}_\omega}\wedge\partial\omega_{n-2} - \|\rho^{2,0}_\omega\|^2_\omega. \tag{2.20}$$

Now, the goal is to compute $\int_X \xi\wedge\overline{\rho^{2,\,0}_\omega}\wedge\partial\omega_{n-2}$ in equation (2.20). We get :

$$\int_X \xi\wedge\overline{\rho^{2,\,0}_\omega}\wedge\partial\omega_{n-2} = \int_X \xi\wedge\overline{\rho^{2,\,0}_\omega}\wedge\partial\omega\wedge\omega_{n-3} = -\int_X \xi\wedge\overline{\rho^{2,\,0}_\omega}\wedge\bar{\partial}\rho^{2,\,0}_\omega\wedge\omega_{n-3}. \tag{2.21}$$

Again, by the Stokes' theorem, $\int\limits_X \bar{\partial}(\xi \wedge \overline{\rho_\omega^{2,\,0}} \wedge \rho_\omega^{2,\,0} \wedge \omega_{n-3}) = 0$ and because $\bar{\partial}\overline{\rho_\omega^{2,\,0}} = 0$, we have

$$0 = \int\limits_X \bar{\partial}\xi \wedge \overline{\rho_\omega^{2,\,0}} \wedge \rho_\omega^{2,\,0} \wedge \omega_{n-3} - \int\limits_X \xi \wedge \overline{\rho_\omega^{2,\,0}} \wedge \bar{\partial}\rho_\omega^{2,\,0} \wedge \omega_{n-3} - \int\limits_X \xi \wedge \overline{\rho_\omega^{2,\,0}} \wedge \rho_\omega^{2,\,0} \wedge \bar{\partial}\omega_{n-3}. \tag{2.22}$$

Therefore, from (2.22) and (2.21) one can deduce the following equation

$$\int\limits_X \xi \wedge \overline{\rho_\omega^{2,\,0}} \wedge \partial\omega_{n-2} = -\int\limits_X \bar{\partial}\xi \wedge \overline{\rho_\omega^{2,\,0}} \wedge \rho_\omega^{2,\,0} \wedge \omega_{n-3} + \int\limits_X \xi \wedge \overline{\rho_\omega^{2,\,0}} \wedge \rho_\omega^{2,\,0} \wedge \bar{\partial}\omega_{n-3}. \tag{2.23}$$

By putting equations (2.19), (2.20) and (2.23) together we see that

$$-2\,\mathrm{Re}\langle\langle \xi,\, \bar{\partial}^\star\omega\rangle\rangle_\omega = 2\,\mathrm{Re}\int\limits_X \bar{\partial}\xi \wedge \overline{\rho_\omega^{2,\,0}} \wedge \rho_\omega^{2,\,0} \wedge \omega_{n-3} - 2\,\mathrm{Re}\int\limits_X \xi \wedge \overline{\rho_\omega^{2,\,0}} \wedge \rho_\omega^{2,\,0} \wedge \bar{\partial}\omega_{n-3} + 2\|\rho_\omega^{2,0}\|^2_\omega. \tag{2.24}$$

By adding $2\,\mathrm{Re}\int\limits_X \xi \wedge \rho_\omega^{2,\,0} \wedge \overline{\rho_\omega^{2,\,0}} \wedge \bar{\partial}\omega_{n-3}$ to equation (2.24) and by using (2.3) with $u = \xi$ we get the formula (2.15). This proves the proposition. □

In formula (2.15), $2\mathrm{Re}\int\limits_X \bar{\partial}\xi \wedge \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,\,0}} \wedge \omega_{n-3}$ is signless in general. However, if it supposes to be non-negative one sees immediately that ω is a Kähler metric whenever it is a critical point for F. In the following proof, we show that if $\bar{\partial}\xi$ is a weakly semi-positive $(1,\,1)$-form then $2\mathrm{Re}\int\limits_X \bar{\partial}\xi \wedge \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge \omega_{n-3}$ is non-negative.

Corollary 2.4.2 *Under the assumptions of Proposition 2.4.1 if*

(i) ω is a critical point for F, and

(ii) the $(2,\,0)$-torsion form $\rho_\omega^{2,\,0} = \partial\xi$ such that $\bar{\partial}\xi$ is weakly semi-positive,

then ω is a Kähler metric on X.

Proof. Since positivity is a pointwise property, one can fix a point $x \in X$ and local coordinates $(z_1, \ldots, z_n)$ centered at x such that ω has the following shape

$$\omega = \sum i dz_i \wedge d\bar{z}_i \qquad \text{at } x.$$

In particular, ω is a strongly strictly positive $(1,\,1)$-form. By Definition 1.4.1 (1) and Proposition 1.4.3, ω_{n-3} is a strongly strictly positive $(n-3,\, n-3)$-form. On the other hand by Example 1.2 of [10], for every $p \in \{1, \ldots, n\}$ and any $(p,\,0)$-form β, the $(p,\,p)$-form $i^{p^2}\beta \wedge \bar{\beta}$ is weakly strictly positive. Hence the $(2,\,2)$-form

$$i^4\rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} = \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}}$$

is weakly strictly positive.

Since $\bar{\partial}\xi$ is a weakly semi-positive $(1,\,1)$-form, there exist real non negative functions $c_1, \ldots, c_n$ and $(1,\,0)$-forms $\alpha_1, \ldots, \alpha_n$ such that

$$\bar{\partial}\xi = \sum c_k i\alpha_k \wedge \bar{\alpha}_k.$$

Therefore

$$\begin{aligned}\bar{\partial}\xi \wedge \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge \omega_{n-3} &= \sum c_k i\alpha_k \wedge \bar{\alpha}_k \wedge \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge \omega_{n-3}\\ &= \sum c_k \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge i\alpha_k \wedge \bar{\alpha}_k \wedge \omega_{n-3}.\end{aligned}$$

Note that by Definition 1.4.1 $\alpha_k \wedge \bar{\alpha}_k$ is strongly strictly positive $(1,\,1)$-form for all $k \in \{1, \ldots, n\}$. By Definition 1.4.1 (2), $c_k \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge i\alpha_k \wedge \bar{\alpha}_k \wedge \omega_{n-3}$ is a weakly semi-positive $(n,\,n)$-form. Hence

$$2\mathrm{Re}\int\limits_X \sum c_k \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge i\alpha_k \wedge \bar{\alpha}_k \wedge \omega_{n-3} = \sum 2\mathrm{Re}\int\limits_X c_k \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge i\alpha_k \wedge \bar{\alpha}_k \wedge \omega_{n-3} \geqslant 0$$

This proves the Corollary. $\square$

Chapitre 3

Functionals for the Study of LCK Metrics on Compact Complex Manifolds

3.1 Introduction

Let X be an n-dimensional compact complex manifold with $n \geq 2$. In this chapter, we propose a variational approach to the existence of locally conformally Kähler (lcK) metrics on X by introducing and analysing a functional in each of the cases $n = 2$ and $n \geq 3$. This functional, defined on the non-empty set $\mathcal{H}_X$ of all the Hermitian metrics on X, assumes non-negative values and vanishes precisely on the lcK metrics. We compute the first variation of our functional on both surfaces and higher-dimensional manifolds.

We will identify a Hermitian metric on X with the associated C^∞ positive definite (1, 1)-form ω. The set $\mathcal{H}_X$ of all these metrics is a non-empty open convex cone in the infinite-dimensional real vector space $C^\infty_{1,1}(X, \mathbb{R})$ of all the real-valued smooth (1, 1)-forms on X. As is well known, a Hermitian metric ω is called Kähler if $d\omega = 0$ and a complex manifold X is said to be Kähler if there exists a Kähler metric thereon. Meanwhile, the notion of **locally conformally Kähler (lcK)** manifold originates with I. Vaisman in [42]. There are several equivalent definitions of lcK manifolds. The one adopted in this chapter stipulates that a complex manifold X is lcK if there exists an lcK metric thereon, while a Hermitian metric ω on X is said to be lcK if there exists a C^∞ 1-form θ on X such that $d\theta = 0$ and

$$d\omega = \omega \wedge \theta.$$

When it exists, the 1-form θ is unique and is called the Lee form of ω. For equivalent definitions of lcK manifolds, the reader is referred e.g. to Definitions 3.18 and 3.29 of [29].

One of the early results in the theory of lcK manifolds is Vaisman's theorem according to which any lcK metric on a compact Kähler manifold is, in fact, globally conformally Kähler. This theorem was extended to compact complex spaces with singularities by Preda and Stanciu in [33].

The question of when lcK metrics exist on a given compact complex manifold X has been extensively studied. For example, Otiman characterised the existence of such metrics with prescribed Lee form in terms of currents : given a d-closed 1-form θ on X and considering the associated twisted operator $d_\theta = d + \theta \wedge \cdot$, Theorem 2.1 in [32] stipulates that X admits an lcK metric whose Lee form is θ if and only if there are no non-trivial positive $(1, 1)$-currents on X that are $(1, 1)$-components of

d_θ-boundaries.

On the other hand, Istrati investigated the relation between the existence of special lcK metrics on a compact complex manifold and the group of biholomorphisms of the manifold. Specifically, according to Theorem 0.2 in [20], a compact lcK manifold X admits a Vaisman metric if the group of biholomorphisms of X contains a torus $\mathbb{T}$ that is not purely real. A compact torus $\mathbb{T}$ of biholomorphisms of a compact complex manifold $(X,\, J)$ is said to be *purely real* (in the sense of (1) of Definition 0.1. in [20]) if its Lie algebra $\mathfrak{t}$ satisfies the condition $\mathfrak{t} \cap J\mathfrak{t} = 0$, where J is the complex structure of X. Recall that an lcK metric ω is said to be a *Vaisman metric* if $\nabla^\omega\theta = 0$, where θ is the Lee form of ω and ∇^ω is the Levi-Civita connection determined by ω.

The approach we propose in this chapter to the issue of the existence of lcK metrics on a compact complex n-dimensional manifold X is analytic. Given an arbitrary Hermitian metric ω on X, the Lefschetz decomposition

$$d\omega = (d\omega)_{prim} + \omega \wedge \theta_\omega$$

of $d\omega$ into a uniquely determined ω-primitive part and a part divisible by ω with a uniquely determined quotient 1-form θ_ω (the Lee form of ω) gives rise to the following dichotomy (cf. Lemma 3.2.2) :

(i) either $n=2$, in which case $(d\omega)_{prim} = 0$ but the Lee form θ_ω need not be d-closed, so the lcK condition on ω is equivalent to $d\theta_\omega = 0$. This turns out to be equivalent to $\partial\theta_\omega^{1,\,0} = 0$. Therefore, we define our functional $L : \mathcal{H}_X \longrightarrow [0,\, +\infty)$ in this case to be

$$L(\omega) = ||\partial\theta_\omega^{1,\,0}||^2_\omega,$$

namely its value at every Hermitian metric ω on X is defined to be the squared L^2_ω-norm of $\partial\theta_\omega^{1,\,0}$.

(ii) or $n \geq 3$, in which case the lcK condition on ω is equivalent to the vanishing condition $(d\omega)_{prim} = 0$. This is further equivalent to the vanishing of either $(\partial\omega)_{prim}$ or $(\bar\partial\omega)_{prim}$. We, therefore, define our functional $L : \mathcal{H}_X \longrightarrow [0,\, +\infty)$ in this case to be

$$L(\omega) = ||(\bar\partial\omega)_{prim}||^2_\omega,$$

namely its value at every Hermitian metric ω on X is defined to be the squared L^2_ω-norm of the ω-primitive part of the $(1,\, 2)$-form $\bar\partial\omega$.

The main results of the paper are the computations of the first variation of our functional L in each of the cases $n=2$ (cf. Theorem 3.4.4) and $n \geq 3$ (cf. Theorem 3.5.1).

While the functional L is scaling-invariant when $n=2$, this fails to be the case when $n \geq 3$. In this latter case, we obtain two proofs – one as a corollary of the formula for the first variation of our functional (cf. Proposition 3.5.4), the other as a direct consequence of the behaviour of our functional in the scaling direction (cf. Proposition 3.6.2) – for the equivalence :

ω is a **critical point** for the functional L if and only if ω is **lcK**

Still in the case $n \geq 3$, we introduce in Definition 3.6.5 a normalised version $\widetilde{L}_\rho$ of the functional L depending on an arbitrary background Hermitian metric ρ. The first variation of $\widetilde{L}_\rho$ is then deduced in Proposition 3.6.6 from the analogous computation for L obtained in Theorem 3.5.1. One motivation for the normalisation we propose in terms of a (possibly balanced and possibly moving) metric ρ stems from the conjecture predicting that the simultaneous existence of a balanced metric and of an lcK metric on a compact complex manifold ought to imply the existence of a Kähler metric. We hope to be able to develop this line of thought in future work.

At the end of §.3.6, we use our scaling-invariant functionals L (in the case of compact complex surfaces) and $\widetilde{L}_\rho$ (in the case of higher-dimensional compact complex manifolds) to produce positive $(1,\,1)$-currents whose failure to be either C^∞ forms or strictly positive provides possible obstructions to the existence of lcK metrics.

3.2 Preliminaries

In this section, we recast some standard material in the language of primitive forms and make a few observations that will be used in the next sections.

Let X be a complex manifold with $\dim_{\mathbb{C}}X = n$. We will denote by :

(i) $C^\infty_k(X,\,\mathbb{C})$, resp. $C^\infty_{p,\,q}(X,\,\mathbb{C})$, the space of C^∞ differential forms of degree k, resp. of bidegree $(p,\,q)$ on X. When these forms α are real (in the sense that $\overline{\alpha}=\alpha$), the corresponding spaces will be denoted by $C^\infty_k(X,\,\mathbb{R})$, resp. $C^\infty_{p,\,q}(X,\,\mathbb{R})$.

(ii) $\Lambda^kT^\star X$, resp. $\Lambda^{p,\,q}T^\star X$, the vector bundle of differential forms of degree k, resp. of bidegree $(p,\,q)$, as well as the spaces of such forms considered in a pointwise way.

For any $(1,\,1)$-form $\rho\geq 0$, we will also use the following notation :

$$\rho_k := \frac{\rho^k}{k!}, \qquad 1\leq k\leq n.$$

When $\rho=\omega$ is C^∞ and positive definite (i.e. ω is a Hermitian metric on X), it can immediately be checked that

$$d\omega_k = \omega_{k-1}\wedge d\omega \quad \text{and} \quad \star_\omega\,\omega_k = \omega_{n-k}$$

for all $1\leq k\leq n$, where $\star=\star_\omega$ is the Hodge star operator induced by ω.

Recall the following standard

Definition 3.2.1 *A C^∞ positive definite $(1,\,1)$-form (i.e. a Hermitian metric) ω on a complex manifold X is said to be* **locally conformally Kähler (lcK)** *if*

$$d\omega=\omega\wedge\theta \quad \textit{for some } C^\infty \ 1\textit{-form } \theta \textit{ satisfying } d\theta=0.$$

The 1-form θ is uniquely determined, is real and is called the **Lee form** *of ω.*

(A) Hermitian-geometric preliminaries

• Recall that for any $k\leq n$ and any Hermitian metric ω on X, the multiplication map

$$L^l_\omega=\omega^l\wedge\cdot : \Lambda^kT^\star X\longrightarrow\Lambda^{k+2l}T^\star X$$

defined at every point of X is an **isomorphism** if $l = n - k$, is **injective** (but in general not surjective) for every $l < n-k$ and is **surjective** (but in general not injective) for every $l > n-k$. A k-form is said to be ω-**primitive** if it lies in the kernel of the multiplication map L_ω^{n-k+1}. Equivalently, the ω-primitive k-forms are precisely those that lie in the kernel of $\Lambda_\omega : \Lambda^k T^\star X \longrightarrow \Lambda^{k-2}T^\star X$, the adjoint w.r.t. the pointwise inner product $\langle\cdot, \cdot\rangle_\omega$ (hence also w.r.t. the L^2-inner product $\langle\langle\cdot, \cdot\rangle\rangle_\omega$) of the Lefschetz operator $L_\omega = \omega \wedge \cdot : \Lambda^k T^\star X \longrightarrow \Lambda^{k+2}T^\star X$.

• Also recall that for every $k \leq n$, every k-form α admits a unique $\langle\ ,\ \rangle_\omega$-orthogonal pointwise splitting (called the **Lefschetz decomposition**) :

$$\alpha = \alpha_{prim} + \omega \wedge \beta^{(1)}_{prim} + \omega^2 \wedge \beta^{(2)}_{prim} + \cdots + \omega^r \wedge \beta^{(r)}_{prim}, \tag{3.1}$$

where r is the largest non-negative integer such that $2r \leq k$, α_{prim}, $\beta^{(1)}_{prim}, \dots, \beta^{(r)}_{prim}$ are ω-primitive forms of respective degrees k, $k-2, \dots, k-2r \geq 0$, and $\langle\ ,\ \rangle_\omega$ is the pointwise inner product defined by ω. We will call α_{prim} the *primitive part* of α.

• The following general formula (cf. e.g. [Voi02, Proposition 6.29, p. 150]) that holds for any *primitive* form v of arbitrary bidegree (p, q) on any complex n-dimensional manifold will be of great use :

$$\star\, v = (-1)^{k(k+1)/2}\, i^{p-q}\, \omega_{n-p-q} \wedge v, \quad \text{where } k := p+q, \tag{3.2}$$

• We will often use the standard notation $[A,\, B] := AB - (-1)^{ab}BA$, where A and B are arbitrary linear operators of degrees a, resp. b, acting on the differential forms of X. The following formula (see e.g. [Dem97, VI, §5.2, Corollary 5.9]) will come in handy several times :

$$[\Lambda_\omega,\, L_\omega] = (n-k)\,\mathrm{Id} \tag{3.3}$$

when acting on k-forms on X.

• Finally, recall the Hermitian commutation relation :

$$i[\Lambda_\omega,\, \partial] = -(\bar\partial^\star_\omega + \bar\tau^\star_\omega) \tag{3.4}$$

proved in [Dem84], where $\tau_\omega := [\Lambda_\omega,\, \partial\omega \wedge \cdot]$ is the torsion operator of order 0 and bidegree $(1, 0)$. This definition of τ_ω yields

$$\bar\tau^\star_\omega \omega = [(\bar\partial\omega \wedge \cdot)^\star, L_\omega](\omega) = (\bar\partial\omega \wedge \cdot)^\star(\omega^2).$$

• On the other hand, if $\alpha^{1,0}$ is any $(1, 0)$-form on X, let $\xi_{\alpha^{1,0}}$ be the $(1, 0)$-vector field defined by the requirement $\bar\xi_{\alpha^{1,0}} \lrcorner \omega = \alpha^{1,0}$. If we set $\alpha^{0,1} := \overline{\alpha^{1,0}}$, we have $\bar\xi_{\alpha^{1,0}} = \xi_{\alpha^{0,1}}$, where $\xi_{\alpha^{0,1}}$ is the $(0, 1)$-vector field defined by the requirement $\bar\xi_{\alpha^{0,1}} \lrcorner \omega = \alpha^{0,1}$.

It is easily checked in local coordinates chosen about a given point x such that the metric ω is defined by the identity matrix at x, that the adjoint w.r.t. $\langle\ ,\ \rangle_\omega$ of the contraction operator by $\bar\xi_{\alpha^{1,0}} = \xi_{\alpha^{0,1}}$ is given by the formula

$$(\bar\xi_{\alpha^{1,0}} \lrcorner \cdot)^\star = (\xi_{\alpha^{0,1}} \lrcorner \cdot)^\star = -i\alpha^{0,1} \wedge \cdot, \quad \text{ or equivalently } \quad -i\bar\xi_{\alpha^{1,0}} \lrcorner \cdot = -i\xi_{\alpha^{0,1}} \lrcorner \cdot = (\alpha^{0,1} \wedge \cdot)^\star. \tag{3.5}$$

Taking conjugates, we get :

$$(\xi_{\alpha^{1,0}}\lrcorner\cdot)^\star = i\alpha^{1,0}\wedge\cdot, \quad \text{or equivalently} \quad i\xi_{\alpha^{1,0}}\lrcorner\cdot = (\alpha^{1,0}\wedge\cdot)^\star. \tag{3.6}$$

Explicitly, if $\alpha^{0,1} = \sum_k \bar{a}_k d\bar{z}_k$ on a neighbourhood of x, then $-i\bar{\xi}_\alpha\lrcorner\cdot = (\alpha^{0,1}\wedge\cdot)^\star = \sum_k a_k \frac{\partial}{\partial\bar{z}_k}\lrcorner\cdot$ at x. Hence, $-i\bar{\xi}_\alpha\lrcorner\alpha^{0,1} = \sum_k |a_k|^2 = |\alpha^{0,1}|^2_\omega$ at x. We have just got the pointwise formula :

$$-i\bar{\xi}_\alpha\lrcorner\alpha^{0,1} = |\alpha^{0,1}|^2_\omega = |\alpha^{1,0}|^2_\omega \tag{3.7}$$

at every point of X.

(B) lcK-geometric preliminaries

Now, suppose that $d\omega = \omega\wedge\theta_\omega$ for some (necessarily real) 1-form θ_ω. Then, $\bar\partial\omega = \omega\wedge\theta_\omega^{0,1}$, so $(\bar\partial\omega\wedge\cdot)^\star = -i\Lambda_\omega(\bar{\xi}_\theta\lrcorner\cdot)$, where $\bar{\xi}_\theta := \bar{\xi}_\alpha$ with $\alpha^{1,0} = \theta_\omega^{1,0}$. The above formula for $\bar\tau_\omega^\star\omega$ translates to

$$\bar\tau_\omega^\star\omega = -i\Lambda_\omega(\bar{\xi}_\theta\lrcorner\omega^2) = -2i\Lambda_\omega(\omega\wedge(\bar{\xi}_\theta\lrcorner\omega)) = -2i[\Lambda_\omega,\, L_\omega](\bar{\xi}_\theta\lrcorner\omega) = -2i(n-1)\theta_\omega^{1,0}$$

The conclusion of this discussion is that, when $d\omega = \omega\wedge\theta_\omega$, formula (3.10) (which will be proved as part of Lemma 3.2.2 below) translates to

$$\theta_\omega^{1,0} = \frac{1}{n-1}\,\Lambda_\omega(\partial\omega) = \frac{1}{n-1}\,[\Lambda_\omega,\,\partial](\omega) = \frac{1}{n-1}\,i\bar\partial_\omega^\star\omega + \frac{1}{n-1}\,i\bar\tau_\omega^\star\omega = \frac{1}{n-1}\,i\bar\partial_\omega^\star\omega + 2\theta_\omega^{1,0},$$

which amounts to $\theta_\omega^{1,0} = -\frac{1}{n-1}\,i\bar\partial_\omega^\star\omega$. This proves (3.11) for an arbitrary n, hence also (3.9) when $n=2$, if the other statements in Lemma 3.2.2 have been proved.

The obstruction to a given Hermitian metric ω being lcK depends on whether $n=2$ or $n\geq 3$.

Lemma 3.2.2 *Let X be a complex manifold with $dim_{\mathbb{C}}X = n$.*

(i) If $n=2$, for any Hermitian metric ω there exists a unique, possibly non-closed, C^∞ 1-form $\theta = \theta_\omega$ such that $d\omega = \omega\wedge\theta$. Therefore, ω is **lcK** *if and only if θ_ω is d-***closed***.*

*Moreover, for any Hermitian metric ω, the 2-form $d\theta_\omega$ is ω-***primitive***, i.e. $\Lambda_\omega(d\theta_\omega) = 0$, or equivalently, $\omega\wedge d\theta_\omega = 0$, while the Lee form is real and is explicitly given by the formula :*

$$\theta_\omega = \Lambda_\omega(d\omega). \tag{3.8}$$

Alternatively, if $\theta_\omega = \theta_\omega^{1,0} + \theta_\omega^{0,1}$ is the splitting of θ_ω into components of pure types, we have

$$\theta_\omega^{1,0} = \Lambda_\omega(\partial\omega) = -i\bar\partial^\star\omega \tag{3.9}$$

and the analogous formulae for $\theta_\omega^{0,1} = \overline{\theta_\omega^{1,0}}$ obtained by taking conjugates.

(ii) If $n\geq 3$, for any Hermitian metric ω there exists a unique ω-primitive C^∞ 3-form $(d\omega)_{prim}$ and a unique C^∞ 1-form $\theta = \theta_\omega$ such that $d\omega = (d\omega)_{prim} + \omega\wedge\theta$. The Lee form is real and is explicitly given by the formula

$$\theta_\omega = \frac{1}{n-1}\,\Lambda_\omega(d\omega). \tag{3.10}$$

Moreover, ω is **lcK** *if and only if $(d\omega)_{prim} = 0$.*

If ω is lcK, then

$$\theta_\omega^{1,\,0} = \frac{1}{n-1}\,\Lambda_\omega(\partial\omega) = -\frac{i}{n-1}\,\bar\partial^\star\omega \tag{3.11}$$

and the analogous formulae obtained by taking conjugates hold for $\theta_\omega^{0,\,1} = \overline{\theta_\omega^{1,\,0}}$.

Proof. (i) When $n=2$, the map $\omega\wedge\cdot : \Lambda^1T^\star X \longrightarrow \Lambda^3T^\star X$ is an isomorphism at every point of X. In particular, the 3-form $d\omega$ is the image of a unique 1-form θ under this map.

To see that $d\theta$ is primitive, we apply d to the identity $d\omega = \omega\wedge\theta$ to get

$$0 = d^2\omega = d\omega\wedge\theta + \omega\wedge d\theta.$$

Meanwhile, multiplying the same identity by θ, we get $d\omega\wedge\theta = \omega\wedge\theta\wedge\theta = 0$ since $\theta\wedge\theta = 0$ due to the degree of θ being 1. Therefore, $\omega\wedge d\theta = 0$, which means that the 2-form $d\theta$ is ω-primitive.

To prove formula (3.8), we apply Λ_ω to the identity $d\omega = \omega\wedge\theta$ to get

$$\Lambda_\omega(d\omega) = [\Lambda_\omega,\, L_\omega](\theta) = \theta,$$

where we used the identities $\Lambda_\omega(\theta) = 0$ (for bidegree reasons) and (3.3) (with $k=1$ and $n=2$).

(ii) The splitting $d\omega = (d\omega)_{prim} + \omega\wedge\theta$ is the Lefschetz decomposition of $d\omega$ w.r.t. the metric ω. Applying Λ_ω, we get $\Lambda_\omega(d\omega) = [\Lambda_\omega,\, L_\omega](\theta) = (n-1)\,\theta$ (having applied (3.3) with $k=1$ to get the latter identity), which proves (3.10).

The implication "ω lcK $\implies$ $(d\omega)_{prim} = 0$" follows at once from the definitions. To prove the reverse implication, suppose that $(d\omega)_{prim} = 0$. We have to show that θ is d-closed. The assumption means that $d\omega = \omega\wedge\theta$, so $d\omega\wedge\theta = \omega\wedge\theta\wedge\theta = 0$ and $0 = d^2\omega = d\omega\wedge\theta + \omega\wedge d\theta$. Consequently, $\omega\wedge d\theta = 0$. Now, the multiplication of k-forms by ω^l is injective whenever $l\leq n-k$. When $n\geq 3$, if we choose $l=1$ and $k=2$ we get that the multiplication of 2-forms by ω is injective. Hence, the identity $\omega\wedge d\theta = 0$ implies $d\theta = 0$, so ω is lcK. □

Another standard observation is that the Lefschetz decomposition transforms nicely, hence the lcK property is preserved, under conformal rescaling.

Lemma 3.2.3 *Let ω be an arbitrary Hermitian metric and let f be any smooth real-valued function on a compact complex n-dimensional manifold X. If $d\omega = (d\omega)_{prim} + \omega\wedge\theta_\omega$ is the Lefschetz decomposition of $d\omega$ w.r.t. the metric ω (with the understanding that $(d\omega)_{prim} = 0$ when $n=2$), then*

$$d(e^f\omega) = e^f\,(d\omega)_{prim} + e^f\,\omega\wedge(\theta_\omega + df) \tag{3.12}$$

is the Lefschetz decomposition of $d(e^f\omega)$ w.r.t. the metric $\widetilde\omega := e^f\omega$.

Consequently, ω is lcK if and only if any conformal rescaling $e^f\omega$ of ω is lcK, while the Lee form transforms as $\theta_{e^f\omega} = \theta_\omega + df$. In particular, when the lcK metric ω varies in a fixed conformal class, the Lee form θ_ω varies in a fixed De Rham 1-class $\{\theta_\omega\}_{DR}\in H^1(X,\,\mathbb{R})$ called the **Lee De Rham class** *associated with the given conformal class. Moreover, the map $\omega\mapsto\theta_\omega$ defines a bijection from the set of lcK metrics in a given conformal class to the set of elements of the corresponding Lee De Rham 1-class.*

Proof. Differentiating, we get $d(e^f\omega)=e^f d\omega+e^f\omega\wedge df=e^f(d\omega)_{prim}+e^f\omega\wedge(\theta_\omega+df)$. Meanwhile, it can immediately be checked that

$$\Lambda_{e^f\omega}=e^{-f}\Lambda_\omega,$$

so $\ker\Lambda_{e^f\omega}=\ker\Lambda_\omega$. Thus, the ω-primitive forms coincide with the $\widetilde{\omega}$-primitive forms. Since $\Lambda_{\widetilde{\omega}}$ commutes with the multiplication by any real-valued function, $e^f(d\omega)_{prim}$ is $\widetilde{\omega}$-primitive, so (3.12) is the Lefschetz decompostion of $d\widetilde{\omega}$ w.r.t. $\widetilde{\omega}$. □

When X is compact, we know from [Gau77] that every Hermitian metric ω on X admits a (unique up to a positive multiplicative constant) conformal rescaling $\widetilde{\omega}:=e^f\omega$ that is a **Gauduchon metric**. These metrics are defined (cf. [Gau77]) by the requirement that $\partial\bar\partial\widetilde{\omega}^{n-1}=0$, where n is the complex dimension of X. This fact, combined with Lemma 3.2.3, shows that no loss of generality is incurred in the study of the existence of lcK metrics on compact complex manifolds if we confine ourselves to Gauduchon metrics.

We end this review of known material with the following characterisation (cf. [AD15, Lemma 2.5]) of Gauduchon metrics on surfaces in terms of their Lee forms. It appears that, in any dimension, a metric ω is Gauduchon if and only if $d^\star_\omega\theta_\omega=0$ [1], but we confine ourselves to the 2-dimensional case.

Lemma 3.2.4 *Let ω be a Hermitian metric on a complex* **surface** *X. The following equivalence holds :*

$$\partial\bar\partial\omega=0\quad\textit{(i.e. ω is a Gauduchon metric)}\quad\iff\quad\bar\partial^\star_\omega\theta^{0,\,1}_\omega=0,$$

where $\theta^{0,\,1}_\omega$ is the component of type $(0,\,1)$ of the Lee form θ_ω of ω.

Proof. We give a proof different from the one in [AD15] by making use of the Hermitian commutation relations. By applying ∂ to the identity $\bar\partial\omega=\omega\wedge\theta^{0,\,1}_\omega$ and using the identity $\partial\omega=\omega\wedge\theta^{1,\,0}_\omega$, we get

$$\partial\bar\partial\omega=\partial\omega\wedge\theta^{0,\,1}_\omega+\omega\wedge\partial\theta^{0,\,1}_\omega=\omega\wedge(\theta^{1,\,0}_\omega\wedge\theta^{0,\,1}_\omega+\partial\theta^{0,\,1}_\omega).$$

Taking Λ_ω, we get

$$\Lambda_\omega(\partial\bar\partial\omega)=[\Lambda_\omega,\,L_\omega](\theta^{1,\,0}_\omega\wedge\theta^{0,\,1}_\omega+\partial\theta^{0,\,1}_\omega)+\omega\wedge\Lambda_\omega(\theta^{1,\,0}_\omega\wedge\theta^{0,\,1}_\omega+\partial\theta^{0,\,1}_\omega)=\Lambda_\omega(\theta^{1,\,0}_\omega\wedge\theta^{0,\,1}_\omega+\partial\theta^{0,\,1}_\omega)\,\omega,$$

where the second identity follows from $[\Lambda_\omega,\,L_\omega]=-(2-2)\,\mathrm{Id}=0$ on 2-forms on complex surfaces. Now, $\Lambda_\omega(\theta^{1,\,0}_\omega\wedge\theta^{0,\,1}_\omega+\partial\theta^{0,\,1}_\omega)$ is a function, so from the above identities we get the equivalences

$$\begin{aligned}\Lambda_\omega(\partial\bar\partial\omega)=0\quad&\iff\quad\Lambda_\omega(\theta^{1,\,0}_\omega\wedge\theta^{0,\,1}_\omega+\partial\theta^{0,\,1}_\omega)=0\iff\theta^{1,\,0}_\omega\wedge\theta^{0,\,1}_\omega+\partial\theta^{0,\,1}_\omega\ \text{is }\omega\text{-primitive}\\&\iff\quad\omega\wedge(\theta^{1,\,0}_\omega\wedge\theta^{0,\,1}_\omega+\partial\theta^{0,\,1}_\omega)=0\iff\partial\bar\partial\omega=0.\end{aligned}$$

We remember the equivalence $\partial\bar\partial\omega=0\iff\Lambda_\omega(\theta^{1,\,0}_\omega\wedge\theta^{0,\,1}_\omega)+\Lambda_\omega(\partial\theta^{0,\,1}_\omega)=0$. Since $\Lambda_\omega(i\theta^{1,\,0}_\omega\wedge\theta^{0,\,1}_\omega)=|\theta^{1,\,0}_\omega|^2_\omega$ (immediate verification) and $\Lambda_\omega\theta^{0,\,1}_\omega=0$ (for bidegree reasons), we get the equivalence :

$$\partial\bar\partial\omega=0\iff|\theta^{1,\,0}_\omega|^2_\omega+i[\Lambda_\omega,\,\partial]\,\theta^{0,\,1}_\omega=0.$$

1. The authors are grateful to the referee for pointing this fact out to them.

The Hermitian commutation relation $i[\Lambda_\omega,\,\partial]=-(\bar\partial^\star_\omega+\bar\tau^\star_\omega)$ (cf. (3.4), see [Dem84]) transforms the last equivalence into

$$\partial\bar\partial\omega=0 \iff |\theta^{1,\,0}_\omega|^2_\omega-(\bar\partial^\star_\omega\theta^{0,\,1}_\omega+\bar\tau^\star_\omega\theta^{0,\,1}_\omega)=0. \tag{3.13}$$

On the other hand, $\bar\tau^\star_\omega=[(\bar\partial\omega\wedge\cdot)^\star,\,\omega\wedge\cdot]$. From this we get

Formula 3.2.5 *For any Hermitian metric ω on a complex surface, we have*

$$\bar\tau^\star_\omega\theta^{0,\,1}_\omega=|\theta^{0,\,1}_\omega|^2_\omega.$$

Proof of Formula 3.2.5. Since $(\bar\partial\omega\wedge\cdot)^\star\theta^{0,\,1}_\omega=0$ for bidegree reasons, we get $\bar\tau^\star_\omega\theta^{0,\,1}_\omega=(\bar\partial\omega\wedge\cdot)^\star(\omega\wedge\theta^{0,\,1}_\omega)$. Since $\bar\partial\omega=\omega\wedge\theta^{0,\,1}_\omega$, we have $(\bar\partial\omega\wedge\cdot)^\star=-i\Lambda_\omega(\bar\xi_\theta\lrcorner\cdot)$ (see (3.7) and the discussion there below), where $\bar\xi_\theta$ is the $(0,\,1)$-vector field defined by the requirement $\bar\xi_\theta\lrcorner\omega=\theta^{1,\,0}_\omega$. Hence

$$\bar\tau^\star_\omega\theta^{0,\,1}_\omega=-i\Lambda_\omega(\theta^{1,\,0}_\omega\wedge\theta^{0,\,1}_\omega)-i\Lambda_\omega[\omega\wedge(\bar\xi_\theta\lrcorner\theta^{0,\,1}_\omega)].$$

Since $-i\bar\xi_\theta\lrcorner\theta^{0,\,1}_\omega=|\theta^{0,\,1}_\omega|^2_\omega$ (cf. (3.7)), we infer that

$$\bar\tau^\star_\omega\theta^{0,\,1}_\omega=-\Lambda_\omega(i\theta^{1,\,0}_\omega\wedge\theta^{0,\,1}_\omega)+2\,|\theta^{0,\,1}_\omega|^2_\omega,$$

since $\Lambda_\omega(\omega)=n=2$. Meanwhile, $\theta^{1,\,0}_\omega=\overline{\theta^{0,\,1}_\omega}$, so we get $\Lambda_\omega(i\theta^{1,\,0}_\omega\wedge\theta^{0,\,1}_\omega)=|\theta^{1,\,0}_\omega|^2_\omega=|\theta^{0,\,1}_\omega|^2_\omega$ (immediate verification in local coordinates). Formula 3.2.5 is now proved. $\square$

End of proof of Lemma 3.2.4. Formula 3.2.5 transforms equivalence (3.13) into

$$\partial\bar\partial\omega=0 \iff (|\theta^{1,\,0}_\omega|^2_\omega-|\theta^{0,\,1}_\omega|^2_\omega)-\bar\partial^\star_\omega\theta^{0,\,1}_\omega=0 \iff \bar\partial^\star_\omega\theta^{0,\,1}_\omega=0$$

and we are done $\square$

3.3 An enerygy functional for the study of lcK metrics

In what follows, we will restrict attention to the set

$$\mathcal{H}_X:=\{\omega\in C^\infty_{1,\,1}(X,\,\mathbb{R})\;|\;\omega>0\}$$

of all Hermitian metrics on X. This is a non-empty open cone in the infinite-dimensional vector space $C^\infty_{1,\,1}(X,\,\mathbb{R})$ of all smooth real $(1,\,1)$-forms on X. It will be called the **Hermitian cone** of X.

Building on Lemma 3.2.2, we introduce the following energy functional. By $||\;||_\omega$, respectively $|\;|_\omega$, we mean the L^2-norm, respectively the pointwise norm, defined by ω.

Definition 3.3.1 *Let X be a compact complex manifold with $\dim_{\mathbb{C}}X=n$.*

(i) If $n=2$, let $L:\mathcal{H}_X\longrightarrow[0,\,+\infty)$ be defined by

$$L(\omega):=\int\limits_X\partial\theta^{1,\,0}_\omega\wedge\bar\partial\theta^{0,\,1}_\omega=||\partial\theta^{1,\,0}_\omega||^2_\omega,$$

where θ_ω is the Lee form of ω.

(ii) If $n\geq 3$, let $L:\mathcal{H}_X\longrightarrow[0,\,+\infty)$ be defined by

$$L(\omega):=\int\limits_X i(\bar\partial\omega)_{prim}\wedge\overline{(\bar\partial\omega)_{prim}}\wedge\omega_{n-3}=||(\bar\partial\omega)_{prim}||^2_\omega,$$

where $(\bar\partial\omega)_{prim}$ is the ω-primitive part of $\bar\partial\omega$ in its Lefschetz decomposition (3.1).

This definition is justified by the following observation.

Lemma 3.3.2 *In the setup of Definition 3.3.1, for every metric $\omega\in\mathcal{H}_X$ the following equivalence holds :*

$$\omega\quad\text{is an lcK metric}\iff L(\omega)=0.$$

Proof. • In the case $n=2$, we know from (i) of Lemma 3.2.2 that ω is lcK if and only if $d\theta_\omega=0$. This condition is equivalent to $\mathcal{L}(\omega)=0$, where we set

$$\mathcal{L}(\omega):=||d\theta_\omega||^2_\omega=\int\limits_X d\theta_\omega\wedge\star(d\bar\theta_\omega).$$

We also know from (i) of Lemma 3.2.2 that $d\theta_\omega$ is ω-primitive, so we get

$$0=\Lambda_\omega(d\theta_\omega)=\Lambda_\omega(\partial\theta^{1,\,0}_\omega)+\Lambda_\omega(\partial\theta^{0,\,1}_\omega+\bar\partial\theta^{1,\,0}_\omega)+\Lambda_\omega(\bar\partial\theta^{0,\,1}_\omega)=\Lambda_\omega(\partial\theta^{0,\,1}_\omega+\bar\partial\theta^{1,\,0}_\omega),$$

where the last identity follows from the previous one for bidegree reasons. We infer that the $(1,\,1)$-form $\partial\theta^{0,\,1}_\omega+\bar\partial\theta^{1,\,0}_\omega$ is ω-primitive. But so are $\partial\theta^{1,\,0}_\omega$ and $\bar\partial\theta^{0,\,1}_\omega$ for bidegree reasons, so we can apply the standard formula (3.2) to get $\star(d\theta_\omega)=\partial\theta^{1,\,0}_\omega-(\partial\theta^{0,\,1}_\omega+\bar\partial\theta^{1,\,0}_\omega)+\bar\partial\theta^{0,\,1}_\omega$. We infer that

$$\begin{aligned}d\theta_\omega\wedge\star(d\bar\theta_\omega) &= [\partial\theta^{1,\,0}_\omega+(\partial\theta^{0,\,1}_\omega+\bar\partial\theta^{1,\,0}_\omega)+\bar\partial\theta^{0,\,1}_\omega]\wedge[\partial\theta^{1,\,0}_\omega-(\partial\theta^{0,\,1}_\omega+\bar\partial\theta^{1,\,0}_\omega)+\bar\partial\theta^{0,\,1}_\omega]\\ &= 2\,\partial\theta^{1,\,0}_\omega\wedge\bar\partial\theta^{0,\,1}_\omega-(\partial\theta^{0,\,1}_\omega+\bar\partial\theta^{1,\,0}_\omega)^2\end{aligned}$$

and finally that

$$\mathcal{L}(\omega)=2\,L(\omega)-\int\limits_X(\partial\theta^{0,\,1}_\omega+\bar\partial\theta^{1,\,0}_\omega)^2. \tag{3.14}$$

On the other hand, the Stokes formula implies the first of the following identities

$$\begin{aligned}0 &= \int\limits_X d\theta_\omega\wedge d\theta_\omega=\int\limits_X[\partial\theta^{1,\,0}_\omega+(\partial\theta^{0,\,1}_\omega+\bar\partial\theta^{1,\,0}_\omega)+\bar\partial\theta^{0,\,1}_\omega]\wedge[\partial\theta^{1,\,0}_\omega+(\partial\theta^{0,\,1}_\omega+\bar\partial\theta^{1,\,0}_\omega)+\bar\partial\theta^{0,\,1}_\omega]\\ &= 2\,L(\omega)+\int\limits_X(\partial\theta^{0,\,1}_\omega+\bar\partial\theta^{1,\,0}_\omega)^2. \qquad (3.15)\end{aligned}$$

We conclude from (3.14) and (3.15) that $\mathcal{L}(\omega)=0$ if and only if $L(\omega)$. Thus, we have proved that ω is lcK if and only if $L(\omega)=0$, as claimed.

The identity $L(\omega) = ||\partial\theta_\omega^{1,\,0}||^2_\omega$ follows at once from the general formula (3.2) applied to the primitive (2, 0)-form $\partial\theta_\omega^{1,\,0}$. Indeed, $\star\partial\theta_\omega^{1,\,0} = \partial\theta_\omega^{1,\,0}$, hence $\partial\theta_\omega^{1,\,0}\wedge\bar\partial\theta_\omega^{0,\,1} = \partial\theta_\omega^{1,\,0}\wedge\star\overline{(\partial\theta_\omega^{1,\,0})} = |\partial\theta_\omega^{1,\,0}|^2_\omega\,dV_\omega$.

• In the case $n\geq 3$, we know from (ii) of Lemma 3.2.2 that ω is lcK if and only if $(d\omega)_{prim}=0$. Now, $(d\omega)_{prim} = (\partial\omega)_{prim} + (\bar\partial\omega)_{prim}$ and the forms $(\partial\omega)_{prim}$ and $(\bar\partial\omega)_{prim}$ are conjugate to each other and of different pure types ((2, 1), respectively (1, 2)), so the vanishing of $(d\omega)_{prim}$ is equivalent to the vanishing of $(\bar\partial\omega)_{prim}$.

Meanwhile, the standard formula (3.2) applied to the primitive (2, 1)-form $\overline{(\bar\partial\omega)_{prim}} = (\partial\omega)_{prim}$ spells :

$$\star\,\overline{(\bar\partial\omega)_{prim}} = i\,\overline{(\bar\partial\omega)_{prim}}\wedge\omega_{n-3}.$$

This proves the identity $L(\omega) = ||(\bar\partial\omega)_{prim}||^2_\omega$.

Putting these pieces of information together, we get the following equivalences :

$$\omega \quad \text{lcK} \iff (d\omega)_{prim}=0 \iff (\bar\partial\omega)_{prim}=0 \iff L(\omega)=0.$$

The proof is complete. $\square$

3.4 First variation of the functional : case of complex surfaces

Let S be a compact complex surface. (So, we set $X=S$ when $n=2$.) We will compute the differential of the functional $L:\mathcal{H}_S\longrightarrow[0,\,+\infty)$ defined on the Hermitian cone of S. Let $\omega\in\mathcal{H}_S$. Then, $T_\omega\mathcal{H}_S = C^\infty_{1,\,1}(S,\,\mathbb{R})$, so we will compute the differential

$$d_\omega L : C^\infty_{1,\,1}(S,\,\mathbb{R})\longrightarrow\mathbb{R}$$

by computing the derivative of $L(\omega+t\gamma)$ w.r.t. $t\in(-\varepsilon,\,\varepsilon)$ at $t=0$ for any given real (1, 1)-form γ.

Lemma 3.4.1 *The differential at ω of the map $\mathcal{H}_S\ni\omega\mapsto\theta_\omega^{0,\,1} = \Lambda_\omega(\bar\partial\omega)$ is given by*

$$(d_\omega\theta_\omega^{0,\,1})(\gamma) = \frac{d}{dt}_{|t=0}\Lambda_{\omega+t\gamma}(\bar\partial\omega + t\,\bar\partial\gamma) = \star(\gamma\wedge\star\bar\partial\omega) + \Lambda_\omega(\bar\partial\gamma),$$

while the differential at ω of L is given by

$$(d_\omega L)(\gamma) = 2\,Re\int\limits_S \partial\theta_\omega^{1,\,0}\wedge\bar\partial\bigg(\star(\gamma\wedge\star\bar\partial\omega) + \Lambda_\omega(\bar\partial\gamma)\bigg),$$

for every form $\gamma\in C^\infty_{1,\,1}(S,\,\mathbb{R})$, where $\star=\star_\omega$ is the Hodge star operator defined by the metric ω.

Before giving the proof of this lemma, we recall the following result from [DP22] that will be used several times in the sequel.

Lemma 3.4.2 *([DP22], Lemmas 3.5 and 3.3) For any complex manifold X of any dimension $n\geq 2$, for any bidegree $(p,\,q)$ and any C^∞ family $(\alpha_t)_{t\in(-\varepsilon,\,\varepsilon)}$ of forms $\alpha_t\in C^\infty_{p,\,q}(X,\,\mathbb{C})$ with $\varepsilon>0$ so small that $\omega+t\gamma>0$ for all $t\in(-\varepsilon,\,\varepsilon)$, the following formulae hold :*

$$\frac{d}{dt}\bigg|_{t=0}(\Lambda_{\omega+t\gamma}\alpha_t) = \Lambda_\omega\bigg(\frac{d\alpha_t}{dt}\bigg|_{t=0}\bigg) - (\gamma\wedge\cdot)^\star_\omega\,\alpha_0 = \Lambda_\omega\bigg(\frac{d\alpha_t}{dt}\bigg|_{t=0}\bigg) + (-1)^{p+q+1}\,\star_\omega(\gamma\wedge\star_\omega\alpha_0).$$

The former of the above equalities appears as such in Lemma 3.5 of [DP22], while the latter equality follows from the former and from formula (27) of Lemma 3.3 of [DP22] which states that $\star_\omega(\eta\wedge\cdot) = (\overline{\eta}\wedge\cdot)^\star_\omega\,\star_\omega$ for any $(1,\,1)$-form η on X. Indeed, in our case, taking $\eta=\gamma$ we get $\bar\eta=\gamma$ since γ is real. Moreover, composing with $\star_\omega$ on the right and using the standard equality $\star_\omega\star_\omega = (-1)^{p+q}\,\mathrm{Id}$ on $(p,\,q)$-forms, we get $\star_\omega(\gamma\wedge\cdot)\star_\omega = (-1)^{p+q}\,(\gamma\wedge\cdot)^\star_\omega$ on $(p,\,q)$-forms.

Proof of Lemma 3.4.1. The formula for $(d_\omega\theta^{0,\,1}_\omega)(\gamma)$ is an immediate consequence of Lemma 3.4.2 applied with $\alpha_t = \bar\partial\omega + t\,\bar\partial\gamma$ (hence also with $(p,\,q)=(1,\,2)$). We further get :

$$\begin{aligned}(d_\omega L)(\gamma) &= \frac{d}{dt}_{|t=0}L(\omega+t\gamma) = \frac{d}{dt}_{|t=0}\int\limits_S\partial\theta^{1,\,0}_{\omega+t\gamma}\wedge\bar\partial\theta^{0,\,1}_{\omega+t\gamma}\\ &= \int\limits_S\partial\bigg(\star\,(\gamma\wedge\star\partial\omega)+\Lambda_\omega(\partial\gamma)\bigg)\wedge\bar\partial\theta^{0,\,1}_\omega + \int\limits_S\partial\theta^{1,\,0}_\omega\wedge\bar\partial\bigg(\star\,(\gamma\wedge\star\bar\partial\omega)+\Lambda_\omega(\bar\partial\gamma)\bigg).\end{aligned}$$

This is the stated formula for $(d_\omega L)(\gamma)$ since the two terms of the r.h.s. expression are mutually conjugated. $\square$

We will now simplify the above expression of $(d_\omega L)(\gamma)$ starting with a preliminary observation.

Lemma 3.4.3 *Let $(X,\,\omega)$ be an n-dimensional complex Hermitian manifold and let $\star=\star_\omega$ be the Hodge star operator defined by ω.*

(i) For every $(0,\,1)$-form α on X, we have :

$$\star(\alpha\wedge\omega) = i\Lambda_\omega(\alpha\wedge\omega_{n-1}).$$

Moreover, if $n=2$, then $\star(\alpha\wedge\omega)=i\alpha$ for any $(0,\,1)$-form α on X.

(ii) If $n=2$, then $\star(\gamma\wedge\alpha)=i\Lambda_\omega(\gamma\wedge\alpha)$ for any $(1,\,1)$-form γ and any $(0,\,1)$-form α on X. In particular, $\star\bar\partial\omega = i\theta^{0,\,1}_\omega$ for any Hermitian metric ω on a complex surface.

(iii) In arbitrary dimension n, for any $(1,\,1)$-form γ and any $(0,\,1)$-form α on X, we have :

$$\Lambda_\omega(\gamma\wedge\alpha) = (\Lambda_\omega\gamma)\,\alpha + i\,\xi_\alpha\lrcorner\gamma,$$

where ξ_α is the (unique) vector field of type $(1,\,0)$ defined by the requirement

$$\xi_\alpha\lrcorner\omega = i\alpha.$$

Proof. (i) From the standard formula $\star\Lambda_\omega = L_\omega\star$ (cf. e.g. [Dem97, VI, §.5.1]) we get

$\Lambda_\omega = \star L_\omega\star$ on even-degreed forms and $\Lambda_\omega = -\star L_\omega\star$ on odd-degreed forms.

Consequently, $\star(\alpha\wedge\omega) = \star L_\omega\alpha = -(\star L_\omega\star)\star\alpha = \Lambda_\omega(\star\alpha) = \Lambda_\omega(-(1/i)\,\alpha\wedge\omega^{n-1}/(n-1)!)$, where we used the fact that $\star\star=-1$ on odd-degreed forms and the standard formula (3.2) applied to the (necessarily primitive) $(0,\,1)$-form α.

When $n=2$, we get $\star(\alpha\wedge\omega) = i\Lambda_\omega(\alpha\wedge\omega) = i[\Lambda_\omega,\,L_\omega]\,\alpha = -i(1-2)\,\alpha = i\alpha$ after using the general formula $[L_\omega,\,\Lambda_\omega]=(k-n)$ on k-forms on n-dimensional complex manifolds.

(ii) If $n=2$, the map $\omega\wedge\cdot:\Lambda^1T^\star X\longrightarrow\Lambda^3T^\star X$ is an isomorphism at every point of X. Since $\gamma\wedge\alpha$ is a 3-form, there exists a unique 1-form β (necessarily of type $(0,1)$) such that $\gamma\wedge\alpha=\omega\wedge\beta$. Moreover, $\beta=\Lambda_\omega(\gamma\wedge\alpha)$ because $\omega\wedge\Lambda_\omega(\gamma\wedge\alpha)=[L_\omega,\,\Lambda_\omega](\gamma\wedge\alpha)=\gamma\wedge\alpha$. Indeed, $\omega\wedge(\gamma\wedge\alpha)=0$ for bidegree reasons (here $n=2$) and $[L_\omega,\,\Lambda_\omega]=(k-n)$ on k-forms.

Thus, $\gamma\wedge\alpha=\omega\wedge\Lambda_\omega(\gamma\wedge\alpha)$. So, applying (i) for the second identity below, we get :

$$\star(\gamma\wedge\alpha)=\star\bigg(\omega\wedge\Lambda_\omega(\gamma\wedge\alpha)\bigg)=i\Lambda_\omega(\gamma\wedge\alpha).$$

To get the last equality, we used (i) for $n=2$ with α replaced by $\Lambda_\omega(\gamma\wedge\alpha)$.

In order to prove the formula for $\star\bar\partial\omega$, recall that $\bar\partial\omega=\omega\wedge\theta_\omega^{0,\,1}$, so we get

$$\star\bar\partial\omega=\star(\omega\wedge\theta_\omega^{0,\,1})=i\theta_\omega^{0,\,1},$$

where we used again (i) for $n=2$ with α replaced by $\theta_\omega^{0,\,1}$.

(iii) Since the claimed identity is pointwise and involves only zero-th order operators, we fix an arbitrary point $x\in X$ and choose local holomorphic coordinates about x such that at x we have

$$\omega=\sum_{a=1}^n idz_a\wedge d\bar z_a \quad\text{ and }\quad \gamma=\sum_{j=1}^n\gamma_{j\bar j}\,idz_j\wedge d\bar z_j.$$

Then, $\Lambda_\omega=-i\sum\limits_{j=1}^n\frac{\partial}{\partial\bar z_j}\lrcorner\frac{\partial}{\partial z_j}\lrcorner\cdot$ at x. If we set $\alpha=\sum\limits_{j=1}^n\alpha_j\,d\bar z_j$ (at any point), we get $\xi_\alpha=\sum\limits_{j=1}^n\alpha_j\,\frac{\partial}{\partial z_j}$ (at x) and the following equalities (at x) :

$$\begin{aligned}\Lambda_\omega(\gamma\wedge\alpha) &= -i\sum_{j=1}^n\frac{\partial}{\partial\bar z_j}\lrcorner\frac{\partial}{\partial z_j}\lrcorner(\gamma\wedge\alpha)\stackrel{(a)}{=}-i\sum_{j=1}^n\frac{\partial}{\partial\bar z_j}\lrcorner\bigg(\bigg(\frac{\partial}{\partial z_j}\lrcorner\gamma\bigg)\wedge\alpha\bigg)\\ &= -i\sum_{j=1}^n\bigg(\frac{\partial}{\partial\bar z_j}\lrcorner\frac{\partial}{\partial z_j}\lrcorner\gamma\bigg)\wedge\alpha+i\sum_{j=1}^n\bigg(\frac{\partial}{\partial z_j}\lrcorner\gamma\bigg)\wedge\bigg(\frac{\partial}{\partial\bar z_j}\lrcorner\alpha\bigg)\\ &\stackrel{(b)}{=} \bigg(\sum_{j=1}^n\gamma_{j\bar j}\bigg)\,\alpha-\sum_{j=1}^n\alpha_j\gamma_{j\bar j}\,d\bar z_j=(\Lambda_\omega\gamma)\,\alpha+i\xi_\alpha\lrcorner\gamma,\end{aligned}$$

where (a) follows from $(\partial/\partial z_j)\lrcorner\alpha=0$ for bidegree reasons and (b) follows from $(\partial/\partial z_j)\lrcorner\gamma=i\gamma_{j\bar j}\,d\bar z_j$ and from $(\partial/\partial\bar z_j)\lrcorner\alpha=\alpha_j$.

This proves the desired equality at x, hence at any point since x was arbitrary. $\square$

We can now derive a simplified form of the first variation of the functional L.

Theorem 3.4.4 *Let S be a compact complex* **surface** *on which a Hermitian metric ω has been fixed.*

(i) *The differential at $\omega\in\mathcal{H}_S$ of the functional $L:\mathcal{H}_S\longrightarrow[0,+\infty)$ evaluated at any form*

$\gamma \in C^\infty_{1,1}(S, \mathbb{R})$ *is given by any of the following three formulae :*

$$\begin{aligned}(d_\omega L)(\gamma) &= -2\,Re\int\limits_S \Lambda_\omega(\gamma)\,\partial\theta_\omega^{1,0}\wedge\bar\partial\theta_\omega^{0,1} - 2\,Re\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial\Lambda_\omega(\gamma)\wedge\theta_\omega^{0,1} + 2\,Re\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial\Lambda_\omega(\bar\partial\gamma)\\ &\quad -2\,Re\int\limits_S i\partial\theta_\omega^{1,0}\wedge\bar\partial(\xi_{\theta_\omega^{0,1}}\lrcorner\gamma) \qquad (3.16)\\ &= -2\,Re\int\limits_S \Lambda_\omega(\gamma)\,|\partial\theta_\omega^{1,0}|^2_\omega\,dV_\omega - 2\,Re\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial\Lambda_\omega(\gamma)\wedge\theta_\omega^{0,1} - 2\,Re\,i\langle\langle\partial\bar\partial\theta_\omega^{1,0},\,\partial\gamma\rangle\rangle_\omega\\ &\quad -2\,Re\int\limits_S i\partial\theta_\omega^{1,0}\wedge\bar\partial(\xi_{\theta_\omega^{0,1}}\lrcorner\gamma) \qquad (3.17)\\ &= -2\,Re\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial\Lambda_\omega(\gamma\wedge\theta_\omega^{0,1}) - 2\,Re\,i\langle\langle\partial\bar\partial\theta_\omega^{1,0},\,\partial\gamma\rangle\rangle_\omega, \qquad (3.18)\end{aligned}$$

where $\star = \star_\omega$ *is the Hodge star operator defined by the metric* ω *and* $\xi_{\theta_\omega^{0,1}}$ *is the vector field of type* $(1,0)$ *defined by the requirement* $\xi_{\theta_\omega^{0,1}}\lrcorner\omega = i\theta_\omega^{0,1}$.

(ii) *In particular, for any given* $\omega \in \mathcal{H}_S$, *if we choose* $\gamma = \partial\theta_\omega^{0,1} + \bar\partial\theta_\omega^{1,0}$, *we have*

$$(d_\omega L)(\gamma) = -2\,Re\int\limits_S i\partial\theta_\omega^{1,0}\wedge\bar\partial\Big(\xi_{\theta_\omega^{0,1}}\lrcorner\gamma\Big) = -2\,Re\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial\Lambda_\omega(\gamma\wedge\theta_\omega^{0,1}).$$

Proof. (i) From (ii) and (iii) of Lemma 3.4.3 applied with $\alpha := i\theta_\omega^{0,1}$, we get

$$\star(\gamma\wedge\star\bar\partial\omega) = \star(\gamma\wedge i\theta_\omega^{0,1}) = i\,\Lambda_\omega(\gamma\wedge i\theta_\omega^{0,1}) = -\Lambda_\omega(\gamma)\,\theta_\omega^{0,1} - i\xi_{\theta_\omega^{0,1}}\lrcorner\gamma.$$

Formula (3.16) follows from this and from Lemma 3.4.1.

To get (3.17), we first notice that $\bar\partial\theta_\omega^{0,1} = \star\bar\partial\theta_\omega^{0,1}$ by the standard formula (3.2) applied to the (necessarily primitive) $(0,2)$-form $\bar\partial\theta_\omega^{0,1}$. This accounts for the first term on the r.h.s. of (3.17). Then, we transform the third term on the right-hand side of (3.16) as follows :

$$\begin{aligned}2\,\mathrm{Re}\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial\Lambda_\omega(\bar\partial\gamma) &\overset{(a)}{=} -2\,\mathrm{Re}\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial\star L_\omega\star(\bar\partial\gamma) \overset{(b)}{=} 2\,\mathrm{Re}\int\limits_S \bar\partial\partial\theta_\omega^{1,0}\wedge\star(\omega\wedge\star(\bar\partial\gamma))\\ &\overset{(c)}{=} 2\,\mathrm{Re}\,i\int\limits_S \bar\partial\partial\theta_\omega^{1,0}\wedge\star(\bar\partial\gamma) \overset{(d)}{=} 2\,\mathrm{Re}\,i\int\limits_S \langle\bar\partial\partial\theta_\omega^{1,0},\,\partial\bar\gamma\rangle_\omega\,dV_\omega,\end{aligned}$$

where we used the standard identity $\Lambda_\omega = -\star L_\omega\star$ on odd-degreed forms to get (a), Stokes to get (b), part (i) of Lemma 3.4.3 to get (c), and the definition of $\star$ to get (d). Finally, we recall that $\bar\gamma = \gamma$ since γ is real.

Finally, (3.18) follows from Lemma 3.4.1 after using the equality $\star(\gamma\wedge\star\bar\partial\omega) = -\Lambda_\omega(\gamma\wedge\theta_\omega^{0,1})$ (seen above in the proof of (3.16)) and after transforming the third term in (3.16) as we did above in the proof of (3.17).

(ii) The stated choice of γ means that γ is the component $(d\theta_\omega)^{1,1}$ of type $(1, 1)$ of the primitive 2-form $d\theta_\omega$. (See (i) of Lemma 3.2.2 for the primitivity statement.) Since $\Lambda_\omega((d\theta_\omega)^{2,0}) = 0$ and $\Lambda_\omega((d\theta_\omega)^{0,2}) = 0$ for bidegree reasons, we infer that

$$\Lambda_\omega(\gamma) = \Lambda_\omega((d\theta_\omega)^{1,1}) = \Lambda_\omega(d\theta_\omega) = 0.$$

Therefore, the first two integrals on the r.h.s. of (3.17) vanish.

Meanwhile, to handle the third integral on the r.h.s. of (3.17), we notice that $\partial\bar\gamma = \partial\bar\partial\theta_\omega^{1,0}$ and this gives the second equality below :

$$2\,\mathrm{Re}\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial\Lambda_\omega(\bar\partial\gamma) = 2\,\mathrm{Re}\, i\int\limits_S \langle\bar\partial\partial\theta_\omega^{1,0},\, \partial\bar\gamma\rangle_\omega\, dV_\omega = -2\,\mathrm{Re}\, i||\bar\partial\partial\theta_\omega^{1,0}||_\omega^2 = 0,$$

where the first equality above followed from the proof of (3.17).

Thus, the r.h.s. of formula (3.17) for $(d_\omega L)(\gamma)$ reduces to its last integral for this choice of γ. This proves the first claimed equality.

For the same reason as above, the latter term on the r.h.s. of formula (3.18) for $(d_\omega L)(\gamma)$ vanishes. This proves the second claimed equality. $\square$

As a first application of (i) of Theorem 3.4.4, we deduce the Euler-Lagrange equation for our functional in dimension 2. The next result can be compared with formula (2.15) of [Vai90].

Corollary 3.4.5 *Let S be a compact complex* **surface**. *The* **Euler-Lagrange equation** *for the functional L introduced in (i) of Definition 3.3.1 is*

$$\xi_{\theta_\omega^{1,0}}\lrcorner\bar\partial\partial\theta_\omega^{1,0} + \xi_{\theta_\omega^{0,1}}\lrcorner\partial\bar\partial\theta_\omega^{0,1} - i\partial^\star\partial\bar\partial\theta_\omega^{1,0} + i\bar\partial^\star\bar\partial\partial\theta_\omega^{0,1} = 0.$$

Proof. • We will use formula (3.18). The second term on its r.h.s. reads

$$-2\mathrm{Re}\, i\langle\langle\partial\bar\partial\theta_\omega^{1,0},\, \partial\gamma\rangle\rangle_\omega = \langle\langle -i\partial^\star\partial\bar\partial\theta_\omega^{1,0} + i\bar\partial^\star\bar\partial\partial\theta_\omega^{0,1},\, \gamma\rangle\rangle_\omega \tag{3.19}$$

for every real-valued $(1, 1)$-form γ.

• The integral in the first term on the r.h.s. of (3.18) reads

$$\begin{aligned}\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial\Lambda_\omega(\gamma\wedge\theta_\omega^{0,1}) &\overset{(a)}{=} \int\limits_S \partial\theta_\omega^{1,0}\wedge\star\Big(\bar\partial\Lambda_\omega(\gamma\wedge\theta_\omega^{0,1})\Big) = \langle\langle\partial\theta_\omega^{1,0},\, \partial\Lambda_\omega(\gamma\wedge\theta_\omega^{1,0})\rangle\rangle_\omega \\ &\overset{(b)}{=} -\langle\langle\star\bar\partial\star\partial\theta_\omega^{1,0},\, \Lambda_\omega(\gamma\wedge\theta_\omega^{1,0})\rangle\rangle_\omega \overset{(c)}{=} -\langle\langle\star\bar\partial\partial\theta_\omega^{1,0},\, \Lambda_\omega(\gamma\wedge\theta_\omega^{1,0})\rangle\rangle_\omega,\end{aligned} \tag{3.20}$$

where (a) and (c) followed from the standard formula (3.2) applied to a $(0, 2)$-form, resp. a $(2, 0)$-form, while (b) followed from the standard identity $\partial^\star = -\star\bar\partial\star$.

Now, $\bar\partial\partial\theta_\omega^{1,0}$ is a $(2, 1)$-form on the 2-dimensional complex manifold S. Since the pointwise map $L_\omega = \omega\wedge\cdot : \Lambda^{1,0}T^\star S \longrightarrow \Lambda^{2,1}T^\star S$ is bijective (cf. (A) of §3.2), for any $(2, 1)$-form Γ on S there exists a unique $(1, 0)$-form α such that $\Gamma = \omega\wedge\alpha$. The standard formula (3.2) applied to α yields $\star\alpha = -i\omega\wedge\alpha = -i\Gamma$. Taking $\star$ in the last equality, we get :

$$\alpha = i\,\star(\omega\wedge\alpha). \tag{3.21}$$

On the other hand,

$$\Lambda_\omega(\omega\wedge\alpha) = [\Lambda_\omega,\, L_\omega]\,\alpha = \alpha, \tag{3.22}$$

where we applied (3.3) with $n=2$ and $k=1$ to get the last equality. Putting (3.21) and (3.22) together, we conclude that

$$\star(\omega\wedge\alpha) = -i\,\Lambda_\omega(\omega\wedge\alpha)$$

for any $(1,\,0)$-form α on a Hermitian complex surface $(S,\,\omega)$.

In our case, considering the $(2,\,1)$-form $\Gamma := \bar\partial\partial\theta_\omega^{1,\,0}$ and the unique $(1,\,0)$-form α such that $\Gamma = \omega\wedge\alpha$, we conclude that $\star\bar\partial\partial\theta_\omega^{1,\,0} = \star(\omega\wedge\alpha) = -i\,\Lambda_\omega(\omega\wedge\alpha)$. Hence, (3.20) becomes :

$$\begin{aligned}\int\limits_S \partial\theta_\omega^{1,\,0}\wedge\bar\partial\Lambda_\omega(\gamma\wedge\theta_\omega^{0,\,1}) &= i\,\langle\langle\Lambda_\omega(\omega\wedge\alpha),\,\Lambda_\omega(\gamma\wedge\theta_\omega^{1,\,0})\rangle\rangle_\omega = i\,\langle\langle[L_\omega,\,\Lambda_\omega](\omega\wedge\alpha),\,\gamma\wedge\theta_\omega^{1,\,0}\rangle\rangle_\omega \\ &\stackrel{(a)}{=} -\langle\langle\bar\partial\partial\theta_\omega^{1,\,0},\,i\theta_\omega^{1,\,0}\wedge\gamma\rangle\rangle_\omega \stackrel{(b)}{=} -\langle\langle\xi_{\theta_\omega^{1,\,0}}\lrcorner\bar\partial\partial\theta_\omega^{1,\,0},\,\gamma\rangle\rangle_\omega,\end{aligned}$$

where (a) follows from $\omega\wedge\alpha = \bar\partial\partial\theta_\omega^{1,\,0}$ and from formula (3.3) applied for $k=3$ and $n=2$, while (b) follows from formula (3.6).

Thus, the first term on the r.h.s. of (3.18) reads :

$$-2\,\mathrm{Re}\int\limits_S \partial\theta_\omega^{1,\,0}\wedge\bar\partial\Lambda_\omega(\gamma\wedge\theta_\omega^{0,\,1}) = \langle\langle\xi_{\theta_\omega^{1,\,0}}\lrcorner\bar\partial\partial\theta_\omega^{1,\,0} + \xi_{\theta_\omega^{0,\,1}}\lrcorner\partial\bar\partial\theta_\omega^{0,\,1},\,\gamma\rangle\rangle_\omega, \tag{3.23}$$

for every real-valued $(1,\,1)$-form γ.

• Formulae (3.19) and (3.23) prove the contention. $\square$

As another application of (i) of Theorem 3.4.4, we will now see that the differential $d_\omega L$ vanishes on all the real $(1,\,1)$-forms γ that are ω-*anti-primitive* (in the sense that γ is $\langle\ ,\ \rangle_\omega$-orthogonal to all the ω-primitive $(1,\,1)$-forms, a condition which is equivalent to γ being a function multiple of ω). Since $(d_\omega L)(f\omega)$ computes the variation of L in a conformal class, the following statement also follows with no computations from Proposition 3.6.1 which shows that the functional L is conformally invariant when $n=2$. However, we prefer giving a direct proof at this point.

Corollary 3.4.6 *Let S be a compact complex* **surface** *on which a Hermitian metric ω has been fixed. For any real-valued C^∞ function f on X, we have*

$$(d_\omega L)(f\omega) = 0.$$

In particular, for any real $(1,\,1)$-form γ on S we have

$$(d_\omega L)(\gamma) = (d_\omega L)(\gamma_{prim}),$$

where γ_{prim} is the ω-primitive component of γ in its Lefschetz decomposition.

Proof. Applying formula (3.16) with $\gamma = f\omega$ and using the obvious equalities $\Lambda_\omega(f\omega) = 2f$ (recall that $\dim_{\mathbb{C}} S = 2$) and $\xi_{\theta_\omega^{0,1}} \lrcorner (f\omega) = f\,(i\theta_\omega^{0,1})$, we get :

$$\begin{aligned}(d_\omega L)(f\omega) &= -4\,\mathrm{Re}\int\limits_S f\,\partial\theta_\omega^{1,0}\wedge\bar\partial\theta_\omega^{0,1} - 4\,\mathrm{Re}\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial f\wedge\theta_\omega^{0,1} \\ &\quad +2\,\mathrm{Re}\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial\Lambda_\omega(f\bar\partial\omega + \bar\partial f\wedge\omega) - 2\,\mathrm{Re}\int\limits_S i\partial\theta_\omega^{1,0}\wedge(if\bar\partial\theta_\omega^{0,1} + i\bar\partial f\wedge\theta_\omega^{0,1}) \\ &= T_1 + T_2 + T_3 + T_4,\end{aligned} \tag{3.24}$$

where T_1, T_2, T_3 and T_4 stand for the four terms, listed in order, on the r.h.s. of the above expression for $(d_\omega L)(f\omega)$.

Computing T_3, we get :

$$T_3 = 2\,\mathrm{Re}\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial(f\,\theta_\omega^{0,1}) + 2\,\mathrm{Re}\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial\Big([\Lambda_\omega,\,L_\omega](\bar\partial f)\Big),$$

where we used the equalities $\Lambda_\omega(\bar\partial\omega) = \theta_\omega^{0,1}$ (see (3.8)) and $\Lambda_\omega(\bar\partial f) = 0$ (which leads to $\Lambda_\omega(\bar\partial f\wedge\omega) = [\Lambda_\omega,\,L_\omega](\bar\partial f)$). Now, it is standard that $[\Lambda_\omega,\,L_\omega] = (n-k)\,\mathrm{Id}$ on k-forms on an n-dimensional complex manifold, so in our case we get $[\Lambda_\omega,\,L_\omega](\bar\partial f) = \bar\partial f$ since $n = 2$ and $k = 1$. We conclude that $\bar\partial([\Lambda_\omega,\,L_\omega](\bar\partial f)) = \bar\partial^2 f = 0$, hence

$$T_3 = 2\,\mathrm{Re}\int\limits_S f\,\partial\theta_\omega^{1,0}\wedge\bar\partial\theta_\omega^{0,1} + 2\,\mathrm{Re}\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial f\wedge\theta_\omega^{0,1} = T_4,$$

where the last equality follows at once from the definition of T_4.

Thus, formula (3.24) translates to

$$\begin{aligned}(d_\omega L)(f\omega) &= T_1 + T_2 + T_3 + T_4 \\ &= (-4+4)\,\mathrm{Re}\int\limits_S f\,\partial\theta_\omega^{1,0}\wedge\bar\partial\theta_\omega^{0,1} + (-4+4)\,\mathrm{Re}\int\limits_S \partial\theta_\omega^{1,0}\wedge\bar\partial f\wedge\theta_\omega^{0,1} \\ &= 0.\end{aligned}$$

This proves the first statement.

The second statement follows at once from the first, from the linearity of the map $d_\omega L$ and from the Lefschetz decomposition $\gamma = \gamma_{prim} + (1/2)\,\Lambda_\omega(\gamma)\,\omega$. $\square$

We hope that it will be possible in the future to prove that any Hermitian metric ω on a compact complex surface that is a critical point for the functional L is actually an lcK metric.

3.5 First variation of the functional : case of dimension ≥ 3

In this section, we suppose that the complex dimension of X is $n \geq 3$. The goal is to compute the differential of the energy functional L introduced in Definition 3.3.1-(ii). Let ω be a Hermitian metric on X and let γ be a real $(1,1)$-form. The latter can bee seen as a tangent vector to $\mathcal{H}_X$ at ω.

Theorem 3.5.1 *For any Hermitian metric ω and any real $(1,1)$-form γ, we have :*

$$\begin{aligned}(d_\omega L)(\gamma) &= \int_X i(\bar\partial\omega)_{prim}\wedge\overline{(\bar\partial\omega)_{prim}}\wedge\gamma\wedge\omega_{n-4}\\ &+2Re\,\langle\langle(\bar\partial\omega)_{prim},\,(\bar\partial\gamma)_{prim}\rangle\rangle_\omega - 2Re\,\langle\langle\theta^{0,1}_\omega\wedge\gamma,\,(\bar\partial\omega)_{prim}\rangle\rangle_\omega.\end{aligned}\tag{3.25}$$

Proof. Recall (cf. the conjugate of (3.11)) that $(n-1)\,\theta^{0,1}_\omega = \Lambda_\omega(\bar\partial\omega)$ for any Hermitian metric ω. Now, for any real t sufficiency close to 0, $\omega+t\gamma$ is again a Hermitian metric on X. Taking $\alpha_t = \bar\partial\omega + t\,\bar\partial\gamma$ in Lemma 3.4.2, we get the second equality below :

$$(n-1)\frac{d}{dt}\bigg|_{t=0}\theta^{0,1}_{\omega+t\gamma} = \frac{d}{dt}\bigg|_{t=0}\Lambda_{\omega+t\gamma}(\bar\partial\omega+t\bar\partial\gamma) = \Lambda_\omega(\bar\partial\gamma) - (\gamma\wedge\cdot)^\star_\omega\,(\bar\partial\omega).\tag{3.26}$$

On the other hand, taking $(d/dt)_{|t=0}$ in the expression for $L(\omega+t\gamma)$ given in (ii) of Definition 3.3.1 (with $\omega+t\gamma$ in place of ω), we get :

$$(d_\omega L)(\gamma) = \frac{d}{dt}\bigg|_{t=0}L(\omega+t\gamma) = \frac{d}{dt}\bigg|_{t=0}\int_X i(\bar\partial\omega+t\bar\partial\gamma)_{prim}\wedge\overline{(\bar\partial\omega+t\bar\partial\gamma)_{prim}}\wedge(\omega+t\gamma)_{n-3},\tag{3.27}$$

where the subscript *prim* indicates the $(\omega+t\gamma)$-primitive part of the form to which it is attached.

Now, consider the Lefschetz decompositions (cf. (3.1)) of $\bar\partial\omega$ and $\bar\partial\gamma$ with respect to ω :

$$\begin{aligned}\bar\partial\omega &= (\bar\partial\omega)_{prim} + \theta^{0,1}_\omega\wedge\omega\\ \bar\partial\gamma &= (\bar\partial\gamma)_{prim} + \theta^{0,1}_\gamma\wedge\omega\end{aligned}$$

and the Lefschetz decomposition of $\bar\partial\omega+t\bar\partial\gamma$ with respect to $\omega+t\gamma$:

$$\bar\partial\omega+t\bar\partial\gamma = (\bar\partial\omega+t\bar\partial\gamma)_{prim} + \theta^{0,1}_{\omega+t\gamma}\wedge(\omega+t\gamma).$$

By the above equations we get :

$$(\bar\partial\omega+t\bar\partial\gamma)_{prim} = (\bar\partial\omega)_{prim} + \theta^{0,1}_\omega\wedge\omega + t\,(\bar\partial\gamma)_{prim} + t\,\theta^{0,1}_\gamma\wedge\omega - \theta^{0,1}_{\omega+t\gamma}\wedge(\omega+t\gamma),\tag{3.28}$$

where primitivity is construed w.r.t. the metric $\omega+t\gamma$ in the case of the left-hand side term and w.r.t. the metric ω in the case of $(\bar\partial\omega)_{prim}$ and $(\bar\partial\gamma)_{prim}$.

Thanks to (3.28), equality (3.27) becomes :

$$\begin{aligned}(d_\omega L)(\gamma) &= \frac{d}{dt}\bigg|_{t=0}\int_X i\bigg((\bar\partial\omega)_{prim} + \theta^{0,1}_\omega\wedge\omega + t\,(\bar\partial\gamma)_{prim} + t\,\theta^{0,1}_\gamma\wedge\omega - \theta^{0,1}_{\omega+t\gamma}\wedge(\omega+t\gamma)\bigg)\\ &\wedge\bigg(\overline{(\bar\partial\omega)_{prim}} + \overline{\theta^{0,1}_\omega}\wedge\omega + t\,\overline{(\bar\partial\gamma)_{prim}} + t\,\overline{\theta^{0,1}_\gamma}\wedge\omega - \overline{\theta^{0,1}_{\omega+t\gamma}}\wedge(\omega+t\gamma)\bigg)\wedge(\omega+t\gamma)_{n-3}.\end{aligned}$$

Now,

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\bigg(\theta^{0,1}_{\omega+t\gamma}\wedge(\omega+t\gamma)\bigg) &= \theta^{0,1}_\omega\wedge\gamma + \bigg(\frac{d}{dt}\bigg|_{t=0}\theta^{0,1}_{\omega+t\gamma}\bigg)\wedge\omega\\ &= \theta^{0,1}_\omega\wedge\gamma + \frac{1}{n-1}\bigg(\Lambda_\omega(\bar\partial\gamma) - (\gamma\wedge\cdot)^\star_\omega(\bar\partial\omega)\bigg)\wedge\omega,\end{aligned}$$

where formula (3.26) was used to get the last equality. Using this, straightforward computations yield :

$$(d_\omega L)(\gamma) = I_1 + \overline{I_1} + I_2, \tag{3.29}$$

where

$$\begin{aligned} I_2 &= \int_X i\bigg((\bar\partial\omega)_{prim} + \theta_\omega^{0,\,1}\wedge\omega - \theta_\omega^{0,\,1}\wedge\omega\bigg)\wedge\bigg(\overline{(\bar\partial\omega)_{prim}} + \overline{\theta_\omega^{0,\,1}}\wedge\omega - \overline{\theta_\omega^{0,\,1}}\wedge\omega\bigg)\wedge\omega_{n-4}\wedge\gamma \\ &= \int_X i(\bar\partial\omega)_{prim}\wedge\overline{(\bar\partial\omega)_{prim}}\wedge\omega_{n-4}\wedge\gamma \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} I_1 &= \int_X i\bigg[(\bar\partial\gamma)_{prim} + \theta_\gamma^{0,\,1}\wedge\omega - \theta_\omega^{0,\,1}\wedge\gamma - \frac{1}{n-1}\bigg(\Lambda_\omega(\bar\partial\gamma) - (\gamma\wedge\cdot)^\star_\omega(\bar\partial\omega)\bigg)\wedge\omega\bigg]\wedge(\partial\omega)_{prim}\wedge\omega_{n-3} \\ &= \int_X i(\bar\partial\gamma)_{prim}\wedge(\partial\omega)_{prim}\wedge\omega_{n-3} - \int_X i\,\theta_\omega^{0,\,1}\wedge\gamma\wedge(\partial\omega)_{prim}\wedge\omega_{n-3}, \end{aligned} \tag{3.31}$$

where the last equality follows from $(\partial\omega)_{prim}\wedge\omega_{n-2} = 0$ (a consequence of the ω-primitivity of the 3-form $(\partial\omega)_{prim}$) which leads to the vanishing of the products of the second and the fourth terms (that are multiples of ω) inside the large parenthesis with $(\partial\omega)_{prim}\wedge\omega_{n-3}$ in the integral on the first line of (3.31).

Now, due to the ω-primitivity of the 3-form $(\partial\omega)_{prim}$, the standard formula (3.2) yields :

$$\star(\partial\omega)_{prim} = i\,(\partial\omega)_{prim}\wedge\omega_{n-3}, \tag{3.32}$$

where $\star = \star_\omega$ is the Hodge star operator induced by ω. Thus, (3.31) translates to

$$\begin{aligned} I_1 &= \int_X (\bar\partial\gamma)_{prim}\wedge\star\overline{(\bar\partial\omega)_{prim}} - \int_X \theta_\omega^{0,\,1}\wedge\gamma\wedge\star\overline{(\bar\partial\omega)_{prim}} \\ &= \langle\langle(\bar\partial\gamma)_{prim},\,(\bar\partial\omega)_{prim}\rangle\rangle_\omega - \langle\langle\theta_\omega^{0,\,1}\wedge\gamma,\,(\bar\partial\omega)_{prim}\rangle\rangle_\omega. \end{aligned}$$

This last formula for I_1, together with (3.29) and (3.30), proves the contention. □

The first application of Theorem 3.5.1 that we give is the computation of the Euler-Lagrange equation for our energy functional L in dimension $n > 2$.

Corollary 3.5.2 *Let X be a compact complex manifold with $dim_{\mathbb{C}}X = n \geq 3$. The* **Euler-Lagrange equation** *for the functional L introduced in (ii) of Definition 3.3.1 is*

$$\star\bigg(i(\bar\partial\omega)_{prim}\wedge\overline{(\bar\partial\omega)_{prim}}\wedge\omega_{n-4}\bigg) + \bigg(\bar\partial^\star + i\xi_{\theta_\omega^{0,\,1}}\lrcorner\,\cdot\bigg)(\bar\partial\omega)_{prim} + \bigg(\partial^\star - i\xi_{\theta_\omega^{1,\,0}}\lrcorner\,\cdot\bigg)(\partial\omega)_{prim} = 0,$$

where the Hodge star operator $\star$, the adjoints and the primitive parts are computed w.r.t. the Hermitian metric ω, the unknown of the equation.

Proof. Using the general formula $\alpha\wedge\beta=\star\alpha\wedge\star\beta$ given in Lemma 5.1. of [Pop22] for any differential forms such that $\deg\alpha+\deg\beta=2n$, the first term on the r.h.s. of (3.25) transforms as

$$\begin{aligned}\int_X i(\bar\partial\omega)_{prim}\wedge\overline{(\bar\partial\omega)_{prim}}\wedge\gamma\wedge\omega_{n-4} &= \int_X\star\bigg(i(\bar\partial\omega)_{prim}\wedge\overline{(\bar\partial\omega)_{prim}}\wedge\omega_{n-4}\bigg)\wedge\star\overline{\gamma}\\ &= \bigg\langle\bigg\langle\star\bigg(i(\bar\partial\omega)_{prim}\wedge\overline{(\bar\partial\omega)_{prim}}\wedge\omega_{n-4}\bigg),\,\gamma\bigg\rangle\bigg\rangle \qquad (3.33)\end{aligned}$$

for any real (1, 1)-form γ.

The second term on the r.h.s. of (3.25) transforms as

$$\begin{aligned}2\,\mathrm{Re}\,\langle\langle(\bar\partial\omega)_{prim},\,(\bar\partial\gamma)_{prim}\rangle\rangle_\omega &= 2\mathrm{Re}\,\langle\langle(\bar\partial\omega)_{prim},\,\bar\partial\gamma\rangle\rangle_\omega=2\,\mathrm{Re}\,\langle\langle\bar\partial^\star(\bar\partial\omega)_{prim},\,\gamma\rangle\rangle_\omega\\ &= \langle\langle\bar\partial^\star(\bar\partial\omega)_{prim}+\partial^\star(\partial\omega)_{prim},\,\gamma\rangle\rangle_\omega \qquad (3.34)\end{aligned}$$

for any real (1, 1)-form γ. The first equality above followed from the Lefschetz decomposition $\bar\partial\gamma=(\bar\partial\gamma)_{prim}+\omega\wedge u$ (with some (0, 1)-form u) and from $\Lambda_\omega((\bar\partial\omega)_{prim})=0$.

The third term on the r.h.s. of (3.25) transforms, for any real (1, 1)-form γ, as

$$\begin{aligned}2\,\mathrm{Re}\,\langle\langle\theta^{0,\,1}_\omega\wedge\gamma,\,(\bar\partial\omega)_{prim}\rangle\rangle_\omega &= 2\,\mathrm{Re}\,\langle\langle(\bar\partial\omega)_{prim},\,\theta^{0,\,1}_\omega\wedge\gamma\rangle\rangle_\omega=2\,\mathrm{Re}\,\langle\langle-i\xi_{\theta^{0,\,1}_\omega}\lrcorner(\bar\partial\omega)_{prim},\,\gamma\rangle\rangle_\omega\\ &= \langle\langle-i\xi_{\theta^{0,\,1}_\omega}\lrcorner(\bar\partial\omega)_{prim}+i\xi_{\theta^{1,\,0}_\omega}\lrcorner(\partial\omega)_{prim},\,\gamma\rangle\rangle_\omega, \qquad (3.35)\end{aligned}$$

where we used formula (3.5) to get the last equality on the first line.

The contention follows from Theorem 3.5.1 by putting together (3.33), (3.34) and (3.35). □

Recall that we are interested in the set of critical points of L. We now notice that a suitable choice of γ in the previous result leads to an explicit description of this set. Since equation (3.25) is valid for all real (1, 1)-forms γ, the choice $\gamma=\omega$ is licit, as any other choice. We get the following

Corollary 3.5.3 *Let X be a compact complex manifold with $\dim_{\mathbb{C}}X=n\geq 3$ and let L be the functional defined in 3.3.1-(ii). For any Hermitian metric ω on X, we have :*

$$(d_\omega L)(\omega)=(n-1)\,\|(\bar\partial\omega)_{prim}\|^2_\omega=(n-1)\,L(\omega). \qquad (3.36)$$

Proof. Taking $\gamma=\omega$ in equation (3.25), we get :

$$\begin{aligned}(d_\omega L)(\omega) &= \int_X i(\bar\partial\omega)_{prim}\wedge\overline{(\bar\partial\omega)_{prim}}\wedge\omega\wedge\omega_{n-4}+2\mathrm{Re}\,\langle\langle(\bar\partial\omega)_{prim},\,(\bar\partial\omega)_{prim}\rangle\rangle_\omega\\ &\quad -2\mathrm{Re}\,\langle\langle\overline{\theta^{0,\,1}_\omega}\wedge\omega,\,\overline{(\bar\partial\omega)_{prim}}\rangle\rangle_\omega\\ &= (n-3)i\int_X(\bar\partial\omega)_{prim}\wedge\overline{(\bar\partial\omega)_{prim}}\wedge\omega_{n-3}+2\,\|(\bar\partial\omega)_{prim}\|^2_\omega-2\mathrm{Re}\,\langle\langle\overline{\theta^{0,\,1}_\omega},\,\Lambda_\omega((\partial\omega)_{prim})\rangle\rangle_\omega\\ &= (n-1)\|(\bar\partial\omega)_{prim}\|^2_\omega,\end{aligned}$$

where the last equality followed from $\overline{(\bar\partial\omega)_{prim}}\wedge\omega_{n-3}=-i\,\star\,\overline{(\bar\partial\omega)_{prim}}$ (see (3.32)) and from $\Lambda_\omega((\partial\omega)_{prim}))=0$ (due to any ω-primitive form lying in the kernel of Λ_ω).

□

An immediate consequence of Corollary 3.5.3 is the following

Proposition 3.5.4 *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 3$ and let ω be a Hermitian metric on X.*

If ω is a critical point for the functional L defined in 3.3.1-(ii), then ω is lcK.

Proof. If ω is a critical point for L, then $(d_\omega L)(\gamma) = 0$ for any real $(1, 1)$-form γ on X. Taking $\gamma = \omega$ and using (3.36), we get $(\bar\partial\omega)_{prim} = 0$. By (ii) of Lemma 3.2.2, this is equivalent to ω being lcK. □

The converse follows trivially from what we already know. Indeed, if ω is an lcK metric, $L(\omega) = 0$ (by Lemma 3.3.2), so L achieves its minimum at ω since $L \geq 0$. Any minimum is, of course, a critical point.

3.6 Normalised energy functionals when $\dim_{\mathbb{C}} X \geq 3$

We start with the immediate observation that the functional introduced in (i) of Definition 3.3.1 in the case of compact complex surfaces is conformally invariant. In particular, it is scaling-invariant, so it does not need normalising.

Proposition 3.6.1 *Let S be a compact complex surface.*

(i) For any Hermitian metric ω on S and any C^∞ function $\lambda : S \longrightarrow (0, +\infty)$, the following formula holds :

$$\theta^{1,0}_{\lambda\omega} = \theta^{1,0}_\omega + \frac{1}{\lambda}\,\partial\lambda.$$

(ii) The functional $L : \mathcal{H}_S \longrightarrow [0, +\infty)$, $L(\omega) = \int_X \partial\theta^{1,0}_\omega \wedge \bar\partial\theta^{0,1}_\omega$, has the property :

$$L(\lambda\omega) = L(\omega)$$

for every C^∞ function $\lambda : S \longrightarrow (0, +\infty)$ and every Hermitian metric ω on S.

Proof. (i) Recall (cf. (3.9)) that $\theta^{1,0}_\omega = \Lambda_\omega(\partial\omega)$ and $\theta^{0,1}_\omega = \Lambda_\omega(\bar\partial\omega)$.

On the other hand, for any function $\lambda > 0$ and any form α of any bidegree (p, q), we have :

$$\Lambda_{\lambda\omega}\alpha = \frac{1}{\lambda}\,\Lambda_\omega\alpha,$$

as can be checked right away. Therefore, we get :

$$\theta^{1,0}_{\lambda\omega} = \Lambda_{\lambda\omega}\bigg(\partial(\lambda\omega)\bigg) = \frac{1}{\lambda}\,\Lambda_\omega(\lambda\,\partial\omega) + \frac{1}{\lambda}\,\Lambda_\omega(\partial\lambda\wedge\omega) = \theta^{1,0}_\omega + \frac{1}{\lambda}\,[\Lambda_\omega,\, L_\omega](\partial\lambda) = \theta^{1,0}_\omega + \frac{1}{\lambda}\,\partial\lambda,$$

where we applied (3.3) with $n = 2$ and $k = 1$ to get the last equality.

(ii) Taking ∂ in the formula proved under (i), we get :

$$\partial\theta^{1,0}_{\lambda\omega} = \partial\theta^{1,0}_\omega - \frac{1}{\lambda^2}\,\partial\lambda\wedge\partial\lambda = \partial\theta^{1,0}_\omega.$$

By conjugation, we also get $\bar\partial\theta^{0,1}_{\lambda\omega} = \bar\partial\theta^{0,1}_\omega$ and the contention follows. □

By contrast, the functional $L:\mathcal{H}_X\longrightarrow[0,\,+\infty)$ introduced in (ii) of Definition 3.3.1 in the case of compact complex manifolds X with $\mbox{dim}_\mathbb{C}X=n\geq 3$ is not scaling-invariant. Indeed, it follows at once from its definition that

$$L(\lambda\omega)=\lambda^{n-1}\,L(\omega) \tag{3.37}$$

for every constant $\lambda>0$ and every Hermitian metric ω on X.

This homogeneity property of L can be used to derive a short proof of the main property of L that was deduced in §.3.5 from the result of the computation of the first variation of L, namely from Theorem 3.5.1.

Proposition 3.6.2 *(Proposition 3.5.4 revisited) Let X be a compact complex manifold with $\mbox{dim}_\mathbb{C}X=n\geq 3$ and let ω be a Hermitian metric on X. The following equivalence holds :*

ω is a **critical point** *for the functional L defined in 3.3.1-(ii) if and only if ω is* **lcK**.

Proof. Suppose ω is a critical point for L. This means that $(d_\omega L)(\gamma)=0$ for every real $(1,\,1)$-form γ on X. Taking $\gamma=\omega$, we get the first eqsuality below :

$$0=(d_\omega L)(\omega)=\frac{d}{dt}\bigg|_{t=0}L(\omega+t\omega)=\frac{d}{dt}\bigg|_{t=0}\bigg((1+t)^{n-1}\,L(\omega)\bigg)=(n-1)\,L(\omega).$$

Thus, whenever ω is a critical point for L, $L(\omega)=0$. This last fact is equivalent to the metric ω being lcK thanks to Lemma 3.3.2.

Conversely, if ω is lcK, it is a minimum point for L, hence also a critical point, because $L(\omega)=0$ by Lemma 3.3.2. $\square$

On the other hand, recall the following by now standard

Observation 3.6.3 *Let ω be a Hermitian metric on a complex manifold X with $\mbox{dim}_\mathbb{C}X=n\geq 2$. If ω is both* **lcK** *and* **balanced**, *ω is* **Kähler**.

Proof. The Lefschetz decomposition of $d\omega$ spells $d\omega=(d\omega)_{prim}+\omega\wedge\theta$, where $(d\omega)_{prim}$ is an ω-primitive 3-form and θ is a 1-form on X.

We saw in Lemma 3.2.2 that ω is lcK if and only if $(d\omega)_{prim}=0$. On the other hand, the following equivalences hold :

$$\omega\,\,\mbox{is balanced}\,\,\,\iff d\omega^{n-1}=0\iff\omega^{n-2}\wedge d\omega=0\iff d\omega\,\,\mbox{is}\,\,\omega\mbox{-primitive}\,\,\iff d\omega=(d\omega)_{prim}.$$

We infer that, if ω is both lcK and balanced, $d\omega=0$, so ω is Kähler. $\square$

It is tempting to conjecture the existence of a Kähler metric in the more general situation where the lcK and balanced hypotheses are spread over possibly different metrics.

Conjecture 3.6.4 *Let X be a compact complex manifold with $\mbox{dim}_\mathbb{C}X\geq 3$. If an lcK metric ω and a balanced metric ρ exist on X, there exists a Kähler metric on X.*

Together with the behaviour of L under rescaling (see (3.37)), this conjecture suggests a natural normalisation for our functional L when $n \geq 3$.

Definition 3.6.5 *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 3$. Fix a Hermitian metric ρ on X. We define the ρ-dependent functional acting on the Hermitian metrics of X :*

$$\widetilde{L}_\rho : \mathcal{H}_X \to [0, +\infty), \qquad \widetilde{L}_\rho(\omega) := \frac{L(\omega)}{\bigg(\int_X \omega \wedge \rho_{n-1}\bigg)^{n-1}}, \tag{3.38}$$

where L is the functional introduced in (ii) of Definition 3.3.1.

It follows from (3.37) that the normalised functional $\widetilde{L}_\rho$ is scaling-invariant :

$$\widetilde{L}_\rho(\lambda\,\omega) = \widetilde{L}_\rho(\omega)$$

for every constant $\lambda > 0$. Moreover, thanks to Lemma 3.3.2, $\widetilde{L}_\rho(\omega) = 0$ if and only of ω is an lcK metric on X.

We now derive the formula for the first variation of the normalised functional $\widetilde{L}_\rho$ in terms of the similar expression for the unnormalised functional L that was computed in Theorem 3.5.1.

Proposition 3.6.6 *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 3$. Fix a Hermitian metric ρ on X. Then, for any Hermitian metric ω and any real $(1,1)$-form γ on X, we have :*

$$(d_\omega \widetilde{L}_\rho)(\gamma) = \frac{1}{\bigg(\int_X \omega \wedge \rho_{n-1}\bigg)^{n-1}} \Bigg((d_\omega L)(\gamma) - (n-1)\,\frac{\int_X \gamma \wedge \rho_{n-1}}{\int_X \omega \wedge \rho_{n-1}}\, L(\omega)\Bigg), \tag{3.39}$$

where $(d_\omega L)(\gamma)$ is given by formula (3.25) in Theorem 3.5.1.

Proof. Straightforward computations yield :

$$\begin{aligned}(d_\omega \widetilde{L}_\rho)(\gamma) &= \frac{d}{dt}\Bigg[\frac{1}{\bigg(\int_X (\omega + t\gamma) \wedge \rho_{n-1}\bigg)^{n-1}}\, L(\omega + t\gamma)\Bigg]_{t=0} = \frac{1}{\bigg(\int_X \omega \wedge \rho_{n-1}\bigg)^{n-1}}\,(d_\omega L)(\gamma) \\ &\quad - \frac{1}{\bigg(\int_X \omega \wedge \rho_{n-1}\bigg)^{2(n-1)}}\,(n-1)\,\bigg(\int_X \omega \wedge \rho_{n-1}\bigg)^{n-2}\,\bigg(\int_X \gamma \wedge \rho_{n-1}\bigg)\, L(\omega).\end{aligned}$$

This is formula (3.39). $\square$

A natural question is whether the critical points of any (or some) of the normalised functionals $\widetilde{L}_\rho$ are precisely the lcK metrics (if any) on X. The following result goes some way in this direction.

Corollary 3.6.7 *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 3$. Fix a Hermitian metric ρ on X. Suppose a Hermitian metric ω is a* **critical point** *for $\widetilde{L}_\rho$. Then :*

*(i) for every ρ-***primitive** *real $(1,1)$-form γ, $(d_\omega L)(\gamma) = 0$.*

(ii) if the metric ρ is **Gauduchon**, *$(d_\omega L)(i\partial\bar\partial\varphi) = 0$ for any real-valued C^2 function φ on X.*

Proof. (i) If γ is ρ-primitive, then $\gamma \wedge \rho_{n-1} = 0$, so formula (3.39) reduces to

$$(d_\omega \widetilde{L}_\rho)(\gamma) = \frac{(d_\omega L)(\gamma)}{\left(\int_X \omega \wedge \rho_{n-1}\right)^{n-1}}.$$

Meanwhile, $(d_\omega \widetilde{L}_\rho)(\gamma) = 0$ for every real $(1,\,1)$-form γ since ω is a critical point for $\widetilde{L}_\rho$. The contention follows.

(ii) Choose $\gamma := \omega + i\partial\bar\partial\varphi$ for any function φ as in the statement. We get :

$$0 \stackrel{(a)}{=} \left(\int_X \omega \wedge \rho_{n-1}\right)^{n-1} (d_\omega \widetilde{L}_\rho)(\omega + i\partial\bar\partial\varphi) \stackrel{(b)}{=} (d_\omega L)(\omega) - (n-1)\,L(\omega) + (d_\omega L)(i\partial\bar\partial\varphi) \stackrel{(c)}{=} (d_\omega L)(i\partial\bar\partial\varphi),$$

where ω being a critical point for $\widetilde{L}_\rho$ gave (a), formula (3.39) and the metric ρ being Gauduchon (the latter piece of information implying $\int_X i\partial\bar\partial\varphi \wedge \rho_{n-1} = 0$ thanks to the Stokes theorem) gave (b), while Corollary 3.5.3 gave (c). $\square$

As in the case of surfaces, our hope is that it will be possible in the future to prove that any Hermitian metric ω on a compact complex manifold of dimension ≥ 3 that is a critical point for one (or all) of the normalised functionals $\widetilde{L}_\rho$ is actually an lcK metric.

Concluding remarks.

(a) Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 3$. Fix a Hermitian metric ρ on X and consider the set U_ρ of ρ-normalised Hermitian metrics ω on X such that

$$\int\limits_X \omega \wedge \rho_{n-1} = 1.$$

By Definition 3.6.5, we have $\widetilde{L}_\rho(\omega) = L(\omega)$ for every $\omega \in U_\rho$. Moreover, since $\widetilde{L}_\rho$ is scaling-invariant, it is completely determined by its restriction to U_ρ. Let

$$c_\rho := \inf_{\omega \in \mathcal{H}_X} \widetilde{L}_\rho(\omega) = \inf_{\omega \in U_\rho} \widetilde{L}_\rho(\omega) = \inf_{\omega \in U_\rho} L(\omega) \geq 0.$$

For every $\varepsilon > 0$, there exists a Hermitian metric $\omega_\varepsilon \in U_\rho$ such that $c_\rho \leq L(\omega_\varepsilon) < c_\rho + \varepsilon$. Since U_ρ is a relatively compact subset of the space of positive $(1,\,1)$-currents equipped with the weak topology of currents, there exists a subsequence $\varepsilon_k \downarrow 0$ and a positive (see e.g. the terminology of [Dem97, III-1.B.]) $(1,\,1)$-current $T_\rho \geq 0$ on X such that the sequence $(\omega_{\varepsilon_k})_k$ converges weakly to T_ρ as $k \to +\infty$. By construction, we have :

$$\int\limits_X T_\rho \wedge \rho_{n-1} = 1.$$

The possible failure of the current $T_\rho \geq 0$ to be either a C^∞ form or strictly positive (for example in the sense that it is bounded below by a positive multiple of a Hermitian metric on X) constitutes an obstruction to the existence of minimisers for the functional $\widetilde{L}_\rho$. If it eventually turns out that the critical points of $\widetilde{L}_\rho$, if any, are precisely the lcK metrics of X, if any, they will further coincide with the minimisers of $\widetilde{L}_\rho$. In that case, the currents T_ρ will provide obstructions to the existence of lcK metrics on X.

(b) The same discussion as in the above (a) can be had on a compact complex surface S using the (already scaling-invariant) functional L introduced in (i) of Definition 3.3.1 if one can prove that its critical points coincide with the lcK metrics on S.

Chapitre 4

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