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Introduction

The classification of compact complex manifolds is based on the study of algebraic and analytic properties.

Algebraic aspects of compact complex manifolds can be described using cohomology groups; in particular, the most used are the de Rham's, the Dolbeualt's, the Bott-Chern's and the Aeppli's cohomology groups. The de Rham's cohomology is associated to the complex of differentiable forms and it depends only on the topological structure of the manifold. Conversely, since the Dolbeault's cohomology is associated to the complex of holomorphic forms, it depends both on the topology and on the complex structure of the manifold. Although the definitions of those cohomology groups are similar, in general there is no direct relation between them. This is one of the reasons because, in [Sch07], Schweitzer introduced the other two cohomology groups. They provide, with the homomorphisms induced by the identity, a bridge between the de Rham's and the Dolbeault's cohomology. Other important algebraic tools in the study of relations between cohomology groups are the $\partial\overline{\partial}$-Lemma, the Frölicher spectral sequence and the Varouchas spaces. The first is a result of Deligne, Griffiths, Morgan and Sullivan ([DGMS75]) that provides a necessary and sufficient condition such that all the cohomology groups (of the same bi-degree) are isomorphic. The second is a sequence of complexes such that the first term is the complex of $\mathcal{C}^\infty$-forms associated to the operator $\overline{\partial}$. At every successive step we take the cohomology of the previous complex. Then, at the second step, we obtain the Dolbeault's cohomology associated to the operator $\overline{\partial}$. By the construction of the Frölicher spectral sequence, after a finite number k of steps, every complex is isomorphic to the successives and we said that the sequence *degenerates* at the k-th level. We have that the k-th term of this sequence is isomorphic to the de Rham's cohomology. The last tool was introduced in [Var], those auxiliary groups are used to construct exact sequences that involve the Dolbeualt's, Bott-Chern's and Aeppli's cohomology groups.

Among the analytic aspects of complex geometry there is the existence of special Hermitian metrics on a compact complex manifold. A Hermitian metric g is called *special* if its fundamental form ω satisfies certain differential conditions. The most famous among these metrics are the *Kähler*, i.e., Hermitian metrics such that $d\omega = 0$. Such metrics are the natural generalization of the flat metric in the Euclidean space; moreover the presence of a Kähler metric has a strong influence on the topology of the manifold: for example the odds Betti's numbers are even. More recently non-Kähler metrics with important properties have been studied. One of the first results is due to Michelson: in [Mic82] she defined the *balanced* metrics. Such metrics are characterized by the vanishing of the $(1,0)$ torsion tensor and have been studied by several other authors ([AB93], [Sil]). Another important class of metrics is the class of *strong Kähler with torsion* metrics (shortly SKT). They were introduced by Bismut in [Bis89] and have found great applications in physics ([HP88], [Str86]). Moreover, in [FPS04], it has been proved a characterization result for solvmanifolds that admits SKT metrics.

Many authors have studied relations between algebraic and analytic properties of compact complex manifolds. For example, every Kähler manifold satisfies the $\partial\overline{\partial}$-Lemma. Another important aspect is the isomorphism between the cohomology groups and the kernel of suitable differential operators. A similar result is due to Popovici, in [Pop16] he proved that the second step of the Frölicher spectral sequence is isomorphic to the kernel of an elliptic pseudo-differential operator $\tilde{\Delta}$.

Two are the main topics of this PhD thesis: the study of the cones of SKT and super SKT metrics and the stability, under small deformations of the complex structure, of the degeneration at the second step of the Frölicher spectral sequence. First of all we prove, using Varouchas spaces, that the equality of those two cones can be characterized in terms of cohomology groups (see Theorem 1). Moreover we apply our result to the case of 6-dimensional solvmanifolds (see Theorem 2). Concerning the second topic of our work, we deals with the theory of pseudo-differential operators and the theory of $\mathcal{C}^\infty$ families of differential operators (see [Kod06]). More precisely, we have proved an *a priori estimate* for pseudo-differential operators (see Theorem 3). Starting from this result, we have proved that the dimension of the kernels of a $\mathcal{C}^\infty$ family of pseudo-differential operators varies upper-semicontinuously. Then we have considered the family of pseudo-differential operators $\{\tilde{\Delta}_t\}$ over a family of compact complex manifolds $\{(M, J_t)\}$ and we show that, if the dimension of the Dolbeault's cohomology is independent on t, then $\{\tilde{\Delta}_t\}$ is a $\mathcal{C}^\infty$ family. Thus the dimension of the kernel is an upper-semicontinuous function of t. This result implies that, under the hypothesis of the independence of the dimension of the Dolbeault's cohomology from t, the degeneration at the second step of the Frölicher spectral sequence is a property stable under small deformations of the complex structure (see Theorem 4). Finally we have computed the Frölicher spectral sequence of a suitable family of compact complex manifold and we have showed that the Frölicher spectral sequence degenerates at second step only on the central fiber. This example shows that the hypothesis of the independence of the dimension of the Dolbeault's cohomology groups from t is necessary.

This thesis is structured in the following way.

In the first chapter we make a short introduction on compact complex manifolds. Following Egidi and Popovici, we recall some basic definitions and results on forms and currents. Moreover we discuss about special Hermitian metrics giving some motivations to the study of such metrics.

Chapter 2 is devoted to the study of the cones of SKT and super SKT metrics. We start with the basic notions about cohomology groups. We also recall the construction of the Frölicher spectral sequence, such construction will be used in the next chapters. In the last part, using the Varouchas spaces and the theory of cones of metrics, we prove the following.

Theorem 1. *Let M be a compact complex manifold admitting a SKT metric. Then the following facts are equivalent:*

1. $s\mathcal{S} = \mathcal{S}$;
2. $\operatorname{Ker} T = H^{1,1}_A(M)$;
3. $c^{2,1}(M) = h^{2,1}_A(M) - h^{2,1}_{\overline{\partial}}(M)$;
4. $A^{2,1}(M) \simeq B^{2,1}(M)$;
5. $a^{2,1}(M) = b^{2,1}(M)$;
6. *every smooth d-closed ∂-exact $(2,1)$-form on M is $\overline{\partial}$-exact;*
7. *every SKT metric g is super SKT.*

We conclude this chapter applying this theorem to the case of complex nilmanifolds showing the following

Theorem 2. *Let M be a non-torus compact complex 6-dimensional nilmanifold with Lie algebra different from $\mathfrak{h}_7$. If $\mathcal{S} \neq \emptyset$ then $s\mathcal{S} \neq \mathcal{S}$.*

In Chapter 3 we discuss the theory of elliptic operators on complex manifolds. This chapter is divided in three parts. In the first part we recall the theory of differential operators. Using this as a starting point, we prove those results also for pseudo-differential operators. To be more precise we obtain the following *a priori estimate.*

Theorem 3. *Let $A(x,D)$ be an elliptic pseudo-differential operator of order m. Then for any $k \in \mathbb{Z}$ there exists a constant C depending only on k, δ and $M_{|k|}$ such that for any $f \in \mathcal{C}^\infty(M;\mathbb{C})$*

$$\|f\|_{k+m} \leq C\left(\|A(x,D)f\|_k + \|f\|_k\right). \tag{1}$$

In the last section we report the results of Popovici about the relation between the second step of the Frölicher spectral sequence and the kernel of a suitable pseudo-differential operator. This will be used in the next chapter to prove a theorem of stability for the degeneration of the Frölicher spectral sequence.

In the last chapter we recall the theory of deformations of the complex structures of compact complex manifolds. Like the previous chapter we start by recalling the standard theory for families of differential operators and, with those results in our mind, we prove similar theorems using families of pseudo-differential operators. In particular we prove that the dimension of the kernel of a $\mathcal{C}^\infty$ family of pseudo-differential operators is an upper-semicontinuous function. Using this result we prove the following.

Theorem 4. *Let (M, J_t) be a family of complex manifolds and suppose that the dimension of* $\operatorname{Ker}\Delta_{\overline{\partial}_t} \cap \Lambda^{p,q}(M, J_t)$ *is independent of t for every $(p, q) \in \mathbb{Z}^2$. Then the degeneration at the second step of the Frölicher spectral sequence is stable under small deformations of the complex structure.*

In the last section, starting with the completely solvable Nakamura manifold and taking the deformations of the complex structure studied in [TT14], we show, by direct computations, that if the dimension of the kernel of $\Delta_{\overline{\partial}_t}$ varies, then the degeneration at second step of the Frölicher spectral sequence is not stable under small deformations.

Chapter 1

Preliminaries on Complex Geometry

In this first chapter we recall some basic definitions and well-known results on the theory of compact complex manifolds. We begin with a brief introduction about the consequences of the presence of a complex structure on a compact manifolds: we give the definition of Hermitian metric and we recall the orthogonal decomposition of the tangent bundle. Then we give a description of the spaces of (p, q)-forms and of (p, q)-currents. We recall some basic properties of the spaces of metrics and currents that will be used in the next chapters.

Finally we examine in depth Hermitian metrics. In particular we focus on special metrics such as Kähler or balanced metrics recalling some well-known facts and motivations for their study.

1.1 Compact Complex Manifolds

A *compact complex manifold* is a pair (M, J) where M is a compact differentiable manifold and J is a complex structure, i.e., J is a $(1,1)$ integrable tensor field such that $J^2 = -Id$. By [NN57], the integrable condition is equivalent to the vanishing of the Nijenhuis tensor N_J given by

$$N_J(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY], \tag{1.1}$$

where $X, Y \in TM$ and $[\bullet, \bullet]$ is the usual Lee bracket. The presence of the complex structure J forced M to be orientable and even-dimensional.

Since M is orientable, it admits a Riemannian metric g, i.e., a family of positive defined inner product $\{g_p\}_{p \in M}$ such that, for all $X, Y \in TM$, the map

$$p \mapsto g_p(X(p), Y(p))$$

is $\mathcal{C}^\infty$. Moreover, using g and J, we can define an Hermitian metric h on M, i.e., positive defined Hermitian product on T_pM varying smoothly with respect to p. Namely

$$h_p(X(p), Y(P)) = g(X(p), Y(p)) - \sqrt{-1}g(JX(p), Y(p))$$

is an Hermitian metric. We will discuss further about Hermitian metrics in section 1.4.

Finally we recall the decomposition of TM. Since $J^2 = -Id$ it has only $\sqrt{-1}$ and $-\sqrt{-1}$ as eigenvalue, thus it induces the following decomposition

$$TM^{\mathbb{C}} = T_{1,0}M \oplus T_{0,1}M, \tag{1.2}$$

where $TM^{\mathbb{C}} := TM \otimes_{\mathbb{R}} \mathbb{C}$, $T_{1,0}M$ is the $\sqrt{-1}$-eigenspace and $T_{0,1}M$ is the $-\sqrt{-1}$-eigenspace. Using local coordinates we denote a generic element of the standard basis of $T_{1,0p}M$ (resp. $T_{0,1p}M$) as $\frac{\partial}{\partial z_i^k}$ (resp. $\frac{\partial}{\partial \bar{z}_i^k}$).

As notation, when there is no ambiguity, we write M instead of (M, J).

1.2 (p, q)-Forms

The decomposition (1.2) induces by duality the following decomposition of the cotangent bundle T^*M

$$T^*M^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M,$$

where, as before, $T^*M^{\mathbb{C}}$ is the complexification of T^*M and $T^{1,0}M$ (resp. $T^{0,1}M$) is the $\sqrt{-1}$ (resp.$-\sqrt{-1}$) eigenspace of the application induced by J by duality and denoted again with J. We denote with $\{dz_i^1, \ldots, dz_i^n\}$ (resp. $\{d\bar{z}_i^1, \ldots, d\bar{z}_i^n\}$) the standard local basis of $T_x^{1,0}M$ (resp. $T_x^{0,1}M$) on the open coordinate U_i. $\{dz_i^1, \ldots, dz_i^n\}$ (resp. $\{d\bar{z}_i^1, \ldots, d\bar{z}_i^n\}$) is the dual basis of $\{\frac{\partial}{\partial z_i^1}, \ldots, \frac{\partial}{\partial z_i^n}\}$ (resp. $\frac{\partial}{\partial \bar{z}_i^1}, \ldots, \frac{\partial}{\partial \bar{z}_i^n}\}$).

Let $\Lambda^k(M)$ be the bundle of $\mathcal{C}^\infty$ k-forms, then we have

$$\Lambda^k(M) = \oplus_{p+q=k}\Lambda^{p,q}(M),$$

where

$$\Lambda^{p,q}(M) := \underbrace{T^{1,0}M \wedge \cdots \wedge T^{1,0}M}_{\text{p times}} \wedge \underbrace{T^{0,1}M \wedge \cdots \wedge T^{0,1}M}_{\text{q times}}$$

is the space of $\mathcal{C}^\infty$ (p, q)-forms. As notation, if A and B are multi-indexes of length a and b respectively, we put

$$dz_i^{A\overline{B}} := dz_i^{A_1} \wedge \cdots \wedge dz_i^{A_a} \wedge d\bar{z}_i^{B_1} \wedge \cdots \wedge d\bar{z}_i^{B_b}.$$

Moreover if $\{\phi^i\}$ is a set of forms indexed over $\mathbb{N}$ and A is a multi-index of length a, then we write

$$\phi^A := \phi^{A_1} \wedge \cdots \wedge \phi^{A_a}. \tag{1.3}$$

Given a (p,q)-form ϕ , we can express it in local coordinates on U_i as a $\mathcal{C}^\infty$-combination of $\{dz^{A\overline{B}}\}$, namely

$$\phi_{|_{U_i}} = \sum_{|A|=p}\sum_{|B|=q} \phi_{A\overline{B}} dz_i^{A\overline{B}}, \tag{1.4}$$

where $\phi_{A\overline{B}} : U_i \to \mathbb{C}$ are $\mathcal{C}^\infty$ functions.

Finally, also the external differential d splits in two component $\partial := \pi^{p+1,q} \circ d : \Lambda^{p,q}(M) \to \Lambda^{p+1,q}(M)$ and $\overline{\partial} := \pi^{p,q+1} \circ d : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)$. The integrable condition of J is equivalent to the splitting of the external differential $d = \partial + \overline{\partial}$. Since $d^2 = 0$ we have the following equations due to bi-degree

$$\begin{cases} \partial^2 = 0 \\ \partial\overline{\partial} = -\overline{\partial}\partial \\ \overline{\partial}^2 = 0 \end{cases} \tag{1.5}$$

According to Demally [Dem97], there is a standard topology on the space of differential (p,q)-forms on M. Let $\phi \in \Lambda^{p,q}(M)$ and let $U \subset M$ be a coordinate open subset, then, using (1.4), for every $K \subset U$ and every $s \in \mathbb{N}$ we associate the semi-norm

$$p_L^s(\phi) := \sup_{x\in K} \max_{|\alpha|\leq s} \max_{A,B} |D^\alpha \phi_{A\overline{B}}| , \tag{1.6}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ is a multi-index and D^α is a derivation of order $|\alpha|$ in the local variables $z_i^1, \ldots, z_i^n, \overline{z}_i^1, \ldots, \overline{z}_i^n$. Since M is a compact manifold the topology of $\Lambda^{p,q}(M)$ is induced by a finite number of those semi-norms, hence $\Lambda^{p,q}(M)$ is a Banach space.

We recall that the bundle of $\mathcal{C}^\infty$ (p,q)-forms on M can be equipped with an L^2 product. To define such a product we need the following definition

Definition 1 (Positive form)**.** *A (p,p)-form ϕ is said to be* positive *if, for every point $p \in M$, there exist an open coordinate U_i and a local basis $\{dz_i^a\}$ such that*

$$\phi_{|_{U_i}} = \sqrt{-1}dz_i^1 \wedge d\overline{z}_i^1 \wedge \cdots \wedge \sqrt{-1}dz_i^p \wedge d\overline{z}_i^p.$$

An (n,n) positive form is called volume form*.*

In [Mic82], Michelson proved the following

Theorem 5. *There exists a one-one correspondence between the $(1,1)$ and the $(n-1,n-1)$ positive forms.*

The proof of this theorem is constructive, so we have also the notion of the $n-1$th root of a $(n-1,n-1)$ positive form.

In Section 1.4 we will see that to every Hermitian metric g there exists a $(1,1)$ positive form ω called *fundamental form of* g. The n-th power Ω of ω is a volume form and it defines the following L^2 product on $\Lambda^{p,q}(M)$

$$< \phi, \psi >:= \sum_i \sum_{|A|=p}\sum_{|B|=q} \int_{\mathcal{U}_i} \eta_i \phi_{A\overline{B}} \overline{\psi}_{A\overline{B}} \Omega, \tag{1.7}$$

where $\{U_i\}$ is a finite open covering of M made by open coordinates, $\{\eta_i\}$ is a partition of unity subordinate to $\{U_i\}$ and $\phi_{A\overline{B}}$, $\psi_{A\overline{B}}$ are the $\mathcal{C}^\infty$ function defined in 1.4. Shortly we write

$$< \phi, \psi >= \int_M \phi \wedge *\psi, \tag{1.8}$$

where $*$ is the Hodge star operator with respect to the metric g.

There is also an algebraic structure on the whole space of $\mathcal{C}^\infty$ forms on M. If we consider the double complex $\left(\Lambda^{p,q}(M), \partial, \overline{\partial}\right)$, then it has a structure of differential bi-graded algebra. For forms of type (p,q) we define the sum as follows: if $\phi = \sum_{|I|=p} \sum_{|K|=q} \phi_{I\overline{K}} dz^{I\overline{K}}$ and $\psi = \sum_{|I|=p} \sum_{|K|=q} \psi_{I\overline{K}} dz^{I\overline{K}}$ are (p,q)-forms, then we put

$$\phi + \psi = \sum_{|I|=p} \sum_{|K|=q} (\phi_{I\overline{K}} + \psi_{I\overline{K}}) dz^{I\overline{K}} \tag{1.9}$$

If ϕ and ψ are forms of type (p,q) and (r,s) respectively, then we define the external product or wedge product as

$$\phi \wedge \psi = \sum \phi_{I\overline{K}} \psi_{A\overline{B}} dz^{I\overline{K}A\overline{B}}. \tag{1.10}$$

Finally we have the two differential operators ∂ and $\overline{\partial}$ of bi-degree $(1,0)$ and $(0,1)$ respectively, that are the exterior derivatives.

Using the definitions above we have that $(\Lambda^k(M), d)$ has a structure of differential graded algebra and that $(\Lambda^{p,q}(M), \partial, \overline{\partial})$ has a structure of differential bi-graded algebra.

1.3 Currents

The notion of current was introduced by Georges de Rham in [dR73]. It generalizes the notion of distribution. In analysis, a distribution is an element of the dual of a certain space of function (usually this space is $\mathcal{C}_c^\infty$, the space of $\mathcal{C}^\infty$ function with compact support).

Definition 2 (Current)**.** *A* current *of bi-degree (p,q) (or bi-dimension $(n-p, n-q)$) on M is a linear continuous form $T : \Lambda^{p,q}(M) \to \mathbb{C}$. We denote with $\mathcal{D}'^{p,q}(M)$ the space of (p,q)-currents on M.*

Since $\Lambda^{p,q}(M)$ is a Banach space, also $\mathcal{D}^{p,q}(M)$ is a Banach space with the dual scalar product. A (p,q)-distribution T can be view as a (p,q)-form with coefficient distribution. In fact let $\phi \in \Lambda^{p,q}(M)$ and $T \in \mathcal{D}'^{p,q}$ then we have

$$T(\phi) =< T, \phi >= \int_M T \wedge *\overline{\phi}. \tag{1.11}$$

If $\{\phi_1, \ldots, \phi_r\}$ is a basis for $\Lambda^{p,q}(M)$ and $f_1, \ldots, f_r$ are distributions on M, we write

$$T := \sum_{i=1}^r f_i \phi_i.$$

Definition 3. *Given a (p,q)-current T on M, the support of T* $\mathrm{supp}(T)$ *is subset of M such that for every (p,q)-form ϕ on M with* $\mathrm{supp}(\phi) \subset M \backslash \mathrm{supp}(T)$*,* $T(\phi) = 0$.

Usually, when someone uses currents over manifolds, he requires that either the currents or the test forms are compactly supported. Since we will work with compact manifold both condition are obviously satisfied.

From a geometric point of view, currents can be seen as integration over submanifolds.

Proposition 1. *If $N \subset M$ is a complex submanifold of complex dimension p. Then the map*

$$\begin{aligned} [N]: \quad & \Lambda^{p,p}(M) \to \mathbb{C} \\ & \phi \mapsto \int_N \phi \end{aligned} \tag{1.12}$$

defines an element of $\mathcal{D}'^{p,p}$.

If $T, S \in \mathcal{D}'^{p,q}(M)$ then $T + S \in \mathcal{D}'^{p,q}(M)$ and for every (p,q)-form ϕ over M we have

$$(T + S)(\phi) = T(\phi) + S(\phi). \tag{1.13}$$

If $T \in \mathcal{D}'^{p,q}(M)$ then $\partial T \in \mathcal{D}'^{p-1,q}(M)$ and, for every $\phi \in \Lambda^{p-1,q}(M)$, we have

$$(\partial T)(\phi) = T(\partial\phi). \tag{1.14}$$

Similarly for $\overline{\partial}$ and d. Thus the space $\mathcal{D}'(M) := \oplus \mathcal{D}'^{p,q}(M)$ has a structure of bi-graded differential algebra.

Conversly to case of forms, the wedge product between two currents is not well defined. However it can be defined between a currents and form. If $T \in \mathcal{D}'^{p,q}(M)$ and $\psi \in \Lambda^{r,s}(M)$ than $T \wedge \psi \in \mathcal{D}'^{p+r,q+s}(M)$ and, for every $\phi \in \Lambda^{p-r,q-s}(M)$, we have

$$(T \wedge \psi)(\phi) = (-1)^{(r+s)(2n-r-s)} T(*\overline{\psi} \wedge \phi). \tag{1.15}$$

1.4 Hermitian Metrics on Complex Manifolds

Let M be a compact complex manifold of complex dimension n and let TM be the tangent vector bundle. As we said in previous section, $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$, a *Hermitian metric* on M is a smoothly varying positive-definite Hermitian form on each fiber of TM. Such a metric can be written as a smooth section

$$g \in \Gamma(T_{1,0}M \otimes T_{0,1}M)$$

such that, for every $X, Y \in TM$

$$g(JX, JY) = g(X, Y), \quad g(X, JX) > 0 \text{ if } X \neq 0.$$

Given an Hermitian metric g over M, there is associated the $(1,1)$-form

$$\omega(X,Y) := g(X,JY),$$

such form is called *Kähler form* of g (some authors refer at it as fundamental form of g). Using local holomorphic coordinates $\{dz^k\}$, g can be written as

$$g = \sum_{a,b=1}^{n} g_{a\bar{b}} dz^a \otimes d\bar{z}^b,$$

where $(g_{a\bar{b}})_{a,b=1,\ldots,n}$ is a positive-definite Hermitian matrix. In same coordinates, ω can be expressed as

$$\omega = \frac{\sqrt{-1}}{2} \sum_{a,b=1}^{n} g_{a\bar{b}} dz^a \wedge d\bar{z}^b.$$

Combining the Hermitian metric g with its Kähler form ω one gets the *complex hermitian metric*

$$h = g + \sqrt{-1}\omega.$$

Observation 1. $h(JX,Y) = \sqrt{-1}h(X,Y) = -h(X,JY)$.

A *connection* on M is a map $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ such that, for every $X, Y \in TM$ and for every $f \in \mathcal{C}^\infty(M)$, the following equalities

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y,$$
$$\nabla_{fX} Y = f\nabla_X Y$$

hold. The *torsion tensor* $T^\nabla \in \Gamma(\Lambda^2(M) \otimes TM)$ associated to a connection ∇ is defined as

$$T^\nabla_{X,Y} := \nabla_X Y - \nabla_Y X - [X,Y],$$

for every $X, T \in TM$.

We recall the following theorems that characterize special connections.

Theorem 6. *Given a Hermitian metric g, there exists a unique connection ∇ over M such that:*

- *it preserves g, i.e., for every vector fields $X, Y, Z \in TM$, $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$;*
- *its torsion vanishes.*

This unique connection is called Levi-Civita connection *and it is denoted with ∇^{LC}.*

Theorem 7. *Given a Hermitian metric g, there exists a unique connection ∇ over M such that:*

- *it preserves g, i.e., for every vector fields $X, Y, Z \in TM$, $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$;*

- *it preserves the complex structure J, i.e., for every vector fields $X, Y \in TM$ $\nabla_X(JY) = J(\nabla_X Y)$;*
- *the $(1,1)$-component of the torsion tensor vanishes.*

This unique connection is called Hermitian connection *and it is denoted with ∇^H.*

Proposition 2 ([Mic82])**.** *The $(2,0)$-component of the torsion associated to the Hermitian connection of the metric $h = \sum h_{j\overline{k}} dz_j \otimes d\overline{z}_k$ can be written as*

$$T = \sum T^l_{j\overline{k}} dz_j \wedge d\overline{z}_k \otimes \frac{\partial}{\partial z_l}$$

where

$$T^l_{j\overline{k}} = \sum \left(\frac{\partial h_{k\overline{\alpha}}}{\partial z_j} h^{\overline{\alpha}l} - \frac{\partial h_{j\overline{\alpha}}}{\partial z_k} h^{\overline{\alpha}l} \right).$$

Definition 4. *The* torsion $(1,0)$-form *of a complex hermitian metric h is defined as $\tau = \sum \tau_k dz_k$, where*

$$\tau_k = \sum T^j_{k\overline{j}}.$$

The last part of this introduction is dedicated to the *Hodge $*$ operator*. We start with the following

Definition 5. *The Hodge $*$ operator on a complex manifold M, of complex dimension n, related to a hermitian metric g is an anti-linear operator*

$$* : \Lambda^{p,q}(M) \to \Lambda^{n-q,n-p}(M)$$

such that, for every (p,q)-forms ϕ and ψ,

$$\phi \wedge *\overline{\psi} = (\phi, \psi)\omega^n,$$

where ω^n is the volume form associated to g and (ϕ, ψ) is the scalar product induced by g.

Proposition 3. *The Hodge $*$ operator induces and isomorphism between $\Lambda^{p,q}(M)$ and $\Lambda^{n-q,n-p}(M)$. Moreover $*^2 = (-1)^{(p+q)(n-p-q)} Id$.*

Definition 5 is formal; it is possible to provide an explicit formula for the Hodge $*$ operator in following way: let $(p+q)!\phi = \sum \phi_{A\overline{B}} dz^{A\overline{B}}$ be a (p,q)-form, then

$$*\phi = \frac{1}{k!(n-k)!} \varepsilon_{AA'} \varepsilon_{BB'} \sqrt{|det(g)|} \phi_{A\overline{B}} dz^{A'\overline{B'}},$$

where

- A, B, A', B' are multi-indexes such that $A' = (1, 2, \dots, n) \setminus A$ and $B' = (1, 2, \dots, n) \setminus B$;
- $\varepsilon_{AA'}$ is the sign of the permutation $(1, 2, \dots, n) \mapsto AA'$.

Using the Hodge operator it is possible to define the adjoint of the usual differential operators, namely

$$\partial^* := - * \overline{\partial} * : \Lambda^{p,q}(M) \to \Lambda^{p,q-1}(M)$$

and

$$\overline{\partial}^* := - * \partial * : \Lambda^{p,q}(M) \to \Lambda^{p-1,q}(M).$$

As we will see in the next chapters, they are used to define differential operators over manifolds.

1.4.1 Special Metrics

In this section we will discuss about Hermitian metrics with particular properties. Such metrics provide information about the geometry of the manifold or on the complex structure. We begin providing a list of special metrics. Let h be a Hermitian metric and let ω be its fundamental form. Then we call h

- *Kähler* if $d\omega = 0$;
- *Balanced* if $d\omega^{n-1} = 0$;
- *SKT* if $\partial\overline{\partial}\omega = 0$;
- *Gauduchon* if $\partial\overline{\partial}\omega^{n-1} = 0$;
- *super SKT* if $\partial\omega$ is $\overline{\partial}$-exact;
- *strongly Gauduchon* if $\partial\omega^{n-1}$ is $\overline{\partial}$-exact.

A manifold M is said to be Kähler (resp. Balanced, SKT,...) if it admits a Kähler (resp. Balanced, SKT,...) metric. Moreover we have the following relations between metrics

$$\begin{tikzcd}
& \text{super SKT} \arrow[rr, Rightarrow] & & \text{SKT} \arrow[dr, Rightarrow] & \\
\text{Kähler} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & & \text{Gauduchon} \\
& \text{Balanced} \arrow[r, Rightarrow] & \text{strongly Gauduchon} \arrow[urr, Rightarrow] & &
\end{tikzcd}$$

We recall some results that characterize such manifolds in terms of currents:

Theorem 8. *A compact complex manifold is Kähler if and only if there exists no non-zero positive $(1,1)$-current which is the $(1,1)$-component of a boundary.*

Theorem 9. *A compact complex manifold M of complex dimension n is balanced if and only if there exists no positive non-zero current T of degree $(1,1)$ which is the component of a boundary (i.e., there exists a current S such that $T = dS + dS$).*

Theorem 10. *A compact complex manifold M is SKT if and only if there exists no non-zero positive $(1,1)$-current which is $\partial\overline{\partial}$-exact.*

We conclude this section giving some motivation for the study of the various type of special metrics. Probably the most important class of metrics is the class of Kähler metrics. Such metrics are similar to the flat metric in the euclidean space and they have been studied by several authors ([HL83]). The Kähler condition is quite strong and it reflects on the topology of M, in fact we have

Theorem 11. *Let M be a compact Kähler manifold. Then the odds Betti's numbers of M are even.*

This theorem follows directly from the Hodge decomposition (see [GH14]). Moreover, on compact Kähler manifolds, we have the so called Kähler identity:

Theorem 12. *Let M be a compact Kähler manifold. Let $\Delta := dd^* + d^*d$ and $\Delta_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ be the Laplacian operators associated to the cohomology groups (see Chapter 2). Then $\Delta = 2\Delta_{\overline{\partial}}$.*

Another important results in Kähler geometry is the following.

Theorem 13 (Hard Lefschetz Theorem)**.** *If (M, g) is a Kähler manifold of complex dimension n, then the homomorphism*

$$\begin{aligned} L^r: \quad H^{n-r}_{dR}(M;\mathbb{C}) &\to H^{n+r}_{dR}(M;\mathbb{C}) \\ [\alpha] &\mapsto [\omega^r \wedge \alpha] \end{aligned} \tag{1.16}$$

is an isomorphism.

The next class of metric is the class of Balanced metrics. In [Mic82], Micheleson introduced the notion of balanced metric. It first was defined in terms of the vanishing of the $(1,0)$ torsion tensor of the metric. Our definition was given in same paper as equivalent condition and it is now generally assumed as the natural definition. The exstistence of balanced metrics is related to modifications of Kähler manifold.

Definition 6. *Let $\tilde{M}$ and M be compact complex manifolds. A* modification *$f : \tilde{M} \to M$ is proper bimeromorphic map that is holomorphic outside an analytic subset of $\tilde{M}$.*

If M is Kähler manifold, $\tilde{M}$ is not necessarly Kähler but we have the following results

Theorem 14. *Let $\pi : \tilde{M} \to M$ be a proper modification of a compact complex manifold M. Then M admits a balanced metric if and only if $\tilde{M}$ admits a balanced metric.*

The unique connection ∇ satisfying $\nabla g = 0$ and $\nabla J = 0$ for which $g(X, T(Y, Z))$ ([Gau97] [Yan65]) is totally skew-symmetric and it was used by Bismut in [Bis89] to prove a local index formula in non-Kähler geometry. If $Jd\omega$ is closed but not zero, then g is a strong Kähler with torsion and have applications in type II string theory [GHR84][HP88][Str86]. The presence of balanced or SKT metrics provides no topological obstruction, but we have the following results

Theorem 15. *Let g be a Hermitian metric. Then g is Kähler if and only if it is both balanced and SKT.*

Moreover Popovici made the following conjecture

Conjecture 1. *Let M be a compact complex manifold. Then M is a Kähler manifold if and only if it is both balanced and SKT.*

Gauduchon metrics have the property that they exists on every compact manifold. Moreover

Theorem 16. *There exists a Gauduchon metric (unique up to normalization) in the conformal class of any Hermitian metric on M.*

Strongly Gauduchon metrics were firstly introduced in [Pop09], Popovici used such metrics to study the holomorphic deformation limit of projective manifolds. Let $\phi : \mathcal{M} \to B(0,1)$ be a family of compact complex manifolds over the unit disk, Popovici proved that if the fiber M_t is projective for $t \neq 0$ and the center fiber M_0 is a strongly Gauduchon manifold, then M_0 must be a Moishezon manifold. Thus strongly Gauduchon metrics are useful in the study of deformation limits of projective manifolds.

Theorem 17. *[Pop13] Let $\mu : \tilde{M} \to M$ be a proper modification, then $\tilde{M}$ is strongly Gauduchon if and only if M is strongly Gauduchon.*

The last type of metrics is the most recent. It was introduced in order to study Conjecture 1. Super SKT metrics are used in place of Hermitian symplectic metrics because the former property is not preserved in the same cohomology class. We will use super SKT metrics in our study in the next chapter.

Chapter 2

Cohomology

The aim of this chapter is to provide a characterization in cohomological terms of the coincidence of certain cones of Hermitian metrics.

Cohomology is a very important tool in the study and classification of compact complex manifolds. The most famous cohomology group is the de Rham's and it is the cohomology group associated to differential forms. This group depends only from the differentiable structure (i.e. the topology) of the manifold and it is independent from the complex structure. In [Dol53], Dolbeault introduced the cohomology group that was named after him. It is the group associated to the complex of holomorphic forms and it depends both from the differentiable and the complex structures. Those two groups are very similar in their construction, but there are no direct relations between them. This is one of the reasons because, in [Sch07], Schweitzer introduced other two cohomology groups: the Bott-Chern's and the Aeppli's. They provide, with the homomorphisms induced by the identity, a bridge between the de Rham's and the Dolbeault's. All this four groups can be described algebraically, as the cohomology of a short sequence, or analytically, as the kernel of a suitable self-adjoint elliptic differential operator.

Moreover, in this chapter, we recall three tools that relate the previous groups: the Frölicher spectral sequence, the $\partial\overline{\partial}$-lemma and the Varouchas' spaces. The first was introduced in [Frö55] and it is a sequence of sequences that provides a link between the Dolbeault's and the de Rham's chomology. The second is a result of Deligne, Griffiths, Morgan and Sullivan ([DGMS75]) that provides a necessary and sufficient condition such that all the cohomology groups (of the same bi-degree) are isomorphic. Finally, the Varouchas' spaces, introduced in [Var], are auxiliary groups that are used to construct exact sequences that involve the Dolbeualt's, Bott-Chern's and Aeppli's cohomology groups.

In the last section we introduce the cones of Hermitian metrics. In particular we exhibit our work on the cones of SKT and super SKT metrics and we provide necessary and sufficient conditions, in terms of cohomology groups, of the coincidence of such cones.

2.1 de Rham Cohomology

The first cohomology group we recall is the de Rham's. It was introduced by de Rham and it is the cohomology of the complex $\left(\Lambda^k(M;\mathbb{C}), d\right)$ and it is associated to the sequence

$$\cdots \to \Lambda^{k-1}(M;\mathbb{C}) \xrightarrow{d} \Lambda^k(M;\mathbb{C}) \xrightarrow{d} \Lambda^{k+1}(M;\mathbb{C}) \to \dots \tag{2.1}$$

Definition 7. *The k-th* de Rham cohomology group *is defined as*

$$H^k_{dR}(M,\mathbb{C}) := \frac{\operatorname{Ker} d : \Lambda^k(M) \to \Lambda^{k+1}(M)}{\operatorname{Im} d : \Lambda^{k-1}(M) \to \Lambda^k(M)}. \tag{2.2}$$

It is very important because it is not associated to the complex structure and it is a topological invariant.

Fixed a Hermitian metric g on M, the elliptic operator associated to the de Rham cohomology is

$$\Delta_d := dd^* + d^*d, \tag{2.3}$$

where d is the external differential operator and $d^* = *d*$ is the adjoint operator with respect to the metric g.

Theorem 18. *Δ_d is an elliptic self-adjiont differential operator of the 2-nd order.*

By Theorem 18 and standard results about elliptic differential operators, we have the following decomposition

$$\Lambda^k(M) = (\operatorname{Ker}\Delta_d)^k \oplus (\operatorname{Im}\Delta_d)^k, \tag{2.4}$$

where with $(\operatorname{Ker}\Delta_d)^k$, resp. $(\operatorname{Im}\Delta_d)^k$, we mean $\operatorname{Ker}\Delta_d \cap \Lambda^k(M)$, resp. $\operatorname{Im}\Delta_d \cap \Lambda^k(M)$. Moreover, we have that, for every $k \in \mathbb{Z}$ $\dim_{\mathbb{C}}(\operatorname{Ker}\Delta_d)^k < \infty$ and we denote $(\operatorname{Ker}\Delta_d)^k$ with $\mathcal{H}^k_d$ calling it the space of *d-harmonic* k-th forms.

$\mathcal{H}^k$ is characterized in the following way: let $\phi \in \mathcal{H}^k_d$, using the L^2 product on $\Lambda^k(M)$ induced by g we obatain

$$0 =< \Delta_d\phi, \phi >= \|d\phi\|^2 + \|d^*\phi\|^2 ; \tag{2.5}$$

so if a k-form is d-harmonic then it is both d-closed and d^*-closed. Since the converse is trivially true, we have that

$$\mathcal{H}^k_d = \operatorname{Ker} d \cap \operatorname{Ker} d^* \cap \Lambda^k(M). \tag{2.6}$$

Similarly,

$$(\operatorname{Im}\Delta_d)^k = (\operatorname{Im} d \oplus \operatorname{Im} d^*) \cap \Lambda^k(M). \tag{2.7}$$

So

$$\Lambda^k(M) = \mathcal{H}^k_d \oplus (\operatorname{Im} d)^k \oplus (\operatorname{Im} d^*)^k. \tag{2.8}$$

Since the kernel of the d operator is orthogonal to the image of the d^* operator, we have that $(\operatorname{Ker} d)^k = \mathcal{H}_d^k \oplus (\operatorname{Im} d)^k$. Hence

$$H_{dR}^k(M,\mathbb{C}) \simeq \mathcal{H}_d^k. \tag{2.9}$$

As a notation, we denote with $b_k := \dim H_{dr}^k(M,\mathbb{C})$ the k-th *Betti's number* of M. Moreover, the Hodge star operator $*$ induces the following isomorphism

$$H_{dR}^k(M,\mathbb{C}) \simeq H_{dR}^{2n-k}(M,\mathbb{C}).$$

2.2 Dolbeault Cohomology

The Dolbeault cohomology is the cohomology of the complex $\left(\Lambda^{p,q}(M;\mathbb{C}), \overline{\partial}\right)$ and it is associated to the sequence

$$\cdots \to \Lambda^{p,q-1}(M) \xrightarrow{\overline{\partial}} \Lambda^{p,q}(M) \xrightarrow{\overline{\partial}} \Lambda^{p,q+1}(M;\mathbb{C}) \to \dots \tag{2.10}$$

Definition 8. *The (p,q)-th* Dolbeault cohomology group *is defined as*

$$H_{\overline{\partial}}^{p,q}(M) := \frac{\operatorname{Ker} \overline{\partial} : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)}{\operatorname{Im} \overline{\partial} : \Lambda^{p,q-1}(M) \to \Lambda^{p,q}(M)}. \tag{2.11}$$

Contrary to de Rham cohomology, Dolbeault's dependes on the complex structure.

Fixed a Hermitian metric g on M, the elliptic operator associated to the Dolbeault cohomology is

$$\Delta_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}, \tag{2.12}$$

where $\overline{\partial}^* = - * \partial *$ is the adjoint operator of $\overline{\partial}$.

Theorem 19. *$\Delta_{\overline{\partial}}$ is an elliptic self-adjiont differential operator of the 2-nd order.*

By Theorem 19, we have the following decomposition

$$\Lambda^{p,q}(M) = (\operatorname{Ker} \Delta_{\overline{\partial}})^{p,q} \oplus (\operatorname{Im} \Delta_{\overline{\partial}})^{p,q}, \tag{2.13}$$

where with $(\operatorname{Ker} \Delta_{\overline{\partial}})^{p,q}$, resp. $(\operatorname{Im} \Delta_{\overline{\partial}})^{p,q}$, we mean $\operatorname{Ker} \Delta_{\overline{\partial}} \cap \Lambda^{p,q}(M)$, resp. $\operatorname{Im} \Delta_{\overline{\partial}} \cap \Lambda^{p,q}(M)$. Moreover we have that, for every $(p,q) \in \mathbb{Z}^2$ $\dim_{\mathbb{C}}(\operatorname{Ker} \Delta_{\overline{\partial}})^{p,q} < \infty$. We also denote $(\operatorname{Ker} \Delta_{\overline{\partial}})^{p,q}$ with $\mathcal{H}_{\overline{\partial}}^{p,q}$ calling it the space of $\overline{\partial}$*-harmonic* (p,q)-th forms. Let $\phi \in \mathcal{H}_{\overline{\partial}}^{p,q}$, using the L^2 product on $\Lambda^{p,q}(M)$ we obatain

$$0 =< \Delta_{\overline{\partial}}\phi, \phi >= \left\|\overline{\partial}\phi\right\|^2 + \left\|\overline{\partial}^*\phi\right\|^2; \tag{2.14}$$

so if a (p,q)-form is $\overline{\partial}$-harmonic then it is both $\overline{\partial}$-closed and $\overline{\partial}^*$-closed. Since the converse is trivially true, we have that

$$\mathcal{H}_{\overline{\partial}}^{p,q} = \operatorname{Ker} \overline{\partial} \cap \operatorname{Ker} \overline{\partial}^* \cap \Lambda^{p,q}(M). \tag{2.15}$$

Similarly,

$$(\operatorname{Im}\Delta_{\overline{\partial}})^{p,q}=\left(\operatorname{Im}\overline{\partial}\oplus\operatorname{Im}\overline{\partial}^{*}\right)\cap\Lambda^{p,q}(M). \tag{2.16}$$

So

$$\Lambda^{p,q}(M)=\mathcal{H}^{p,q}_{\overline{\partial}}\oplus(\operatorname{Im}\overline{\partial})^{p,q}\oplus(\operatorname{Im}\overline{\partial}^{*})^{p,q}. \tag{2.17}$$

Since the kernel of the $\overline{\partial}$ operator is orthogonal to the image of the $\overline{\partial}^{*}$ operator, we have that $(\operatorname{Ker}\overline{\partial})^{k}=\mathcal{H}^{p,q}_{\overline{\partial}}\oplus(\operatorname{Im}\overline{\partial})^{p,q}$. Hence

$$H^{p,q}_{\overline{\partial}}(M)\simeq\mathcal{H}^{p,q}_{\overline{\partial}}. \tag{2.18}$$

As notation, we denote with $h^{p,q}_{\overline{\partial}}:=\dim H^{p,q}_{\overline{\partial}}(M)$ the (p,q)-th *Hodge's number* of M. Moreover, the Hodge star operator $*$ and the conjiugation induce the following isomorphisms

$$H^{p,q}_{\overline{\partial}}(M)\simeq H^{n-q,n-p}_{\overline{\partial}}(M)$$

and

$$H^{p,q}_{\overline{\partial}}(M)\simeq H^{q,p}_{\partial}(M),$$

where $H^{p,q}_{\partial}(M)$ is the (p,q)-th cohomology group obtained by substituie in equation (2.11) the operator $\overline{\partial}$ with the operator ∂.

2.3 Bott-Chern Cohomology

The Bott-Chern cohomology is not a cohomology in classical sense, it is associated to the short sequence

$$\Lambda^{p-1,q-1}(M)\xrightarrow{\partial\overline{\partial}}\Lambda^{p,q}(M)\xrightarrow{\partial+\overline{\partial}}\Lambda^{p+1,q}(M)\oplus\Lambda^{p,q+1}(M) \tag{2.19}$$

Definition 9. *The* (p,q)*-th* Bott-Chern cohomology group *is defined as*

$$H^{p,q}_{BC}(M):=\frac{\left(\operatorname{Ker}\partial:\Lambda^{p,q}(M)\to\Lambda^{p+1,q}(M)\right)\cap\left(\operatorname{Ker}\overline{\partial}:\Lambda^{p,q}(M)\to\Lambda^{p,q+1}(M)\right)}{\operatorname{Im}\partial\overline{\partial}:\Lambda^{p-1,q-1}(M)\to\Lambda^{p,q}(M)}. \tag{2.20}$$

This cohomology was introduced to provide a link between the de Rham's and the Dolbeault's. Fixed a Hermitian metric g on M, the elliptic operator associated to the Bott-Chern cohomology is

$$\Delta_{BC}:=(\partial\overline{\partial})(\partial\overline{\partial})^{*}+(\partial\overline{\partial})^{*}(\partial\overline{\partial})+\left(\overline{\partial}^{*}\partial\right)\left(\overline{\partial}^{*}\partial\right)^{*}+\left(\overline{\partial}^{*}\partial\right)^{*}\left(\overline{\partial}^{*}\partial\right)+\overline{\partial}^{*}\overline{\partial}+\partial^{*}\partial. \tag{2.21}$$

This differential operator was firstly introduced in [KS60] for the study of the stability of Kähler metrics under small deformation. As the other differential operators introduced in the previous sections

Theorem 20. *Δ_{BC} is an elliptic self-adjoint differential operator of the 4-th order.*

By Theorem 20, we have the following decomposition

$$\Lambda^{p,q}(M) = (\operatorname{Ker}\Delta_{BC})^{p,q} \oplus (\operatorname{Im}\Delta_{BC})^{p,q}, \tag{2.22}$$

where with $(\operatorname{Ker}\Delta_{BC})^{p,q}$, resp. $(\operatorname{Im}\Delta_{BC})^{p,q}$, we mean $\operatorname{Ker}\Delta_{BC}\cap\Lambda^{p,q}(M)$, resp. $\operatorname{Im}\Delta_{BC}\cap\Lambda^{p,q}(M)$. Moreover, for every $(p,q)\in\mathbb{Z}^2$, $\dim_{\mathbb{C}}(\operatorname{Ker}\Delta_{BC})^{p,q}<\infty$. We also denote $(\operatorname{Ker}\Delta_{BC})^{p,q}$ with $\mathcal{H}^{p,q}_{BC}$ calling it the space of *BC-harmonic* (p,q)-th forms. Let $\phi\in\mathcal{H}^{p,q}_{BC}$, using the L^2 product on $\Lambda^{p,q}(M)$ we obatain

$$0 =< \Delta_{BC}\phi,\phi >= \left\|\partial\overline{\partial}\phi\right\|^2 + \left\|(\partial\overline{\partial})^*\phi\right\|^2 + \left\|\overline{\partial}\phi\right\|^2 + \left\|\partial\phi\right\|^2; \tag{2.23}$$

so if a (p,q)-form is BC-harmonic then it is ∂-closed, $\overline{\partial}$-closed and $(\partial\overline{\partial})^*$-closed. Since the converse is trivially true, we have that

$$\mathcal{H}^{p,q}_{BC} = \operatorname{Ker}\partial\cap\operatorname{Ker}\overline{\partial}\cap\operatorname{Ker}(\partial\overline{\partial})^*\cap\Lambda^{p,q}(M). \tag{2.24}$$

Similarly,

$$(\operatorname{Im}\Delta_{BC})^{p,q} = \left(\operatorname{Im}\partial\overline{\partial}\oplus\left(\operatorname{Im}\partial^* + \operatorname{Im}\overline{\partial}^*\right)\right)\cap\Lambda^{p,q}(M). \tag{2.25}$$

So

$$\Lambda^{p,q}(M) = \mathcal{H}^{p,q}_{BC}\oplus(\operatorname{Im}\partial\overline{\partial})^{p,q}\oplus\left((\operatorname{Im}\partial^*)^{p,q} + (\operatorname{Im}\overline{\partial}^*)^{p,q}\right). \tag{2.26}$$

We have that $(\operatorname{Ker} d)^{p,q} = \mathcal{H}^{p,q}_{BC}\oplus(\operatorname{Im}\partial\overline{\partial})^{p,q}$. Hence

$$H^{p,q}_{BC}(M)\simeq\mathcal{H}^{p,q}_{BC}. \tag{2.27}$$

As notation, we denote with $h^{p,q}_{BC} := \dim H^{p,q}_{BC}(M)$ the (p,q)-th *Hodge's number* of M. Moreover, the conjiugation induces the following isomorphism

$$H^{p,q}_{BC}(M)\simeq H^{q,p}_{BC}(M).$$

2.4 Aeppli Cohomology

The Aeppli cohomology is the dual of the Bott-Chern's. It is associated to the short sequence

$$\Lambda^{p-1,q}(M)\oplus\Lambda^{p,q-1}(M)\xrightarrow{\partial+\overline{\partial}}\Lambda^{p,q}(M)\xrightarrow{\partial\overline{\partial}}\Lambda^{p+1,q+1}(M) \tag{2.28}$$

Definition 10. *The* (p,q)*-th* Aeppli cohomology group *is defined as*

$$H^{p,q}_{A}(M) := \frac{\operatorname{Ker}\partial\overline{\partial}:\Lambda^{p,q}(M)\to\Lambda^{p+1,q+1}(M)}{(\operatorname{Im}\partial:\Lambda^{p-1,q}(M)\to\Lambda^{p,q}(M)) + \left(\operatorname{Im}\overline{\partial}:\Lambda^{p,q-1}(M)\to\Lambda^{p,q}(M)\right)}. \tag{2.29}$$

Fixed a Hermitian metric g on M, the elliptic operator associated to the Bott-Chern cohomology is

$$\Delta_A := (\partial\overline{\partial})(\partial\overline{\partial})^* + (\partial\overline{\partial})^*(\partial\overline{\partial}) + \left(\overline{\partial}\partial^*\right)\left(\overline{\partial}\partial^*\right)^* + \left(\overline{\partial}\partial^*\right)^*\left(\overline{\partial}\partial^*\right) + \overline{\partial}\overline{\partial}^* + \partial\partial^*. \tag{2.30}$$

Theorem 21. *Δ_A is an elliptic self-adjiont differential operator of the 4-th order.*

By Theorem 21, we have the following decomposition

$$\Lambda^{p,q}(M) = (\operatorname{Ker}\Delta_A)^{p,q} \oplus (\operatorname{Im}\Delta_A)^{p,q}, \tag{2.31}$$

where with $(\operatorname{Ker}\Delta_A)^{p,q}$, resp. $(\operatorname{Im}\Delta_A)^{p,q}$, we mean $\operatorname{Ker}\Delta_A \cap \Lambda^{p,q}(M)$, resp. $\operatorname{Im}\Delta_A \cap \Lambda^{p,q}(M)$. Moreover, always by Theorem 21, we have that, for every $(p,q) \in \mathbb{Z}^2$ $\dim_{\mathbb{C}}(\operatorname{Ker}\Delta_A)^{p,q} < \infty$. We also denote $(\operatorname{Ker}\Delta_A)^{p,q}$ with $\mathcal{H}_A^{p,q}$ calling it the space of *A-harmonic* (p,q)-th forms. Let $\phi \in \mathcal{H}_A^{p,q}$, using the L^2 product on $\Lambda^{p,q}(M)$ we obatain

$$0 =< \Delta_A\phi, \phi >= \left\|\partial\overline{\partial}\phi\right\|^2 + \left\|(\partial\overline{\partial})^*\phi\right\|^2 + \left\|\overline{\partial}^*\phi\right\|^2 + \|\partial^*\phi\|^2\,; \tag{2.32}$$

so if a (p,q)-form is BC-harmonic then it is ∂^*-closed, $\overline{\partial}^*$-closed and $\partial\overline{\partial}$-closed. Since the converse is trivially true, we have that

$$\mathcal{H}_A^{p,q} = \operatorname{Ker}\partial^* \cap \operatorname{Ker}\overline{\partial}^* \cap \operatorname{Ker}\partial\overline{\partial} \cap \Lambda^{p,q}(M). \tag{2.33}$$

Similarly,

$$(\operatorname{Im}\Delta_A)^{p,q} = \left(\operatorname{Im}(\partial\overline{\partial})^* \oplus \left(\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}\right)\right) \cap \Lambda^{p,q}(M). \tag{2.34}$$

So

$$\Lambda^{p,q}(M) = \mathcal{H}_A^{p,q} \oplus (\operatorname{Im}(\partial\overline{\partial})^*)^{p,q} \oplus \left((\operatorname{Im}\partial)^{p,q} + (\operatorname{Im}\overline{\partial})^{p,q}\right). \tag{2.35}$$

We have that $(\operatorname{Ker} d)^{p,q} = \mathcal{H}_A^{p,q} \oplus (\operatorname{Im}\partial + \operatorname{Im}\overline{\partial})^{p,q}$. Hence

$$H_A^{p,q}(M) \simeq \mathcal{H}_A^{p,q}. \tag{2.36}$$

As notation, we denote with $h_A^{p,q} := \dim H_A^{p,q}(M)$ the (p,q)-th *Hodge's number* of M. Moreover, the Hodge star operator $*$ and the conjiugation induce the following isomorphisms

$$H_{BC}^{p,q}(M) \simeq H_A^{n-q,n-p}(M)$$

and

$$H_A^{p,q}(M) \simeq H_A^{q,p}(M).$$

2.5 Frölicher Spectral Sequence

The *Frölicher spectral sequence* $(E_r^{p,q}(M), d_r)$ is the spectral sequence associated to the double complex $\left(\Lambda^{p,q}(M), \partial, \overline{\partial}\right)$. It was introduced in [Frö55] as a link between the Dolbeault's and the de Rham's cohomology groups. It is defined in the following way, let

$$E_0^{p,q}(M) := \Lambda^{p,q}(M) \quad \text{and} \quad d_0 := \overline{\partial}. \tag{2.37}$$

Then $E_{r+1}^{p,q}(M)$ is defined inductively as the (p,q)-th cohomology group of the complex

$$\cdots \to E_r^{p-r,q+r-1}(M) \xrightarrow{d_r} E_r^{p,q}(M) \xrightarrow{d_r} E_r^{p+r,q-r+1}(M) \to \ldots \tag{2.38}$$

and the differential d_r is of type $(r,-r+1)$. In [COUV16], it is provided an explicit description of both $E_r^{p,q}$ and d_r, namely they proved

Theorem 22. *Let M be a complex manifold. Then*

$$E_r^{p,q}(M) \simeq \frac{X_r^{p,q}(M)}{Y_r^{p,q}(M)}, \tag{2.39}$$

where

$$X_1^{p,q}(M) = \{\alpha \in \Lambda^{p,q}(M) | \overline{\partial}\alpha = 0\}, \quad Y_1^{p,q}(M) = \overline{\partial}\Lambda^{p,q}(M), \tag{2.40}$$

and for $r \geq 2$

$$\begin{aligned} X_r^{p,q}(M) = \quad & \{\alpha_{p,q} \in \Lambda^{p,q}(M) | \overline{\partial}\alpha = 0 \textit{ and there exist} \\ & \alpha_{p+i,q-i} \in \Lambda^{p+i,q-i}(M) \textit{ such that} \\ & \partial\alpha_{p+i-1,q-i+1} + \overline{\partial}\alpha_{p+i,q-i} = 0, 0 \leq i \leq r-1\}, \end{aligned} \tag{2.41}$$

$$\begin{aligned} Y_r^{p,q}(M) = \quad & \{\partial\beta_{p-1,q} + \overline{\partial}\beta_{p,q-1} \in \Lambda^{p,q}(M) | \textit{ there exist} \\ & \beta_{p-i,q+i-1} \in \Lambda^{p-i,q+i-1}(M), 2 \leq i \leq r-1, \\ & \textit{satisfying } \partial\beta_{p-i,q+i-1} + \overline{\partial}\beta_{p-i+1,q+i-2} = 0, \\ & \overline{\partial}\beta_{p-r+1,q+r-2} = 0\}. \end{aligned} \tag{2.42}$$

and

Theorem 23. *For $r \geq 1$ the map $d_r : E_r^{p,q}(M) \to E^{p+rq-r+1}(M)$ is given by*

$$d_r[\alpha_{p,q}] = [\partial\alpha_{p+r-1,q-r+1}], \tag{2.43}$$

for $[\alpha_{p,q}] \in E_r^{p,q}(M)$. Furthermore,

$$E_{r+1}^{p,q}(M) = \frac{X_{r+1}^{p,q}(M)}{Y_{r+1}^{p,q}(M)} = \frac{\operatorname{Ker} d_r : E_r^{p,q}(M) \to E_r^{p+r,q-r+1}(M)}{\operatorname{Im} d_r : E_r^{p-r,q+r-1}(M) \to E_r^{p,q}(M)}. \tag{2.44}$$

By definition we have that, for every $(p,q) \in \mathbb{Z}^2$ and every $r \in \mathbb{N}$, $E_r^{p,q}(M)$ is a $\mathbb{C}$-vector space and $E_{r+1}^{p,q}(M)$ is a subspace of $E_r^{p,q}(M)$. Moreover, since $E_r^{p,q}(M) = H_{\overline{\partial}}^{p,q}(M)$ is finite dimensional, the dimension of $E_r^{p,q}(M)$ is a non increasing function of r. We say that the Frölicher spectral sequence of M *degenerates at the step* r if r is the smallest integer such that, for every $(p,q) \in \mathbb{Z}^2$ and every $r' \geq r$, $E_{r'}^{p,q}(M) \simeq E_{r'+1}^{p,q}$, when this happens we denote $E_r^{p,q}(M)$ with $E_\infty^{p,q}(M)$ and we have the following isomorphism (see [Frö55])

$$H_{dR}^k(M) \simeq \oplus_{p+q=k} E_\infty^{p,q}. \tag{2.45}$$

In this sense we say that $(E_r^{p,q}(M), d_r)$ measure the distance between the Dolbeault and the de Rham chomologies. In particular we have that the complex dimension of the k-th group of the de Rham cohomology of any compact complex manifold is less or equal then the sum of the dimension of $H^{p,q}_{\overline{\partial}}(M)$ with $p + q = k$. The equality holds if and only if the Frölicher spectral sequence degenerates at the first step. This happens, for example, for compact Kähler manifolds or, more in general, for manifolds that satisfy the $\partial\overline{\partial}$-Lemma.

2.6 The $\partial\overline{\partial}$-Lemma

We recall a celebrated result due to Deligne, Grffiths, Morgan and Sullivan: the $\partial\overline{\partial}$-Lemma.

Theorem 24. *Let $(K^{\bullet,\bullet}, d', d'')$ be a bounded double complex of vector spaces, and let $(K^\bullet, d)$ be the associated simplex complex ($d = d' + d''$). For each integer n, the following conditions are equivalent:*

- $\operatorname{Ker} d' \cap \operatorname{Ker} d'' \cap \operatorname{Im} d = \operatorname{Im} d'd''$;
- $\operatorname{Ker} d'' \cap \operatorname{Im} d' = \operatorname{Im} d'd'' = \operatorname{Ker} d' \cap \operatorname{Im} d''$;
- $\operatorname{Ker} d' \cap \operatorname{Ker} d'' \cap (\operatorname{Im} d' + \operatorname{Im} d'') = \operatorname{Im} d'd''$;
- $\operatorname{Im} d' + \operatorname{Im} d'' + \operatorname{Ker} d = \operatorname{Ker} d'd''$;
- $\operatorname{Im} d' + \operatorname{Ker} d'' = \operatorname{Ker} d'd'' = \operatorname{Im} d'' + \operatorname{Ker} d'$;
- $\operatorname{Im} d' + \operatorname{Im} d'' + (\operatorname{Ker} d' \cap \operatorname{Ker} d'') = \operatorname{Ker} d'd''$.

The importance of this theorem is due to the following fact: consider the following diagram

$$\begin{tikzcd} & H^{p,q}_{BC}(M) \arrow[dl] \arrow[d] \arrow[dr] & \\ H^{p,q}_{\overline{\partial}}(M) \arrow[dr] & H^k_{dR}(M, \mathbb{C}) \arrow[d] & H^{p,q}_{\partial}(M) \arrow[dl] \\ & H^{p,q}_A(M) & \end{tikzcd} \tag{2.46}$$

where $k = p + q$ and the arrows denote the maps induced by the identity. In general those maps are neither injective nor surjective, but if one of those is injective, then all of them are isomorphisms (see [DGMS75]). Such a result is equivalent, for example, to the first point of Theorem 24. We say that a manifold M *satisfies the $\partial\overline{\partial}$-lemma* if

$$\operatorname{Ker} \partial \cap \operatorname{Ker} \overline{\partial} \cap \operatorname{Im} d = \operatorname{Im} \partial\overline{\partial}. \tag{2.47}$$

Among the manifolds that verify this property there are, for example, the Kähler manifolds. However those two classes of manifolds does not coincide.

Many authors have studied the manifolds that satisfy the $\partial\overline{\partial}$-lemma, among the results of characterization we recall the following

Theorem 25 ([AT13]). *Let M be a compact complex manifold of complex dimension n. Then, for every $(p,q) \in \mathbb{Z}^2$, the following inequality holds:*

$$h^{p,q}_{BC} + h^{p,q}_{A} \geq h^{p,q}_{\overline{\partial}} + h^{p,q}_{\partial}. \tag{2.48}$$

In particular, for evey $k \in \mathbb{N}$, the following inequality holds:

$$h^{k}_{BC} + h^{k}_{A} \geq 2b^k, \tag{2.49}$$

where $h^k_{BC} = \sum_{p+q=k} h^{p,q}_{BC}$ and $h^k_A = \sum_{p+q=k} h^{p,q}_A$. Moreover, the equality

$$h^{k}_{BC} + h^{k}_{A} = 2b^k \tag{2.50}$$

holds if and only if M satisfies the $\partial\overline{\partial}$-Lemma.

2.7 Varouchas spaces

The classical cohomology groups cited in the previous section are not enough for our purposes. We will need some useful tools introduced by Varouchas in [Var].

In this section we recall the definitions of certain $\mathbb{C}$-vector spaces and exact sequences related to cohomology groups and we prove some relations between their dimensions.

The *Varouchas spaces* are finite dimensional $\mathbb{C}$-vector spaces defined as follow:

$$A^{p,q}(M) := \frac{\left(\operatorname{Im}\partial : \Lambda^{p-1,q}(M) \to \Lambda^{p,q}(M)\right) \cap \left(\operatorname{Im}\overline{\partial} : \Lambda^{p,q-1}(M) \to \Lambda^{p,q}(M)\right)}{\operatorname{Im}\partial\overline{\partial} : \Lambda^{p-1,q-1}(M) \to \Lambda^{p,q}(M)};$$
$$B^{p,q}(M) := \frac{\left(\operatorname{Im}\partial : \Lambda^{p-1,q}(M) \to \Lambda^{p,q}(M)\right) \cap \left(\operatorname{Ker}\overline{\partial} : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)\right)}{\operatorname{Im}\partial\overline{\partial} : \Lambda^{p-1,q-1}(M) \to \Lambda^{p,q}(M)};$$
$$D^{p,q}(M) := \frac{\left(\operatorname{Ker}\partial : \Lambda^{p,q}(M) \to \Lambda^{p+1,q}(M)\right) \cap \left(\operatorname{Im}\overline{\partial} : \Lambda^{p,q-1}(M) \to \Lambda^{p,q}(M)\right)}{\operatorname{Im}\partial\overline{\partial} : \Lambda^{p-1,q-1}(M) \to \Lambda^{p,q}(M)};$$
$$C^{p,q}(M) := \frac{\operatorname{Ker}\partial\overline{\partial} : \Lambda^{p,q}(M) \to \Lambda^{p+1,q+1}(M)}{\left(\operatorname{Im}\partial : \Lambda^{p-1,q}(M) \to \Lambda^{p,q}(M)\right) + \left(\operatorname{Ker}\overline{\partial} : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)\right)};$$
$$E^{p,q}(M) := \frac{\operatorname{Ker}\partial\overline{\partial} : \Lambda^{p,q}(M) \to \Lambda^{p+1,q+1}(M)}{\left(\operatorname{Ker}\partial : \Lambda^{p,q}(M) \to \Lambda^{p+1,q}(M)\right) + \left(\operatorname{Im}\overline{\partial} : \Lambda^{p,q-1}(M) \to \Lambda^{p,q}(M)\right)};$$
$$F^{p,q}(M) := \frac{\operatorname{Ker}\partial\overline{\partial} : \Lambda^{p,q}(M) \to \Lambda^{p+1,q+1}(M)}{\left(\operatorname{Ker}\partial : \Lambda^{p,q}(M) \to \Lambda^{p+1,q}(M)\right) + \left(\operatorname{Ker}\overline{\partial} : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)\right)}.$$

Using the previous spaces, Varouchas proved that the sequences

$$0 \to A^{p,q}(M) \to B^{p,q}(M) \to H^{p,q}_{\overline{\partial}}(M) \to H^{p,q}_{A}(M) \to C^{p,q}(M) \to 0 \qquad (2.51)$$

and

$$0 \to D^{p,q}(M) \to H^{p,q}_{BC}(M) \to H^{p,q}_{\overline{\partial}}(M) \to E^{p,q}(M) \to F^{p,q}(M) \to 0 \qquad (2.52)$$

are exact.

Proposition 4. *For every $p, q \in \{0, ..., n\}$, the conjugation and the $*$-Hodge linear operator give rise the following non-canonical isomorphisms:*

$$1) A^{p,q}(M) \simeq A^{q,p}(M); \quad 2) B^{p,q}(M) \simeq D^{q,p}(M);$$

$$3) C^{p,q}(M) \simeq E^{q,p}(M); \quad 4) F^{p,q}(M) \simeq F^{q,p}(M).$$

Moreover Varouchas proved the following:

Proposition 5. *The differential operator ∂ induces an isomorphism between $E^{p,q}(M)$ and $B^{p+1,q}(M)$ and the differential operator $\overline{\partial}$ induces an isomorphism between $C^{p,q}(M)$ and $D^{p,q+1}(M)$.*

For the sake of completeness, we give the proof.

Proof. We prove only the first statement due to its similarity of the second proof. Using the same name we consider the operator

$$\partial : E^{p,q}(M) \to B^{p+1,q}(M)$$

defined as

$$\partial[\alpha]_E := [\partial\alpha]_B.$$

The map ∂ is well defined. Let α be a $\partial\overline{\partial}$-closed (p, q)-form, then $\partial\alpha$ is a ∂-exact, $\overline{\partial}$-closed $(p+1, q)$-form therefore it defines a class in $B^{p+1,q}(M)$. Moreover if we consider another form in the same class in $E^{p,q}(M)$ of α, namely $\alpha+\beta+\overline{\partial}\gamma$ with β and γ of suitable bi-degrees and $\partial\beta = 0$, then we have that $\partial(\alpha + \beta + \overline{\partial}\gamma) = \partial\alpha + \partial\overline{\partial}\gamma$. Thus it defines the same class of $\partial\alpha$ in $B^{p+1,q}(M)$.

The map ∂ is surjective. Let β be a representative of a class in $B^{p+1,q}(M)$, then $\beta = \partial\alpha$ for some (p, q)-form α. Now, by the $\overline{\partial}$-closeness of β, $\partial\overline{\partial}\alpha = 0$ therefore α defines a class in $E^{p,q}(M)$.

The map ∂ is injective. Take a representative β of the zero class of $B^{p+1,q}(M)$, then $\beta = \partial\overline{\partial}\alpha$ for some $(p, q-1)$-form α. Moreover $[\overline{\partial}\alpha]_E = [0]_E$ and $\partial[\overline{\partial}\alpha]_E = [\beta]_B$. This concludes the proof. □

From now on we denote with the lower case characters denote the dimension of the respective Varouchas space, e.g. $a^{p,q} := \dim A^{p,q}(M)$.

In addition to the equalities given by the isomorphisms above, we have the following:

Proposition 6. *Let M be a compact complex manifold. Then $c^{p,q} = d^{n-p,n-q}$ and $b^{p,q} = e^{n-p,n-q}$.*

Proof. *Step one.* We observe that if the theorem holds for a fixed pair $(p_0, q_0) \in \mathbb{N}^2$ then it holds for the pair $(n - q_0, n - p_0)$: in fact, using the isomorphisms (7) and (6) of the Remark, we have the following:

$$c^{n-q_0,n-p_0} = e^{n-p_0,n-q_0} = b^{p_0,q_0} = d^{q_0,p_0}.$$

The same argument stands for the case $b^{n-q,n-p} = e^{q,p}$ and so this step is proved.

Step two. We prove by induction over $k = 0, 1, ..., 2n$ that, if $p + q = k$ we have $c^{p,q} = d^{n-p,n-q}$ and $b^{p,q} = e^{n-p,n-q}$. If $k = 0$, from the definitions of the spaces, we have that $b^{0,0} = c^{n,n} = d^{0,0} = e^{n,n} = 0$. By applying step one to $c^{n,n} = d^{0,0}$ we have that $c^{0,0} = d^{n,n}$ and thus we complete the base of induction.

Let $k > 0$ and suppose the theorem holds for every $(p, q) \in \mathbb{N}^2$ such that $p + q < k$. Let $(p_0, q_0) \in \mathbb{N}^2$ such that $p_0 + q_0 = k$ then, using the isomorphisms of the Remark and Proposition (8), we have:

$$\begin{aligned} b^{p_0,q_0} &= e^{p_0-1,q_0} \\ &= c^{q_0,p_0-1} \\ &= d^{n-q_0,n-p_0+1} \\ &= c^{n-q_0,n-p_0} \\ &= e^{n-p_0,n-q_0}. \end{aligned}$$

With the same argument one can prove the $c^{p,q} = d^{n-p,n-q}$ case. □

We can summarize the results above in the following:

Corollary 1. *Let M be a compact complex manifold. Then*

$$e^{q,p} = c^{p,q} = d^{n-p,n-q} = b^{n-q,n-p} = e^{n-q-1,n-p} = c^{n-p,n-q-1} = d^{p,q+1} = b^{q+1,p}.$$

Similar relations can also be found between the dimensions of $A^{p,q}(M)$ and $F^{p,q}(M)$. More precisely:

Proposition 7. *Let M be a compact complex manifold. Then*

$$a^{p,q} = f^{n-p,n-q}$$

and

$$a^{p,q} + f^{q,p} = h^{p,q}_A + h^{p,q}_{BC} - h^{p,q}_{\overline{\partial}} - h^{q,p}_{\overline{\partial}}.$$

Proof. By the exact sequences (2.51) and (2.52) we have that, for every $p, q \in \{0, ..., n\}$:

$$a^{p,q} - b^{p,q} + h^{p,q}_{\overline{\partial}} - h^{p,q}_A + c^{p,q} = 0; \tag{2.53}$$

$$d^{p,q} - h^{p,q}_{BC} + h^{p,q}_{\overline{\partial}} - e^{p,q} + f^{p,q} = 0. \tag{2.54}$$

The first equality is obtained by subtracting the second equation in the case $(n - p, n - q)$ from the first equation in the case (p, q). The second equality is obtained by summing the two equation, the first in the case (p, q) and the second in the case (q, p). □

2.8 Cones of Hermitian Metrics

In this section we report our study about the cones of SKT and super SKT metrics. Before doing that we recall some results due to other authors about cones of metrics. Given a special Hermitian metric as in Section 1.4, a cone of such metric is a subset of a suitable cohomology group composed by classes that have at least a positive form as a representative. We briefly recall the following definitions

Definition 11. *Let M be a compact complex manifold of complex dimension n. Let g an Hermitian metric on M and let ω be its fundamental form. Then g is said to be:*

- *Kähler, if $d\omega = 0$;*
- *Gauduchon, if ω^{n-1} is $\partial\overline{\partial}$-closed;*
- *strongly Gauduchon, if $\partial\omega^{n-1}$ is $\overline{\partial}$-exact;*
- *SKT, if ω is $\partial\overline{\partial}$-closed;*
- *sSKT, if $\partial\omega$ is $\overline{\partial}$-exact.*

Since ω is a $(1,1)$-form the Kähler condition is equivalent to impose $\partial\omega = \overline{\partial}\omega = 0$. The *Kähler cone* $\mathcal{K}$ was firstly studied by Demailly and Paun (see [DP04] or [BDPP04]) and is defined as the subset of classes in $H^{1,1}_{BC}(M,\mathbb{R})$ that contain at least one positive form. Namely

$$\mathcal{K} := \{[\phi]_{BC} \in H^{1,1}_{BC}(M,R) \mid \exists \omega \in [\phi]_{BC} \text{ s.t. } \omega > 0\}.$$

In [PU14] the cones $\mathcal{G}$ of Gauduchon metrics and $\mathcal{SG}$ of strongly Gauduchon metrics were introduced. Namely

$$\mathcal{G} := \{[\phi]_A \in H^{n-1,n-1}_A(M,\mathbb{R}) | \exists \omega^{n-1,n-1} \in [\phi]_A \text{ s.t. } \omega > 0\}$$

and

$$\mathcal{SG} := \mathcal{G} \cap \operatorname{Ker} T,$$

where $T : H^{n-1,n-1}_A(M) \to H^{n,n-1}_{\overline{\partial}}(M)$ is defined as $T([\phi]_A) := [\partial\phi]_{\overline{\partial}}$. That is, they are the cones of classes in $H^{n-1,n-1}_A(M)$ that contain the $(n-1)$-th power of the fundamental of a Gauduchon (resp. strongly Gauduchon) metric. In their work, Popovici and Ugarte called a manifold an *sGG manifold* if $\mathcal{SG} = \mathcal{G}$. Moreover, since the kernel of T is a vector subspace of $H^{n-1,n-1}_A(M)$, its intersection with the open convex cone $\mathcal{G}$ leave this latter unchanged if and only if T vanished identically. Using this observation, they proved the following results of characterization.

Lemma 1. *Let M be a compact complex manifold of complex dimension n. Then the following statements are equivalent:*

- *M is an sGG manifold;*

- *the map T vanished identically;*
- *the following spacial case of the $\partial\overline{\partial}$-lemma holds: for every d-closed $(n, n-1)$-form ϕ on M, if ϕ is ∂-exact then ϕ is also $\partial\overline{\partial}$-exact;*
- *every Gauduchon metric on M is strongly Gauduchon.*

Theorem 26. *Let M be a compact complex manifold of complex dimension n. The well-defined canonical $\mathbb{C}$-linear map induced by the identity*

$$S : H^{n,n-1}_{\partial}(M, \mathbb{C}) \to H^{n,n-1}(M, \mathbb{C})$$

is surjective and we have an exact sequence

$$H^{n-1,n-1}_{A}(M, \mathbb{C}) \xrightarrow{T} H^{n,n-1}_{\partial}(M, \mathbb{C}) \xrightarrow{S} H^{n,n-1}_{A}(M, \mathbb{C}) \to 0.$$

In particular, M is an sGG manifold if and only if S is injective (i.e. if and only if S is bijective).

Theorem 27. *Let M be a compact complex manifold of complex dimension n. Then there is a well-defined canonical $\mathbb{C}$-linear map*

$$F : H^{1}_{dR}(M, \mathbb{C}) \to H^{0,1}_{\overline{\partial}}(M, \mathbb{C}) \oplus \overline{H^{0,1}_{\overline{\partial}}}(M, \mathbb{C}),$$

such that $F([\phi]_{dR}) := ([\phi^{0,1}]_{\overline{\partial}}, \overline{[\overline{\phi}^{1,0}]_{\overline{\partial}}})$. Moreover, the map F is injective. Consequently, the following inequality

$$b_1 \leq 2h^{0,1}_{\overline{\partial}}$$

holds on every compact complex manifold. The following equivalence holds

$$X \text{ is an sGG manifold} \Leftrightarrow F \text{ is surjective.}$$

This last result is the crucial point in the proof of the openness under small deformations of the condition "being an sGG manifold".

Now we move to our work on the SKT and sSKT metrics. With the same notation, we put

$$T : H^{1,1}_{A}(M) \to H^{2,1}_{\overline{\partial}}(M)$$

as $T([\alpha]_A) = [\partial\alpha]_{\overline{\partial}}$. This map is well defined: let α be a representative of an Aeppli cohomology class and consider $\alpha + \partial\beta + \overline{\partial}\gamma$, for some β and γ of suitable bi-degrees. Now

$$T([\alpha + \partial\beta + \overline{\partial}\gamma]_A) = [\partial(\alpha + \partial\beta + \overline{\partial}\gamma)]_{\overline{\partial}} = [\partial\alpha - \overline{\partial}\partial\gamma]_{\overline{\partial}} = [\partial\alpha]_{\overline{\partial}} = T([\alpha]_A).$$

We define the cones

$$\mathcal{S} := \{[\phi]_A \in H^{1,1}_{A}(M) | \exists\omega \in [\phi]_A, \omega > 0\} \tag{2.55}$$

and

$$s\mathcal{S} := \mathcal{S} \cap \operatorname{Ker} T \tag{2.56}$$

of SKT and sSKT metrics respectively.

We want to study when $s\mathcal{S} = \mathcal{S}$ .From now on, we will assume that M admits at least one SKT metric; otherwise $\mathcal{S} = \emptyset$ and the equality is trivially true. The following proposition follows from the above consideration about the intersection of the convex cone $\mathcal{S}$ and the subspace $\operatorname{Ker} T$.

Proposition 8. *Let M be a compact complex manifold admitting a SKT metric, i.e., such that $\mathcal{S} \neq \emptyset$. Then we have $s\mathcal{S} = \mathcal{S} \cap \operatorname{Ker} T$. Moreover $s\mathcal{S} = \mathcal{S}$ if and only if $\operatorname{Ker} T = H_A^{1,1}(M)$ if and only if $T \equiv 0$.*

Proof. The first statement follows directly from the definitions. For the second one we observe that $s\mathcal{S}$ is the intersection between a cone and a vector space, so the equality $s\mathcal{S} = \mathcal{S}$ is equivalent to the $\operatorname{Ker} T$ being the whole space. □

Now, since we are interested in the study of $\operatorname{Ker} T$, we want to construct a suitable exact sequence, we consider the following:

Proposition 9. *Let M be a compact complex manifold. Then the sequence*

$$H_A^{1,1}(M) \xrightarrow{T} H_{\overline{\partial}}^{2,1}(M) \xrightarrow{i} H_A^{2,1}(M) \xrightarrow{j} C^{2,1}(M) \to 0, \tag{2.57}$$

is exact, where i and j are the maps induced in cohomology by the identity.

Proof. We only need to prove the exactness at $H_{\overline{\partial}}^{2,1}(M)$. If $[\alpha]_A \in H_A^{1,1}(M)$, then $[\partial\alpha]_{\overline{\partial}} \mapsto [\partial\alpha]_A = [0]_A \in H_A^{2,1}(M)$, so $\operatorname{Im} T \subset \operatorname{Ker} i$. Conversely, in order to prove that $\operatorname{Ker} i \subset \operatorname{Im} T$, we consider a $(2,1)$-form α such that: $\overline{\partial}\alpha = 0$ and $\alpha = \partial\beta + \overline{\partial}\gamma$, for suitable forms β and γ. The two conditions mean respectively that α defines a class in the Dolbeault cohomology and $i([\alpha]_{\overline{\partial}}) = [0]_A$. Now $[\alpha]_{\overline{\partial}} = [\partial\beta + \overline{\partial}\gamma]_{\overline{\partial}} = [\partial\beta]_{\overline{\partial}}$ so $[\alpha]_{\overline{\partial}}$ has a ∂-exact representative; moreover $\partial\overline{\partial}\beta = -\overline{\partial}\alpha = 0$ thus β defines a class in the Aeppli cohomology. Hence $\operatorname{Ker} T \subset \operatorname{Im} T$. □

We can now prove the following:

Theorem 28. *Let M be a compact complex manifold admitting a SKT metric. Then the following facts are equivalent:*

1. $s\mathcal{S} = \mathcal{S}$;
2. $\operatorname{Ker} T = H_A^{1,1}(M)$;
3. $c^{2,1}(M) = h_A^{2,1}(M) - h_{\overline{\partial}}^{2,1}(M)$;
4. $A^{2,1}(M) \simeq B^{2,1}(M)$;
5. $a^{2,1}(M) = b^{2,1}(M)$;

6. *every smooth d-closed ∂-exact $(2,1)$-form on M is $\overline{\partial}$-exact;*

7. *every SKT metric g is super SKT.*

Proof. $1 \Leftrightarrow 2$: this is proved in Proposition 8.

$2 \Rightarrow 3$: if $\operatorname{Ker} T = H^{1,1}_A(M)$, then sequence (2.57) becomes

$$0 \to H^{2,1}_{\overline{\partial}}(M) \to H^{2,1}_A(M) \to C^{2,1}(M) \to 0.$$

Hence $c^{2,1}(M) = h^{2,1}_A(M) - h^{2,1}_{\overline{\partial}}(M)$, i.e. 3 holds.

$3 \Rightarrow 2$ we recall that all the spaces in (8) are finite-dimensional so, by the exactness of the sequence, we have:

$$\begin{aligned}\dim(\operatorname{Im} T) &= \dim(\operatorname{Ker} i) = h^{2,1}_{\overline{\partial}} - \dim(\operatorname{Im} i)\\ &= h^{2,1}_{\overline{\partial}} - \dim(\operatorname{Ker} j) = h^{2,1}_{\overline{\partial}} - h^{2,1}_A + \dim(\operatorname{Im} j)\\ &= h^{2,1}_{\overline{\partial}} - h^{2,1}_A + c^{2,1}.\end{aligned}$$

But, by hypothesis, $h^{2,1}_{\overline{\partial}} - h^{2,1}_A + c^{2,1} = 0$, then $\dim(\operatorname{Im} T) = 0$, that implies $\operatorname{Ker} T = H^{1,1}_A(M)$.

$2 \Leftrightarrow 4$: if 2 holds we have that the map i in (5) is injective, so the sequence (2.51) can be written as

$$0 \to A^{2,1}(M) \to B^{2,1}(M) \to 0 \to H^{2,1}_{\overline{\partial}}(M) \to H^{2,1}_A(M) \to C^{2,1}(M) \to 0.$$

Hence we have the fourth statement. The converse still holds by the same argument.

$4 \Rightarrow 5$: trivial.

$5 \Rightarrow 4$: using sequence (2.51), we have an injective map from $A^{2,1}$ to $B^{2,1}$. Since they have the same dimension that map must be an isomorphism.

$4 \Rightarrow 6$: let ϕ be a $(2,1)$-form such that $\phi \in \operatorname{Ker} \overline{\partial} \cap \operatorname{Im} \partial$ then it defines a class in $B^{2,1}$. By the assumption it also defines a class in $A^{2,1}$ and therefore it is $\overline{\partial}$-exact.

$6 \Rightarrow 4$: let $[\phi] \in B^{2,1}$, then $\phi \in \operatorname{Ker} \overline{\partial} \cap \operatorname{Im} \partial$. Thus $\phi \in \operatorname{Im} \overline{\partial}$ and consequently ϕ defines a class in $A^{2,1}$.

$1 \Rightarrow 7$: given an SKT metric g, its fundamental form ω defines a class in the cone $\mathcal{S}$. Since $s\mathcal{S} = \mathcal{S}$, there exists $\alpha \in [\omega]_A$ such that $\alpha > 0$ and $\partial\alpha$ is $\overline{\partial}$-exact. Thus $\omega = \alpha + \partial\beta + \overline{\partial}\gamma$, for suitable β and γ. As a consequence $\partial\omega = \partial\alpha - \overline{\partial}\partial\gamma$ is $\overline{\partial}$-exact, therefore ω is the fundamental form of a super SKT metric.

$7 \Rightarrow 1$: given a class $[\alpha]_A \in \mathcal{S}$, there exists a (1,1)-form $\omega \in [\alpha]_A$ such that $\omega > 0$ and $\partial\overline{\partial}\omega = 0$, namely ω is the fundamental form of a SKT metric g. By the assumption, g is a super SKT metric, thus $\partial\omega$ is $\overline{\partial}$-exact and therefore $[\alpha]_A = [\omega]_A \in s\mathcal{S}$.

□

We conclude this section by focusing on the case of compact complex surfaces, namely compact complex manifolds of complex dimension 2.

Theorem 29. *Let M be a compact complex surface admitting a SKT metric. Then $s\mathcal{S} = \mathcal{S}$ if and only if $h_A^{2,1} = h_{\overline{\partial}}^{2,1}$.*

Proof. Using Theorem 28, we have that $s\mathcal{S} = \mathcal{S}$ if and only $h_{\overline{\partial}}^{2,1} - h_A^{2,1} + c^{2,1} = 0$. By Corollary 1, $c^{2,1} = c^{0,0}$. By definition $c^{0,0} = 0$ because $C^{0,0}$ is a group of $\mathcal{C}^\infty$ function that are $\partial\overline{\partial}$-closed, i.e., of constant function, quotiented by the $\overline{\partial}$-closed $\mathcal{C}^\infty$ function (the ∂-exact are excluded for reason of bi-degree). But every constant function is $\overline{\partial}$-closed thus $C^{0,0} = \{0\}$. □

One interesting aspect in the case of the surfaces is that, in complex dimension 2, SKT and super SKT metrics satisfy respectively the Gauduchon and strongly Gauduchon conditions.

Since, for $n = 2$, a metric is SKT (resp. sSKT) if and only if it is Gauduchon (resp. strongly Gauduchon), we have that every compact complex surface admits a SKT metric (since every compact complex manifold admits a Gauduchon metric) and we can omit this hypothesis from Theorem 29. Furthermore the $\mathcal{S}$ and $s\mathcal{S}$ cones coincide respectively with the Gauduchon and strongly Gauduchon cones studied in [PU14], thus Theorem 29 can be view as a special case of

Theorem 30. *[PU14, Theorem 1.3] On any compact complex manifold M we have $h_{BC}^{0,1} \leq h_{\overline{\partial}}^{0,1}$. Moreover, M is an sGG manifold if and only if $h_{BC}^{0,1} = h_{\overline{\partial}}^{0,1}$.*

Here an sGG manifold is a manifold on which the cone of Gauduchon metrics coincides with the cone of strongly Gauduchon metrics. Finally, using [PU14, Theorem 1.5], we have the following

Remark: let M be a compact complex surface. M is Kähler if and only if $s\mathcal{S} = \mathcal{S}$.

2.8.1 The equivalence $s\mathcal{S} = \mathcal{S}$ on Nilmanifolds

We asked ourselves if the equivalence of the cones $s\mathcal{S}$ and $\mathcal{S}$ is satisfied by every compact complex manifold and we found out that this is not true. We will provide a family of examples of compact complex manifolds such that $s\mathcal{S} \neq \mathcal{S}$. As explicit examples we will use Theorem 28 in the case of 6-dimensional nilmanifold. A complex nilmanifold is a compact quotient $\Gamma \setminus G$ of a connected simply-connected nilpotent Lie group G by a co-compact discrete sub-group Γ, endowed with a G-left-invariant complex structure J. In [FPS04] Fino, Parton and Salamon classify 6-dimensional complex nilmanfolds admitting invariant SKT metrics; moreover Angella and Kasuya (see [AK12, Theorem 1.3]) proved that, in the case of 6-dimensional nilmanifolds endowed with SKT metrics, it is possible to compute Bott-Chern cohomology groups using only invariant forms. First we recall the result by Angella and Kasuya in its most general form:

Theorem 31. *Let $\left(A^{\bullet,\bullet}, \partial, \overline{\partial}\right)$ be a bounded double complex of $\mathbb{C}$-vector spaces, and let $\left(C^{\bullet,\bullet}, \partial, \overline{\partial}\right) \hookrightarrow \left(A^{\bullet,\bullet}, \partial, \overline{\partial}\right)$ be a sub-complex. Fix $(p, q) \in \mathbb{Z}^2$. Suppose that:*

- *for every* $r \in \mathbb{Z}$ *the induced map* $\left(C^{r,\bullet}, \overline{\partial}\right) \hookrightarrow \left(A^{r,\bullet}, \overline{\partial}\right)$ *is a quasi-isomorphism;*
- *for every* $s \in \mathbb{Z}$ *the induced map* $(C^{\bullet,s}, \partial) \hookrightarrow (A^{\bullet,}, \partial)$ *is a quasi-isomorphism;*
- *the induced map*

$$\frac{\operatorname{Ker}(d : Tot^{p+q}(C^{\bullet,\bullet}) \to Tot^{p+q+1}(C^{\bullet,\bullet}))}{\operatorname{Im}(d : Tot^{p+q-1}(C^{\bullet,\bullet}) \to Tot^{p+q}(C^{\bullet,\bullet}))} \to \frac{\operatorname{Ker}(d : Tot^{p+q}(A^{\bullet,\bullet}) \to Tot^{p+q+1}(A^{\bullet,\bullet}))}{\operatorname{Im}(d : Tot^{p+q-1}(A^{\bullet,\bullet}) \to Tot^{p+q}(A^{\bullet,\bullet}))}$$

is surjective, (here $Tot^k(A^{\bullet,\bullet}) := \oplus_{p+q=k} A^{p,q}$*).*

Then the induced map

$$\left(C^{p-1,q-1} \xrightarrow{\partial\overline{\partial}} C^{p,q} \xrightarrow{\partial+\overline{\partial}} C^{p+1,q} \oplus C^{p,q+1}\right) \hookrightarrow \left(A^{p-1,q-1} \xrightarrow{\partial\overline{\partial}} A^{p,q} \xrightarrow{\partial+\overline{\partial}} A^{p+1,q} \oplus A^{p,q+1}\right)$$

of complexes induces a surjective map in cohomology.

The previous theorem allows us to compute the cohomology groups in terms of invariant forms.

From now on, we will use the following notation: $\phi^{i\overline{j}}$ stands for $\phi^i \wedge \overline{\phi}^j$; more generally, if I and J are multi-index of length l and m respectively, with $\phi^{I\overline{J}}$ we denote $\phi^{i_1} \wedge \cdots \wedge \phi^{i_l} \wedge \overline{\phi}^{j_1} \wedge \cdots \wedge \overline{\phi}^{j_m}$.

Then we recall a result by Fino, Parton and Salamon about the characterization of 6-dimensional nilmanifold admitting SKT metrics (see [FPS04, Theorem 1.2]):

Theorem 32. *Let* $M = \Gamma \backslash G$ *be a 6-dimensional nilmanifold with an invariant complex structure* J*. Then the SKT condition is satisfied by either all invariant Hermitian metrics* g *or by none. Indeed, it is satisfied if and only if* J *has a basis* (ϕ^i) *of (1,0)-forms such that:*

$$\begin{cases} d\phi^1 = 0 \\ d\phi^2 = 0 \\ d\phi^3 = A\phi^{\overline{1}2} + B\phi^{\overline{2}2} + C\phi^{1\overline{1}} + D\phi^{1\overline{2}} + E\phi^{12} \end{cases} \tag{2.58}$$

where A, B, C, D, E *are complex numbers such that*

$$|A|^2 + |D|^2 + |E|^2 + 2Re(\overline{B}C) = 0. \tag{2.59}$$

Using Theorems 28 and 31, 32 we will compute cohomology groups of compact complex nilmanifolds of dimension 6 and we prove that:

Theorem 33. *Let* M *be a non-torus compact complex 6-dimensional nilmanifold with Lie algebra different from* $\mathfrak{h}_7$*. If* $\mathcal{S} \neq \emptyset$ *then* $s\mathcal{S} \neq \mathcal{S}$*.*

Proof. We will show that $h_A^{2,1} - c^{2,1} - h_{\overline{\partial}}^{2,1} \neq 0$. Therefore, by Theorem 28, we obtain that $s\mathcal{S} \neq \mathcal{S}$. First we will compute $h_{BC}^{2,1}$ which is equal to $h_A^{2,1}$ (see Remark in section 4). By Theorem 31, we only need to compute it using invariant forms. By Theorem 32, we obtain the following table (see [AFR15] or

[LUV14] for complete computation of cohomology groups of compact complex manifolds)

$(2,1)$-form invariant	∂	$\overline{\partial}$
$\phi^{12\overline{1}}$	0	0
$\phi^{12\overline{2}}$	0	0
$\phi^{12\overline{3}}$	0	$\overline{E}\phi^{12\overline{12}}$
$\phi^{13\overline{1}}$	0	$-B\phi^{12\overline{12}}$
$\phi^{13\overline{2}}$	0	$A\phi^{12\overline{12}}$
$\phi^{13\overline{3}}$	$-\overline{B}\phi^{123\overline{2}}+\overline{D}\phi^{123\overline{1}}$	$A\phi^{12\overline{13}}+B\phi^{12\overline{23}}+\overline{E}\phi^{13\overline{12}}$
$\phi^{23\overline{1}}$	0	$-D\phi^{12\overline{12}}$
$\phi^{23\overline{2}}$	0	$C\phi^{12\overline{12}}$
$\phi^{23\overline{3}}$	$\overline{A}\phi^{123\overline{2}}-\overline{C}\phi^{123\overline{1}}$	$C\phi^{12\overline{13}}+D\phi^{12\overline{23}}+\overline{E}\phi^{23\overline{12}}$

Moreover we have that $\operatorname{Im}\partial\overline{\partial}:\Lambda^{1,0}_{inv}\to\Lambda^{2,1}_{inv}=\{0\}$. As a consequence $H^{2,1}_{BC}(M)\simeq\operatorname{Ker}\partial\cap\operatorname{Ker}\overline{\partial}$.

Obviously both $\phi^{12\overline{1}}$ and $\phi^{12\overline{2}}$ define a class in $H^{2,1}_{BC}(M)$. Other d-closed invariant forms of bidegree (2,1) are

$$B\phi^{12\overline{3}}+\overline{E}\phi^{13\overline{1}};\quad A\phi^{12\overline{3}}-\overline{E}\phi^{13\overline{2}};\quad D\phi^{12\overline{3}}+\overline{E}\phi^{23\overline{1}};\quad C\phi^{12\overline{3}}-\overline{E}\phi^{23\overline{2}};\quad A\phi^{13\overline{1}}+B\phi^{13\overline{2}}$$

$$D\phi^{13\overline{1}}-B\phi^{23\overline{1}};\quad C\phi^{13\overline{1}}+B\phi^{23\overline{2}};\quad D\phi^{13\overline{2}}+A\phi^{23\overline{1}};\quad C\phi^{13\overline{2}}-A\phi^{23\overline{2}};\quad C\phi^{23\overline{1}}+D\phi^{23\overline{2}}.$$

Since M is not a torus, at least one coefficient between A,B,C,D,E is not zero, in particular both B and C must be different from zero, so the $\mathbb{C}$-vector space generated by the above (2,1) forms has complex dimension 4. In fact, $B\phi^{12\overline{3}}+\overline{E}\phi^{13\overline{1}}$, $A\phi^{13\overline{1}}+B\phi^{13\overline{2}}$, $D\phi^{13\overline{1}}-B\phi^{23\overline{1}}$ and $C\phi^{13\overline{1}}+B\phi^{23\overline{2}}$ are $\mathbb{C}$-linearly indipendent and every other forms can be written as a linear combination of two of them, e.g., $C\phi^{23\overline{1}}+D\phi^{23\overline{2}}=+\frac{D}{B}(C\phi^{13\overline{1}}+B\phi^{23\overline{2}})-\frac{C}{B}(D\phi^{13\overline{1}}-B\phi^{23\overline{1}})$.

Finally, considering $\phi^{13\overline{3}}+\lambda\phi^{23\overline{3}}$, we obtain that

- if $E\neq 0$ then $h^{2,1}_A=6$;
- if $E=0$
  - and there exists $\lambda\in\mathbb{C}$ such that $B=\overline{\lambda}A=-\lambda D=-|\lambda|^2C$, then $h^{2,1}_A=7$;
  - otherwise $h^{2,1}_A=6$.

Now we compute $c^{2,1}$. By Corollary (1), $c^{2,1}=b^{2,1}$. The latter is much easier to compute, in fact every class in $B^{2,1}(M)$ is a class in $H^{2,1}_{BC}(M)$, so we only need to understand which class in $H^{2,1}_{BC}(M)$ has ∂-exact representative:

$$\operatorname{Im}(\partial:\Lambda^{1,1}_{inv}(M)\to\Lambda^{2,1}_{inv}(M))=\mathbb{C}<\overline{D}\phi^{12\overline{1}}-\overline{B}\phi^{12\overline{2}},-\overline{C}\phi^{12\overline{1}}+\overline{A}\phi^{12\overline{2}},$$
$$E\phi^{12\overline{1}},E\phi^{12\overline{2}},E\phi^{12\overline{3}}+\overline{A}\phi^{13\overline{2}}+\overline{B}\phi^{23\overline{2}}-\overline{D}\phi^{23\overline{1}}-\overline{C}\phi^{13\overline{1}}>.$$

Then we have:

- if $E \neq 0$ then $c^{2,1} = 3$;
- if $E = 0$ and $AD - BC \neq 0$ then $c^{2,1} = 3$;
- if $E = 0$ and $AD - CB = 0$ then $c^{2,1} = 2$;

Finally we give a lower bound of $h^{2,1}_{\overline{\partial}}$ using the third column of the table and

$$\mathrm{Im}(\overline{\partial} : \Lambda^{2,0}_{inv}(M) \to \Lambda^{2,1}_{inv}(M)) = \mathbb{C} < A\phi^{12\overline{1}} + B\phi^{12\overline{2}}, C\phi^{12\overline{1}} + D\phi^{12\overline{2}} > .$$

We have five cases:

- if $E \neq 0$ and
  - $AD - BC \neq 0$, then $h^{2,1}_{\overline{\partial}} \geq 4$;
  - $A = C = -B = -D$, then $h^{2,1}_{\overline{\partial}} \geq 6$;
  - otherwise $h^{2,1}_{\overline{\partial}} \geq 5$;
- if $E = 0$ and
  - $AD - BC \neq 0$, then $h^{2,1}_{\overline{\partial}} \geq 4$;
  - $AD - BC \neq 0$, then $h^{2,1}_{\overline{\partial}} \geq 6$.

Summing up results above, we have that $c^{2,1} - h^{2,1}_{A} + h^{2,1}_{\overline{\partial}} \neq 0$, in every case. Therefore, by Theorem 28, this implies that $s\mathcal{S} \neq \mathcal{S}$. □

It is also possible to provide a metric $\omega \in \mathcal{S} \backslash s\mathcal{S}$. If we consider the (1,1)-form $\phi^{3\overline{3}}$ we have that

$$\partial\phi^{3\overline{3}} = E\phi^{12\overline{3}} + \overline{A}\phi^{13\overline{2}} + \overline{B}\phi^{23\overline{2}} - \overline{C}\phi^{13\overline{1}} - \overline{D}\phi^{23\overline{1}} \tag{2.60}$$

and

$$\partial\overline{\partial}\phi^{3\overline{3}} = (|A|^2 + |D|^2 + |E|^2 + 2Re(\overline{B}C))\phi^{12\overline{12}} = 0 \tag{2.61}$$

So $\phi^{3\overline{3}}$ is $\partial\overline{\partial}$-closed, but $\partial\phi^{3\overline{3}}$ is not $\overline{\partial}$-exact, as a consequence we have that $\frac{i}{2}(\phi^{1\overline{1}} + \phi^{2\overline{2}} + \phi^{3\overline{3}}) \in \mathcal{S} \setminus s\mathcal{S}$.

Moreover, using the symmetrization process described in [Bel00], it is possible to adapt the arguments in [FG04, Theorem 2.1] to the complex case. In particular it is easy to check that the second statement holds even if we change the operator d with the operator $d^c := J^{-1} \circ d \circ J$, where J is the complex structure of our manifold. As an immediate consequence, the statement holds if we substitute the operator d with the operators ∂ or $\overline{\partial}$. Thus we have a result anologous to [COUV16, Proposition 5.1], in fact we proved the following:

Proposition 10. *Let M be a compact complex 6-dimensional nilmanifold. If M admits a super SKT metric then it also admits an invariant super SKT metric.*

Using the proposition above we prove the following

Theorem 34. *Let M be a compact complex 6-dimensional nilmanifold. Then M admits super SKT metrics if and only if M is a torus.*

Proof. First of all we observe that if M does not admit a SKT metric then it does not admit a super SKT metric, and thus we can restrict the proof of the Theorem to the case described by Theorem 32.

By Proposition 10 we only need to study the invariant case. Finally, by [Uga07] we have that the fundamental form of a invariant Hermitian metric is given by

$$\omega = r\phi^{1\overline{1}} + s\phi^{2\overline{2}} + t\phi^{3\overline{3}} + u\phi^{1\overline{2}} - \overline{u}\phi^{2\overline{1}} + v\phi^{1\overline{3}} - \overline{v}\phi^{3\overline{1}} + w\phi^{2\overline{3}} - \overline{w}\phi^{3\overline{2}}, \quad (2.62)$$

where r, s and t are positive real numbers and u, v, and w are complex numbers which satisfy the following

$$\begin{cases} rs > |u|^2 \\ rt > |v|^2 \\ st > |w|^2 \\ rst + 2\Re(i\overline{u}\overline{v}z) > t|u|^2 + r|w|^2 + s|v|^2 \end{cases}$$

By direct computation follows

$$\partial\omega = t\partial\phi^{3\overline{3}} + (v\overline{D} - \overline{v}E - w\overline{C})\phi^{12\overline{1}} + (w\overline{A} - v\overline{B} - \overline{w}E)\phi^{12\overline{2}}. \quad (2.63)$$

While, for every fixed A, B, C, D and E, it is always possible to find two complex numbers v and w such that the sum of the second and third terms of (2.63) is $\overline{\partial}$-exact, $\partial\phi^{3\overline{3}}$ is $\overline{\partial}$-exact if and only if $A = B = C = D = E = 0$. That is equivalent to M being a torus.

$\square$

Chapter 3

Elliptic operators on Manifolds

One of the subjects of this PhD thesis is the study of the stability of the degeneration at the second step of the Frölicher spectral sequence (see §2.5). The theory of deformations was introduced in Kodaira and Spencer in [KS60]. Moreover, in [Kod06], Kodaira presented a study of $\mathcal{C}^\infty$ families of differential operators on compact manifolds. In the appendices of the aforementioned book, there are proved the properties of differential operators on manifolds that are needed to develope such theory. We recall those definitions and theorems in the first section of this chapter.

In [Pop16], Popovici proved that there is an isomorphism between the second step of the Frölicher spectral sequence and the kernel of a suitable pseudo-differential operator $\tilde{\Delta}$. For this reason we decided to study the theory of pseudo-differential operators in order to provide the preliminary results needed to apply the theory of deformations to $\mathcal{C}^\infty$ families of pseudo-differential operators. We begin the second section giving the definition of pseudo-differential operator and we recall some results, e.g., the Garding inequality and the finiteness of the dimension of the kernel of a pseudo-differential operator. Moreover, we prove an *a priori estimate* for this wider class of operators. This theorem will be crucial in the next chapter in the proof of the upper-semicontinuity of the dimension of the kernel of a $\mathcal{C}^\infty$ family of pseudo-differential operators.

In the last section, after having recalled the Hodge theory, we provide the construction of $\tilde{\Delta}$.

3.1 Sobolev's Norm

We assume the following notations. Let M be a compact orientable differentiable manifold of real dimension n (in this chapter we do not need the complex structure so we assume only the existence of a volume form). Let $\{U_j, \phi_j\}_{j=1,\dots m}$ be a finite covering of M made of coordinate open sets and let $V_j := \phi_j(U_j) \subset \mathbb{R}^n$.

We observe that, since M is compact, V_j is a limited open set of $\mathbb{R}^n$. Let $\{\eta_j\}$ be a partition of unity subordinate to $\{U_j\}$. We fix a Riemannian metric g on M and we denote with ω^n its volume form.

We recall the construction of $\mathcal{C}^\infty$ functions and $\mathcal{C}^\infty$ forms on M. A complex-valued $\mathcal{C}^\infty$ function $f : M \to \mathbb{C}$ is defined as

$$f(\bullet) = \sum_{j=1}^{n} \eta_j(\bullet)(f_j \circ \phi_j)(\bullet), \tag{3.1}$$

where $f_j \in \mathcal{C}^\infty(V_j; \mathbb{C})$. Since the behavior of f_j at the boundary of V_j has no influence in this construction, we assume that f_j is defined and differentiable on $\overline{V_j}$ which is compact. An l-form α on M can be locally defined as

$$\alpha_{|U_j} = \sum_{|I|=l} f_j^I dx_j^I, \tag{3.2}$$

where $I = (i_1, ..., i_l)$ is a multi-index, f_j^I is a $\mathcal{C}^\infty$ function and $dx_j^I = dx_j^{i_1} \wedge \cdots \wedge dx_j^{i_l}$.

In Chapter 1 we gave the definition of the L^2-product associated to an Hermitian metric. Such a product does not involve the derivatives of the forms. In this chapter we want to study the behavior of differential operators, thus we need the following norm.

Definition 12 (Sobolev's norm)**.** *Let $f \in \mathcal{C}^\infty(M, \mathbb{C})$ and let k be a non negative integer. Then the k-th* Sobolev's norm *of f is*

$$\|f\|_k^2 := \sum_{j=1}^{n} \sum_{|\alpha| \le k} \int_{U_j} \left|\eta_j(p) D_j^\alpha (f_j \circ \phi_j)(p)\right|^2 \omega^n_{|U_j}, \tag{3.3}$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ and $D_j^\alpha := \frac{\partial^{\alpha_1}}{\partial x_{j1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_{jn}^{\alpha_n}}$ is a derivation in the local coordinates $\{x_{ji}\}$.

Let $\phi \in \Lambda^l(M)$, using the local expression (3.2), we put

$$\left\|\phi_{|U_j}\right\|_k^2 = \sum_{|I|=l} \left\|f_j^I\right\|_k^2 \tag{3.4}$$

and

$$\|\phi\|_k^2 = \sum_j \left\|\phi_{|U_j}\right\|_k^2. \tag{3.5}$$

We recall the following.

Proposition 11. *Let $f \in \mathcal{C}^\infty(M, \mathbb{C})$ and let $s'' < s' < s$ then the following interpolation inequalities hold for any $t > 0$.*

$$\|f\|_{s'} \le \|f\|_s^{(s'-s'')/(s-s'')} \|f\|_{s''}^{(s-s')(s-s'')}, \tag{3.6}$$

$$\|f\|_{s'}^2 \le \frac{s'-s''}{s-s''} t^{(s-s')/(s'-s'')} \|f\|_s^2 + \frac{s-s'}{s-s''} t^{-(s-s'')/(s-s')} \|f\|_{s''}^2. \tag{3.7}$$

3.2 Differential operators

In this section we recall some notions about differential operators on manifolds (for a complete disseretation see [Kod06]). We consider only operators acting on $\mathcal{C}^\infty$ functions, because the construction and properties of operators acting on forms follow directly from (3.2). Since M is compact, we can assume always that the functions we are using are compactly supported.

Definition 13 (Differential operator)**.** *A* linear partial differential operator *of order* m $A(x,D) : \mathcal{C}^\infty(M,\mathbb{C}) \to \mathcal{C}^\infty(M,\mathbb{C})$ *is defined as following: for every* $f \in \mathcal{C}^\infty(M,\mathbb{C})$ *expressed as in (3.1),*

$$A(x,D)f(x) := \sum_j \sum_{|\alpha|\le m} a_{j\alpha}(x)\eta_j(x)D_j^\alpha(f_j \circ \phi_j)(x), \tag{3.8}$$

where $a_{j\alpha} \in \mathcal{C}^\infty(U_j,\mathbb{C})$ *and* α *and* D_j^α *are as in Definition 12.*

Shortly we write

$$A(x,D)f(x) = \sum_{|\alpha|\le l} a_\alpha(x)D^\alpha f(x). \tag{3.9}$$

From Definition 13, $A(x,D)$ is a polynomial function in the variable D with $\mathcal{C}^\infty$ coefficient. We call *principal part of* $A(x,D)$ the homogeneous polynomial of maximum degree. Namely, if $A(x,D)$ is of order m,

$$A_m(x,D) = \sum_{|\alpha|=m} a_\alpha(x)D^\alpha \tag{3.10}$$

is its principal part. The principal part of an operator carries important informations, for example the following definition relies only on $A_m(x,D)$.

Definition 14. $A(x,D)$ *is said to be elliptic if for any* $x \in M$ *there exists a positive constant* δ *such that for any* $\zeta_x \in \mathbb{R}^n$ *with* $\zeta_x \neq 0$ *and for any* $z \in \mathbb{C}$ *the inequality*

$$[(A_m(x,\zeta_x)z)_j]^2 \ge \delta^2|z|^2. \tag{3.11}$$

The supremum of such δ *is called the constant of ellipticity of* $A(p,D)$ *and we denote it with* δ_0.

Since elliptic operators are related to cohomology groups (see [Voi03]), they have been largely studied in literature (see for example [BT13] or [WGP80]). We recall some of the most important properties of elliptic operator (for complete proofs see [Kod06]). First of all we define the following constant: for every $k \in \mathbb{N}$ let

$$M_k = \sum_j \sum_\alpha \sum_{|\beta|\le k} sup_{x_j\in U_j} \left| D_j^\beta a_{j\alpha}(x_j)\right|. \tag{3.12}$$

Using M_k one can prove the following

Theorem 35 (L^2 a priori Estimate)**.** *Let $A(x, D)$ be an elliptic partial differential operator of order m. For any $k \in \mathbb{N}$, there exists a positive constant C depending only on n, m, k, δ and M_k such that for every $f \in \mathcal{C}^\infty(M, \mathbb{C})$, the inequality*

$$\|f\|_{k+m} \leq C(\|A(p, D)f\|_k + \|f\|_k) \tag{3.13}$$

holds.

Theorem 35 will be a fundamental tool in the next chapter since it provide an upper estimate of the Sobolev's norm of f. Other useful estimates are the followings.

Lemma 2. *Let $A(x, D)$ be a linear differential operator of order m. Then, for every $k \in \mathbb{N}$, there exists a constant C such that for any $f \in \mathcal{C}^\infty(M, \mathbb{C})$ the following inequality holds*

$$\|A(x, D)f\|_k \leq C \|f\|_{k+m}.$$

Lemma 3. *Let $A(x, D)$ be a linear partial differential operator of order $m + l$. Then there exists a positive constant C determined by m and l such that for any f, g*

$$|< A(p, D)f, g >| \leq CM_l \|f\|_m \|g\|_l \tag{3.14}$$

holds, where $< A(p, D)f, g >$ denotes the standard L^2 product associated to the fixed metric.

Let $A(x, D)$ be a linear partial differential operator of order $2m$. For any $x \in M$ and any $\zeta_x \in T^*_x M$ with $\zeta_x \neq 0$, the principal part $A_{2m}(x, \zeta_x)$ of $A(x, D)$ is a linear map $\mathbb{C} \to \mathbb{C}$

Definition 15. *A linear partial differential operator $A(x, D)$ of order $2m$ is said to be strongly elliptic if there exists a positive constant δ such that for any $p \in M$, any $\zeta \in T^*_x M$ with $\zeta \neq 0$ and any $z \in \mathbb{C} \setminus \{0\}$*

$$(-1)^m \Re\left(A_{2m}(x, \zeta_x)z\right) \overline{z} \geq \delta^2 |z||\zeta_x|^{2m} \tag{3.15}$$

holds. The supremum of such δ is called constant of strong ellipticity of $A(x, D)$ and it is denoted with δ_0.

Since all the operators we use are of this type, this stronger property provides no burden in our study. Moreover, we have the following stronger estimate ([Eva10]).

Theorem 36 (Garding's inequality)**.** *Let $A(m, D)$ be a strongly elliptic partial differential operator of order $2m$. Then there are positive constants δ_1, δ_2 depending on m, n, δ and M_m such that for any $f \in \mathcal{C}^\infty(M, \mathbb{C})$*

$$\Re(A(x, D)f, f) + \delta_1(f, f) \geq \delta_2 \|f\|^2_m \tag{3.16}$$

holds.

Theorem 35 and 36 are the estimates we need in our study. Given an operator $A(x, D)$ we want to construct the adjoint and the Green operators associated to it. Fixed an Hermitian metric g the following Lemma guarantees the existence of the adjoint operator.

Lemma 4. *For a linear partial differential operator $A(x, D)$ of order m, there exists a unique linear partial differential operator $A(x, D)^*$ such that for any $f, g \in \mathcal{C}^\infty(M, \mathbb{C})$, the following inequality holds*

$$< A(x, D)f, g >=< f, A(x, D)^* g > . \tag{3.17}$$

Moreover $A(x, d)^*$ is of order m. $A(x, D)^*$ is called the formal adjoint *of $A(x, D)$.*

We also know how to construct $A(x, D)^*$ from $A(x, D)$.

Lemma 5. *Let $x \in M$ and $\zeta_x \in T_x^*M$ with $\zeta_x \neq 0$. If we write the principal symbol of the differential operator $A(x, D)^*$ as $A_l(x, \zeta_x)^*$, then it is the adjoint of the linear map determined by the principal symbol of $A(x, D)$ with respect to the metric g_x.*

There is also a more abstract construction of the adjoint starting from the following operator

Definition 16. *We define a linear operator A in the Hilbert space $L^2(M, \mathbb{C})$ as follows: The domain $D(A)$ of A is given by*

$$D(A) = \{u \in L^2(M, \mathbb{C}) | A(p, D)u \in L^2(M, \mathbb{C})\} \tag{3.18}$$

and for any $u \in D(A)$, we put $Au := A(x, D)u$.

The operator A has the following properties.

Theorem 37. *$D(A) = W^l(M, \mathbb{C})$, and the topology of $D(A)$ defined by the graph norm coincides with the standard topology of $W^l(M\mathbb{C})$.*

Theorem 38. *A is a closed operator.*

Theorem 39. *$\mathcal{C}^\infty(M, \mathbb{C})$ is dense in $D(A)$ with respect to the graph norm, that is, for $u \in D(A)$ there is a sequence $\{\phi_k\} \subset \mathcal{C}^\infty(M, \mathbb{C})$ such that ϕ_k converges to u in $L^2(M, \mathbb{C})$ and $A(x, D)\phi_k$ converges to Au.*

A can be used to construct the adjoint of $A(x, D)$ in the following way

Theorem 40. *Let A^* be the adjoint of A in the sense of the operator on Hilbert space. Then the domain $D(A^*)$ is given by $D(A^*) = W^l(M, \mathbb{C})$ and, for $v \in D(A^*)$, we have*

$$A^* v = A(x, D)^* v \tag{3.19}$$

In particular, if $A(x, D)$ is formally self adjoint, A is self adjoint.

We recall a couple of other properties of A and A^*.

Theorem 41. *Let $A(x, D)$ be an elliptic partial differential operator of order m. We define the operators A and its adjoint A^* as in Definition (16). Then we have the following*

- *Both* $\operatorname{Ker} A$ *and* $\operatorname{Ker} A^*$ *are finite dimensional subspaces of* $\mathcal{C}^\infty(M, \mathbb{C})$*;*
- *the range $R(A)$ of A and the range $R(A^*)$ of A^* are closed subspaces of $L^2(M, \mathbb{C})$;*
- $R(A) = (\operatorname{Ker} A^*)^\perp$ *and* $R(A^*) = (\operatorname{Ker} A)^\perp$.

Lemma 6. *Let $A(x, D)$ be an elliptic linear partial differential operator of order m. We define the operator A as in Definition (16). Then*

- $\operatorname{Ker} A$ *is a closed subspace of* $L^2(M, \mathbb{C})$.
- *For any integer $k \geq 0$, there exists a positive constant C_k such that for any $u \in W^{k+l}(M, \mathbb{C}) \cap (\operatorname{Ker} A)^\perp$, the following estimate holds:*

$$\|u\|_{l+k} \leq C_k \|Au\|_k . \tag{3.20}$$

One of the most important result in the theory of differential operators over manifolds is the Hodge decomposition theorem ([Sch95]) that we will enunciate in section. This theory involves both the kernel and the range of suitable elliptic differential operators. We have already discussed about the kernel of an operator, now we want to study its range. In particular we want to solve

$$A(x, D)u(x) = f(x) \tag{3.21}$$

Formally we want find

$$u(x) = A^{-1}(x, D)f(x). \tag{3.22}$$

The role of $A^{-1}(x, D)$ is fulfilled by the Green operator $G(x, D)$ of $A(x, D)$.

Definition 17 (Green operator)**.** *Let $D(A) \cap (\operatorname{Ker} A)^\perp = \mathcal{H}$ and let Q be the orthogonal projection onto* $\operatorname{Ker} A^*$*. Since A is a bijection of $\mathcal{H}$ into $R(A)$, let $\tilde{G}$ be its inverse. $\tilde{G}$ is a bijection of $R(A)$ into $\mathcal{H}$. We define*

$$G = \tilde{G}(1 + Q), \tag{3.23}$$

and call G the Green operator *of A. G is a linear map of $L^2(M, \mathbb{C})$ to $\mathcal{H}$ which coincides with $\tilde{G}$ on $R(A)$ and vanishes on* $\operatorname{Ker} A^*$.

Theorem 42. *The Green operator has the following properties*

- *G is defined on $L^2(M, \mathbb{C})$ and its range $R(G)$ is given by*

$$R(G) = W^l(M, \mathbb{C}) \cap (\operatorname{Ker} A)^\perp. \tag{3.24}$$

For any $u \in L^2(M, \mathbb{C})$, $AGu = (I - Q)u$ and, for any $v \in W^l(M, \mathbb{C})$, $GAv = (I - P)v$;

- *for any integer $k \geq 0$ there exists a positive constant C_k such that, for any $u \in W^k(M, \mathbb{C})$, $Gu \in W^{k+l}(M, \mathbb{C})$ and*

$$\|Gu\|_{k+l} \leq C_k \|u\|_k \tag{3.25}$$

  holds;

- *let F be the projection map from $L^2(M)$ into the Kernel of A then for ever $f \in L^2(M)$*

$$AG(f) = GA(f) = f - Ff \tag{3.26}$$

  holds.

Equation (3.21) is a classical problem. In order to solve it, the first step is to find a weak solution. Namely,

Definition 18 (Weak solution)**.** *Let $f \in W^k(M, \mathbb{C})$. $u \in W^k(M, \mathbb{C})$ is said to be a* weak solution *of (3.21) if for any $\phi \in \mathcal{C}^\infty(M, \mathbb{C})$,*

$$(u, A(x, D)^*\phi) = (f, \phi) \tag{3.27}$$

holds

Given a weak solution we want to regularize it. This is a standard process in analysis involving the Lax-Milgram theorem. We recall the main passages that prove that, if there exists a weak solution, then there exists also a regular one.

Lemma 7. *Let $A(x, D)$ be a partial differential operator of order l. Then, by Lemma 2, it is extended to a continuous map of $W^k(M, \mathbb{C})$ to $W^{k-l}(M, \mathbb{C})$. if u is a weak solution, in this sense, we have*

$$A(x, D)u = f \tag{3.28}$$

in $W^{k-l}(M, \mathbb{C})$.

Theorem 43 (Lax-Milgram [Sho13])**.** *Let $\mathcal{H}$ be a complex Hilbert Space, let $(\bullet, \bullet)$ be its inner product and let $|\bullet|$ be its norm. Suppose that $B(x, y)$ is a Hermitian form on $\mathcal{H}$ satisfying the following condition: there exist positive constants $C_1 \leq C_2$ such that for any $x, y \in \mathcal{H}$*

$$|B(x, y)| \leq C_2 |x||y|, \tag{3.29}$$

$$\Re B(x, x) \geq C_1 |x|^2. \tag{3.30}$$

Then, for any continuous conjugate line $f(x)$ on $\mathcal{H}$, there exists a unique element F_B of $\mathcal{H}$ with

$$B(F_B, z) = f(z) \tag{3.31}$$

Lemma 8. *Let δ_1 and δ_2 be the positive constants given in Theorem 36. Then $B(\phi, \psi)$ defined on $\mathcal{C}^\infty(M, \mathbb{C})$ extends uniquely by continuity to a continuous Hermitian form on $W^m(M, \mathbb{C})$ which we denote by the same notation $B(\phi, \psi)$.*

Moreover there exists a positive constant C_1 determined by n, m and M_m such that if $\Re\lambda > \delta_1$, for any $\phi, \psi \in W^m(M, \mathbb{C})$ the following inequalities hold

$$|B(\phi, \psi)|_{\leq} \delta_2 \left\|\phi\right\|_m \left\|\psi\right\|_m , \tag{3.32}$$

$$\Re B(\phi, \phi) \geq C_1 \left\|\phi\right\|_m^2 . \tag{3.33}$$

Theorem 44. *Let $A(p, D)$ be an elliptic linear partial differential operator of order m with $\mathcal{C}^\infty$ coefficients. Suppose that for some integers s, k, $f \in W^{s-m+k}(M, \mathbb{C})$, then for a weak solution $u \in W^s(M, \mathbb{C})$ of the equation*

$$A(x, D)u = f \tag{3.34}$$

there exists a positive constant C such that

$$\left\|u\right\|_{s+k} \leq C(\left\|f\right\|_{s+k-m} + \left\|u\right\|_s) \tag{3.35}$$

holds, where C is independent of f and u.

Lemma 9. *Let $A(x, D)$ be an elliptic linear partial differential operator of order m. Suppose that for $f \in \mathcal{C}^\infty(M, \mathbb{C})$, $u \in W^s(M, \mathbb{C})$ is a weak solution of*

$$A(x, D)u = f. \tag{3.36}$$

Then there exists $v \in \mathcal{C}^\infty(M, \mathbb{C})$ such that in $L^2(M, \mathbb{C})$

$$u - v = 0 \tag{3.37}$$

holds.

Moreover, for elliptic operators, we have a result of existence of the solution.

Theorem 45. *Let $A(p, D)$ be a strongly elliptic linear partial differential operator of order $2m$, and let δ_1, δ_2 be the positive constants given in Theorem 36. If $\Re\lambda > \delta_1$, for any $w \in L^2(M, \mathbb{C})$, there exists a weak solution of the equation*

$$A(x, D)u + \lambda u = f \tag{3.38}$$

contained in $W^m(M, \mathbb{C})$. Moreover the weak solution of this equation is unique.

Finally we study the spectrum of an ellptic operator. The spectrum of an operator $A(x, D)$ is the set $\{\lambda \in \mathbb{C} | \exists f \in \mathcal{C}^\infty(M, \mathbb{C}) \text{ s.t. } A(x, D)f = \lambda f\}$.

This is important in the thoery of deformations (see [KS60]) since it is the basic tool to prove the upper-semicontinuity of the dimension of the kernel of a $\mathcal{C}^\infty$ family of elliptic differential operators. The main theorems follow from the construction of the Green operator associated to a suitable perturbation of $A(x, D)$. In particular we have

Theorem 46. *Let $A(x, D)$ be a strongly elliptic partial differential operator and let δ_1 as in Theorem 36. If $\Re\lambda > \delta_1$, then $A + \lambda$ is a linear isomorphism of $W^{2m}(M, \mathbb{C})$ onto $L^2(M, \mathbb{C})$.*

Fix $\mu \geq \delta_1$ and put

$$G_\mu := (A + \mu)^{-1} \tag{3.39}$$

From Rellich's theorem [Rel30], we have

Theorem 47. *G_μ is a compact linear map of $L^2(M, \mathbb{C})$ to itself.*

Theorem 48. *A complex number λ is contained in the spectrum of A if and only if $\zeta^{-1} = (\lambda - \mu)^{-1}$ is contained in the spectrum of G_μ.*

Since G_μ is a compact operator, we have the following results.

Theorem 49. *Let $A(x, D)$ be a strongly elliptic linear partial differential operator. Then the spectrum of A is contained in the half space $\Re\lambda > -\delta_1$ and it consists only of the point spectrum which has no finite accumulation point. Furthermore the generalized eigenspace belonging to each eigenvalue is finite dimensional.*

Theorem 50. *Let $A(x, D)$ be a strongly elliptic self adjoint operator. Then we can choose eigenfunctions $\{e_j\}$ of A, with $Ae_j = \lambda_j e_j$, such that*

- *$\{e_j\}$ form a complete orthonormal system of $L^2(M)$;*
- *$\lambda_j \in \mathbb{R}$, $\lambda_j \leq \lambda_{j+1}$ and $\lambda_j \xrightarrow{j \to \infty} +\infty$.*

3.3 Pseudo differential operators

Pseudo-differential operators are the natural generalization of classical differential operators. They have non-polynomial behavior with respect to the derivation. In [Pop16], Popovici showed that, for a suitable pseudo-differential operator $\tilde{\Delta}$, it is possible to develop an Hodge theory and he proved that the kernel of $\tilde{\Delta}$ is isomorphic to the second step of the Frölicher spectral sequence. Moreover, pseudo-differential operators could be used to study Varouchas spaces (see [Var]).

3.3.1 Abstract Theory

We begin with the theory of Fredholm operators since they are the abstract representation of pseudo-differential operators. Fredholm operators were studied by Hormander in order to prove the Atiyah-Singer theorem [AS63]. Here we report only the few properties useful for our work. For more detailed information see [Hör85].

Definition 19. *Let B_1 and B_2 be Banach spaces. A linear operator $T \in \mathcal{L}(B_1; B_2)$ is called a Fredholm operator is $\dim \operatorname{Ker} T$ is finite and $T(B_1) \subset B_2$ is closed and has finite co-dimension. In that case we define*

$$\operatorname{ind}(T) = \dim \operatorname{Ker} T - \dim \operatorname{Coker} T. \tag{3.40}$$

In the definition the condition that the image of T must be closed is unnecessary, in fact we have the following.

Proposition 12. *Il $T \in \mathcal{L}(B_1; B_2)$ and the range $T(B_1)$ has finite co-dimension in B_2, then TB_1 is closed.*

We can also give a characterization of the Fredhom condition in terms of the topology of B_1 and B_2.

Proposition 13. *If $T \in \mathcal{L}(B_1; B_2)$, then the following conditions are equivalent:*

- $\dim \operatorname{Ker} T < \infty$ *and $T(B_1)$ is closed;*
- *Every sequence $\{f_j\} \subset B_1$ such that $\{Tf_j\}$ is convergent and f_j is bounded has a convergent subsequence.*

The following theorem is a result of stability for Fredhom operators.

Theorem 51. *If T satisfies the conditions in Proposition 13 and $S \in \mathcal{L}(B_1; B_2)$ has sufficiently small norm, then* $\dim \operatorname{Ker}(T+S) \leq \dim \operatorname{Ker} T$, *$T+S$ has closed range and* $\operatorname{ind}(T+S) = \operatorname{ind}(T)$.

We conclude this survey about Fredhom operators with the following consequences of Theorem 51. In particular Corollary 2 will be useful in the last part of this chapter since it guaranties that $\tilde{\Delta}$ is a pseudo-differential operator.

Corollary 2. *If $T \in \mathcal{L}(B_1; B_2)$ is a Fredholm operator and $K \in \mathcal{L}(B_1; B_2)$ is compact, then $T + K$ is a Fredholm operator and* $\operatorname{ind}(T+K) = \operatorname{ind}(T)$.

Corollary 3. *If $T \in \mathcal{L}(B_1; B_2)$ and $S_1, S_2 \in \mathcal{L}(B_2; B_1)$ are such that $TS_2 = Id + K_2$ and $S_1 T = Id + K_1$, where K_j are compact operators, then T, S_1 and S_2 are Fredholm operators and* $\operatorname{ind}(T) = -\operatorname{ind}(S_j)$.

3.3.2 Concrete construction

Now we proceed with a concrete construction of pseudo-differential operators using symbols. Symbols are $\mathcal{C}^\infty$ function that replace the role of the polynomial function in (3.9). This construction starts with the Fourier transformation and its properties related to the differentiation (see [Rud87]). As always we use the theory in $\mathbb{R}^n$ and then we translate it to the manifold M using local charts. We recall that the Fourier transform of a function f is defined as

Definition 20 (Fourier transform)**.**

$$\hat{f}(\xi) := \frac{1}{(2\pi)^n} \int f(x) e^{-ix\dot{\xi}} dx. \tag{3.41}$$

Definition 21 (Fourier inverse formula)**.**

$$f(x) = \int \hat{f}(\xi) e^{ix\dot{\xi}} d\xi \tag{3.42}$$

If $p(\circ, \bullet)$ is a $\mathcal{C}^\infty$ function, polynomial with respect to the second variable, then we can describe a classical differential operator of order m as

$$p(x, D) := \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \tag{3.43}$$

where α is a multi-index, $D^\alpha := D_1^{\alpha_1} \dots D_n^{\alpha_n}$ and $D_j := \frac{\partial}{\partial x_j}$. Using (3.41) we have

$$p(x, D) f(x) = \int p(x, \xi) \hat{f}(\xi) e^{ix\dot{\xi}} d\xi. \tag{3.44}$$

We want to generalize and study what happens when $p(\circ, \bullet)$ is not a polynomial function.

Definition 22. *Let $m \in \mathbb{R}$ then the* space of symbols of order m $S^m = S^m(\mathbb{R}^n \times \mathbb{R}^n)$ *is the set of all $a \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that, for every multi-indexes α and β the following inequality*

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} (1 + |\xi|^2)^{\frac{m - \delta|\beta|}{2}} \tag{3.45}$$

holds.

S^m is a Frechet space with semi-norms given by the smallest constant which can be used in (3.45).

As for standard differential operators, we want to give the definition of principal part. First we need the following.

Proposition 14. *Let $a_j \in S^{m_j}$, $j = 0, 1, \dots,$ and assume that $m_j \to -\infty$ as $j \to \infty$. Let $m'_k := max_{j \geq k} m_j$. Then one can find $a \in S^{m_0}$ such that* $\operatorname{supp} a \subset \cup \operatorname{supp} a_j$ *and for every k*

$$a - \sum_{j<k} a_j \in S^{m'_k}. \tag{3.46}$$

The function a is uniquely determined modulo $S^{-\infty} := \cap S^m$ and has the same property relative to any rearrangement of the series $\sum a_j$; we write

$$a \sim \sum a_j.$$

Given a symbol $a \in S^m$ we want to define the pseudo-differential operator associated to it. The following theorem ensure us that the construction made in (3.44) for ordinary differential operators can be used also when the symbol is not of polynomial type. In fact we have

Theorem 52. *Let $a \in S^m$ and $u \in \mathcal{C}^\infty(R^n)$, then*

$$A(x, D) u(x) := (2\pi)^{-n} \int e^{i<x;\xi>} a(x, \xi) \hat{u}(\xi) d\xi \tag{3.47}$$

defines a function $A(x, D)u(x) \in \mathcal{C}^\infty(R^n)$. Moreover, the bilinear map $(a, u) \mapsto A(x, D)u(x)$ is continuous. One calls $A(x, D)$ a pseudo-differential operator *of order m.*

Definition 23 (Principal part)**.** *Let $A(x,D)$ be a pseudo-differential operator of order m and let a be its symbol. The* principal part *of $A(x,D)$ is the pseudo-differential operator $A_m(x,D)$ associated to the symbol a_m, where $a - a_m$ is a symbol of order less than m.*

Definition 24. *A* pseudo-differential operator *of order m on a $\mathcal{C}^\infty$ manifold M is a continuous linear map $A(x,D) : \mathcal{C}_0^\infty(M) \to \mathcal{C}^\infty(M)$ such that for every local chart U with coordinate map $\phi : U \to V \subset \mathbb{R}^n$ and all $f, g \in \mathcal{C}_0^\infty(V)$, the map*

$$u \mapsto f(\phi^{-1})^* A(x,D)\phi^*(gu)$$

is in OpS^m. We shall then write $A(x,D) \in \Psi^m(M)$ and extend $A(x,D)$ to a map from $\mathcal{E}'(M) \to \mathcal{D}'$.

Definition 25. *Let E and F be complex $\mathcal{C}^\infty$ vector bundles over the $\mathcal{C}^\infty$ manifold M. Then a* pseudo-differential operator *of order m from section of E to sections F is a continuous linear map*

$$A : \mathcal{C}_0^\infty(M,E) \to \mathcal{C}_0^\infty(M,F)$$

such that for every open $N \subset M$ where E and F are trivialized by

$$\phi_E : E\mid_N \to N \times \mathbb{C}^e, \quad \phi_F : F\mid_N \to N \times \mathbb{C}^f,$$

there is a $f \times e$ matrix of pseudo-differential operators $A_{ij} \in \Psi^m(N)$ such that, for every $u \in \mathcal{C}_0^\infty(N,E)$,

$$(\phi_F(Au)\mid_Y)_i = \sum_j A_{ij}(\phi_E u).$$

We shall write $A \in \Psi^m(M;E,F)$.

We have proved that there is a (1,1) correspondence between symbols and operators. Proceeding in same order of the previous section, we want to define the adjoint operator of $A(x,D)$. We recall the L^2 product

$$(u,v) = \int uv dx.$$

Then we define the adjoint of $A(x,D)$ in the usual way:

Definition 26 (Formal adjoint)**.** *Let $A(x,D)$ and $A^*(x,D)$ be two pseudo-differential operators of order m. Then $A^*(x,D)$ is called* formal adjoint *of $A(x,D)$ if*

$$(A(x,D)f_1, f_2) = (f_1, A^*(x,D)f_2)\,. \tag{3.48}$$

From the definition we can recover the construction of the adjoint operator. Using the Fubini's theorem we have

$$\begin{aligned}(A(x,D)f_1, f_2) = &\int\int e^{i<x,\xi>} a(x,\xi)\hat{f}_1(\xi)\overline{f_2(x)} d\xi dx \qquad (3.49)\\ = &\int \hat{f}_1(\xi)\overline{\int e^{-i<x,\xi>}\overline{a(x,\xi)} f_2(x) dx} d\xi.\end{aligned}$$

Then, using the Fourier inversion formula, we obtain

$$(A^*(x,D)f)(x) = \int\int e^{i<x-y,\xi>}\overline{a(y,\xi)}f(y)dyd\xi, \tag{3.50}$$

which is a pseudo-differential operator of order m.

After the adjoint we construct the Green operator. The following Theorem provide sufficient and necessary conditions to its existence.

Theorem 53. *Let $a \in S^m$ and $b \in S^{-m}$. Then the conditions*

- $A(x,D)B(x,D) - Id \in OpS^{-\infty}$*;*
- $B(x,D)A(x,D) - Id \in OpS^{-\infty}$

are equivalent and a determines b mod S^{-m}. They imply

$$a(x,\xi)b(x,\xi) - 1 \in S^{-1} \tag{3.51}$$

which implies in turn that for some positive constants c and C

$$|a(x,\xi)| > c|\xi|^m, \text{ if } |\xi| > C. \tag{3.52}$$

Conversely, if the last inequality is fulfilled then one can find $b \in S^{-m}$ satisfying the other three conditions.

Now that we have recalled the basic notions, we focus on operators of elliptic type.

Definition 27. *A symbol $a \in S^m$ is called* elliptic *if there exist two positive constant C and R such that, for every $|\xi| \geq R$ and every $x \in \mathbb{R}^n$, the inequality*

$$|a(x,\xi)| \geq C|\xi|^m \tag{3.53}$$

holds. An operator associated to an elliptic symbol is called elliptic*.*

The following is a characterization of elliptic operators

Theorem 54. *Let $a \in S^m$. Then the following conditions are equivalent:*

- *a is elliptic;*
- *there exist $b \in S^{-m}$ and $r \in S^{-1}$ such that $A(x,D)B(x,D) = Id+R(x,D)$;*
- *there exist $b \in S^{-m}$ and $r \in S^{-1}$ such that $B(x,D)A(x,D) = Id+R(x,D)$,*

where the capital letters denote the associated pseudo-differential operator.

The following theorem, that generalizes the Garding inequality, holds for pseudo-differential operators of elliptic type.

Theorem 55 (Garding inequality)**.** *If $a(x,D) \in S^m$ and $\Re a(x,\xi) \geq C|\xi|^m$ for $|\xi|$ large enough. Then, for any $s \in \mathbb{R}$, there exist constant C_0, C_1 such that*

$$\Re\left(a(x,D)f,f\right) + C_1 \left\|f\right\|_0^2 \geq \left\|f\right\|_m^2. \tag{3.54}$$

We recall that in the next chapter we will need two main results: the decomposition of the spectrum and the *a priori estimate*. We begin with the first result.

Theorem 56 ([**?**, Lemma 1.6.3]). *Let $A(x,D) : \mathcal{C}^\infty(M,\mathbb{C}) \to \mathcal{C}^\infty(M,V)$ be an elliptic self-adjoint pseudo-differential operator of order m¿ 0. Then*

- *We can find a complete orthonormal basis $\{\psi_n\}_{n=1}^\infty$ for $L^2(M)$ of eigenvectors of $A(x,D)$. $A(x,D)\psi_n = \lambda_n\psi_n$.*
- *The eigenvectors ψ_n are smooth and $\lim_{n\to\infty}|\lambda_n| = \infty$.*
- *If we order the eigenvalues $|\lambda_1| \leq |\lambda_2| \leq \ldots$ then there exists a constant $C > 0$ and an exponent $\delta > 0$ such that $|\lambda_n| \geq Cn\delta$ if $n > n_0$ is large.*

This theorem follows directly from this two Lemmas.

Lemma 10. *A complex number λ is contained in the spectrum of $A(x,D)$ if and only if $\zeta^{-1} = (\lambda-\mu)^{-1}$ is contained in the spectrum of $G_\mu := (A(x,D)+\mu)^{-1}$.*

Lemma 11. *G_μ is a compact linear map of $L^2(M,\mathbb{C})$ into itself.*

The second lemma is a consequence of the Rellich's theorem.

In order to prove the *a priori estimate* we need some preliminay results.

Lemma 12. *If $a(x,\xi) \in S^m$ then the commutators with D_j and multiplication by x_j are*

$$[A(x,D), D_j] = i\frac{\partial}{\partial x_j}A(x,D) \tag{3.55}$$

and

$$[A(x,D), x_j] = -i\frac{\partial}{\partial D_j}A(x,D). \tag{3.56}$$

The proof of this lemma can be found in [Hör85]. We have generalized the previous lemma and we have proved the following.

Lemma 13. *If $a(x,\xi) \in S^m$ and $f \in \mathcal{C}^\infty(U_i,\mathbb{C})$, where U_i is an open coordinate of M, then the commutator $[A(x,D), f]$ defines a pseudo-differential operator of order $m-1$.*

Proof. Let $\phi_i : U_i \to V_i \subset \mathbb{R}^n$ be the coordinate map related to U_i, then $f\circ\phi_i^{-1} : V_i \to \mathbb{C}$ is a $\mathcal{C}^\infty$ function. Assuming that $0 \in V_i$, we use the Taylor formula with Lagrange's remainder of $f\circ\phi_i^{-1}$ and we obtain

$$f\circ\phi_i^{-1}(y) = \sum_{|l|=0}^{L}\frac{(f\circ\phi_i^{-1})^{(l)}(0)}{|l|!}y^l + \sum_{|l|=L+1}\frac{(f\circ\phi_i^{-1})^{(l)}(\zeta)}{(L+1)!}y^l, \tag{3.57}$$

where $l = (l_1,\ldots,l_n)$ is a multi-index, $(f\circ\phi_i^{-1})^{(l)} = \frac{\partial^{|l|}(f\circ\phi_i^{-1})}{\partial y_1^{l_1}\ldots\partial y_n^{l_n}}$, $y = (y_1,\ldots,y_n) \in \mathbb{R}^n$, $y^l = y_1^{l_1}y_2^{l_2}\ldots y_n^{l_n}$ and $\zeta \in V_i$. Putting (3.57) in the commutator formula, by the linearity of $A(x, D)$ we have

$$[A(x,D), f] = [A(x,D), f \circ \phi_i^{-1}(0)] + \sum_{|l|=1}^{L} [A(x,D), \frac{(f \circ \phi_i^{-1})^{(l)}(0)}{|l|!} (\phi_i(x))^l] + \sum_{|l|=L+1} [A(x,D), \frac{(f \circ \phi_i^{-1})^{(l)}(\zeta)}{(L+1)!} (\phi_i(x))^l].$$

The first term of the right-hand side is zero, while the others are, up to a constant, in the form $[A(x, D), x^l]$. Then, using Lemma (12), we have that $[A(x, D), f]$ is a pseudo-differential operator of order $m - 1$. □

Another preliminary estimate we have proved is the following.

Lemma 14. *Let $A_m(D)$ be an elliptic pseudo-differential operator of order m, which consists of only its principal part and does not depend on the variable x. Let δ_0 be its constant of ellipticity. Then for any $k \in \mathbb{N}$ and any $f \in \mathcal{C}^\infty(M, \mathbb{C})$, the following inequality holds:*

$$\|A_m(D)f\|_k^2 + \delta_0^2 \|f\|_k^2 \geq 2^{-m} \delta_0^2 \|f\|_{k+m}^2 . \tag{3.58}$$

Proof. Let

$$\tilde{f} := \sum_{\xi} f_\xi e^{i<x,\xi>} \tag{3.59}$$

be the Fourier expansion of f. Then

$$\begin{aligned} \|A_m(D)f\|_k^2 &= \sum_{U_i} \sum_{|\alpha|=0}^{k} \sum_{\xi} \int_{U_i} \left| D^\alpha A_m(D) f_\xi e^{i<x,\xi>} \right|^2 dX \\ &= \sum_{U_i} \sum_{|\alpha|=0}^{k} \sum_{\xi} \int_{U_i} \left| f_\xi A_m(D) \xi^\alpha e^{i<x,\xi>} \right|^2 dX \\ &\geq \sum_{U_i} \sum_{|\alpha|=0}^{k} \sum_{\xi} \int_{U_i} \delta_0^2 (1+|\xi|^2)^m \left| f_\xi e^{i<x,\xi>} \right|^2 dX \\ &\geq \delta_0^2 (2^{-m} \|f\|_{k+m}^2 - \|f\|_k^2 . \end{aligned} \tag{3.60}$$

□

We are ready to prove the following.

Theorem 57 (a priori estimates)**.** *Let $A(x, D)$ be an elliptic pseudo-differential operator of order m. Then for any $k \in \mathbb{Z}$ there exists a constant C depending only on k, δ and $M_{|k|}$ such that for any $f \in \mathcal{C}^\infty(M; \mathbb{C})$*

$$\|f\|_{k+m} \leq C \left(\|A(x,D)f\|_k + \|f\|_k \right) . \tag{3.61}$$

Proof. Let $\{U_i\}$, $j = 1, \dots, I$, be a finite open covering of M and let $\eta_i \in \mathcal{C}^\infty(M, \mathbb{C})$ such that $\operatorname{supp} \eta_i \subset U_i$ and, for every $x \in M$, $\sum \eta_i^2(x) = 1$. For every $f \in \mathcal{C}^\infty(M, \mathbb{C})$ there exists an index i such that

$$I^{1/2} \|f\|_{k+m} \leq \|\eta_i f\|_{k+m} .$$

Fix $f \in \mathcal{C}^\infty(M, \mathbb{C})$ and let $f_i := \eta_i f$. Let $\omega_i \in \mathcal{C}^\infty(M, \mathbb{C})$ such that $\operatorname{supp} \omega_i \subset U_i$, $\sum \omega_i^2(x) = 1$ and $\omega_i = 1$ in some neighborhood of $\operatorname{supp} \eta_i$. Then we have

$$\begin{aligned} \|A(x,D)f\|_k &\geq \|\eta_i A(p,D) f\|_k \\ &= \|\eta_i A(p,D)\omega_i f\|_k \\ &\geq \|A(p,D) f_i\|_k - \|[A(p,D), \eta_i]\omega_i f\|_k . \end{aligned} \tag{3.62}$$

By Lemma (14), we have that there exists a constant $C_1 > 0$ such that

$$\|[A(p,D), \eta_i]\omega_i f\|_k \leq C_1 \|\omega_i f\|_{k+m-1} . \tag{3.63}$$

We want to estimate $\|A(p,D)f_i\|_k$. Let $\varepsilon > 0$. Cover V_i with open balls $B_1, ..., B_L$ of radius ε and take a partition of unity $\{\omega_l\}$ subordinate to $\{B_l\}$. We set $f_l := \omega_l f_i$, then there exists an integer l such that

$$L \|f_i\|_{k+m} \geq \|f_l\|_{k+m} .$$

Let p_l be the center of the ball B_l. Let $A_m(x,D)$ be the principal part of $A(x,D)$ and let $A_m(p_l,D)$ be the pseudo-differential operator obtained by computing $A_l(x,D)$ at the point p_l. By Lemma 14 we have that there exists a constant C_ε such that

$$\|(A_m(x,D) - A_m(p_l,D))f_l\|_k \leq 2\varepsilon M_1 \|f_l\|_{k+m} + C_\varepsilon M_{2k} \|f_l\|_{k+m-1} .$$

Moreover, by Lemma 14, we have that

$$\begin{aligned} \|A(x,D)f_l\|_k &\leq \|\omega_l A(x,D) f_i\|_k + \|[A(x,D), \omega_l] f_i\|_k \\ &\leq C_3 \|A(x,D) f_i\|_k + C_2 M_k \|f_i\|_{k+m-1} . \end{aligned}$$

Therefore

$$\begin{aligned} \|A_m(p_l,D)f_i\|_k \leq{}& \|A(x,D)f_i\|_k + \|(A_m(x,D) - A(x,D))f_i\|_k \\ &+ \|(A_m(p_l,D) - A_m(x,D))f_i\|_k \\ \leq{}& C_3 \|A(x,D)f_i\|_k + C_1 \|f_i\|_{k+m} + C_4 \|f_i\|_{k+m-1} . \end{aligned}$$

By Lemma 14, we have that

$$2^{-1/2}\delta_0 \|f_i\|_{k+m} - \delta_0 \|f_i\|_k \leq \|A_m(p_l,D)f_i\|_k . \tag{3.64}$$

Then, since for ε small enough we can take $C_1 < 2^{-1/2}\delta_0$, we obtain

$$C_5 \|f_i\|_{k+m} \leq C_3 \|A(x,D)f_i\|_k + C_1 \|f_i\|_{k+m} + C_4 \|f_i\|_{k+m-1} . \tag{3.65}$$

The thesis follows by using Proposition 11 to replace $\|f_i\|_{k+m-1}$ with $C_6 \|f_i\|_{k+m} + C_7 \|f_i\|_k$. $\square$

We conclude this section with the following auxiliary results.

Definition 28. *The pseudo-differential operator $A(x, D)$ in M is said to be properly supported if both projections from the support of the kernel in $M \times M$ to M are proper maps, that is, for every compact set $K \subset M$ there is a compact set $K' \subset M$ such that*

$$suppu \subset K \Rightarrow \operatorname{supp} A(x, D)u \subset K'; u = 0 \text{ at } K' \Rightarrow A(x, D)u = 0 \text{ at } K.$$

Theorem 58. *If $A_j \in \Psi^{m_j}(M)$ are properly supported for $j = 1, 2$, then $A = A_1 A_2 \in \Psi^{m_1+m_2}(M)$ is properly supported and the principal symbol is the product of those of A_1 and of A_2.*

Theorem 59. *If $A \in \Psi^m(M)$ is properly supported and elliptic in the sense that the principal symbol $a \in S^m(T^*(M))/S^{m-1}(T^*(M))$ has an inverse in $S^{-m}(T^*(M))/S^{-m-1}(T^*(M))$ then one can find $B \in \Psi^{-m}(M)$ properly supported such that*

$$BA - Id \in \Psi^{-\infty}(M), \quad AB - Id\Psi^{-\infty}(M).$$

One calls B a parametrix for A.

Theorem 60. *If $P \in \Psi^m(X; E \otimes \Omega^{\frac{1}{2}}, F \otimes \Omega^{\frac{1}{2}})$ is elliptic, then P defines a Fredholm operator operator from $H_{(s)}(X; E \otimes \Omega^{\frac{1}{2}})$ to $H_{(s-m)}(X; F \otimes \Omega^{\frac{1}{2}})$ with Kernel contained in $\mathcal{C}^\infty(X; E \otimes \Omega^{\frac{1}{2}})$ and with the Kernel of the adjoint $P^* \in \Psi^m(X; F^* \otimes \Omega^{\frac{1}{2}}, E^* \otimes \Omega^{\frac{1}{2}})$ contained in $\mathcal{C}^\infty(X; F^* \otimes \Omega^{\frac{1}{2}})$. The range is the orthogonal space of* $\operatorname{Ker} P^*$. *Thus those spaces are independent of s and the index of P is equal to the index of P as operator from $\mathcal{C}^\infty(X; E \otimes \Omega^{\frac{1}{2}})$ to $\mathcal{C}^\infty(X; F \otimes \Omega^{\frac{1}{2}})$ (or from $\mathcal{D}'(X; E \otimes \Omega^{\frac{1}{2}})$ to $\mathcal{D}'(X; E \otimes \Omega^{\frac{1}{2}})$) and it depends only on the class of P modulo Ψ^{m-1}. If $E = F^*$ and $P - P^* \in \Psi^{m-1}$ then* $\operatorname{ind}(P) = 0$.

3.4 Hodge Theory

One of the main results in the theory of differential operators over manifolds is the following

Theorem 61 (Hodge decomposition)**.** *Let M be a compact oriented Riemannian manifold and let $\Delta := dd^* + d^*d$ be the Laplace operator. Then, with respect to the given metric, we have the following orthogonal decomposition*

$$\Lambda^k(M) = (\operatorname{Ker}\Delta \oplus \operatorname{Im} d \oplus \operatorname{Im} d^*) \cap \Lambda^k(M).$$

Moreover $\operatorname{Ker}\Delta \cap \Lambda^k(M)$ *is a finite dimensional $\mathbb{R}$-vector space for all $k \in \mathbb{N}$.*

A similar result has been proved for the Beltrami-Laplace operator, namely

Theorem 62. *Let M be a compact complex Hermitian manifold and let $\Delta_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ be the Laplace-Beltrami operator. Then, with respect to the given metric, we have the following orthogonal decomposition*

$$\Lambda^{p,q}(M) = \left(\operatorname{Ker}\Delta_{\overline{\partial}} \oplus \operatorname{Im}\overline{\partial} \oplus \operatorname{Im}\overline{\partial}^*\right) \cap \Lambda^{p,q}(M).$$

Moreover $\operatorname{Ker}\Delta_{\overline{\partial}} \cap \Lambda^{p,q}(M)$ *is a finite dimensional $\mathbb{C}$-vector space for all $p, q \in \mathbb{N}$.*

Also operators as Δ_{BC} and Δ_A (see [Sch07]) have a decomposition theorem.

Theorem 63. *Let M be a compact complex Hermitian manifold and let $\Delta_{BC} := (\partial\overline{\partial})(\partial\overline{\partial})^* + (\partial\overline{\partial})^*(\partial\overline{\partial}) + (\overline{\partial}^*\partial)(\overline{\partial}^*\partial)^* + (\overline{\partial}^*\partial)^*(\overline{\partial}^*\partial) + \overline{\partial}^*\overline{\partial} + \partial^*\partial$. Then, with respect to the given metric, we have the following orthogonal decomposition*

$$\Lambda^{p,q}(M) = \left(\operatorname{Ker}\Delta_{BC} \oplus (\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}) \oplus \operatorname{Im}\partial^*\overline{\partial}^*\right) \cap \Lambda^{p,q}(M).$$

Moreover $\operatorname{Ker}\Delta_{BC} \cap \Lambda^{p,q}(M)$ *is a finite dimensional $\mathbb{C}$-vector space for all $p, q \in \mathbb{N}$.*

Theorem 64. *Let M be a compact complex Hermitian manifold and let $\Delta_A := (\partial\overline{\partial})(\partial\overline{\partial})^* + (\partial\overline{\partial})^*(\partial\overline{\partial}) + (\overline{\partial}\partial^*)(\overline{\partial}\partial^*)^* + (\overline{\partial}\partial^*)^*(\overline{\partial}\partial^*) + \overline{\partial}^*\overline{\partial} + \partial^*\partial$ be the Laplace-Beltrami operator. Then, with respect to the given metric, we have the following orthogonal decomposition*

$$\Lambda^{p,q}(M) = \left(\operatorname{Ker}\Delta_A \oplus \operatorname{Im}\partial\overline{\partial} \oplus (\operatorname{Im}\partial^* + \operatorname{Im}\overline{\partial}^*)\right) \cap \Lambda^{p,q}(M).$$

Moreover $\operatorname{Ker}\Delta_A \cap \Lambda^{p,q}(M)$ *is a finite dimensional $\mathbb{C}$-vector space for all $p, q \in \mathbb{N}$.*

For pseudo-differential operators a decomposition result has been proved by Popovici in a very special case. We recall the construction of this operator and its properties, in particular we want to focus on the connection with the Frölicher spectral sequence.

From Theorem 62, we know that $\operatorname{Ker}\Delta_{\overline{\partial}}$ is finite dimensional as $\mathbb{C}$-vector space. Let $p'' : \Lambda^{p,q}(M) \to \operatorname{Ker}\Delta_{\overline{\partial}} \cap \Lambda^{p,q}(M)$ be the orthogonal projection. We define

$$\tilde{\Delta} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} + \partial p''\partial^* + \partial^* p''\partial. \tag{3.66}$$

The following Lemmas proved that $\tilde{\Delta}$ is a pseudo-differential operator.

Lemma 15. *Let P and Q be the orthogonal projection to* $\operatorname{Ker} A$ *and* $\operatorname{Ker} A^*$ *in $L^2(M, \mathbb{C})$ respectively. Then for any integer $k \geq 0$ there exists a constant C_k such that for any $u \in L^2(M, \mathbb{C})$, the following estimates hold.*

$$\|Pu\|_k \leq C_k \|u\| \tag{3.67}$$

$$\|Qu\|_k \leq C_k \|u\| \tag{3.68}$$

Lemma 16. *Let $A(p, D)$ be a formally self adjoint elliptic partial differential operator, P the orthogonal projiection to $\operatorname{Ker} A$, G the Green operator of A. If $u \in \mathcal{C}^\infty(M, \mathbb{C})$, then $Gu \in \mathcal{C}^\infty(M, \mathbb{C})$ and the equality*

$$u = Pu + AGu \tag{3.69}$$

holds. By this, the orthogonal decomposition

$$\mathcal{C}^\infty(M, \mathbb{C}) = \ker A \oplus A\mathcal{C}^\infty(M, \mathbb{C}) \tag{3.70}$$

is given, where $A\mathcal{C}^\infty(M, \mathbb{C})$ denotes the image of $\mathcal{C}^\infty(M, \mathbb{C})$ by $A(p, D)$.

In [Pop16], Popovi introduced the following pseudo-differential operator:

$$\tilde{\Delta} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} + \partial p''\partial^* + \partial^* p''\partial,$$

where p'' is the natural projection of $\Lambda^{p,q}(M)$ onto $\operatorname{Ker} \Delta_{\overline{\partial}}$. Thus $\tilde{\Delta}$ is the sum of a pseudo-differential regularizing operator and an elliptic differential operator of order two (the classical $\overline{\partial}$-Laplacian $\Delta_{\overline{\partial}}$).

Theorem 65. *For all p, q, $\tilde{\Delta} : \Lambda^{p,q}(M) \to \Lambda^{p,q}(M)$ behaves like an elliptic self-adjoint differential operator in the sense that $\operatorname{Ker} \tilde{\Delta}$ is a finite dimensional $\mathbb{C}$-vector space, $\operatorname{Im} \tilde{\Delta}$ is closed and finite co-dimensional in $\Lambda^{p,q}(M)$, there is an orthogonal (for the L^2 inner product induced by g) 2-dimensional decomposition*

$$\Lambda^{p,q}(M) = \operatorname{Ker} \tilde{\Delta} \oplus \operatorname{Im} \tilde{\Delta}$$

giving rise to an orthogonal 3-space decomposition

$$\Lambda^{p,q}(M) = \operatorname{Ker} \tilde{\Delta} \oplus \left(\operatorname{Im} \overline{\partial} + \partial_{|_{\operatorname{Ker} \overline{\partial}}}\right) \oplus \Big(\operatorname{Im} \overline{\partial}^* + \operatorname{Im}(\partial^* \circ p'')\Big)$$

in which

$$\begin{aligned}
&\operatorname{Ker} \tilde{\Delta} \oplus \left(\operatorname{Im} \overline{\partial} + \partial_{|_{\operatorname{Ker} \overline{\partial}}}\right) = \operatorname{Ker}(p'' \circ \partial) \cap \operatorname{Ker} \overline{\partial};\\
&\operatorname{Ker} \tilde{\Delta} \oplus \Big(\operatorname{Im} \overline{\partial}^* + \operatorname{Im}(\partial^* \circ p'')\Big) = \operatorname{Ker}(p'' \circ \partial^*) \cap \operatorname{Ker} \overline{\partial}^*;\\
&\left(\operatorname{Im} \overline{\partial} + \partial_{|_{\operatorname{Ker} \overline{\partial}}}\right) \oplus \Big(\operatorname{Im} \overline{\partial}^* + \operatorname{Im}(\partial^* \circ p'')\Big) = \operatorname{Im} \tilde{\Delta}.
\end{aligned}$$

Moreover, $\tilde{\Delta}$ has a compact resolvent which is a pseudo-differential operator G of order -2, the Green's operator of $\tilde{\Delta}$, hence the spectrum of $\tilde{\Delta}$ is discrete and consists of non-negative eigenvalues that tend to $+\infty$.

Proposition 15. *For every $(p, q) \in \mathbb{N}^2$ let*

$$\tilde{H}^{p,q}(M, \mathbb{C}) := \frac{\operatorname{Ker}(p'' \circ \partial) \cap \operatorname{Ker} \overline{\partial}}{\operatorname{Im} \overline{\partial} + \operatorname{Im}(\partial_{|_{\operatorname{Ker} \overline{\partial}}})}.$$

Then, for every p, q, the following linear map

$$\begin{array}{rccc}
T = T^{p,q} : & \tilde{H}^{p,q}(M, \mathbb{C}) & \to & E_2^{p,q}(M)\\
& [\tilde{\alpha}] & \mapsto & \big[[\alpha]_{\overline{\partial}}\big]_{d_1}
\end{array}$$

is an isomorphism.

The operator $\tilde{\Delta}$ is related to the Frölicher spectral sequence by the following.

Theorem 66. *Let (M,g) be an Hermitian manifold with $\operatorname{Dim}_{\mathbb{C}} M = n$. For every $p,q \in \mathbb{N}$, let $\tilde{\mathcal{H}}^{p,q}$ be the kernel of $\tilde{\Delta}$ acting on (p,q)-forms. Then the map*

$$S = S^{p,q} : \tilde{\mathcal{H}}^{p,q}(M,\mathbb{C}) \to \tilde{H}^{p,q}(M,\mathbb{C}), \tag{3.71}$$

is an isomorphism, where

$$\tilde{H}^{p,q}(M,\mathbb{C}) := \frac{\operatorname{Ker} p'' \circ \partial \cap \operatorname{Ker} \overline{\partial}}{\operatorname{Im} \overline{\partial} + \operatorname{Im} \partial \mid_{\operatorname{Ker} \overline{\partial}}} \tag{3.72}$$

and $S(\phi) = [\tilde{\phi}]$. In particular, its composition with the isomorphism $T : \tilde{H}^{p,q}(M,\mathbb{C}) \to E_2^{p,q}$ defined in Proposition 15 yields the Hodge isomorphism

$$T \circ S = T^{p,q} \circ S^{p,q} : \tilde{\mathcal{H}}^{p,q}(M,\mathbb{C}) \to E_2^{p,q}. \tag{3.73}$$

Thus, every class $[[\phi]_{\overline{\partial}}]_{d_1} \in E_2^{p,q}$ contains a unique $\tilde{\Delta}$-harmonic representative ϕ.

Chapter 4

Deformation Theory

In this chapter we discuss deformations of complex structure on a compact complex manifold. Such theory was born with the celebrated work of Kodaira and Spencer [KS60], in which they proved that the condition of being Kähler is stable under small deformations of the complex structure.

We start with a brief recap of the work of Kodaira [KS60] about $\mathcal{C}^\infty$ family of elliptic differential operators.

The second section is devoted to our work on $\mathcal{C}^\infty$ families of elliptic pseudo-differential operators. We prove that most of the properties described in [Kod06] still hold if we consider such more general case.

Then we recall when a property $\mathcal{P}$ is open (or close) under small deformations of the complex structure and we recall some properties that verify such conditions. In the last part we expose our work on the degeneration at the second step of the Frölicher spectral sequence and we prove that such property is open under small deformation only if the dimension of the Dolbeault cohomology groups is independent on t (see Theorem 82). We conclude this chapter computing explicitly the first two step of the Frölicher spectral sequence for a suitable curve compact complex manifolds obtained as deformations of the Nakamura manifold; we show that in such case the dimension of the Dolbeault cohomology is not constant and the degeneration at the second step is not preserved along the curve.

4.1 Introduction

We start recalling the Kodaira and Spencer's theory of deformations. As for the previous chapter, we prove the theorems only for $\mathcal{C}^\infty$ function. The general case of forms can be obtained using (3.2). Let $\mathcal{M}$ be a differentiable manifold, B a domain of $\mathbb{R}^m$ and $\pi : \mathcal{M} \to B$ a $\mathcal{C}^\infty$ map. Suppose that

- the rank of the Jacobian matrix of π is equal to m at every point of $\mathcal{M}$;
- for each $t \in B$, $\pi^{-1}(t)$ is a compact connected subset of $\mathcal{M}$;

- there exists a locally finite open covering $\{\mathcal{U}_j\}$ of $\mathcal{M}$ and there exist complex-valued $\mathcal{C}^\infty$ functions $\{z_j^i\}$, $i = 1, ..., n$, defined on $\mathcal{U}_j$ such that for each $t \in B$

$$\{p \to (z_j^1(p), ..., z_j^n(p)) | \mathcal{U}_j \cap \pi^{-1}(t) \neq \emptyset\} \tag{4.1}$$

  form a system of local complex coordinates of $\pi^{-1}(t)$.

Then we call $\mathcal{M}$ a $\mathcal{C}^\infty$ *family of compact complex manifolds*.

The first two conditions imply that $\pi^{-1}(t)$ is a compact differentiable manifold of the same dimension for every $t \in B$. The third condition tells us that $\pi^{-1}(t)$ admits a structure of complex manifold. Then we can denote $\pi^{-1}(t)$ with the couple (M_t, J_t). Moreover, for every $t \in B$, M_t is diffeomorphic to a differentiable manifold M that does not depends on t. As a consequence, we use the notation (M, J_t) instead of $\mathcal{M}$ and we think that the underlying manifold M does not change while we take a $\mathcal{C}^\infty$ family $\{J_t\}$ of complex structures on it.

Given a differentiable manifold M, a $\mathcal{C}^\infty$ *family of differential operators* $\{A_t(x, D)\}$ on M is a collection of differential operators

$$\begin{array}{rlcl} A_t(x,D): & \mathcal{C}^\infty(M,\mathbb{C}) & \to & \mathcal{C}^\infty(M,\mathbb{C}) \\ & f(x) & \mapsto & \sum_{|\alpha|\le l} a_\alpha(x,t) D^\alpha f(x), \end{array} \tag{4.2}$$

where we use the same notation of the previous chapter with the only exception that $a_\alpha(x, t)$ is a differentiable function of (x, t).

A classical example of a $\mathcal{C}^\infty$ family of differential operators is given by the family $\{\Delta_{\overline{\partial}_t}\}$ of Beltrami-Laplace operators. In this case the variation of the complex structure induces a different decomposition of the cotangent bundle and, as a consequence, a different expression, in terms of the real coordinates, of the differential $\overline{\partial}_t$.

In the previous chapters we have seen that there exist isomorphisms between the cohomology groups of a complex manifold and the kernels of suitable differential operators; then it is natural to study the behavior of such kernels under small deformation of the complex structure in order to understand how the cohomology groups change.

Let $\{A_t(x, D)\}$ be a $\mathcal{C}^\infty$ family of strongly elliptic formally self-adjoint differential operators. For every $t \in B$, let $\lambda_h(t)$ be the h-th element (take in non decreasing order and counted with its multiplicity) of the spectrum of $A_t(x, D)$. The first result we recall is the following

Theorem 67. *$\lambda_h(t)$ is a continuous function of t.*

Let $\mathbb{F}_t := \{f \in \mathcal{C}^\infty(M, \mathbb{C}) | A_t(x, D)f = 0\}$ be the kernel of $A_t(x, D)$. Then the theorem above implies the following

Theorem 68. *$\dim \mathbb{F}_t$ is upper-semicontinuous in t.*

This is a very important result since it tells us that also the dimensions of the Dolbeault, Bott-Chern and Aeppli cohomology are upper-semicontinuous in t. Moreover, let F_t be the orthogonal projection of $\mathcal{C}^\infty(M, \mathbb{C})$ on $\mathbb{F}_t$, then we have

Theorem 69. *If* $\dim \mathbb{F}_t$ *is independent of* t *for every* t*, then* F_t *is* $\mathcal{C}^\infty$ *differentiable in* t*.*

The proofs of those theorems are based on the construction of a suitable Jordan curve C around the origin of the complex plane, i.e., the first eigenvalue of $A_0(x, D)$. To do so we need some preliminary results. The first is a lower estimate for the Sobolev's norm of $\{A_t(x, D)\}$ due to Friedrichs [Fri53].

Lemma 17. *Let* $k \in \mathbb{N}$*. Then there exists a constant* c_k *independent of* t *such that the inequality*

$$\|f\|_{k+m}^2 \leq c_k \left(\|A_t f\|_k^2 + \|f\|_k^2 \right) \tag{4.3}$$

holds for every $f \in \mathcal{C}^\infty(M, \mathbb{C})$*.*

Using this lemma it is possible to prove that also the Green operators of certain $\mathcal{C}^\infty$ family of differential operators form a $\mathcal{C}^\infty$ family.

Theorem 70. *Assume that* $A_t : \mathcal{C}^\infty(M, \mathbb{C}) \to \mathcal{C}^\infty(M, \mathbb{C})$ *is bijective for every* $t \in B$*. If there exists a constant* $c > 0$ *independent of* t*, such that for every* $f \in \mathcal{C}^\infty(M, \mathbb{C})$

$$\|A_t f\|_0 \geq c \|f\|_0 , \tag{4.4}$$

then A_t^{-1} *is a* $\mathcal{C}^\infty$ *family differentiable in* t

Now we want to show that, changing a bit the family $\{A_t(x, D)\}$, one can obtain a family that satisfies the hypothesis of the previous theorem. Let $\zeta \in \mathbb{C}$ be complex number different from every eigenvalue of $A_t(x, D)$ for a given t. We define

$$A_t(x, D, \zeta) := A_t(x, D) - \zeta. \tag{4.5}$$

We have the following

Proposition 16. $A_t(x, D, \zeta)$ *is a strongly elliptic differential operator acting bijectively on* $\mathcal{C}^\infty(M, \mathbb{C})$*.*

Moreover, for suitable ζ, the family $\{A_t(x, D, \zeta)\}$ satisfies the hypothesis of Theorem 70. In fact we have

Lemma 18. *Suppose that are given* $t_0 \in \Delta$ *and* $\zeta_0 \in \mathbb{C}$ *with* ζ_0 *different from every eigenvalue of* A_{t_0}*. If we take a sufficiently small* $\delta > 0$*, there exists a constant* $c > 0$ *such that, for* $|t - t_0| < \delta$ *and* $|\zeta - \zeta_0| < \delta$*, the following inequality*

$$\|A_t(\zeta) f\|_0 \geq c \|f\|_0 \tag{4.6}$$

holds for every $f \in \mathcal{C}^\infty(M, \mathbb{C})$*.*

We fix $\zeta \neq \lambda_h(0)$ for every $h \in \mathbb{N}$. Then, by this Lemma, we have that the family $A_t(x, D, \zeta)$ is bijective and, by Theorem 70, the operators $G_t(\zeta) := A_t^{-1}(x, D, \zeta)$ form a $\mathcal{C}^\infty$ family. We take a Jordan curve C on $\mathbb{C}$ which does not pass through any of the $\lambda_h(0)$. We denote with $((C))$ the interior of C. For any t sufficiently small, we define the linear operator $F_t(C)$ acting on $\mathcal{C}^\infty(M,\mathbb{C})$ by

$$F_t(C)f := \sum_{\lambda_h(t)\in((C))} (f, e_{th})e_{th}, \tag{4.7}$$

where e_{th} is an eigenfunction relative to the eigenvalue $\lambda_h(t)$ and $\{e_{th}\}$ form a complete orthonormal system of $\mathcal{C}^\infty(M,\mathbb{C})$. We put $\mathbb{F}_t(C)$ as the image through $F_t(C)$ of $\mathcal{C}^\infty(M,\mathbb{C})$. $\mathbb{F}_t(C)$ is a finite dimensional subspace of $\mathcal{C}^\infty(M,\mathbb{C})$ and $F_t(C)$ is the orthogonal projection of $\mathcal{C}^\infty(M,\mathbb{C})$ onto $\mathbb{F}_t(C)$.

Proposition 17. *The operator $F_t(C)$ can be written as*

$$F_t(C)f = -\frac{1}{2\pi}\int_C G_t(\zeta)d\zeta. \tag{4.8}$$

Lemma 19. *$F_t(C)$ is $\mathcal{C}^\infty$ differentiable in t for t close enough to t_0.*

Lemma 20. *$\dim \mathbb{F}_t(C)$ is independent of t for t close enough to t_0.*

4.2 Deformation of Pseudo-differential operators

In this section we generalize the theory of deformations to pseudo-differential operators. We prove the theorems of the previous section for this ampler class of operators; most of the prove are similar to the classic case, but we do them because there are some important details.

Lemma 21. *Let $k \in \mathbb{N}$. Then there exists a constant c_k independent of t such that the inequality*

$$\|f\|^2_{k+m} \le c_k \left(\|A_t f\|^2_k + \|f\|^2_0\right) \tag{4.9}$$

holds.

Proof. By Theorem 35, for every $t \in B$ there exists a positive constant $C_{k,t}$ such that

$$\|f\|^2_{k+m} \le C_{k,t}(\|A_t(x,D)f\|^2_k + \|f\|^2_k). \tag{4.10}$$

Since $\{A_t(x,D)\}$ is differentiable in t, we can assume that, up to shrinking B, the constant $C_{k,t}$ can be taken independent on t.

We proceed by induction on k. For $k = 0$, (4.10) is exactly our thesis. Now suppose that the thesis holds for $k-1$, then we have

$$\begin{aligned}\|f\|^2_{k+m} &\le C_k(\|A_t(x,D)f\|^2_k + \|f\|^2_k) \le C_k(\|A_t(x,D)f\|^2_k + \|f\|^2_{k-1+m})\\ &\le C_k(\|A_t(x,D)f\|^2_k + c_{k-1}(\|A_t(x,D)f\|^2_{k-1} + \|f\|^2_0))\\ &\le C_k(c_{k-1}+1)(\|A_t(x,D)f\|^2_k + \|f\|^2_0).\end{aligned}$$

$\square$

Theorem 71. *Assume that $A_t(x,D) : \mathcal{C}^\infty(M,\mathbb{C}) \to \mathcal{C}^\infty(M,\mathbb{C})$ is bijective for every $t \in B$. If there exists a constant $c > 0$ independent of t, such that for every $f \in \mathcal{C}^\infty(M,\mathbb{C})$*

$$\|A_t(x,D)f\|_0 \geq c\,\|f\|_0\,, \tag{4.11}$$

then $A_t^{-1}(x,D)$ is a $\mathcal{C}^\infty$ family differentiable in t

Proof. Let $\{f(x,t)\} \subset \mathcal{C}^\infty(M,\mathbb{C})$ be $\mathcal{C}^\infty$ differentiable in t. We prove by induction on $r \in \mathbb{N}$ that, putting $g_t := A_t^{-1}(x,D)f_t \in \mathcal{C}^\infty(M,\mathbb{C})$, $\{g_t\}$ is C^r differentiable in t.

For $r = 0$, by the previous Lemma and the Sobolev's inequality, we have that, for every multi-index l, we can choose an integer $k > l - m + n/2$ and a positive constant c such that

$$\begin{aligned} c|D^l(g_t(x) - g_s(x))| &\leq \|A_t(x,D)(g_t(x) - g_s(X))\|_k \\ &\leq \|A_t(x,D)g_t(x) - A_s(x,D)g_s\|_k + \|(A_t(x,D) - A_s(x,D))g_s\|_k \\ &= \|f_t(x) - f_s(x)\|_k + \|(A_t(x,D) - A_s(x,D))g_s\|_k\,. \end{aligned} \tag{4.12}$$

Since $f_t(x)$ and the coefficients of $A_t(x,D)$ are $\mathcal{C}^\infty$ differentiable in t, the last row converges uniformly to zero as $t \to s$. So $D^l g_t(x)$ converges to $D^l g_s(x)$ uniformly in x.

For $r = 1$, formally we have that, if $g_t(x)$ is C^1 differentiable in t, then by differentiating we obtain

$$\frac{\partial f_t}{\partial t} = A_t(x,D)\frac{\partial g_t(x)}{\partial t} + \frac{\partial A_t(x,D)}{\partial t} g_t(x). \tag{4.13}$$

Thus we need to prove that

$$\lim_{h\to 0}\left\| A_{t+h}(x,D)\left(\frac{1}{h}(g_{t+h}(x) - g_t(x)) - A_t^{-1}(x,D)\left(\frac{\partial f_t}{\partial t} - \frac{\partial A_t(x,D)}{\partial t} g_t\right)\right)\right\|_k = 0. \tag{4.14}$$

By direct computation we have that

$$\begin{aligned} &A_{t+h}(x,D)\left(\frac{1}{h}(g_{t+h}(x) - g_t(x)) - A_t^{-1}(x,D)\left(\frac{\partial f_t}{\partial t} - \frac{\partial A_t(x,D)}{\partial t} g_t\right)\right) \\ &\quad = \frac{1}{h}(f_{t+h}(x) - f_t(x)) - \frac{\partial f_t}{\partial t} - \frac{1}{h}(A_{t+h}(x,D) - A_t(x,D))g_t(x) \\ &\quad + \frac{\partial A_t(x,D)}{\partial t} g_t(x) - (A_{t+h}(x,D) - A_t(x,D))\phi_t(x), \end{aligned}$$

where we put

$$\phi_t(x) = A_t^{-1}(x,D)\left(\frac{\partial f_t}{\partial t} - \frac{\partial A_t(x,D)}{\partial t} g_t\right).$$

Since $f_t(x)$ and the coefficients of $A_t(x,D)$ are $\mathcal{C}^\infty$ in t we have that

$$\left\|\frac{1}{h}(f_{t+h}(x) - f_t(x)) - \frac{\partial f_t}{\partial t}\right\|_k \xrightarrow[h\to 0]{} 0, \tag{4.15}$$

$$\left\| \frac{1}{h}(A_{t+h}(x,D) - A_t(x,D))g_t(x) - \frac{\partial A_t(x,D)}{\partial t} g_t(x) \right\|_k \xrightarrow[h \to 0]{} 0 \tag{4.16}$$

and

$$\|(A_{t+h}(x,D) - A_t(x,D))\phi_t(x)\|_k \xrightarrow[h \to 0]{} 0 \tag{4.17}$$

uniformly in x. The thesis follows immediately by putting (4.15), (4.16), (4.17) in (4.14).

Suppose now by induction that $g_t(x)$ is C^r differentiable. To prove that it is C^{r+1} differentiable we consider the function $h_t := \frac{\partial^r g_t}{\partial t^r}$, then we have

$$A_t(x,D)h_t(x) = \frac{\partial^r f_t}{\partial t^r}(x) - \sum_{l=0}^{r-1} \frac{\partial^l A_t(x,D)}{\partial t^l} \frac{\partial^{r-l} g_t}{\partial t^{r-l}}(x).$$

By induction g_t is C^r differentiable in t, so the right-hand side is C^1 differentiable in t. Thus, by the previous case, h_t is C^1 differentiable in t. □

Lemma 22. *Let $\zeta_0 \in \mathbb{C}$, with ζ_0 different from every eigenvalue of $A_0(x,D)$. If we take a sufficiently small $\delta > 0$, there exists a constant $c > 0$ such that, for $|t| < \delta$ and $|\zeta - \zeta_0| < \delta$, the following inequality*

$$\|A_t(\zeta)f\|_0 \geq c\,\|f\|_0 \tag{4.18}$$

holds for every $f \in \mathcal{C}^\infty(M,\mathbb{C})$.

Proof. Suppose that, for any $\delta > 0$, there exists no such constant. Then, for every $q \in \mathbb{N}$ there exist $t_q \in B$, $\zeta_q \in \mathbb{C}$ and $f_q \in \mathcal{C}^\infty(M,\mathbb{C})$ such that

$$|t_q| < \tfrac{1}{q}, \quad |\zeta_q - \zeta_0| < \tfrac{1}{q}, \quad \left\|A_{t_q}(\zeta_q)f_q\right\|_0 < \tfrac{1}{q}, \quad \|f_q\|_0 = 1.$$

Hence we have that $\left\|A_{t_q}(\zeta_q)f_q\right\|_0 \to 0$ as $q \to +\infty$ and, by construction of $A_{t_q}(\zeta_q)$,

$$\left\|A_{t_q}(\zeta_q)f_q - A_0(\zeta_0)f_q\right\|_0 \to 0. \tag{4.19}$$

Then $\|A_0(\zeta_0)f_q\|_0 \to 0$, but, by Lemma 21, $\left\|A_{t_q}(\zeta_q)f\right\|_0 \geq c_0\,\|f\|_0$ for every $f \in \mathcal{C}^\infty(M,\mathbb{C})$. So $\|f_q\|_0 \to 0$, which contradicts $\|f_q\|_0 = 1$. □

We re-propose the construction of Jordan curve around the eigenvalues of $A_0(x,D)$. We fix $\zeta \neq \lambda_h(0)$ for every $h \in \mathbb{N}$. Then, by this Lemma, we have that the family $A_t(x,D,\zeta)$ is bijective and, by Theorem 70, the operators $G_t(\zeta) := A_t^{-1}(x,D,\zeta)$ form a $\mathcal{C}^\infty$ family. We take a Jordan curve C on $\mathbb{C}$ which does not pass through any of the $\lambda_h(0)$. We denote with $((C))$ the interior of C. For any t sufficiently small, we define the linear operator $F_t(C)$ acting on $\mathcal{C}^\infty(M,\mathbb{C})$ by

$$F_t(C)f := \sum_{\lambda_h(t)\in((C))} (f, e_{th})e_{th}, \tag{4.20}$$

where e_{th} is an eigenfunction relative to the eigenvalue $\lambda_h(t)$ and $\{e_{th}\}$ form a complete orthonormal system of $\mathcal{C}^\infty(M, \mathbb{C})$. We put $\mathbb{F}_t(C)$ as the image through $F_t(C)$ of $\mathcal{C}^\infty(M, \mathbb{C})$. $\mathbb{F}_t(C)$ is a finite dimensional subspace of $\mathcal{C}^\infty(M, \mathbb{C})$ and $F_t(C)$ is the orthogonal projection of $\mathcal{C}^\infty(M, \mathbb{C})$ onto $\mathbb{F}_t(C)$.

Proposition 18. *The operator $F_t(C)$ can be written as*

$$F_t(C)f = -\frac{1}{2\pi}\int_C G_t(\zeta)d\zeta. \tag{4.21}$$

Lemma 23. *$F_t(C)$ is $\mathcal{C}^\infty$ differentiable in t for t close enough to t_0.*

Proof. This follows immediately from (4.21) since both $f_t(x)$ and $G_t(\zeta)$ are $\mathcal{C}^\infty$ differentiable in t. □

Lemma 24. $\dim \mathbb{F}_t(C)$ *is independent of t for t close enough to t_0.*

Proof. Put $\mathrm{Dim}\, \mathbb{F}_0(C) = d$ and let $\{e_1, \dots, e_d\}$ be a basis of $\mathbb{F}_0(C)$ such that $F_0(C)(e_k) = e_k$. Since $F_t(C)$ is $\mathcal{C}^\infty$ differentiable in t, for sufficiently small $\delta > 0$ and $|t| < \delta$, we have that $\{F_t(C)(e_1), \dots, F_t(C)(e_d)\}$ are linearly independent. Suppose that we can find a sequence t_q, $q = 1, 2, \dots$, with $|t| < 1/q$, such that $\mathrm{Dim}\, \mathcal{F}_{\sqcup_{\rm II}}(\mathcal{C}) > d$. Then at least $d+1$ eigenvalues $\lambda_h(t_q)$ must lie in the interior of C. Then for each derivative D^l we have

$$|D^l e_{r,q}(x)|^2 \leq C_l(1 + \sum_{\alpha=1}^{k} |\lambda_h(t_q)|^{2\alpha}), \tag{4.22}$$

where k is an integer greater than $m + 1 + n/2$. Since $\lambda_h(t_q)$ is bounded, then $\{D^l e_{r,q}\}_q$ is uniformly bounded in M, hence it is equicontinuous. So we can find a uniformly convergent subsequence. Suppose that we have already taken such a subsequence, then we have

$$\lim_{q\to\infty} D^l e_{r,q}(x) = D^l e_r(x).$$

Since $(e_{r,q}, e_{s,q}) = \delta_{rs}$, also $(e_r, e_s) = \delta_{rs}$. Moreover

$$\lim_{q\to\infty} A_{t_q}(x, D)e_{r,q}(x) = A_0(x, D)e_r(x) = \lambda_r e_r(x),$$

where $\lambda_r = \lim \lambda_{r,q}$. So there are at least $d + 1$ linearly independent functions in $\mathcal{F}_0(C)$, which contradicts the hypothesis. □

Theorem 72. *$\lambda_h(t)$ is a continuous function of t.*

Proof. We proceed by induction on h. If $h = 1$, then by Theorem 56, there exists $\beta \in \mathbb{R}$ such that, for every $t \in B$, $\lambda_1(t) > \beta$. Moreover, for every $\varepsilon > 0$, consider the circle C_ε with center $\lambda_1(t_0)$ and radius ε and a Jordan curve which intersects the real plane only in β and $\lambda_1(t_0) - \varepsilon$. By Theorem 56, $\mathrm{Dim}\, \mathcal{F}_t(C) = 0$, then $\lambda_1(t) > \lambda_1(t_0) - \varepsilon$. Since C_ε is a Jordan curve and

$\operatorname{Dim} \mathcal{F}_{t_0}(C_\varepsilon) > 0$, by Lemma 24, there exists $k \in \mathbb{N}$ such that $\lambda_k(t) \in C_\varepsilon$. Then we have $\lambda_1(t_0) - \varepsilon < \lambda_1(t) \leq \lambda_k(t) < \lambda_1(T_0) + \varepsilon$, so $\lim_{t \to t_0} \lambda_1(t) = \lambda_1(t_0)$.

For $h > 1$ we have two cases, first suppose that $\lambda_1(t_0) = \cdots = \lambda_h(t_0)$. Then $\operatorname{Dim} \mathcal{F}_{t_0}(C_\varepsilon)$ is at least h so, by the arguments above, $\lambda_1(t), \ldots, \lambda_h(t)$ are in the interior of C_ε. Hence $\lim_{t \to t_0} \lambda_h(t) = \lambda_h(t_0)$.

Otherwise, suppose that l is the largest integer such that $\lambda_h(t_0) > \lambda_l(t_0)$. Then, for ε small enough, the circle C_ε with center $\lambda_h(t_0)$ and radius ε contains only the eigenvalues of $A_{t_0}(x, D)$ equal to $\lambda_h(t_0)$. By hypothesis, for t close enough to t_0, we can assume that $\lambda_k(t)$ lies in the exterior of C_ε for $k = 1, \ldots, l$. Then we can use the arguments of the case $h = 1$ and find that $\lambda_h(t) \to \lambda_h(t_0)$ as $t \to t_0$. □

Theorem 73. $\dim \mathbb{F}_t$ *is upper-semicontinuous in* t.

Proof. Let $\lambda_h \neq 0$ be the eigenvalue of $A_{t_0}(x, D)$ closest to zero. Since the spectrum of $A_{t_0}(x, D)$ is composed only by isolated points, $|\lambda_h| > \varepsilon$ for some $\varepsilon > 0$. Consider the circle C_ε of center 0 and radius ε, then $\operatorname{Dim} \mathbb{F}_{t_0} = \operatorname{Dim} \mathbb{F}_{t_0}(C_\varepsilon)$. By Lemma 24, $\operatorname{Dim} \mathbb{F}_{t_0}(C_\varepsilon) = \operatorname{Dim} \mathbb{F}_t(C_\varepsilon)$ for t close enough to t_0. Then, by the definition of $\mathbb{F}_t$, $\operatorname{Dim} \mathbb{F}_t \leq \operatorname{Dim} \mathbb{F}_t(C_\varepsilon) = \operatorname{Dim} \mathbb{F}_{t_0}(C_\varepsilon) = \operatorname{Dim} \mathbb{F}_{t_0}$. □

Theorem 74. *If* $\dim \mathbb{F}_t$ *is independent of* t *for every* t*, then* F_t *is* $\mathcal{C}^\infty$ *differentiable in* t.

Proof. Let C_ε as in the theorem above. Then $\operatorname{Dim} \mathbb{F}_{t_0} = \operatorname{Dim} \mathbb{F}_{t_0}(C_\varepsilon) = \operatorname{Dim} \mathbb{F}_t(C_\varepsilon)$ for t close enough to t_0. Since, by hypothesis, $\operatorname{Dim} \mathbb{F}_t$ is independent of t, $\mathbb{F}_t = \mathbb{F}_t(C_\varepsilon)$.

By definition F_t is the orthogonal projection of $\mathcal{C}^\infty(M, \mathbb{C})$ onto $\mathbb{F}_t$. Since $\mathbb{F}_t = \mathbb{F}_t(C_\varepsilon)$, $F_t = F_t(C_\varepsilon)$ that we know, by Lemma 23, to be $\mathcal{C}^\infty$ differentiable in t. □

4.3 Stability of Properties

Several authors have studied the behavior of properties under small deformation of the complex structure. Let $\mathcal{P}$ be a property of complex manifold, e.g. being a Kähler manifold.

Definition 29. $\mathcal{P}$ *is said to be* open under small deformations of complex structure *if for every* $\mathcal{C}^\infty$ *family* (M, J_t) *of compact complex manifolds the implication*

$$(M, J_0) \text{ satisfies } \mathcal{P} \Rightarrow (M, J_t) \text{ satisfies } \mathcal{P}$$

holds for every $t \in B$.

Definition 30. $\mathcal{P}$ *is said to be* close under small deformations of complex structure *if for every* $\mathcal{C}^\infty$ *family* (M, J_t) *of compact complex manifolds the implication*

$$(M, J_t) \text{ satisfies } \mathcal{P} \text{ for every } t \in B \setminus \{0\} \Rightarrow (M, J_0) \text{ satisfies } \mathcal{P}$$

holds.

We briefly recall some of the most results about the stability of properties.

Theorem 75. *[KS60] Admitting a Kähler metric is a property open under small deformations.*

Theorem 76. *Satisfying the $\partial\overline{\partial}$-lemma is a property open under small deformations.*

Theorem 77. *[Hir62] Admitting a Kähler metric is not a property closed under small deformations.*

Theorem 78. *[Pop11] Admitting a strongly Gaudouchon metric is a property open under small deformations.*

Theorem 79. *[AB90] Admitting a Balanced metric is a not property open under small deformations.*

Theorem 80. *[KS60] The degeneration at the first step of the Frölicher spectral sequence is a property open under small deformations.*

Theorem 81. *[ES$^+$93] The degeneration at the first step of the Frölicher spectral sequence is not a property closed under small deformations.*

4.3.1 Degeneration of the Frölicher spectral sequence

During this PhD we have studied the degeneration at the second step of the Frölicher spectral sequence. In particular we have analysed the behavior of such property under small deformations of the complex structure and we have proved the following

Theorem 82. *Let (M, J_t) be a family of complex manifolds and suppose that the dimension of $\operatorname{Ker}\Delta_{\overline{\partial}_t} \cap \Lambda^{p,q}(M, J_t)$ is independent of t for every $(p, q) \in \mathbb{Z}^2$. Then the degeneration at the second step of the Frölicher spectral sequence is stable under small deformations of the complex structure.*

The proof of this theorem follows from Theorems 66 and 73. We only need to prove the following.

Proposition 19. *If $\operatorname{Dim}\operatorname{Ker}\Delta_{\overline{\partial}_t}$ is independent of t, then $\{\tilde{\Delta}_t\}$ is a $\mathcal{C}^\infty$ family of pseudo differential operators.*

Proof. From [Kod06], we have that all the derivative operators depend $\mathcal{C}^\infty$ with respect to t, hence we only need to prove that if $\{\phi_t\}$ and $\{\psi_t\}$ are $\mathcal{C}^\infty$ family of (p, q) forms over M and if $\{g_t\}$ is a $\mathcal{C}^\infty$ family of Hermitian metrics over M, then the scalar product $(\psi_t, \phi_t)_t$ varies in a $\mathcal{C}^\infty$ way respect to t. By definition

$$(\psi_t, \phi_t)_t = \int_M \psi_t \wedge *_t\overline{\phi}_t. \tag{4.23}$$

Let $\{U_j\}$ be a finite covering of M made by open coordinate neighborhood and let $\{\eta_j\}$ be a partition of unity subordinate to $\{U_j\}$. Then, for every $t \in B$ we have

$$\int_M \psi_t \wedge *_t\overline{\phi}_t = \sum_j \int_{U_j} \eta_j \psi_t \wedge *_t\overline{\phi}_t. \tag{4.24}$$

Now, locally we have $\psi_t = \sum_{A_p,B_q} \psi_t^{A_p\overline{B}_q} dz^{A_p\overline{B}_q}$ and

$$*_t\overline{\phi}_t(z) := (i)^n(-1)^k \sum_{A_p,B_q} sgn\begin{pmatrix} A_p & A_{n-p} \\ B_q & B_{n-q}\end{pmatrix} g_t(z)\overline{\phi}_t^{A_p\overline{B}_q}(z) dz^{B_{n-p}\overline{A_{n-q}}}. \tag{4.25}$$

Then we have

$$\int_{U_j} \eta_j\psi_t\wedge *_t\overline{\phi}_t = \sum_{A_p,B_q} \sigma_{A_pB_p}\int_{U_j} \eta_j \psi_t^{A_p\overline{B}_q}\overline{\phi}_t^{A_p\overline{B}_q} g_t dz^1\wedge\cdots\wedge dz^n\wedge d\overline{z}^1\wedge\cdots\wedge d\overline{z}^n, \tag{4.26}$$

where $\sigma_{A_pB_p}$ is the sign of the permutation.

In order to prove that the scalar product is $\mathcal{C}^\infty$, it suffices to show that it is $\mathcal{C}^k$ for every $k \in \mathbb{N}$. We begin proving that it is $\mathcal{C}^0$: by the continuity of the integral operator, the coefficients of ψ_t, ϕ_t and g_t and since $\eta_j\psi_t \wedge *_t\overline{\phi}$ is continuous and compactly supported in U_j , we have the continuity of the scalar product.

Now we prove by induction over $r \in \mathbb{N}$ that (4.23) is $\mathcal{C}^r$. Let $r = 1$ and consider the following

$$\frac{(\psi_t,\phi_t)_t - (\psi_s,\phi_s)_s}{t-s} = \frac{1}{t-s}\int_M \psi_t\wedge *_t\overline{\phi}_t - \psi_s\wedge *_s\overline{\phi}_s. \tag{4.27}$$

Locally we can rewrite the integral above as

$$\frac{1}{t-s}\sum_{A_p,B_q}\int_{U_j}\left(\eta_j\psi_t^{A_p\overline{B}_q}\overline{\phi}_t^{A_p\overline{B}_q} g_t - \eta_j\psi_s^{A_p\overline{B}_q}\overline{\phi}_s^{A_p\overline{B}_q} g_s\right) dz^1\wedge\cdots\wedge dz^n\wedge d\overline{z}^1\wedge\cdots\wedge d\overline{z}^n. \tag{4.28}$$

Now, for every multi-indexes A_p and B_q, we consider the following construction

$$\begin{aligned}
&\frac{1}{t-s}\eta_j\left(\psi_t^{A_p\overline{B}_q}\overline{\phi}_t^{A_p\overline{B}_q} g_t - \psi_s^{A_p\overline{B}_q}\overline{\phi}_s^{A_p\overline{B}_q} g_s\right) = \\
&= \frac{1}{t-s}\eta_j\left(\psi_t^{A_p\overline{B}_q}\overline{\phi}_t^{A_p\overline{B}_q}(g_t-g_s)\right) \\
&+ \frac{1}{t-s}\eta_j\left(\psi_t^{A_p\overline{B}_q}(\overline{\phi}_t^{A_p\overline{B}_q} - \overline{\phi}_s^{A_p\overline{B}_q})g_s + (\psi_t^{A_p\overline{B}_q} - \psi_s^{A_p\overline{B}_q})\overline{\phi}_s^{A_p\overline{B}_q} g_s\right) = \\
&= \eta_j\left(\psi_t^{A_p\overline{B}_q}\overline{\phi}_t^{A_p\overline{B}_q}\frac{g_t-g_s}{t-s} + \psi_t^{A_p\overline{B}_q}\frac{\overline{\phi}_t^{A_p\overline{B}_q} - \overline{\phi}_s^{A_p\overline{B}_q}}{t-s} g_s + \frac{\psi_t^{A_p\overline{B}_q} - \psi_s^{A_p\overline{B}_q}}{t-s}\overline{\phi}_s^{A_p\overline{B}_q} g_s\right).
\end{aligned}$$

When t tends to s, we obtain that

$$\frac{d(\psi_t,\phi_t)_t}{dt}\,|_{t=s}=(\psi_s',\phi_s)_s+(\psi_s,\phi_s')_s + \sum_j\sum_{A_p,B_q}\int_{U_j}\eta_j\psi_s^{A_p\overline{B}_q}\overline{\phi}_s^{A_p\overline{B}_q}g_s'dz^1\wedge\cdots\wedge dz^n\wedge d\overline{z}^1\wedge\cdots\wedge d\overline{z}^n, \tag{4.29}$$

where ϕ_s' and ψ_s' denote the derivative along t of the $\mathcal{C}^\infty$ forms $\phi(z,t)$ and $\psi(z,t)$ respectively. Using the same arguments of the $\mathcal{C}^0$ case, we have the derivative is continuous.

Suppose, by induction, that (4.23) is $\mathcal{C}^r$. By reiteration of (4.29) we have that the r-th derivative of (4.23) is made by two type of components

i) $(\psi_t^{(i)},\phi_t^{(j)})_t$;

ii) $\int_M g_t^{(k)}\psi_t^{(i)}\wedge\phi_t^{(j)}$, for some $k\in\mathbb{N}$.

In either case, using the same argument as above, we have the existence and the continuity of the derivative of the r-th derivative of $(\phi(z,t),\psi(z,t))_t$.

□

Since we are in the hypotesis of Theorem 73, the dimension of the kernel of $\tilde{\Delta}_t$ is an upper-semicontiuos function of t. Hence, by Theorem 66, we have the following.

Proof of Theorem 82. We denote with b_k the dimension of $H^k_{dR}(M;\mathbb{C})$, with $\tilde{h}^{p,q}_t$ the complex dimension of $\operatorname{Ker}\tilde{\Delta}_t\cap\Lambda^{p,q}(M,J_t)$ and with $e_2^{p,q}(t)$ the dimension of $E_2^{p,q}(M,J_t)$. We recall the degeneration at the second step of $\{E_r^{p,q}(M,J_t)\}$ is equivalent to

$$b_k=\sum_{p+q=k}e_2^{p,q}(t); \tag{4.30}$$

by Theorem [Pop16, Theorem 3.4] we have

$$\sum_{p+q=k}\tilde{h}^{p,q}_t=\sum_{p+q=k}e_2^{p,q}(t); \tag{4.31}$$

finally, by Theorem 73, we know that $\tilde{h}^{p,q}_t$ is an upper-semi continuous function of t.

Suppose that the Frölicher spectral sequence of (M,J_0) degenerates at the second step, then, summing up all the previous considerations, we have

$$b_k=\sum_{p+q=k}e^{p,q}_{2,0}=\sum_{p+q=k}\tilde{h}^{p,q}_0\geq\sum_{p+q=k}\tilde{h}^{p,q}_t=\sum_{p+q=k}e^{p,q}_{2,t}\geq b_k. \tag{4.32}$$

Thus

$$b_k=\sum_{p+q=k}\tilde{h}^{p,q}_t=\sum_{p+q=k}e^{p,q}_{2,t}, \tag{4.33}$$

that means that, for t small enough, the Frölicher spectral sequence of M_t degenerates at the second step. □

4.3.2 Example

In this section we provide an example of $\mathcal{C}^\infty$ curve of compact complex manifolds such that the Frölicher spectral sequence degenerate at the second step for one of them and at higher steps for for the others. In this example the dimension of $\operatorname{Ker}\Delta_{\overline{\partial}_t}$ is not constant with respect to t. Let X be the Nakamura manifold, namely a compact complex three-dimensional holomorphically parallelizable solvmanifold constructed in the following way: let G be the Lie group defined as $G := \mathbb{C} \ltimes_\phi \mathbb{C}^2$, where

$$\phi(z) := \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}. \tag{4.34}$$

Let $\Gamma := \Gamma' \ltimes_\phi \Gamma''$ be a lattice in G, where Γ'' is a lattice in $\mathbb{C}^2$ and $\Gamma' := \mathbb{Z}(a+ib)+\mathbb{Z}(c+id)$ is such that it is a lattice in $\mathbb{C}$ and $\phi(a+ib)$ and $\phi(c+id)$ are conjugate elements in $SL(4,\mathbb{Z})$. Then $X := \Gamma \setminus G$.

We consider the following deformation of X: let $t \in \mathbb{C}$ and consider the $(0,1)$-form on X with value in $T^1, 0X$ defined by

$$\phi_t = te^{z_1} d\overline{z}_1 \otimes \frac{\partial}{\partial z_2}.$$

For $|t| < \epsilon$, let X_t be the small deformation of X associated to ϕ_t. We prove the following

Theorem 9. *The Frölicher spectral sequence of X_t degenerates at the second step for $t = 0$, while it degenerates at higher steps for $t \neq 0$.*

First we recall that the Betti's number of X are the following

$$b_0 = b_6 = 1, \quad b_1 = b_5 = 2, \quad b_2 = b_4 = 5, \quad b_3 = 8. \tag{4.35}$$

Moreover it is a well-known fact that they do not change under deformations of the complex structure. Now we explicitly compute $E_1^{\bullet,\bullet}(X_t)$ and $E_2^{\bullet,\bullet}(X_t)$ and we show that

$$b_k = \sum_{p+q=k} \operatorname{Dim} E_2^{p,q}(X_0) \tag{4.36}$$

for every $k = 1, ..., 6$, while the equality is false for $t \neq 0$.

We proceed in the following way: we begin with the computation of the Dolbeault cohomology of X_t since, as we recalled is Section **??**, the first step of the Frölicher spectral sequence is isomorphic to the Dolbeault cohomology, namely $E_1^{p,q}(X_t) \simeq H_{\overline{\partial}}^{p,q}(X_t)$ for every $(p,q) \in \mathbb{Z}^2$. By applying [AK12, Theorem 1.3], in [TT14] Tomassini and Torelli found the Δ_t''-harmonic forms of X_t; for every $(p,q) \in \mathbb{Z}$, those forms are a basis for $H_{\overline{\partial}}^{p,q}(X_t)$ as $\mathbb{C}$ vector space. Then, as proved in [CFUG97], $E_2^{p,q}(X_t)$ can be described as

$$E_2^{p,q} = \frac{X_2^{p,q}(X_t)}{Y_2^{p,q}(X_t)}, \tag{4.37}$$

where

$$X_2^{p,q}(X_t) := \left\{\alpha \in \Lambda^{p,q}(X_t) | \overline{\partial}\alpha = 0 \text{ and } \exists \beta \in \Lambda^{p+1,q-1}(X_0) \text{ s.t. } \partial\alpha + \overline{\partial}\beta = 0\right\}, \tag{4.38}$$

$$Y_2^{p,q}(X_t) := \left\{\partial\alpha + \overline{\partial}\beta \in \Lambda^{p,q}(X_0) | \overline{\partial}\alpha = 0\right\}. \tag{4.39}$$

Namely, if $\alpha \in X_2^{p,q}(X_t)$, $[\alpha]_2 \in E_2^{p,q}(X_t)$ can be written as

$$[\alpha]_2 = \{\alpha + \partial\beta + \overline{\partial}\gamma | \overline{\partial}\beta = 0\}. \tag{4.40}$$

On the other hand, $E_2^{p,q}(X_0)$ is the cohomology group of the complex

$$E_1^{p-1,q}(X_0) \xrightarrow{d_1} E_1^{p,q}(X_0) \xrightarrow{d_1} E_1^{p+1,q}(X_t) \tag{4.41}$$

where d_1 is the operator $[\alpha] \mapsto [\partial\alpha]$. Since it is well defined and every $\overline{\partial}$-closed form α belongs to a class in the Dolbeault cohomology, we need only to work with Δ_t''-harmonic form. In fact if an harmonic form ϕ is such that $\partial\phi$ is $\overline{\partial}$-exact, then every form $\alpha = \phi + \overline{\partial}\beta$ is such that $\partial\alpha$ is $\overline{\partial}$-exact. Moreover, from the decomposition

$$\operatorname{Ker} \overline{\partial}_t = \mathcal{H}^{\bullet,\bullet}(X_t) \oplus \operatorname{Im} \overline{\partial}_t \tag{4.42}$$

we have that if ϕ is ∂-exact then it is the the image through ∂ of another harmonic form.

Now we proceed with the computation. We recall that in [TT14], the Dolbeault cohomology of X_t was computed using a suitable sub-complex $(B_t, \overline{\partial}_t) \subset (\Gamma^{\bullet,\bullet}X_t, \overline{\partial}_t)$. Namely, let

$$B_t := \wedge^{\bullet,\bullet}(\mathbb{C} < \phi_1^{1,0}(t), \phi_2^{1,0}(t), \phi_3^{1,0}(t) > \oplus \mathbb{C} < \phi_1^{0,1}(t), \phi_2^{0,1}(t), \phi_3^{0,1}(t) >,$$

where

$$\begin{array}{lll} \phi_1^{1,0}(t) := dz_1, & \phi_2^{1,0}(t) := e^{-z_1}dz_2 - td\bar{z}_1, & \phi_3^{1,0}(t) := e^{z_1}dz_3, \\ \phi_1^{0,1}(t) := d\bar{z}_1, & \phi_2^{1,0}(t) := \bar{t}e^{\bar{z}_1 - z_1}dz_1 - e^{-z_1}d\bar{z}_2, & \phi_2^{1,0}(t) := e^{z_1}d\bar{z}_3. \end{array}$$

We have the following structure equations

$$\begin{cases} d\phi_1^{1,0}(t) = 0 \\ d\phi_2^{1,0}(t) = -\phi_1^{1,0}(t) \wedge \phi_2^{1,0}(t) - t\phi_1^{1,0}(t) \wedge \phi_1^{0,1}(t) \\ d\phi_3^{1,0}(t) = \phi_1^{1,0}(t) \wedge \phi_3^{1,0}(t) \\ d\phi_1^{0,1}(t) = 0 \\ d\phi_2^{0,1}(t) = -\phi_1^{1,0}(t) \wedge \phi_2^{0,1}(t) + \bar{t}e^{\bar{z}_1 - z_1}\phi_1^{1,0}(t) \wedge \phi_1^{0,1}(t) \\ d\phi_3^{0,1}(t) = \phi_1^{1,0}(t) \wedge \phi_3^{0,1}(t) \end{cases}$$

Then $(B_t, \overline{\partial}_t)$ is a finite sub-complex of $(\Gamma^{\bullet,\bullet}X_t, \overline{\partial}_t)$, which is smooth on $X \times B(0,\epsilon)$, closed with respect to the $\mathbb{C}$-anti-linear Hodge star operator $*_t$ associated to the Hermitian metric $g_t := \sum_{i=1}^3 \phi_i^{1,0}(t) \odot \overline{\phi_i^{1,0}(t)}$ and such that $H_{\overline{\partial}}^{\bullet,\bullet}(B_0) \simeq H_{\overline{\partial}}^{\bullet,\bullet}(X)$.

We start with the computation of $E_1^{p,q}(X_0)$. For $t = 0$ we have that every (p, q)-form of the type

$$\wedge^{p,q}\left(\mathbb{C} < \phi_1^{1,0}(0), \phi_2^{1,0}(0), \phi_3^{1,0}(0) > \oplus \mathbb{C} < \phi_1^{0,1}(0), \phi_2^{0,1}(0), \phi_3^{0,1}(0) >\right)$$

is $\Delta_{\overline{\partial}}$-harmonic. Thus they form a basis for $E_1^{p,q}(X_0)$. Next we compute the image through ∂ of every harmonic form. In Table 4.1 we report only the harmonic forms that are not ∂-closed.

With the same Table, we also know which harmonic form is ∂-exact and so represents the zero class in $E_2^{p,q}(X_0)$. Let ϕ be a harmonic (p, q)-form in Table 4.1, we want to know if $\partial\phi$ is $\overline{\partial}$-exact. To do so we consider the scalar product of $\partial\phi$ with every $(p + 1, q)$ harmonic form and, by straightforward computation, we have that it is never zero. Then, again by the decomposition $\operatorname{Ker}\overline{\partial}_t = \mathcal{H}_{\overline{\partial}}^{\bullet,\bullet}(X_0) \oplus \operatorname{Im}\overline{\partial}_t$, they are not such that $\partial\phi$ is $\overline{\partial}$-exact. In Table 4.2 is listed a basis for $E_2^{p,q}(X_0)$.

Instead, for $t \neq 0$, the Δ''-harmonic forms are listed in Table 4.3

Using the same arguments as in the case $t = 0$, we obtain that $E_2^{p,q}(X_t)$ is generated by the forms listed in Table 4.4

Let $h^{p,q}(t) := \operatorname{Dim} H_{\overline{\partial}}^{p,q}(X_t)$ and $e_2^{p,q}(t) := \operatorname{Dim} E_2^{p,q}(X_t)$.

Then we have that, for $t = 0$ and $k \in \mathbb{Z}$, $\sum_{p+q=k} e_2^{p,q}(0) = b_k$ and this is equivalent to the degeneration at the second step of $(E_r^{\bullet,\bullet}(X_0), d_r)$. While, for $t \neq 0$ and $k = 2, 3$, $b_k < \sum_{p+q=k} e_2^{p,q}(t)$ (see Table 4.5).

So we have proved Theorem 9.

Observation 2. *The dimensions of $E_2^{p,q}(X_t)$ do not vary, in general, nor upper semi-continuously neither lower semi-continuously with respect to t. For example $e_2^{2,1}(t)$ is a lower semi-continuous function of t and $e_2^{2,2}(t)$ is upper semi-continuous.*

ϕ	$\partial\phi$
$\phi_2^{1,0}$	$-\phi_{12}^{1,0}$
$\phi_3^{1,0}$	$\phi_{13}^{1,0}$
$\phi_2^{0,1}$	$-\phi_1^{1,0} \wedge \phi_2^{0,1}$
$\phi_3^{0,1}$	$\phi_1^{1,0} \wedge \phi_3^{0,1}$
$\phi_2^{1,0} \wedge \phi_1^{0,1}$	$-\phi_{12}^{1,0} \wedge \phi_1^{0,1}$
$\phi_2^{1,0} \wedge \phi_2^{0,1}$	$-2\phi_{12}^{1,0} \wedge \phi_2^{0,1}$
$\phi_3^{1,0} \wedge \phi_1^{0,1}$	$\phi_{13}^{1,0} \wedge \phi_1^{0,1}$
$\phi_3^{1,0} \wedge \phi_3^{0,1}$	$\phi_{13}^{1,0} \wedge \phi_3^{0,1}$
$\phi_{23}^{1,0} \wedge \phi_2^{0,1}$	$-\phi_{123}^{1,0} \wedge \phi_2^{0,1}$
$\phi_{23}^{1,0} \wedge \phi_3^{0,1}$	$\phi_{123}^{1,0} \wedge \phi_3^{0,1}$
$\phi_{12}^{0,1}$	$-\phi_1^{1,0} \wedge \phi_{12}^{0,1}$
$\partial\phi_{13}^{0,1}$	$\phi_1^{1,0} \wedge \phi_{13}^{0,1}$
$\phi_2^{1,0} \wedge \phi_{12}^{0,1}$	$-2\phi_{12}^{1,0} \wedge \phi_{12}^{0,1}$
$\phi_2^{1,0} \wedge \phi_{23}^{0,1}$	$-\phi_{12}^{1,0} \wedge \phi_{23}^{0,1}$
$\phi_3^{1,0} \wedge \phi_{13}^{0,1}$	$2\phi_{13}^{1,0} \wedge \phi_{13}^{0,1}$
$\phi_3^{1,0} \wedge \phi_{23}^{0,1}$	$\phi_{13}^{1,0} \wedge \phi_{23}^{0,1}$
$\phi_{23}^{1,0} \wedge \phi_{12}^{0,1}$	$-\phi_{123}^{1,0} \wedge \phi_{12}^{0,1}$
$\phi_{23}^{1,0} \wedge \phi_{13}^{0,1}$	$\phi_{123}^{1,0} \wedge \phi_{13}^{0,1}$
$\phi_2^{1,0} \wedge \phi_{123}^{0,1}$	$-\phi_{12}^{1,0} \wedge \phi_{123}^{0,1}$
$\phi_3^{1,0} \wedge \phi_{123}^{0,1}$	$\phi_{13}^{1,0} \wedge \phi_{123}^{0,1}$

Table 4.1: Image through ∂ of $\Delta_{\overline{\partial}}$-harmonic forms of X_0.

(p,q)	Basis for $E_2^{p,q}(X_0)$
$(0,0)$	1
$(1,0)$	$\phi_1^{1,0}$
$(0,1)$	$\phi_1^{0,1}$
$(2,0)$	$\phi_{23}^{1,0}$
$(1,1)$	$\phi_1^{1,0} \wedge \phi_1^{0,1}$, $\phi_2^{1,0} \wedge \phi_3^{0,1}$, $\phi_3^{1,0} \wedge \phi_2^{0,1}$
$(0,2)$	$\phi_{23}^{0,1}$
$(3,0)$	$\phi_{123}^{1,0}$
$(2,1)$	$\phi_{12}^{1,0} \wedge \phi_3^{0,1}$, $\phi_{13}^{1,0} \wedge \phi_2^{0,1}$, $\phi_{23}^{1,0} \wedge \phi_1^{0,1}$
$(1,2)$	$\phi_1^{1,0} \wedge \phi_{23}^{0,1}$, $\phi_2^{1,0} \wedge \phi_{13}^{0,1}$, $\phi_3^{1,0} \wedge \phi_{12}^{0,1}$
$(0,3)$	$\phi_{123}^{0,1}$
$(3,1)$	$\phi_{123}^{1,0} \wedge \phi_1^{0,1}$
$(2,2)$	$\phi_{12}^{1,0} \wedge \phi_{13}^{0,1}$, $\phi_{13}^{1,0} \wedge \phi_{12}^{0,1}$, $\phi_{23}^{1,0} \wedge \phi_{23}^{0,1}$
$(1,3)$	$\phi_1^{1,0} \wedge \phi_{123}^{0,1}$
$(3,2)$	$\phi_{123}^{1,0} \wedge \phi_{23}^{0,1}$
$(2,3)$	$\phi_{23}^{1,0} \wedge \phi_{123}^{0,1}$
$(3,3)$	$\phi_{123}^{1,0} \wedge \phi_{123}^{0,1}$

Table 4.2: Basis for $E_2^{p,q}(X_0)$.

Bi-degree	Δ''-Harmonic form
(0,0)	1
(1,0)	$\phi_1^{1,0}(t)$, $\phi_3^{1,0}(t)$
(2,0)	$\phi_{12}^{1,0}(t)$, $\phi_{13}^{1,0}(t)$
(3,0)	$\phi_{123}^{1,0}(t)$
(0,1)	$\phi_1^{0,1}(t)$, $\phi_2^{0,1}(t)$, $\phi_3^{0,1}(t)$
(1,1)	$\phi_1^{1,0}(t) \wedge \phi_2^{0,1}(t)$, $\phi_1^{1,0}(t) \wedge \phi_3^{0,1}(t)$, $\phi_2^{1,0}(t) \wedge \phi_1^{0,1}(t)$
	$\phi_3^{1,0}(t) \wedge \phi_1^{0,1}(t)$, $\phi_3^{1,0}(t) \wedge \phi_2^{0,1}(t)$, $\phi_3^{1,0}(t) \wedge \phi_3^{0,1}(t)$
(2,1)	$\phi_{12}^{1,0}(t) \wedge \phi_1^{0,1}(t)$, $\phi_{12}^{1,0}(t) \wedge \phi_2^{0,1}(t)$, $\phi_{12}^{1,0}(t) \wedge \phi_3^{0,1}(t)$ $\phi_{13}^{1,0}(t) \wedge \phi_2^{0,1}(t)$, $\phi_{13}^{1,0}(t) \wedge \phi_3^{0,1}(t)$, $\phi_{23}^{1,0}(t) \wedge \phi_3^{0,1}(t)$
(3,1)	$\phi_{123}^{1,0}(t) \wedge \phi_1^{0,1}(t)$, $\phi_{123}^{1,0}(t) \wedge \phi_2^{0,1}(t)$, $\phi_{123}^{1,0}(t) \wedge \phi_3^{0,1}(t)$
(0,2)	$\phi_{12}^{0,1}(t)$, $\phi_{13}^{0,1}(t)$, $\phi_{23}^{0,1}(t)$
(1,2)	$\phi_1^{1,0}(t) \wedge \phi_{23}^{0,1}(t)$, $\phi_2^{1,0}(t) \wedge \phi_{12}^{0,1}(t)$, $\phi_2^{1,0}(t) \wedge \phi_{13}^{0,1}(t)$ $\phi_3^{1,0}(t) \wedge \phi_{12}^{0,1}(t)$, $\phi_3^{1,0}(t) \wedge \phi_{13}^{0,1}(t)$, $\phi_3^{1,0}(t) \wedge \phi_{23}^{0,1}(t)$
(2,2)	$\phi_{12}^{1,0}(t) \wedge \phi_{12}^{0,1}(t)$, $\phi_{12}^{1,0}(t) \wedge \phi_{13}^{0,1}(t)$, $\phi_{12}^{1,0}(t) \wedge \phi_{23}^{0,1}(t)$ $\phi_{13}^{1,0}(t) \wedge \phi_{23}^{0,1}(t)$, $\phi_{23}^{1,0}(t) \wedge \phi_{12}^{0,1}(t)$, $\phi_{23}^{1,0}(t) \wedge \phi_{13}^{0,1}(t)$
(3,2)	$\phi_{123}^{1,0}(t) \wedge \phi_{12}^{0,1}(t)$, $\phi_{123}^{1,0}(t) \wedge \phi_{13}^{0,1}(t)$, $\phi_{123}^{1,0}(t) \wedge \phi_{23}^{0,1}(t)$
(0,3)	$\phi_{123}^{0,1}(t)$
(1,3)	$\phi_2^{1,0}(t) \wedge \phi_{123}^{0,1}(t)$, $\phi_3^{1,0}(t) \wedge \phi_{123}^{0,1}(t)$
(2,3)	$\phi_{12}^{1,0}(t) \wedge \phi_{123}^{0,1}(t)$, $\phi_{23}^{1,0}(t) \wedge \phi_{123}^{0,1}(t)$
(3,3)	$\phi_{123}^{1,0}(t) \wedge \phi_{123}^{0,1}(t)$

Table 4.3: $\Delta_{\overline{\partial}}$-Harmonic forms for X_t with $t \neq 0$.

(p,q)	Basis for $E_2^{p,q}(X_t)$
$(0,0)$	1
$(1,0)$	$\phi_1^{1,0}(t)$
$(0,1)$	$\phi_1^{0,1}(t)$
$(2,0)$	$\phi_{12}^{1,0}(t)$
$(1,1)$	$\phi_3^{1,0}(t) \wedge \phi_1^{0,1}(t)$, $\phi_3^{1,0}(t) \wedge \phi_2^{0,1}(t)$
$(0,2)$	$\phi_{12}^{0,1}(t)$, $\phi_{13}^{0,1}(t)$, $\phi_{23}^{0,1}(t)$
$(3,0)$	$\phi_{123}^{1,0}(t)$
$(2,1)$	$\phi_{12}^{1,0}(t) \wedge \phi_2^{0,1}(t)$, $\phi_{12}^{1,0}(t) \wedge \phi_3^{0,1}(t)$, $\phi_{13}^{1,0}(t) \wedge \phi_2^{0,1}(t)$
$(1,2)$	$\phi_1^{1,0}(t) \wedge \phi_{23}^{0,1}(t)$, $\phi_2^{1,0}(t) \wedge \phi_{13}^{0,1}(t)$, $\phi_3^{1,0}(t) \wedge \phi_{12}^{0,1}(t)$, $\phi_3^{1,0}(t) \wedge \phi_{13}^{0,1}(t)$
$(0,3)$	$\phi_{123}^{0,1}(t)$
$(3,1)$	$\phi_{123}^{1,0}(t) \wedge \phi_1^{0,1}(t)$, $\phi_{123}^{1,0}(t) \wedge \phi_2^{0,1}(t)$
$(2,2)$	$\phi_{12}^{1,0}(t) \wedge \phi_{13}^{0,1}(t)$, $\phi_{12}^{1,0}(t) \wedge \phi_{23}^{0,1}(t)$
$(1,3)$	$\phi_3^{1,0}(t) \wedge \phi_{123}^{0,1}(t)$
$(3,2)$	$\phi_{123}^{1,0}(t) \wedge \phi_{23}^{0,1}(t)$
$(2,3)$	$\phi_{23}^{1,0}(t) \wedge \phi_{123}^{0,1}(t)$
$(3,3)$	$\phi_{123}^{1,0}(t) \wedge \phi_{123}^{0,1}(t)$

Table 4.4: Basis for $E_2^{p,q}(X_t)$ with $t \neq 0$.

Bi-degree	b_k	$t=0$ $h^{p,q}(0)$	$e_2^{p,q}(0)$	$t\neq 0$ $h^{p,q}(t)$	$e_2^{p,q}(t)$
$(0,0)$	1	1	1	1	1
$(1,0)$	2	3	1	2	1
$(0,1)$		3	1	3	1
$(2,0)$	5	3	1	2	1
$(1,1)$		9	3	6	2
$(0,2)$		3	1	3	3
$(3,0)$	8	1	1	1	1
$(2,1)$		9	3	6	3
$(1,2)$		9	3	6	4
$(0,3)$		1	1	1	1
$(3,1)$	5	3	1	3	2
$(2,2)$		9	3	6	2
$(1,3)$		3	1	2	1
$(3,2)$	2	3	1	3	1
$(2,3)$		3	1	2	1
$(3,3)$	1	1	1	1	1

Table 4.5: Comparison between dimensions of $E_r^{\bullet,\bullet}(X_0)$ and $E_r^{\bullet,\bullet}(X_t)$, with $r=1,2$ and $t\neq 0$.

Bibliography

[AB90] L. Alessandrini and G. Bassanelli. Small deformations of a class of compact non-Kähler manifolds. *Proceedings of the American Mathematical Society*, 109(4):1059–1062, 1990.

[AB93] L. Alessandrini and G. Bassanelli. Metric properties of manifolds bimeromorphic to compact Kähler spaces. *J. Differential Geom.*, 37(1):95–121, 1993.

[AFR15] D. Angella, M. G. Franzini, and F. A. Rossi. Degree of non-Kählerianity for 6-dimensional nilmanifolds. *Manuscripta Math.*, 148(1-2):177–211, 2015.

[AK12] D. Angella and H. Kasuya. Bott-Chern cohomology of solvmanifolds. *arXiv preprint arXiv:1212.5708*, 2012.

[AS63] M. Atiyah and I. Singer. The index of elliptic operators on compact manifolds. *Bull. Amer. Math. Soc.*, 69(3):422–433, 1963.

[AT13] D. Angella and A. Tomassini. On the $\partial\overline{\partial}$-lemma and Bott-Chern cohomology. *Invent. Math.*, 192(1):71–81, 2013.

[BDPP04] S. Boucksom, J.P. Demailly, M. Paun, and T. Peternell. The pseudo-effective cone of a compact Kähler manifold and varieties of negative kodaira dimension. *arXiv preprint math/0405285*, 2004.

[Bel00] F. A. Belgun. On the metric structure of non-Kähler complex surfaces. *Math. Ann.*, 317(1):1–40, 2000.

[Bis89] J.-M. Bismut. A local index theorem for non Kähler manifolds. *Math. Ann.*, 284(4):681–699, 1989.

[BT13] R. Bott and L. Tu. *Differential forms in algebraic topology*, volume 82. Springer Science & Business Media, 2013.

[CFUG97] L. A. Cordero, M. Fernández, L. Ugarte, and A. Gray. A general description of the terms in the Frölicher spectral sequence. *Differential Geom. Appl.*, 7(1):75–84, 1997.

[COUV16] M. Ceballos, A. Otal, L. Ugarte, and R. Villacampa. Invariant complex structures on 6-nilmanifolds: classification, Frölicher spectral sequence and special Hermitian metrics. *J. Geom. Anal.*, 26(1):252–286, 2016.

[Dem97] J. P. Demailly. *Complex analytic and differential geometry.* Université de Grenoble I Grenoble, 1997.

[DGMS75] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan. Real homotopy theory of Kähler manifolds. *Invent. Math.*, 29(3):245–274, 1975.

[Dol53] P. Dolbeault. Sur la cohomologie des variétés analytiques complexes. *CR Acad. Sci. Paris*, 236(2):175–177, 1953.

[DP04] J.P. Demailly and M. Paun. Numerical characterization of the Kähler cone of a compact Kähler manifold. *Ann. of Math.*, pages 1247–1274, 2004.

[dR73] G. de Rham. Varietes differentiables, publ. inst. math. nancago iii. *Hermann, Paris*, 1973.

[ES$^+$93] M. Eastwood, M. Singer, et al. The Fröhlicher spectral sequence on a twistor space. *J. Diff. Geom.*, 38(3):653–669, 1993.

[Eva10] L. Evans. *Partial differential equations.* American Mathematical Society, 2010.

[FG04] A. Fino and G. Grantcharov. Properties of manifolds with skew-symmetric torsion and special holonomy. *Adv. Math.*, 189(2):439–450, 2004.

[FPS04] A. Fino, M. Parton, and S. Salamon. Families of strong KT structures in six dimensions. *Comment. Math. Helv.*, 79(2):317–340, 2004.

[Fri53] K. O. Friedrichs. On the differentiability of the solutions of linear elliptic differential equations. *Comm. Pure App. Math.*, 6(3):299–326, 1953.

[Frö55] A. Frölicher. Relations between the cohomology groups of Dolbeault and topological invariants. *Proc. Natl. Acad. Sci.*, 41(9):641–644, 1955.

[Gau97] P. Gauduchon. Hermitian connections and Dirac operators. *Boll. Unione Mat. Ital.*, (2):257–288, 1997.

[GH14] P. Griffiths and J. Harris. *Principles of algebraic geometry.* John Wiley & Sons, 2014.

[GHR84] S James Gates, Chris M Hull, and Martin Rocek. Twisted multiplets and new supersymmetric non-linear σ-models. *Nuclear Physics B*, 248(1):157–186, 1984.

[Hir62] H. Hironaka. An example of a non-Kahlerian complex-analytic deformation of Kahlerian complex structures. *Ann. Math.*, pages 190–208, 1962.

[HL83] R. Harvey and H.B. Lawson. An intrinsic characterization of Kähler manifolds. *Invent. Math.*, 74(2):169–198, 1983.

[Hör85] L. Hörmander. The analysis of linear partial differential operators., 1985.

[HP88] P.S. Howe and G Papadopoulos. Further remarks on the geometry of two-dimensional non-linear σ-models. *Classical Quantum Gravity*, 5(12):1647, 1988.

[Kod06] K. Kodaira. *Complex manifolds and deformation of complex structures.* Springer, 2006.

[KS60] K. Kodaira and D. C. Spencer. On deformations of complex analytic structures, iii. stability theorems for complex structures. *Ann. Math.*, pages 43–76, 1960.

[LUV14] A. Latorre, L. Ugarte, and R. Villacampa. On the Bott-Chern cohomology and balanced Hermitian nilmanifolds. *Internat. J. Math.*, 25(06):1450057, 2014.

[Mic82] M.L. Michelsohn. On the existence of special metrics in complex geometry. *Acta Math.*, 149(1):261–295, 1982.

[NN57] A. Newlander and L. Nirenberg. Complex analytic coordinates in almost complex manifolds. *Ann. Math.*, pages 391–404, 1957.

[Pop11] D. Popovici. Deformation openness and closedness of various classes of compact complex manifolds; examples. *arXiv preprint arXiv:1102.1687*, 2011.

[Pop13] Dan Popovici. Stability of strongly Gauduchon manifolds under modifications. *J. Geom. Anal.*, pages 1–7, 2013.

[Pop16] D. Popovici. Degeneration at E2 of certain spectral sequences. *Int. J. Math.*, 27(14):1650111, 2016.

[PU14] D. Popovici and L. Ugarte. The sGG class of compact complex manifolds. *arXiv preprint arXiv:1407.5070*, 2014.

[Rel30] F. Rellich. Ein satz über mittlere konvergenz. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1930:30–35, 1930.

[Rud87] W. Rudin. *Real and complex analysis.* Tata McGraw-Hill Education, 1987.

[Sch95] G. Schwarz. The hodge decomposition. In *Hodge Decomposition. A Method for Solving Boundary Value Problems*, pages 59–112. Springer, 1995.

[Sch07] M. Schweitzer. Autour de la cohomologie de Bott-Chern. *arXiv preprint arXiv:0709.3528*, 2007.

[Sho13] R. E. Showalter. *Monotone operators in Banach space and nonlinear partial differential equations*, volume 49. American Mathematical Soc., 2013.

[Sil] A. Silva. Closed positive currents and special metrics on compact complex manifolds, complex analysis and geometry (trento, 1993). *Lecture Notes in Pure and Applied Mathematics*, 173:377–441.

[Str86] A. Strominger. Superstrings with torsion. *Nuclear Phys B*, 274(2):253–284, 1986.

[TT14] A. Tomassini and S. Torelli. On Dolbeault formality and small deformations. *Int. J. Math.*, 25(11):1450111, 2014.

[Uga07] L. Ugarte. Hermitian structures on six-dimensional nilmanifolds. *Transform. Groups*, 12(1):175–202, 2007.

[Var] J. Varouchas. Propriétés cohomologiques d'une classe de variétés analytiques complexes compactes, séminaire d'analyse p. lelong-p. dolbeault-h. skoda, années 1983/1984. *Lect. Notes in Math*, 1198:233–243.

[Voi03] C. Voisin. *Hodge theory and complex algebraic geometry II*, volume 2. Cambridge University Press, 2003.

[WGP80] R. Wells and O. García-Prada. *Differential analysis on complex manifolds*, volume 21980. Springer Berlin, Heidelberg, New York, 1980.

[Yan65] K. Yano. *Differential geometry on complex and almost complex spaces*, volume 10. Pergamon press Oxford, 1965.