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Deux notions d'hyperbolicité en codimension complexe un pour les variétés complexes compactes

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Abstract

The goal of this thesis is to introduce two new notions of hyperbolicity for compact complex manifolds and two new notions of positivity for real De Rham cohomology classes of degree 2 on such manifolds. According to S. Kobayashi, a complex manifold, that need not be either Kähler or compact, is said to be hyperbolic if its Kobayashi pseudo-distance is a distance. Using the distance-decreasing property of holomorphic maps, one shows that any holomorphic map from the complex plane $\mathbb{C}$ to a Kobayashi -hyperbolic manifold is constant. Conversely, Brody proved that a compact complex manifold X is hyperbolic in the sense of Kobayashi if any holomorphic map from $\mathbb{C}$ to X is constant. The Kobayshi-Lang conjecture predicts that the canonical bundle K_X of any Kobayashi-hyperbolic compact complex manifold X ought to be ample. In particular, the manifold X would be projective ecao in this case by Kodaira's Embedding Theorem. On the other hand, Gromov introduced the notion of Kähler hyperbolicity for compact Kähler manifolds X. Whenever X is Kähler hyperbolic, he proved that X is also Kobayashi hyperbolic and its canonical bundle K_X is big. It is now known that K_X is even ample, hence X is projective, The main goal of this thesis is to propose a hyperbolicity theory for compact complex manifolds in which entire maps are replaced by non-degenerate holomorphic maps from $\mathbb{C}^{n-1}$ to n-dimensional compact complex manifolds X and differential forms of bidegree $(1,1)$ on X are replaced by $(n-1,n-1)$-forms. We start by introducing the notion of balanced hyperbolicity as a generalisation of Kähler hyperbolicity. We go on to introduce the notion of divisorial hyperbolicity as a generalisation of the Brody hyperbolicity. Our first main result asserts that any balanced hyperbolic compact complex manifold is also divisorially hyperbolic. We also introduce the notions of divisorially Kähler and divisorially nef De Rham cohomology classes of degree 2 and study their properties. They are intended to kickstart a positivity theory in bidegree $(n-1,n-1)$ for the compact complex n-dimensional manifolds that are hyperbolic in our two new senses. In particular, we conjecture that the canoncal bundle K_X of any such hyperbolic manifold X ought to be at least divisorially nef.
Keywords: Compact manifolds, Universal covering, Balanced metric, Kähler hyperbolicity, Brody hyperbolicity, Nef line bundel, Positivity, Hodge theory, Lefschetz theorem, Primitive forms, Cohomology, Balanced hyperbolicity, Divisorial hyperbolicity, Cohomology De Rahm class divisorially nef.

Résumé

Cette thèse est consacrée à l'introduction de deux notions nouvelles d'hyperbolicité pour les variétés complexes compactes lisses, ainsi qu'à l'introduction de deux notions nouvelles de positivité pour les classes de cohomologie de De Rham.

D'aprés S. Kobayashi, toute variété complexe, qui n'est a priori supposée ni kählérienne ni compacte, est appelée hyperbolique si sa pseudo-distance de Kobayashi est une distance. En utilisant la propriété de contraction des distances, on voit que toute application holomorphe du plan complexe $\mathbb{C}$ dans une variété Kobayashi-hyperbolique est constante. Réciproquement, Brody a démontré qu'une variété complexe compacte X est hyperbolique au sens de Kobayashi si toute application holomorphe de $\mathbb{C}$ dans X est constante. La conjecture de Kobayshi-Lang prédit que le fibré canonique K_X de toute variété complexe compacte Kobayashi-hyperbolique X devrait être ample. En particulier, grâce au théoréme de plongement de Kodaira, toute telle variété devrait être projective, donc aussi kählérienne.

M. Gromov a introduit en **1991** la notion de variété kählérienne hyperbolique en demandant l'existence d'une métrique kählérienne dont le relévement au revêtement universel est une forme différentielle d-exacte ayant un potentiel borné. Gromov montre, entre autres, que toute variété kählérienne hyperbolique est Kobayashi hyperbolique. De plus, il est maintenant connu que toute variété kählérienne hyperbolique au sens de Gromov est projective. En prenant comme point de départ l'observation de phénomènes d'hyperbolicité sur de nombreux exemples de variétés complexes compactes non kählériennes, l'objectif principal de cette thése est d'étendre la théorie de l'hyperbolicité au contexte non kählérien. Plus précisément, nous proposons une théorie dans laquelle les courbes entiéres sont remplacées par des applications holomorphes de $\mathbb{C}^{n-1}$ dans les variétés complexes n-dimensionnelles X et les formes différentielles de bidegré $(1,1)$ sur X sont remplacées par de telles formes de bidegré $(n-1,n-1)$. Ainsi, on commence par introduire la notion d'hyperbolicité équilibrée généralisant l'hyperbolicité kählérienne de Gromov au moyen des métriques équilibrées introduites par Gauduchon en **1977**. On généralise ensuite l'hyperbolicité au sens de Brody en introduisant la notion d'hyperbolicité divisorielle. Pour ce faire, nous dégageons une notion de croissance sous-exponentielle pour les applications holomorphes de $\mathbb{C}^{n-1}$ dans les variétés complexes compactes n-dimensionnelles X que nous considérons comme l'une de nos principales observations. Notre premier résultat principal affirme que toute variété équilibrée hyperbolique est divisoriellement hyperbolique. L'introduction de deux notions de positivité pour les classes de cohomologie de De Rham de degré 2, que nous appelons classes divisoriellement kählériennes et divisoriellement nef et dont nous étudions les propriétés de base, a pour but d'initier la construction d'une théorie de la positivité pour les variétés complexes compactes hyperboliques dans un ou l'autre de nos deux sens nouveaux introduits dans cette thèse. En particulier, nous conjecturons que le fibré canonique K_X de toute variété hyperbolique X devrait être au moins divisoriellement nef.
Mots clés: Variétés compactes, Revêtement universel, Métrique équilibrée, Hyperbolicité kählérienne, Brody hyperbolicité, Fibré en droite nef, Positivité, Théorie de Hodge, Théorème de Lefschetz, Forme primitive, Cohomologie, Hyperbolicité équilibrée, Hyperbolicité divisorielle, Classe de cohomologie de De Rahm divisoriellement nef.

General Introduction

The main objective of this work is to introduce and study two notions of hyperbolicity for not necessarily Kähler compact complex manifolds.

S. Kobayashi called a complex manifold X, that need not be either Kähler or compact, *hyperbolic* if the pseudo-distance he had introduced on X is actually a distance. Using the mapping decreasing proprety of this distance, one can show that every holomorphic map from the complex plane $\mathbb{C}$ to a *Kobayashi hyperbolic* manifold is *constant*. Conversely, Brody observed that a compact complex manifold X is Kobayashi hyperbolic if every holomorphic map from $\mathbb{C}$ to X is constant. The long standing Kobayashi-Lang conjecture predicts that, for a compact Kähler manifold X, if X is Kobayashi hyperbolic then its canonical bundle K_X is *ample*. In particular, the manifold X is *projective* in this case by Kodaira's embedding theorem.

Gromov introduced the notion of *Kähler hyperbolicty* for a compact Kähler manifold X. The manifold X is called *Kähler hyperbolic* if X admits a Kähler metric ω whose lift $\widetilde{\omega}$ to the universal cover $\tilde{X}$ of X can be expressed as

$$\widetilde{\omega} = d\alpha$$

for a *bounded* 1-form α on $\tilde{X}$. As pointed out by Gromov, it is not hard to see that the Kähler hyperbolicity implies the Kobayashi hyperbolicity.

The first result of this thesis generalizes this observation by Gromov to its analogue in dimension $n-1$ on a compact complex n-dimensional manifold X. To achieve this, we start by introducing two new notions of hyperbolicity suitable for this context.

The Kähler hyperbolicity is generalized to what we call **balanced hyperbolicity**. This is done by replacing the Kähler metric in the Kähler hyperbolicity by a *balanced metric*. Recall that a Hermitian metric on a complex n-dimensional manifold is called a balanced metric if $d\omega^{n-1} = 0$.

The Brody hyperbolicity is replaced by what we call **divisorial hyperbolicity**. A compact complex manifold X is called *divisorially hyperbolic* if there exists no non-trivial holomorphic map from $\mathbb{C}^{n-1}$ to X satisfying certain *subexponential volume growth condition*. The indispensability of this volume growth condition is one of our main findings in this thesis and emphasises the contrast between holomorphic maps from $\mathbb{C}$ and holomorphic maps from $\mathbb{C}^{n-1}$. In particular, the divisorial hyperbolicity allows the existence of a nondegenerate holomorphic map from $\mathbb{C}^{n-1}$ to X as long as this map does note have a subexponential volume growth.

We also introduce a notions of **divisorially Kähler** and **divisorially nef** De Rahm cohomology classes of degree 2 as generalizations of Kähler, respectively nef cohomology classes. We then go on to establish a number of basic properties of these notions and raise a number of questions.

This thesis is structured as follows.

In the **first chapter**, several basic definitions and useful results on different concepts that we need in the next two chapters are recalled. Outstanding details upon special Hermitian metrics are provided. Afterwardes, some basic definitions and pertinent results on forms and currents, as well as on the different concepts of positivity are displayed. Subsequently, we address briefly Kähler hyperbolicity in the sense of Gromov and Brody's hyperbolicity and we close this chapter with some reminders on simple and semi-simple Lie algebras and Lie groups.

In the **second chapter**, two notions of hyperbolicity for not necessarily Kähler n-dimensional compact complex manifolds X are introduced. The first, called *balanced hyperbolicity*, generalises Gromov's Kähler hyperbolicity by means of Gauduchon's balanced metrics. The second, called *divisorial hyperbolicity*, generalises the Brody hyperbolicity by ruling out the existence of non-degenerate holomorphic maps from $\mathbb{C}^{n-1}$ to X having what we call a subexponential growth. Our main result in the first part of this chapter asserts that every balanced hyperbolic X is also divisorially hyperbolic, and therfore the following implication holds:

$$
\begin{array}{lcl}
X \text{ is } \textbf{Kähler hyperbolic} & \Longrightarrow & X \text{ is } \textbf{Kobayashi/Brody hyperbolic} \\
\Downarrow & & \Downarrow \\
X \text{ is } \textbf{balanced hyperbolic} & \Longrightarrow & X \text{ is } \textbf{divisorially hyperbolic} \\
\Uparrow & & \\
X \text{ is } \textbf{degenerate balanced} & &
\end{array}
$$

where our main result proves the bottom horizontal implication. Next, we provide certain expressive examples of Kähler and no Kähler manifold that are balanced hyperbolic and hence divisorial hyperbolic, in addition to some examples which are not divisorial hyperbolic. Further more, we identify and describe various properties of these manifolds.

The second part of this chapter is devoted to introducing notions of divisorially Kähler and nef classes enacting an analogy with Kähler and nef classes in complex codimension 1.

We elaborate two definitions of projectively divisorially and divisorially nef coho-

mology class, in the case of projective manifolds and in the case of arbitrary compact complex manifolds, where in the particular case of projective manifold, a class $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$ is projectively divisorially nef whenever $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$ is divisorially nef.
In the projective case, *A cohomology class $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$ is said to be* **projectively divisorially nef** *if*

$$P(\{\alpha\}_{DR}).\{[D]\}_{BC} := \int\limits_D (\alpha^{n-1})^{n-1,\,n-1} \geq 0$$

for all effective **divisors** *$D \geq 0$ on X and some (hence any) representative $\alpha \in C^\infty_2(X, \mathbb{R})$ of $\{\alpha\}_{DR}$*, where the map:

$$P = P^{n-1}_{n-1,\,n-1} : H^2_{DR}(X, \mathbb{R}) \longrightarrow H^{n-1,\,n-1}_A(X, \mathbb{R}), \qquad \{\alpha\}_{DR} \longmapsto \{(\alpha^{n-1})^{n-1,\,n-1}\}_A,$$

is **well defined** in the sense that it is independent of the choice of a C^∞ representative α of its De Rham cohomology class, where $(\alpha^{n-1})^{n-1,\,n-1}$ is the component of bidegree $(n-1,\, n-1)$ of the $(2n-2)$-form α^{n-1}.
In an arbitrary compact complex n-dimensional manifolds, *a cohomology class $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$ is said to be* **divisorially nef** *if $P(\{\alpha\}_{DR}) \in \overline{\mathcal{G}}_X$, where $\overline{\mathcal{G}}_X$ is the closure of the Gauduchon cone in $H^{n-1,\,n-1}_A(X, \mathbb{R})$ that was defined in [Pop15a] as:*

$$\mathcal{G}_X := \left\{ \{\omega^{n-1}\}_A \in H^{n-1,\,n-1}_A(X, \mathbb{R}) \mid \omega \textit{ is a Gauduchon metric on } X \right\} \subset H^{n-1,\,n-1}_A(X, \mathbb{R}).$$

It will be said **divisorially Kähler** *if $P(\{\alpha\}_{DR}) \in \mathcal{G}_X$.*
We obtain an altenative definition of nef class provided in [Dem92] as the following:
A class $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$ is **divisorially nef** *if and only if for every constant $\varepsilon > 0$, there exists a representative $\Omega_\varepsilon \in C^\infty_{n-1,\,n-1}(X, \mathbb{R})$ of the class $P(\{\alpha\}_{DR})$ such that*

$$\Omega_\varepsilon \geq -\varepsilon\, \omega^{n-1},$$

where $\omega > 0$ is an arbitrary Hermitian metric on a compact complex n-dimensional manifold X fixed beforehand. Therfore, we establish a number of basic properties thereof, and raise a number of questions.

In the **third chapter**, we proceed the study of hyperbilic balanced manifolds. Indeed, we corroborate several vanishing theorems for the cohomology of balanced hyperbolic manifolds that we have already introduced in the previous chapter and for the L^2 harmonic spaces on the universal cover of these manifolds. Other prominent results involve a Hard Lefschetz-type theorem for certain compact complex balanced manifolds and the non-existence of certain L^1 currents on the universal covering space of a balanced hyperbolic manifold. We obtained a **Hard Lefschetz Isomorphism** between the De Rahm cohomologies of degree 1 and $2n-1$. Our first main result in this section is the following:

Theorem 0.0.1. Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.

(i) If ω is a **balanced** *metric on X, the linear map:*

$$\{\omega_{n-1}\}_{DR} \wedge \cdot : H^1_{DR}(X, \mathbb{C}) \longrightarrow H^{2n-1}_{DR}(X, \mathbb{C}), \quad \{u\}_{DR} \longmapsto \{\omega_{n-1} \wedge u\}_{DR} \tag{1}$$

is well defined and depends only on the cohomology class $\{\omega_{n-1}\}_{DR} \in H^{2n-2}_{DR}(X, \mathbb{C})$.

(ii) If, moreover, X has the following additional property: for every form $v \in C^\infty_{1,1}(X, \mathbb{C})$ such that $dv = 0$, the following implication holds:

$$v \in Im\partial \implies v \in Im(\partial\bar{\partial}),$$

the map (1) is an **isomorphism**.

Afterwards, the central focus of our work is upon a few studies on the universal cover of balanced manifolds where we obtain vanishing theorems for the L^2 harmonic cohomology on the univesal cover. Our first main result in degree 1 and the dual $2n-1$ is expressed as follows:

Theorem 0.0.2. *Let X be a compact complex* **balanced hyperbolic** *manifold with $\dim_{\mathbb{C}} X = n$. Let $\pi : \widetilde{X} \longrightarrow X$ be the universal cover of X and $\widetilde{\omega} := \pi^\star\omega$, the lift to $\widetilde{X}$ of a balanced hyperbolic metric ω on X.*

There are no non-zero $\Delta_{\widetilde{\omega}}$-harmonic $L^2_{\widetilde{\omega}}$-forms of pure types and of degrees 1 and $2n-1$ on $\widetilde{X}$:

$$\mathcal{H}^{1,0}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C}) = \mathcal{H}^{0,1}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C}) = 0 \quad \text{and} \quad \mathcal{H}^{n,n-1}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C}) = \mathcal{H}^{n-1,n}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C}) = 0,$$

where $\Delta_{\widetilde{\omega}} := dd^\star_{\widetilde{\omega}} + d^\star_{\widetilde{\omega}} d$ is the d-Laplacian induced by the metric $\widetilde{\omega}$.

The second main result in this section in degree 2 is indicated as follows:

Theorem 0.0.3. *Let X be a compact complex* **balanced hyperbolic** *manifold with $\dim_{\mathbb{C}} X = n$. Let $\pi : \widetilde{X} \longrightarrow X$ be the universal cover of X and $\widetilde{\omega} := \pi^\star\omega$, the lift to $\widetilde{X}$ of a balanced hyperbolic metric ω on X.*

There are no non-zero semi-positive $\Delta_{\widetilde{\tau}}$-harmonic $L^2_{\widetilde{\omega}}$-forms of pure type $(1, 1)$ on $\widetilde{X}$:

$$\left\{\alpha^{1,1} \in \mathcal{H}^{1,1}_{\Delta_{\widetilde{\tau}}}(\widetilde{X}, \mathbb{C}) \mid \alpha^{1,1} \geq 0\right\} = \{0\},$$

where $\widetilde{\tau} = \widetilde{\tau}_{\widetilde{\omega}} := [\Lambda_{\widetilde{\omega}}, \partial\widetilde{\omega} \wedge \cdot]$.

Eventually, the closing part of our thesis displays an appendix chapter which incorporates some promising results which shall be invested in the third chapter.

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Contents

Chapter 1

Preliminaries

1.1 Special Hermitian Metrics on Compact Complex Manifolds

1.1.1 Hermitian metrics

Definition 1.1.1. *A* ***Hermitian metric*** ω *on* X *is a family* $(\langle\cdot,\cdot\rangle_{\omega(x)})_{x\in X}$*, where, for every point* $x \in X$*,*

$$\langle\cdot,\cdot\rangle_{\omega(x)} : T_x^{1,0}X \times T_x^{1,0}X \to \mathbb{C}$$

is an inner product on the holomorphic tangent space to X *at* x*, such that the inner products* $\langle\cdot,\cdot\rangle_{\omega(x)}$ *depend in a* C^∞ *way on* $x \in X$*.*

By an inner product on a $\mathbb{C}$-vector space we mean a positive definite sesquilinear map. It isstandard that any Hermitian metric ω on X identifies canonically with a unique C^∞ $(1,1)$-form ω (denoted henceforth by the same letter) that is positive definite at every point $x \in X$. Some authors call it the Kähler form associated with the Hermitian metric and denote these two objects differently, but we will not use this terminology. In local holomorphic coordinates $z_1, ..., z_n$ on some open subset $U \subset X$, any such object is of the shape

$$\omega = \sum_{j,k}^{n} \omega_{j,\bar{k}} i dz_j \wedge d\bar{z}_k, \qquad (1,1)$$

where the coefficients $\omega_{j,\bar{k}} : U \to \mathbb{C}$ are C^∞ functions such that the matrix $(\omega_{j,\bar{k}}(x))_{j,\bar{k}}$ is positive definite (equivalently, its eigenvalues are all positive) at every point $x \in X$.

In fact, an equivalent definition for a Hermitian metric ω on X is as a family $(\omega^{(\alpha)})_{\alpha\in\Lambda})$ of locally defined, positive definite C^∞ forms $\omega^{(\alpha)}$, defined respectively by the analogues of (1.1) on open coordinate subsets $U_\alpha \subset X$ that cover X, such that $\omega^{(\alpha)} = \omega^{(\beta)}$ on $U_\alpha \cap U_\beta$ whenever this intersectionis non-empty. In particular, Hermitian metrics always exist on any given X. Indeed, take any open cover of X by coordinate patches, take any locally defined Hermitian metrics on these patches and glue them together into a global Hermitian metric on X using a partition of unity.

1.1.2 Gauduchon metrics

Definition 1.1.2. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. A C^∞ positive definite $(1,1)$-form ω on X is said to be a Gauduchon metric if $\partial\overline{\partial}\omega^{n-1} = 0$.*

The fundamental fact of life about Gauduchon metrics is the following statement that is a special case of Gauduchon's main result in *[Gau77]*

Theorem 1.1.3. *Every compact complex manifold carries Gauduchon metrics.*

1.1.3 Kähler metrics and manifolds

Definition 1.1.4. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. A C^∞ positive definite $(1,1)$-form ω on X is said to be a Kähler metric if $d\omega = 0$. If X carries such a metric, X is said to be a **Kähler manifold**.*

1.1.4 Balanced metrics and manifolds

The notion that will be discussed in this section was introduced by Gauduchon in *[Gau77]* under the name of semi-Kähler metric. These metrics were renamed balanced by Michelsohn in *[Mic83]* and this latter terminology is now widely used in the literature.

Definition 1.1.5. ([Gau77], [Mic83]) *Let X be a complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$.*

1. *A C^∞ positive definite $(1,1)$-form ω on X is said to be a balanced metric if $d\omega^{n-1} = 0$.*

2. *If X carries such a metric, X is said to be a **balanced manifold**.*

Obviously, when $n = 2$, balanced metrics coincide with Kähler metric.

Lemma 1.1.6. *Let ω be a Hermitian metric on a complex manifold X with $\dim_{\mathbb{C}} X = n \geq 2$.*

1. *If ω is Kähler, then ω is balanced.*

2. *The metric ω is balanced if and only if it is co-closed. Specifically, the following equivalences hold:*

$$\omega \text{ is balanced } \Leftrightarrow d^\star\omega = 0 \Leftrightarrow \partial^\star\omega = 0 \Leftrightarrow \overline{\partial}^\star\omega = 0$$

Theorem 1.1.7. *Let X be a complex n-dimensional manifold, and let ω be C^∞ positive definite $(1,1)$-form on X. Then ω is a balanced metric if and only if, for every $x \in X$, there is a holomorphic coordinate system $(z_1, ..., z_n)$ centered at x such that*

$$\omega^{n-1} = \sigma_{n-1} \sum_{i,j=1}^{n} a_{i,\bar{j}} \, \widehat{dz_i} \wedge \widehat{d\overline{z_j}}$$

with

(i) $a_{i,\bar{j}}(0) = \delta_{i,j}$

(ii) $a_{i,\bar{j}}(z)$ *does not contain linear terms involving* $z_i, \quad z_j, \quad \overline{z_i}, \quad \overline{z_j}$

(iii) $d(tr(a_{i,\bar{j}})(0) = 0.$

1.1.5 SKT metrics and manifolds

The definition of SKT metric was introduced by Streets and Tian.

Definition 1.1.8. *A* C^∞ *positive definite* $(1,1)$*-form* ω *on a complex* n*-dimensional manifold* X *is said to be a* ***SKT*** *metric if* $\partial\overline{\partial}\omega = 0$. *If* X *carries such a metric,* X *is said to be a* ***SKT manifold****.*

Proposition 1.1.9. *If* ω *is both* SKT *and balanced, then* ω *is Kähler.*

Conjecture 1.1.10. ***(Streets and Tian)*** *A compact complex manifold cannot admit a balanced metric and an* SKT *metric unless it is a Kähler manifold.*

1.2 Basic Concepts of Positivity

1.2.1 Positive and Strongly Positive Forms

Let V be a complex vector space of dimension n and $(z_1, ..., z_n)$ coordinates on V. We denote by $(\partial/\partial z_1, ..., \partial/\partial z_n)$ the corresponding basis of V, by $(dz_1, ..., dz_n)$ its dual basis in $V^\star$ and consider the exterior algebra

$$\Lambda V^\star_{\mathbb{C}} = \bigoplus \Lambda^{p,q} V^\star, \qquad \Lambda^{p,q} V^\star = \Lambda^p V^\star \otimes \Lambda^q \overline{V^\star}$$

We are of course especially interested in the case where $V = T_x X$ is the tangent space to a complex manifold X, but we want to emphasize here that our considerations only involve linear algebra. Let us first observe that V has a canonical orientation, given by the $(n,n)-$form

$$\tau(z) = i dz_1 \wedge d\overline{z_1} \wedge ... \wedge i dz_n \wedge d\overline{z_n} = 2^n dx_1 \wedge dy_1 \wedge ... \wedge dx_n \wedge dy_n$$

where $z_j = x_j + iy_j$. In fact, if $(w_1, ..., w_n)$ are other coordinates, we find

$$\begin{aligned} dw_1 \wedge \ldots \wedge dw_n &= det(\partial w_j/\partial z_k) dz_1 \wedge \ldots \wedge dz_n, \\ \tau(w) &= |det(\partial w_j/\partial z_k)|^2 \tau(z) \end{aligned}$$

In particular, a complex manifold always has a canonical orientation. More generally, natural positivity concepts for $(p,p)-$forms can be defined.

Definition 1.2.1. *A* $(p,p)-$*forms* $u \in \Lambda^{p,p} V^\star$ *is said to be positive if for all* $\alpha_j \in V^\star, \quad 1 \le j \le q = n-p$, *then*

$$u \wedge i\alpha_1 \wedge \overline{\alpha_1} \wedge \ldots \wedge i\alpha_q \wedge \overline{\alpha_q}$$

is a positive $(n,n)-$*form. A* $(q,q)-$*form* $v \in \Lambda^{p,q}V^\star$ *is said to be strongly positive if* v *is a convex combination*

$$V = \sum \gamma_s i\alpha_{s,1} \wedge \overline{\alpha}_{s,1} \wedge \ldots \wedge i\alpha_{s,q} \wedge \overline{\alpha}_{s,q}$$

where $\alpha_{s,j} \in V^\star$ *and* $\gamma_s \geq 0$.

Example 1.2.2. *Since* $i^p(-1)^{p(p-1)/2} = i^{p^2}$*, we have the commutation rules*

$$\begin{aligned} i\alpha_1 \wedge \overline{\alpha_1} \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha_p} &= i^{p^2}\alpha \wedge \overline{\alpha}, \qquad \forall \alpha = \alpha_1 \wedge \ldots \wedge \alpha_p \in \Lambda^{p,0}V^\star, \\ i^{p^2}\beta \wedge \overline{\beta} \wedge i^{m^2}\gamma \wedge \overline{\gamma} &= i^{(p+m)^2}\beta \wedge \gamma \wedge \overline{\beta \wedge \gamma}, \qquad \forall \beta \in \Lambda^{p,0}V^\star, \quad \gamma \in \Lambda^{m,0}V^\star. \end{aligned}$$

Take $m = q$ *to be the complementary degree of* p*. Then* $\beta \wedge \gamma = \lambda dz_1 \wedge \ldots \wedge dz_n$ *for some* $\lambda \in \mathbb{C}$ *and* $i^{n^2}\beta \wedge \gamma \wedge \overline{\beta \wedge \gamma} = |\lambda|^2\tau(z)$*. If we set* $\gamma = \alpha_1 \wedge \ldots \wedge \alpha_q$*, we find that* $i^{n^2}\beta \wedge \overline{\beta}$ *is a positive* (p,p)*-form for every* $\beta \in \Lambda^{p,0}V^\star$ *; in particular, strongly positive forms are positive.*

The sets of positive and strongly positive forms are closed convex cones, i.e.closed and stable under convex combinations. By definition, the positive cone is dual to the strongly positive cone via the pairing

$$\begin{aligned} \Lambda^{p,p}V^\star \times \Lambda^{q,q}V^\star &\longrightarrow \mathbb{C} \\ (u,v) &\longmapsto u \wedge v/\tau. \end{aligned} \tag{1.1}$$

that is, $u \in \Lambda^{p,q}V^\star$ is positive if and only if $u \wedge v$ all strongly positive forms $v \in \Lambda^{p,q}V^\star$. Since the bidual of an arbitrary convex cone Γ is equal to its closure $\overline{\Gamma}$, we also obtain that v is strongly positive if and only if $v \wedge u = u \wedge v$ is ≥ 0 for all positive forms u. Later on, we will need the following elementary lemma.

Lemma 1.2.3. *Let* $(z_1, \ldots, z_n)$ *be arbitrary coordinates on* V*. Then admits* $\Lambda^{p,q}V^\star$ *a basis consisting of strongly positive forms*

$$\beta_s = i\beta_{s,1} \wedge \overline{\beta}_{s,1} \wedge \ldots \wedge i\beta_{s,p} \wedge \overline{\beta}_{s,p} \qquad 1 \leq s \leq (\mathcal{C}_p^n)^2$$

where each $\beta_{s,1}$ *is of the type* $dz_j \pm dz_k$ *or* $dz_j \pm idz_k$*,* $1 \leq j,k \leq n$.

Remarks 1.2.4. *1. All positive forms* u *are real, i.e. satisfy* $u = \overline{u}$*. In terms of coordinates, if* $u = i^{p^2}\sum_{|I|=|J|=p} u_{I,J}dz_I \wedge d\overline{dz}_J$*, then the coefficients satisfy the hermitian symmetry relation* $\overline{u}_{I,J} = u_{I,J}$.

2. *A from* $u = i\sum_{j,k} u_{jk}dz_I \wedge d\overline{z}_J$ *of bidegree* $(1,1)$ *is positive if and only if* $\xi \mapsto \sum u_{jk}\xi_j\overline{\xi}_k$ *is a semi-positive hermitian form on* $\mathbb{C}$.

3. *The notions of positive and strongly positive* (p,p)*-forms coincide for* $p = 0, 1, n-1, n$.

4. *If* $u_1, \ldots, u_s$ *are positive forms, all of them strongly positive (resp. all except perhaps one), then* $u_1 \wedge \ldots \wedge u_s$ *is strongly positive (resp. positive).*

5. *if* $\Phi : W \longrightarrow V$ *is a complex linear map and* $u \in \Lambda^{p,q}V^\star$ *is (strongly) positive, Then* $\Phi^\star u \in \Lambda^{p,q}W^\star$ *is (strongly) positive.*

1.2.2 Positive Currents

The duality between the positive and strongly positive cones of forms can be used to define corresponding positivity notions for currents.

Definition 1.2.5. *A current $T \in \mathcal{D}'_{p,p}(X)$ is said to be positive (resp. strongly positive) if $\langle T, u\rangle \geq 0$ for all test forms $u \in \mathcal{D}_{p,p}(X)$ that are strongly positive (resp. positive) at each point.*

Another way of stating Definition 1.2.5 is: T is positive (strongly positive) if and only if $T \wedge u \in \mathcal{D}'_{p,p}(X)$ is a positive measure for all strongly positive (positive) forms $u \in \mathcal{C}^{\infty}_{p,p}(X)$. This is so because a distribution $S \in \mathcal{D}'(X)$ such that $S(f) \geq 0$ for every non-negative function $f \in \mathcal{D}(X)$ is a positive measure

Proposition 1.2.6. *Every positive current $T = i^{(n-p)^2} \sum T_{I,J} dz_I \wedge d\overline{z}_J$ in $\mathcal{D}'^{+}_{p,p}(X)$ is real and of order 0, i.e. its coefficients $T_{I,J}$ are complex measures and satisfy $\overline{T_{I,J}} = T_{J,I}$ for all multi-indices $|I| = |J| = n - p$. Moreover $T_{I,I} \geq 0$, and the absolute values $|T_{I,J}|$ of the measures $T_{I,J}$ satisfy the inequality*

$$\lambda_I \lambda_J |T_{I,J}| \leq 2^p \sum_m \lambda_M^2 T_{M,M}, \qquad I \cap J \subset M \subset I \cup J$$

where $\lambda_k \geq 0$ are arbitrary coefficients and $\lambda_I = \prod_{k\in I} \lambda_k$.

Example 1.2.7. *Let X be a complex manifold and $u \in Psh(X) \cap L^1_{loc}(X)$ a plurisubharmonic function then,*

$$T = i\partial\overline{\partial} u = i \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} dz_j \wedge d\overline{z}_k$$

is a positive current of bidegree $(1,1)$.

1.2.3 Hermitian Vector Bundles and Connections

Let X be a C^∞ differentiable manifold of dimension n and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ be the scalar field. A (real, complex) vector bundle of rank r over X is a C^∞ manifold E together with

i) a C^∞ map $\pi : E \to X$ called the projection,

ii) a $\mathbb{K}$-vector space structure of dimension r on each fiber $E_x = \pi^{-1}(x)$

such that there exists an open covering $(V_\alpha)_{\alpha\in I}$ of X and C^∞ diffeomorphisms called trivializations

$$\phi_\alpha : E_{|V_\alpha} \to V_\alpha \times \mathbb{K}, \text{ where } E_{|V_\alpha} = \pi^{-1}(V_\alpha)$$

such that for every $x \in V_\alpha$ the map $\phi_\alpha : E_x \to \{x\} \times \mathbb{K} \to \mathbb{K}$ is a linear isomorphism.

A holomrophic vector bundle is a complex vector bundle over a complex manifold X such that the totale space E is a complex manifold, the projective mup $\pi : E \to X$ is holomorphic, there exists an open covering $(V_\alpha)_{\alpha\in I}$ of X and a family of holomorphic trivializations $\phi_\alpha : E_{|V_\alpha} \to V_\alpha \times \mathbb{C}^r$.

Example 1.2.8. *if $\varphi_\alpha : V_\alpha \to X$ is a collection of coordinate charts on X, then $\phi_\alpha = \pi \times d\phi_\alpha : TX_{|V\alpha} \to V_\alpha \times \mathbb{R}$ define trivializations of the tangent bundle TX and the transition matrices are given by $g_{\alpha\beta}(x) = d\varphi_{\alpha\beta}(x^\beta)$ where $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ and $x^\beta = \varphi_\beta(x)$. The dual $T^\star X$ of TX is called the cotangent bundle and the p-th exterior power $\Lambda^p T^\star X$ is called the bundle of differential forms of degree p on X.*

A section of a vector bundle $\pi : E \to X$ is a map $s : X \to E$ such that $\pi \circ s = Id_X$. This section is said to be continuous, resp. differentiable, resp. C^k differentiable, if s is continuous, resp. differentiable, resp.

A holomorphic section of a holomorphic vector bundle $\pi : E \to X$ over an open set V of X is a section $s : X \to E$ of π which is a holomorphic map. For example, a holomorphic local trivialisation ϕ of E as above is given by the choice of a family of holomorphic sections of E, whose values at each point x of V_j form a basis of the fibre E_x over $\mathbb{C}$.

Definition 1.2.9. *A (real, complex, holomorphic) line bundle is a (real, complex, holomorphic) vector bundle of rank* 1.

Let $\phi : E_{|V} \to V \times \mathbb{K}$ be a trivialization of $E_{|V}$. To ϕ, we associate the $\mathbb{C}^\infty$ frame $(e_1, ..., e_r)$ of $E_{|V}$ defined by

$$e_\lambda(x) = \phi^{-1}(x, \xi_\lambda), \qquad x \in V,$$

where (ξ_λ) is the standard basis of $\mathbb{K}$. A section $s \in \mathbb{C}^k(V, E)$ can then be represented in terms of its components $\phi(s) = \sigma = (\sigma_1, ..., \sigma_r)$ by

$$s = \sum_{\lambda=1}^{r} \sigma_\lambda e_\lambda \qquad \text{on } V \qquad \sigma_\lambda \in \mathbb{C}^k(V, \mathbb{K}).$$

Let ϕ_α be a family of trivializations relative to a covering (V_α) of X. Given a global section $s \in C^\infty(E, X)$, the components $\phi_\alpha(s) = \sigma^\alpha = (\sigma_1^\alpha, ..., \sigma_r^\alpha)$ satisfy the transition relations

$$\sigma^\alpha = g_{\alpha\beta}\sigma^\beta \qquad \text{on } V_\alpha \cap V_\beta.$$

Conversely, any collection of vector valued functions $\sigma^\alpha : V_\alpha \to \mathbb{K}^r$ satisfying the transition relations defines a global section s of E.

A complex vector bundle E is said to be hermitian if a positive definite hermitian form $|\ \ |^2$ is given on each fiber E_x in such a way that the map $E \to \mathbb{R}^+$, $\xi \to |\xi|^2$ is smooth. Let $\phi : E_{|V} \to V \times \mathbb{C}^r$ be a trivialization and let $(e_1, ..., e_r)$ be the corresponding frame of $E_{|V}$. The associated inner product of E is given by a positive definite hermitian matrix $(h_{\lambda\mu})$ with C^∞ coefficients on V, such that

$$\langle e_\lambda(x), e_\mu(x) \rangle = h_{\lambda_\mu}(x), \forall x \in V.$$

Definition 1.2.10. *A (linear) connection D on the bundle E is a linear differential operator of order* 1 *acting on $\mathbb{C}^\infty_\bullet(X, E)$ and satisfying the following properties:*

$$D : C^\infty_q(X, E) \to C^\infty_{q+1}(X, E) \tag{1.2}$$

$$D(f \wedge s) = df \wedge s + (-1)^p f \wedge Ds \tag{1.3}$$

for any $f \in C^\infty_p(X, \mathbb{K})$ and $s \in C^\infty_q(X, E)$, where df stands for the usual exterior derivative of f.

The curvature of a connection D on $E \to X$ is a 2-forme $\Theta(D) \in C_2^\infty(X, Hom(E,E))$ such that

$$D^2 s = \Theta(D) \wedge s.$$

given with respect to any trivialization ϕ by

$$\Theta(D) \simeq_\phi dA + A \wedge A$$

where $A = (a_{\lambda\mu}) \in C_1^\infty(V, Hom(\mathbb{K}, \mathbb{K}^r))$.

Let X be a complex manifold, $\dim_{\mathbb{C}} X = n$ and E a C^∞ vector bundle of rank r over X, a connection of type $(1,0)$ on E is a differential operator D' of order 1 acting on $\mathbb{C}^\infty_{\bullet,\bullet}(X,E)$ and satisfying the following two properties:

$$D' : C^\infty_{p,q}(X,E) \to C^\infty_{p+1,q}(X,E) \tag{1.4}$$
$$D'(f \wedge s) = \partial f \wedge s + (-1)^{degf} f \wedge \partial s \tag{1.5}$$

for any $f \in C^\infty_{p_1,q_1}(X,C)$, $\quad s \in C^\infty_{p_2,q_2}(X,E)$. The definition of a connection D'' of type $(0,1)$ is similar.

Proposition 1.2.11. *Let D_0'' be a given $(0,1)$-connection on a hermitian bundle $\pi : E \to X$. Then there exists a unique hermitian connection $D = D' + D''$ such that $D_0'' = D''$*

The unique hermitian connection D such that $D'' = \overline{\partial}$ is called the Chern connection of E. The curvature tensor of this connection will be denoted by $\Theta(E)$ and is called the Chern curvature tensor of E.

Theorem 1.2.12. *Let X be an arbitrary complex manifold.*

1. *For any hermitian line bundle E over X, the Chern curvature form $\frac{i}{2\pi}\Theta(E)$ is a closed real (1,1)-form whose De Rham cohomology class is the image of an integral class.*

1.2.3.1 Positivity Concepts for Vector Bundles

Definition 1.2.13. *Let E be a hermitian holomorphic vector bundle of rank r over X, where $\dim_{\mathbb{C}} X = n$.*

1. *E is said to be Nakano positive (resp. Nakano semi- negative) if $\Theta(E)$ is positive definite (resp. semi-negative) as a hermitian form on $TX \otimes E$, i.e. if for every $u \in TX \otimes E$, $u \neq 0$, we have*

$$\Theta(E)(u,u) > 0 \quad \textit{(resp. } \leq 0).$$

2. *E is said to be Griffiths positive (resp. Griffiths semi-negative) if for all $\xi \in T_xX, \xi \neq 0$ and $s \in E_x, s \neq 0$ we have*

$$\Theta(E)(\xi \otimes s, \xi \otimes s) > 0 \quad \textit{(resp. } \leq 0).$$

It is clear that Nakano positivity implies Griffiths positivity and that both concepts coincide if r = 1. In the case of a line bundle, E is merely said to be positive.

Definition 1.2.14. *Let X be a compact complex manifold with a fixed hermitian metric ω. A line bundle L over X is nef if for every $\varepsilon > 0$ there exists a smooth hermitian metric h_ε on L such that the curvature satisfies*

$$\Theta_{h_\varepsilon} \geq -\varepsilon\omega.$$

Clearly a nef line bundle L satisfies $L.C \geq 0$ for all curves $C \subset X$, but the converse is not true. For projective algebraic X both notions coincide.

1.3 Brody's Criterion for Hyperbolicity and Kähler Hyperbolicity in the sense of Gromove

1.3.1 Brody's and Kobayashi's Hyperbolicity

Let X be a complex manifold. We shall say that X is Brody hyperbolic if every holomorphic map $f : \mathbb{C} \to X$ is constant. Similarly, if X is a subset of a complex manifold Y, we say that X is Brody hyperbolic in Y (or relative to Y) if every holomorphic map $f : \mathbb{C} \to Y$ whose image is contained in X is constant.

Lemma 1.3.1. *(Brody's reparametrization lemma). Let X be a subset of a complex manifold with a length function. Let*

$$f : D_r \to X$$

be holomorphic. Let $c > 0$, and for $0 \leq t \leq 1$ let

$$f_t(z) = f(tz)$$

1. *If $|df(0)| > c$ then there exists $t < 1$ and an automorphism h of D such that if we put*

   $$g = f_t \circ h$$

   then

   $$sup_{z \in D_r} |dg(z)| = |dg(0)| = c$$

2. *If $|df(0)| = c$ then we get the same conclusion allowing $t \leq 1$.*

Proof. Let $m_t : D_r \to D_r$ be multiplication by t, so that f_t can be factored

$$D_r \to^{m_t} D_r \to^f X$$

Then $dm_t(z)v = tv$ so

$$|df_t(z)| = |df(tz)| t \frac{1 - |z|^2/r^2}{1 - |tz|^2/r^2}$$

Let

$$s(t) = sup_{z \in D_r} |df_t(z)|$$

Note that if $t < 1$ then $|df_t(z)| \to 0$ for $|z| \to r$ so $|df_t(z)|$ has a maximum for $z \in D_r$ and thus for $t < 1$,

$$s(t) = max_{z \in D_r} |df_t(z)|.$$

We have $s(0) = 0$. Also we can write $tz = w$. Taking the sup for $z \in D_r$ amounts to taking the sup for $w \in tD_r$. If $t < 1$, we can even take the sup over the closure $t\overline{D}_r$. It follows that $s(t)$ is continuous for

$$0 \leq t < 1.$$

Also $s(t) \to s(1)$ as $t \to 1$, even if $s(1) = \infty$. By assumption in the first part, $|df(0)| > c$, and hence $s(1) > c$. Hence there exists $0 \leq t < 1$ such that $s(t) = c$. Hence there is some $z_0 \in t\overline{D}_r$ such that $|df_t(z_0)| = c$. Now let $h : D_r \to D_r$ be an automorphism such that $h(0) = z_0$ and let $g = f_t \circ h$. Then

$$|dg(0)| = |df_t(z_0)||dh(0)| = |df_t(z_0)| = c,$$

thus proving the first part. The second part is proved similarly, allowing $t = 1$. This concludes the proof of the Lemma. □

Theorem 1.3.2. *Let X be a complex manifold with a length function. Let $f : \mathbb{C} \to X$ be a holomorphic non-constant map, whose image is contained in a relatively compact subset S. Then there exists a non-constant holomorphic map $g : \mathbb{C} \to X$ whose image is contained in the closure of S, and such that*

$$|dg(0)| = sup|dg(z)| = 1.$$

For each disc D_{r_n} with increasing radius r_n , we let f_n be the restriction of f to this disc. Then the same proof as above works. We observe that in Brody's reparametrization, the image of g_n is contained in the image of f_n and so the image of the limit is contained in $\overline{S}$. This proves the theorem.

Remark 1.3.3. *Suppose that the length function on X comes from a hermitian metric on some ambient complex manifold, and let ω be the associated positive $(1,1)-form$. Then we see that*

$$\int_{D_r} g^\star \omega \leq \pi r^2$$

This is usually expressed by saying that g is of order ≤ 2, and in particular is of finite order.

Definition 1.3.4. *Let X be a complex manifold. Given an arbitrary holomorphic tangent vector $\xi \in T_x X$, the infinitesimal Kobayashi-Royden pseudometric is the pseudo-Finsler metric defined as*

$$k_X(\xi) = inf\{\lambda > 0; \exists f : \delta \to X, f(0) = x, \lambda f'(0) = \xi\}.$$

Given a piecewise smooth C^1 curve : $\gamma : [0,1] \to X$, the length of γ is:

$$l_X(\gamma) = \int_0^1 K_X(\gamma'(t))dt.$$

The Kobayashi pseudometric is the integrated form of the infinitesimal Kobayashi-Royden pseudometric.

$$d_X(p,q) = \inf\{l_X(\gamma), \gamma(0) = p, \gamma(1) = q\}$$

A complex manifold X is said to be Kobayashi hyperbolic,if d_X is a distance.

Recall that the Kobayashi-Royden pseudometric is identically zero in $\mathbb{C}$. Moreover, if X is a compact complex manifold then, X is Kobayashi hyperbolic if and only if, there is no non-constant holomorphic function $f : \mathbb{C} \to X$.

1.3.2 Kähler Hyperbolicity in the sense of Gromov

Definition 1.3.5 (Gro91). *A form α on a complex manifold (X, ω) is called* $\tilde{d}$-bounded *if the lift $\tilde{\alpha}$ of α to the universal covering $\tilde{X} \to X$ is* d-bounded *over $(\tilde{X}, \pi^*(\omega))$.*

Recall that if X is compact, then every smooth form α is bounded and α is *d-bounded* if and only if it is d-exact. However, if X is non-compact, then an exact bounded form is not necessarily *d-bounded*.

Definition 1.3.6. *A compact Kähler complex n-dimensional manifold (X, ω) is called* ***Kähler hyperbolic*** *if ω is* $\tilde{d}$-bounded.

1.4 Simple and Semisimple Lie Algebras and Lie Groups

Definition 1.4.1. *A subset $\mathbf{H}$ of a Lie algebra $\mathbf{G}$ is a Lie subalgebra if it is a subspace of $\mathbf{G}$ (as a vector space) and if it is closed under the bracket operation on $\mathbf{G}$.*

Definition 1.4.2.

1. *A subalgebra $\mathbf{H}$ of $\mathbf{G}$ is abelian if $[x, y] = 0$ for all $x, y \in \mathbf{H}$.*

2. *An ideal in $\mathbf{G}$ is a Lie subalgebra $\mathbf{H}$ such that*

$$[h, g] \in \mathbf{H}, \textit{ for all } h \in \mathbf{H}, \quad \textit{and all } g \in \mathbf{G}$$

   The center $\mathcal{Z}(\mathbf{G})$ of a Lie algebra $\mathbf{G}$ is the set of all elements $u \in \mathbf{G}$ such that $[u, v] = 0$ for all $v \in \mathbf{G}$, or equivalently, such that $\boldsymbol{ad}(u) = 0$

3. *A Lie algebra $\mathbf{G}$ is simple if it is non-abelian and if it has no ideal other than (0) and $\mathbf{G}$.*

4. *A Lie algebra $\mathbf{G}$ is semisimple if it has no abelian ideal other than (0).*

5. *A Lie group is simple (resp. semisimple) if its Lie algebra is simple (resp. semisimple).*

Clearly, the trivial subalgebras (0) and $\mathbf{G}$ itself are ideals, and the center of a Lie algebra is an abelian ideal. It follows that the center $\mathcal{Z}(\mathbf{G})$ of a semisimple Lie algebra must be the trivial ideal (0). Given two subsets $\mathbf{a}$ and $\mathbf{b}$ of a Lie algebra $\mathbf{G}$, we let $[\mathbf{a}, \mathbf{b}]$ be the subspace of $\mathbf{G}$ consisting of all linear combinations $[a, b]$, with $a \in \mathbf{a}$ and $b \in \mathbf{G}$. If $\mathbf{a}$ and $\mathbf{b}$ are ideals in $\mathbf{G}$, then $\mathbf{a}+\mathbf{b}$, $\mathbf{a}\cap\mathbf{b}$, and $[\mathbf{a}, \mathbf{b}]$,are also ideals (for $[\mathbf{a}, \mathbf{b}]$, use the Jacobi identity). In particular, $[\mathbf{G}, \mathbf{G}]$ is an ideal in $\mathbf{G}$ called the commutator ideal $\mathbf{G}$. The commutator ideal $[\mathbf{G}, \mathbf{G}]$ is also denoted by $\mathfrak{D}^1\mathbf{G}$ (or $\mathfrak{D}\mathbf{G}$). If $\mathbf{G}$ is a simple Lie algebra, then $[\mathbf{G}, \mathbf{G}] = \mathbf{G}$.

Definition 1.4.3. *The derived series (or commutator series) ($\mathfrak{D}^k\mathbf{G}$) of $\mathbf{G}$ is defined as follows:*

$$\begin{aligned} \mathfrak{D}^0\mathbf{G} &= \mathbf{G} \\ \mathfrak{D}^{k+1}\mathbf{G} &= [\mathfrak{D}^k\mathbf{G}, \mathfrak{D}^k\mathbf{G}],\ k \geq 0 \end{aligned}$$

We have a decreasing sequence

$$\mathbf{G} = \mathfrak{D}^0\mathbf{G} \supseteq \mathfrak{D}^1 \supseteq \mathfrak{D}^2 \supseteq \dots$$

We say that $\mathbf{G}$ is solvable iff $\mathfrak{D}^k\mathbf{G} = (0)$ for some k.

If $\mathbf{G}$ is abelian, then $[\mathbf{G}, \mathbf{G}] = 0$, so $\mathbf{G}$ is solvable.
Observe that a nonzero solvable Lie algebra has a nonzero abelian ideal, namely, the last nonzero $\mathfrak{D}^j\mathbf{G}$.
As a consequence, a Lie algebra is semisimple iff it has no nonzero solvable ideal.

It can be shown that every Lie algebra $\mathbf{G}$ has a largest solvable ideal $\mathfrak{r}$, called the radical of $\mathbf{G}$.
The radical of $\mathbf{G}$ is also denoted rad $\mathbf{G}$.
Then a Lie algebra is semisimple iff $\mathrm{rad}\mathbf{G} = (0)$.

Definition 1.4.4. *1. The lower central series ($\mathfrak{C}^k\mathbf{G}$) of $\mathbf{G}$ is defined as follows:*

$$\begin{aligned} \mathfrak{C}^0\mathbf{G} &= \mathbf{G} \\ \mathfrak{C}^{k+1}\mathbf{G} &= [\mathbf{G}, \mathfrak{C}^k\mathbf{G}],\ k \geq 0 \end{aligned}$$

We have a decreasing sequence

$$\mathbf{G} = \mathfrak{C}^0\mathbf{G} \supseteq \mathfrak{C}^1 \supseteq \mathfrak{C}^2 \supseteq ...$$

2. We say that $\mathbf{G}$ is nilpotent if $\mathfrak{C}^k\mathbf{G} = (0)$ for some k. By induction, it is easy to show that

$$\mathfrak{D}^k\mathbf{G} \subseteq \mathfrak{C}^k\mathbf{G} \qquad k \geq 0.$$

Consequently, every nilpotent Lie algebra is solvable. Note that, by definition, simple and semisimple Lie algebras are non-abelian, and a simple algebra is a semisimple algebra.
It turns out that a Lie algebra $\mathbf{G}$ is semisimple iff it can be expressed as a direct sum of ideals $\mathbf{g_i}$, with each $\mathbf{g_i}$ a simple algebra.

As a consequence, if $\mathbf{G}$ is semisimple, then we also have $[\mathbf{G}, \mathbf{G}] = \mathbf{G}$.

If we drop the requirement that a simple Lie algebra be non-abelian, thereby allowing one dimensional Lie algebras to be simple, we run into the trouble that a simple Lie algebra is no longer semisimple, and the above theorem statement for this reason. Thus, it seems technically advantageous to require that simple Lie algebras be non-abelian. Nevertheless, in certain situations, it is desirable to drop the requirement that a simple Lie algebra be non-abelian and this is what Milnor does in his paper because it is more convenient for one of his proofs. This is a minor point but it could be confusing for uninitiated readers.

Proposition 1.4.5. *Let $\mathbf{G}$ be a Lie algebra with an inner product such that the linear map $ad(u)$ is skew-adjoint for every $u \in \mathbf{G}$. Then, the orthogonal complement $\mathbf{a}^\perp$ of any ideal $\mathbf{a}$ is itself an ideal. Consequently, $\mathbf{G}$ can be expressed as an orthogonal direct sum*

$$\mathbf{G} = \mathbf{G}_1 \oplus \mathbf{G}_2 \oplus ... \oplus \mathbf{G}_k$$

where each $\mathbf{G}_i$ is either a simple ideal or a one-dimensional abelian ideal ($\mathbf{g_i} \cong \mathbb{R}$).

Chapter 2

Balanced Hyperbolic and Divisorially Hyperbolic Compact Complex Manifolds

2.1 Introduction

We propose a hyperbolicity theory in which curves are replaced by divisors and the bidegree (1, 1) is replaced by the bidegree $(n-1,\, n-1)$ on n-dimensional compact complex manifolds. The notions we introduce are weaker, hence more inclusive, than their classical counterparts. In particular, the setting need not be projective or even Kähler. Recall that, by contrast, the classical notions of Kähler and Brody/Kobayashi hyperbolicity cannot occur in the context of compact manifolds, at least conjecturally for the latter, outside the projective context. Our motivation stems from the existence of many interesting examples of non-Kähler compact complex manifolds that display hyperbolicity features in the generalised sense that we now set out to explain.

(I) Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$. Recall that

(a) X is said to be *Kähler hyperbolic* in the sense of Gromov (see [Gro91]) if there exists a Kähler metric ω on X whose lift $\widetilde{\omega}$ to the universal covering space $\widetilde{X}$ of X is d-exact with an $\widetilde{\omega}$-bounded d-potential on $\widetilde{X}$.

Meanwhile, it is well known that balanced metrics (i.e. Hermitian metrics ω on X such that $d\omega^{n-1} = 0$) may exist on certain compact complex n-dimensional manifolds X such that ω^{n-1} is even d-exact on X. These manifolds are called *degenerate balanced*. There is no analogue of this phenomenon in the Kähler setting. Building on this fact, we propose in Definition 2.2.1 the balanced analogue of Gromov's Kähler hyperbolicity by requiring the existence of a balanced metric ω on X such that $\widetilde{\omega}^{n-1}$ is d-exact with an $\widetilde{\omega}$-bounded d-potential on $\widetilde{X}$. We call any compact manifold X admitting such a metric ω a **balanced hyperbolic** manifold. We immediately get our first examples: any degenerate balanced manifold X is automatically balanced hyperbolic.

(b) X is said to be *Kobayashi hyperbolic* (see e.g. [Kob70]) if the Kobayashi pseudo-distance on X is actually a distance. By Brody's Theorem 4.1. in [Bro78], this is equivalent to the non-existence of entire curves in X, namely the non-existence of non-constant holomorphic maps $f : \mathbb{C} \longrightarrow X$. This latter property has come to be known as the *Brody hyperbolicity* of X. Thus, a compact manifold X is Brody hyperbolic if and only if it is Kobayashi hyperbolic. (The equivalence is known to fail when X is non-compact.) In Definition 2.2.7, we propose the $(n-1)$-dimensional analogue of the Brody hyperbolicity, that we call **divisorial hyperbolicity**. However, due to the absence of a higher-dimensional analogue of Brody's Reparametrisation Lemma [Bro78, Lemma 2.1.], there is a surprising twist: to ensure that the *balanced hyperbolicity* of X implies its *divisorial hyperbolicity*, it does not suffice to rule out the existence of non-degenerate (at some point) holomorphic maps $f : \mathbb{C}^{n-1} \longrightarrow X$ in Definition 2.2.7 when $\dim_{\mathbb{C}} X = n \geq 3$, but the non-existence has to be confined to such maps of *subexponential growth* in the sense of Definition 2.2.3.

This restriction is also warranted by the trivial existence of a non-degenerate (at some point) holomorphic map $f : \mathbb{C}^{n-1} \longrightarrow X$ whenever $X = G/\Gamma$ is the quotient of an n-dimensional *complex* Lie group G by a discrete co-compact subgroup Γ. Indeed, such a map is obtained by composing two obvious maps with the *exponential* $\exp : \mathfrak{g} \longrightarrow G$ from the Lie algebra $\mathfrak{g}$ of G. However, we will see that some of these quotients $X = G/\Gamma$ deserve to be called *divisorially hyperbolic*.

To put our *balanced hyperbolicity* and *divisorial hyperbolicity* into perspective, we sum up below the relations among the various hyperbolicity notions mentioned above.

Theorem 2.1.1. *Let X be a compact complex manifold. The following implications hold:*

$$\begin{array}{ccc}
X \textit{ is } \textbf{Kähler hyperbolic} & \Longrightarrow & X \textit{ is } \textbf{Kobayashi/Brody hyperbolic} \\
\Downarrow & & \Downarrow \\
X \textit{ is } \textbf{balanced hyperbolic} & \Longrightarrow & X \textit{ is } \textbf{divisorially hyperbolic} \\
\Uparrow & & \\
X \textit{ is } \textbf{degenerate balanced} & &
\end{array}$$

The vertical implications in Theorem 2.1.1 are obvious, while the top horizontal implication has been known since [Gro91, 0.3.B.]. (See also [CY17, Theorem 4.1.] for a proof.) Our main result of §.2.2 is Theorem 2.2.8 proving the bottom horizontal implication of Theorem 2.1.1.

We now collect a few results from §.2.2.3. They contain various examples and counter-examples illustrating our *balanced hyperbolicity* and *divisorial hyperbolicity* notions, as well as a method for constructing new examples from existing ones.

Theorem 2.1.2. *(i) The following two classes of compact complex manifolds X consist exclusively of* **degenerate balanced** *(hence also* **balanced hyperbolic***, hence also* **divisorially hyperbolic***) manifolds:*

(a) the connected sums $X = \sharp_k(S^3 \times S^3)$ of k copies (with $k \geq 2$) of $S^3 \times S^3$ endowed with the Friedman-Lu-Tian complex structure J_k constructed via conifold transitions, where S^3 is the 3-sphere;

(b) the Yachou manifolds $X = G/\Gamma$ arising as the quotient of any **semi-simple** *complex Lie group G by a lattice $\Gamma \subset G$.*

(ii) If X_1 and X_2 are **balanced hyperbolic** *manifolds, so is $X_1 \times X_2$.*

(iii) The **non-divisorially hyperbolic** *manifolds include: all the complex projective spaces $\mathbb{P}^n$, all the complex tori $\mathbb{C}^n/\Gamma$, the 3-dimensional Iwasawa manifold and all the 3-dimensional Nakamura solvmanifolds.*

A key feature of the Friedman-Lu-Tian manifolds $(X = \sharp_k(S^3 \times S^3, J_k)$ and the Yachou manifolds $X = G/\Gamma$ is that their canonical bundle K_X is *trivial*. By Kobayashi's Conjecture 2.1.3, this is not expected to be the case in the classical context of Kobayashi/Brody hyperbolic compact complex manifolds X. We hope that examples of *projective* balanced hyperbolic or divisorially hyperbolic manifolds X with K_X *trivial* can be found in the future. Such an extension of the hyperbolicity theory into Calabi-Yau territory would be one of the spin-offs of our approach.

(II) A fundamental problem in complex geometry is to prove positivity properties of various objects, notably the canonical bundle K_X, associated with a compact hyperbolic manifold X, as a way of emphasising the links between the complex analytic and the metric aspects of the theory.

In this vein, Gromov showed in [Gro91, Corollary 0.4.C] that K_X is a *big* line bundle whenever X is a compact Kähler hyperbolic manifold. Building on Gromov's result and on several classical results in birational geometry (including Mori's *Cone Theorem* implying that K_X is nef whenever X contains no rational curves, the Kawamata-Reid-Shokurov *Base-Point-Free Theorem* to the effect that K_X is semi-ample whenever it is big and nef, and the *Relative Cone Theorem for log pairs*), Chen and Yang showed in [CY17, Theorem 2.11] that K_X is even *ample* under the Kähler hyperbolicity assumption on the compact X.

This answers affirmatively, in the special case of a compact Kähler hyperbolic manifold X, the following

Conjecture 2.1.3. *(Kobayashi, see e.g. [CY17, Conjecture 2.8] or Lang's survey cited therein)*

If X is a **Kobayashi hyperbolic** *compact complex manifold, its canonical bundle K_X is* **ample**.

Our undertaking in §2.3 is motivated by a desire to prove positivity properties of K_X under the weaker hyperbolicity assumptions on X introduced in this chapter, the *balanced hyperbolicity* and the *divisorial hyperbolicity*. Since there are quite a few *non-projective* and even *non-Kähler* compact complex manifolds X that are hyperbolic in our two senses (see e.g. Theorem 2.1.2), K_X cannot be positive in the usual big/ample senses for those X's. Therefore, it seems natural to introduce positivity concepts relative to the complex codimension 1 (rather than the usual complex dimension 1) that hopefully match our codimension-1 hyperbolicity notions.

This is precisely what we propose in §.2.3. Given a compact complex n-dimensional manifold X, recall that the Bott-Chern and Aeppli cohomology groups of any bidegree $(p,\, q)$ of X are classically defined, using the spaces $C^{r,\, s}(X) = C^{r,\, s}(X,\, \mathbb{C})$ of smooth $\mathbb{C}$-valued $(r,\, s)$-forms on X, as

$$\begin{aligned} H^{p,\,q}_{BC}(X,\,\mathbb{C}) &= \frac{\ker(\partial : C^{p,\,q}(X)\to C^{p+1,\,q}(X))\cap\ker(\bar\partial : C^{p,\,q}(X)\to C^{p,\,q+1}(X))}{\operatorname{Im}(\partial\bar\partial : C^{p-1,\,q-1}(X)\to C^{p,\,q}(X))} \\ H^{p,\,q}_{A}(X,\,\mathbb{C}) &= \frac{\ker(\partial\bar\partial : C^{p,\,q}(X)\to C^{p+1,\,q+1}(X))}{\operatorname{Im}(\partial : C^{p-1,\,q}(X)\to C^{p,\,q}(X)) + \operatorname{Im}(\bar\partial : C^{p,\,q-1}(X)\to C^{p,\,q}(X))}. \end{aligned}$$

We will use the Serre-type duality (see e.g. [Sch07]):

$$H^{1,\,1}_{BC}(X,\,\mathbb{C})\times H^{n-1,\,n-1}_{A}(X,\,\mathbb{C})\longrightarrow\mathbb{C}, \qquad (\{u\}_{BC},\,\{v\}_A)\mapsto\{u\}_{BC}.\{v\}_A := \int\limits_X u\wedge v, \tag{2.1}$$

as well as the *strongly Gauduchon (sG) cone* $\mathcal{SG}_X$ and the *Gauduchon cone* $\mathcal{G}_X$ of X that were defined in [Pop15a] as:

$$\mathcal{SG}_X := \bigg\{\{\omega^{n-1}\}_A\in H^{n-1,\,n-1}_A(X,\,\mathbb{R})\;\mid\;\omega \text{ is an sG metric on } X\bigg\}\subset H^{n-1,\,n-1}_A(X,\,\mathbb{R});$$

$$\mathcal{G}_X := \bigg\{\{\omega^{n-1}\}_A\in H^{n-1,\,n-1}_A(X,\,\mathbb{R})\;\mid\;\omega \text{ is a Gauduchon metric on } X\bigg\}\subset H^{n-1,\,n-1}_A(X,\,\mathbb{R}).$$

Recall that a Hermitian metric ω on X is said to be a *Gauduchon metric* (cf. [Gau77a]), resp. a *strongly Gauduchon (sG) metric* (cf. [Pop13]), if $\partial\bar\partial\omega^{n-1}=0$, resp. if $\partial\omega^{n-1}\in\operatorname{Im}\bar\partial$. Obviously, $\mathcal{SG}_X\subset\mathcal{G}_X$.

Now, given a real De Rham cohomology class $\{\alpha\}\in H^2_{DR}(X,\,\mathbb{R})$ (not necessarily of type $(1,\,1)$), we say (see Definition 2.3.6) that $\{\alpha\}$ is **divisorially Kähler**, resp. **divisorially nef**, if its image under the canonically defined map (see (2.28) of Lemma 2.3.1):

$$P : H^2_{DR}(X,\,\mathbb{R})\longrightarrow H^{n-1,\,n-1}_A(X,\,\mathbb{R}), \qquad \{\alpha\}_{DR}\longmapsto\{(\alpha^{n-1})^{n-1,\,n-1}\}_A,$$

lies in the *Gauduchon cone* $\mathcal{G}_X$ of X, respectively in the closure $\overline{\mathcal{G}}_X$ of this cone in $H^{n-1,\,n-1}_A(X,\,\mathbb{R})$.

We say that a C^∞ complex line bundle L on X is *divisorially nef* if its first Chern class $c_1(L)$ is. An example of result in the special projective setting is the following immediate consequence of Propositions 2.3.4 and 2.3.8 (see (4) of §.2.3.3):

Proposition 2.1.4. *Let L be a holomorphic line bundle on an n-dimensional* **projective** *manifold X. The following implication holds:*

$$L \;\; \textit{is } \textbf{divisorially nef} \quad\Longrightarrow\quad L^{n-1}.D\geq 0 \;\;\textit{for all effective divisors}\;\; D\geq 0 \;\;\textit{on } X,$$

where

$$L^{n-1}.D := \int\limits_D\bigg(\frac{i}{2\pi}\Theta_h(L)\bigg)^{n-1}$$

and $(i/2\pi)\,\Theta_h(L)$ is the curvature form of L with respect to any Hermitian fibre metric h.

If L satisfies the last property above, we say that L is *projectively divisorially nef.* This property is the divisorial analogue of the classical nefness property on projective manifolds X: L is nef $\iff$ $L.C \geq 0$ for every curve $C \subset X$.

We also introduce the **divisorially Kähler cone** $\mathcal{DK}_X$ and the **divisorially nef cone** $\mathcal{DN}_X$ of X in Definition 2.3.6 and discuss various properties of these notions in §.2.3.1 and §.2.3.2. In §.2.3.3, we point out examples of divisorially Kähler and divisorially nef cohomology classes.

Our hope is that we are able to take up the following problem in future work.

Question 2.1.5. *Let X be a compact complex manifold. If X is* **balanced hyperbolic** *or, more generally,* **divisorially hyperbolic***, does it follow that its canonical bundle K_X is* **divisorially nef** *or even* **divisorially Kähler***?*

2.2 Balanced and divisorial hyperbolicity

In this section, we introduce and discuss two hyperbolicity notions that generalise Gromov's Kähler hyperbolicity and the Kobayashi/Brody hyperbolicity respectively.

2.2.1 Balanced hyperbolic manifolds

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Fix an arbitrary Hermitian metric (i.e. a C^∞ positive definite $(1,1)$-form) ω on X. Throughout the text, $\pi_X : \widetilde{X} \longrightarrow X$ will stand for the universal cover of X and $\widetilde{\omega} = \pi_X^\star \omega$ will be the Hermitian metric on $\widetilde{X}$ that is the lift of ω. Recall that a C^∞ k-form α on X is said to be $\widetilde{d}$(bounded) with respect to ω if $\pi_X^\star \alpha = d\beta$ on $\widetilde{X}$ for some C^∞ $(k-1)$-form β on $\widetilde{X}$ that is bounded w.r.t. $\widetilde{\omega}$. (See [Gro91].)

Recall two standard notions introduced by Gauduchon and Gromov respectively.

(1) The metric ω is said to be **balanced** if $d\omega^{n-1} = 0$. The manifold X is said to be *balanced* if it carries a *balanced metric.* (See [Gau77b], where these metrics were called **semi-Kähler**.)

(2) The metric ω is said to be **Kähler hyperbolic** if ω is *Kähler* (i.e. $d\omega = 0$) and $\widetilde{d}$(bounded) with respect to itself. The manifold X is said to be *Kähler hyperbolic* if it carries a *Kähler hyperbolic metric.* (See [Gro91].)

The first notion that we introduce in this work combines the above two classical ones.

Definition 2.2.1. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. A Hermitian metric ω on X is said to be* **balanced hyperbolic** *if ω is balanced and ω^{n-1} is $\widetilde{d}$(bounded) with respect to ω.*

The manifold X is said to be balanced hyperbolic if it carries a balanced hyperbolic metric.

Let us first notice the following implication:

$$X \text{ is } \textit{Kähler hyperbolic} \implies X \text{ is } \textit{balanced hyperbolic.}$$

To see this, besides the obvious fact that every Kähler metric is balanced, we need the following

Lemma 2.2.2. *Let $(X,\,\omega)$ be a compact complex Hermitian manifold with $\dim_{\mathbb{C}}X = n$. Let $k \in \{1,\dots,2n\}$ and $\alpha \in C^\infty_k(X,\,\mathbb{C})$. If α is $\widetilde{d}$(bounded) (with respect to ω), then α^p is $\widetilde{d}$(bounded) (with respect to ω) for every non-negative integer p.*

Proof. By the $\widetilde{d}$(boundedness) assumption on α, $\pi_X^\star\alpha = d\beta$ on $\widetilde{X}$ for some smooth $\widetilde{\omega}$-bounded $(k-1)$-form β on $\widetilde{X}$. Note that $d\beta$ is trivially $\widetilde{\omega}$-bounded on $\widetilde{X}$ since it equals $\pi_X^\star\alpha$ and α is ω-bounded on X thanks to X being *compact*.

We get: $\pi_X^\star\alpha^p = d(\beta\wedge(d\beta)^{p-1})$ on $\widetilde{X}$, where both β and $d\beta$ are $\widetilde{\omega}$-bounded, hence so is $\beta\wedge(d\beta)^{p-1}$. □

2.2.2 Divisorially hyperbolic manifolds

We begin with a few preliminaries. Fix an arbitrary integer $n\geq 2$. For any $r>0$, let $B_r := \{z\in\mathbb{C}^{n-1}\mid |z|<r\}$ and $S_r := \{z\in\mathbb{C}^{n-1}\mid |z|=r\}$ stand for the open ball, resp. the sphere, of radius r centred at $0\in\mathbb{C}^{n-1}$. Moreover, for any $(1,\,1)$-form $\gamma\geq 0$ on a complex manifold and any positive integer p, we will use the notation:

$$\gamma_p := \frac{\gamma^p}{p!}.$$

If X is a compact complex manifold with $\dim_{\mathbb{C}}X = n\geq 2$ and ω is a Hermitian metric on X, for any holomorphic map $f:\mathbb{C}^{n-1}\to X$ that is *non-degenerate* at some point $x\in\mathbb{C}^{n-1}$ (in the sense that its differential map $d_xf:\mathbb{C}^{n-1}\longrightarrow T_{f(x)}X$ at x is of maximal rank), we consider the smooth $(1,\,1)$-form $f^\star\omega$ on $\mathbb{C}^{n-1}$. The assumptions made on f imply that the differential map d_zf is of maximal rank for every point $z\in\mathbb{C}^{n-1}\setminus\Sigma$, where $\Sigma\subset\mathbb{C}^{n-1}$ is an analytic subset. Thus, $f^\star\omega$ is ≥ 0 on $\mathbb{C}^{n-1}$ and is >0 on $\mathbb{C}^{n-1}\setminus\Sigma$. Consequently, $f^\star\omega$ can be regarded as a *degenerate metric* on $\mathbb{C}^{n-1}$. Its degeneration locus, Σ, is empty if f is non-degenerate at every point of $\mathbb{C}^{n-1}$, in which case $f^\star\omega$ is a genuine Hermitian metric on $\mathbb{C}^{n-1}$. However, in our case, Σ will be non-empty in general, so $f^\star\omega$ will only be a genuine Hermitian metric on $\mathbb{C}^{n-1}\setminus\Sigma$.

For a holomorphic map $f:\mathbb{C}^{n-1}\to(X,\,\omega)$ in the above setting and for $r>0$, we consider the $(\omega,\,f)$*-volume* of the ball $B_r\subset\mathbb{C}^{n-1}$:

$$\mathrm{Vol}_{\omega,\,f}(B_r) := \int\limits_{B_r} f^\star\omega_{n-1} > 0.$$

Meanwhile, for $z\in\mathbb{C}^{n-1}$, let $\tau(z):=|z|^2$ be its squared Euclidean norm. At every point $z\in\mathbb{C}^{n-1}\setminus\Sigma$, we have:

$$\frac{d\tau}{|d\tau|_{f^\star\omega}}\wedge\star_{f^\star\omega}\left(\frac{d\tau}{|d\tau|_{f^\star\omega}}\right) = f^\star\omega_{n-1}, \tag{2.2}$$

where $\star_{f^\star\omega}$ is the Hodge star operator induced by $f^\star\omega$. Thus, the $(2n-3)$-form

$$d\sigma_{\omega,\,f} := \star_{f^\star\omega}\left(\frac{d\tau}{|d\tau|_{f^\star\omega}}\right)$$

on $\mathbb{C}^{n-1}\setminus\Sigma$ is the area measure induced by $f^\star\omega$ on the spheres of $\mathbb{C}^{n-1}$. This means that its restriction

$$d\sigma_{\omega,\,f,\,t} := \left(\star_{f^\star\omega}\left(\frac{d\tau}{|d\tau|_{f^\star\omega}}\right)\right)_{|S_t} \tag{2.3}$$

is the area measure induced by the degenerate metric $f^\star\omega$ on the sphere $S_t = \{\tau(z) = t^2\} \subset \mathbb{C}^{n-1}$ for every $t>0$. In particular, the area of the sphere $S_r \subset \mathbb{C}^{n-1}$ w.r.t. $d\sigma_{\omega,\,f,\,r}$ is

$$A_{\omega,\,f}(S_r) = \int\limits_{S_r} d\sigma_{\omega,\,f,\,r} > 0, \qquad r>0.$$

Definition 2.2.3. *Let* $(X,\,\omega)$ *be a* **compact** *complex Hermitian manifold with* $\dim_{\mathbb{C}}X = n\geq 2$ *and let* $f:\mathbb{C}^{n-1}\to X$ *be a holomorphic map that is* **non-degenerate** *at some point* $x\in\mathbb{C}^{n-1}$.

We say that f *has* **subexponential growth** *if the following two conditions are satisfied:*

(i) there exist constants $C_1>0$ *and* $r_0>0$ *such that*

$$\int\limits_{S_t} |d\tau|_{f^\star\omega}\, d\sigma_{\omega,\,f,\,t} \leq C_1 t\; Vol_{\omega,\,f}(B_t), \qquad t>r_0; \tag{2.4}$$

(ii) for every constant $C>0$*, we have:*

$$\limsup_{b\to+\infty}\left(\frac{b}{C} - \log F(b)\right) = +\infty, \tag{2.5}$$

where

$$F(b) := \int\limits_0^b Vol_{\omega,\,f}(B_t)\,dt = \int\limits_0^b\left(\int\limits_{B_t} f^\star\omega_{n-1}\right)dt, \qquad b>0.$$

Note that (i), imposing a relative growth condition of the spheres S_t w.r.t. the balls B_t as measured by the degenerate metric $f^\star\omega$, is of a known type in this context. See, e.g. [dTh10]. The subexponential growth is expressed by condition (ii).

In a bid to shed light on the subexponential growth condition, we now spell out the very particular case where $f^\star\omega$ is the standard Kähler metric (i.e. the Euclidean metric) $\beta = (1/2)\sum\limits_{j=1}^{n-1} i dz_j\wedge d\bar{z}_j$ of $\mathbb{C}^{n-1}$. It will come in handy when we discuss certain examples in §.2.2.3.

Lemma 2.2.4. *Let*

$$d\sigma_\beta := \star_\beta\left(\frac{d\tau}{|d\tau|_\beta}\right)$$

be the $(2n-3)$*-form on* $\mathbb{C}^{n-1}$ *defining the area measure induced by* β *on the spheres of* $\mathbb{C}^{n-1}$*. Then*

$$|d\tau|_\beta\, d\sigma_\beta = 2\sqrt{\tau}\, d\sigma_\beta \qquad \text{on} \quad \mathbb{C}^{n-1}. \tag{2.6}$$

In particular,

$$\int\limits_{S_t} |d\tau|_\beta\, d\sigma_\beta = 2A_{2n-3}\, t^{2n-2}, \qquad t>0, \tag{2.7}$$

where A_{2n-3} *is the area of the unit sphere* $S_1 \subset \mathbb{C}^{n-1}$ *w.r.t. the measure* $(d\sigma_\beta)_{|S_1}$ *induced by the Euclidean metric* β*.*

In particular, any holomorphic map $f : \mathbb{C}^{n-1} \longrightarrow (X,\,\omega)$ *such that* $f^\star\omega = \beta$ *has subexponential growth in the sense of Definition 2.2.3.*

Proof. Since $d\tau = \partial\tau + \bar\partial\tau = \sum\limits_{j=1}^{n-1} \bar{z}_j\, dz_j + \sum\limits_{j=1}^{n-1} z_j\, d\bar{z}_j$ and $\langle dz_j,\, dz_k\rangle_\beta = \langle d\bar{z}_j,\, d\bar{z}_k\rangle_\beta = 2\delta_{jk}$, we get $|d\tau|^2_\beta = 4|z|^2 = 4\tau$. This proves (2.6). Meanwhile, $\tau(z) = |z|^2 = t^2$ for $z \in S_t$, so we get (2.7).

On the other hand, $\mathrm{Vol}_\beta(B_t) = V_{2n-2}\, t^{2n-2}$ for every $t>0$, so, when $f^\star\omega = \beta$, (2.4) amounts to

$$2A_{2n-3}\, t^{2n-2} \leq C_1\, V_{2n-2}\, t^{2n-1}, \qquad t > r_0,$$

which obviously holds for some constants $C_1, r_0 > 0$. Property (2.5) also holds in an obvious way. $\square$

To further demystify condition (i) in Definition 2.2.3, we give an alternative expression for the integral on the sphere $S_t = \{|z| = t\} \subset \mathbb{C}^{n-1}$ featuring on the l.h.s. of (2.4) in terms of integrals on the ball $B_t = \{|z| < t\} \subset \mathbb{C}^{n-1}$.

Lemma 2.2.5. *In the context of Definition 2.2.3, the following identities hold for all* $t>0$*:*

$$\begin{aligned}\int\limits_{S_t} |d\tau|_{f^\star\omega}\, d\sigma_{\omega,\,f,\,t} &= 2\int\limits_{B_t} i\partial\bar\partial\tau \wedge f^\star\omega_{n-2} - \int\limits_{B_t} i(\bar\partial\tau - \partial\tau)\wedge d(f^\star\omega_{n-2}) \qquad (2.8)\\ &= 2\int\limits_{B_t} \Lambda_{f^\star\omega}(i\partial\bar\partial\tau)\, f^\star\omega_{n-1} - \int\limits_{B_t} i(\bar\partial\tau - \partial\tau)\wedge d(f^\star\omega_{n-2}),\end{aligned}$$

where $\Lambda_{f^\star\omega}$ *is the trace w.r.t.* $f^\star\omega$ *or, equivalently, the pointwise adjoint of the operator of multiplication by* $f^\star\omega$*, while*

$$i\partial\bar\partial\tau = i\partial\bar\partial|z|^2 = \sum_{j=1}^{n-1} idz_j \wedge d\bar{z}_j := \beta$$

is the standard metric of $\mathbb{C}^{n-1}$*.*

Proof. The pointwise identity $i\partial\bar\partial\tau \wedge (f^\star\omega)_{n-2} = \Lambda_{f^\star\omega}(i\partial\bar\partial\tau)\,(f^\star\omega)_{n-1}$ is standard on any $(n-1)$-dimensional complex manifold (which happens to be $\mathbb{C}^{n-1}$ in this case), so it suffices to prove the first equality in (2.8).

We saw just above (2.3) that $|d\tau|_{f^\star\omega}\, d\sigma_{\omega,\,f} = \star_{f^\star\omega}(d\tau)$. Meanwhile, $d\tau = \partial\tau + \bar\partial\tau$ and the 1-forms $\partial\tau$ and $\bar\partial\tau$ are primitive w.r.t. to any metric (in particular, w.r.t. $f^\star\omega$), as any 1-form is. Consequently, the standard formula (3.7) yields:

$$\star_{f^\star\omega}(\partial\tau) = -i\partial\tau \wedge f^\star\omega_{n-2} \quad \text{and} \quad \star_{f^\star\omega}\,(\bar\partial\tau) = i\bar\partial\tau \wedge f^\star\omega_{n-2}.$$

Hence, we get the first equality below, where the second one follows from Stokes's theorem:

$$\begin{aligned}\int\limits_{S_t} |d\tau|_{f^\star\omega}\, d\sigma_{\omega,\,f,\,t} &= \int\limits_{S_t} i(\bar\partial\tau - \partial\tau)\wedge f^\star\omega_{n-2} = \int\limits_{B_t} d\Big(i(\bar\partial\tau - \partial\tau)\wedge f^\star\omega_{n-2}\Big) \\ &= \int\limits_{B_t} i\, d(\bar\partial\tau - \partial\tau)\wedge f^\star\omega_{n-2} - \int\limits_{B_t} i\,(\bar\partial\tau - \partial\tau)\wedge d(f^\star\omega_{n-2}),\end{aligned}$$

which is nothing but (2.8). □

Another immediate observation is that, due to X being compact, we have

Lemma 2.2.6. *In the setting of Definition 2.2.3, the subexponential growth condition on f is independent of the choice of Hermitian metric ω on X.*

Proof. Let ω_1 and ω_2 be arbitrary Hermitian metrics on X. Since X is compact, there exists a constant $A > 0$ such that $(1/A)\,\omega_2 \leq \omega_1 \leq A\,\omega_2$ on X. Hence, $(1/A)\,f^\star\omega_2 \leq f^\star\omega_1 \leq A\,f^\star\omega_2$ on $\mathbb{C}^{n-1}$ for any holomorphic map $f : \mathbb{C}^{n-1} \to X$. The contention follows. □

Recall that a holomorphic map $f : \mathbb{C}^{n-1} \to (X,\,\omega)$ is standardly said to be *of finite order* if there exist constants $C_1,\,C_2,\,r_0 > 0$ such that

$$\text{Vol}_{\omega,\,f}(B_r) \leq C_1\, r^{C_2} \quad \text{for all } r \geq r_0. \tag{2.9}$$

By the proof of Lemma 2.2.6, f being of finite order does not depend on the choice of Hermitian metric ω on X. Moreover, any f of finite order satisfies condition (ii) in the definition 2.2.3 of a subexponential growth. Furthermore, in the special case where $n-1=1$, it is a standard consequence of *Brody's Renormalisation Lemma* [Bro78, Lemma 2.1.] that any non-constant holomorphic map $f : \mathbb{C} \to X$ can be modified to a non-constant holomorphic map $\tilde f : \mathbb{C} \to X$ *of finite order*. (See e.g. [Lan87, Theorem 2.6., p. 72].)

However, one of the key differences between $\mathbb{C}$ and $\mathbb{C}^p$ with $p \geq 2$ is that a holomorphic map $\tilde f$ that is non-degenerate at some point (the higher dimensional analogue of the non-constancy of maps from $\mathbb{C}$) and has subexponential growth need not exist from $\mathbb{C}^{n-1}$ to a given compact n-dimensional X when $n-1 \geq 2$ even if a holomorphic map $f : \mathbb{C}^{n-1} \to X$ that is non-degenerate at some point exists. Based on this observation, we propose the following notion that generalises that of *Kobayashi/Brody hyperbolicity*.

Definition 2.2.7. *Let $n \geq 2$ be an integer. An n-dimensional compact complex manifold X is said to be* **divisorially hyperbolic** *if there is no holomorphic map $f : \mathbb{C}^{n-1} \longrightarrow X$ such that f is* **non-degenerate** *at some point $x \in \mathbb{C}^{n-1}$ and f has* **subexponential growth** *in the sense of Definition 2.2.3.*

The following implication is obvious:

$$X \text{ is } \textit{Kobayashi/Brody hyperbolic} \implies X \text{ is } \textit{divisorially hyperbolic.}$$

Indeed, if there is a holomorphic map $f : \mathbb{C}^{n-1} \longrightarrow X$ (of any growth) such that f is non-degenerate at some point $x \in \mathbb{C}^{n-1}$, the restriction of f to every complex line through x in $\mathbb{C}^{n-1}$ is non-constant.

Meanwhile, the following implication is standard (see [Gro91, 0.3.B.]):

$$X \text{ is } \textit{Kähler hyperbolic} \implies X \text{ is } \textit{Kobayashi/Brody hyperbolic.}$$

Taking our cue from the proof of Theorem 4.1 in [CY17], we now complete the diagram of implications in Theorem 2.1.1 by proving its bottom row.

Theorem 2.2.8. *Every* **balanced hyperbolic** *compact complex manifold is* **divisorially hyperbolic***.*

Proof. Let X be a compact complex manifold, with $\dim_{\mathbb{C}} X = n$, equipped with a *balanced hyperbolic* metric ω. This means that, if $\pi_X : \widetilde{X} \longrightarrow X$ is the universal cover of X, we have

$$\pi_X^\star \omega^{n-1} = d\Gamma \quad \text{on } \widetilde{X},$$

where Γ is an $\widetilde{\omega}$-bounded C^∞ $(2n-3)$-form on $\widetilde{X}$ and $\widetilde{\omega} = \pi_X^\star \omega$ is the lift of the metric ω to $\widetilde{X}$.

Suppose there exists a holomorphic map $f : \mathbb{C}^{n-1} \longrightarrow X$ that is non-degenerate at some point $x \in \mathbb{C}^{n-1}$ and has subexponential growth in the sense of Definition 2.2.3. We will prove that $f^\star \omega^{n-1} = 0$ on $\mathbb{C}^{n-1}$, in contradiction to the non-degeneracy assumption made on f at x.

Since $\mathbb{C}^{n-1}$ is simply connected, there exists a lift $\tilde{f}$ of f to $\widetilde{X}$, namely a holomorphic map $\tilde{f} : \mathbb{C}^{n-1} \longrightarrow \widetilde{X}$ such that $f = \pi_X \circ \tilde{f}$. In particular, $d_x \tilde{f}$ is injective since $d_x f$ is.

The smooth $(n-1, n-1)$-form $f^\star \omega^{n-1}$ is ≥ 0 on $\mathbb{C}^{n-1}$ and > 0 on $\mathbb{C}^{n-1} \setminus \Sigma$, where $\Sigma \subset \mathbb{C}^{n-1}$ is the proper analytic subset of all points $z \in \mathbb{C}^{n-1}$ such that $d_z f$ is not of maximal rank. We have:

$$f^\star \omega_{n-1} = \tilde{f}^\star (\pi_X^\star \omega^{n-1}) = d(\tilde{f}^\star \Gamma) \quad \text{on } \mathbb{C}^{n-1}.$$

With respect to the degenerate metric $f^\star \omega$ on $\mathbb{C}^{n-1}$, we have the following

Claim 2.2.9. *The $(2n-3)$-form $\tilde{f}^\star \Gamma$ is $(f^\star \omega)$-bounded on $\mathbb{C}^{n-1}$.*

Proof of Claim. For any tangent vectors $v_1, \dots, v_{2n-3}$ in $\mathbb{C}^{n-1}$, we have:

$$\begin{aligned} |(\tilde{f}^\star \Gamma)(v_1, \dots, v_{2n-3})|^2 &= |\Gamma(\tilde{f}_\star v_1, \dots, \tilde{f}_\star v_{2n-3})|^2 \overset{(a)}{\leq} C\, |\tilde{f}_\star v_1|^2_{\widetilde{\omega}} \dots |\tilde{f}_\star v_{2n-3}|^2_{\widetilde{\omega}} \\ &= C\, |v_1|^2_{\tilde{f}^\star \widetilde{\omega}} \dots |v_{2n-3}|^2_{\tilde{f}^\star \widetilde{\omega}} \overset{(b)}{=} C\, |v_1|^2_{f^\star \omega} \dots |v_{2n-3}|^2_{f^\star \omega}, \end{aligned}$$

where $C > 0$ is a constant independent of the v_j's that exists such that inequality (a) holds thanks to the $\widetilde{\omega}$-*boundedness* of Γ on $\widetilde{X}$, while (b) follows from $\tilde{f}^\star\widetilde{\omega} = f^\star\omega$. $\square$

End of Proof of Theorem 2.2.8. We use the notation in the preliminaries of this §.2.2.2.

- On the one hand, we have $d\tau = 2t\,dt$ and

$$\mathrm{Vol}_{\omega,f}(B_r) = \int\limits_{B_r} f^\star\omega_{n-1} = \int\limits_0^r \left(\int\limits_{S_t} d\mu_{\omega,f,t}\right) dt = \int_{B_r} d\mu_{\omega,f,t} \wedge \frac{d\tau}{2t}, \tag{2.10}$$

where $d\mu_{\omega,f,t}$ is the positive measure on S_t defined by

$$\frac{1}{2t}\, d\mu_{\omega,f,t} \wedge (d\tau)_{|S_t} = (f^\star\omega_{n-1})_{|S_t}, \qquad t > 0.$$

Comparing with (2.2) and (2.3), this means that the measures $d\mu_{\omega,f,t}$ and $d\sigma_{\omega,f,t}$ on S_t are related by

$$\frac{1}{2t}\, d\mu_{\omega,f,t} = \frac{1}{|d\tau|_{f^\star\omega}}\, d\sigma_{\omega,f,t}, \qquad t > 0. \tag{2.11}$$

Now, the Hölder inequality yields:

$$\int_{S_t} \frac{1}{|d\tau|_{f^\star\omega}}\, d\sigma_{\omega,f,t} \geq \frac{A^2_{\omega,f}(S_t)}{\int_{S_t}|d\tau|_{f^\star\omega}\, d\sigma_{\omega,f,t}},$$

so together with (2.10) and (2.11) this leads to:

$$\begin{aligned}\mathrm{Vol}_{\omega,f}(B_r) &= \int\limits_0^r \left(\int\limits_{S_t} \frac{1}{2t}\, d\mu_{\omega,f,t}\right) d\tau = \int\limits_0^r \left(\int\limits_{S_t} \frac{1}{|d\tau|_{f^\star\omega}}\, d\sigma_{\omega,f,t}\right) d\tau \\ &\geq 2\int\limits_0^r \frac{A^2_{\omega,f}(S_t)}{\int_{S_t}|d\tau|_{f^\star\omega}\, d\sigma_{\omega,f,t}}\, t\, dt, \qquad r > 0. \end{aligned} \tag{2.12}$$

- On the other hand, for every $r > 0$, we have:

$$\mathrm{Vol}_{\omega,f}(B_r) = \int\limits_{B_r} f^\star\omega_{n-1} = \int\limits_{B_r} d(\tilde{f}^\star\Gamma) = \int\limits_{S_r} \tilde{f}^\star\Gamma \overset{(a)}{\leq} C \int\limits_{S_r} d\sigma_{\omega,f} = C\, A_{\omega,f}(S_r), \tag{2.13}$$

where $C > 0$ is a constant that exists such that inequality (a) holds thanks to Claim 2.2.9.

Putting (2.12) and (2.13) together, we get for every $r > r_0$:

$$\begin{aligned}\mathrm{Vol}_{\omega,f}(B_r) &\geq \frac{2}{C^2}\int\limits_0^r \mathrm{Vol}_{\omega,f}(B_t)\, \frac{t\,\mathrm{Vol}_{\omega,f}(B_t)}{\int_{S_t}|d\tau|_{f^\star\omega}\, d\sigma_{\omega,f,t}}\, dt \\ &\overset{(a)}{\geq} \frac{2}{C_1\,C^2}\int\limits_{r_0}^r \mathrm{Vol}_{\omega,f}(B_t)\, dt \overset{(b)}{:=} C_2\, F(r), \end{aligned} \tag{2.14}$$

where (a) follows from the growth assumption (2.4) and (b) is the definition of a function $F : (r_0, +\infty) \longrightarrow (0, +\infty)$ with $C_2 := 2/(C_1\, C^2)$.

By taking the derivative of F, we get for every $r > r_0$:

$$F'(r) = \mathrm{Vol}_{\omega,\, f}(B_r) \geq C_2\, F(r),$$

where the last inequality is (2.14). This amounts to

$$\frac{d}{dt}\bigg(\log F(t)\bigg) \geq C_2, \qquad t > r_0.$$

Integrating this over $t \in [a,\, b]$, with $r_0 < a < b$ arbitrary, we get:

$$-\log F(a) \geq -\log F(b) + C_2\,(b - a), \qquad r_0 < a < b. \tag{2.15}$$

Now, fix an arbitrary $a > r_0$ and let $b \to +\infty$. Thanks to the *subexponential growth* assumption (2.5) made on f, there exists a sequence of reals $b_j \to +\infty$ such that the right-hand side of inequality (2.15) for $b = b_j$ tends to $+\infty$ as $j \to +\infty$. This forces $F(a) = 0$ for every $a > r_0$, hence $\mathrm{Vol}_{\omega,\, f}(B_r) = 0$ for every $r > r_0$. This amounts to $f^\star\omega^{n-1} = 0$ on $\mathbb{C}^{n-1}$, in contradiction to the non-degeneracy assumption made on f at a point $x \in \mathbb{C}^{n-1}$. $\square$

We now adapt in a straightforward way to our context the first part of the proof of [CY17, Proposition 2.11], where the non-existence of rational curves in compact Kähler hyperbolic manifolds was proved, and get the following analogous result.

Proposition 2.2.10. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Suppose that X carries a* **balanced hyperbolic** *metric ω. Then, there is no holomorphic map $f : \mathbb{P}^{n-1} \longrightarrow X$ such that f is non-degenerate at some point $x \in \mathbb{P}^{n-1}$.*

Proof. Let $\pi_X^\star\omega_{n-1} = d\Gamma$ for some smooth $(2n-3)$-form Γ on $\widetilde{X}$, where $\pi_X : \widetilde{X} \to X$ is the universal covering map of X. (We can even choose Γ to be $\widetilde{\omega}$-bounded on $\widetilde{X}$, but we do not need this here.)

Suppose there exists a holomorphic map $f : \mathbb{P}^{n-1} \longrightarrow X$ that is non-degenerate at some point. We will show that $f^\star\omega_{n-1} = 0$ on $\mathbb{P}^{n-1}$, contradicting the non-degeneracy assumption on f.

Let $\tilde{f} : \mathbb{P}^{n-1} \longrightarrow \widetilde{X}$ be a lift of f to $\widetilde{X}$, namely a holomorphic map such that $f = \pi_X \circ \tilde{f}$. From

$$f^\star\omega_{n-1} = \tilde{f}^\star(\pi_X^\star\omega_{n-1}) = d(\tilde{f}^\star\Gamma),$$

we get by integration:

$$\int\limits_{\mathbb{P}^{n-1}} f^\star\omega_{n-1} = \int\limits_{\mathbb{P}^{n-1}} d(\tilde{f}^\star\Gamma) = 0,$$

where the last identity follows from Stokes's theorem.

Meanwhile, $f^\star\omega_{n-1} \geq 0$ at every point of $\mathbb{P}^{n-1}$. Therefore, $f^\star\omega_{n-1} = 0$ on $\mathbb{P}^{n-1}$, a contradiction. $\square$

2.2.3 Examples

(I) The following definition was given in [Pop15a].

Definition 2.2.11. *Let X be an n-dimensional complex manifold.*

A C^∞ positive definite $(1,\,1)$-form ω on X is said to be a **degenerate balanced metric** *if ω^{n-1} is d-exact. Any X carrying such a metric is called a* **degenerate balanced manifold.**

Degenerate balanced manifolds are characterised as follows.

Proposition 2.2.12. *([Pop15a, Proposition 5.4]) Let X be a compact complex manifold with $dim_C X = n$. The following statements are equivalent.*

(i) The manifold X is degenerate balanced.

(ii) There exists no non-zero d-closed $(1,1)$-current $T \geq 0$ on X.

(iii) The Gauduchon cone of X degenerates in the following sense: $\mathcal{G}_X = H_A^{n-1,\,n-1}(X,\,\mathbb{R})$.

We are aware of two classes of **degenerate balanced manifolds:**

(a) connected sums $X_k := \sharp_k(S^3 \times S^3)$ of $k \geq 2$ copies of $S^3 \times S^3$, where S^3 is the unit sphere of $\mathbb{R}^4$ and each X_k is endowed with the complex structure constructed by Friedman in [Fri89] and by Lu and Tian in [LT93] via *conifold transitions*. These complex structures were shown to be *balanced* in [FLY12]. Since $\dim_{\mathbb{C}} X_k = 3$ for every k and since $H^4_{DR}(X_k,\,\mathbb{C}) = 0$, any balanced metric ω_k on X_k is necessarily *degenerate balanced*. Thus, X_k is a *degenerate balanced manifold* for every $k \geq 2$. Note that every such X_k is *simply connected*.

(b) quotients $X = G/\Gamma$ of a *semi-simple* complex Lie group G by a lattice (i.e. a discrete co-compact subgroup) $\Gamma \subset G$. It was shown by Yachou in [Yac98, Propositions 17 and 18] that every left-invariant Hermitian metric on G induces a *degenerate balanced* metric on X. Thus, any such X (henceforth termed a **Yachou manifold**) is a *degenerate balanced manifold*. Note that $X = G/\Gamma$ is *not simply connected* if G is simply connected.

The immediate observation that provides the first class of examples of balanced hyperbolic manifolds is the following

Lemma 2.2.13. *Every* **degenerate balanced** *compact complex manifold is* **balanced hyperbolic**.

Proof. If ω is a degenerate balanced metric on an n-dimensional X, then $\omega^{n-1} = d\Gamma$ for some smooth $(2n-3)$-form Γ on X. Then, $\pi_X^\star(\omega^{n-1}) = d(\pi_X^\star\Gamma)$ on the universal covering manifold $\widetilde{X}$, while $\pi_X^\star\Gamma$ is $\widetilde{\omega}$-bounded on $\widetilde{X}$ since Γ is ω-bounded on the *compact* manifold X as any smooth form is. As usual, $\pi_X : \widetilde{X} \longrightarrow X$ stands for the universal covering map and $\widetilde{\omega} := \pi_X^\star\omega$.

Thus, ω^{n-1} is $\tilde{d}$(bounded) on X, so ω is a balanced hyperbolic metric. $\square$

Thus, for every compact complex manifold X, the following implications hold:

X is *degenerate balanced* $\implies$ X is *balanced hyperbolic* $\implies$ X is *divisorially hyperbolic*.

Recall that a compact complex manifold X is said to be *complex parallelisable* if its holomorphic tangent bundle $T^{1,0}X$ is trivial. By a result of Wang in [Wan54], X is complex parallelisable if and only if X is the compact quotient $X = G/\Gamma$ of a simply connected, connected **complex** Lie group G by a discrete subgroup $\Gamma \subset G$.

Meanwhile, it is standard that no *complex* Lie group G is Brody hyperbolic. Indeed, the complex one-parameter subgroup generated by any given element ξ in the Lie algebra of G provides an example of an entire curve in G. (In other words, take any tangent vector ξ of type $(1, 0)$ in the tangent space T_eG at the identity element $e \in G$ seen as the Lie algebra $\mathfrak{g}$ of G, then compose the linear map from $\mathbb{C}$ to $\mathfrak{g}$ that maps 1 to ξ with the exponential map $\exp : \mathfrak{g} \to G$, which is holomorphic since G is a complex Lie group, to get an entire curve in G.) Together with Wang's result mentioned above, this shows that no *complex parallelisable* compact complex manifold X is *Brody hyperbolic* (or, equivalently, since X is compact, *Kobayashi hyperbolic*).

Now, note that the Yachou manifolds $X = G/\Gamma$ mentioned above are *complex parallelisable* manifolds since G is a *complex* Lie group. So, they are *not Kobayashi hyperbolic*. In particular, they are *not Kähler hyperbolic*. However, they are *degenerate balanced*, hence also *balanced hyperbolic* (by Lemma 2.2.13), hence also *divisorially hyperbolic* (by Theorem 2.2.8). On the other hand, the Yachou manifolds $X = G/\Gamma$ are not Kähler (since, for example, they do not even support non-zero d-closed positive $(1, 1)$-currents, by Proposition 2.2.12). Hence, we get the following observation showing that the notions of *balanced hyperbolic* manifolds and *divisorially hyperbolic* manifolds are new and propose a hyperbolicity theory in the possibly non-Kähler context.

Proposition 2.2.14. *There exist compact complex non-Kähler manifolds that are* **balanced hyperbolic** *but are not* **Kobayashi hyperbolic**.

It seems natural to ask the following

Question 2.2.15. *Which compact quotients $X = G/\Gamma$ of a* **complex** *Lie group G by a lattice Γ are* **balanced hyperbolic** *or, at least,* **divisorially hyperbolic**?

We know from [Yac98] that all these quotients are even *degenerate balanced* (hence also *balanced hyperbolic*, hence also *divisorially hyperbolic*) when G is *semi-simple*. On the other hand, there is always a *holomorphic map* $f : \mathbb{C}^{n-1} \to X = G/\Gamma$, non-degenerate at some point $x \in \mathbb{C}^{n-1}$, whenever G is an n-dimensional complex Lie group and $\Gamma \subset G$ is a discrete co-compact subgroup. Indeed, let $\xi_1, \dots, \xi_{n-1} \in T_eG = \mathfrak{g}$ be $\mathbb{C}$-linearly independent vectors of type $(1, 0)$ in the Lie algebra of G and let $h : \mathbb{C}^{n-1} \to \mathfrak{g}$ be the $\mathbb{C}$-linear map that takes the vectors $e_1, \dots, e_{n-1}$ forming the canonical basis of $\mathbb{C}^{n-1}$ to $\xi_1, \dots, \xi_{n-1}$ respectively. The desired map $f : \mathbb{C}^{n-1} \to X = G/\Gamma$ is obtained by composing h with the exponential map $\exp : \mathfrak{g} \to G$ (which is holomorphic, due to G being a complex Lie group, and non-degenerate at least at $0 \in \mathfrak{g}$, hence at least on a neighbourhood of it, since its differential map at 0 is the identity map) and with the projection map $G \to G/\Gamma$.

Thus, part of Question 2.2.15 reduces to determining the complex Lie groups G and their lattices Γ for which no map as above that also has *subexponential growth* in the sense of Definition 2.2.3 exists. Meanwhile, we point to (a), (b), (c) under (VI) in this §.2.2.3 for examples of non-hyperbolic compact quotients G/Γ of a complex Lie group by a lattice.

(II) The other obvious class of *balanced hyperbolic manifolds* consists of all the *Kähler hyperbolic manifolds.* (See §.2.2.)

(III) We shall now point out examples of *balanced hyperbolic* manifolds that are neither degenerate balanced, nor Kähler hyperbolic. We first recall the following result of Michelsohn.

Proposition 2.2.16. *([Mic83, Proposition 1.9]) Let X and Y be complex manifolds.*

(i) If X and Y are balanced, the product manifold $X \times Y$ is balanced.

(ii) Let σ_X and σ_Y be the projections of $X \times Y$ onto X, resp. Y. If ω_X and ω_Y are balanced metrics on X, resp. Y, the induced product metric $\omega = \sigma_X^\star\omega_X + \sigma_Y^\star\omega_Y$ is a balanced metric on $X \times Y$.

Using this, we notice the following simple way of producing new balanced hyperbolic manifolds from existing ones.

Proposition 2.2.17. *The Cartesian product of* **balanced hyperbolic** *manifolds is* **balanced hyperbolic***.*

Proof. Let (X_1, ω_1) and (X_2, ω_2) be balanced hyperbolic manifolds of respective dimensions n and m, and let $\pi_1 : \widetilde{X_1} \longrightarrow X_1$, $\pi_2 : \widetilde{X_2} \longrightarrow X_2$ be their respective universal covers. By hypothesis, we have:

• ω_1^{n-1} is $\tilde{d}$*(bounded)* on (X_1, ω_1), so there exists an $\widetilde{\omega}_1$-bounded $(2n-3)$-form Θ_1 on $\widetilde{X}_1$ such that $\pi_1^*(\omega_1^{n-1}) = d\Theta_1$, where $\widetilde{\omega}_1 := \pi_1^\star\omega_1$;

• ω_2^{m-1} is $\tilde{d}$*(bounded)* on (X_2, ω_2), so there exists an $\widetilde{\omega}_2$-bounded $(2m-3)$-form Θ_2 on $\widetilde{X}_2$ such that $\pi_2^*(\omega_2^{m-1}) = d\Theta_2$, where $\widetilde{\omega}_2 := \pi_2^\star\omega_2$.

The product map

$$\pi := \pi_1 \times \pi_2 : \widetilde{X_1} \times \widetilde{X_2} \longrightarrow X_1 \times X_2$$

is the universal cover of $X_1 \times X_2$. Meanwhile, by (ii) of Proposition 2.2.16, the product metric

$$\omega = \sigma_1^\star\omega_1 + \sigma_2^\star\omega_2$$

on $X_1 \times X_2$ is balanced, where $\sigma_1 : X_1 \times X_2 \to X_1$ and $\sigma_2 : X_1 \times X_2 \to X_2$ are the projections on the two factors. From the equality

$$\omega^{n+m-1} = \binom{n+m-1}{n-1}\sigma_1^\star\omega_1^{n-1} \wedge \sigma_2^\star\omega_2^m + \binom{n+m-1}{n}\sigma_1^\star\omega_1^n \wedge \sigma_2^\star\omega_2^{m-1}$$

on $X_1 \times X_2$, we infer the following equalities on $\widetilde{X_1} \times \widetilde{X_2}$:

$$
\begin{aligned}
\pi^\star(\omega^{n+m-1}) &= \binom{n+m-1}{n-1}\,\pi^\star(\sigma_1^\star(\omega_1^{n-1}))\wedge\pi^\star(\sigma_2^\star(\omega_2^{m}))+\binom{n+m-1}{n}\,\pi^\star(\sigma_1^\star(\omega_1^{n}))\wedge\pi^\star(\sigma_2^\star(\omega_2^{m-1}))\\
&\overset{(a)}{=} \binom{n+m-1}{n-1}\,\widetilde{\sigma_1}^\star(\pi_1^\star(\omega_1^{n-1}))\wedge\widetilde{\sigma_2}^\star(\pi_2^\star(\omega_2^{m}))+\binom{n+m-1}{n}\,\widetilde{\sigma_1}^\star(\pi_1^\star(\omega_1^{n}))\wedge\widetilde{\sigma_2}^\star(\pi_2^\star(\omega_2^{m-1}))\\
&= \binom{n+m-1}{n-1}\,\widetilde{\sigma_1}^\star(d\Theta_1)\wedge\widetilde{\sigma_2}^\star(\pi_2^\star(\omega_2^{m}))+\binom{n+m-1}{n}\,\widetilde{\sigma_1}^\star(\pi_1^\star(\omega_1^{n}))\wedge\widetilde{\sigma_2}^\star(d\Theta_2)\\
&= \binom{n+m-1}{n-1}\,d(\widetilde{\sigma_1}^\star\Theta_1\wedge\widetilde{\sigma_2}^\star(\pi_2^\star(\omega_2^{m})))+\binom{n+m-1}{n}\,d\,(\widetilde{\sigma_1}^\star(\pi_1^\star(\omega_1^{n}))\wedge\widetilde{\sigma_2}^\star\Theta_2)\\
&= d\left[\binom{n+m-1}{n-1}\,\widetilde{\sigma_1}^\star\Theta_1\wedge\widetilde{\sigma_2}^\star(\pi_2^\star(\omega_2^{m}))+\binom{n+m-1}{n}\,\widetilde{\sigma_1}^\star(\pi_1^\star(\omega_1^{n}))\wedge\widetilde{\sigma_2}^\star\Theta_2\right]
\end{aligned}
\tag{2.16}
$$

where $\widetilde{\sigma_1}:\widetilde{X_1}\times\widetilde{X_2}\to\widetilde{X_1}$ and $\widetilde{\sigma_2}:\widetilde{X_1}\times\widetilde{X_2}\to\widetilde{X_2}$ are the projections on the two factors. Note that the equalities $\sigma_j\circ\pi=\pi_j\circ\widetilde{\sigma_j}$ for $j=1,\,2$ were used to get equality (a) in (2.16).

Now, for every $j\in\{1,\,2\}$, Θ_j is $\widetilde{\omega}_j$-bounded on $\widetilde{X}_j$. Therefore, $\widetilde{\sigma_j}^\star\Theta_j$ is $\widetilde{\sigma_j}^\star\widetilde{\omega}_j$-bounded, hence also $\widetilde{\omega}$-bounded, on $\widetilde{X_1}\times\widetilde{X_2}$, where $\widetilde{\omega}$ is the product metric

$$\widetilde{\omega}=\widetilde{\sigma}_1^\star\widetilde{\omega}_1+\widetilde{\sigma}_2^\star\widetilde{\omega}_2$$

on $\widetilde{X}_1\times\widetilde{X}_2$. We infer that the forms $\widetilde{\sigma_1}^\star\Theta_1\wedge\widetilde{\sigma_2}^\star(\pi_2^\star(\omega_2^m))$ and $\widetilde{\sigma_1}^\star(\pi_1^\star(\omega_1^n))\wedge\widetilde{\sigma_2}^\star\Theta_2$ are $\widetilde{\omega}$-bounded on $\widetilde{X_1}\times\widetilde{X_2}$, hence so is their linear combination featuring in the d-potential of the form on the r.h.s. of the last line in (2.16).

Consequently, the form ω^{n+m-1} is $\tilde{d}$*(bounded)* on $(X_1\times X_2,\,\omega)$. (Note that $\pi^\star\omega=\widetilde{\omega}$.) This means that the metric ω of the $(n+m)$-dimensional complex manifold $X_1\times X_2$ is balanced hyperbolic. Hence, the manifold $X_1\times X_2$ is balanced hyperbolic. □

In particular, using the above result, we can construct examples of *non-Kähler* balanced hyperbolic manifolds.

Corollary 2.2.18. *If X_1 is any* **degenerate balanced** *manifold and X_2 is any* **Kähler hyperbolic** *manifold, $X_1\times X_2$ is a* **balanced hyperbolic** *manifold that need not be either degenerate balanced, or Kähler hyperbolic, or even Kähler.*

On the other hand, we can also construct examples of *Kähler* balanced hyperbolic manifolds that are neither Kähler hyperbolic, nor degenerate balanced.

Proposition 2.2.19. *Let X_1 be a* **Kähler hyperbolic** *manifold with $\dim_{\mathbb{C}}X_1=n>1$ and let X_2 be a* **compact Kähler** *manifold. Then, $X_1\times X_2$ is a* **balanced hyperbolic** *Kähler manifold.*

Proof. Let ω_1 be a Kähler hyperbolic metric on X_1, ω_2 a Kähler metric on X_2 and $m=\dim_{\mathbb{C}}X_2$. We will keep the notation used in the proof of Proposition 2.2.17, except for the differences that will be pointed out.

Since ω_1 is $\tilde{d}$(*bounded*), so are ω_1^{n-1} and ω_1^n, by Lemma 2.2.2. Thus, there exist $\widetilde{\omega}_1$-bounded forms Θ_1 and Γ_1 on $\widetilde{X}_1$, of respective degrees $(2n-3)$ and $(2n-1)$, such that

$$\pi_1^*(\omega_1^{n-1}) = d\Theta_1 \quad \text{and} \quad \pi_1^*(\omega_1^n) = d\Gamma_1.$$

The only differences from the proof of Proposition 2.2.17 are the disappearance of Θ_2 and the appearance of Γ_1, together with the different properties that ω_1 and ω_2 now have. Running the equalities (2.16) with these differences incorporated, we get on $\widetilde{X_1} \times \widetilde{X_2}$:

$$\pi^\star(\omega^{n+m-1}) \;=\; d\left[\binom{n+m-1}{n-1}\widetilde{\sigma_1}^\star\Theta_1 \wedge \widetilde{\sigma_2}^\star(\pi_2^\star(\omega_2^m)) + \binom{n+m-1}{n}\widetilde{\sigma_1}^\star\Gamma_1 \wedge \widetilde{\sigma_2}^\star(\pi_2^\star(\omega_2^{m-1}))\right]$$

after using the fact that $d\omega_2 = 0$ (the Kähler assumption on ω_2).

We conclude in the same way as in the proof of Proposition 2.2.17 that the d-potential on the r.h.s. of the last line above is $\widetilde{\omega}$-bounded on $\widetilde{X_1} \times \widetilde{X_2}$. Thus, the form ω^{n+m-1} is $\tilde{d}$*(bounded)* on $(X_1 \times X_2,\, \omega)$, so ω is a balanced hyperbolic metric on $X_1 \times X_2$. $\square$

(IV) We now discuss in some detail the case of the semi-simple complex Lie group $G = SL(2,\, \mathbb{C})$, where several of the above constructions can be described explicitly.

As a complex manifold, $G = SL(2,\, \mathbb{C})$ is of complex dimension 3. Its complex structure is described by three holomorphic $(1,\, 0)$-forms α, β, γ that satisfy the structure equations:

$$d\alpha = \beta \wedge \gamma, \qquad d\beta = \gamma \wedge \alpha, \qquad d\gamma = \alpha \wedge \beta. \tag{2.17}$$

Moreover, the dual of the Lie algebra $\mathfrak{g} = T_eG$ of G is generated, as an $\mathbb{R}$-vector space, by these forms and their conjugates:

$$(T_eG)^\star = \langle \alpha,\, \beta,\, \gamma,\, \overline{\alpha},\, \overline{\beta},\, \overline{\gamma} \rangle.$$

The C^∞ positive definite $(1,\, 1)$-form

$$\omega := \frac{i}{2}\alpha \wedge \overline{\alpha} + \frac{i}{2}\beta \wedge \overline{\beta} + \frac{i}{2}\gamma \wedge \overline{\gamma}$$

defines a left-invariant (under the action of G on itself) Hermitian metric on G. From this, using (2.17), we get

$$\omega^2 = \frac{1}{2}\, d(\alpha \wedge d\overline{\alpha} + \beta \wedge d\overline{\beta} + \gamma \wedge d\overline{\gamma}) \in \operatorname{Im} d.$$

So, ω is a *degenerate balanced* metric on G (see Definition 2.2.11). Since it is left-invariant under the G-action, ω descends to a *degenerate balanced metric* on the compact quotient $X = G/\Gamma$ of G by any lattice Γ. In particular, this example illustrates Yachou's result [Yac98, Propositions 17 and 18] in the special case of $G = SL(2,\, \mathbb{C})$.

Now, consider the holomorphic map

$$f : \mathbb{C}^2 \to G = SL(2,\, \mathbb{C}), \qquad f(z_1,\, z_2) = \begin{pmatrix} e^{z_1} & z_2 \\ 0 & e^{-z_1} \end{pmatrix}. \tag{2.18}$$

This map is non-degenerate at every point $z = (z_1, z_2) \in \mathbb{C}^2$, as can be seen at once. However, f is *not of subexponential growth* in the sense of Definition 2.2.3, as we will now see. Actually, there is no non-degenerate holomorphic map $g : \mathbb{C}^2 \to X = G/\Gamma$ of subexponential growth thanks to X being *degenerate balanced* (hence also *balanced hyperbolic*) and to our Theorem 2.2.8.

Lemma 2.2.20. [1] *In* $\mathbb{C}^2$, *we have*

$$\begin{aligned} f^\star\omega &= (|z_2|^2 e^{-2Re\, z_1} + 2)\, idz_1 \wedge d\bar{z}_1 + e^{-2Re\, z_1}\, idz_2 \wedge d\bar{z}_2 \\ &+ z_2\, e^{-2Re\, z_1}\, idz_1 \wedge d\bar{z}_2 + \bar{z}_2\, e^{-2Re\, z_1}\, idz_2 \wedge d\bar{z}_1. \end{aligned}$$

Proof. • *Calculations at* $(0, 0) \in \mathbb{C}^2$.

$$\begin{aligned} f_\star\left(\frac{\partial}{\partial z_1}_{|(0,0)}\right) &= \frac{d}{dt}_{|t=0} f(t, 0) = \frac{d}{dt}_{|t=0}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} := H \\ f_\star\left(\frac{\partial}{\partial z_2}_{|(0,0)}\right) &= \frac{d}{dt}_{|t=0} f(0, t) = \frac{d}{dt}_{|t=0}\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} := X. \end{aligned}$$

Note that $H, X \in T_{I_2}SL(2, \mathbb{C}) = sl(2, \mathbb{C})$. A basis of the Lie algebra $sl(2, \mathbb{C})$ is given by $\{H, X, Y\}$, where

$$Y := \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}.$$

The Lie brackets linking the elements of this basis are

$$[X, Y] = H; \quad [Y, H] = 2Y; \quad [X, H] = -2X. \tag{2.19}$$

Rather than being dual to the basis $\{H, X, Y\}$, the basis $\{\alpha, \beta, \gamma\}$ of left-invariant $(1, 0)$-forms that satisfy equations (2.17) is dual to the following basis of tangent vectors of type $(1, 0)$ at $I_2 \in SL(2, \mathbb{C})$:

$$A := \frac{i}{2}(X+Y) = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \quad B := \frac{1}{2}(X-Y) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \quad C := \frac{i}{2}H = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}.$$

This amounts to

$$X = -iA + B, \quad Y = -iA - B, \quad H = -2iC.$$

From (2.19), we get

$$[A, B] = -C; \quad [A, C] = B; \quad [B, C] = -A. \tag{2.20}$$

(To see this, observe, for example, that $[A, B] = -(i/2)\,[X, Y] = -(i/2)\,H = -C$.)

• *Calculations at an arbitrary point* $(z_1^0, z_2^0) \in \mathbb{C}^2$.

[1] The authors are very grateful to L. Ugarte for pointing out to them the calculations leading to the result of this lemma and most of the discussion of $SL(2, \mathbb{C})$ preceding it under (IV).

Since $T_{I_2}SL(2,\,\mathbb{C}) = \langle H,\, X,\, Y\rangle = \langle A,\, B,\, C\rangle$, for every $g \in SL(2,\,\mathbb{C})$, the tangent space at g is generated as

$$T_gSL(2,\,\mathbb{C}) = \langle (L_g)_\star H,\, (L_g)_\star X,\, (L_g)_\star Y\rangle = \langle (L_g)_\star A,\, (L_g)_\star B,\, (L_g)_\star C\rangle,$$

where $L_g : G \to G$ is the left translation by g, namely $L_g(h) = gh$ for every $h \in G$. Since L_g is a linear map, $(L_g)_\star = L_g$. Therefore, for every

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\,\mathbb{C}) \quad (\text{with } ad - bc = 1),$$

we get:

$$\begin{aligned}
(L_g)_\star A &= (L_g)A = \frac{i}{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} b & a \\ d & c \end{pmatrix} := A_g \\
(L_g)_\star B &= (L_g)B = \frac{1}{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -b & a \\ -d & c \end{pmatrix} := B_g \\
(L_g)_\star C &= (L_g)C = \frac{i}{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} a & -b \\ c & -d \end{pmatrix} := C_g.
\end{aligned}$$

We now fix an arbitrary point $(z_1^0,\, z_2^0) \in \mathbb{C}^2$ and we let

$$g_0 := f(z_1^0,\, z_2^0) = \begin{pmatrix} e^{z_1^0} & z_2^0 \\ 0 & e^{-z_1^0} \end{pmatrix} \in SL(2,\,\mathbb{C}).$$

(Thus, for g_0, we have: $a = e^{z_1^0}$, $b = z_2^0$, $c = 0$, $d = e^{-z_1^0}$.) We get:

$$\begin{aligned}
f_\star\left(\frac{\partial}{\partial z_1}_{|(z_1^0,\, z_2^0)}\right) &= \frac{d}{dt}_{|t=0} f((z_1^0,\, z_2^0) + (t,\, 0)) = \frac{d}{dt}_{|t=0}\begin{pmatrix} e^{z_1^0+t} & z_2^0 \\ 0 & e^{-z_1^0-t} \end{pmatrix} \\
&= \begin{pmatrix} e^{z_1^0} & 0 \\ 0 & -e^{-z_1^0} \end{pmatrix} = \begin{pmatrix} e^{z_1^0} & -z_2^0 \\ 0 & -e^{-z_1^0} \end{pmatrix} + e^{-z_1^0} z_2^0 \begin{pmatrix} 0 & e^{z_1^0} \\ 0 & 0 \end{pmatrix} \\
&= -2iC_{g_0} + e^{-z_1^0} z_2^0\,(-iA_{g_0} + B_{g_0}).
\end{aligned} \tag{2.21}$$

Similarly, we get

$$\begin{aligned}
f_\star\left(\frac{\partial}{\partial z_2}_{|(z_1^0,\, z_2^0)}\right) &= \frac{d}{dt}_{|t=0} f((z_1^0,\, z_2^0) + (0,\, t)) = \frac{d}{dt}_{|t=0}\begin{pmatrix} e^{z_1^0} & z_2^0 + t \\ 0 & e^{-z_1^0} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e^{-z_1^0}\begin{pmatrix} 0 & e^{z_1^0} \\ 0 & 0 \end{pmatrix} = e^{-z_1^0}\,(-iA_{g_0} + B_{g_0}).
\end{aligned} \tag{2.22}$$

We now use the general formula $(f^\star\omega)(V,\, W) = \omega(f_\star V,\, f_\star W)$ (for all vector fields V, W) to deduce expressions for $(f^\star\omega)(\partial/\partial z_j,\, \partial/\partial \bar{z}_k)$ (for all $j, k = 1, 2$) from (2.21) and (2.22).

From (2.21), we get:

$$
\begin{aligned}
& (f^\star\omega)\bigg(\frac{\partial}{\partial z_1}_{|(z_1^0,\,z_2^0)},\,\frac{\partial}{\partial \bar{z}_1}_{|(z_1^0,\,z_2^0)}\bigg)\\
= \;& \frac{i}{2}\,(\alpha\wedge\bar\alpha+\beta\wedge\bar\beta+\gamma\wedge\bar\gamma)\bigg(-ie^{-z_1^0}z_2^0\,A_{g_0}+e^{-z_1^0}z_2^0\,B_{g_0}-2iC_{g_0},\,ie^{-\overline{z_1^0}}\overline{z_2^0}\,\bar{A}_{g_0}+e^{-\overline{z_1^0}}\overline{z_2^0}\,\bar{B}_{g_0}+2i\bar{C}_{g_0}\bigg)\\
= \;& i\,(|z_2^0|^2\,e^{-2\,\mathrm{Re}\,(z_1^0)}+2).
\end{aligned} \tag{2.23}
$$

From (2.22), we get:

$$
\begin{aligned}
& (f^\star\omega)\bigg(\frac{\partial}{\partial z_2}_{|(z_1^0,\,z_2^0)},\,\frac{\partial}{\partial \bar{z}_2}_{|(z_1^0,\,z_2^0)}\bigg)\\
= \;& \frac{i}{2}\,(\alpha\wedge\bar\alpha+\beta\wedge\bar\beta+\gamma\wedge\bar\gamma)\bigg(-ie^{-z_1^0}\,A_{g_0}+e^{-z_1^0}\,B_{g_0},\,ie^{-\overline{z_1^0}}\,\bar{A}_{g_0}+e^{-\overline{z_1^0}}\,\bar{B}_{g_0}\bigg)=i\,e^{-2\,\mathrm{Re}\,(z_1^0)}.
\end{aligned} \tag{2.24}
$$

From (2.21) and (2.22), we get:

$$
\begin{aligned}
& (f^\star\omega)\bigg(\frac{\partial}{\partial z_1}_{|(z_1^0,\,z_2^0)},\,\frac{\partial}{\partial \bar{z}_2}_{|(z_1^0,\,z_2^0)}\bigg)\\
= \;& \frac{i}{2}\,(\alpha\wedge\bar\alpha+\beta\wedge\bar\beta+\gamma\wedge\bar\gamma)\bigg(-ie^{-z_1^0}z_2^0\,A_{g_0}+e^{-z_1^0}z_2^0\,B_{g_0}-2iC_{g_0},\,ie^{-\overline{z_1^0}}\,\bar{A}_{g_0}+e^{-\overline{z_1^0}}\,\bar{B}_{g_0}\bigg)\\
= \;& i\,z_2^0\,e^{-2\,\mathrm{Re}\,(z_1^0)}.
\end{aligned} \tag{2.25}
$$

Finally, from (2.21) and (2.22) we also get:

$$
\begin{aligned}
& (f^\star\omega)\bigg(\frac{\partial}{\partial z_2}_{|(z_1^0,\,z_2^0)},\,\frac{\partial}{\partial \bar{z}_1}_{|(z_1^0,\,z_2^0)}\bigg)\\
= \;& \frac{i}{2}\,(\alpha\wedge\bar\alpha+\beta\wedge\bar\beta+\gamma\wedge\bar\gamma)\bigg(-ie^{-z_1^0}\,A_{g_0}+e^{-z_1^0}\,B_{g_0},\,ie^{-\overline{z_1^0}}\,\overline{z_2^0}\,\bar{A}_{g_0}+e^{-\overline{z_1^0}}\,\overline{z_2^0}\,\bar{B}_{g_0}+2i\,\overline{C}_{g_0}\bigg)\\
= \;& i\,\overline{z_2^0}\,e^{-2\,\mathrm{Re}\,(z_1^0)}.
\end{aligned} \tag{2.26}
$$

All that is left to do is to put (2.23), (2.24), (2.25) and (2.26) together and get the contention. □

Based on this, an elementary calculation, spelt out in the proof of the following statement, shows that f is not of subexponential growth.

Lemma 2.2.21. *The map f defined in (2.18) has the following property:*

$$\log\int\limits_0^b Vol_{\omega,\,f}(B_t)\,dt\geq\sqrt{2}\,b,\qquad b\gg 1.$$

Proof. Taking squares in the expression for $f^\star\omega$ of Lemma 2.2.20, we get in $\mathbb{C}^2\simeq\mathbb{R}^4$:

$$f^\star\omega^2=4e^{-2\mathrm{Re}\,z_1}\,idz_1\wedge d\bar{z}_1\wedge idz_2\wedge d\bar{z}_2=16e^{-2x_1}\,dx_1\wedge dy_1\wedge dx_2\wedge dy_2,$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Passing to spherical coordinates $(\rho,\, \theta_1,\, \theta_2,\, \theta_3)$ in $\mathbb{R}^4$, with $\rho \geq 0$, $\theta_1, \theta_2 \in [0,\, \pi]$, $\theta_3 \in [0,\, 2\pi)$, such that $x_1 = \rho\cos\theta_1$, $y_1 = \rho(\sin\theta_1)(\cos\theta_2)$, $x_2 = \rho(\sin\theta_1)(\sin\theta_2)(\cos\theta_3)$ and $y_2 = \rho(\sin\theta_1)(\sin\theta_2)(\sin\theta_3)$, we get:

$$\begin{aligned}\mathrm{Vol}_{\omega,\,f}(B_t) = \frac{1}{2}\int\limits_{B_t} f^\star\omega^2 &= 8(2\pi^2)\int\limits_0^\pi\bigg(\int\limits_0^t e^{-2\rho\cos\theta_1}\,d\rho\bigg)\,d\theta_1 \\ &\geq 16\pi^2\int\limits_{\frac{3\pi}{4}}^\pi\bigg(\int\limits_0^t e^{-2\rho\cos\theta_1}\,d\rho\bigg)\,d\theta_1 = -8\pi^2\int\limits_{\frac{3\pi}{4}}^\pi\frac{e^{-2t\cos\theta_1}}{\cos\theta_1}\,d\theta_1 + a,\end{aligned}$$

where $a \in \mathbb{R}$ is independent of t. Since $-1 \leq \cos\theta_1 \leq -\sqrt{2}/2$ (hence also $1 \leq -1/\cos\theta_1 \leq \sqrt{2}$) for $\theta_1 \in [3\pi/4,\, \pi]$, we get:

$$\mathrm{Vol}_{\omega,\,f}(B_t) = \frac{1}{2}\int\limits_{B_t} f^\star\omega^2 \geq 8\pi^2\frac{\pi}{4}\,e^{\sqrt{2}\,t} + a, \qquad t > 0.$$

Integrating over $t \in [0,\, b]$, with $b > 0$, we get:

$$\int\limits_0^b \mathrm{Vol}_{\omega,\,f}(B_t)\,dt \geq \frac{2\pi^3}{\sqrt{2}}\,(e^{\sqrt{2}\,b} - 1) + a\,b \geq e^{\sqrt{2}\,b}, \qquad b \gg 1.$$

This proves the contention. $\square$

We conclude that, for any constant $C > 0$, we have:

$$\frac{b}{C} - \log\int\limits_0^b \mathrm{Vol}_{\omega,\,f}(B_t)\,dt \leq \bigg(\frac{1}{C} - \sqrt{2}\bigg)\,b \longrightarrow -\infty \qquad \text{as } b \to +\infty$$

if the constant C is chosen such that $C > 1/\sqrt{2}$.

Thus, for any lattice $\Gamma \subset G = SL(2,\, \mathbb{C})$, the map $f : \mathbb{C}^2 \to X = G/\Gamma$ is *not of subexponential growth* in the sense of Definition 2.2.3.

(V) We now discuss the rather curious example of the n-dimensional complex projective space $\mathbb{P}^n$ for an arbitrary integer $n \geq 2$. We will see that an obvious map $\mathbb{C}^{n-1} \to \mathbb{P}^n$ easily satisfies condition (ii) but may not satisfy condition (i) in the definition 2.2.3 of the subexponential growth.

Let $j : \mathbb{C}^{n-1} \to \mathbb{P}^n$ be the holomorphic embedding obtained by composing the inclusions $\mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ and $\mathbb{C}^n \hookrightarrow \mathbb{P}^n$ given respectively by $(z_1, \dots,\, z_{n-1}) \mapsto (z_1, \dots,\, z_{n-1},\, 0)$ and $(z_1, \dots,\, z_n) \mapsto [1 : z_1 : \cdots : z_n]$.

As is well known, the restriction to $\mathbb{C}^n$ of the Fubini-Study metric ω_{FS} of $\mathbb{P}^n$ under the above inclusion $\mathbb{C}^n \hookrightarrow \mathbb{P}^n$ is

$$\omega_{FS} = i\partial\bar\partial\log(1 + |z|^2) = \frac{1}{1 + |z|^2}\,i\partial\bar\partial|z|^2 - \frac{1}{(1 + |z|^2)^2}\,i\partial|z|^2 \wedge \bar\partial|z|^2,$$

where $|z|^2 = \sum_{l=1}^n |z_l|^2$. Since $i\partial|z|^2 \wedge \bar\partial|z|^2 \geq 0$ as a $(1,\,1)$-form and $\omega_0 := (i/2)\,\partial\bar\partial|z|^2$ is the Euclidean metric on $\mathbb{C}^n$, we get:

$$\omega_{FS} \leq \frac{2}{1+|z|^2}\,\omega_0 \leq 2\,\omega_0 \quad \text{ on } \mathbb{C}^n.$$

Restricting to $\mathbb{C}^{n-1}$ under the above inclusion $\mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$, we get

$$j^\star\omega_{FS} \leq 2\beta_0 \quad \text{ on } \mathbb{C}^{n-1},$$

where $\beta_0 = (\omega_0)_{|\mathbb{C}^{n-1}}$ is the Euclidean metric on $\mathbb{C}^{n-1}$. Hence, the $(\omega_{FS},\,j)$-volume of the ball $B_r \subset \mathbb{C}^{n-1}$ of radius r centred at 0 is estimated as:

$$\mathrm{Vol}_{\omega_{FS},\,j}(B_r) = \frac{1}{(n-1)!}\int\limits_{B_r} j^\star\omega_{FS}^{n-1} \leq \frac{2^{n-1}}{(n-1)!}\int\limits_{B_r}\beta_0^{n-1} = c_n\,r^{2n-2}, \qquad r>0,$$

where $c_n > 0$ is a constant depending only on n.

This shows that the embedding $j : \mathbb{C}^{n-1} \to (\mathbb{P}^n,\,\omega_{FS})$ is of *finite order*, so it satisfies property (ii) in Definition 2.2.3.

As for property (i) in Definition 2.2.3, $j^\star\omega_{FS}$ is a Kähler metric on $\mathbb{C}^{n-1}$, hence the second term in formula (2.8) for $\int_{S_t}|d\tau|_{j^\star\omega_{FS}}\,d\sigma_{\omega_{FS},\,j,\,t}$ vanishes. To compute the first term, we deduce from

$$j^\star\omega_{FS} = \frac{1}{(1+|z|^2)^2}\sum_{j,\,k}\bigg(\delta_{jk}\,(1+|z|^2) - \bar{z}_j\,z_k\bigg)\,idz_j\wedge d\bar{z}_k \quad \text{ on } \mathbb{C}^{n-1}$$

that

$$\Lambda_{j^\star\omega_{FS}}(i\partial\bar\partial\tau) = (1+|z|^2)^2\sum_{j=1}^{n-1}\frac{1}{1+|z|^2-|z_j|^2}.$$

Hence, using (2.8) for the equality below, we get:

$$\begin{aligned}\int\limits_{S_t}|d\tau|_{j^\star\omega_{FS}}\,d\sigma_{\omega_{FS},\,j,\,t} &= 2\int\limits_{B_t}(1+|z|^2)^2\left(\sum_{j=1}^{n-1}\frac{1}{1+|z|^2-|z_j|^2}\right)(j^\star\omega_{FS})_{n-1} \\ &\leq 2(n-1)(1+t^2)^2\,\mathrm{Vol}_{\omega_{FS},\,j}(B_t), \qquad t>0.\end{aligned}$$

The last inequality falls far short of the required (2.4), so we cannot say anything at this stage about whether $\mathbb{P}^n$ is divisorially hyperbolic or not.

(VI) (a) A prototypical example of a compact complex manifold that is *not divisorially hyperbolic* is any *complex torus* $X = \mathbb{C}^n/\Gamma$, where $\Gamma \subset (\mathbb{C}^n,\,+)$ is any lattice. Any Hermitian metric with constant coefficients on $\mathbb{C}^n$ (for example, the Euclidean metric $\beta = (1/2)\sum_j idz_j\wedge d\bar{z}_j$) defines a Kähler metric ω on X: $\pi^\star\omega = \beta$, where $\pi : \mathbb{C}^n \to X$ is the projection. If $j : \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^n$ is the obvious inclusion $(z_1,\dots,,z_{n-1}) \mapsto (z_1,\dots,,z_n)$, the non-degenerate holomorphic map $f = \pi\circ j : \mathbb{C}^{n-1} \to X$ has subexponential growth thanks to Lemma 2.2.4 because $f^\star\omega = j^\star\beta = \beta_0$, where β_0 is the

Euclidean metric of $\mathbb{C}^{n-1}$. This shows that the complex torus $X = \mathbb{C}^n/\Gamma$ is not divisorially hyperbolic.

(b) Similarly, the *Iwasawa manifold* $X = G/\Gamma$ is *not divisorially hyperbolic*, where $G = (\mathbb{C}^3, \star)$ is the nilpotent complex Lie group (called the Heisenberg group) whose group operation is defined as

$$(\zeta_1,\, \zeta_2,\, \zeta_3) \star (z_1,\, z_2,\, z_3) = (\zeta_1 + z_1,\, \zeta_2 + z_2,\, \zeta_3 + z_3 + \zeta_1\, z_2),$$

while the lattice $\Gamma \subset G$ consists of the elements $(z_1,\, z_2,\, z_3) \in G$ with $z_1,\, z_2,\, z_3 \in \mathbb{Z}[i]$. (See e.g. [Nak75].)

Indeed, the holomorphic $(1,\,0)$-forms $dz_1,\, dz_2,\, dz_3 - z_1\, dz_2$ on $\mathbb{C}^3$ induce holomorphic $(1,\,0)$-forms $\alpha,\, \beta,\, \gamma$ on X. The Hermitian metric

$$\omega_0 = i\alpha\wedge\bar\alpha + i\beta\wedge\bar\beta + i\gamma\wedge\bar\gamma$$

on X lifts to the Hermitian metric

$$\omega = \pi^\star\omega_0 = idz_1\wedge d\bar z_1 + (1+|z_1|^2)\, idz_2\wedge d\bar z_2 + idz_3\wedge d\bar z_3 - \bar z_1\, idz_3\wedge d\bar z_2 - z_1\, idz_2\wedge d\bar z_3$$

on $G = \mathbb{C}^3$, where $\pi : G \to X$ is the projection.

Considering the non-degenerate holomorphic map $f = \pi\circ j : \mathbb{C}^2 \longrightarrow X$, where $j : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$ is the obvious inclusion $(z_1,\, z_2) \mapsto (z_1,\, z_2,\, 0)$, we get

$$f^\star\omega_0 = j^\star\omega = \omega_{|\mathbb{C}^2} = idz_1\wedge d\bar z_1 + (1+|z_1|^2)\, idz_2\wedge d\bar z_2$$

on $\mathbb{C}^2$. Hence,

$$f^\star\omega_0^2 = 2(1+|z_1|^2)\, dV_0$$

on $\mathbb{C}^2$, where we put $dV_0 := idz_1\wedge d\bar z_1\wedge idz_2\wedge d\bar z_2$.. Thus, for the ball $B_r \subset \mathbb{C}^2$ of radius r centred at 0, we get

$$\mathrm{Vol}_{\omega_0,\, f}(B_r) = \frac{1}{2}\int\limits_{B_r} f^\star\omega_0^2 = \int\limits_{B_r}(1+|z_1|^2)\, dV_0 \leq c_2\, r^4(1+r^2), \qquad r>0, \tag{2.27}$$

where $c_2 > 0$ is a constant independent of r. This shows that f is of *finite order*, hence f satisfies property (ii) in Definition 2.2.3.

To show that f has subexponential growth, it remains to check that it also satisfies property (i) in Definition 2.2.3. We will first compute the integral on the left of (2.4) in this case. Recall that $n = 3$. Then, note that

$$d(f^\star\omega_0) = \partial|z_1|^2\wedge idz_2\wedge d\bar z_2 + \bar\partial|z_1|^2\wedge idz_2\wedge d\bar z_2 = (\partial\tau + \bar\partial\tau)\wedge idz_2\wedge d\bar z_2.$$

Thus, we get the following equalities on $\mathbb{C}^2$:

$$\begin{aligned} i(\bar\partial\tau - \partial\tau)\wedge d(f^\star\omega_0) &= i(\bar\partial\tau - \partial\tau)\wedge(\partial\tau + \bar\partial\tau)\wedge idz_2\wedge d\bar z_2 = -2i\partial\tau\wedge\bar\partial\tau\wedge idz_2\wedge d\bar z_2 \\ &= -2|z_1|^2\, idz_1\wedge d\bar z_1\wedge idz_2\wedge d\bar z_2 = -2|z_1|^2\, dV_0, \end{aligned}$$

where we put $dV_0 := idz_1\wedge d\bar z_1\wedge idz_2\wedge d\bar z_2$.

On the other hand, since $i\partial\bar\partial\tau = idz_1\wedge d\bar{z}_1 + idz_2\wedge d\bar{z}_2$, we have

$$\Lambda_{f^\star\omega_0}(i\partial\bar\partial\tau) = 1 + \frac{1}{1+|z_1|^2}.$$

Therefore, the integral on the left of (2.4) reads in this case:

$$\begin{aligned}\int\limits_{S_t}|d\tau|_{f^\star\omega_0}\,d\sigma_{\omega_0,\,f,\,t} &= 2\int\limits_{B_t}\left(1+\frac{1}{1+|z_1|^2}\right)(1+|z_1|^2)\,dV_0 + 2\int\limits_{B_t}|z_1|^2\,dV_0\\ &= 4\int\limits_{B_t}(1+|z_1|^2)\,dV_0 = 4\,\mathrm{Vol}_{\omega_0,\,f}(B_t), \qquad t>0,\end{aligned}$$

where the last equality was seen in (2.27).

This proves that f satisfies property (i) in Definition 2.2.3. We conclude that the map f has *subexponential growth*, proving that the 3-dimensional Iwasawa manifold X is not divisorially hyperbolic.

(c) Finally, we point out that no *Nakamura manifold* $X = G/\Gamma$ is divisorially hyperbolic, where $G = (\mathbb{C}^3,\,\star)$ is the solvable, non-nilpotent complex Lie group whose group operation is defined as

$$(\zeta_1,\,\zeta_2,\,\zeta_3)\star(z_1,\,z_2,\,z_3) = (\zeta_1+z_1,\,\zeta_2+e^{-\zeta_1}z_2,\,\zeta_3+e^{\zeta_1}z_3),$$

while $\Gamma\subset G$ is a lattice. (See e.g. [Nak75].)

We equip X with the metric

$$\omega_0 = i\eta_1\wedge\bar\eta_1 + i\eta_2\wedge\bar\eta_2 + i\eta_3\wedge\bar\eta_3,$$

where η_1,η_2,η_3 are the holomorphic $(1,\,0)$-forms on X induced respectively by the left-invariant holomorphic $(1,\,0)$-forms $dz_1,\,e^{-z_1}dz_2,\,e^{z_1}dz_3$ on G. If $\pi: G\longrightarrow X$ is the projection, we see that ω_0 lifts to the Hermitian metric

$$\omega = \pi^\star\omega_0 = idz_1\wedge d\bar{z}_1 + e^{-2\mathrm{Re}(z_1)}\,idz_2\wedge d\bar{z}_2 + e^{2\mathrm{Re}(z_1)}\,idz_3\wedge d\bar{z}_3$$

on $G=\mathbb{C}^3$.

Let $j:\mathbb{C}^2\longrightarrow G\simeq\mathbb{C}^3$ be the obvious inclusion $(z_2,\,z_3)\mapsto(0,\,z_2,\,z_3)$. Then, for the non-degenerate holomorphic map $f=\pi\circ j:\mathbb{C}^2\longrightarrow X$, we get

$$f^\star\omega_0 = j^\star(\pi^\star\omega_0) = idz_2\wedge d\bar{z}_2 + idz_3\wedge d\bar{z}_3$$

on $\mathbb{C}^2$. Thus, $f^\star\omega_0$ is the Euclidean metric of $\mathbb{C}^2$, so f has *subexponential growth* by Lemma 2.2.4, proving that the Nakamura manifold X is *not divisorially hyperbolic*.

2.3 Divisorially Kähler and divisorially nef classes

The starting point is the following simple, but key observation.

Lemma 2.3.1. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The map:*

$$P = P^{n-1}_{n-1,\,n-1} : H^2_{DR}(X,\,\mathbb{R}) \longrightarrow H^{n-1,\,n-1}_A(X,\,\mathbb{R}), \qquad \{\alpha\}_{DR} \longmapsto \{(\alpha^{n-1})^{n-1,\,n-1}\}_A, \tag{2.28}$$

is **well defined** *in the sense that it is independent of the choice of a C^∞ representative α of its De Rham cohomology class, where $(\alpha^{n-1})^{n-1,\,n-1}$ is the component of bidegree $(n-1,\,n-1)$ of the $(2n-2)$-form α^{n-1}.*

This follows from the following

Lemma 2.3.2. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.*

(i) For any $k \in \{0, \dots, 2n\}$, any form $\alpha \in C^\infty_k(X,\,\mathbb{C})$ such that $d\alpha = 0$ and any bidegree $(p,\,q)$ with $p+q=k$, we have

$$\partial\bar\partial\alpha^{p,\,q} = 0,$$

where $\alpha^{p,\,q}$ is the $(p,\,q)$-type component of α.

In particular, for every 2-form α such that $d\alpha = 0$, we have $\partial\bar\partial(\alpha^{n-1})^{n-1,\,n-1} = 0$, so $(\alpha^{n-1})^{n-1,\,n-1}$ defines an Aeppli cohomology class $\{(\alpha^{n-1})^{n-1,\,n-1}\}_A \in H^{n-1,\,n-1}_A(X,\,\mathbb{C})$.

(ii) For any 2-forms α_1 and α_2 such that $d\alpha_1 = d\alpha_2 = 0$ and $\alpha_1 = \alpha_2 + d\beta$ for some 1-form β, we have

$$\{(\alpha_1^{n-1})^{n-1,\,n-1}\}_A = \{(\alpha_2^{n-1})^{n-1,\,n-1}\}_A.$$

Proof. (i) Writing the decomposition $\alpha = \sum\limits_{r+s=k}\alpha^{r,\,s}$ of α into pure-type forms, we see that the hypothesis $d\alpha = 0$ is equivalent to $\partial\alpha^{r,\,s} + \bar\partial\alpha^{r+1,\,s-1} = 0$ for all $(r,\,s)$. Applying $\bar\partial$, this implies that $\partial\bar\partial\alpha^{r,\,s} = 0$ for all $(r,\,s)$.

(ii) Taking the $(n-1)$-st power in $\alpha_1 = \alpha_2 + d\beta$ and using the fact that $d\alpha_2 = 0$, we get:

$$\alpha_1^{n-1} = \alpha_2^{n-1} + \sum_{k=1}^{n-1}\binom{n-1}{k}\, d(\alpha_2^{n-1-k}\wedge\beta\wedge(d\beta)^{k-1}).$$

Hence, $\alpha_1^{n-1} - \alpha_2^{n-1} \in \operatorname{Im} d$, which implies that $(\alpha_1^{n-1})^{n-1,\,n-1} - (\alpha_2^{n-1})^{n-1,\,n-1} \in \operatorname{Im}\partial + \operatorname{Im}\bar\partial$. □

2.3.1 Case of projective manifolds

Let X be a **projective** manifold with $\dim_{\mathbb{C}} X = n$. As is well known (see e. g. [Ha70]), a holomorphic line bundle $L \longrightarrow X$ is said to be **nef** if

$$L.C := \int\limits_C c_1(L) \geq 0$$

for every *curve* $C \subset X$, where $c_1(L) = \{\frac{i}{2\pi}\Theta_h(L)\}_{DR} \in H^{1,\,1}(X,\,\mathbb{R}) \cap H^2(X,\,\mathbb{Z})$ is the first Chern class of L, namely the De Rham cohomology class of the curvature form of L with respect to any Hermitian metric h on L.

We now generalise this notion in the context of divisors (rather than curves) and of possibly non-integral and non-type (1, 1) real De Rham cohomology classes using the Serre-type duality (3.6).

Definition 2.3.3. *Let X be a* **projective** *manifold with $dim_{\mathbb{C}} X = n$. A cohomology class $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$ is said to be* **projectively divisorially nef** *if*

$$P(\{\alpha\}_{DR}).\{[D]\}_{BC} := \int\limits_D (\alpha^{n-1})^{n-1,\,n-1} \geq 0$$

for all effective **divisors** *$D \geq 0$ on X and some (hence any) representative $\alpha \in C^\infty_2(X, \mathbb{R})$ of $\{\alpha\}_{DR}$.*

As is well known, the current of integration $[D]$ on an effective divisor D is a closed positive $(1, 1)$-current. (By a $(1, 1)$-current we mean a current of *bidegree* $(1, 1)$.) However, not every such current T is the current of integration on an effective divisor D. Nevertheless, we notice that Definition 2.3.3 does not change if divisors are replaced by currents whose cohomology classes lie in the real vector space $NS_{\mathbb{R}}(X)$ spanned by *integral* $(1, 1)$-cohomology classes, known as the Neron-Severi subspace of $H^2_{DR}(X, \mathbb{R})$.

Proposition 2.3.4. *Let X be a* **projective** *manifold with $dim_{\mathbb{C}} X = n$. Fix any cohomology class $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$. Then, $\{\alpha\}_{DR}$ is* **projectively divisorially nef** *if and only if*

$$\{(\alpha^{n-1})^{n-1,\,n-1}\}_A.\{T\}_{BC} := \int\limits_X (\alpha^{n-1})^{n-1,\,n-1} \wedge T \geq 0$$

for all closed positive $(1, 1)$-currents $T \geq 0$ on X such that $\{T\}_{DR} \in NS_{\mathbb{R}}(X)$ and some (hence any) representative $\alpha \in C^\infty_2(X, \mathbb{R})$ of $\{\alpha\}_{DR}$.

Proof. It is proved in (b) of Proposition 6.6. in [Dem00], as a consequence of Nadel's Vanishing Theorem, that the integral part of the pseudo-effective cone of X (i.e. the set of cohomology classes of closed positive $(1, 1)$-currents that are linear combinations with real coefficients of integral classes) is the closure of the *effective cone* of X (i.e. the set of cohomology classes of effective divisors). This means that, for every closed positive $(1, 1)$-current $T \geq 0$ on X such that $\{T\}_{DR} \in NS_{\mathbb{R}}(X)$, the class $\{T\}_{BC}$ is a limit of classes $\{[D_j]\}_{BC}$ with $(D_j)_{j\in\mathbb{N}}$ effective divisors on X. This suffices to conclude. □

We now observe a useful property of the set of projectively divisorially nef classes.

Proposition 2.3.5. *Let X be a* **projective** *manifold. The set*

$$\mathcal{PDN}_X := \left\{\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R}) \mid \{\alpha\}_{DR} \textit{ is projectively divisorially nef}\right\}$$

is a **closed** *cone in $H^2_{DR}(X, \mathbb{R})$.*

Proof. To show that $\mathcal{PDN}_X$ is a cone, we have to show that it is stable under multiplications by non-negative reals, which is obvious.

To show closedness, let $\{\alpha\}_{DR} \in H^2_{DR}(X,\,\mathbb{R})$ be a limit of classes $\{\alpha_j\}_{DR} \in \mathcal{PDN}_X$. Then, the class $\{\alpha_j\}_{DR} - \{\alpha\}_{DR}$ converges to 0 in $H^2_{DR}(X,\,\mathbb{R})$ as $j \to +\infty$. By the definition of the quotient topology of $H^2_{DR}(X,\,\mathbb{R})$, there exists a sequence of C^∞ d-closed 2-forms $\beta_j \in \{\alpha_j\}_{DR} - \{\alpha\}_{DR}$ such that $\beta_j \longrightarrow 0$ in the C^∞ topology as $j \to +\infty$. Now, pick an arbitrary C^∞ representative α of the class $\{\alpha\}_{DR}$. We infer that $\alpha_j := \alpha + \beta_j$ represents the class $\{\alpha_j\}_{DR}$ for every j and $\lim_{j\to+\infty} \alpha_j = \alpha$ in the C^∞ topology.

Thus, for every effective divisor D on X, we have:

$$\int\limits_D (\alpha_j^{n-1})^{n-1,\,n-1} \geq 0 \quad \text{for all} \quad j \in \mathbb{N} \quad \text{and} \quad \lim_{j\to+\infty} \int\limits_D (\alpha_j^{n-1})^{n-1,\,n-1} = \int\limits_D (\alpha^{n-1})^{n-1,\,n-1},$$

where the first inequality follows from the assumption $\{\alpha_j\}_{DR} \in \mathcal{PDN}_X$ for all j. Therefore, $\int_D (\alpha^{n-1})^{n-1,\,n-1} \geq 0$ for every effective divisor D, proving that the class $\{\alpha\}_{DR}$ is projectively divisorially nef. $\square$

2.3.2 Case of arbitrary compact complex manifolds

Let X be a compact complex n-dimensional manifold. Recall the obvious inclusion $\mathcal{SG}_X \subset \mathcal{G}_X$ of the *strongly Gauduchon (sG) cone* of X in the *Gauduchon cone*. (See the introduction for a reminder of the definitions.) The equality $\mathcal{SG}_X = \mathcal{G}_X$ is equivalent to every Gauduchon metric on X being sG (see [PU18, Lemma 1.3]). Manifolds X with this property are called **sGG manifolds**; they were studied in [PU18].

The definition of divisorially nef classes given in (ii) of the next Definition 2.3.6 on an arbitrary compact complex manifold will be shown in (b) of Proposition 2.3.8 to imply the projectively divisorially nef property defined on projective manifolds in Definition 2.3.3.

Definition 2.3.6. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ and let*

$$P : H^2_{DR}(X,\,\mathbb{R}) \longrightarrow H^{n-1,\,n-1}_A(X,\,\mathbb{R}), \qquad \{\alpha\}_{DR} \longmapsto \{(\alpha^{n-1})^{n-1,\,n-1}\}_A,$$

be the map of Lemma 2.3.1.

(i) A cohomology class $\{\alpha\}_{DR} \in H^2_{DR}(X,\,\mathbb{R})$ is said to be **divisorially Kähler** *if $P(\{\alpha\}_{DR}) \in \mathcal{G}_X$. The set*

$$\mathcal{DK}_X := \left\{ \{\alpha\}_{DR} \in H^2_{DR}(X,\,\mathbb{R}) \mid \{\alpha\}_{DR} \text{ is divisorially Kähler} \right\}$$

is called the **divisorially Kähler cone** *of X.*

(ii) A cohomology class $\{\alpha\}_{DR} \in H^2_{DR}(X,\,\mathbb{R})$ is said to be **divisorially nef** *if $P(\{\alpha\}_{DR}) \in \overline{\mathcal{G}}_X$, where $\overline{\mathcal{G}}_X$ is the closure of the Gauduchon cone in $H^{n-1,\,n-1}_A(X,\,\mathbb{R})$. The set*

$$\mathcal{DN}_X := \left\{ \{\alpha\}_{DR} \in H^2_{DR}(X,\,\mathbb{R}) \mid \{\alpha\}_{DR} \text{ is divisorially nef} \right\}$$

is called the **divisorially nef cone** *of X.*

Note that $\mathcal{DK}_X$ and $\mathcal{DN}_X$ are *cones* in $H^2_{DR}(X,\mathbb{R})$ in the sense that they are stable under multiplications by positive reals. However, they are not convex and are not stable under additions since the map P is not linear.

Proposition 2.3.7. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The* **divisorially Kähler cone** *of X can be described as*

$$\mathcal{DK}_X = P^{-1}(\mathcal{SG}_X) = P^{-1}(\mathcal{G}_X) \subset H^2_{DR}(X,\mathbb{R}). \tag{2.29}$$

In particular, $\mathcal{DK}_X$ is **open** *in $H^2_{DR}(X,\mathbb{R})$ and the following implication holds:*

$$\mathcal{DK}_X \neq \emptyset \implies X \text{ is an sG manifold.} \tag{2.30}$$

Proof. The identity $\mathcal{DK}_X = P^{-1}(\mathcal{G}_X)$ holds by the definition of $\mathcal{DK}_X$. Meanwhile, $P^{-1}(\mathcal{SG}_X) \subset P^{-1}(\mathcal{G}_X)$ since $\mathcal{SG}_X \subset \mathcal{G}_X$. So, it suffices to prove the inclusion $\mathcal{DK}_X \subset P^{-1}(\mathcal{SG}_X)$.

Let $\{\alpha\}_{DR} \in \mathcal{DK}_X$. Pick an arbitrary smooth representative $\alpha \in \{\alpha\}_{DR}$. Since $P(\{\alpha\}_{DR}) \in \mathcal{G}_X$, there exists a Gauduchon metric ω on X such that

$$(\alpha^{n-1})^{n-1,\,n-1} = \omega^{n-1} + \partial u^{n-2,\,n-1} + \bar\partial u^{n-1,\,n-2}$$

for some smooth forms $u^{n-2,\,n-1}$ and $u^{n-1,\,n-2}$ of the displayed bidegrees. These forms can be chosen to be conjugate to each other since α and ω are real. We get:

$$(\alpha^{n-1})^{n-1,\,n-1} = \omega^{n-1} + (d(u^{n-2,\,n-1} + u^{n-1,\,n-2}))^{n-1,\,n-1},$$

so ω^{n-1} is the $(n-1,\,n-1)$-component of the smooth real d-closed $(2n-2)$-form $\alpha^{n-1} - d(u^{n-2,\,n-1} + u^{n-1,\,n-2})$. This proves that ω is strongly Gauduchon (see [Pop13, Proposition 4.2.]). Since $\{\omega^{n-1}\}_A = \{(\alpha^{n-1})^{n-1,\,n-1}\}_A = P(\{\alpha\}_{DR})$, we infer that $P(\{\alpha\}_{DR}) \in \mathcal{SG}_X$. This proves the inclusion $\mathcal{DK}_X \subset P^{-1}(\mathcal{SG}_X)$.

The openness of $\mathcal{DK}_X$ in $H^2_{DR}(X,\mathbb{R})$ follows from the openness of $\mathcal{G}_X$ in $H^{n-1,\,n-1}_A(X,\mathbb{R})$ and from the continuity of the map P.

Finally, to prove implication (2.30), suppose there exists $\{\alpha\}_{DR} \in \mathcal{DK}_X$. Then $P(\{\alpha\}_{DR}) \in \mathcal{SG}_X$, so $\mathcal{SG}_X \neq \emptyset$. The last piece of information is equivalent to X being an sG manifold. $\square$

Part (c) of the following result gives, on any compact complex manifold, an alternative definition of a *divisorially nef* class that is analogous to the classical analytic definition of a nef class given in [Dem92].

Proposition 2.3.8. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.*

(a) The divisorially nef cone $\mathcal{DN}_X$ is **closed** *in $H^2_{DR}(X,\mathbb{R})$. In particular,*

$$\overline{\mathcal{DK}}_X \subset \mathcal{DN}_X, \tag{2.31}$$

where $\overline{\mathcal{DK}}_X$ is the closure of the divisorially Kähler cone in $H^2_{DR}(X,\mathbb{R})$.

(b) For every class $\{\alpha\}_{DR} \in H^2_{DR}(X,\mathbb{R})$, the following equivalence holds:

$$\{\alpha\}_{DR} \in \mathcal{DN}_X \iff P(\{\alpha\}_{DR}).\{T\}_{BC} \geq 0 \quad \textit{for every } \{T\}_{BC} \in \mathcal{E}_X, \tag{2.32}$$

where $\mathcal{E}_X \subset H^{1,1}_{BC}(X, \mathbb{R})$ is the pseudo-effective cone of X consisting of the Bott-Chern cohomology classes of all closed positive $(1, 1)$-currents $T \geq 0$ on X.

In particular, if X is projective, a class $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$ is projectively divisorially nef in the sense of Definition 2.3.3 whenever $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$ is divisorially nef in the sense of Definition 2.3.6.

(c) A class $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$ is **divisorially nef** *if and only if for every constant $\varepsilon > 0$, there exists a representative $\Omega_\varepsilon \in C^\infty_{n-1,\,n-1}(X, \mathbb{R})$ of the class $P(\{\alpha\}_{DR})$ such that*

$$\Omega_\varepsilon \geq -\varepsilon\, \omega^{n-1},$$

where $\omega > 0$ is an arbitrary Hermitian metric on X fixed beforehand.

Proof. Since X is compact, any two Hermitian metrics on X are comparable. Meanwhile, a Gauduchon metric always exists on X by [Gau77a], so we may assume that the background metric ω on X is actually Gauduchon.

(a) The first statement is an immediate consequence of the continuity of P and of the identity $\mathcal{DN}_X = P^{-1}(\overline{\mathcal{G}}_X)$ defining the divisorially nef cone. Inclusion (2.31) follows from the closedness of $\mathcal{DN}_X$ and from the obvious inclusion $\mathcal{DK}_X \subset \mathcal{DN}_X$.

(b) The first statement follows from the duality between the pseudo-effective cone $\mathcal{E}_X \subset H^{1,1}_{BC}(X, \mathbb{R})$ and the closure of the Gauduchon cone $\overline{\mathcal{G}}_X \subset H^{n-1,\,n-1}_A(X, \mathbb{R})$ under duality (3.6) between $H^{1,1}_{BC}(X, \mathbb{C})$ and $H^{n-1,\,n-1}_A(X, \mathbb{C})$. This cone duality, observed in [Pop15b] as a reformulation of Lamari's duality Lemma 3.3 in [Lam99], implies that, given any class $\mathfrak{c}^{n-1,\,n-1}_A \in H^{n-1,\,n-1}_A(X, \mathbb{R})$, the following equivalence holds:

$$\mathfrak{c}^{n-1,\,n-1}_A \in \overline{\mathcal{G}}_X \iff \mathfrak{c}^{1,1}_{BC}.\mathfrak{c}^{n-1,\,n-1}_A \geq 0 \quad \text{for every class } \mathfrak{c}^{1,1}_{BC} \in \mathcal{E}_X.$$

In our case, it suffices to apply this duality to $\mathfrak{c}^{n-1,\,n-1}_A := P(\{\alpha\}_{DR})$ to get equivalence (2.32).

When X is projective, the second statement follows from (2.32) and from Proposition 2.3.4.

(c) "$\Longleftarrow$" Fix a *Gauduchon* metric ω on X and a class $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$. Suppose that, for every $\varepsilon > 0$, the class $P(\{\alpha\}_{DR})$ can be represented by a form $\Omega_\varepsilon \in C^\infty_{n-1,\,n-1}(X, \mathbb{R})$ such that $\Omega_\varepsilon \geq -\varepsilon\, \omega^{n-1}$. Then, $\Omega_\varepsilon + 2\varepsilon\, \omega^{n-1} \geq \varepsilon\, \omega^{n-1} > 0$ and $\Omega_\varepsilon + 2\varepsilon\, \omega^{n-1}$ is $\partial\bar\partial$-closed, so it is the $(n-1)$-st power of a Gauduchon metric. (See [Mic83] for the existence of a unique $(n-1)$-st root for any positive definite $(n-1,\, n-1)$-form on an n-dimensional complex manifold.) Hence,

$$\mathfrak{c}_\varepsilon := \{\Omega_\varepsilon\}_A + 2\varepsilon\, \{\omega^{n-1}\}_A = P(\{\alpha\}_{DR}) + 2\varepsilon\, \{\omega^{n-1}\}_A \in \mathcal{G}_X, \qquad \varepsilon > 0,$$

so $\mathfrak{c}_\varepsilon \longrightarrow P(\{\alpha\}_{DR})$ as $\varepsilon \to 0$. Therefore, $P(\{\alpha\}_{DR}) \in \overline{\mathcal{G}}_X$, which amounts to $\{\alpha\}_{DR}$ being divisorially nef.

"$\Longrightarrow$" Suppose $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{R})$ is divisorially nef. Then $P(\{\alpha\}_{DR}) \in \overline{\mathcal{G}}_X$, so there exists a sequence of classes $\mathfrak{c}_k \in \mathcal{G}_X$ such that $\mathfrak{c}_k \longrightarrow P(\{\alpha\}_{DR})$ as $k \to +\infty$. Thus, $P(\{\alpha\}_{DR}) - \mathfrak{c}_k \longrightarrow 0$ in $H^{n-1,\,n-1}_A(X, \mathbb{R})$, so, from the definition of the quotient topology, we infer the existence of a sequence of real C^∞ representatives $\Gamma_k \in P(\{\alpha\}_{DR}) - \mathfrak{c}_k$ such that $\Gamma_k \longrightarrow 0$ in the C^∞ topology (hence also in the C^0

topology) as $k \to +\infty$. This implies that, for every $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\Gamma_k \geq -\varepsilon\,\omega^{n-1}, \qquad k \geq k_\varepsilon,$$

where ω is an arbitrarily fixed Gauduchon metric on X.

On the other hand, for every $k \in \mathbb{N}$, pick a Gauduchon metric ω_k on X such that $\omega_k^{n-1} \in \mathfrak{c}_k$. (This is possible since $\mathfrak{c}_k \in \mathcal{G}_X$.) We infer that

$$P(\{\alpha\}_{DR}) \ni \Omega_\varepsilon := \Gamma_{k_\varepsilon} + \omega_{k_\varepsilon}^{n-1} \geq -\varepsilon\,\omega^{n-1}, \qquad \varepsilon > 0.$$

This proves the contention. $\square$

Question 2.3.9. *If $\mathcal{DK}_X \neq \emptyset$, is (2.31) an equality?*

2.3.3 Examples

(1) Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Suppose there exists a *Hermitian-symplectic (H-S) structure* on X. According to [ST10, Definition 1.5], this is a real C^∞ d-closed 2-form $\widetilde{\omega}$ on X whose component ω of type $(1,\,1)$ is positive definite. It is easy to see that the $(n-1,\,n-1)$-component $(\widetilde{\omega}^{n-1})^{n-1,\,n-1}$ of the $(n-1)$-st power of $\widetilde{\omega}$ is *positive definite* on X. (See [YZZ19, §.2, Lemma 1] or [DP20, Proposition 2.1].) Since it is also $\partial\bar\partial$-closed, it defines an element $P(\{\widetilde{\omega}\}_{DR}) = \{(\widetilde{\omega}^{n-1})^{n-1,\,n-1}\}_A$ in the Gauduchon cone $\mathcal{G}_X$. Thus, we get

Proposition 2.3.10. *If $\widetilde{\omega}$ is a* **Hermitian-symplectic** *structure on a compact complex manifold X, the cohomology class $\{\widetilde{\omega}\}_{DR} \in H^2_{DR}(X,\,\mathbb{R})$ is* **divisorially Kähler**.

(2) From the equivalence of (i) and (iii) in Proposition 2.2.12, we deduce the following

Proposition 2.3.11. *If X is a* **degenerate balanced** *compact complex manifold, $\mathcal{DK}_X = H^2_{DR}(X,\,\mathbb{R})$.*

(3) Let $L \longrightarrow X$ be a C^∞ (not necessarily holomorphic) complex line bundle over an n-dimensional compact complex manifold X. For any C^∞ Hermitian metric h on L, the curvature form $\alpha = \frac{i}{2\pi}\Theta_h(L)$ is a C^∞ real d-closed 2-form on X which represents the first Chern class $c_1(L) \in H^2(X,\,\mathbb{Z})$ of X. Moreover, $c_1(L)$ is of type $(1,\,1)$ if and only if L is *holomorphic*. We say that L is divisorially nef if $c_1(L)$ is.

Similarly, when X is *projective*, we say that L is projectively divisorially nef if $c_1(L)$ is. In this case, we have:

$$L \;\text{ is }\textbf{projectively divisorially nef} \iff P(c_1(L)).\{[D]\}_{BC} := \int\limits_D \left(\left(\frac{i}{2\pi}\Theta_h(L)\right)^{n-1}\right)^{n-1,\,n-1} \geq 0$$

for all effective divisors D on X, where $P : H^2_{DR}(X,\,\mathbb{R}) \longrightarrow H^{n-1,\,n-1}_A(X,\,\mathbb{R})$ is the map (2.28).

(4) If $L \longrightarrow X$ is a *holomorphic* line bundle over an n-dimensional compact complex manifold X, then its curvature form $\frac{i}{2\pi}\Theta_h(L)$ with respect to any C^∞ Hermitian metric h on L is of type (1, 1). Hence, if X is *projective*, we have:

$$L \quad \text{is } \textbf{projectively divisorially nef} \quad \iff c_1(L)^{n-1}.\{[D]\}_{BC} := \int\limits_D \left(\frac{i}{2\pi}\Theta_h(L)\right)^{n-1} \geq 0$$

for all effective divisors $D \geq 0$ on X.

A well-known result (see e.g. [Ha70, §.6, p. 34-36]) tells us that projectively divisorially nef holomorphic line bundles are, indeed, generalisations of nef such bundles.

Theorem 2.3.12. *Let $L \longrightarrow X$ be a holomorphic line bundle over a* **projective** *manifold X. The following implication holds:*

$$L \quad \textit{is nef} \quad \Longrightarrow \quad L \quad \textit{is projectively divisorially nef.}$$

Proof. This follows at once from Kleiman's Theorem 6.1. in [Ha70, p. 34-36] which states that L being nef is equivalent to $L^p.Y := \int_Y c_1(L)^p \geq 0$ for every p-dimensional subvariety $Y \subset X$, for all $1 \leq p \leq \dim_{\mathbb{C}} X$. $\square$

Chapter 3

Some Properties of Balanced Hyperbolic Compact Complex Manifolds

3.1 Introduction

In this chapter, we continue the study of compact complex *balanced hyperbolic* manifolds that we talk about in the previous chapter as generalisations in the possibly non-projective and even non-Kähler context of the classical notions of *Kähler hyperbolic* (in the sense of Gromov) and *Kobayashi/Brody hyperbolic* manifolds.

We now outline the specific properties of these classes of manifolds.

(I) Case of balanced and degenerate balanced manifolds

In the first part of the chapter, we obtain some general results on compact complex manifolds carrying *balanced* metrics (and, in some cases, results on *Gauduchon* metrics) and then we use them to infer vanishing results for *degenerate balanced* manifolds. See §.3.2.1 for a reminder of the terminology used in what follows.

(a) Our first main result, obtained as a consequence of the computation in Lemma 3.2.1, is a **Hard Lefschetz Isomorphism** between the De Rham cohomologies of **degrees 1** and **2n-1** that holds on any compact complex *balanced* manifold satisfying a mild $\partial\bar\partial$-type condition.

Theorem 3.1.1. *Let X be a compact complex manifold with $dim_{\mathbb{C}}X = n$.*

(i) If ω is a **balanced** *metric on X, the linear map:*

$$\{\omega_{n-1}\}_{DR}\wedge\cdot : H^1_{DR}(X,\,\mathbb{C}) \longrightarrow H^{2n-1}_{DR}(X,\,\mathbb{C}), \qquad \{u\}_{DR} \longmapsto \{\omega_{n-1}\wedge u\}_{DR}, \tag{3.1}$$

is well defined and depends only on the cohomology class $\{\omega_{n-1}\}_{DR} \in H^{2n-2}_{DR}(X,\,\mathbb{C})$, where $\omega_{n-1} := \omega^{n-1}/(n-1)!$.

(ii) If, moreover, X has the following additional property: for every form $v \in C^\infty_{1,1}(X,\,\mathbb{C})$ such that $dv = 0$, the following implication holds:

$$v \in Im\,\partial \implies v \in Im\,(\partial\bar\partial), \tag{3.2}$$

the map (3.1) is an **isomorphism**.

As a consequence of this discussion, we obtain the following vanishing properties for the cohomology of degenerate balanced manifolds.

Proposition 3.1.2. *Let X be a compact* **degenerate balanced** *manifold.*

(i) The Bott-Chern cohomology groups of types $(1,\,0)$ *and* $(0,\,1)$ *of X vanish:* $H^{1,\,0}_{BC}(X,\,\mathbb{C})=0$ *and* $H^{0,\,1}_{BC}(X,\,\mathbb{C})=0$.

(ii) If, moreover, X satisfies hypothesis (3.2), its De Rham cohomology group of degree 1 *vanishes:* $H^1_{DR}(X,\,\mathbb{C})=0$.

Note that degenerate balanced manifolds that satisfy hypothesis (3.2) do exist. Indeed, Friedman showed in [Fri17] that the manifolds $X=\sharp_k(S^3\times S^3)$, with $k\geq 2$, endowed with the Friedman-Lu-Tian complex structure constructed via conifold transitions ([Fri89], [LT93], [FLY12]) are even $\partial\bar\partial$-manifolds.

(b) Our study of the cohomology of **degree 2** in this setting centres on seeking out possible positivity properties of balanced hyperbolic manifolds. As an alternative to question 2.1.5, wondering about possible positivity properties, in the senses defined therein, of the canonical bundle K_X of any balanced hyperbolic manifold X, we concentrate this time on whether there are "many" (in a sense to be determined) closed positive currents T of bidegree $(1,\,1)$ on such a manifold.

The starting point of this investigation is Proposition 5.4 in [Pop15a], reproduced as Proposition 2.2.12 a compact complex manifold X is *degenerate balanced* if and only if there exists no non-zero d-closed $(1,1)$-current $T\geq 0$ on X. In other words, the compact *degenerate balanced* manifolds X are characterised by their pseudo-effective cone $\mathcal{E}_X$ (namely, the set of Bott-Chern cohomology classes of d-closed positive $(1,1)$-currents on X) being reduced to the zero class.

This prompts one to ask the following

Question 3.1.3. *Let X be a compact complex manifold. Is it true that X is* **balanced hyperbolic** *if and only if its pseudo-effective cone $\mathcal{E}_X$ is* **small** *(in a sense to be determined)?*

In §.3.2.3 and §.3.2.4 we provide some evidence for this by first showing that both the *balanced* hypothesis on a given Hermitian metric ω (see Lemma and Definition 3.2.2) and the *Gauduchon* hypothesis (see Lemma and Definition 3.2.12) enable one to define a notion of *ω-primitive* De Rham cohomology classes of degree 2 (resp. *ω-primitive* Bott-Chern cohomology classes of bidegree $(1,\,1)$). For example, if ω is *balanced*, we set

$$H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim}:=\ker\left(\{\omega_{n-1}\}_{DR}\wedge\cdot\right)\subset H^2_{DR}(X,\,\mathbb{C}),$$

after we have showed that the linear map:

$$\{\omega_{n-1}\}_{DR}\wedge\cdot:H^2_{DR}(X,\,\mathbb{C})\longrightarrow H^{2n}_{DR}(X,\,\mathbb{C})\simeq\mathbb{C},\qquad\{\alpha\}_{DR}\longmapsto\{\omega_{n-1}\wedge\alpha\}_{DR},$$

is well defined and depends only on the cohomology class $\{\omega_{n-1}\}_{DR} \in H^{2n-2}_{DR}(X,\,\mathbb{C})$. We go on to show that a class $\mathfrak{c} \in H^2_{DR}(X,\,\mathbb{C})$ is ω-primitive if and only if it can be represented by an ω-primitive form (cf. Lemma 3.2.3), a fact that does not seem to hold in the Gauduchon context of §.3.2.4. We then show that, when the balanced metric ω is *not degenerate balanced*, the ω-primitive classes form a complex hyperplane $H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim}$ in $H^2_{DR}(X,\,\mathbb{C})$ that depends only on the balanced class $\{\omega_{n-1}\}_{DR}$. (See Corollary 3.2.5.) Finaly, we are able to define a *positive side* $H^2_{DR}(X,\,\mathbb{R})^+_\omega$ and a *negative side* $H^2_{DR}(X,\,\mathbb{R})^-_\omega$ of the hyperplane $H^2_{DR}(X,\,\mathbb{R})_{\omega\text{-}prim} := H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim} \cap H^2_{DR}(X,\,\mathbb{R})$ in $H^2_{DR}(X,\,\mathbb{R})$ and get a *partition* of $H^2_{DR}(X,\,\mathbb{R})$:

$$H^2_{DR}(X,\,\mathbb{R}) = H^2_{DR}(X,\,\mathbb{R})^+_\omega \cup H^2_{DR}(X,\,\mathbb{R})_{\omega\text{-}prim} \cup H^2_{DR}(X,\,\mathbb{R})^-_\omega.$$

A similar study of the case where ω is only a Gauduchon metric in §.3.2.4 leads to the characterisation of the pseudo-effective cone as the intersection of the *non-negative sides* of all the hyperplanes $H^{1,\,1}_{BC}(X,\,\mathbb{R})_{\omega\text{-}prim}$ determined by Aeppli cohomology classes $[\omega_{n-1}]_A$ of Gauduchon metrics ω on X:

$$\mathcal{E}_X = \bigcap_{[\omega_{n-1}]_A \in \mathcal{G}_X} H^{1,\,1}_{BC}(X,\,\mathbb{R})^{\geq 0}_\omega, \tag{3.3}$$

where $\mathcal{G}_X$ is the *Gauduchon cone* of X (introduced in [Pop15a] as the set of all such Aeppli cohomology classes, see §.3.2.1 for a reminder of the terminology).

In §.3.3.1, we answer a version of Question 3.1.3 on the universal covering space of a balanced hyperbolic manifold in the following form. Throughout the chapter, $L^p_{\widetilde{\omega}}$, L^p_ω, resp. L^p_g will stand for the space of objects that are L^p with respect to the metric $\widetilde{\omega}$, ω, resp. g.

Proposition 3.1.4. *Let $(X,\,\omega)$ be a* **balanced hyperbolic manifold** *and let $\pi : \widetilde{X} \longrightarrow X$ be the universal cover of X. There exists no non-zero d-closed positive $(1,\,1)$-current $\widetilde{T} \geq 0$ on $\widetilde{X}$ such that $\widetilde{T}$ is $L^1_{\widetilde{\omega}}$, where $\widetilde{\omega} := \pi^\star\omega$ is the lift of ω to $\widetilde{X}$.*

This result provides the link with the second part of this chapter that we now briefly outline.

(II) Case of balanced hyperbolic manifolds

The results in the second part of the chapter mirror, to some extent, those in the first part. The main difference is that the stage changes from X to its universal covering space $\widetilde{X}$. Specifically, when X is supposed to be balanced hyperbolic, we obtain vanishing theorems for the L^2 harmonic cohomology of $\widetilde{X}$.

(a) In this setting, our main result in **degree 1** and its dual **degree** $2n-1$ is the following

Theorem 3.1.5. *Let X be a compact complex* **balanced hyperbolic** *manifold with $\dim_{\mathbb{C}} X = n$. Let $\pi : \widetilde{X} \longrightarrow X$ be the universal cover of X and $\widetilde{\omega} := \pi^\star\omega$ the lift to $\widetilde{X}$ of a balanced hyperbolic metric ω on X.*

There are no non-zero $\Delta_{\widetilde\omega}$-harmonic $L^2_{\widetilde\omega}$-forms of pure types and of degrees 1 *and* $2n-1$ *on* $\widetilde X$*:*

$$\mathcal{H}^{1,\,0}_{\Delta_{\widetilde\omega}}(\widetilde X,\,\mathbb{C}) = \mathcal{H}^{0,\,1}_{\Delta_{\widetilde\omega}}(\widetilde X,\,\mathbb{C}) = 0 \quad \text{and} \quad \mathcal{H}^{n,\,n-1}_{\Delta_{\widetilde\omega}}(\widetilde X,\,\mathbb{C}) = \mathcal{H}^{n-1,\,n}_{\Delta_{\widetilde\omega}}(\widetilde X,\,\mathbb{C}) = 0,$$

where $\Delta_{\widetilde\omega} := dd^\star_{\widetilde\omega} + d^\star_{\widetilde\omega}d$ *is the d-Laplacian induced by the metric* $\widetilde\omega$*.*

The differential operators $d, d^\star_{\widetilde\omega}, \Delta_{\widetilde\omega}$ and all the similar ones are considered as *closed* and *densely defined* unbounded operators on the spaces $L^2_k(\widetilde X,\,\mathbb{C})$ of $L^2_{\widetilde\omega}$-forms of degree k on the *complete* complex manifold $(\widetilde X,\,\widetilde\omega)$. (See reminder of some basic results on complete Riemannian manifolds and unbounded operators in §.3.3.1.)

(b) To introduce our results in **degree 2**, we start by reminding the reader of the following facts of [Dem84] (see also [Dem97, VII, §.1]). For any Hermitian metric ω on a complex manifold X with $\dim_{\mathbb{C}}X = n$, one defines the *torsion operator* $\tau = \tau_\omega := [\Lambda_\omega,\,\partial\omega\wedge\cdot]$ of order 0 and of type (1, 0) acting on the differential forms of X, where Λ_ω is the adjoint of the multiplication operator $\omega\wedge\cdot$ w.r.t. the pointwise inner product $\langle\ ,\ \rangle_\omega$ defined by ω. The Kähler commutation relations generalise to the arbitrary Hermitian context as

$$i[\Lambda_\omega,\,\bar\partial] = \partial^\star + \tau^\star \tag{3.4}$$

and the three other relations obtained from this one by conjugation and/or adjunction. (See [Dem97, VII, §.1, Theorem 1.1.].) Moreover, considering the torsion-twisted Laplacians

$$\Delta_\tau := [d+\tau,\,d^\star+\tau^\star] \quad \text{and} \quad \Delta'_\tau := [\partial+\tau,\,\partial^\star+\tau^\star],$$

the following formula holds (see [Dem97, VII, §.1, Proposition 1.16.]):

$$\Delta_\tau = \Delta'_\tau + \Delta''. \tag{3.5}$$

When the metric ω is Kähler, one has $\tau = 0$ and one recovers the classical formula $\Delta = \Delta' + \Delta''$. However, we will deal with a more general, possibly non-Kähler, case.

In the context of balanced hyperbolic manifolds, our main result in **degree 2** is the following

Theorem 3.1.6. *Let X be a compact complex* **balanced hyperbolic** *manifold with* $\dim_{\mathbb{C}}X = n$*. Let* $\pi : \widetilde X \longrightarrow X$ *be the universal cover of X and* $\widetilde\omega := \pi^\star\omega$ *the lift to* $\widetilde X$ *of a balanced hyperbolic metric ω on X.*

There are no non-zero semi-positive $\Delta_{\widetilde\tau}$-harmonic $L^2_{\widetilde\omega}$-forms of pure type (1, 1) *on* $\widetilde X$*:*

$$\left\{\alpha^{1,\,1}\in\mathcal{H}^{1,\,1}_{\Delta_{\widetilde\tau}}(\widetilde X,\,\mathbb{C})\ \mid\ \alpha^{1,\,1}\geq 0\right\} = \{0\},$$

where $\widetilde\tau = \widetilde\tau_{\widetilde\omega} := [\Lambda_{\widetilde\omega},\,\partial\widetilde\omega\wedge\cdot]$

As a piece of notation that will be used throughout the text, whenever u is a k-form and $(p,\,q)$ is a bidegree with $p+q=k$, $u^{p,\,q}$ will stand for the component of u of bidegree $(p,\,q)$.

3.2 Properties of degenerate balanced manifolds

In this section, we investigate the effect of the balanced condition on the cohomology of degrees 1 and 2, while pointing out the peculiarities of the degenerate balanced case.

3.2.1 Background

Given a complex manifold X with $\dim_{\mathbb{C}} X = n \geq 2$ and a Hermitian metric ω (identified with its underlying C^∞ positive definite $(1,\,1)$-form ω) on X, we will put $\omega_r := \omega^r/r!$ for $r = 1, \dots, n$. Moreover, we denote by $C^{r,\,s}(X) = C^{r,\,s}(X,\,\mathbb{C})$ the space of smooth $\mathbb{C}$-valued $(r,\,s)$-forms on X for $r, s = 1, \dots, n$. If X is compact, recall the classical definitions of the *Bott-Chern* and *Aeppli* cohomology groups of X of any bidegree $(p,\,q)$:

$$H^{p,\,q}_{BC}(X,\,\mathbb{C}) = \frac{\ker(\partial : C^{p,\,q}(X) \to C^{p+1,\,q}(X)) \cap \ker(\bar\partial : C^{p,\,q}(X) \to C^{p,\,q+1}(X))}{\mathrm{Im}\,(\partial\bar\partial : C^{p-1,\,q-1}(X) \to C^{p,\,q}(X))}$$

$$H^{p,\,q}_{A}(X,\,\mathbb{C}) = \frac{\ker(\partial\bar\partial : C^{p,\,q}(X) \to C^{p+1,\,q+1}(X))}{\mathrm{Im}\,(\partial : C^{p-1,\,q}(X) \to C^{p,\,q}(X)) + \mathrm{Im}\,(\bar\partial : C^{p,\,q-1}(X) \to C^{p,\,q}(X))}.$$

We will use the Serre-type duality (see e.g. [Sch07]):

$$H^{1,\,1}_{BC}(X,\,\mathbb{C}) \times H^{n-1,\,n-1}_{A}(X,\,\mathbb{C}) \longrightarrow \mathbb{C}, \qquad (\{u\}_{BC},\,\{v\}_A) \mapsto \{u\}_{BC}.\{v\}_A := \int\limits_X u \wedge v, \tag{3.6}$$

as well as the *pseudo-effective cone* of X introduced in [Dem92, Definition 1.3] as the set

$$\mathcal{E}_X := \left\{[T]_{BC} \in H^{1,\,1}_{BC}(X,\,\mathbb{R})\,/\,T \geq 0 \ \ d\text{-closed } (1,\,1)\text{-current on } X\right\}.$$

Recall that a Hermitian metric ω on X is said to be a *Gauduchon metric* (cf. [Gau77]) if $\partial\bar\partial\omega^{n-1} = 0$. For any such metric ω, ω^{n-1} defines an Aeppli cohomology class and the set of all these cohomology classes is called the *Gauduchon cone* of X (cf. [Pop15a]):

$$\mathcal{G}_X := \left\{\{\omega^{n-1}\}_A \in H^{n-1,\,n-1}_{A}(X,\,\mathbb{R}) \ \mid \ \omega \text{ is a Gauduchon metric on } X\right\} \subset H^{n-1,\,n-1}_{A}(X,\,\mathbb{R}).$$

The main link between the cones $\mathcal{G}_X$ and $\mathcal{E}_X$ on a compact n-dimensional X is provided by the following reformulation observed in [Pop15b] of a result of Lamari's from [Lam99]. The pseudo-effective cone $\mathcal{E}_X \subset H^{1,\,1}_{BC}(X,\,\mathbb{R})$ and the closure of the Gauduchon cone $\overline{\mathcal{G}}_X \subset H^{n-1,\,n-1}_{A}(X,\,\mathbb{R})$ are **dual** to each other under the duality (3.6). This means that the following two statements hold.

(1) Given any class $\mathfrak{c}^{1,\,1}_{BC} \in H^{1,\,1}_{BC}(X,\,\mathbb{R})$, the following equivalence holds:

$$\mathfrak{c}^{1,\,1}_{BC} \in \mathcal{E}_X \iff \mathfrak{c}^{1,\,1}_{BC}.\mathfrak{c}^{n-1,\,n-1}_{A} \geq 0 \quad \text{for every class } \mathfrak{c}^{n-1,\,n-1}_{A} \in \mathcal{G}_X.$$

(2) Given any class $\mathfrak{c}_A^{n-1,\,n-1} \in H_A^{n-1,\,n-1}(X,\,\mathbb{R})$, the following equivalence holds:

$$\mathfrak{c}_A^{n-1,\,n-1} \in \overline{\mathcal{G}}_X \iff \mathfrak{c}_{BC}^{1,\,1}.\mathfrak{c}_A^{n-1,\,n-1} \geq 0 \quad \text{for every class } \mathfrak{c}_{BC}^{1,\,1} \in \mathcal{E}_X.$$

Finally, recall that a compact complex manifold X is said to be a $\partial\bar\partial$-**manifold** (see [DGMS75] for the notion, [Pop14] for the name) if for any d-closed *pure-type* form u on X, the following exactness properties are equivalent:

$$u \text{ is } d\text{-exact} \iff u \text{ is } \partial\text{-exact} \iff u \text{ is } \bar\partial\text{-exact} \iff u \text{ is } \partial\bar\partial\text{-exact}.$$

On a complex manifold X with $\dim_{\mathbb{C}}X = n$, we will often use the following standard formula (cf. e.g. [Voi02, Proposition 6.29, p. 150]) for the Hodge star operator $\star = \star_\omega$ of any Hermitian metric ω applied to ω-*primitive* forms v of arbitrary bidegree $(p,\,q)$:

$$\star\, v = (-1)^{k(k+1)/2}\, i^{p-q}\, \omega_{n-p-q} \wedge v, \quad \text{where } k := p+q. \tag{3.7}$$

Recall that, for any $k = 0, 1, \dots, n$, a k-form v is said to be (ω)-*primitive* if $\omega_{n-k+1}\wedge v = 0$ and that this condition is equivalent to $\Lambda_\omega v = 0$, where Λ_ω is the adjoint of the operator $\omega\wedge\cdot$ (of multiplication by ω) w.r.t. the pointwise inner product $\langle\ ,\ \rangle_\omega$ defined by ω.

We will also often deal with C^∞ $(1,\,1)$-forms α. If $\alpha = \alpha_{prim} + f\omega$ is the *Lefschetz decomposition*, where α_{prim} is *primitive* and f is a smooth function on X, we get $\Lambda_\omega\alpha = nf$, hence

$$\alpha = \alpha_{prim} + \frac{1}{n}\,(\Lambda_\omega\alpha)\,\omega. \tag{3.8}$$

We will often write $(1,\,1)$-forms in this form.

On the other hand, we will often indicate the metric with respect to which certain operators are calculated. For example, $d^\star_\omega$ and $\Delta_\omega := dd^\star_\omega + d^\star_\omega d$ are the adjoint of d, resp. the d-Laplacian, induced by the metric ω.

3.2.2 Case of degree 1

The starting point is the following

Lemma 3.2.1. *Let ω be a Hermitian metric on a complex manifold X with $\dim_{\mathbb{C}}X = n$. Fix a form $u = u^{1,\,0} + u^{0,\,1} \in C^\infty_1(X,\,\mathbb{C})$.*

(i) The following formula holds:

$$\begin{aligned} d^\star(\omega_{n-1}\wedge u) &= i(\partial u^{1,\,0} - \bar\partial u^{0,\,1})\wedge\omega_{n-2} + i\bigg((\partial u^{0,\,1})_{prim} - (\bar\partial u^{1,\,0})_{prim}\bigg)\wedge\omega_{n-2} \\ &+ \frac{i}{n}\bigg(\Lambda_\omega(\bar\partial u^{1,\,0}) - \Lambda_\omega(\partial u^{0,\,1})\bigg)\,\omega_{n-1}, \end{aligned} \tag{3.9}$$

where $d^\star = d^\star_\omega$ is the formal adjoint of d w.r.t. the L^2_ω inner product, while the subscript prim *indicates the ω-primitive part in the Lefschetz decomposition of the form to which it is applied.*

In particular, if $du^{1,0}=0$ and $du^{0,1}=0$, we get

$$d^\star(\omega_{n-1}\wedge u)=0.$$

(ii) If ω is **balanced** *and $du^{1,0}=0$ and $du^{0,1}=0$, then*

$$\Delta(\omega_{n-1}\wedge u)=0,$$

where $\Delta=\Delta_\omega=dd^\star+d^\star d$ is the d-Laplacian induced by ω.

(iii) If X is compact, ω is **degenerate balanced** *and $du^{1,0}=du^{0,1}=0$, then $u=0$.*

Proof. (i) All 1-forms are primitive, so from the standard formula (3.7) we get: $\star\, u^{1,0}=-i\omega_{n-1}\wedge u^{1,0}$, hence $\star\,(\omega_{n-1}\wedge u^{1,0})=-iu^{1,0}$. Meanwhile, $d^\star=-\star\, d\star$, so applying $-\star\, d$ to the previous identity and writing the $(1,\,1)$-form $\bar\partial u^{1,0}$ in the form (3.8), we get the first line below:

$$\begin{aligned} d^\star(\omega_{n-1}\wedge u^{1,0}) &= i\star\partial u^{1,0}+i\star(\bar\partial u^{1,0})_{prim}+\frac{i}{n}\,(\Lambda_\omega\bar\partial u^{1,0})\star\omega \\ &= i\partial u^{1,0}\wedge\omega_{n-2}-i(\bar\partial u^{1,0})_{prim}\wedge\omega_{n-2}+\frac{i}{n}\,(\Lambda_\omega\bar\partial u^{1,0})\,\omega_{n-1}, \end{aligned}$$

where the second line follows from the standard formula (3.7).

Running the analogous computations for $u^{0,1}$ or taking conjugates, we get:

$$d^\star(\omega_{n-1}\wedge u^{0,1})=-i\bar\partial u^{0,1}\wedge\omega_{n-2}+i(\partial u^{0,1})_{prim}\wedge\omega_{n-2}-\frac{i}{n}\,(\Lambda_\omega\partial u^{0,1})\,\omega_{n-1}.$$

Formula (3.9) follows by adding up the above expressions for $d^\star(\omega_{n-1}\wedge u^{1,0})$ and $d^\star(\omega_{n-1}\wedge u^{0,1})$.

(ii) If ω is balanced, we get $d(\omega_{n-1}\wedge u)=\omega_{n-1}\wedge du=0$ since $du=0$ under the assumptions. Since we also have $d^\star(\omega_{n-1}\wedge u)=0$ by (i), the contention follows.

(iii) If ω is degenerate balanced, there exists a smooth $(2n-3)$-form Γ such that $\omega^{n-1}=d\Gamma$. Hence, $\omega^{n-1}\wedge u=d(\Gamma\wedge u)\in\operatorname{Im} d$ because we also have $du=0$ by our assumptions. However, $\omega^{n-1}\wedge u\in\ker\Delta$ by (ii) and $\ker\Delta\perp\operatorname{Im} d$ by the compactness assumption on X. Thus, the form $\omega^{n-1}\wedge u\in\ker\Delta\cap\operatorname{Im} d=\{0\}$ must vanish.

On the other hand, the pointwise map $\omega_{n-1}\wedge\cdot:\Lambda^1T^\star X\longrightarrow\Lambda^{2n-1}T^\star X$ is bijective, so we get $u=0$ from $\omega^{n-1}\wedge u=0$. $\square$

We now use Lemma 3.2.1 to infer its consequences announced in the introduction.

• **Proof of (i) of Proposition 3.1.2.** This follows at once from (iii) of Lemma 3.2.1. $\square$

Another consequence of Lemma 3.2.1 is that the balanced condition, combined with the mild $\partial\bar\partial$-type condition in (ii) of Theorem 3.1.1, enables one to get a *Hard Lefschetz Isomorphism* between the De Rham cohomology spaces of degrees 1 and $2n-1$.

• **Proof of Theorem 3.1.1.** (i) Lemma 3.2.1 is not needed here. Let u be a smooth 1-form. Since $d\omega_{n-1}=0$, $d(\omega_{n-1}\wedge u)=0$ whenever $du=0$, while $\omega_{n-1}\wedge u = d(f\omega_{n-1})$ whenever $u=df$ for some smooth function f on X. This proves the well-definedness of the map (3.1).

Similarly, if $\omega_{n-1} = \gamma_{n-1} + d\Gamma$ for some smooth $(2n-2)$-form γ_{n-1} and some smooth $(2n-3)$-form Γ, then $\omega_{n-1}\wedge u = \gamma_{n-1}\wedge u + d(\Gamma\wedge u)$ for every d-closed 1-form u. Hence, $\{\omega_{n-1}\wedge u\}_{DR} = \{\gamma_{n-1}\wedge u\}_{DR}$ whenever $\{\omega_{n-1}\}_{DR} = \{\gamma_{n-1}\}_{DR}$, so the map (3.1) depends only on $\{\omega_{n-1}\}_{DR}$.

(ii) Since $H^1_{DR}(X,\,\mathbb{C})$ and $H^{2n-1}_{DR}(X,\,\mathbb{C})$ have equal dimensions, by Poincaré duality, it suffices to prove that the map (3.1) is injective.

Let u be an arbitrary smooth d-closed 1-form on X. We start by showing that there exists a smooth function $f: X\to\mathbb{C}$ such that $\partial u^{0,\,1} = \partial\bar\partial f$ on X. To see this, notice that the property $du=0$ translates to the following three relations holding:

$$(a)\ \partial u^{1,\,0}=0;\qquad (b)\ \partial u^{0,\,1}+\bar\partial u^{1,\,0}=0;\qquad (c)\ \bar\partial u^{0,\,1}=0. \tag{3.10}$$

Thus, the $(1,\,1)$-form $\partial u^{0,\,1}$ is d-closed (since it is $\bar\partial$-closed by (c) of (3.10)) and ∂-exact. Thanks to hypothesis (3.2), we infer that $\partial u^{0,\,1}$ is $\partial\bar\partial$-exact. Thus, there exists a smooth function f as stated.

Using (b) of (3.10), we further infer that $\bar\partial u^{1,\,0} = -\partial u^{0,\,1} = -\partial\bar\partial f$, so $\bar\partial(u^{1,\,0}-\partial f) = 0$. From the identities $\partial(u^{0,\,1}-\bar\partial f)=0$ and $\bar\partial(u^{1,\,0}-\partial f)=0$ and from (a) and (c) of (3.10), we get:

$$d(u^{1,\,0}-\partial f)=0 \quad \text{and} \quad d(u^{0,\,1}-\bar\partial f)=0.$$

This means that

$$(u-df)^{1,\,0}\in\ker d \quad \text{and} \quad (u-df)^{0,\,1}\in\ker d.$$

From this and from (i) of Lemma 3.2.1, we deduce that

$$\omega_{n-1}\wedge(u-df)\in\ker d^\star. \tag{3.11}$$

On the other hand, if ω is balanced and if $\{\omega_{n-1}\wedge u\}_{DR} = 0\in H^{2n-1}_{DR}(X,\,\mathbb{C})$ (i.e. $\omega_{n-1}\wedge u\in\operatorname{Im} d$), then

$$\omega_{n-1}\wedge(u-df)\in\operatorname{Im} d. \tag{3.12}$$

From (3.11), (3.12) and $\ker d^\star \perp \operatorname{Im} d$, we infer that $\omega_{n-1}\wedge(u-df)=0$. Since $u-df$ is a smooth 1-form on X and the pointwise-defined linear map:

$$\omega_{n-1}\wedge\cdot : C^\infty_1(X,\,C)\longrightarrow C^\infty_{2n-1}(X,\,C), \qquad \alpha\mapsto\omega_{n-1}\wedge\alpha,$$

is bijective, we finally get $u-df=0$, so $\{u\}_{DR}=0\in H^1_{DR}(X,\,\mathbb{C})$.

This proves the injectivity of the map (3.1) whenever ω is balanced and X satisfies hypothesis (3.2). □

In the degenerate balanced case, we get the vanishing of the first Betti number of the manifold.

• **Proof of (ii) of Proposition 3.1.2.** If ω is degenerate balanced, the map (3.1) vanishes identically. Meanwhile, by Theorem 3.1.1, the map (3.1) is an isomorphism. We get $H^1_{DR}(X,\,\mathbb{C})=0$. □

3.2.3 Case of degree 2: De Rham cohomology

The balanced property of a metric enables one to define a notion of primitivity for 2-forms.

Lemma and Definition 3.2.2. *Let ω be a* **balanced** *metric on a compact complex manifold X with $dim_{\mathbb{C}}X = n$. The linear map:*

$$\{\omega_{n-1}\}_{DR}\wedge\cdot : H^2_{DR}(X,\,\mathbb{C}) \longrightarrow H^{2n}_{DR}(X,\,\mathbb{C})\simeq\mathbb{C}, \quad \{\alpha\}_{DR}\longmapsto\{\omega_{n-1}\wedge\alpha\}_{DR}, \tag{3.13}$$

is **well defined** *and depends only on the cohomology class $\{\omega_{n-1}\}_{DR}\in H^{2n-2}_{DR}(X,\,\mathbb{C})$. We set:*

$$H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim} := \ker\bigg(\{\omega_{n-1}\}_{DR}\wedge\cdot\bigg)\subset H^2_{DR}(X,\,\mathbb{C})$$

and we call its elements **(ω-)primitive** *De Rham 2-classes.*

Proof. Since $d\omega_{n-1}=0$, for every d-closed (resp. d-exact) 2-form α, $\omega_{n-1}\wedge\alpha$ is d-closed (resp. d-exact). This proves the well-definedness of the map.

Meanwhile, if $\Omega\in C^\infty_{n-1,\,n-1}(X,\,\mathbb{C})$ is such that $\Omega=\omega_{n-1}+d\Gamma$ for some smooth $(2n-3)$-form Γ, then, for every d-closed 2-form α, $\Omega\wedge\alpha=\omega_{n-1}\wedge\alpha+d(\Gamma\wedge\alpha)$. Hence, $\{\Omega\wedge\alpha\}_{DR}=\{\omega_{n-1}\wedge\alpha\}_{DR}$ whenever $\{\Omega\}_{DR}=\{\omega_{n-1}\}_{DR}$. This proves the independence of the map $\{\omega_{n-1}\}_{DR}\wedge\cdot$ of the choice of representative of the class $\{\omega_{n-1}\}_{DR}$. □

We now observe a link between *primitive* 2-classes and *primitive* 2-forms.

Lemma 3.2.3. *Let ω be a* **balanced** *metric on a compact complex manifold X with $dim_{\mathbb{C}}X=n$. For any class $\mathfrak{c}\in H^2_{DR}(X,\,\mathbb{C})$, the following equivalence holds:*

$$\mathfrak{c}\ \textit{is}\ \omega\textbf{-primitive} \iff \exists\,\alpha\in\mathfrak{c}\quad\textit{such that}\quad\alpha\ \textit{is}\ \omega\textbf{-primitive}.$$

Proof. "$\Longleftarrow$" Suppose $\alpha\in C^\infty_2(X,\,\mathbb{C})$ such that $d\alpha=0$, $\alpha\in\mathfrak{c}$ and α is ω-primitive. Then, $\omega_{n-1}\wedge\alpha=0$, hence $\{\omega_{n-1}\wedge\alpha\}_{DR}=0$. This means that the class $\mathfrak{c}=\{\alpha\}_{DR}$ is ω-primitive.

"$\Longrightarrow$" Suppose the class $\mathfrak{c}$ is ω-primitive. Pick an arbitrary representative $\beta\in\mathfrak{c}$. The ω-primitivity of $\mathfrak{c}=\{\beta\}_{DR}$ translates to $\{\omega_{n-1}\wedge\beta\}_{DR}=0\in H^{2n}_{DR}(X,\,\mathbb{C})$. This, in turn, is equivalent to the existence of a form $\Gamma\in C^\infty_{2n-1}(X,\,\mathbb{C})$ such that $\omega_{n-1}\wedge\beta=d\Gamma$.

Meanwhile, we know from the general theory that the map

$$\omega_{n-1}\wedge\cdot : C^\infty_1(X,\,\mathbb{C})\longrightarrow C^\infty_{2n-1}(X,\,\mathbb{C})$$

is an isomorphism. Hence, there exists a unique $u\in C^\infty_1(X,\,\mathbb{C})$ such that $\Gamma=\omega_{n-1}\wedge u$. We get:

$$\omega_{n-1}\wedge\beta=d\Gamma=\omega_{n-1}\wedge du,$$

where the last identity follows from the balanced property of ω. Consequently,

$$\omega_{n-1}\wedge(\beta-du)=0,$$

proving that $\alpha:=\beta-du$ is a *primitive* representative of the class $\mathfrak{c}=\{\beta\}_{DR}$. □

Finally, we can characterise the *degenerate balanced* property of a given *balanced* metric in terms of primitivity for 2-classes.

Lemma 3.2.4. *Let ω be a* **balanced** *metric on a compact complex manifold X with $dim_{\mathbb{C}} X = n$. The following equivalence holds:*

$$H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim} = H^2_{DR}(X,\,\mathbb{C}) \iff \omega \textit{ is } \textbf{degenerate balanced}.$$

Proof. "$\Longleftarrow$" Suppose that ω is degenerate balanced. Then ω_{n-1} is d-exact, hence $\omega_{n-1}\wedge\alpha$ is d-exact (or equivalently $\{\omega_{n-1}\wedge\alpha\}_{DR} = 0 \in H^{2n}_{DR}(X,\,\mathbb{C})$) for every d-closed 2-form α. This means that the map $\{\omega_{n-1}\}_{DR}\wedge\cdot : H^2_{DR}(X,\,\mathbb{C}) \longrightarrow H^{2n}_{DR}(X,\,\mathbb{C})$ vanishes identically, so $H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim} = H^2_{DR}(X,\,\mathbb{C})$.

"$\Longrightarrow$" Suppose that $H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim} = H^2_{DR}(X,\,\mathbb{C})$. This translates to

$$\omega_{n-1}\wedge\alpha \in \operatorname{Im} d, \qquad \forall\,\alpha\in C^\infty_2(X,\,\mathbb{C})\cap\ker d. \tag{3.14}$$

Since both ω_{n-1} and α are d-closed, they both have unique L^2_ω-orthogonal decompositions:

$$\omega_{n-1} = (\omega_{n-1})_h + d\Gamma \qquad \text{and} \qquad \alpha = \alpha_h + du,$$

where $(\omega_{n-1})_h$ and α_h are Δ_ω-harmonic, while Γ and u are smooth forms of respective degrees $2n-3$ and 1. We get:

$$\omega_{n-1}\wedge\alpha = (\omega_{n-1})_h\wedge\alpha_h + d\bigg((\omega_{n-1})_h\wedge u + \Gamma\wedge\alpha_h + \Gamma\wedge du\bigg), \qquad \forall\,\alpha\in C^\infty_2(X,\,\mathbb{C})\cap\ker d.$$

Together with (3.14), this implies that

$$(\omega_{n-1})_h\wedge\alpha_h \in \operatorname{Im} d, \qquad \forall\,\alpha_h\in\ker\Delta_\omega\cap C^\infty_2(X,\,\mathbb{C}). \tag{3.15}$$

Meanwhile, since $(\omega_{n-1})_h$ is Δ_ω-harmonic (and real), $\star_\omega(\omega_{n-1})_h$ is again Δ_ω-harmonic (and real). Hence,

$$\operatorname{Im} d \ni (\omega_{n-1})_h\wedge\star_\omega(\omega_{n-1})_h = |(\omega_{n-1})_h|^2_\omega\, dV_\omega \geq 0,$$

where the first relation follows from (3.15) by choosing $\alpha_h = \star_\omega(\omega_{n-1})_h$. Consequently, from Stokes's Theorem we get:

$$\int\limits_X |(\omega_{n-1})_h|^2_\omega\, dV_\omega = 0,$$

hence $(\omega_{n-1})_h = 0$. This implies that ω_{n-1} is d-exact, which means that ω is degenerate balanced. □

Corollary 3.2.5. *Let ω be a* **balanced** *metric on a compact complex manifold X with $dim_{\mathbb{C}} X = n$. The following dichotomy holds:*

(a) if ω is **not degenerate balanced**, *$H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim}$ is a complex hyperplane in $H^2_{DR}(X,\,\mathbb{C})$ depending only on the balanced class $\{\omega_{n-1}\}_{DR}$;*

(b) if ω is **degenerate balanced**, *$H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim} = H^2_{DR}(X,\,\mathbb{C})$.*

Proof. This follows immediately from Lemma and Definition 3.2.2, from Lemma 3.2.4 and from $H^{2n}_{DR}(X,\,\mathbb{C})\simeq\mathbb{C}$. $\square$

We shall now get a *Lefschetz-type decomposition* of $H^2_{DR}(X,\,\mathbb{C})$, induced by an arbitrary balanced metric ω, with $H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim}$ as a direct factor. Recall that the balanced condition $d\omega^{n-1}=0$ is equivalent to $d^\star_\omega\omega=0$.

Thanks to the orthogonal 3-space decompositions:

$$C^\infty_k(X,\,\mathbb{C})=\ker\Delta_\omega\oplus\operatorname{Im}d\oplus\operatorname{Im}d^\star_\omega,\qquad k\in\{0,\dots,2n\},$$

where $\ker\Delta_\omega\oplus\operatorname{Im}d=\ker d$ and $\ker\Delta_\omega\oplus\operatorname{Im}d^\star_\omega=\ker d^\star_\omega$, applied with $k=2$ and $k=2n-2$, we get unique decompositions of ω, resp. ω_{n-1}:

$$\ker d^\star_\omega\ni\omega=\omega_h+d^\star_\omega\eta_\omega\quad\text{ and }\quad\ker d\ni\omega_{n-1}=(\omega_{n-1})_h+d\Gamma_\omega,\tag{3.16}$$

where $\omega_h\in\ker\Delta_\omega$ as a 2-form, $(\omega_{n-1})_h\in\ker\Delta_\omega$ as a $(2n-2)$-form, while η_ω and Γ_ω are smooth forms of respective degrees 3 and $2n-3$. Since ω and ω_{n-1} are real, so are their harmonic components ω_h and $(\omega_{n-1})_h$.

Moreover, it is well known that $\star_\omega\omega=\omega_{n-1}$ and that the Hodge star operator $\star_\omega$ maps d-exact forms to $d^\star_\omega$-exact forms and vice-versa. Hence, we get:

$$\star_\omega\,\omega_h=(\omega_{n-1})_h\quad\text{ and }\quad\star_\omega\,(d^\star_\omega\eta_\omega)=d\Gamma_\omega.\tag{3.17}$$

Thus, ω_h is uniquely determined by ω and is d-closed (because it is even Δ_ω-harmonic). Therefore, it represents a class in $H^2_{DR}(X,\,\mathbb{R})$.

Definition 3.2.6. *For any balanced metric ω on a compact complex manifold X, the De Rham cohomology class $\{\omega_h\}_{DR}\in H^2_{DR}(X,\,\mathbb{R})$ is called the* **cohomology class** *of ω.*

Of course, if ω is Kähler, $\omega_h=\omega$, so $\{\omega_h\}_{DR}$ is the usual Kähler class $\{\omega\}_{DR}$.

Lemma 3.2.7. *Suppose there exists a balanced metric ω on a compact complex manifold X. Then, for every $\alpha\in C^\infty_2(X,\,\mathbb{C})$ such that $d\alpha=0$ and $\{\alpha\}_{DR}\in H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim}$, we have:*

$$\langle\langle\omega_h,\,\alpha\rangle\rangle_\omega=0,$$

where $\langle\langle\ ,\ \rangle\rangle_\omega$ is the L^2 inner product induced by ω.

Proof. Since $\{\alpha\}_{DR}\in H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim}$, there exists a form $\Omega\in C^\infty_{2n-1}$ such that $\omega_{n-1}\wedge\alpha=du$. We get:

$$\begin{aligned}\langle\langle\alpha,\,\omega_h\rangle\rangle_\omega &= \int\limits_X\alpha\wedge\star_\omega\omega_h\overset{(a)}{=}\int\limits_X\alpha\wedge(\omega_{n-1})_h\overset{(b)}{=}\int\limits_X\alpha\wedge(\omega_{n-1}-d\Gamma_\omega)\\ &= \int\limits_X\alpha\wedge\omega_{n-1}=\int\limits_X du=0,\end{aligned}$$

where Stokes implies two of the last three equalities (note that $\alpha\wedge d\Gamma_\omega=d(\alpha\wedge\Gamma_\omega)$), while (a) follows from (3.17) and (b) follows from (3.16). $\square$

Conclusion 3.2.8. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Suppose there exists a non-degenerate* **balanced metric** *ω on X. Then, the De Rham cohomology space of degree 2 has a* **Lefschetz-type** *L^2_ω-orthogonal* **decomposition***:*

$$H^2_{DR}(X,\,\mathbb{C}) = H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim} \oplus \mathbb{C}\cdot\{\omega_h\}_{DR}, \tag{3.18}$$

where the ω-primitive subspace $H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim}$ is a complex hyperplane of $H^2_{DR}(X,\,\mathbb{C})$ depending only on the cohomology class $\{\omega_{n-1}\}_{DR} \in H^{2n-2}_{DR}(X,\,\mathbb{C})$, while ω_h is the Δ_ω-harmonic component of ω and the complex line $\mathbb{C}\cdot\{\omega_h\}_{DR}$ depends on the choice of the balanced metric ω.

If ω is Kähler, the Lefschetz-type decomposition (3.18) depends only on the *Kähler class* $\{\omega\}_{DR} \in H^2_{DR}(X,\,\mathbb{C})$ since $\omega_h = \omega$ in that case.

Lemma 3.2.9. *The assumptions are the same as in Conclusion 3.2.8. For every $\alpha \in C^\infty_2(X,\,\mathbb{C}) \cap \ker d$, the coefficient of $\{\omega_h\}_{DR}$ in the Lefschetz-type decomposition of $\{\alpha\}_{DR} \in H^2_{DR}(X,\,\mathbb{C})$ according to (3.18), namely in*

$$\{\alpha\}_{DR} = \{\alpha\}_{DR,\,prim} + \lambda\,\{\omega_h\}_{DR}, \tag{3.19}$$

is given by

$$\lambda = \lambda_\omega(\{\alpha\}_{DR}) = \frac{\{\omega_{n-1}\}_{DR}.\{\alpha\}_{DR}}{||\omega_h||^2_\omega} = \frac{1}{||\omega_h||^2_\omega}\int\limits_X \alpha\wedge\omega_{n-1}. \tag{3.20}$$

Proof. Since $\{\alpha\}_{DR,\,prim} \in H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim}$, we have $\{\omega_{n-1}\}_{DR}.\{\alpha\}_{DR,\,prim} = 0$, so

$$\begin{aligned}\{\omega_{n-1}\}_{DR}.\{\alpha\}_{DR} &= \lambda\int\omega_{n-1}\wedge\omega_h = \lambda\int(\omega_{n-1})_h\wedge\omega_h = \lambda\int(\omega_{n-1})_h\wedge\star_\omega(\omega_{n-1})_h \\ &= \lambda\,||(\omega_{n-1})_h||^2_\omega = \lambda\,||\omega_h||^2_\omega.\end{aligned}$$

This gives (3.20). □

Formula (3.20) implies that $\lambda_\omega(\{\alpha\}_{DR})$ is *real* if the class $\{\alpha\}_{DR} \in H^2_{DR}(X,\,\mathbb{R})$ is real. This enables one to define a *positive side* and a *negative side* of the hyperplane $H^2_{DR}(X,\,\mathbb{R})_{\omega\text{-}prim} := H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim} \cap H^2_{DR}(X,\,\mathbb{R})$ in $H^2_{DR}(X,\,\mathbb{R})$ by

$$\begin{aligned}H^2_{DR}(X,\,\mathbb{R})^+_\omega &:= \bigg\{\{\alpha\}_{DR} \in H^2_{DR}(X,\,\mathbb{R}) \mid \lambda_\omega(\{\alpha\}_{DR}) > 0\bigg\}, \\ H^2_{DR}(X,\,\mathbb{R})^-_\omega &:= \bigg\{\{\alpha\}_{DR} \in H^2_{DR}(X,\,\mathbb{R}) \mid \lambda_\omega(\{\alpha\}_{DR}) < 0\bigg\}.\end{aligned} \tag{3.21}$$

These open subsets of $H^2_{DR}(X,\,\mathbb{R})$ depend only on the cohomology class $\{\omega_{n-1}\}_{DR} \in H^{2n-2}_{DR}(X,\,\mathbb{R})$.

Since $\{\alpha\}_{DR}$ is ω-primitive if and only if $\lambda_\omega(\{\alpha\}_{DR}) = 0$, we get a *partition* of $H^2_{DR}(X,\,\mathbb{R})$:

$$H^2_{DR}(X,\,\mathbb{R}) = H^2_{DR}(X,\,\mathbb{R})^+_\omega \cup H^2_{DR}(X,\,\mathbb{R})_{\omega\text{-}prim} \cup H^2_{DR}(X,\,\mathbb{R})^-_\omega \tag{3.22}$$

depending only on the cohomology class $\{\omega_{n-1}\}_{DR} \in H^{2n-2}_{DR}(X, \mathbb{R})$.

The next (trivial) observation is that the ω-primitive hyperplane $H^2_{DR}(X, \mathbb{C})_{\omega\text{-}prim} \subset H^2_{DR}(X, \mathbb{C})$ depends only on the ray $\mathbb{R}_{>0} \cdot \{\omega_{n-1}\}_{DR}$ generated by the De Rham cohomology class of ω_{n-1} in the De Rham version of the *balanced cone* $\mathcal{B}_{X,\,DR} \subset H^{2n-2}_{DR}(X, \mathbb{R})$ of X. (We denote by $\mathcal{B}_{X,\,DR}$ the set of De Rham cohomology classes $\{\omega_{n-1}\}_{DR}$ induced by balanced metrics ω.)

Lemma 3.2.10. *Let X be a compact complex non-degenerate balanced manifold with $dim_{\mathbb{C}}X = n$. Let ω and γ be* **balanced metrics** *on X. The following equivalence holds:*

$$H^2_{DR}(X, \mathbb{C})_{\omega\text{-}prim} = H^2_{DR}(X, \mathbb{C})_{\gamma\text{-}prim} \iff \exists\, c > 0 \textit{ such that } \{\omega_{n-1}\}_{DR} = c\,\{\gamma_{n-1}\}_{DR}.$$

Proof. "$\Longleftarrow$" This implication follows from proportional linear maps having the same kernel.

"$\Longrightarrow$" This implication follows from the following elementary fact. Suppose $T, S : E \longrightarrow \mathbb{C}$ are $\mathbb{C}$-linear maps on a $\mathbb{C}$-vector space E such that $\ker T = \ker S \subset E$ is of $\mathbb{C}$-codimension 1 in E. Then, there exists $c \in \mathbb{C}\setminus\{0\}$ such that $T = cS$. To see this, let $\{e_j \mid j \in J\}$ be a $\mathbb{C}$-basis of $\ker T = \ker S$ and let $e \in E$ such that $\{e\} \cup \{e_j \mid j \in J\}$ is a $\mathbb{C}$-basis of E. Then, $T(e)$ and $S(e)$ are non-zero complex numbers, so there exists a unique $c \in \mathbb{C} \setminus \{0\}$ such that $T(e) = c\,S(e)$. Now, fix an arbitrary $u \in E$. We will show that $T(u) = c\,S(u)$. There is a unique choice of $\lambda \in \mathbb{C}$ and $v \in \ker T = \ker S$ such that $u = \lambda\, e + v$. Hence, $T(u) = \lambda\, T(e) = c\,(\lambda\, S(e)) = c\, S(u)$.

In our case, the assumption $H^2_{DR}(X, \mathbb{C})_{\omega\text{-}prim} = H^2_{DR}(X, \mathbb{C})_{\gamma\text{-}prim}$ amounts to $\ker(\{\omega_{n-1}\}_{DR}\wedge\cdot) = \ker(\{\gamma_{n-1}\}_{DR}\wedge\cdot)$. Hence, by the above elementary fact, it amounts to the exietence of a constant $c \in \mathbb{C} \setminus \{0\}$ such that $\{\omega_{n-1}\}_{DR} \wedge \cdot = c\,\{\gamma_{n-1}\}_{DR} \wedge \cdot$ as $\mathbb{C}$-linear maps on $H^2_{DR}(X, \mathbb{C})$. By the non-degeneracy of the Poincaré duality $H^2_{DR}(X, \mathbb{C}) \times H^{2n-2}_{DR}(X, \mathbb{C}) \longrightarrow \mathbb{C}$, this further amounts to the existence of a constant $c \in \mathbb{C} \setminus \{0\}$ such that $\{\omega_{n-1}\}_{DR} = c\,\{\gamma_{n-1}\}_{DR}$.

Now, since the forms ω_{n-1} and γ_{n-1} are real, the constant c can be chosen real. (Replace c with $(c+\bar{c})/2$ if necessary.) Since the balanced metric ω is non-degenerate, $c \neq 0$. If $c < 0$, then $\omega_{n-1} - c\,\gamma_{n-1} > 0$ would be the d-exact $(n-1)$-st power of a balanced metric. This balanced metric would then be degenerate balanced, contradicting the assumption on X. Thus, c must be positive. □

We will now see that not only do proportional balanced classes $\{\omega_{n-1}\}_{DR}$ and $\{\gamma_{n-1}\}_{DR}$ induce the same hyperplane of primitive classes in $H^2_{DR}(X, \mathbb{C})$, but they can be made to also induce the same Lefschetz-type decomposition (3.18). This is fortunate since, in general, the complex line $\mathbb{C} \cdot \{\omega_h\}_{DR}$ depends on the choice of the balanced metric ω, unlike $H^2_{DR}(X, \mathbb{C})_{\omega\text{-}prim}$ which depends only on the balanced class $\{\omega_{n-1}\}_{DR} \in H^{2n-2}_{DR}(X, \mathbb{C})$.

Lemma 3.2.11. *Let X be a compact complex non-degenerate balanced manifold with $dim_{\mathbb{C}}X = n$.*

(i) If ω and γ are **balanced metrics** *on X such that $\omega_{n-1} = c\,\gamma_{n-1}$ for some constant $c > 0$, there exists a constant $a > 0$ such that $\omega_h = a\,\gamma_h$.*

(ii) For every ray $\mathbb{R}_{>0}\cdot\{\omega_{n-1}\}_{DR}$ in the De Rham version of the balanced cone $\mathcal{B}_{X,\,DR}\subset H^{2n-2}_{DR}(X,\,\mathbb{R})$ of X, the balanced metrics representing the classes on this ray can be chosen such that they induce the same Lefschetz-type decomposition (3.18).

Proof. (i) Since $\omega = c^{\frac{1}{n-1}}\,\gamma$, we get $\star_\omega = const\cdot\star_\gamma$ and $d^\star_\omega = const\cdot d^\star_\gamma$. The latter identity implies $\Delta_\omega = const\cdot\Delta_\gamma$, hence $\ker\Delta_\omega = \ker\Delta_\gamma$. In particular, $(\omega_{n-1})_h = c\,(\gamma_{n-1})_h$ and thus

$$\omega_h = \star_\omega(\omega_{n-1})_h = const\cdot\star_\gamma(\gamma_{n-1})_h = const\cdot\gamma_h,$$

where in all the above identities *const* stands for a positive constant that may change from one occurrence to another.

(ii) Fix a balanced De Rham class $\{\gamma_{n-1}\}_{DR}\in\mathcal{B}_{X,\,DR}\subset H^{2n-2}_{DR}(X,\,\mathbb{R})$ and fix a balanced metric γ (whose choice is arbitrary) such that γ_{n-1} represents the class $\{\gamma_{n-1}\}_{DR}$. For every constant $c>0$, the balanced class $c\,\{\gamma_{n-1}\}_{DR}$ can be represented by the form $\omega_{n-1} := c\,\gamma_{n-1}$ which is induced by the balanced metric $\omega := c^{\frac{1}{n-1}}\,\gamma$. From (i), we get $\mathbb{C}\,\{\omega_h\}_{DR} = \mathbb{C}\,\{\gamma_h\}_{DR}$. Since we also have $H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim} = H^2_{DR}(X,\,\mathbb{C})_{\gamma\text{-}prim}$ by Lemma 3.2.10, the contention follows. $\square$

The proof of (ii) of the above Lemma 3.2.11 shows that the line $\mathbb{C}\,\{\omega_h\}_{DR}$ in the Lefschetz-type decomposition (3.18) induced by a given ray $\mathbb{R}_{>0}\cdot\{\omega_{n-1}\}_{DR}$ in the De Rham version of the *balanced cone* $\mathcal{B}_{X,\,DR}\subset H^{2n-2}_{DR}(X,\,\mathbb{R})$ of X still depends on the arbitrary choice of a balanced metric γ such that γ_{n-1} represents a given class $\{\gamma_{n-1}\}_{DR}$ on this ray. To tame this dependence, we can fix an arbitrary Hermitian (not necessarily balanced) metric ρ on X and make all the choices of harmonic representatives and projections be induced by ρ. Thus, we get L^2_ρ-orthogonal decompositions:

$$\omega = \omega_{h,\,\rho} + d^\star_\rho\,\eta_{\omega,\,\rho} \quad\text{ and }\quad \omega_{n-1} = (\omega_{n-1})_{h,\,\rho} + d\,\Gamma_{\omega,\,\rho}, \tag{3.23}$$

where $\omega_{h,\,\rho}\in\ker\Delta_\rho$ as a 2-form, $(\omega_{n-1})_{h,\,\rho}\in\ker\Delta_\rho$ as a $(2n-2)$-form, while $\eta_{\omega,\,\rho}$ and $\Gamma_{\omega,\,\rho}$ are smooth forms of respective degrees 3 and $2n-3$. Since ω and ω_{n-1} are real, so are their Δ_ρ-harmonic components $\omega_{h,\,\rho}$ and $(\omega_{n-1})_{h,\,\rho}$.

In this way, every *non-zero balanced class* $\{\omega_{n-1}\}_{DR}$ induces a Lefschetz-type decomposition analogous to (3.18) that depends only on the class $\{\omega_{n-1}\}_{DR}$ and on the background metric ρ:

$$H^2_{DR}(X,\,\mathbb{C}) = H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim}\oplus\mathbb{C}\cdot\{\omega_{h,\,\rho}\}_{DR}, \tag{3.24}$$

where the hyperplane $H^2_{DR}(X,\,\mathbb{C})_{\omega\text{-}prim}$ depends only on the class $\{\omega_{n-1}\}_{DR}$.

In other words, we remove the dependence of the line $\mathbb{C}\,\{\omega_h\}_{DR}$ in the Lefschetz-type decomposition (3.18) on a representative of the class $\{\omega_{n-1}\}_{DR}$ and replace it with the dependence on a fixed background metric ρ.

3.2.4 Case of degree 2: Bott-Chern and Aeppli cohomologies

Let us finally point out that the theory developed in §.3.2.3 in the context of the Poincaré duality for the De Rham cohomology spaces of degrees 2 and $2n-2$ can be rerun in the context of the duality (3.6) between the Bott-Chern and Aeppli cohomology spaces of bidegrees $(1,\,1)$, resp. $(n-1,\,n-1)$.

Since all the results and constructions of §.3.2.3, except for Lemma 3.2.3, have analogues in the new context with very similar proofs, we will leave most of these proofs to the reader.

In fact, the new cohomological setting allows for the theory of §.3.2.3 to be repeated in the more general context of *Gauduchon* (not necessarily balanced) *metrics* and the Aeppli cohomology classes they define in $H^{n-1,\,n-1}_A(X,\,\mathbb{R})$. We start with the following analogue of Lemma and Definition 3.2.2.

Lemma and Definition 3.2.12. *Let ω be a* **Gauduchon** *metric on a compact complex manifold X with $\dim_{\mathbb{C}}X=n$. The linear map:*

$$[\omega_{n-1}]_A\wedge\cdot : H^{1,\,1}_{BC}(X,\,\mathbb{C})\longrightarrow H^{n,\,n}_A(X,\,\mathbb{C})\simeq\mathbb{C},\qquad [\alpha]_{BC}\longmapsto[\omega_{n-1}\wedge\alpha]_A,\tag{3.25}$$

is **well defined** *and depends only on the cohomology class $[\omega_{n-1}]_A\in H^{n-1,\,n-1}_A(X,\,\mathbb{C})$. We set:*

$$H^{1,\,1}_{BC}(X,\,\mathbb{C})_{\omega\text{-}prim}:=\ker\bigg([\omega_{n-1}]_A\wedge\cdot\bigg)\subset H^{1,\,1}_{BC}(X,\,\mathbb{C})$$

and we call its elements **(ω-)primitive** *Bott-Chern $(1,\,1)$-classes.*

Proof. The well-definedness follows at once from the identities:

$$\begin{aligned}\partial\bar\partial(\omega_{n-1}\wedge\alpha) &= \partial\bar\partial\omega_{n-1}\wedge\alpha=0, \qquad \alpha\in C^\infty_{1,\,1}(X,\,\mathbb{C})\cap\ker d,\\ \omega_{n-1}\wedge\partial\bar\partial\varphi &= \partial(\omega_{n-1}\wedge\bar\partial\varphi)+\bar\partial(\varphi\,\partial\omega_{n-1})\in\operatorname{Im}\partial+\operatorname{Im}\bar\partial, \qquad \varphi\in C^\infty_{0,\,0}(X,\,\mathbb{C}),\end{aligned}$$

where the latter takes into account the fact that $\partial\bar\partial\omega_{n-1}=0$.

That the map $[\omega_{n-1}]_A\wedge\cdot$ depends only on the Aeppli cohomology class $[\omega_{n-1}]_A$ follows from:

$$(\omega_{n-1}+\partial\bar\Gamma+\bar\partial\Gamma)\wedge\alpha-\omega_{n-1}\wedge\alpha=\partial(\bar\Gamma\wedge\alpha)+\bar\partial(\Gamma\wedge\alpha)\in\operatorname{Im}\partial+\operatorname{Im}\bar\partial,\qquad \alpha\in C^\infty_{1,\,1}(X,\,\mathbb{C})\cap\ker d.$$

□

The following result is the analogue of Lemma 3.2.4.

Lemma 3.2.13. *Let ω be a* **Gauduchon** *metric on a compact complex manifold X with $\dim_{\mathbb{C}}X=n$. The following equivalence holds:*

$$H^{1,\,1}_{BC}(X,\,\mathbb{C})_{\omega\text{-}prim}=H^{1,\,1}_{BC}(X,\,\mathbb{C}) \iff \omega_{n-1}\in Im\,\partial+Im\,\bar\partial \quad (i.e.\ \omega_{n-1}\ \textbf{is Aeppli-exact}).$$

Proof. “$\Longleftarrow$” If $\omega_{n-1}\in\operatorname{Im}\partial+\operatorname{Im}\bar\partial$, $[\omega_{n-1}]_A=0$, so the map $[\omega_{n-1}]_A\wedge\cdot$ vanishes identically.

“$\Longrightarrow$” Suppose that $H^{1,\,1}_{BC}(X,\,\mathbb{C})_{\omega\text{-}prim}=H^{1,\,1}_{BC}(X,\,\mathbb{C})$. This translates to

$$\omega_{n-1}\wedge\alpha\in\operatorname{Im}\partial+\operatorname{Im}\bar\partial,\qquad \forall\,\alpha\in C^\infty_{1,\,1}(X,\,\mathbb{C})\cap\ker d.\tag{3.26}$$

Since ω_{n-1} is $(\partial\bar\partial)$-closed, it has a unique L^2_ω-orthogonal decomposition:

$$\omega_{n-1}=(\omega_{n-1})_h+(\partial\bar\Gamma_\omega+\bar\partial\Gamma_\omega),$$

with an $(n-1,\,n-1)$-form $(\omega_{n-1})_h\in\ker\Delta_{A,\,\omega}$ and an $(n-1,\,n-2)$-form Γ_ω. (See (3.28) below.)

On the other hand, α is d-closed, so it has a unique L^2_ω-orthogonal decomposition:

$$\alpha=\alpha_h+\partial\bar\partial\varphi,$$

where α_h is $\Delta_{BC,\,\omega}$-harmonic and φ is a smooth function on X. (See again (3.28) below.)

Thus, for every $\alpha\in C^\infty_{1,\,1}(X,\,\mathbb{C})\cap\ker d$, we get:

$$\begin{aligned}\omega_{n-1}\wedge\alpha &= (\omega_{n-1})_h\wedge\alpha+\partial(\bar\Gamma_\omega\wedge\alpha)+\bar\partial(\Gamma_\omega\wedge\alpha)\\ &= (\omega_{n-1})_h\wedge\alpha_h+\partial\bigg((\omega_{n-1})_h\wedge\bar\partial\varphi\bigg)+\bar\partial\bigg(\varphi\,\partial(\omega_{n-1})_h\bigg)+\partial(\bar\Gamma_\omega\wedge\alpha)+\bar\partial(\Gamma_\omega\wedge\alpha),\end{aligned}$$

where for the last identity we used the fact that $\bar\partial\partial(\omega_{n-1})_h=0$.

Thanks to assumption (3.26), the last identity implies that

$$(\omega_{n-1})_h\wedge\alpha_h\in\operatorname{Im}\partial+\operatorname{Im}\bar\partial,\qquad\forall\,\alpha_h\in C^\infty_{1,\,1}(X,\,\mathbb{C})\cap\ker\Delta_{BC,\,\omega}.\tag{3.27}$$

Meanwhile, since $(\omega_{n-1})_h$ is $\Delta_{A,\,\omega}$-harmonic (and real), $\star_\omega(\omega_{n-1})_h$ is $\Delta_{BC,\,\omega}$-harmonic (and real). Hence,

$$\operatorname{Im}\partial+\operatorname{Im}\bar\partial\ni(\omega_{n-1})_h\wedge\star_\omega(\omega_{n-1})_h=|(\omega_{n-1})_h|^2_\omega\,dV_\omega\geq 0,$$

where the first relation follows from (3.27) by choosing $\alpha_h=\star_\omega(\omega_{n-1})_h$. Consequently, from Stokes's Theorem we get:

$$\int\limits_X|(\omega_{n-1})_h|^2_\omega\,dV_\omega=0,$$

hence $(\omega_{n-1})_h=0$. This implies that $\omega_{n-1}\in\operatorname{Im}\partial+\operatorname{Im}\bar\partial$ and we are done. $\square$

The analogue in this context of Corollary 3.2.5 is the following

Corollary 3.2.14. *Let ω be a* **Gauduchon** *metric on a compact complex manifold X with $\dim_{\mathbb{C}}X=n$. The following dichotomy holds:*

(a) if ω_{n-1} is **not Aeppli exact***, $H^{1,\,1}_{BC}(X,\,\mathbb{C})_{\omega\text{-}prim}$ is a complex hyperplane of $H^{1,\,1}_{BC}(X,\,\mathbb{C})$ depending only on the Aeppli-Gauduchon class $\{\omega_{n-1}\}_{DR}\in\mathcal{G}$;*

(b) if ω_{n-1} is **Aeppli exact***, $H^{1,\,1}_{BC}(X,\,\mathbb{C})_{\omega\text{-}prim}=H^{1,\,1}_{BC}(X,\,\mathbb{C})$.*

To get a *Lefschetz-type decomposition* of $H^{1,\,1}_{BC}(X,\,\mathbb{C})$ induced by an arbitrary Gauduchon metric ω, we use the orthogonal 3-space decompositions featuring the Aeppli-, resp. Bott-Chern-Laplacians induced by the metric ω:

$$\begin{aligned}C^\infty_{n-1,\,n-1}(X,\,\mathbb{C}) &= \ker\Delta_{A,\,\omega}\oplus(\operatorname{Im}\partial+\operatorname{Im}\bar\partial)\oplus\operatorname{Im}(\partial\bar\partial)^\star,\\ C^\infty_{1,\,1}(X,\,\mathbb{C}) &= \ker\Delta_{BC,\,\omega}\oplus\operatorname{Im}(\partial\bar\partial)\oplus(\operatorname{Im}\partial^\star+\operatorname{Im}\bar\partial^\star),\end{aligned}\tag{3.28}$$

where $\ker\Delta_{A,\,\omega}\oplus(\operatorname{Im}\partial+\operatorname{Im}\bar\partial)=\ker(\partial\bar\partial)$ and $\ker\Delta_{BC,\,\omega}\oplus(\operatorname{Im}\partial^\star+\operatorname{Im}\bar\partial^\star)=\ker(\partial\bar\partial)^\star$. Thus, we get unique decompositions of ω, resp. ω_{n-1}:

$$\ker(\partial\bar\partial)^\star\ni\omega=\omega_h+(\partial^\star_\omega\bar u_\omega+\bar\partial^\star_\omega u_\omega)\quad\text{and}\quad\ker(\partial\bar\partial)\ni\omega_{n-1}=(\omega_{n-1})_h+(\partial\bar\Gamma_\omega+\bar\partial\Gamma_\omega),\tag{3.29}$$

where $\omega_h\in\ker\Delta_{BC,\,\omega}$ as a $(1,\,1)$-form, $(\omega_{n-1})_h\in\ker\Delta_{A,\,\omega}$ as an $(n-1,\,n-1)$-form, while u_ω and Γ_ω are smooth forms of respective bidegrees $(1,\,2)$ and $(n-1,\,n-2)$. Since ω and ω_{n-1} are real, so are their harmonic components ω_h and $(\omega_{n-1})_h$. Since $\star_\omega\omega=\omega_{n-1}$ and since the Hodge star operator $\star_\omega$ maps Aeppli-harmonic forms to Bott-Chern-harmonic forms and vice-versa, we get:

$$\star_\omega\,\omega_h=(\omega_{n-1})_h\quad\text{and}\quad\star_\omega\,(\partial^\star_\omega\bar u_\omega+\bar\partial^\star_\omega u_\omega)=\partial\bar\Gamma_\omega+\bar\partial\Gamma_\omega.\tag{3.30}$$

Thus, ω_h is uniquely determined by ω and is d-closed (because it is even $\Delta_{BC,\,\omega}$-harmonic). Therefore, it represents a class in $H^{1,\,1}_{BC}(X,\,\mathbb{R})$.

Definition 3.2.15. *For any Gauduchon metric ω on a compact complex manifold X, the Bott-Chern cohomology class $[\omega_h]_{BC}\in H^{1,\,1}_{BC}(X,\,\mathbb{R})$ is called the* **cohomology class** *of ω.*

Of course, if ω is Kähler, $\omega_h=\omega$, so $\{\omega_h\}_{BC}$ is the usual Bott-Chern Kähler class $\{\omega\}_{BC}$.

The analogue of Lemma 3.2.7 is the following

Lemma 3.2.16. *Suppose there exists a Gauduchon metric ω on a compact complex manifold X. Then, for every $\alpha\in C^\infty_{1,\,1}(X,\,\mathbb{C})$ such that $d\alpha=0$ and $[\alpha]_{BC}\in H^{1,\,1}_{BC}(X,\,\mathbb{C})_{\omega\text{-}prim}$, we have:*

$$\langle\langle\omega_h,\,\alpha\rangle\rangle_\omega=0,$$

where $\langle\langle\ ,\ \rangle\rangle_\omega$ is the L^2 inner product induced by ω.

The analogue in this context of Conclusion 3.2.8 is the following

Conclusion 3.2.17. *Let X be a compact complex manifold with $\dim_{\mathbb{C}}X=n$. Let ω be a* **Gauduchon metric** *on X such that ω_{n-1} is not Aeppli-exact. Then, the Bott-Chern cohomology space of bidegree $(1,\,1)$ has a* **Lefschetz-type** *L^2_ω-orthogonal* **decomposition***:*

$$H^{1,\,1}_{BC}(X,\,\mathbb{C})=H^{1,\,1}_{BC}(X,\,\mathbb{C})_{\omega\text{-}prim}\oplus\mathbb{C}\cdot[\omega_h]_{BC},\tag{3.31}$$

where the ω-primitive subspace $H^{1,\,1}_{BC}(X,\,\mathbb{C})_{\omega\text{-}prim}$ is a complex hyperplane of $H^{1,\,1}_{BC}(X,\,\mathbb{C})$ depending only on the cohomology class $[\omega_{n-1}]_A\in H^{n-1,\,n-1}_A(X,\,\mathbb{C})$, while ω_h is the $\Delta_{BC,\,\omega}$-harmonic component of ω and the complex line $\mathbb{C}\cdot[\omega_h]_{BC}$ depends on the choice of the Gauduchon metric ω.

We also have the following analogue of Lemma 3.2.9.

Lemma 3.2.18. *The assumptions are the same as in Conclusion 3.2.17. For every $\alpha \in C^\infty_{1,1}(X, \mathbb{C}) \cap \ker d$, the coefficient of $[\omega_h]_{BC}$ in the Lefschetz-type decomposition of $[\alpha]_{BC} \in H^{1,1}_{BC}(X, \mathbb{C})$ according to (3.31), namely in*

$$[\alpha]_{BC} = [\alpha]_{BC,\,prim} + \lambda\,[\omega_h]_{BC}, \tag{3.32}$$

is given by

$$\lambda = \lambda_\omega([\alpha]_{BC}) = \frac{[\omega_{n-1}]_A.[\alpha]_{BC}}{||\omega_h||^2_\omega} = \frac{1}{||\omega_h||^2_\omega}\int\limits_X \alpha\wedge\omega_{n-1}. \tag{3.33}$$

As in §.3.2.3, formula (3.33) implies that $\lambda_\omega([\alpha]_{BC})$ is *real* if the class $[\alpha]_{BC} \in H^{1,1}_{BC}(X, \mathbb{C})$ is real. Thus, we can define a *positive side* and a *negative side* of the hyperplane $H^{1,1}_{BC}(X, \mathbb{R})_{\omega\text{-}prim} := H^{1,1}_{BC}(X, \mathbb{C})_{\omega\text{-}prim} \cap H^{1,1}_{BC}(X, \mathbb{R})$ in $H^{1,1}_{BC}(X, \mathbb{R})$ by

$$\begin{aligned} H^{1,1}_{BC}(X, \mathbb{R})^+_\omega &:= \left\{[\alpha]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R}) \;\mid\; \lambda_\omega([\alpha]_{BC}) > 0\right\}, \\ H^{1,1}_{BC}(X, \mathbb{R})^-_\omega &:= \left\{[\alpha]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R}) \;\mid\; \lambda_\omega([\alpha]_{BC}) < 0\right\}. \end{aligned} \tag{3.34}$$

These are open subsets of $H^{1,1}_{BC}(X, \mathbb{R})$ that depend only on the cohomology class $[\omega_{n-1}]_A \in H^{n-1,\,n-1}_A(X, \mathbb{R})$.

Since $[\alpha]_{BC}$ is ω-primitive if and only if $\lambda_\omega([\alpha]_{BC}) = 0$, we get a *partition* of $H^{1,1}_{BC}(X, \mathbb{R})$:

$$H^{1,1}_{BC}(X, \mathbb{R}) = H^{1,1}_{BC}(X, \mathbb{R})^+_\omega \cup H^{1,1}_{BC}(X, \mathbb{R})_{\omega\text{-}prim} \cup H^{1,1}_{BC}(X, \mathbb{R})^-_\omega \tag{3.35}$$

depending only on the cohomology class $[\omega_{n-1}]_A \in H^{n-1,\,n-1}_A(X, \mathbb{R})$.

As a consequence of these considerations, we get

Proposition 3.2.19. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The* **pseudo-effective cone** *$\mathcal{E}_X \subset H^{1,1}_{BC}(X, \mathbb{R})$ of X is the intersection of all the* **non-negative sides**

$$H^{1,1}_{BC}(X, \mathbb{R})^{\geq 0}_\omega := H^{1,1}_{BC}(X, \mathbb{R})^+_\omega \cup H^{1,1}_{BC}(X, \mathbb{R})_{\omega\text{-}prim}$$

of hyperplanes $H^{1,1}_{BC}(X, \mathbb{R})_{\omega\text{-}prim}$ determined by Aeppli-Gauduchon classes $[\omega_{n-1}]_A \in \mathcal{G}_X$:

$$\mathcal{E}_X = \bigcap_{[\omega_{n-1}]_A \in \mathcal{G}_X} H^{1,1}_{BC}(X, \mathbb{R})^{\geq 0}_\omega, \tag{3.36}$$

Proof. By the duality between the pseudo-effective cone $\mathcal{E}_X$ and the closure $\overline{\mathcal{G}}_X$ of the Gauduchon cone (see §.3.2.1), we know that a given class $[T]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R})$ lies in $\mathcal{E}_X$ (i.e. $[T]_{BC}$ can be represented by a closed *semi-positive* $(1,\,1)$-current) if and only if

$$\int\limits_X T\wedge\omega_{n-1} \geq 0 \qquad \text{for all } [\omega_{n-1}]_A \in \mathcal{G}_X.$$

The last condition is equivalent to $\lambda_\omega([T]_{BC}) \geq 0$, hence to $[T]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R})^{\geq 0}_\omega$, for all $[\omega_{n-1}]_A \in \mathcal{G}_X$, so the contention follows. $\square$

Question 3.2.20. *Is it true that the pseudo-effective cone $\mathcal{E}_X$ is* **small** *(in a sense to be determined) if (and only if) X is* **balanced hyperbolic***?*

3.3 Properties of balanced hyperbolic manifolds

The discussion of balanced hyperbolic manifolds featured in this section will mirror that of degenerate balanced manifolds of the previous section.

3.3.1 Background and L^1 currents on the universal cover

It is a classical fact due to Gaffney [Gaf54] that certain basic facts in the Hodge Theory of compact Riemannian manifolds remain valid on *complete* such manifolds. The main ingredient in the proof of this fact is the following *cut-off trick* of Gaffney's that played a key role in [Gro91, §.1]. It also appears in [Dem97, VIII, Lemma 2.4].

Lemma 3.3.1. *([Gaf54]) Let $(X,\, g)$ be a Riemannian manifold. Then, $(X,\, g)$ is* **complete** *if and only if there exists an exhaustive sequence $(K_\nu)_{\nu\in\mathbb{N}}$ of* **compact** *subsets of X:*

$$K_\nu\subset\mathring{K}_{\nu+1} \quad \textit{for all } \nu\in\mathbb{N} \quad \textit{and} \quad X=\bigcup_{\nu\in\mathbb{N}}K_\nu,$$

and a sequence $(\psi_\nu)_{\nu\in\mathbb{N}}$ of C^∞ functions $\psi_\nu : X\longrightarrow [0,\, 1]$ satisfying, for every $\nu\in\mathbb{N}$, the conditions:

$$\psi_\nu = 1 \textit{ in a neighbourhood of } K_\nu, \qquad \textit{Supp}\,\psi_\nu\subset\mathring{K}_{\nu+1} \qquad \textit{and}$$

$$||d\psi_\nu||_{L^\infty_g} := \sup_{x\in X}|(d\psi_\nu)(x)|_g\leq\varepsilon_\nu,$$

for some constants $\varepsilon_\nu>0$ such that $\varepsilon_\nu\downarrow 0$ as ν tends to $+\infty$.

In particular, the *cut-off functions* ψ_ν are *compactly supported.* One can choose $\varepsilon_\nu = 2^{-\nu}$ for each ν (see e.g. [Dem97, VIII, Lemma 2.4]), but this will play no role here.

An immediate consequence of Gaffney's cut-off trick is the following classical generalisation of Stokes's Theorem to possibly non-compact, but complete Riemannian manifolds when the forms involved are L^1.

Lemma 3.3.2. *([Gro91, Lemma 1.1.A.]) Let $(X,\, g)$ be a* **complete** *Riemannian manifold of real dimension m. Let η be an L^1_g-form on X of degree $m-1$ such that $d\eta$ is again L^1_g. Then*

$$\int_X d\eta = 0.$$

By the form η being L^1 with respect to the Riemannian metric g (L^1_g for short) we mean that its L^1-norm is *finite*:

$$||\eta||_{L^1_g} := \int_X|\eta(x)|_g\,dV_g(x)<+\infty,$$

where dV_g is the volume form induced by g.

Proof of Lemma 3.3.2. Let $(\psi_\nu)_{\nu\in\mathbb{N}}$ be a sequence of cut-off functions as in Lemma 3.3.1 whose existence is guaranteed by the completeness of $(X,\, g)$. The $(m-1)$-form $\psi_\nu\eta$ is *compactly supported* for every $\nu\in\mathbb{N}^\star$, so the usual Stokes's Theorem yields:

$$\int_X d(\psi_\nu\eta) = 0, \qquad \nu\in\mathbb{N}^\star.$$

Meanwhile, $d(\psi_\nu\eta) = d\psi_\nu\wedge\eta + \psi_\nu\, d\eta$, so we get:

$$\bigg|\int_X \psi_\nu\, d\eta\bigg| = \bigg|\int_X d\psi_\nu\wedge\eta\bigg| \leq ||d\psi_\nu||_{L^\infty_g}\, ||\eta||_{L^1_g} \leq \varepsilon_\nu\, ||\eta||_{L^1_g}, \qquad \nu\in\mathbb{N}, \tag{3.37}$$

for some sequence of constants $\varepsilon_\nu\downarrow 0$.

Since η is L^1_g, $\varepsilon_\nu\, ||\eta||_{L^1_g}\downarrow 0$ as $\nu\to+\infty$. On the other hand, since $d\eta$ is L^1_g, the properties of the functions ψ_ν imply that

$$\lim_{\nu\to+\infty}\int_X \psi_\nu d\eta = \int_X d\eta.$$

Together with (3.37), these arguments yield $\int_X d\eta = 0$, as desired. □

We now apply this standard cut-off function technique to prove Proposition 3.1.4 stated in the introduction. It is an analogue in our more general context of *balanced hyperbolic* manifolds of Proposition 5.4 in [Pop15a] according to which compact *degenerate balanced* manifolds are characterised by the absence of non-zero d-closed positive $(1,\,1)$-currents.

Note that, due to X being *compact*, any pair of Hermitian metrics ω_1 and ω_2 on X are *comparable* in the sense that there exists a constant $C>0$ such that $(1/C)\,\omega_1\leq\omega_2\leq C\,\omega_1$. Thus, their lifts $\widetilde\omega_1 := \pi^\star\omega_1$ and $\widetilde\omega_2 := \pi^\star\omega_2$ are again comparable on $\widetilde X$ by means of the same constant: $(1/C)\,\widetilde\omega_1\leq\widetilde\omega_2\leq C\,\widetilde\omega_1$. Therefore, the $L^1_{\widetilde\omega}$-assumption on $\widetilde T$ is independent of the choice of Hermitian metric on $\widetilde X$ if this metric is obtained by lifting a metric on X. However, the L^1-condition changes for metrics on $\widetilde X$ that are not lifts of metrics on X. But we will not deal with the latter type of metrics.

Proof of Proposition 3.1.4. Let $n=\dim_{\mathbb{C}}X$. The balanced hyperbolic assumption on X means that $\pi^\star\omega_{n-1} = d\widetilde\Gamma$ on $\widetilde X$ for some smooth $L^\infty_{\widetilde\omega}$-form $\widetilde\Gamma$ of degree $(2n-3)$ on $\widetilde X$.

If a current $\widetilde T$ as in the statement existed on $\widetilde X$, we would have

$$0 < \int\limits_{\widetilde X}\widetilde T\wedge\pi^\star\omega_{n-1} = \int\limits_{\widetilde X} d(\widetilde T\wedge\widetilde\Gamma) = 0, \tag{3.38}$$

which is contradictory.

The last identity in (3.38) follows from Lemma 3.3.2 applied on the complete manifold $(\widetilde X,\,\widetilde\omega)$ to the $L^1_{\widetilde\omega}$-current $\eta := \widetilde T\wedge\widetilde\Gamma$ of degree $2n-1$ whose differential $d\eta = \widetilde T\wedge\pi^\star\omega_{n-1}$ is again $L^1_{\widetilde\omega}$. That η is $L^1_{\widetilde\omega}$ follows from $\widetilde T$ being $L^1_{\widetilde\omega}$ (by hypothesis) and from $\widetilde\Gamma$ being $L^\infty_{\widetilde\omega}$, while $d\eta$ being $L^1_{\widetilde\omega}$ follows from $\widetilde T$ being $L^1_{\widetilde\omega}$ and from $\pi^\star\omega_{n-1}$

being L^∞_ω (as a lift of the smooth, hence bounded, form ω_{n-1} on the *compact* manifold X). $\square$

We now recall the following standard result saying that some further key facts in the Hodge Theory of compact Riemannian manifolds remain valid on *complete* such manifolds X when the differential operators involved (e.g. d, $d^\star$, Δ) are considered as *closed* and *densely defined* unbounded operators on the spaces $L^2_k(X,\,\mathbb{C})$ of L^2-forms of degree k on X. The only major property that is lost in passing to complete manifolds is the closedness of the images of these operators. As usual, any differential operator P originally defined on $C^\infty_\bullet(X,\,\mathbb{C})$ is extended to an unbounded operator on $L^2_\bullet(X,\,\mathbb{C})$ by defining its domain $\mathrm{Dom}\,P$ as the space of L^2-forms u such that Pu, computed in the sense of distributions, is again L^2.

Theorem 3.3.3. *(see e.g. [Dem97, VIII, Theorem 3.2.]) Let $(X,\,g)$ be a* **complete** *Riemannian manifold of real dimension m. Then:*

(a) The space $\mathcal{D}_\bullet(X,\,\mathbb{C})$ of compactly supported C^∞ forms of any degree (indicated by a $\bullet$) on X is **dense** *in the domains $Dom\,d$, $Dom\,d^\star$ and in $Dom\,d\cap Dom\,d^\star$ for the respective graph norms:*

$$u\mapsto ||u||+||du||,\qquad u\mapsto ||u||+||d^\star u||,\qquad u\mapsto ||u||+||du||+||d^\star u||.$$

(b) The extension $d^\star$ of the formal adjoint of d to the L^2-space coincides with the Hilbert space adjoint of the extension of d.

(c) The d-Laplacian $\Delta=\Delta_g:=dd^\star+d^\star d$ has the following property :

$$\langle\langle\Delta u,\,u\rangle\rangle=||du||^2+||d^\star u||^2 \tag{3.39}$$

for every form $u\in Dom\,\Delta$. In particular, $Dom\,\Delta\subset Dom\,d\cap Dom\,d^\star$ and $\ker\Delta=\ker d\cap\ker d^\star$.

*(d) There are L^2-***orthogonal decompositions** *in every degree (indicated by a $\bullet$):*

$$\begin{aligned}L^2_\bullet(X,\,\mathbb{C}) &= \mathcal{H}^\bullet_\Delta(X,\,\mathbb{C})\oplus\overline{Im\,d}\oplus\overline{Im\,d^\star}\\ \ker d &= \mathcal{H}^\bullet_\Delta(X,\,\mathbb{C})\oplus\overline{Im\,d}\quad \textit{and}\quad \ker d^\star=\mathcal{H}^\bullet_\Delta(X,\,\mathbb{C})\oplus\overline{Im\,d^\star},\end{aligned} \tag{3.40}$$

*where $\mathcal{H}^\bullet_\Delta(X,\,\mathbb{C}):=\{u\in L^2_\bullet(X,\,\mathbb{C})\mid\Delta u=0\}$ is the space of Δ-***harmonic** *L^2-forms, while*

$$Im\,d:=L^2_\bullet(X,\,\mathbb{C})\cap d(L^2_{\bullet-1}(X,\,\mathbb{C}))\quad \textit{and}\quad Im\,d^\star:=L^2_\bullet(X,\,\mathbb{C})\cap d^\star(L^2_{\bullet+1}(X,\,\mathbb{C})).$$

An immediate consequence of (3.39) applied in degree 0 is that on a connected *complete* Riemannian manifold $(X,\,g)$, every **Δ-harmonic** L^2-function is **constant**:

$$\mathcal{H}^0_\Delta(X,\,\mathbb{C})\subset\mathbb{C}. \tag{3.41}$$

3.3.2 Harmonic L^2-forms of degree 1 on the universal cover of a balanced hyperbolic manifold

Let X be a possibly non-compact complex manifold with $\dim_{\mathbb{C}} X = n$, supposed to carry a *complete balanced metric* ω. In subsequent applications, the roles of X and ω will be played by $\widetilde{X}$, the universal cover $\pi : \widetilde{X} \longrightarrow X$ of a compact balanced hyperbolic manifold (X, ω), resp. $\widetilde{\omega} := \pi^\star\omega$.

A well-known consequence of the Kähler commutation relations is the fact that, if ω is *Kähler*, the induced d-Laplacian $\Delta = \Delta_\omega$ commutes with the multiplication operator $\omega^l \wedge \cdot$ acting on differential forms of any degree on X, for every l.

We will see that, when ω is merely *balanced*, the commutation of Δ with the multiplication operator $\omega^{n-1} \wedge \cdot$ acting on differential forms no longer holds. However, we will now compute this commutation defect on 1-forms.

The computation will continue that of (i) in Lemma 3.2.1. For the sake of generality and for a reason that will become apparent later on, we will work with the more general operators

$$d_h := h\partial + \bar\partial, \qquad h \in \mathbb{C}^\star,$$

acting on $\mathbb{C}$-valued differential forms on X and the associated Laplacians $\Delta_h := d_h d_h^\star + d_h^\star d_h$.

The first stages of the computation lead to the following result in which no completeness assumption is necessary.

Lemma 3.3.4. *Let X be a complex manifold with $\dim_{\mathbb{C}} X = n$. Suppose there exists a* **balanced metric** *ω on X. Then, for any $h \in \mathbb{C}^\star$ and any* **1-form** *φ on X, the following identity holds:*

$$[\Delta_h,\, L_{\omega_{n-1}}]\varphi = \bigg(|h|^2 d_{-\frac{1}{\bar h}} d^\star_{-\frac{1}{\bar h}} - d_h^\star d_h\bigg)\varphi\wedge\omega_{n-1} - i\bar h\, d_{-\frac{1}{\bar h}}\varphi\wedge d_h\omega_{n-2} - i(|h|^2+1)\,\partial\bar\partial\varphi\wedge\omega_{n-2} \tag{3.42}$$

Proof. • The Jacobi identity yields:

$$[[d_h,\, d_h^\star],\, L_{\omega_{n-1}}] - [[d_h^\star,\, L_{\omega_{n-1}}],\, d_h] + [[L_{\omega_{n-1}},\, d_h],\, d_h^\star] = 0.$$

Since ω is *balanced*, $[L_{\omega_{n-1}},\, d_h] = 0$. Writing $\Delta_h = [d_h,\, d_h^\star]$, the above equality reduces to

$$[\Delta_h,\, L_{\omega_{n-1}}] = [d_h^\star,\, L_{\omega_{n-1}}]\, d_h + d_h\, [d_h^\star,\, L_{\omega_{n-1}}]. \tag{3.43}$$

• Note also the following formula for the **formal adjoint** of d_h involving the Hodge star operator:

$$d_h^\star = -\bar h \star d_{\frac{1}{\bar h}} \star. \tag{3.44}$$

Indeed, $d_h^\star = (h\partial + \bar\partial)^\star = \bar h\,(-\star\bar\partial\star) + (-\star\partial\star) = -\bar h \star (\frac{1}{\bar h}\,\partial + \bar\partial)\star = -\bar h \star d_{\frac{1}{\bar h}}\star$. No assumption on ω is needed here.

• As an application of (3.44), we observe the following formula for every $(1,\,1)$-form α:

$$d_h^\star(\omega_{n-1}\wedge\alpha) = -i\bar h\, d_{-\frac{1}{\bar h}}(\Lambda_\omega\alpha)\wedge\omega_{n-1}. \tag{3.45}$$

Again, no assumption on ω is needed.

To see this, we first multiply the Lefschetz decomposition (3.8) of α by ω_{n-1} and we get: $\omega_{n-1}\wedge\alpha = (\Lambda_\omega\alpha)\,\omega_n$. Hence, $\star(\omega_{n-1}\wedge\alpha) = \Lambda_\omega\alpha$, so we get the first equality below:

$$-\bar{h}\,\star d_{\frac{1}{\bar{h}}}\star(\omega_{n-1}\wedge\alpha) = -\bar{h}\,\star d_{\frac{1}{\bar{h}}}(\Lambda_\omega\alpha) = -\bar{h}\,\star\left(\frac{1}{\bar{h}}\,\partial(\Lambda_\omega\alpha)\right) - \bar{h}\,\star\bar\partial(\Lambda_\omega\alpha).$$

Applying (3.44) to the l.h.s. term above and the standard formula (3.7) to the r.h.s. term, we get:

$$d_h^\star(\omega_{n-1}\wedge\alpha) = i\partial(\Lambda_\omega\alpha)\wedge\omega_{n-1} - i\bar{h}\,\bar\partial(\Lambda_\omega\alpha)\wedge\omega_{n-1}.$$

Since $i\partial - i\bar{h}\,\bar\partial = -i\bar{h}\,d_{-\frac{1}{\bar{h}}}$, the above equality is nothing but (3.45).

- *Computation of the first term on the r.h.s. of (3.43) on 1-forms* $\varphi = \varphi^{1,0} + \varphi^{0,1}$.

Using formula (3.45) with $\alpha := h\partial\varphi^{0,1} + \bar\partial\varphi^{1,0}$, we get the second equality below:

$$\begin{aligned}[d_h^\star,\, L_{\omega_{n-1}}]\,d_h\varphi &= d_h^\star(\omega_{n-1}\wedge(h\partial\varphi^{0,1} + \bar\partial\varphi^{1,0})) - \omega_{n-1}\wedge d_h^\star d_h\varphi \\ &= -i\bar{h}\,d_{-\frac{1}{\bar{h}}}\left(h\Lambda_\omega(\partial\varphi^{0,1}) + \Lambda_\omega(\bar\partial\varphi^{1,0})\right)\wedge\omega_{n-1} - d_h^\star d_h\varphi\wedge\omega_n.\end{aligned} \tag{3.46}$$

On the other hand, the standard formula (3.7) yields:

$$\star\varphi = i(\varphi^{0,1} - \varphi^{1,0})\wedge\omega_{n-1}.$$

Since ω is balanced, this implies the first equality on each of the two rows below:

$$\begin{aligned}\partial\star\varphi &= i\partial(\varphi^{0,1} - \varphi^{1,0})\wedge\omega_{n-1} = i\partial\varphi^{0,1}\wedge\omega_{n-1} = i\Lambda_\omega(\partial\varphi^{0,1})\,\omega_n \\ \bar\partial\star\varphi &= i\bar\partial(\varphi^{0,1} - \varphi^{1,0})\wedge\omega_{n-1} = -i\bar\partial\varphi^{1,0}\wedge\omega_{n-1} = -i\Lambda_\omega(\bar\partial\varphi^{1,0})\,\omega_n.\end{aligned}$$

Taking $-\star$ in each of the above two equalities and using the standard identities $-\star\partial\star = \bar\partial^\star$, $-\star\bar\partial\star = \partial^\star$, we get:

$$\bar\partial^\star\varphi = -i\Lambda_\omega(\partial\varphi^{0,1}) \quad \text{and} \quad \partial^\star\varphi = i\Lambda_\omega(\bar\partial\varphi^{1,0}). \tag{3.47}$$

Putting together (3.46) and (3.47), we get:

$$\begin{aligned}[d_h^\star,\, L_{\omega_{n-1}}]\,d_h\varphi &= -i\bar{h}\,d_{-\frac{1}{\bar{h}}}(ih\bar\partial^\star\varphi - i\partial^\star\varphi)\wedge\omega_{n-1} - d_h^\star d_h\varphi\wedge\omega_{n-1} \\ &= h\bar{h}\,d_{-\frac{1}{\bar{h}}}d^\star_{-\frac{1}{\bar{h}}}\varphi\wedge\omega_{n-1} - d_h^\star d_h\varphi\wedge\omega_{n-1}.\end{aligned}$$

We have thus obtained the following formula:

$$[d_h^\star,\, L_{\omega_{n-1}}]\,d_h\varphi = \left(|h|^2 d_{-\frac{1}{\bar{h}}}d^\star_{-\frac{1}{\bar{h}}} - d_h^\star d_h\right)\varphi\wedge\omega_{n-1} \tag{3.48}$$

for every smooth 1-form φ whenever the metric ω is *balanced*.

- *Computation of the second term on the r.h.s. of (3.43) on 1-forms* $\varphi = \varphi^{1,0} + \varphi^{0,1}$.

We start by computing

$$[d_h^\star,\, L_{\omega_{n-1}}]\,\varphi = d_h^\star(\omega_{n-1}\wedge\varphi) - (d_h^\star\varphi)\,\omega_{n-1}. \tag{3.49}$$

Since $\omega_{n-1}\wedge\varphi^{1,\,0} = i\star\varphi^{1,\,0}$ and $\omega_{n-1}\wedge\varphi^{0,\,1} = -i\star\varphi^{0,\,1}$, formula (3.44) yields the first line below:

$$\begin{aligned} d_h^\star(\omega_{n-1}\wedge\varphi) &= i\bar{h}\star d_{\frac{1}{\bar{h}}}(\varphi^{1,\,0}-\varphi^{0,\,1}) = i\star(\partial\varphi^{1,\,0}+\bar{h}\bar\partial\varphi^{1,\,0}-\partial\varphi^{0,\,1}-\bar{h}\bar\partial\varphi^{0,\,1})\\ &= \frac{i}{n}\,\Lambda_\omega(\bar{h}\bar\partial\varphi^{1,\,0}-\partial\varphi^{0,\,1})\,\omega_{n-1} - i(\bar{h}\bar\partial\varphi^{1,\,0}-\partial\varphi^{0,\,1})_{prim}\wedge\omega_{n-2}\\ &+ i(\partial\varphi^{1,\,0}-\bar{h}\bar\partial\varphi^{0,\,1})\wedge\omega_{n-2}, \end{aligned} \tag{3.50}$$

where we used the Lefschetz decomposition (3.8) of the (1, 1)-form $\bar{h}\bar\partial\varphi^{1,\,0}-\partial\varphi^{0,\,1}$ and then the standard formula (3.7) to express the value of $\star$ on the primitive forms $\partial\varphi^{1,\,0}$ (of type (2, 0)), $\bar\partial\varphi^{0,\,1}$ (of type (0, 2)) and $(\bar{h}\bar\partial\varphi^{1,\,0}-\partial\varphi^{0,\,1})_{prim}$ (of type (1, 1)) and got $\star(\partial\varphi^{1,\,0}) = \partial\varphi^{1,\,0}\wedge\omega_{n-2}$ and

$$\star(\bar\partial\varphi^{0,\,1}) = \bar\partial\varphi^{0,\,1}\wedge\omega_{n-2}, \qquad \star(\bar{h}\bar\partial\varphi^{1,\,0}-\partial\varphi^{0,\,1})_{prim} = -(\bar{h}\bar\partial\varphi^{1,\,0}-\partial\varphi^{0,\,1})_{prim}\wedge\omega_{n-2}.$$

On the other hand, we get

$$\begin{aligned} d_h^\star\varphi &= -\bar{h}\star d_{\frac{1}{\bar{h}}}(\star\varphi^{1,\,0}+\star\varphi^{0,\,1}) = -\bar{h}\star(\frac{1}{\bar{h}}\partial+\bar\partial)(-i\varphi^{1,\,0}\wedge\omega_{n-1}+i\varphi^{0,\,1}\wedge\omega_{n-1})\\ &\overset{(i)}{=} -\bar{h}\star\bigg(-\frac{i}{\bar{h}}\,\partial\varphi^{1,\,0}\wedge\omega_{n-1}+\frac{i}{\bar{h}}\,\partial\varphi^{0,\,1}\wedge\omega_{n-1}-i\bar\partial\varphi^{1,\,0}\wedge\omega_{n-1}+i\bar\partial\varphi^{0,\,1}\wedge\omega_{n-1}\bigg)\\ &\overset{(ii)}{=} -\bar{h}\star\bigg(i\bigg(\frac{1}{\bar{h}}\,\partial\varphi^{0,\,1}-\bar\partial\varphi^{1,\,0}\bigg)\wedge\omega_{n-1}\bigg) = -\bar{h}\star\bigg(i\Lambda_\omega\bigg(\frac{1}{\bar{h}}\,\partial\varphi^{0,\,1}-\bar\partial\varphi^{1,\,0}\bigg)\,\omega_n\bigg)\\ &= i\Lambda_\omega(\bar{h}\bar\partial\varphi^{1,\,0}-\partial\varphi^{0,\,1}), \end{aligned} \tag{3.51}$$

where the balanced assumption on ω was used to get (i) and the equalities $\partial\varphi^{1,\,0}\wedge\omega_{n-1} = \bar\partial\varphi^{0,\,1}\wedge\omega_{n-1} = 0$, that hold for bidegree reasons, were used to get (ii).

Noticing that the last term in (3.51) also features within the first term on the second line in (3.50), the conclusion of (3.50) can be re-written as

$$d_h^\star(\omega_{n-1}\wedge\varphi) = \frac{1}{n}\,(d_h^\star\varphi)\,\omega_{n-1} - i(\bar{h}\bar\partial\varphi^{1,\,0}-\partial\varphi^{0,\,1})_{prim}\wedge\omega_{n-2} + i(\partial\varphi^{1,\,0}-\bar{h}\bar\partial\varphi^{0,\,1})\wedge\omega_{n-2} \tag{3.52}$$

From this and from (3.49), we get:

$$[d_h^\star,\, L_{\omega_{n-1}}]\,\varphi = \bigg(\frac{1}{n}-1\bigg)\,(d_h^\star\varphi)\,\omega_{n-1} - i(\bar{h}\bar\partial\varphi^{1,\,0}-\partial\varphi^{0,\,1})_{prim}\wedge\omega_{n-2} + i(\partial\varphi^{1,\,0}-\bar{h}\bar\partial\varphi^{0,\,1})\wedge\omega_{n-2}.$$

Hence, using the balanced hypothesis $d_h\omega = 0$, we get the first two lines below:

$$\begin{aligned} d_h[d_h^\star,\, L_{\omega_{n-1}}]\,\varphi &= \bigg(\frac{1}{n}-1\bigg)\,dd_h^\star\varphi\wedge\omega_{n-1} - id_h\bigg((\bar{h}\bar\partial\varphi^{1,\,0}-\partial\varphi^{0,\,1})\wedge\omega_{n-2}\bigg)\\ &+ \frac{n-1}{n}i\,d_h\bigg(\Lambda_\omega(\bar{h}\bar\partial\varphi^{1,\,0}-\partial\varphi^{0,\,1})\bigg)\wedge\omega_{n-1}\\ &+ i(\partial\varphi^{1,\,0}-\bar{h}\bar\partial\varphi^{0,\,1})\wedge d_h\omega_{n-2} - i(|h|^2\,\partial\bar\partial\varphi^{0,\,1}+\partial\bar\partial\varphi^{1,\,0})\wedge\omega_{n-2}. \end{aligned}$$

Now, formula (3.51) shows that the term on the second line above equals minus the first term on the r.h.s. of the first line. Hence, the sum of these two terms vanishes and we get:

$$\begin{aligned} d_h[d_h^\star,\, L_{\omega_{n-1}}]\,\varphi &= i\,\bigg(\partial\varphi^{1,\,0} + (\partial\varphi^{0,\,1} - \bar{h}\bar\partial\varphi^{1,\,0}) - \bar{h}\bar\partial\varphi^{0,\,1}\bigg)\wedge d_h\omega_{n-2} - i(|h|^2+1)\,\partial\bar\partial\varphi\wedge\omega_{n-2} \\ &= -i\bar{h}\,d_{-\frac{1}{\bar{h}}}\varphi\wedge d_h\omega_{n-2} - i(|h|^2+1)\,\partial\bar\partial\varphi\wedge\omega_{n-2}. \end{aligned} \tag{3.53}$$

- *Conclusion.*

Putting together (3.43), (3.48) and (3.53), we get (3.42). The proof of Lemma 3.3.4 is complete. □

Recall that for any Hermitian metric ω on an n-dimensional complex manifold X, the pointwise *Lefschetz map*:

$$L_{\omega_{n-1}} : \Lambda^1 T^\star X \longrightarrow \Lambda^{2n-1}T^\star X, \qquad \varphi \longmapsto \psi := \omega_{n-1}\wedge\varphi,$$

is *bijective* and a *quasi-isometry* (in the sense of Lemma 4.0.14).

We will now integrate the result of Lemma 3.3.4 expressing the commutation defect between Δ_h and $L_{\omega_{n-1}}$ on 1-forms. We need to assume our balanced metric ω to be *complete* to ensure that the two meanings of $d_h^\star$ coincide and the L^2_ω-inner products can be handled as in the compact case (see (b) and (c) of Theorem 3.3.3).

Proposition 3.3.5. *Let X be a complex manifold with $dim_{\mathbb{C}}X = n$. Suppose there exists a* **complete balanced metric** *ω on X.*

Then, for any $h\in\mathbb{C}^\star$ and any **1-form** *$\varphi\in Dom\,(\Delta_{-\frac{1}{\bar{h}}})$ on X, the following identity holds:*

$$\langle\langle\Delta_h(\omega_{n-1}\wedge\varphi),\, \omega_{n-1}\wedge\varphi\rangle\rangle = |h|^2\,\langle\langle\Delta_{-\frac{1}{\bar{h}}}\varphi,\, \varphi\rangle\rangle. \tag{3.54}$$

Proof. Throughout the proof, φ will stand for an arbitrary smooth 1-form on X.

- We first notice that $d_hd_{-\frac{1}{\bar{h}}}\varphi = ((|h|^2+1)/\bar{h})\,\partial\bar\partial\varphi$, hence

$$d_h\bigg(-i\bar{h}\,d_{-\frac{1}{\bar{h}}}\varphi\wedge\omega_{n-2}\bigg) = -i\bar{h}\,d_{-\frac{1}{\bar{h}}}\varphi\wedge d_h\omega_{n-2} - i(|h|^2+1)\,\partial\bar\partial\varphi\wedge\omega_{n-2}.$$

These are the last two terms of formula (3.42).

Putting $\psi := \omega_{n-1}\wedge\varphi$ and using (3.42) with its last two terms transformed as above, we get:

$$\begin{aligned} \langle\langle\Delta_h\psi,\, \psi\rangle\rangle &= \langle\langle\Delta_h\varphi\wedge\omega_{n-1},\, \varphi\wedge\omega_{n-1}\rangle\rangle + \langle\langle(|h|^2 d_{-\frac{1}{\bar{h}}}d^\star_{-\frac{1}{\bar{h}}} - d_h^\star d_h)\,\varphi\wedge\omega_{n-1},\, \varphi\wedge\omega_{n-1}\rangle\rangle \\ &\qquad -i\bar{h}\,\langle\langle d_{-\frac{1}{\bar{h}}}\varphi\wedge\omega_{n-2},\, d_h^\star(\omega_{n-1}\wedge\varphi)\rangle\rangle \\ &= \langle\langle d_hd_h^\star\varphi\wedge\omega_{n-1},\, \varphi\wedge\omega_{n-1}\rangle\rangle + \langle\langle|h|^2 d_{-\frac{1}{\bar{h}}}d^\star_{-\frac{1}{\bar{h}}}\varphi\wedge\omega_{n-1},\, \varphi\wedge\omega_{n-1}\rangle\rangle \\ &\qquad -i\bar{h}\,\langle\langle d_{-\frac{1}{\bar{h}}}\varphi\wedge\omega_{n-2},\, d_h^\star(\omega_{n-1}\wedge\varphi)\rangle\rangle \\ &\stackrel{(i)}{=} \langle\langle d_hd_h^\star\varphi,\, \varphi\rangle\rangle + |h|^2\,\langle\langle d_{-\frac{1}{\bar{h}}}d^\star_{-\frac{1}{\bar{h}}}\varphi,\, \varphi\rangle\rangle - i\bar{h}\,\langle\langle d_{-\frac{1}{\bar{h}}}\varphi\wedge\omega_{n-2},\, d_h^\star(\omega_{n-1}\wedge\varphi)\rangle\rangle \\ &= ||d_h^\star\varphi||^2 + |h|^2\,||d^\star_{-\frac{1}{\bar{h}}}\varphi||^2 - i\bar{h}\,\langle\langle d_{-\frac{1}{\bar{h}}}\varphi\wedge\omega_{n-2},\, d_h^\star(\omega_{n-1}\wedge\varphi)\rangle\rangle, \end{aligned} \tag{3.55}$$

where (i) followed from Lemma 4.0.12 applied to the (necessarily primitive) 1-forms φ, $d_h d_h^\star\varphi$ and $d_{-\frac{1}{\bar h}} d^\star_{-\frac{1}{\bar h}}\varphi$.

• We now transform the last term in (3.55), namely $T(\varphi) := -i\bar{h}\,\langle\langle d_{-\frac{1}{\bar h}}\varphi\wedge\omega_{n-2},\, d_h^\star(\omega_{n-1}\wedge\varphi)\rangle\rangle$.

Since the multiplication map $\omega_{n-2}\wedge\cdot : \Lambda^2T^\star X \longrightarrow \Lambda^{2n-2}T^\star X$ is *bijective*, there exists a unique 2-form β such that $d_h^\star(\omega_{n-1}\wedge\varphi) = \omega_{n-2}\wedge\beta$. Thus, using (3.52) for the second equality below, we get:

$$\omega_{n-2}\wedge\beta = d_h^\star(\omega_{n-1}\wedge\varphi) = \omega_{n-2}\wedge\bigg(\frac{1}{n(n-1)}\,(d_h^\star\varphi)\,\omega - i(\bar{h}\bar\partial\varphi^{1,\,0} - \partial\varphi^{0,\,1})_{prim} + i(\partial\varphi^{1,\,0} - \bar{h}\bar\partial\varphi^{0,\,1})\bigg).$$

The uniqueness of β implies that

$$\beta = \bigg(-i(\bar{h}\bar\partial\varphi^{1,\,0} - \partial\varphi^{0,\,1})_{prim} + i(\partial\varphi^{1,\,0} - \bar{h}\bar\partial\varphi^{0,\,1})\bigg) + \frac{1}{n(n-1)}\,(d_h^\star\varphi)\,\omega. \quad (3.56)$$

In particular, the primitive part β_{prim} of β in the Lefschetz decomposition is the form inside the large parenthesis and $\Lambda_\omega\beta = \frac{1}{n-1}\,d_h^\star\varphi$.

On the other hand, we have

$$d_{-\frac{1}{\bar h}}\varphi = -\frac{1}{\bar h}\,\partial\varphi^{1,\,0} + (-\frac{1}{\bar h}\,\partial\varphi^{0,\,1} + \bar\partial\varphi^{1,\,0})_{prim} + \bar\partial\varphi^{0,\,1} + \frac{1}{ni\bar{h}}\,(d_h^\star\varphi)\,\omega, \quad (3.57)$$

where the value of the last term follows from formula (3.51). This implies that β and $-i\bar{h}\,d_{-\frac{1}{\bar h}}\varphi$ have the same primitive part:

$$\beta_{prim} = -i\bar{h}\,(d_{-\frac{1}{\bar h}}\varphi)_{prim}. \quad (3.58)$$

We get:

$$\begin{aligned}\langle d_{-\frac{1}{\bar h}}\varphi\wedge\omega_{n-2},\, d_h^\star(\omega_{n-1}\wedge\varphi)\rangle &= \langle d_{-\frac{1}{\bar h}}\varphi\wedge\omega_{n-2},\, \beta\wedge\omega_{n-2}\rangle \\ &= \langle (d_{-\frac{1}{\bar h}}\varphi)_{prim},\, \beta_{prim}\rangle + (n-1)^2 n\,\bigg\langle\frac{1}{ni\bar{h}}\,d_h^\star\varphi,\, \frac{1}{n(n-1)}\,d_h^\star\varphi\bigg\rangle,\end{aligned}$$

where the last equality follows from formula (4.6) in the Appendix.

From this and from (3.56)-(3.58), we get:

$$\begin{aligned}T(\varphi) &= -i\bar{h}\,\langle\langle d_{-\frac{1}{\bar h}}\varphi\wedge\omega_{n-2},\, d_h^\star(\omega_{n-1}\wedge\varphi)\rangle\rangle \\ &= ||\partial\varphi^{1,\,0}||^2 + ||(\partial\varphi^{0,\,1} - \bar{h}\,\bar\partial\varphi^{1,\,0})_{prim}||^2 + |h|^2\,||\bar\partial\varphi^{0,\,1}||^2 - (1-\frac{1}{n})\,||d_h^\star\varphi||^2. \quad (3.59)\end{aligned}$$

• Putting (3.55) and (3.59) together and writing $\frac{1}{n}\,||d_h^\star\varphi||^2 = |h|^2\,\frac{1}{n}\,||\frac{1}{i\bar h}\,d_h^\star\varphi||^2$, we get:

$$\begin{aligned}\langle\langle\Delta_h\psi,\,\psi\rangle\rangle &= |h|^2\,||d^\star_{-\frac{1}{\bar h}}\varphi||^2 + |h|^2\,\frac{1}{n}\,||\frac{1}{i\bar h}\,d_h^\star\varphi||^2 \\ &+ |h|^2\bigg(||-\frac{1}{\bar h}\,\partial\varphi^{1,\,0}||^2 + ||(-\frac{1}{\bar h}\,\partial\varphi^{0,\,1} + \bar\partial\varphi^{1,\,0})_{prim}||^2 + ||\bar\partial\varphi^{0,\,1}||^2\bigg).\end{aligned}$$

Thanks to the expression of $d_{-\frac{1}{h}}\varphi$ obtained in (3.57), this translates to

$$\langle\langle\Delta_h\psi,\,\psi\rangle\rangle = |h|^2\left(||d^\star_{-\frac{1}{h}}\varphi||^2 + ||d_{-\frac{1}{h}}\varphi||^2\right) = |h|^2\,\langle\langle\Delta_{-\frac{1}{h}}\varphi,\,\varphi\rangle\rangle,$$

which is (3.54).

Proposition 3.3.5 is proved. $\square$

An immediate consequence of Proposition 3.3.5 is the following *Hard Lefschetz*-type result for spaces of harmonic L^2_ω-forms induced by a given *complete balanced metric* ω and different operators $\Delta_{-\frac{1}{h}}$ and Δ_h. Note that $h \neq -\frac{1}{h}$ for all $h \in \mathbb{C}^\star$. This is the price we have to pay in the non-Kähler balanced context to get this kind of results.

Corollary 3.3.6. *Let X be a complex manifold with $\dim_{\mathbb{C}} X = n$. Suppose there exists a* **complete balanced metric** *ω on X. Then, for any $h \in \mathbb{C}^\star$, the map*

$$\omega_{n-1}\wedge\cdot : \mathcal{H}^1_{\Delta_{-\frac{1}{h}}}(X,\,\mathbb{C}) \longrightarrow \mathcal{H}^{2n-1}_{\Delta_h}(X,\,\mathbb{C}), \qquad \varphi \longmapsto \omega_{n-1}\wedge\varphi,$$

is **well-defined** *and an* **isomorphism**.

Proof. The well-definedness, namely the fact that this map takes $\Delta_{-\frac{1}{h}}$-harmonic L^2_ω-forms to Δ_h-harmonic L^2_ω-forms, follows at once from Proposition 3.3.5 and from the form ω_{n-1} being ω-bounded. The fact that this map is an isomorphism follows from the standard fact that the corresponding pointwise map is bijective. $\square$

Corollary 3.3.7. *Let X be a complex manifold with $\dim_{\mathbb{C}} X = n$. Suppose there exists a* **complete balanced metric** *ω on X such that $\omega_{n-1} = d\Gamma$ for an ω-***bounded** *smooth $(2n-3)$-form Γ. Then*

$$\langle\langle\Delta\psi,\,\psi\rangle\rangle \geq \frac{1}{4||\Gamma||^2_{L^\infty_\omega}}\,||\psi||^2 \tag{3.60}$$

for every **pure-type** *form $\psi \in Dom(\Delta)$ of degree $2n-1$.*

Proof. Taking $h = 1$ in Proposition 3.3.5, (3.54) gives:

$$\langle\langle\Delta\psi,\,\psi\rangle\rangle = \langle\langle\Delta_{-1}\varphi,\,\varphi\rangle\rangle = ||(\partial-\bar\partial)\varphi||^2 + ||(\partial-\bar\partial)^\star\varphi||^2 \geq ||(\partial-\bar\partial)\varphi||^2,$$

for every $(2n-1)$-form ψ, where φ is the unique 1-form such that $\psi = \omega_{n-1}\wedge\varphi$. (See isomorphism (4.1) for $r = 1$.) Meanwhile, ψ is of pure type (either $(n,\,n-1)$ or $(n-1,\,n)$) if and only if φ is of pure type (respectively, either $(1,\,0)$ or $(0,\,1)$). In this case, $\partial\varphi$ and $\bar\partial\varphi$ are of different pure types, hence orthogonal to each other, hence $||(\partial-\bar\partial)\varphi||^2 = ||(\partial+\bar\partial)\varphi||^2$. Thus, we get:

$$\langle\langle\Delta\psi,\,\psi\rangle\rangle \geq ||d\varphi||^2, \tag{3.61}$$

for every **pure-type** $(2n-1)$-form $\psi \in \mathrm{Dom}(\Delta)$.

To complete the proof, we adapt the proof of Theorem 1.4.A. in [Gro91] to our context.

Since any 1-form φ is primitive, Lemma 4.0.12 gives: $|\psi|^2 = |\omega_{n-1}\wedge\varphi|^2 = |\varphi|^2$. In particular,

$$||\psi|| = ||\varphi||. \tag{3.62}$$

Meanwhile, we have: $\psi = \omega_{n-1}\wedge\varphi = d\Gamma\wedge\varphi = d(\Gamma\wedge\varphi) + \Gamma\wedge d\varphi$. In other words,

$$\psi = d\theta + \psi', \qquad \text{where } \theta := \Gamma\wedge\varphi \quad \text{and} \quad \psi' := \Gamma\wedge d\varphi. \tag{3.63}$$

To estimate θ, we write:

$$||\theta|| \leq ||\Gamma||_{L^\infty_\omega}\,||\varphi|| = ||\Gamma||_{L^\infty_\omega}\,||\psi||, \tag{3.64}$$

where (3.62) was used to get the last equality.

To estimate ψ', we write:

$$||\psi'|| \leq ||\Gamma||_{L^\infty_\omega}\,||d\varphi|| \leq ||\Gamma||_{L^\infty_\omega}\,\langle\langle\Delta\psi,\,\psi\rangle\rangle^{\frac{1}{2}}, \tag{3.65}$$

where (3.61) and the fact that φ is of pure type were used to get the last inequality.

To find an upper bound for $||\psi||$, we write:

$$||\psi||^2 = \langle\langle\psi,\,d\theta+\psi'\rangle\rangle \leq |\langle\langle\psi,\,d\theta\rangle\rangle| + |\langle\langle\psi,\,\psi'\rangle\rangle|, \tag{3.66}$$

where (3.63) was used to get the first equality.

For the first term on the r.h.s. of (3.66), we get:

$$|\langle\langle\psi,\,d\theta\rangle\rangle| = |\langle\langle d^\star\psi,\,\theta\rangle\rangle| \leq ||d^\star\psi||\,||\theta|| \leq \langle\langle\Delta\psi,\,\psi\rangle\rangle^{\frac{1}{2}}\,||\Gamma||_{L^\infty_\omega}\,||\psi||, \tag{3.67}$$

where (3.64) was used to get the last inequality.

For the second term on the r.h.s. of (3.66), we get:

$$|\langle\langle\psi,\,\psi'\rangle\rangle| \leq ||\psi'||\,||\psi|| \leq ||\Gamma||_{L^\infty_\omega}\,\langle\langle\Delta\psi,\,\psi\rangle\rangle^{\frac{1}{2}}\,||\psi||, \tag{3.68}$$

where (3.65) was used to get the last inequality.

Adding up (3.67) and (3.68) and using (3.66), we get

$$||\psi|| \leq 2\,||\Gamma||_{L^\infty_\omega}\,\langle\langle\Delta\psi,\,\psi\rangle\rangle^{\frac{1}{2}},$$

which is (3.60). The proof is complete. $\square$

For the record, if we do not assume ψ to be of pure type and use the full force of (3.54) rather than (3.61), we can run the argument in the proof of Corollary 3.3.7 with minor modifications starting from the observation that $\omega_{n-1} = d_{-\frac{1}{h}}\Gamma_{-\frac{1}{h}}$, where $\Gamma_h := h\,\Gamma^{n,\,n-3} + \Gamma^{n-1,\,n-2} + (1/h)\,\Gamma^{n-2,\,n-1} + (1/h^2)\,\Gamma^{n-3,\,n}$ for every $h\in\mathbb{C}^\star$ and the $\Gamma^{p,\,q}$'s are the pure-type components of Γ. Then, we get the following analogue of (3.60):

$$||\psi|| \leq C_h\,||\Gamma_{-\frac{1}{h}}||\left(\langle\langle\Delta_h\psi,\,\psi\rangle\rangle^{\frac{1}{2}} + \langle\langle\Delta_{-\frac{1}{h}}\psi,\,\psi\rangle\rangle^{\frac{1}{2}}\right) \tag{3.69}$$

for every form $\psi \in \mathrm{Dom}(\Delta_h) \cap \mathrm{Dom}(\Delta_{-\frac{1}{\bar{h}}})$ (not necessarily of pure type) of degree $2n-1$, where $C_h := \max(1, 1/|h|)$.

The occurrence of two different Laplacians on the r.h.s. of (3.69) (recall that $h \neq -\frac{1}{\bar{h}}$ for every $h \in \mathbb{C}^\star$) is the downside of that estimate that we avoided in Corollary 3.3.7 by restricting attention to pure-type forms. The advantage of dealing with a single Laplacian is demonstrated by Theorem 3.1.5 in the introduction that we now prove as a consequence of the above discussion.

Proof of Theorem 3.1.5. The pair $(\widetilde{X}, \widetilde{\omega})$ satisfies the hypotheses of Corollary 3.3.7 (playing the role of the pair (X, ω) therein). When applied to $(n, n-1)$-forms and to $(n-1, n)$-forms $\psi \in \mathrm{Dom}(\Delta_{\widetilde{\omega}})$, inequality (3.60) gives the following implication:

$$\Delta_{\widetilde{\omega}}\psi = 0 \implies \psi = 0.$$

This proves the vanishing of $\mathcal{H}^{n,\,n-1}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C})$ and $\mathcal{H}^{n-1,\,n}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C})$.

Meanwhile, the Hodge star operator $\star = \star_{\widetilde{\omega}}$ commutes with $\Delta_{\widetilde{\omega}}$, so it induces isomorphisms

$$\star_{\widetilde{\omega}} : \mathcal{H}^{1,\,0}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C}) \longrightarrow \mathcal{H}^{n,\,n-1}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C}) \quad \text{and} \quad \star_{\widetilde{\omega}} : \mathcal{H}^{0,\,1}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C}) \longrightarrow \mathcal{H}^{n-1,\,n}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C}).$$

Therefore, the spaces $\mathcal{H}^{1,\,0}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C})$ and $\mathcal{H}^{0,\,1}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C})$ must vanish as well. □

3.3.3 Harmonic L^2-forms of degree 2 on the universal cover of a balanced hyperbolic manifold

We will discuss 2-forms in a way analogous to the discussion of 1-forms we had in §.3.3.2. The context and the notation are the same. The analogue of Lemma 3.3.4 is

Lemma 3.3.8. *Let X be a complex manifold with $\dim_{\mathbb{C}} X = n$. Suppose there exists a* **balanced metric** *ω on X. Then, for any $h \in \mathbb{C}^\star$ and any* **2-form** *α on X, the following identity holds:*

$$[\Delta_h,\, L_{\omega_{n-1}}]\alpha = -(|h|^2 + 1)\, i\partial\bar\partial(\Lambda_\omega \alpha) \wedge \omega_{n-1} - \omega_{n-1} \wedge \Delta_h \alpha. \tag{3.70}$$

Proof. We compute separately the two terms applied to α on the r.h.s. of the consequence (3.43) of the Jacobi identity and the balanced hypothesis on ω.

The first term is

$$[d_h^\star,\, L_{\omega_{n-1}}]\, d_h\alpha = -\omega_{n-1} \wedge d_h^\star d_h \alpha, \tag{3.71}$$

since $d_h^\star(\omega_{n-1} \wedge d_h\alpha) = 0$ owing to the vanishing of $\omega_{n-1} \wedge d_h\alpha$ for degree reasons.

To compute $d_h\, [d_h^\star,\, L_{\omega_{n-1}}]\alpha$, we notice that

$$[d_h^\star,\, L_{\omega_{n-1}}]\alpha = d_h^\star(\omega_{n-1} \wedge \alpha) - \omega_{n-1} \wedge d_h^\star \alpha = -i\bar{h}\, d_{-\frac{1}{\bar{h}}}(\Lambda_\omega \alpha) \wedge \omega_{n-1} - \omega_{n-1} \wedge d_h^\star \alpha,$$

where the last identity follows from (3.45). Thus, using the balanced hypothesis on ω, we get:

$$d_h\, [d_h^\star,\, L_{\omega_{n-1}}]\alpha = -i\bar{h} d_h\, d_{-\frac{1}{\bar{h}}}(\Lambda_\omega \alpha) \wedge \omega_{n-1} - \omega_{n-1} \wedge d_h d_h^\star \alpha. \tag{3.72}$$

Finally, $d_h\, d_{-\frac{1}{\bar{h}}} = ((|h|^2+1)/\bar{h})\,\partial\bar\partial$, so (3.70) follows from (3.71) and (3.72). $\square$

We now deduce the following analogue of Proposition 3.3.5.

Proposition 3.3.9. *Let $(X,\,\omega)$ be a* **complete balanced** *manifold, $dim_{\mathbb{C}}X = n \geq 2$. For any $h\in\mathbb{C}^\star$ and any* 2**-form** *$\varphi \in Dom\,(\Delta_h)$ on X, the following identity holds:*

$$\langle\langle\Delta_h(\omega_{n-1}\wedge\alpha),\,\omega_{n-1}\wedge\alpha\rangle\rangle = (|h|^2+1)\,||\bar\partial(\Lambda_\omega\alpha)||^2. \tag{3.73}$$

Proof. An immediate consequence of (3.70) is the identity

$$\Delta_h(\omega_{n-1}\wedge\alpha) = -(|h|^2+1)\,i\partial\bar\partial(\Lambda_\omega\alpha)\wedge\omega_{n-1}.$$

Taking the pointwise inner product (w.r.t. ω) against $\omega_{n-1}\wedge\alpha$ and using the Lefschetz decomposition $\alpha^{1,\,1} = \alpha^{1,\,1}_{prim} + (1/n)\,(\Lambda_\omega\alpha^{1,\,1})\,\omega$ of the $(1,\,1)$-type component of α, its analogue for the $(1,\,1)$-form $i\partial\bar\partial(\Lambda_\omega\alpha)$ and the fact that the product of any primitive 2-form with ω_{n-1} vanishes, we get:

$$\begin{aligned}\langle\Delta_h(\omega_{n-1}\wedge\alpha),\,\omega_{n-1}\wedge\alpha\rangle &= -(|h|^2+1)\,\langle\widetilde\Delta_\omega(\Lambda_\omega\alpha)\,\omega_n,\,(\Lambda_\omega\alpha)\,\omega_n\rangle \\ &= -(|h|^2+1)\,\langle\widetilde\Delta_\omega(\Lambda_\omega\alpha),\,\Lambda_\omega\alpha\rangle,\end{aligned} \tag{3.74}$$

where $\widetilde\Delta_\omega f := \Lambda_\omega(i\partial\bar\partial f)$ for any function f on X. It is standard that the Laplacian $\widetilde\Delta_\omega$ is a non-positive operator on functions. Identity (4.3) in Lemma 4.0.12 with $k=0$ and $r=n$ was used to get the last equality in (3.74).

Now, we need the following simple observation.

Lemma 3.3.10. *Let $(X,\,\omega)$ be a* **complete balanced** *manifold, $dim_{\mathbb{C}}X = n \geq 2$. For any function $f\in Dom\,(\widetilde\Delta_\omega)$, we have: $\langle\langle\widetilde\Delta_\omega f,\,f\rangle\rangle = -||\bar\partial f||^2$.*

Proof of Lemma 3.3.10. The formula $\partial^\star = -\star\bar\partial\star$ gives the third equality below:

$$\begin{aligned}\langle\langle\widetilde\Delta_\omega f,\,f\rangle\rangle &= \langle\langle\Lambda_\omega(i\partial\bar\partial f),\,f\rangle\rangle = \langle\langle i\bar\partial f,\,\partial^\star(f\omega)\rangle\rangle = -i\,\langle\langle\bar\partial f,\,\star\bar\partial(f\,\omega_{n-1})\rangle\rangle \\ &= -i\,\langle\langle\bar\partial f,\,\star(\bar\partial f\wedge\omega_{n-1})\rangle\rangle,\end{aligned}$$

where we used the balanced hypothesis on ω to get the last equality.

Now, $\bar\partial f$ is a $(0,\,1)$-form, hence *primitive*, so the standard formula (3.7) yields:

$$\star(i\bar\partial f) = -\bar\partial f\wedge\omega_{n-1}, \quad \text{or equivalently} \quad \star\,(\bar\partial f\wedge\omega_{n-1}) = i\bar\partial f,$$

since $\star\star = -\mathrm{Id}$ on forms of odd degree.

The contention follows. $\square$

End of proof of Proposition 3.3.9. Integrating (3.74) and applying Lemma 3.3.10 with $f=\Lambda_\omega\alpha$, we get (3.73). $\square$

The next consequence of the above discussion can be conveniently worded in terms of Demailly's torsion operator $\tau = \tau_\omega := [\Lambda_\omega,\,\partial\omega\wedge\cdot]$ and the induced Laplacian $\Delta_\tau := [d+\tau,\,d^\star+\tau^\star]$ mentioned in the introduction.

Corollary 3.3.11. *Let* $(X,\,\omega)$ *be a connected* **complete balanced** *manifold,* $dim_{\mathbb{C}}X = n \geq 2$. *For any* $(1,\,1)$*-form* $\alpha^{1,\,1} \in Dom\,(\Delta_\tau)$*, the following implication holds:*

$$\Delta_\tau\alpha^{1,\,1} = 0 \implies \Lambda_\omega\alpha^{1,\,1} \text{ is } \textbf{constant}. \tag{3.75}$$

Proof. Thanks to (3.5) and to $\Delta'_\tau \geq 0$ and $\Delta'' \geq 0$, the hypothesis $\Delta_\tau\alpha^{1,\,1} = 0$ translates to $\Delta'_\tau\alpha^{1,\,1} = 0$ and $\Delta''\alpha^{1,\,1} = 0$. Since ω is complete, these conditions are further equivalent to

$$\begin{aligned} &(i) \quad (\partial+\tau)\alpha^{1,\,1} = 0, \qquad &&(iii) \quad \bar\partial\alpha^{1,\,1} = 0 \\ &(ii) \quad (\partial^\star+\tau^\star)\alpha^{1,\,1} = 0, \qquad &&(iv) \quad \bar\partial^\star\alpha^{1,\,1} = 0. \end{aligned} \tag{3.76}$$

Thus, we get:

$$\bar\partial(\Lambda_\omega\alpha^{1,\,1}) = [\bar\partial,\,\Lambda_\omega]\,\alpha^{1,\,1} = i(\partial^\star+\tau^\star)\,\alpha^{1,\,1} = 0,$$

where the first equality follows from (iii) of (3.76), the second equality follows from Demailly's Hermitian commutation relation (3.4) and the third equality follows from (ii) of (3.76).

We conclude that the hypothesis $\Delta_\tau\alpha^{1,\,1} = 0$ implies $\bar\partial(\Lambda_\omega\alpha^{1,\,1}) = 0$. This implies, thanks to Proposition 3.3.9 applied with $h = 1$, that $\Delta(\omega_{n-1}\wedge\alpha^{1,\,1}) = 0$, where $\Delta = \Delta_\omega = dd^\star + d^\star d$ is the d-Laplacian induced by ω. Since $\omega_{n-1}\wedge\alpha^{1,\,1} = (\Lambda_\omega\alpha^{1,\,1})\,\omega_n = \star(\Lambda_\omega\alpha^{1,\,1})$ and since Δ commutes with $\star$, we get $\Delta(\Lambda_\omega\alpha^{1,\,1}) = 0$. By completeness of ω, this means that $d(\Lambda_\omega\alpha^{1,\,1}) = 0$ on X, hence $\Lambda_\omega\alpha^{1,\,1}$ must be constant since X is connected. $\square$

An immediate consequence of Corollary 3.3.11 is that the following linear map is **well defined**:

$$T_{\omega_n} : \mathcal{H}^{1,\,1}_{\Delta_\tau}(X,\,\mathbb{C}) \longrightarrow \mathbb{C}, \qquad \alpha^{1,\,1} \longmapsto \Lambda_\omega\alpha^{1,\,1} = \frac{\alpha^{1,\,1}\wedge\omega_{n-1}}{\omega_n}, \tag{3.77}$$

under those assumptions, where $\mathcal{H}^{1,\,1}_{\Delta_\tau}(X,\,\mathbb{C})$ is the space of Δ_τ-harmonic L^2_ω-forms of type $(1,\,1)$.

Proof of Theorem 3.1.6. The pair $(\widetilde{X},\,\widetilde\omega)$ satisfies the hypotheses of Corollary 3.3.11 (playing the role of the pair $(X,\,\omega)$ therein). By the balanced hyperbolic hypothesis on $(X,\,\omega)$, there exists an $\widetilde\omega$-bounded smooth $(2n-3)$-form $\widetilde\Gamma$ on $\widetilde{X}$ such that $\widetilde\omega_{n-1} = d\widetilde\Gamma$.

Let $\alpha^{1,\,1} \in \mathcal{H}^{1,\,1}_{\Delta_{\tilde\tau}}(\widetilde{X},\,\mathbb{C})$ such that $\alpha^{1,\,1} \geq 0$. Then, $\bar\partial\alpha^{1,\,1} = 0$ (by (iii) of (3.76) and real, hence we also have $\partial\alpha^{1,\,1} = 0$. Thus, $\alpha^{1,\,1}$ is d-closed, so

$$\widetilde\omega_{n-1}\wedge\alpha^{1,\,1} = d(\widetilde\Gamma\wedge\alpha^{1,\,1}) \in \operatorname{Im} d \tag{3.78}$$

because $\widetilde\Gamma\wedge\alpha^{1,\,1}$ is $L^2_{\widetilde\omega}$ and $d(\widetilde\Gamma\wedge\alpha^{1,\,1})$ is again $L^2_{\widetilde\omega}$.

On the other hand,

$$\widetilde\omega_{n-1}\wedge\alpha^{1,\,1} = (\Lambda_{\widetilde\omega}\alpha^{1,\,1})\,\widetilde\omega_n \in \mathcal{H}^{2n}_{\Delta_{\widetilde\omega}}(\widetilde{X},\,\mathbb{C}) \tag{3.79}$$

because $\Lambda_{\widetilde\omega}\alpha^{1,\,1}$ is constant by Corollary 3.3.11.

Since the subspaces $\mathcal{H}^{2n}_{\Delta_{\widetilde\omega}}(\widetilde{X},\,\mathbb{C})$ and $\operatorname{Im} d$ of the space of $L^2_{\widetilde\omega}$-forms of degree $2n$ on $\widetilde{X}$ are orthogonal (see (d) of Theorem 3.3.3), we deduce that $\widetilde\omega_{n-1}\wedge\alpha^{1,\,1} = 0$. Equivalently, $\Lambda_{\widetilde\omega}\alpha^{1,\,1} = 0$. This implies that $\alpha^{1,\,1} = 0$ since $\alpha^{1,\,1} \geq 0$ by hypothesis. $\square$

Chapter 4

Appendix

A key classical fact used by Gromov in [Gro91] is that some of the Lefschetz maps at the level of differential forms are *quasi-isometries* w.r.t. the L^2-inner product. We spell out the equalities involving *pointwise* inner products that lead to more precise statements that were used in earlier parts of our text.

Let ω be an arbitrary Hermitian metric on an arbitrary complex manifold X with $\dim_{\mathbb{C}} X = n$. As usual, for any $r = 1, \dots, n$, we put $\omega_r := \omega^r/r!$. Recall the following standard fact.

For every $k \leq n$ and every $r \leq n-k$, the pointwise Lefschetz operator:

$$L^r_\omega : \Lambda^k T^\star X \longrightarrow \Lambda^{k+2r} T^\star X, \quad L^r_\omega(\varphi) = \omega^r \wedge \varphi, \tag{4.1}$$

is **injective**. *When $r = n-k$, L^{n-k}_ω is even* **bijective**.

We will compare the pointwise inner products $\langle \omega_r \wedge \varphi_1, \, \omega_r \wedge \varphi_2 \rangle_\omega$ and $\langle \varphi_1, \, \varphi_2 \rangle_\omega$ for arbitrary k-forms $\varphi_1, \varphi_2 \in \Lambda^k T^\star X$. We will use the following standard formula (cf. e.g. [Voi02]):

$$[L^r_\omega, \, \Lambda_\omega] = r(k - n + r - 1) \, L^{r-1}_\omega \qquad \text{on } k\text{-forms}, \tag{4.2}$$

for any integer $r \geq 1$, where $\Lambda = \Lambda_\omega = (\omega \wedge \cdot)^\star$ is the adjoint of the Lefschetz operator L_ω w.r.t. the pointwise inner product $\langle \ , \ \rangle_\omega$ induced by ω.

(1) Case of primitive forms

Recall that for any non-negative integer $k \leq n$, a k-form φ is said to be *primitive w.r.t.* ω (or ω-*primitive*, or simply *primitive* when no confusion is likely) if it satisfies any of the following equivalent two conditions:

$$\omega_{n-k+1} \wedge \varphi = 0 \iff \Lambda_\omega \varphi = 0.$$

Lemma 4.0.12. *For every $k \leq n$, every $r \leq n-k$ and any k-forms φ_1, φ_2 one of which is* ω-**primitive**, *the following identity holds:*

$$\langle \omega^r \wedge \varphi_1, \, \omega^r \wedge \varphi_2 \rangle_\omega = (r!)^2 \binom{n-k}{r} \langle \varphi_1, \, \varphi_2 \rangle_\omega. \tag{4.3}$$

In particular, the analogous equality holds for the L^2_ω-inner product $\langle\langle \ , \ \rangle\rangle_\omega$.

Proof. To make a choice, let us suppose that φ_1 is primitive. We get:

$$\begin{aligned}
\langle\omega^r\wedge\varphi_1,\,\omega^r\wedge\varphi_2\rangle_\omega &= \langle\Lambda_\omega(\omega^r\wedge\varphi_1),\,\omega^{r-1}\wedge\varphi_2\rangle_\omega \overset{(i)}{=} \langle[\Lambda_\omega,\,L^r_\omega]\,\varphi_1,\,\omega^{r-1}\wedge\varphi_2\rangle_\omega\\
&\overset{(ii)}{=} r(n-k-r+1)\,\langle\omega^{r-1}\wedge\varphi_1,\,\omega^{r-1}\wedge\varphi_2\rangle_\omega\\
&\vdots\\
&= r(r-1)\dots1\,(n-k-r+1)\,(n-k-r+2)\dots(n-k)\,\langle\varphi_1,\,\varphi_2\rangle_\omega\\
&= r!\,\frac{(n-k)!}{(n-k-r)!}\,\langle\varphi_1,\,\varphi_2\rangle_\omega,
\end{aligned}$$

where (i) follows from φ_1 being primitive, (ii) follows from the standard formula (4.2), the remaining equalities except for the last one follow from analogues of (i) and (ii), while the last equality proves (4.3). $\square$

Let us also notice that, when the powers of ω are distinct, the products involved in the analogue of (4.3) are actually orthogonal to each other.

Lemma 4.0.13. *Let $r,s,k\in\mathbb{N}$ with $s>0$ and $k\leq n$. For any $(k-2s)$-form u and any ω-**primitive** k-form v, the following identity holds:*

$$\langle\omega^{r+s}\wedge u,\,\omega^r\wedge v\rangle_\omega = 0. \tag{4.4}$$

In particular, the analogous equality holds for the L^2_ω-inner product $\langle\langle\ ,\ \rangle\rangle_\omega$.

Proof. We have:

$$\begin{aligned}
\langle\omega^{r+s}\wedge u,\,\omega^r\wedge v\rangle_\omega &= \langle\omega^{r+s-1}\wedge u,\,\Lambda_\omega(\omega^r\wedge v)\rangle_\omega \overset{(i)}{=} \langle\omega^{r+s-1}\wedge u,\,[\Lambda_\omega,\,L^r_\omega]\,v\rangle_\omega\\
&\overset{(ii)}{=} c_1\,\langle\omega^{r+s-1}\wedge u,\,\omega^{r-1}\wedge v\rangle_\omega = \cdots = c_1\dots c_r\,\langle\omega^s\wedge u,\,v\rangle_\omega\\
&= c_1\dots c_r\,\langle\omega^{s-1}\wedge u,\,\Lambda_\omega v\rangle_\omega = 0,
\end{aligned}$$

where (i) follows from v being primitive, (ii) follows from the standard formula (4.2) with the appropriate constant c_1 (whose actual value is irrelevant here), the remaining equalities except for the last one follow from analogues of (i) and (ii) with the appropriate constants $c_2,\dots,c_r$, while the last equality follows again from v being primitive and proves (4.4). $\square$

(2) Case of arbitrary forms

Let φ_1, φ_2 be arbitrary k-forms and let

$$\varphi_1 = \varphi_{1,\,prim} + \omega\wedge\varphi_{1,\,1} + \cdots + \omega^l\wedge\varphi_{1,\,l} \quad\text{and}\quad \varphi_2 = \varphi_{2,\,prim} + \omega\wedge\varphi_{2,\,1} + \cdots + \omega^l\wedge\varphi_{2,\,l} \tag{4.5}$$

be their respective *Lefschetz decompositions*, where l is the non-negative integer defined by requiring $2l=k$ if k is even and $2l=k-1$ if k is odd, while the forms $\varphi_{j,\,prim},\varphi_{j,\,1},\dots,\varphi_{j,\,l}$ are *primitive* of respective degrees $k,k-2,\dots,k-2l$ for every $j\in\{1,2\}$.

The sense in which the Lefschetz operator (4.1) is a *quasi-isometry* for the pointwise inner product (hence also the L^2-inner product) induced by ω is made explicit in the following

Lemma 4.0.14. *Fix integers $0 \leq k \leq n$, $0 \leq r \leq n-k$ and arbitrary k-forms φ_1, φ_2.*

(i) The following identity holds:

$$\langle \omega^r \wedge \varphi_1,\, \omega^r \wedge \varphi_2 \rangle_\omega = (r!)^2 \binom{n-k}{r} \langle \varphi_{1,\,prim},\, \varphi_{2,\,prim} \rangle_\omega + ((r+1)!)^2 \binom{n-k+2}{r+1} \langle \varphi_{1,\,1},\, \varphi_{2,\,1} \rangle_\omega + ((r+l)!)^2 \binom{n-k+2l}{r+l} \langle \varphi_{1,\,l},\, \varphi_{2,\,l} \rangle_\omega \tag{4.6}$$

(ii) Putting $C_{n,\,k,\,r,\,s} := ((r+s)!(n-k+s)!)/(s!(n-k-r+s)!)$ and

$$A_{n,\,k,\,r} := \min_{s=0,\dots,l} C_{n,\,k,\,r,\,s}, \qquad B_{n,\,k,\,r} := \max_{s=0,\dots,l} C_{n,\,k,\,r,\,s},$$

the following inequalities hold:

$$A_{n,\,k,\,r}\, |\varphi|^2_\omega \leq |\omega^r \wedge \varphi|^2_\omega \leq B_{n,\,k,\,r}\, |\varphi|^2_\omega. \tag{4.7}$$

(iii) With the notation of (ii), if $\langle \varphi_{1,\,s},\, \varphi_{2,\,s} \rangle_\omega \geq 0$ for every $s \in \{0, 1, \dots, l\}$, the following inequalities hold:

$$A_{n,\,k,\,r}\, \langle \varphi_1,\, \varphi_2 \rangle_\omega \leq \langle \omega^r \wedge \varphi_1,\, \omega^r \wedge \varphi_2 \rangle_\omega \leq B_{n,\,k,\,r}\, \langle \varphi_1,\, \varphi_2 \rangle_\omega. \tag{4.8}$$

Proof. (i) Using the Lefschetz decompositions (4.5) and Lemma 4.0.13, we get:

$$\langle \omega^r \wedge \varphi_1,\, \omega^r \wedge \varphi_2 \rangle_\omega = \langle \omega^r \wedge \varphi_{1,\,prim},\, \omega^r \wedge \varphi_{2,\,prim} \rangle_\omega + \sum_{s=1}^{l} \langle \omega^{r+s} \wedge \varphi_{1,\,s},\, \omega^{r+s} \wedge \varphi_{2,\,s} \rangle_\omega.$$

Identity (4.6) follows from this and from Lemma 4.0.12.

(ii) and (iii) follow at once from (i) applied twice, with a given $1 \leq r \leq n-k$ and with $r = 0$. $\square$

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