Notions d'hyperbolicité en géométrie complexe et déformations holomorphes

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En guise de préambule, je me permets de reprendre un théorème énoncé par mon cher collègue de bureau Louis dans sa thèse, dont l'esprit trouve naturellement sa place dans le présent travail :

Théorème 0.0.1. *Soit M le présent manuscrit et E l'entourage de son auteur. Alors M est obtenu comme un barycentre des points de E.*

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Abstract

This thesis is devoted to the study of notions of hyperbolicity associated with special metrics and structures, as well as to the analysis of their stability properties under holomorphic deformations.

The first part of this thesis focuses on the study of deformations of Gromov's Kähler hyperbolic structures on compact complex manifolds. Several modified notions of Kähler hyperbolicity are introduced, providing a first step toward the investigation of the openness of Kähler hyperbolicity under holomorphic deformations. In particular, we define the notions of Strong Kähler hyperbolicity, as well as L^p-Kähler hyperbolicity and L^p-SKT hyperbolicity, and establish several deformation openness results for these properties in holomorphic families of compact complex manifolds.

The second part of this thesis is set in the non-Kähler framework and concerns the study of deformation properties of various notions of hyperbolicity associated with balanced metrics and their generalizations. In particular, under certain assumptions, we analyze the stability under modifications and holomorphic deformations of degenerate balanced structures, as well as the deformations of balanced hyperbolicity. Moreover, we introduce new notions of hyperbolicity in intermediate degrees, such as p-SKT hyperbolicity and p-Hermitian-symplectic hyperbolicity, which generalize existing concepts in Hermitian geometry. Connections between these analytic notions and geometric forms of hyperbolicity, notably Brody hyperbolicity and $2p$-cyclic hyperbolicity, are established, and several deformation openness results are proved for these properties.

Keywords. Kähler hyperbolicity, SKT hyperbolicity, L^2-Kähler hyperbolicity, Balanced hyperbolicity, Special metrics and structures, Deformations of complex structures.

Résumé

Cette thèse est consacrée à l'étude des notions d'hyperbolicité associées à des métriques et à des structures spéciales, ainsi qu'à l'étude de leurs propriétés de stabilité sous déformations holomorphes.

La première partie de cette thèse est consacrée à l'étude des déformations de structures Kähler hyperboliques au sens de Gromov sur les variétés complexes compactes lisses. On y introduit plusieurs notions modifiées de Kähler hyperbolicité, visant à franchir un premier pas vers l'étude de l'ouverture par déformations holomorphes de la Kähler hyperbolicité. En particulier, on définit les notions de Kähler hyperbolicité forte ainsi que de L^p-Kähler hyperbolicité et de L^p-SKT hyperbolicité, et l'on établit divers résultats d'ouverture par déformations pour ces propriétés dans des familles holomorphes de variétés complexes compactes lisses.

La seconde partie de cette thèse s'inscrit dans le cadre non-kählérien et porte sur l'étude des propriétés de déformation de diverses notions d'hyperbolicité associées aux métriques équilibrées et à leurs généralisations. On y analyse en particulier, sous certaines hypothèses, la stabilité par modifications et par déformations holomorphes des structures équilibrées dégénérées, ainsi que les déformations de l'hyperbolicité équilibrée. Par ailleurs, on introduit de nouvelles notions d'hyperbolicité en degrés intermédiaires, telles que l'hyperbolicité p-SKT et l'hyperbolicité p-hermitienne-symplectique, qui proviennent de concepts existants en géométrie hermitienne. On établit des liens entre ces notions analytiques et des formes géométriques d'hyperbolicité, notamment l'hyperbolicité à la Brody et l'hyperbolicité cyclique en degré $2p$, et l'on démontre plusieurs résultats d'ouverture par déformations pour ces propriétés.

Mots clés. Kähler hyperbolicité, Hyperbolicité SKT, L^2-Kähler hyperbolicité, Hyperbolicité équilibrée, Métriques et structures spéciales, Déformations de structures complexes.

Contents

Introduction générale

L'étude des différentes notions d'hyperbolicité en géométrie complexe occupe une place centrale à l'interface entre la géométrie différentielle et la géométrie algébrique. Depuis les travaux fondateurs de Gromov sur la Kähler hyperbolicité, cette thématique s'est révélée particulièrement féconde. D'un point de vue conceptuel, la classe des variétés Kähler hyperboliques compactes constitue une sous-classe particulièrement riche et structurée des variétés Kobayashi hyperboliques. Elle fournit un cadre géométrique naturel dans lequel plusieurs propriétés encore conjecturées dans le contexte général de l'hyperbolicité au sens de Kobayashi peuvent être établies, ou du moins étudiées de manière plus précise, en mettant en évidence le rôle des structures métriques et cohomologiques sous-jacentes.

Dans le cadre kählérien, l'hyperbolicité au sens de Gromov repose sur l'existence de métriques kählériennes dont le tiré en arrière au revêtement universel est d-exact avec un potentiel borné. Cette propriété, à la fois géométrique et analytique, implique de fortes contraintes sur la topologie de la variété sous-jacente.

De nombreuses variétés complexes lisses naturelles n'admettent pas de métriques kählériennes, tout en présentant des propriétés d'hyperbolicité au sens large. Cela a conduit au développement de généralisations non-kählériennes de l'hyperbolicité de Gromov, basées sur des métriques hermitiennes plus générales, telles que les métriques équilibrées, SKT ou de Gauduchon.

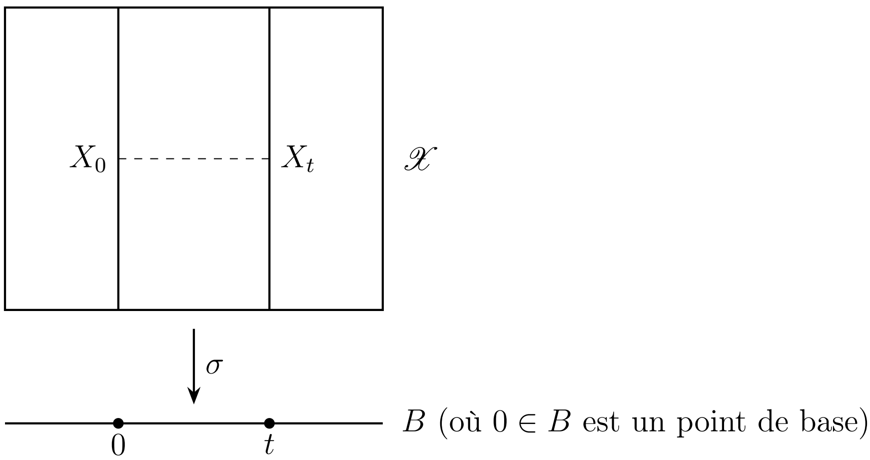
Nous présentons dans cette section l'idée directrice de la thèse, qui se déploie en deux parties principales :

1. L'étude de l'ouverture par déformations de la Kähler hyperbolicité sur les variétés complexes compactes lisses (au sens de Gromov).
2. L'étude de l'ouverture par déformations de notions d'hyperbolicité associées à des métriques/structures complexes non-kählériennes.

La considération des déformations holomorphes des propriétés d'hyperbolicité s'inscrit dans une problématique générale de rigidité de telles structures géométriques complexes, et la théorie des déformations fournit un outil central dans la classification des variétés complexes compactes lisses. Dans ce contexte, nous rappelons brièvement les principaux éléments de la théorie des déformations des structures complexes de Kodaira-Spencer, développée dans leur célèbre série d'articles [KS58a, KS58b, KS60], et qui fournissent le langage et les outils nécessaires à l'étude de la stabilité des propriétés d'hyperbolicité. Une présentation plus détaillée du matériel que nous utiliserons ici se trouve au chapitre 2 de [Pop22].

Une **famille holomorphe** de variétés complexes compactes lisses $\{X_t : t \in B\}$ est la donnée d'une submersion holomorphe propre $\sigma : \mathscr{X} \longrightarrow B$ entre deux variétés complexes lisses $\mathscr{X}$ et B, où $X_t := \sigma^{-1}(t)$ est appelée la **fibre au-dessus** de $t \in B$. Dans ce contexte, on appelle $\mathscr{X}$ **l'espace total** et B **la base** de la famille.

La compacité des variétés $(X_t)_{t\in B}$ est due au fait que σ soit propre. Et comme σ est une submersion, la famille $\{X_t : t \in B\}$ est **équidimensionnelle**, c'est-à-dire que les variétés $(X_t)_{t\in B}$ ont la même dimension, qu'on note par $n = \dim_{\mathbb{C}} X_t$, $\forall t \in B$.



En général, la base B est supposée être une boule ouverte $\mathbb{B}_\varepsilon(0)$ centrée à l'origine et de rayon $\varepsilon > 0$ dans un certain $\mathbb{C}^m$, où $m = \dim_{\mathbb{C}} \mathscr{X} - n$. Si l'on fixe une fibre X_0 au-dessus de l'origine, les fibres voisines X_t sont considérées comme de *petites déformations* de X_0, où $|t| < \varepsilon$ pour $\varepsilon > 0$ est suffisamment petit, et la famille $(J_t)_{t\in\mathbb{B}_\varepsilon(0)}$ de structures complexes sur les fibres $(X_t)_{t\in\mathbb{B}_\varepsilon(0)}$ varie holomorphiquement en fonction de $t \in \mathbb{B}_\varepsilon(0)$.

Comme la base $B = \mathbb{B}_\varepsilon(0)$ est contractile, toutes les fibres $(X_t)_{t\in\mathbb{B}_\varepsilon(0)}$ sont difféomorphes grâce au théorème d'Ehresmann [Ehr47], à une variété différentiable X fixée. En particulier, la cohomologie de De Rham est constante sur la famille $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$, c'est-à-dire que : $H^k_{DR}(X_t, \mathbb{C}) = H^k_{DR}(X, \mathbb{C})$, pour tout $t \in \mathbb{B}_\varepsilon(0)$.

Dans ce cadre, lorsque l'on considère une propriété (P) dépendant de la structure complexe d'une variété complexe compacte, deux questions se posent naturellement :

1. Si la fibre *centrale* X_0 satisfait la propriété (P), les fibres voisines X_t la satisfont-elles également pour $|t| < \varepsilon$, avec $\varepsilon > 0$ suffisamment petit ?

   Lorsque la réponse est affirmative, on dira que la propriété (P) est **ouverte par déformations**.

2. Si toutes les fibres voisines X_t satisfont la propriété (P), avec $t \neq 0$, la fibre *limite* X_0 la satisfait-elle aussi ?

   Si c'est le cas, on dira que la propriété (P) est **fermée par déformations**.

La première notion d'hyperbolicité dont le comportement/ouverture par déformations a été étudié est la Kobayashi hyperbolicité, laquelle est équivalente, pour les variétés complexes compactes lisses, à la *Brody hyperbolicité*, c'est-à-dire à la non-existence de courbes entières non constantes sur la variété considérée (voir [Bro78], Lemme 2.1 et Théorème 3.1).

À ma connaissance, la question de l'ouverture de la Kähler hyperbolicité au sens de Gromov par déformations de structures complexes n'a pas encore été abordée dans la littérature ; elle a été formulée comme conjecture par mon directeur de thèse et constitue donc le thème central des travaux présentés dans cette thèse.

0.1 Déformations des variétés Kähler hyperboliques

L'un des résultats fondamentaux de la théorie de la déformation des structures complexes est le caractère ouvert de la propriété de Kähler, établi par Kodaira et Spencer dans leur article fondateur [KS60]. Plus précisément, soit $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ une famille holomorphe de variétés complexes compactes lisses de dimension complexe n, paramétrée par une boule ouverte $\mathbb{B}_\varepsilon(0) \subset \mathbb{C}^m$ de rayon $\varepsilon > 0$ et de centre l'origine 0. Si la fibre centrale X_0 est kählérienne, alors toutes les fibres voisines X_t sont également kählériennes (pour ε suffisamment petit).

L'idée de la démonstration repose sur l'étude de familles d'opérateurs elliptiques associés aux structures complexes et peut être décomposée en trois étapes principales :

Étape 1. Soit $\{\omega_t\}_{t\in\mathbb{B}_\varepsilon(0)}$ une famille $\mathscr{C}^\infty$ de métriques hermitiennes sur les fibres X_t, vue comme une famille $\mathscr{C}^\infty$ de formes différentielles réelles *positives* de bidegré $(1,1)$. On considère le **laplacien de Bott-Chern** $\Delta_{BC,t}$ associé à la métrique ω_t sur X_t. $\{\Delta_{BC,t}\}_{t\in\mathbb{B}_\varepsilon(0)}$ est une famille $\mathscr{C}^\infty$ d'opérateurs elliptiques et agissent sur les espaces de formes différentielles complexes lisses de type (p,q) sur X_t, pour tout $p,q = 0,\ldots,n$. Si la fibre centrale (X_0,ω_0) est kählérienne, alors pour t suffisamment proche de 0, la dimension des espaces **Bott-Chern harmoniques** $\mathcal{H}^{p,q}_{BC,t}(X,\mathbb{C}) := \operatorname{Ker}\Delta_{BC,t} \cap \mathscr{C}^\infty_{p,q}(X_t,\mathbb{C})$ est constante. Il s'ensuit que la famille des projections orthogonaux pour le produit scalaire $L^2_{\omega_t}$ induites par la famille de métriques hermitiennes $\{\omega_t\}_{t\in\mathbb{B}_\varepsilon(0)}$:

$$F^{p,q}_{BC,t} \colon \mathscr{C}^\infty_{p,q}(X_t,\mathbb{C}) \longrightarrow \mathcal{H}^{p,q}_{BC,t}(X_t,\mathbb{C})$$

varie de manière lisse avec $t \in \mathbb{B}_\varepsilon(0)$.

Étape 2. Une métrique hermitienne ω sur une variété complexe compacte est kählérienne si et seulement si elle est harmonique pour le laplacien de Bott-Chern associé à ω, c'est-à-dire que $\Delta_{BC}\omega = 0$. Cette caractérisation relie la condition différentielle $d\omega = 0$ à la théorie de Hodge pour la cohomologie de Bott-Chern.

Étape 3. Pour t suffisamment proche de 0, la forme :

$$\widetilde{\omega}_t := \frac{1}{2}\big(F^{1,1}_{BC,t}\omega_t + \overline{F^{1,1}_{BC,t}(\omega_t)}\big)$$

définit une métrique kählerienne sur X_t. De plus, la famille de métriques kählériennes $\{\widetilde{\omega}_t\}_{t\in\mathbb{B}_\varepsilon(0)}$ varie de manière $\mathscr{C}^\infty$ en t, avec $\widetilde{\omega}_0 = \omega_0$.

Ces considérations soulèvent naturellement la question de la stabilité par déformations holomorphes de la Kähler hyperbolicité introduite par Gromov.

Rappelons qu'une variété kählérienne compacte lisse X est dite **Kähler hyperbolique** s'il existe une métrique kählérienne ω qui soit $\widetilde{d}$(*bornée*), c'est-à-dire que son tiré en arrière $\widetilde{\omega}$ au revêtement universel $\widetilde{X}$ admette un potentiel borné. Plus précisément $\widetilde{\omega} := \pi^*\omega = d\eta$, où $\pi : \widetilde{X} \longrightarrow X$ est le revêtement universel de X, et η est une 1-forme bornée sur $\widetilde{X}$ pour la métrique $\widetilde{\omega}$.

Cette condition est substantiellement forte, car elle impose en particulier de fortes contraintes sur la géométrie à grande échelle du revêtement universel et entraîne des conséquences topologiques et analytiques majeures, telles que l'amplitude du fibré canonique, l'hyperbolicité au sens de Kobayashi ou encore l'existence d'un trou spectral pour le laplacien de Hodge-De Rham.

L'ouverture par déformations de la Kähler hyperbolicité reste en général un problème ouvert. Une difficulté fondamentale réside dans le caractère global de la définition, qui fait intervenir le revêtement universel et un contrôle uniforme des potentiels sur celui-ci. Afin de contourner cet obstacle, nous introduisant dans [Khe25] une famille de notions intermédiaires, obtenues en affaiblissant soit la condition d'exactitude de $\widetilde{\omega}$, soit le caractère borné uniforme du potentiel.

Soit donc $1 \leq p \leq +\infty$. Nous dirons qu'une variété kählérienne compacte lisse X est

L^p**-Kähler hyperbolique** s'il existe une métrique kählerienne ω telle que $\widetilde{\omega} = d\eta$, où η est une 1-forme définie sur $\widetilde{X}$ satisfaisant la condition de *borne uniforme sur sa norme* L^p suivante : pour un (et donc pour tout) *domaine fondamental* $D \Subset \widetilde{X}$, il existe une constante $C > 0$ telle que :

$$\sup_{\gamma \in \pi_1(X)} \|\eta\|_{L^p(\gamma D)} \leq C.$$

Le cas $p = +\infty$ coïncide exactement avec la notion classique de Kähler hyperbolicité, tandis que les valeurs finies de p permettent d'admettre des potentiels non bornés, mais dont la croissance de l'intégrale d'une puissance est contrôlée. Cette flexibilité supplémentaire s'avère cruciale dans l'étude des déformations.

Dans le même esprit, nous introduisons une version plurifermée de cette notion. Une variété complexe compacte lisse X est dite L^p**-SKT hyperbolique** si elle admet une métrique hermitienne ω *plurifermée* (appelée aussi *SKT*); c'est-à-dire telle que $\partial\bar{\partial}\omega = 0$, dont le tiré en arrière $\widetilde{\omega}$ au revêtement universel s'écrit sous la forme (la condition d'exactitude au sens de la cohomologie d'Aeppli):

$$\widetilde{\omega} = \partial\alpha + \bar{\partial}\beta,$$

où α et β sont des formes de types $(0,1)$ et $(1,0)$ respectivement sur le revêtement universel $\widetilde{X}$, satisfaisant la même condition de borne uniforme sur les normes L^p sur les translatés des domaines fondamentaux. Cette notion généralise celle de **métrique SKT hyperbolique** au sens défini par S. Marouani dans [Mar23].

Le premier résultat principal de cette thèse met en évidence le bon comportement de ces notions affaiblies dans le cadre de la théorie des déformations. Plus précisément :

Théorème 0.1.1. *Soit* $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ *une famille holomorphe de variétés complexes compactes lisses. Si la fibre centrale* X_0 *est* L^p*-Kähler hyperbolique pour un certain* $1 \leq p \leq +\infty$*, alors, pour* t *suffisamment proche de* 0*, les fibres* X_t *sont* L^p*-SKT hyperboliques.*

L'idée de la preuve repose sur l'analyse du laplacien de Bott-Chern dans les familles différentiablement triviales et sur la description des métriques kähleriennes comme formes harmoniques. En projetant convenablement les métriques sur les espaces harmoniques associés aux structures complexes déformées, on obtient une famille $\mathscr{C}^\infty$ de métriques kählériennes dont les $(\partial + \bar{\partial})$-potentiels sur le revêtement universel héritent de contrôles L^p uniformes.

Un cas particulier où nous obtenons l'ouverture par déformations de la Kähler hyperbolicité est le suivant :

Théorème 0.1.2. *Soit* $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ *une famille holomorphe de variétés complexes* *compactes lisses telle que X_0 soit Kähler hyperbolique. Si de plus $h^{2,0}_{BC}(X_0) = 0$, la Kähler hyperbolicité est une propriété ouverte sous déformations holomorphes, c'est-à-dire que les fibres voisines X_t sont également Kähler hyperboliques, pour ε suffisamment petit.*

La preuve consiste à exploiter l'ouverture du cône kählérien dans $H^2_{DR}(X, \mathbb{R})$ (ce qui est assuré par l'annulation du nombre de Hodge $h^{2,0}_{BC}(X_0)$) afin de montrer que la classe de kähler reste $\widetilde{d}$-bornée sous petites déformations, le caractère $\widetilde{d}$-borné étant une propriété cohomologique (et qui ne dépend que du groupe fondamental $\pi_1(X)$ pour les formes différentielles de degré 2).

Dans un second temps, nous étudions la possibilité de renforcer ce résultat afin de retrouver une hyperbolicité kählérienne. Nous montrons donc le résultat suivant :

Théorème 0.1.3. *Si X_0 est une variété Kähler hyperbolique, alors toute petite déformation X_t admet une métrique kählérienne ω_t qui est L^2-Kähler hyperbolique presque partout.*

Ce résultat s'appuie sur l'existence d'un trou spectral uniforme pour le laplacien de Hodge-De Rham sur le revêtement universel, initialement démontré par Gromov, et sur des arguments de résolution d'un système d'équations de type $\bar{\partial}$ avec estimées L^2 (qui a été étudié dans [DP21], Lemme 2.4 dans le cas hermitien compact) sur le revêtement universel (qui est complet, mais **non-compact**).

Enfin, nous introduisons une notion plus rigide, que nous appelons *Kähler hyperbolicité forte*. Une variété kählérienne compacte X est dite **fortement Kähler hyperbolique** si elle admet une métrique kählérienne ω dont le tiré en arrière $\widetilde{\omega}$ au revêtement universel $\widetilde{X}$ est exacte et si son groupe fondamental $\pi_1(X)$ est hyperbolique au sens de Gromov. B-L. Chen et X. Yang ont prouvé dans [CY21], Corollaire 1.7 que si une variété kählérienne compacte lisse est fortement Kähler hyperbolique, alors elle est Kähler hyperbolique (au sens de Gromov).

Théorème 0.1.4. *La propriété de Kähler hyperbolicité forte est ouverte sous déformations holomorphes.*

Ce dernier résultat combine le caractère topologique de l'hyperbolicité du groupe fondamental avec la stabilité analytique dans les familles holomorphes différentiablement triviales fournie par le théorème précédent.

0.2 Déformations d'autres propriétés d'hyperbolicité

L'article [Khe26] s'inscrit dans le programme général visant à étendre l'étude de la stabilité par déformation des notions d'hyperbolicité analytique au-delà du cadre kählérien.

Nous cherchons en particulier à comprendre dans quelle mesure les phénomènes de stabilité de l'hyperbolicité, analysés dans le cas kählérien dans [Khe25], persistent ou se transforment dans des contextes non-kählériens.

Une première classe fondamentale étudiée est celle des **variétés équilibrées hyperboliques**, caractérisées par l'existence d'une métrique Hermitienne ω telle que la forme ω^{n-1} soit d-fermée et $\widetilde{d}$-bornée. Cette notion a été introduite par S. Marouani et D. Popovici dans [MP23]. Inspirée par la définition de l'hyperbolicité kählérienne, cette condition constitue une généralisation stricte de cette notion au bidegré $(n-1, n-1)$: elle autorise l'apparition d'exemples proprement non-kählériens tout en conservant un contrôle analytique suffisant. Parmi ces variétés, les **variétés équilibrées dégénérées** occupent une place privilégiée ; elles sont caractérisées par l'exactitude de ω^{n-1}, ce qui exclut toute structure kählérienne tout en fournissant un cadre naturel pour l'hyperbolicité équilibrée.

Un outil fondamental pour l'analyse de ces variétés est leur caractérisation en termes de courants : une variété compacte complexe est équilibrée dégénérée si et seulement si elle n'admet aucun $(1,1)$-courant positif non nul qui soit d-fermé ([Pop15], Proposition 5.4). Cette équivalence établit un lien profond entre géométrie métrique et théorie des courants, et permet d'utiliser des résultats classiques de support pour les courants positifs afin d'étudier la stabilité des structures équilibrées dégénérées sous diverses opérations géométriques.

En particulier, l'article met en évidence le fait que, si la propriété d'être équilibrée dégénérée n'est pas stable par éclatement, elle est néanmoins stable par descente le long des modifications lisses surjectives. Plus précisément :

Théorème 0.2.1. *Soit $f : \widehat{X} \to X$ une modification lisse surjective entre variétés complexes compactes lisses de dimension $n \geq 3$. Si $\widehat{X}$ est équilibrée dégénérée, alors X est également équilibrée dégénérée.*

Ce résultat repose de manière cruciale sur l'étude du support des courants positifs et sur l'observation que l'existence d'un courant positif d-fermé de bidegré $(1,1)$ sur X se relèverait en un courant positif non nul sur $\widehat{X}$, contredisant la dégénérescence de cette dernière.

Un second axe majeur de l'article est consacré à l'étude de la stabilité de ces notions d'hyperbolicité sous déformations holomorphes. Il est bien connu que la propriété d'être équilibrée n'est, en général, pas stable par déformation, comme l'illustrent les exemples classiques de Nakamura, obtenus à partir de petites déformations de la variété d'Iwasawa. Afin de surmonter cette difficulté, l'article mobilise des outils cohomologiques fins, en particulier les cohomologies de Bott-Chern et d'Aeppli, qui fournissent un cadre naturel pour l'analyse des variations de structures hermitiennes au sein de familles analytiques.

Le rôle central est joué par différentes variantes de la propriété $\partial\bar{\partial}$, formulées en bidegrés spécifiques. Sur une variété complexe compacte lisse X, les propriétés dites :

1. *faible* (p,q)-$\partial\bar{\partial}$: Pour toute forme différentielle $\alpha \in \mathscr{C}^{\infty}_{p,q}(X)$, on a :

$$\alpha \in \operatorname{Im}\partial \cap \operatorname{Im}\bar{\partial} \Longrightarrow \bar{\partial}\alpha \in \operatorname{Im}\partial\bar{\partial}$$

2. *modérée* (p,q)-$\partial\bar{\partial}$: Pour toute forme différentielle $\alpha \in \mathscr{C}^{\infty}_{p,q}(X)$, on a :

$$\alpha \in \operatorname{Ker}\bar{\partial} \cap \operatorname{Im}\partial \Longrightarrow \alpha \in \operatorname{Im}\partial\bar{\partial}$$

3. *forte* (p,q)-$\partial\bar{\partial}$: Pour toute forme différentielle $\alpha \in \mathscr{C}^{\infty}_{p,q}(X)$, on a :

$$\alpha \in \operatorname{Ker} d \cap (\operatorname{Im}\partial + \operatorname{Im}\bar{\partial}) \Longrightarrow \alpha \in \operatorname{Im}\partial\bar{\partial}$$

permettent de résoudre certaines équations de type $\bar{\partial}$ ou de type $\partial\bar{\partial}$, tout en restant compatibles avec des situations non-kählériennes. Ces hypothèses intermédiaires constituent l'un des outils techniques essentiels de l'article.

Rappelons qu'une métrique de Gauduchon ω sur une variété complexe compacte X de dimension $\dim_{\mathbb{C}} X = n$ (c'est-à-dire que $\partial\bar{\partial}\omega^{n-1} = 0$) est dite **Gauduchon hyperbolique** si $\widetilde{\omega}^{n-1} = \partial\alpha + \bar{\partial}\beta$, où α et β sont des formes bornées pour la métrique $\widetilde{\omega}$ et de types $(n-2, n-1)$ et $(n-1, n-2)$ respectivement sur le revêtement universel $\widetilde{X}$.

Dans ce cadre, il est montré que les structures équilibrées hyperboliques se déforment en structures équilibrées Gauduchon hyperboliques, et que la dégénérescence équilibrée est ouverte sous déformations holomorphes dès lors que les fibres voisines satisfont des hypothèses $\partial\bar{\partial}$ appropriées. Plus précisément :

Théorème 0.2.2. *Soit $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ une famille analytique de variétés compactes complexes de dimension n, telle que X_0 soit équilibrée. On suppose que les fibres X_t, pour $t \neq 0$, satisfont la propriété faible $(n-1, n)$-$\partial\bar{\partial}$.*

1. *Si X_0 est équilibrée hyperbolique, alors les fibres X_t admettent, pour t suffisamment petit, des métriques équilibrées qui sont Gauduchon hyperboliques.*
2. *Si, de plus, X_0 est équilibrée dégénérée et si les fibres X_t satisfont la propriété modérée $(n, n-2)$-$\partial\bar{\partial}$, alors les fibres X_t sont équilibrées dégénérées pour t suffisamment petit.*

Lorsque la famille considérée est constituée de $\partial\bar{\partial}$-variétés, ces résultats peuvent être renforcés de manière significative. Il est alors possible de construire des familles lisses de métriques équilibrées qui soient Gauduchon hyperboliques sur les fibres voisines, ayant comme membre initial une métrique équilibrée hyperbolique, et, dans le cas équilibré dégénéré, des familles lisses de d-potentiels pour la famille lisse de métriques équilibrées dégénérées sur $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$, ce qui fournit un contrôle particulièrement précis du comportement de l'hyperbolicité le long des déformations.

L'article introduit ensuite une nouvelle notion d'hyperbolicité de nature métrique, dite **Hermitienne-symplectique hyperbolicité** (ou plus simplement **HS hyperbolicité**). Cette notion repose sur l'existence d'une 2-forme réelle d-fermée Ω qui soit $\widetilde{d}$(bornée), dont la composante $(1,1)$ définit une métrique hermitienne positive, sans imposer que cette forme soit de type pur. Cette approche est motivée par des questions ouvertes concernant l'existence de variétés compactes hermitienne-symplectiques non-kählériennes, et permet d'explorer un territoire encore plus large de la géométrie complexe.

Enfin, on développe un cadre général d'hyperbolicité en degré intermédiaire, s'inspirant des notions de l'hyperbolicité partielle introduites par D. Popovici et H. Kasuya. Pour $1 \leq p \leq n-1$, sont introduites les notions de **p-SKT hyperbolicité** et de **p-HS hyperbolicité**, fondées sur l'existence de formes faiblement strictement positives de type (p,p) satisfaisant des propriétés de caractère borné adaptées. Ces notions généralisent de manière cohérente les cas de degrés extrêmes et s'insèrent dans une hiérarchie incluant la **p-Kähler hyperbolicité** définie par S. Marouani et F. Haggui dans [HM23].

Rappelons que la notion qui généralise la Brody hyperbolicité sur une variété complexe compacte lisse X en degré supérieur est la **p-hyperbolicité cyclique**, qui relève de la non-existence de courbes holomorphes $f : \mathbb{C}^p \longrightarrow X$ non-dégénérées en un point et satisfaisant une condition de *croissance sous-exponentielle* (voir Définition 3.3.20). On établit alors des liens précis entre ces notions analytiques et des notions géométriques d'hyperbolicité.

Théorème 0.2.3. *Soit X une variété compacte complexe lisse de dimension n et $1 \leq p \leq n-1$. Si X est p-HS hyperbolique ou p-SKT hyperbolique, alors X est p-cycliquement hyperbolique.*

Dans ce cadre, on s'intéresse au comportement de ces notions d'hyperbolicité sous déformations holomorphes de la structure complexe. Les résultats obtenus apportent une réponse particulièrement satisfaisante à ce problème. Un premier théorème établit que l'hyperbolicité p-HS est une propriété ouverte sous déformations holomorphes :

Théorème 0.2.4. *Si $\{X_t\}_{t\in B_\varepsilon(0)}$ est une famille holomorphe de variétés complexes compactes lisses et si la fibre centrale X_0 est p-HS hyperbolique, alors toutes les fibres X_t sont p-HS hyperboliques pour t suffisamment proche de zéro.*

Dans un second théorème, on montre que la p-Kähler hyperbolicité se déforme en des structures p-SKT hyperboliques le long de petites déformations de la structure complexe, sous l'hypothèse $\partial\bar{\partial}$:

Théorème 0.2.5. *Si $\{X_t\}_{t\in B_\varepsilon(0)}$ est une famille holomorphe de variétés complexes compactes lisses et si la fibre centrale X_0 est $\partial\bar{\partial}$ et p-Kähler hyperbolique, alors toutes les fibres X_t sont p-SKT hyperboliques pour t suffisamment proche de zéro.*

Ces théorèmes de stabilité constituent l'un des apports majeurs de cette étude. Leur démonstration repose sur des outils de théorie de Hodge spécifiquement adaptés au cadre non-kählérien et à l'étude des familles.

Chapter 1

Preliminaries

The goal of this chapter is to recall the material necessary for the exposition of this thesis. The primary sources for this chapter are [Dem97], [Kob05], and [Pop22]. Throughout this chapter, X will denote a compact complex manifold (unless stated otherwise), with $\dim_{\mathbb{C}} X = n$, and $\pi : \widetilde{X} \longrightarrow X$ its universal covering. For any differential form α of degree $0 \leq k \leq 2n$, we denote by $\widetilde{\alpha} := \pi^*\alpha$ its pullback to the universal cover. We also denote by $\mathbb{B}_\varepsilon(0) \subset \mathbb{C}^m$ the open ball of radius $\varepsilon > 0$ centered at the origin.

1.1 Hodge Theory

In this section, (M, g) will be a compact Riemannian manifold with $\dim_{\mathbb{R}} M = m$. Set $L^2(M, \Lambda^\bullet TM^*)$ to be the Hilbert space of differential forms with measurable coefficients endowed with following inner product:

$$\langle u, v \rangle := \int_M u(x) \cdot v(x) dV_g(x) \,, \quad \forall u, v \in L^2(M, \Lambda^\bullet TM^*)$$

where dV_g is the Riemannian volume form on M induced by g. Then one can show that there is a unique differential operator P^* satisfying:

$$\langle Pu, v \rangle = \langle u, P^*v \rangle \,, \quad \forall u \in \mathscr{C}^\infty_p(M) \,, \forall v \in \mathscr{C}^\infty_q(M)$$

called the **(formal) adjoint** of P. Assume now that M is oriented. Let $(\xi_1, ..., \xi_m)$ be an orthonormal frame of TM over an open subset $U \subset M$ and let $(\xi_1^*, ..., \xi_m^*)$ be the corresponding dual frame of TM^*. Set $\Lambda TM^* = \bigoplus_{0 \leq p \leq m} \Lambda^p TM^*$ as an orthogonal sum, then ΛTM^* has a natural inner product such that:

$$\langle (u_1, ..., u_p), (v_1, ..., v_p) \rangle = \det(u_j \cdot v_k)_{1 \leq j,k \leq p} \,, \quad \forall u_j, v_k \in TM^*, \forall 1 \leq p \leq m$$

Then the covectors $\xi_I = \xi^*_{i_1} \wedge \cdots \wedge \xi^*_{i_p}$ with $i_1 < \cdots < i_p$, form an orthonormal basis of ΛTM^*. The **Hodge $*_g$-operator** (induced by the Riemannian metric g) is the *unique* linear isomorphism $*_g : \Lambda TM^* \xrightarrow{\simeq} \Lambda TM^*$ which descends to an isomorphism $*_g : \Lambda^p TM^* \xrightarrow{\simeq} \Lambda^{m-p} TM^*$ such that:

$$u \wedge *_g v = \langle u, v \rangle \, dV_g \,, \quad \forall u, v \in \Lambda^p TM^*$$

Thus one can write the L^2-product on $L^2_g(M, \Lambda^p TM^*)$ as:

$$\langle\langle u, v \rangle\rangle = \int_M u \wedge *_g v \,, \qquad \forall u, v \in L^2_g(M, \Lambda^p TM^*)$$

It is clear that $*_g$ is an isometry and that $*_g *_g|_{\Lambda^p TM^*} = (-1)^{p(m-p)} \mathrm{Id}|_{\Lambda^p TM^*}$. We define the **co-differential operator** $d^* : \mathscr{C}^\infty_p(M) \longrightarrow \mathscr{C}^\infty_{p-1}(M)$ as: $d^* := (-1)^{m(p+1)+1} *_g d *_g$.

Remark 1.1.1. Strictly speaking, one should write d^*_g and $*_g$, as they depend on the choice of Riemannian metric g on M.

We define the **Hodge-De Rham Laplace operator** to be the (formally) *self-adjoint* operator given by the following expression: $\Delta := dd^* + d^*d$. Notice that:

$$\langle\langle \Delta u, u \rangle\rangle = \|d^* u\|^2 + \|du\|^2 \,, \quad \forall u \in \mathscr{C}^\infty_p(M) \tag{1.1.1}$$

This implies in particular that: $\Delta u = 0 \Longleftrightarrow d^* u = du = 0$. The principal symbol of Δ is given by: $u \longmapsto -\sum\limits_{|I|=p} (\sum\limits_{1 \leq j \leq n} \xi_j^2 u_I) \xi^*_I$. Hence Δ is elliptic. We define the space of **Harmonic p-forms** as : $\mathcal{H}^p_\Delta(M, \mathbb{R}) := \mathrm{Ker}(\Delta : \mathscr{C}^\infty_p(M) \longrightarrow \mathscr{C}^\infty_p(M))$.

Theorem 1.1.2 (The Hodge orthogonal decomposition Theorem)**.** *Let (M, g) be a compact oriented Riemannian manifold. Then $\dim_{\mathbb{R}} \mathcal{H}^p_\Delta(M, \mathbb{R}) < +\infty$ and we have the following orthogonal decomposition:*

$$\mathscr{C}^\infty_p(M) = \mathcal{H}^p_\Delta(M, \mathbb{R}) \oplus Imd \oplus Imd^* \tag{1.1.2}$$

In order to prove this, we need the following known result in Geometric Analysis:

Proposition 1.1.3. *Every elliptic differential operator P on (M, g) is* ***Fredholm****; i.e. ImP is closed and both $KerP$ and $coKerP := \mathscr{C}^\infty_\bullet(M)/ImP$ are finite dimensional vector spaces. Moreover, we have L^2_g-orthogonal decomposition : $\mathscr{C}^\infty_p(M) = ImP \oplus KerP^*$.*

Proof of Theorem 1.1.2. First, note that $\dim_{\mathbb{R}} \mathcal{H}^p_\Delta(M, \mathbb{R}) < +\infty$ by ellipticity of Δ. We shall now prove the orthogonality of these spaces: If $u \in \mathcal{H}^p_\Delta(M, \mathbb{R})$, then for every $v \in \mathscr{C}^\infty_{p-1}(M)$, we have that $\langle\langle u, dv \rangle\rangle = \langle\langle \delta u, v \rangle\rangle = 0$. Similarly, for every $w \in \mathscr{C}^\infty_{p+1}(M)$

we have $\langle\langle u, \delta w\rangle\rangle = \langle\langle du, w\rangle\rangle = 0$. Finally for every $v \in \mathcal{C}^\infty_{p-1}(M)$ and $w \in \mathscr{C}^\infty_{p+1}(M)$ we have $\langle\langle dv, \delta w\rangle\rangle = \langle\langle d^2 v, w\rangle\rangle = 0$, thus all of these spaces are orthogonal. It remains to show that their sum is $\mathscr{C}^\infty_p(M)$. By Proposition 1.1.3 and self-adjointness of Δ we have:

$$\mathscr{C}^\infty_p(M) = \mathcal{H}^p_\Delta(M,\mathbb{R}) \oplus \operatorname{Im}\Delta = \mathcal{H}^p_\Delta(M,\mathbb{R}) \oplus \operatorname{Im}(d^*d + dd^*) \subset \mathcal{H}^p_\Delta(M,\mathbb{R}) \oplus \operatorname{Im} d \oplus \operatorname{Im} d^*$$

□

Corollary 1.1.4 (The Hodge Isomorphism Theorem)**.** $\mathcal{H}^p_\Delta(M,\mathbb{R}) \simeq H^p_{DR}(M,\mathbb{R})$.

Now, we will try to extend what we did previously to the complex case.

Definition 1.1.5. *Let X be a complex manifold of dimension $\dim_{\mathbb{C}} X = n$. A **Hermitian metric** h on X is a $\mathscr{C}^\infty$-family of positive definite inner products $(h_x)_{x\in X}$ on $T^{1,0}_x X$ (where $T^{1,0}_x X$ is the holomorphic tangent space to X at x).*

In a coordinate chart $(z_1, ..., z_n)$, such a form can be written as:

$$h(z) = \sum_{1\le j,k\le n} h_{jk}(z) dz_j \otimes d\bar{z}_k\,,$$

where $(h_{jk})_{1\le j,k\le n}$ is a positive Hermitian matrix with smooth coefficients. The fundamental (1,1)-form associated to h is the positive form of type $(1,1)$:

$$\omega := -\operatorname{Im} h = \frac{i}{2} \sum_{1\le j,k\le n} h_{jk}(z)\, dz_j \wedge d\bar{z}_k$$

The pair (X,h) or (X,ω) is called a **Hermitian manifold**. This induces a volume (n,n)-form on X which coincides with the Hermitian volume element of (X,h). Namely:

$$dV_\omega = \frac{1}{n!}\omega^n \underset{loc}{=} \det(h_{jk}) \bigwedge_{1\le j\le n} \left(\frac{i}{2} dz_j \wedge d\bar{z}_j\right)$$

Let (X,ω) be a compact Hermitian manifold. The Hodge $*$-operator can be extended to $\mathbb{C}$-valued forms by means of the formula:

$$u \wedge *\bar{v} = \langle u, v\rangle\, dV_\omega\,, \quad \forall u, v \in \Lambda^p(\mathbb{C}\otimes TX^*)$$

and it still satisfies the same properties. It can be immediately checked that:

$$d\omega_k = \omega_{k-1} \wedge d\omega \quad \text{and} \quad *\,\omega_k = \omega_{n-k}$$

where $\omega_k = \dfrac{\omega^k}{k!}$, for all $1 \le k \le n$. We define on X the formal adjoints of the ∂ and $\bar\partial$

operators respectively: $\partial^* = - * \bar{\partial} *$, $\bar{\partial}^* = - * \partial *$ and their corresponding **Laplacians** $\Delta' = \partial\partial^* + \partial^*\partial$ and $\Delta'' = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. Let $\mathscr{C}^\infty_{p,q}(X) := \mathscr{C}^\infty(X, \Lambda^{p,q}(TX^*))$ be the set of smooth differential (p,q)-forms. Then we can define the space of **Δ'-harmonic** forms of type (p,q) on X as:

$$\mathcal{H}^{p,q}_{\Delta'}(X) := \mathrm{Ker}(\Delta' : \mathscr{C}^\infty_{p,q}(X) \longrightarrow \mathscr{C}^\infty_{p,q}(X))$$

Similarly, we define the space of **Δ''-harmonic** forms of type (p,q) on X by:

$$\mathcal{H}^{p,q}_{\Delta''}(X) := \mathrm{Ker}(\Delta'' : \mathscr{C}^\infty_{p,q}(X) \longrightarrow \mathscr{C}^\infty_{p,q}(X))$$

An important feature of complex manifolds, which is a priori non-metric in nature, is the *$\partial\bar{\partial}$-property.*

Definition 1.1.6. *A compact complex manifold X is called a **$\partial\bar{\partial}$-manifold** if for any $p,q = 0,\dots,n$, and for any $u \in \mathscr{C}^\infty_{p,q}(X) \cap Ker\ \partial \cap Ker\ \bar{\partial}$, we have:*

$$u \in Im\ d \Longleftrightarrow u \in Im\ \partial \Longleftrightarrow u \in Im\ \bar{\partial} \Longleftrightarrow u \in Im\ \partial\bar{\partial}$$

$\partial\bar{\partial}$-manifolds can be chracterized by strong cohomological properties. For this purpose, we recall the definitions of the *Bott-Chern* cohomology and the *Aeppli* cohomology.

Definition 1.1.7. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.*

1. *The **Bott-Chern cohomology group** of bidegree (p,q) of X is defined as follows:*

$$H^{p,q}_{BC}(X,\mathbb{C}) = \frac{Ker\left(\partial : \mathscr{C}^\infty_{p,q}(X) \longrightarrow \mathscr{C}^\infty_{p+1,q}(X)\right) \cap Ker\left(\bar{\partial} : \mathscr{C}^\infty_{p,q}(X) \longrightarrow \mathscr{C}^\infty_{p,q+1}(X)\right)}{Im\left(\partial\bar{\partial} : \mathscr{C}^\infty_{p-1,q-1}(X) \longrightarrow \mathscr{C}^\infty_{p,q}(X)\right)}$$

   *and the **Bott-Chern numbers** are $h^{p,q}_{BC}(X) := \dim_{\mathbb{C}} H^{p,q}_{BC}(X,\mathbb{C})$.*

2. *The **Aeppli cohomology group** of bidegree (p,q) of X is defined as follows:*

$$H^{p,q}_{A}(X,\mathbb{C}) = \frac{Ker\left(\partial\bar{\partial} : \mathscr{C}^\infty_{p,q}(X) \longrightarrow \mathscr{C}^\infty_{p+1,q+1}(X)\right)}{Im\left(\partial : \mathscr{C}^\infty_{p-1,q}(X) \longrightarrow \mathscr{C}^\infty_{p,q}(X)\right) + Im\left(\bar{\partial} : \mathscr{C}^\infty_{p,q-1}(X) \longrightarrow \mathscr{C}^\infty_{p,q}(X)\right)}$$

   *and the **Aeppli numbers** are $h^{p,q}_{A}(X) := \dim_{\mathbb{C}} H^{p,q}_{A}(X,\mathbb{C})$.*

We recall also the definition of the *Bott-Chern* and *Aeppli* Laplacians.

Definition 1.1.8. *Let (X,ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$.*

1. *The **Bott-Chern Laplacian** is defined as follows:*

$$\Delta_{BC} := \partial^*\partial + \bar{\partial}^*\bar{\partial} + (\partial\bar{\partial})^*(\partial\bar{\partial}) + (\partial\bar{\partial})(\partial\bar{\partial})^* + (\partial^*\bar{\partial})^*(\partial^*\bar{\partial}) + (\partial^*\bar{\partial})(\partial^*\bar{\partial})^*$$

and the ***Bott-Chern harmonic space*** *is* $\mathcal{H}^{p,q}_{BC}(X) = Ker\,\Delta_{BC} \cap \mathscr{C}^\infty_{p,q}(X)$.

2. *The* ***Aeppli Laplacian*** *is defined as follows:*

$$\Delta_A := \partial\partial^* + \bar\partial\bar\partial^* + (\partial\bar\partial)^*(\partial\bar\partial) + (\partial\bar\partial)(\partial\bar\partial)^* + (\partial\bar\partial^*)(\partial\bar\partial^*)^* + (\partial\bar\partial^*)^*(\partial\bar\partial^*)$$

and the and the ***Aeppli harmonic space*** *is* $\mathcal{H}^{p,q}_{A}(X) = Ker\,\Delta_{A} \cap \mathscr{C}^\infty_{p,q}(X)$.

As in the case of the Laplacian Δ, a theorem analogous to Theorem 1.1.2 holds on any compact Hermitian manifold (X, ω): the following L^2_ω**-orthogonal three-space decompositions** of $\mathscr{C}^\infty_{p,q}(X,\mathbb{C})$ hold in every bidegree (p,q), induced by the Laplacians Δ', Δ'', Δ_{BC}, and Δ_A:

$$\begin{aligned}
\mathscr{C}^\infty_{p,q}(X,\mathbb{C}) &= \mathcal{H}^{p,q}_{\Delta'}(X,\mathbb{C}) \oplus \operatorname{Im}\partial \oplus \operatorname{Im}\partial^* \\
\mathscr{C}^\infty_{p,q}(X,\mathbb{C}) &= \mathcal{H}^{p,q}_{\Delta''}(X,\mathbb{C}) \oplus \operatorname{Im}\bar\partial \oplus \operatorname{Im}\bar\partial^* \\
\mathscr{C}^\infty_{p,q}(X,\mathbb{C}) &= \mathcal{H}^{p,q}_{BC}(X,\mathbb{C}) \oplus \operatorname{Im}\partial\bar\partial \oplus (\operatorname{Im}\partial^* + \operatorname{Im}\bar\partial^*) \\
\mathscr{C}^\infty_{p,q}(X,\mathbb{C}) &= \mathcal{H}^{p,q}_{A}(X,\mathbb{C}) \oplus \operatorname{Im}(\partial\bar\partial)^* \oplus (\operatorname{Im}\partial + \operatorname{Im}\bar\partial)
\end{aligned}$$

In particular, we have for any bidegree (p,q):

$$\begin{cases}
\operatorname{Ker}\partial \cap \mathscr{C}^\infty_{p,q}(X,\mathbb{C}) = \mathcal{H}^{p,q}_{\Delta'}(X,\mathbb{C}) \oplus \operatorname{Im}\partial \\
\mathcal{H}^{p,q}_{\Delta'}(X,\mathbb{C}) = \operatorname{Ker}\partial \cap \operatorname{Ker}\partial^* \cap \mathscr{C}^\infty_{p,q}(X,\mathbb{C})
\end{cases}$$
$$\begin{cases}
\operatorname{Ker}\bar\partial \cap \mathscr{C}^\infty_{p,q}(X,\mathbb{C}) = \mathcal{H}^{p,q}_{\Delta''}(X,\mathbb{C}) \oplus \operatorname{Im}\bar\partial \\
\mathcal{H}^{p,q}_{\Delta''}(X,\mathbb{C}) = \operatorname{Ker}\bar\partial \cap \operatorname{Ker}\bar\partial^* \cap \mathscr{C}^\infty_{p,q}(X,\mathbb{C})
\end{cases}$$
$$\begin{cases}
\operatorname{Ker}\partial \cap \operatorname{Ker}\bar\partial \cap \mathscr{C}^\infty_{p,q}(X,\mathbb{C}) = \mathcal{H}^{p,q}_{BC}(X,\mathbb{C}) \oplus \operatorname{Im}\partial\bar\partial \\
\mathcal{H}^{p,q}_{BC}(X,\mathbb{C}) = \operatorname{Ker}\partial \cap \operatorname{Ker}\bar\partial \cap \operatorname{Ker}(\partial\bar\partial)^* \cap \mathscr{C}^\infty_{p,q}(X,\mathbb{C})
\end{cases}$$
$$\begin{cases}
\operatorname{Ker}\partial\bar\partial \cap \mathscr{C}^\infty_{p,q}(X,\mathbb{C}) = \mathcal{H}^{p,q}_{A}(X,\mathbb{C}) \oplus (\operatorname{Im}\partial + \operatorname{Im}\bar\partial) \\
\mathcal{H}^{p,q}_{A}(X,\mathbb{C}) = \operatorname{Ker}\partial^* \cap \operatorname{Ker}\bar\partial^* \cap \operatorname{Ker}(\partial\bar\partial) \cap \mathscr{C}^\infty_{p,q}(X,\mathbb{C})
\end{cases}$$

yielding the **Hodge isomorphisms** (as in Corollary 1.1.4) in every bidegree (p,q):

$$\begin{aligned}
\mathcal{H}^{p,q}_{\Delta'}(X,\mathbb{C}) &\simeq H^{p,q}_{\partial}(X,\mathbb{C}) \\
\mathcal{H}^{p,q}_{\Delta''}(X,\mathbb{C}) &\simeq H^{p,q}_{\bar\partial}(X,\mathbb{C}) \\
\mathcal{H}^{p,q}_{BC}(X,\mathbb{C}) &\simeq H^{p,q}_{BC}(X,\mathbb{C}) \\
\mathcal{H}^{p,q}_{A}(X,\mathbb{C}) &\simeq H^{p,q}_{A}(X,\mathbb{C})
\end{aligned}$$

Definition 1.1.9. *A compact complex manifold X is* ***Kähler*** *if it admits a Hermitian* *metric h whose fundamental form ω is d-closed. such ω is called a* ***Kähler form****.*

Lemma 1.1.10 (The $\partial\bar{\partial}$-Lemma)**.** *Every compact Kähler manifold X is a $\partial\bar{\partial}$-manifold.*

One main ingredient for the proof of this lemma is the **Bochner-Kodaira-Nakano** identity on Kähler manifolds and the Kähler commutation identities in Kähler geometry:

$$\Delta'' = \Delta' = \frac{1}{2}\Delta \tag{1.1.3}$$

$$[\partial, \bar{\partial}^*] = [\bar{\partial}, \partial^*] = 0 \tag{1.1.4}$$

Proof of Lemma 1.1.10. If $u \in \mathscr{C}^\infty_{p,q}(X, \mathbb{C})$, that is $u = \partial\bar{\partial}v$ for some $v \in \mathscr{C}^\infty_{p-1,q-1}(X, \mathbb{C})$, one can write: $u = \partial(\bar{\partial}v) = \bar{\partial}(-\partial v) = d(\bar{\partial}v)$, meaning that $u \in \operatorname{Im}\partial\bar{\partial} \implies u \in \operatorname{Im}\partial$ and $u \in \operatorname{Im}\bar{\partial}$ and $u \in \operatorname{Im} d$. Since u is of pure type (p, q), $du = 0 \implies \partial u = \bar{\partial}u = 0$. Fix a Hermitian metric h on X, and let ω be its fundamental form. The L^2_ω-orthogonal three-space decompositions with respect to Δ, Δ' and Δ'' implies that:

$$\begin{aligned} u \in \operatorname{Im}\partial &\iff u \in (\mathcal{H}^{p,q}_{\Delta'}(X, \mathbb{C}))^\perp \\ u \in \operatorname{Im}\bar{\partial} &\iff u \in (\mathcal{H}^{p,q}_{\Delta''}(X, \mathbb{C}))^\perp \\ u \in \operatorname{Im} d &\iff u \in \left(\mathcal{H}^{p+q}_{\Delta}(X, \mathbb{C})\right)^\perp \end{aligned}$$

Now, suppose that ω is Kähler. Thanks to 1.1.3, the above equivalences imply the following equivalences: $u \in \operatorname{Im}\, d \iff u \in \operatorname{Im}\,\partial \iff u \in \operatorname{Im}\,\bar{\partial}$. So it suffices to prove for instance that: $u \in \operatorname{Im}\bar{\partial} \implies u \in \operatorname{Im}\partial\bar{\partial}$. Suppose $u = \bar{\partial}v$ for some $v \in \mathscr{C}^\infty_{p,q-1}(X, \mathbb{C})$. The L^2_ω-orthogonal three-space decomposition with respect to Δ' yields: $v = v_h + +\partial\alpha + \partial^*\beta$ with $v_h \in \mathcal{H}^{p,q-1}_{\Delta'}(X, \mathbb{C})$, then: $u = \bar{\partial}v = -\partial\bar{\partial}\alpha - \partial^*\bar{\partial}\beta$, since $0 = \Delta' v_h \overset{(1.1.3)}{=} \Delta'' v_h = 0$ which implies that $\bar{\partial}v_h = 0$, and $\bar{\partial}\partial^* \overset{(1.1.4)}{=} -\partial^*\bar{\partial}$. But since $\partial u = 0$, we have: $\partial\partial^*\bar{\partial}\beta = 0$. Meaning that: $\partial^*\bar{\partial}\beta \in \operatorname{Ker}\partial \cap \operatorname{Im}\partial^* = \{0\}$. Hence: $u = \partial\bar{\partial}(-\alpha) \in \operatorname{Im}\partial\bar{\partial}$. □

Another important operator is the **Lefschetz operator** associated to a Hermitian metric ω on X defined as:

$$\begin{aligned} L_\omega : \mathscr{C}^\infty_{p,q}(X, \mathbb{C}) &\longrightarrow \mathscr{C}^\infty_{p+1,q+1}(X, \mathbb{C}) \\ \alpha &\longmapsto L_\omega(\alpha) = \omega \wedge \alpha \end{aligned}$$

The **trace operator** is its formal adjoint $\Lambda_\omega = *^{-1}L_\omega *$ (Λ is even the adjoint of L with respect to the pointwise inner product $\langle\cdot,\cdot\rangle_\omega$, which implies that it is also the formal adjoint with respect to the L^2_ω inner product). Meaning that for any $\alpha \in \mathscr{C}^\infty_{p,q}(X, \mathbb{C})$ and any $\beta \in \mathscr{C}^\infty_{p+1,q+1}(X,\mathbb{C})$, one has: $\langle\langle L_\omega \alpha, \beta\rangle\rangle = \langle\langle \alpha, \Lambda_\omega \beta\rangle\rangle$. One can check that:

$$[L_\omega, \Lambda_\omega] = (p+q-n)\mathrm{Id}_{\mathscr{C}^\infty_{p,q}(X,\mathbb{C})} \quad , \quad *L_\omega = \Lambda_\omega *$$

Definition 1.1.11. *A (p,q)-form α on (X,ω) is called* ***ω-primitive*** *if $\Lambda_\omega \alpha = 0$.*

Note that any form of bidegree $(p,0)$ or $(0,q)$ is ω-primitive.

The following proposition plays an important role in our discussion later:

Proposition 1.1.12. *If $u \in \mathscr{C}^\infty_{p,q}(X,\mathbb{C})$ is ω-primitive then:*

$$*u = (-1)^{\frac{(p+q)^2+p+q}{2}} \frac{i^{p-q}}{(n-p-q)!}\omega^{n-p-q} \wedge u$$

1.2 Special structures on compact complex manifolds

Definition 1.2.1. *Let (X,h) be a compact Hermitian manifold with $\dim_\mathbb{C} X = n$, and let ω be the fundamental $(1,1)$-form associated to h.*

1. *If $d\omega^{n-1} = 0$, we say that ω is a* ***balanced metric****. And if such a metric exists on X, the pair (X,ω) is called a* ***balanced manifold*** *[Gau77a, Mic82].*

2. *If ω^{n-1} is d-exact, we say that ω is a* ***degenerate balanced metric****. And if such a metric exists on X, the pair (X,ω) is called a* ***degenerate balanced manifold*** *[Pop15].*

3. *If $\partial\bar\partial\omega = 0$, we say that ω is a* ***Strong Kähler metric with Torsion*** *(or* ***SKT*** *for short). And if such a metric exists on X, the pair (X,ω) is called a* ***SKT manifold*** *[Str86, Bis89].*

4. *If $\partial\bar\partial\omega^{n-1} = 0$, we say that ω is a* ***Gauduchon metric****. Such a metric always exists on X by a Theorem of Gauduchon [Gau77b].*

5. *If ω is the $(1,1)$-component of a real d-closed 2-form Ω, we say that ω is a* ***Hermitian-symplectic metric*** *(or* ***HS*** *for short). And if such a metric exists on X, the pair (X,ω) is called a* ***HS manifold*** *[ST10].*

6. *If ω^{n-1} is the $(n-1,n-1)$-component of a real d-closed $(2n-2)$-form Ω, we say that ω is a* ***strongly Gauduchon metric*** *(or* ***sG*** *for short). And if such a metric exists on X, the pair (X,ω) is called a* ***sG manifold*** *[Pop13a].*

It is clear from the definitions of these special metrics that the following hierarchy holds:

$$
\begin{tikzcd}
\omega \text{ is Kähler} \arrow[r, Rightarrow, "(1)"] \arrow[d, Rightarrow, "(3)"'] & \omega \text{ is HS} \arrow[r, Rightarrow, "(2)"] & \omega \text{ is SKT} \\
\omega \text{ is balanced} \arrow[r, Rightarrow, "(4)"] & \omega \text{ is sG} \arrow[r, Rightarrow, "(5)"] & \omega \text{ is Gauduchon} \\
\omega \text{ is degenerate balanced} \arrow[u, Rightarrow, "(6)"] & &
\end{tikzcd}
$$

Proposition 1.2.2. *At the level of manifolds, we have:*

$$
\begin{tikzcd}
 & X \textit{ is Kähler} \arrow[r, Rightarrow, "(1)"] \arrow[dr, Rightarrow, "(3)"] & X \textit{ is HS} \arrow[r, Rightarrow, "(2)"] \arrow[dr, Rightarrow, "(4)"] & X \textit{ is SKT} \\
X \textit{ is degenerate balanced} \arrow[rr, Rightarrow, "(5)"'] & & X \textit{ is balanced} \arrow[r, Rightarrow, "(6)"'] & X \textit{ is sG}
\end{tikzcd}
$$

Proof. There is only (4) here to prove. Let ω be a HS metric on X, this means that $\omega > 0$ and $\exists \alpha \in \mathscr{C}^\infty_{2,0}(X)$ such that $\Omega = \alpha + \omega + \bar{\alpha}$ is d-closed. Put $\Gamma := \Omega^{n-1}$, then:

$$
\Gamma = (\alpha + \omega + \bar{\alpha})^{n-1} = \sum_{k=0}^{n-1}\sum_{l=0}^{k} \binom{n-1}{k}\binom{k}{l} \alpha^l \wedge \bar{\alpha}^{k-l} \wedge \omega^{n-k-1}
$$

By taking the bidegree $(n-1, n-1)$ in this expression (which corresponds to $l = k - l$), we get:

$$
\begin{aligned}
\Gamma^{n-1,n-1} &= \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2l}\binom{2l}{l} \alpha^l \wedge \bar{\alpha}^l \wedge \omega^{n-2l-1} \\
&= \omega^{n-1} + 2\binom{n-1}{2} \alpha \wedge \bar{\alpha} \wedge \omega^{n-3} + 6\binom{n-1}{4} \alpha^2 \wedge \bar{\alpha}^2 \wedge \omega^{n-5} + \cdots \\
&\overset{(*)}{\geq} \omega^{n-1} > 0
\end{aligned}
$$

where the inequality $(*)$ comes from the fact that $\alpha^l \wedge \bar{\alpha}^l \wedge \omega^{n-2l-1} \geq 0$ (see Chapter III in [Dem97], Example 1.2 and Proposition 1.11). We conclude thanks to Michelsohn's trick that $\omega' := \sqrt[n-1]{\Gamma^{n-1,n-1}}$ defines a sG metric on X. $\square$

The Fino-Vezzoni conjecture predicts that a compact complex manifold admitting a balanced metric and a SKT metric is actually Kähler. When the same metric is balanced and SKT, we have the following proposition:

Proposition 1.2.3 ([IP01])**.** *Let X be a compact complex manifold admitting a Hermitian metric ω that is both balanced and SKT. Then ω is Kähler.*

Proof. We reproduce here the proof given in [Pop15] of Proposition 1.1. ω being SKT is equivalent to the following:

$$\partial\bar{\partial}\omega = 0 \iff \partial\omega \in \operatorname{Ker}\bar{\partial} \iff *(\partial\omega) \in \operatorname{Ker}\partial^* \tag{1.2.1}$$

where the last equivalence follows from the standard formula $\partial^* = - * \bar{\partial} *$ involving the Hodge-star isomorphism $* = *_\omega$ induced by ω. Meanwhile, the balanced assumption on ω translates to any of the following equivalent properties:

$$d\omega^{n-1} = 0 \iff \omega^{n-2} \wedge \partial\omega = 0 \iff \partial\omega \text{ is primitive}$$

Moreover, since $\partial\omega$ is primitive when ω is balanced, the general formula (1.1.12) yields:

$$*(\partial\omega) = i\frac{\omega^{n-3}}{(n-3)!} \wedge \partial\omega = \frac{i}{(n-2)!}\partial\omega^{n-2} \in \operatorname{Im}\partial \tag{1.2.2}$$

Thus, if ω is both SKT and balanced, we get from (1.2.1) and (1.2.2) that $*(\partial\omega)$ lies in $\operatorname{Ker}\partial^* \cap \operatorname{Im}\partial = \{0\}$. We infer that $\partial\omega = 0$, i.e. ω is Kähler. $\square$

Now, we recall some of the positivity notions for differential forms. Let V be a complex vector space of dimension n and $(z_1, \dots, z_n)$ coordinates on V. We denote by $\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right)$ the corresponding basis of V, and by $(dz_1, \dots, dz_n)$ its dual basis in V^* and consider the exterior algebra

$$\Lambda V^*_{\mathbb{C}} = \bigoplus \Lambda^{p,q}V^*, \qquad \Lambda^{p,q}V^* = \Lambda^p V^* \otimes \Lambda^q V^*.$$

We are of course especially interested in the case where $V = T_xX$ is the tangent space to a complex manifold X, but we want to emphasize here that our considerations only involve linear algebra. Let us first observe that V has a canonical orientation given by the (n,n)-form

$$\tau(z) = i\,dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge i\,dz_n \wedge d\bar{z}_n = 2^n dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

where $z_j = x_j + iy_j$. If $(w_1, \dots, w_n)$ are other coordinates, then

$$dw_1 \wedge \cdots \wedge dw_n = \det\left(\frac{\partial w_j}{\partial z_k}\right) dz_1 \wedge \cdots \wedge dz_n,$$
$$\tau(w) = \left|\det\left(\frac{\partial w_j}{\partial z_k}\right)\right|^2 \tau(z).$$

In particular, a complex manifold always has a canonical orientation. More generally, natural positivity concepts for (p,p)-forms can be defined.

Definition 1.2.4. *1. A (p,p)-form $u \in \Lambda^{p,p}V^*$ is said to be* ***weakly*** *(****strictly****)* ***positive*** *if for all $\alpha_j \in V^*$, $1 \le j \le q = n-p$, the form $u \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_q \wedge \bar{\alpha}_q$ is a (strictly) positive (n,n)-form.*

2. A (q,q)-form $v \in \Lambda^{q,q}V^*$ is said to be ***strongly*** *(****strictly****)* ***positive*** *if it can be written as $v = \sum_s \gamma_s\, i\alpha_{s,1} \wedge \bar{\alpha}_{s,1} \wedge \cdots \wedge i\alpha_{s,q} \wedge \bar{\alpha}_{s,q}$, where $\alpha_{s,j} \in V^*$ and $\gamma_s \ge 0$ ($\gamma_s > 0$ in the strict case).*

Examples 1.2.5. Since $i^p(-1)^{p(p-1)/2} = i^{p^2}$, we have the commutation rules

$$i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_p \wedge \bar{\alpha}_p = i^{p^2} \alpha \wedge \bar{\alpha}, \quad \forall \alpha = \alpha_1 \wedge \cdots \wedge \alpha_p \in \Lambda^{p,0}V^*,$$
$$i^{p^2} \beta \wedge \bar{\beta} \wedge i^{m^2} \gamma \wedge \bar{\gamma} = i^{(p+m)^2} \beta \wedge \gamma \wedge \overline{\beta \wedge \gamma}, \quad \forall \beta \in \Lambda^{p,0}V^*, \quad \forall \gamma \in \Lambda^{m,0}V^*.$$

Take $m = q$ be the complementary degree of p. Then $\beta \wedge \gamma = \lambda\, dz_1 \wedge \cdots \wedge dz_n$ for some $\lambda \in \mathbb{C}$ and $i^{n^2} \beta \wedge \gamma \wedge \overline{\beta \wedge \gamma} = |\lambda|^2 \tau(z)$. If we set $\gamma = \alpha_1 \wedge \cdots \wedge \alpha_q$, we find that $i^{p^2} \beta \wedge \bar{\beta}$ is a wekly positive (p,p)-form for every $\beta \in \Lambda^{p,0}V^*$. In particular, strong positivity implies weak positivity.

Remark 1.2.6. 1. All positive forms u are real, i.e. $u = \bar{u}$.

2. A $(1,1)$-form $u = i\sum_{j,k} u_{jk} dz_j \wedge d\bar{z}_k$ is weakly positive if and only if $\xi \longmapsto \sum u_{jk}\xi_j\bar{\xi}_k$ is a semi-positive Hermitian form.

3. The notions of weak and strong positivity coincide for $p = 0, 1, n-1, n$.

4. If $u_1, \ldots, u_s$ are strongly positive forms (resp. all except possibly one is weakly positive), then $u_1 \wedge \cdots \wedge u_s$ is strongly positive (resp. weakly positive).

5. If $\Phi : W \longrightarrow V$ is a complex linear map and $u \in \Lambda^{p,p}V^*$ is weakly (resp. strongly) positive, then $\Phi^* u$ is weakly (resp. strongly) positive.

6. The sets of weakly positive and strongly positive forms are closed convex cones. By definition, the weakly positive cone is dual to the strongly positive cone via the pairing:
$$\Lambda^{p,p}V^* \times \Lambda^{q,q}V^* \longrightarrow \mathbb{C}, \qquad (u,v) \mapsto \frac{u \wedge v}{\tau}.$$

The duality between the positive and strongly positive cones of forms allows one to define positivity for currents.

Definition 1.2.7. *A current $T \in \mathcal{D}'_{p,p}(X)$ is said to be*

- ***weakly positive*** *if $\langle T, u \rangle \ge 0$ for all strongly positive test forms $u \in \mathcal{D}_{p,p}(X)$,*
- ***strongly positive*** *if $\langle T, u \rangle \ge 0$ for all weakly positive test forms $u \in \mathcal{D}_{p,p}(X)$.*

Proposition 1.2.8. *Every weakly positive current $T = i^{(n-p)^2} \sum T_{I,J}\, dz_I \wedge d\bar{z}_J$ in $\mathcal{D}'_{p,p}(X)$ is real and of order 0, i.e. its coefficients $T_{I,J}$ are complex measures satisfying $T_{I,J} = \overline{T_{J,I}}$. Moreover $T_{I,I} \geq 0$.*

Here, we recall the generalizations of special metrics in degree $2p$, with $1 \leq p \leq n-1$.

Definition 1.2.9 ([Ale11])**.** *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$, and fix $1 \leq p \leq n-1$. Then X is said to be:*

1. ***p-Kähler** if it admits a d-closed weakly strictly positive (p,p)-form Ω. Here, Ω is called a **p-Kähler form**.*
2. ***p-Hermitian-symplectic** (or **p-HS** for short) if it admits a real d-closed $2p$-form Ω such that $\Omega^{p,p}$ is weakly strictly positive. Here, $\Omega^{p,p}$ is called a **p-HS form**.*
3. ***p-SKT** (or **p-pluriclosed**) if it admits a $\partial\bar{\partial}$-closed weakly strictly positive (p,p)-form Ω. Here, Ω is called a **p-SKT form**.*

Remark 1.2.10.
1. 1-Kähler=Kähler, 1-HS=HS and 1-SKT=SKT.
2. $(n-1)$-Kähler=Balanced, $(n-1)$-HS=sG, $(n-1)$-SKT=Gauduchon.
3. For every $1 \leq p \leq n-1$, we have: p-Kähler $\Longrightarrow$ p-HS $\Longrightarrow$ p-SKT.

1.3 Hyperbolicity notions in complex geometry

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$. It is equipped with the **Poincaré metric** $\omega_\Delta := \frac{4i}{(1-|z|^2)^2} dz \wedge d\bar{z}$. Let X be a complex manifold with $\dim_{\mathbb{C}} X = n$.

Definition 1.3.1 (**Kobayashi pseudodistance**, [Kob67a])**.**

1. *A **chain** connecting two points $p, q \in X$ is a collection $(f_i, p_i)_{0 \leq i \leq k}$ where $f_i : \Delta \longrightarrow X$ are holomorphic maps and $p_i \in \Delta$ such that :*
   (a) $p = f_0(0)$, $f_0(p_0) = f_1(0)$.
   (b) $f_{i-1}(p_{i-1}) = f_i(0)$ for all $i = 1, \ldots, k$.
   (c) $q = f_k(p_k)$.

   We denote by $\mathscr{C}_{p,q}$ the set of all chains connecting p and q.
2. *The **Kobayashi pseudodistance** on X is then defined by :*
$$d_X(p,q) := \inf_{\mathscr{C}_{p,q}} \sum_{i=0}^{k} \omega_\Delta(0, p_i)\,, \qquad \textit{for all } p, q \in X$$

Examples 1.3.2. 1. $d_{\mathbb{C}} \equiv 0$, since $\forall \varepsilon > 0$ and $\forall p, q \in \mathbb{C}$, the holomorphic map $z \in \Delta \longmapsto p + \dfrac{q-p}{\varepsilon} z$ sends 0 to p and ε to q. Hence: $0 \leq d_{\mathbb{C}}(p,q) \leq \omega_{\Delta}(0,\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0$.

2. $d_{\Delta} = \omega_{\Delta}$. Indeed, it is obvious that $d_{\Delta} \leq \omega_{\Delta}$, and the reverse inequality follows from the famous **Schwarz-Pick lemma**: $\omega_{\Delta}(f(p), f(q)) \leq \omega_{\Delta}(p,q)$, for any holomorphic map $f : \Delta \longrightarrow \Delta$, $\forall p, q \in \Delta$.

Definition 1.3.3 (**Kobayashi Hyperbolicity**, [Kob67b]). *We say that X is* ***Kobayashi Hyperbolic*** *if d_X is a distance.*

Remark 1.3.4. If X is Kobayashi hyperbolic, then X is **Brody hyperbolic**, i.e. any holomorphic map $f : \mathbb{C} \longrightarrow X$ is constant, since for any $z_1, z_2 \in \mathbb{C}$ we have:

$$0 \leq d_X(f(z_1), f(z_2)) \leq d_{\mathbb{C}}(z_1, z_2) = 0$$

The converse is not true in full generality. A counter example was given by D. Eisenman and L. Taylor (consider $X = \{(z,w) \in \mathbb{C}^2 : |z| < 1, |zw| < 1\} \setminus \{(0,w); |w| \geq 1\}$).

However, if X is **compact**, R. Brody proved that:

Theorem 1.3.5. *X is Kobayashi hyperbolic if and only if X is Brody hyperbolic.*

The proof essentially relies on the following lemma:

Lemma 1.3.6 (Brody's reparametrization lemma)**.** *Let X be a complex manifold equipped with a length function $|\cdot|$. Let $f : \mathbb{D}_r \longrightarrow X$ be a holomorphic map. Let $c > 0$, and for $0 \leq t \leq 1$ let $f_t(z) = f(tz)$.*

1. *If $|f'(0)| > c$, then there exists $t < 1$ and an automorphism h of $\mathbb{D}_r$ such that, if we put $g = f_t \circ h$, then: $\sup_{z \in \mathbb{D}_r} |g'(z)| = |g'(0)| = c$.*

2. *If $|f'(0)| = c$, then the same conclusion holds allowing $t \leq 1$.*

Proof. Let $m_t(z) = tz$, $\forall z \in \mathbb{D}_r$ so that $f_t = f \circ m_t$. Then $m_t'(z)(v) = tv$, hence:

$$|f_t'(z)| = |f'(tz)|\, t\, \frac{1 - |z|^2/r^2}{1 - |tz|^2/r^2}.$$

Set $s(t) = \sup_{z \in \mathbb{D}_r} |f_t'(z)|$. If $t < 1$, then $|f_t'(z)| \longrightarrow 0$ as $|z| \longrightarrow r$, so the supremum is achieved: $s(t) = \max_{z \in \mathbb{D}_r} |f_t'(z)|$. Since $s(0) = 0$ and $s(t)$ is continuous for $0 \leq t < 1$, we have $s(t) \longrightarrow s(1)$ as $t \longrightarrow 1$. Since $|f'(0)| > c$, we have $s(1) > c$, hence there exists $t < 1$ such that $s(t) = c$. Therefore there exists $z_0 \in \mathbb{D}_r$ such that $|f_t'(z_0)| = c$. Let $h : \mathbb{D}_r \longrightarrow \mathbb{D}_r$ be an automorphism such that $h(0) = z_0$ and set $g = f_t \circ h$. Then:

$$|g'(0)| = |f_t'(z_0)|\, |h'(0)| = |f_t'(z_0)| = c,$$

which proves the first statement. The second statement is proved similarly. □

Corollary 1.3.7. *Let (X, ω) be a Hermitian manifold. Then any holomorphic map $f : \mathbb{D}_r \longrightarrow X$ can be reparametrized such that it becomes of* ***order 2****, i.e. $\int_{\mathbb{D}_r} f^*\omega \leq \pi r^2$.*

Let X be a compact complex manifold, and let $\pi : \tilde{X} \longrightarrow X$ be its universal cover.

Definition 1.3.8. *X is* ***Kähler hyperbolic*** *if it admits a Kähler metric ω which is $\widetilde{d}$(**bounded**), i.e : $\exists \eta \in L_1^\infty(\widetilde{X})$ such that $\widetilde{\omega} := \pi^*\omega = d\eta$.*

Examples 1.3.9. The following are examples of Kähler hyperbolic manifolds:

1. Compact Kähler manifolds homotopic to negatively curved Riemannian manifolds.
2. Compact quotients of bounded symmetric domain in $\mathbb{C}^n$.
3. Submanifolds and products of Kähler hyperbolic manifolds.

Theorem 1.3.10 ([Gro91],[CY18])**.** *Let X be a compact Kähler hyperbolic manifold with $\dim_{\mathbb{C}} X = n$. Then the following properties hold:*

1. *X is Kobayashi hyperbolic.*
2. *X and each of its submanifolds are Kähler-Einstein manifolds of general type.*
3. *$\mathcal{H}^p(\widetilde{X}) = 0$, $\forall p \neq n$ and $\dim \mathcal{H}^n(\widetilde{X}) = +\infty$. In particular: $(-1)^n \chi(X) > 0$.*

1.4 Holomorphic Deformations of Complex Structures

Definition 1.4.1. *A* ***holomorphic family*** *of compact complex manifolds is a proper holomorphic submersion $\sigma : \mathscr{X} \longrightarrow B$ between complex manifolds $\mathscr{X}$ and B (B is in general an open ball around the origin in some $\mathbb{C}^N$).*

The two important and commonly used concepts in deformation theory are openness and closedness properties.

Definition 1.4.2. *Suppose $\{X_t\}_{t \in B}$ is a holomorphic family of compact complex manifolds. A given property (P) of a compact complex manifold is said to be:*

1. ***open*** *under holomorphic deformations if:*
$$X_0 \text{ has property } (P) \Longrightarrow X_t \text{ has property } (P), \quad \text{for } t \text{ sufficiently close to } 0.$$
2. ***closed*** *under holomorphic deformations if:*
$$X_t \text{ has property } (P) \text{ for } t \neq 0 \Rightarrow X_0 \text{ has property } (P).$$

The following table shows openness and closedness for some special properties:

Metric	Open	Closed
Kähler	✓	$\times$
SKT	$\times$	$\times$
Balanced	$\times$	$\times$
HS	✓	$\times$
sG	✓	$\times$
$\partial\bar{\partial}$	✓	$\times$

We show for instance that the $\partial\bar{\partial}$ and the Kähler properties are open. First, we mention the upper semicontinuity of the Hodge numbers under deformations.

Theorem 1.4.3. *Let $\{X_t\}_{t\in B}$ be a holomorphic family of compact complex manifolds and fix (p,q). Then the following maps are upper semicontinuous:*

$$t \mapsto h^{p,q}(t) := \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(X_t, \mathbb{C}),$$

$$t \mapsto h^{p,q}_{BC}(t) := \dim_{\mathbb{C}} H^{p,q}_{BC}(X_t, \mathbb{C}),$$

$$t \mapsto h^{p,q}_{A}(t) := \dim_{\mathbb{C}} H^{p,q}_{A}(X_t, \mathbb{C}).$$

One main consequence is the deformation openness of the $\partial\bar{\partial}$-property (which was originally proved by C-C. Wu in [Wu06] then by D. Angella and A. Tomassini in [AT13]).

Theorem 1.4.4. *Suppose that $\{X_t\}_{t\in B}$ is a holomorphic family of compact complex manifolds. If the central fiber X_0 satisfies the $\partial\bar{\partial}$-property, then for all t sufficiently close to 0, X_t also satisfies the $\partial\bar{\partial}$-property.*

Sketch of proof. By D. Angella and A. Tomassini (see [AT13], Theorem B), the $\partial\bar{\partial}$-property on X_0 is equivalent to $\sum_{p+q=k}(h^{p,q}_{BC}(0) + h^{p,q}_{A}(0)) = 2b_k$, where b_k is the k-th Betti number of X. By the previous Theorem, we have $h^{p,q}_{BC}(t) \leq h^{p,q}_{BC}(0)$ and $h^{p,q}_{A}(t) \leq h^{p,q}_{A}(0)$. Moreover, we always have: $\sum_{p+q=k}(h^{p,q}_{BC}(t) + h^{p,q}_{A}(t)) \geq 2b_k$, for every $t \in B$ and $k \in \{0, \ldots, 2n\}$. Putting together all the above inequalities, we obtain:

$$2b_k \leq \sum_{p+q=k}(h^{p,q}_{BC}(t) + h^{p,q}_{A}(t)) \leq \sum_{p+q=k}(h^{p,q}_{BC}(0) + h^{p,q}_{A}(0)) = 2b_k,$$

which proves the deformation openness of the $\partial\bar{\partial}$-property. □

In order to prove the deformation openness of the Kähler property, we need the following key ingredient:

Lemma 1.4.5. *Let ω be a Hermitian metric on a compact complex manifold X. Then:*

$$\omega \text{ is Kähler } \iff \Delta_{BC,\omega}\omega = 0$$

Proof. If $\Delta_{BC,\omega}\omega = 0$, then $\partial\omega = \bar{\partial}\omega = 0$, hence ω is Kähler. Conversly, assume that ω is Kähler, so we need to show that $\omega \in \mathrm{Ker}(\partial\bar{\partial})^*$. By definition we have $*^2 = \mathrm{Id}$ on 2-forms, where $*$ is the Hodge star operator induced by the Kähler metric ω. This yields:

$$(\partial\bar{\partial})^* = \bar{\partial}^*\partial^* = - * \partial * (- * \bar{\partial} *) = - * \partial\bar{\partial} *$$

This implies that:

$$(\partial\bar{\partial})^*\omega = - * \partial\bar{\partial} * \omega = -\frac{1}{(n-1)!} * \partial\bar{\partial}\omega^{n-1} = 0$$

since $\bar{\partial}\omega^{n-1} = (n-1)\omega^{n-2} \wedge \bar{\partial}\omega = 0$ by Kählerianity of ω. Hence $\Delta_{BC,\omega}\omega = 0$. □

Theorem 1.4.6 ([KS60]). *The Kähler property is open under holomorphic deformations.*

Sketch of proof. Suppose that ω_0 is a Kähler metric on X_0. For every $t \in B$, let ω_t be the J_t-type $(1,1)$-components of the 2-form ω_0. Then $(\omega_t)_{t\in B}$ is a $\mathscr{C}^\infty$-family of Hermitian metrics for t sufficiently close to 0. Consider the $L^2_{\omega_t}$-orthogonal projectors:

$$F^{1,1}_{BC,t} \colon \mathscr{C}^\infty_{1,1}(X_t, \mathbb{C}) \longrightarrow \mathcal{H}^{1,1}_{BC,t}(X_t, \mathbb{C}), \quad t \in B$$

Using the fact that $h^{1,1}_{BC}(t)$ is constant for small t, the family $\{F^{1,1}_{BC,t}\}$ is $\mathscr{C}^\infty$ close to 0. Then: $\widetilde{\omega}_t = \frac{1}{2}(F^{1,1}_{BC,t}\omega_t + \overline{F^{1,1}_{BC,t}\omega_t})$ defines a $\mathscr{C}^\infty$-family of Kähler metrics on X_t. □

So far, Kobayashi hyperbolicity is the only notion of hyperbolicity that has been investigated under holomorphic deformations. Let $\{X_t\}_{t\in B}$ be a holomorphic family of compact complex manifolds, and define the function $D : B \longrightarrow (0, +\infty]$ as follows:

$$t \longmapsto D(t) := \sup_{f\in\mathcal{O}(\Delta, X_t)} |f'(0)|$$

Theorem 1.4.7 ([Bro78]).

1. *The function D is continuous.*
2. *$D(t) < +\infty \iff X_t$ is Kobayashi hyperbolic.*

In fact, the upper semicontinuity of D has been proven by M. Kalka using $\bar{\partial}$-methods, so R. Brody proved the lower semicontinuity of D.

Corollary 1.4.8 ([Bro78]). *Kobayashi hyperbolicity is open under deformations.*

Chapter 2

Holomorphic Deformations of Compact Kähler Hyperbolic Manifolds

The goal of this paper is to study the deformations of compact Kähler hyperbolic manifolds. We will propose slightly modified versions of Kähler hyperbolicity as a tool to give a first step towards investigating the deformation openness of Gromov's classical notion of Kähler hyperbolicity.

2.1 Introduction

The first notion of complex hyperbolicity was introduced by S. Kobayashi in 1967. It can be seen as a way of generalizing the well-known Schwarz-Pick lemma in complex analysis to complex manifolds (or complex spaces) by defining a special pseudometric attached to the manifold in question, called the *Kobayashi pseudometric*. When this pseudometric is a genuine metric, the manifold is said to be *Kobayashi hyperbolic* [Kob67b]. In the compact setting, R. Brody showed in [Bro78] that Kobayashi hyperbolicity is equivalent to the non-existence of non-constant entire curves. Subsequently, M. Gromov introduced in one of his seminal papers ([Gro91]), a form of hyperbolicity on complex manifolds in terms of Kähler metrics. There, he defined the notion of *Kähler hyperbolicity*, "probably" inspired by the property of *openness at infinity* introduced by D. Sullivan in [Sul76] to give a condition for the existence of an invariant transversal measure on the leaves of a foliation on a compact complex manifold. Several notions of hyperbolicity have been developed recently by generalizing the non-existence of non-constant entire curves to ruling out non-degenerate holomorphic maps from some $\mathbb{C}^k$ (with $2 \leq k \leq \dim_{\mathbb{C}} X - 1$) into the given complex manifold X with some growth condition, or by replacing the Kähler class with other special metrics. We can cite a few: [MP23], [Mar23], [BDET24],

[Ma24], [KP25], [KPU25].

Recall that a compact complex manifold X is called:

- [Gro91]. **Kähler hyperbolic** if it admits a Kähler metric ω which is $\widetilde{d}$-bounded.
- [Mar23]. **SKT hyperbolic** if it admits an SKT metric ω which is $\widetilde{(\partial + \bar{\partial})}$-bounded.

The main purpose of this article is to take a first step towards the study of the deformation openness of Kähler hyperbolicity by introducing two slightly modified versions. A first one called **Strong Kähler hyperbolicity**, this is an hyperbolicity notion pinched right between *Real hyperbolicity* on Kähler manifolds and Gromov's notion of Kähler hyperbolicity. The second one is defined by weakening the boundedness condition of the d-potential of $\widetilde{\omega}$ to some L^p-condition, for $1 \leq p \leq +\infty$, we shall call a manifold that admits such a metric **L^p-Kähler hyperbolic**. In a similar fashion, we will introduce another variant of SKT hyperbolicity, called **L^p-SKT hyperbolicity**.

We collect our deformation openness results in the following statement:

Main Theorem. *Let $\sigma : \mathscr{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds, where B is an open ball around the origin in some $\mathbb{C}^N$. Let $X_t = \sigma^{-1}(t)$, for $t \in B$.*

1. *If X_0 is L^p-Kähler hyperbolic for some $1 \leq p \leq +\infty$, then the fibers X_t are L^p-SKT hyperbolic for all $t \in B$ sufficiently close to 0.*
2. *If X_0 is Kähler hyperbolic and $h^{2,0}(X_0) = 0$, then the Kähler hyperbolicity property is **open** under small deformations.*
3. *If X admits a SKT hyperbolic Kähler metric ω, then ω is L^2-Kähler hyperbolic "almost evrywhere" (in a sense that will be discussed in Section §2.4.2).*

2.2 Some hyperbolicity notions on complex manifolds

Throughout this paper, X will be a compact connected complex manifold of dimension $\dim_{\mathbb{C}} X = n$ unless otherwise stated, and $\pi : \widetilde{X} \longrightarrow X$ its universal cover. In particular, if ω is some Hermitian metric on X, then $\widetilde{\omega} = \pi^*\omega$ is a *complete* Hermitian metric on $\widetilde{X}$ since π respects bidegrees by holomorphicity, and the pullback of a complete metric is also complete. Let us recall some basic definitions:

Definition 2.2.1. *Let X be a compact Kähler manifold. Then:*

1. *[Li19] A Kähler metric ω on X is said to be **Kähler $\widetilde{d}$-exact** if $\widetilde{\omega}$ is exact, i.e. there exists a (globally defined) smooth 1-form η on $\widetilde{X}$ such that $\widetilde{\omega} = d\eta$. If such a*

*metric exists on X, we say that X is a **Kähler $\widetilde{d}$-exact** manifold.*

2. *[Gro91] If moreover the d-potential η in (1) can be chosen to be $\widetilde{\omega}$-bounded (and not necessarily smooth), then ω is said to be $\widetilde{d}$-**bounded**. If such a metric exists on X, we say that X is **Kähler hyperbolic**.*

Remark 2.2.2. 1. One can drop the smoothness assumption on the d-potential of $\widetilde{\omega}$ in the Kähler $\widetilde{d}$-exactness defintion. Indeed, suppose that X admits a Kähler metric ω such that $\widetilde{\omega} = d\eta$, where η is a 1-from which is not necessarily smooth on $\widetilde{X}$. This means that $[\widetilde{\omega}] = 0 \in H^2_{cur}(\widetilde{X}, \mathbb{R})$, where:

$$H^2_{cur}(\widetilde{X}, \mathbb{R}) = \frac{\{\text{real valued closed 2-currents on } X\}}{\{\text{real valued exact 2-currents on } X\}}$$

But since for every degree k we have (Théorème 14, Chapitre IV in [DR55]):

$$H^k_{cur}(\widetilde{X}, \mathbb{R}) \simeq H^k_{DR}(\widetilde{X}, \mathbb{R})$$

we get: $[\widetilde{\omega}] = 0 \in H^2_{DR}(\widetilde{X}, \mathbb{R})$, which means that ω is Kähler $\widetilde{d}$-exact and $\widetilde{\omega}$ admits a smooth d-potential. Moreover, we can recover a smooth bounded d-potential of $\widetilde{\omega}$ in the Kähler hyperbolicity definition using the **heat equation method** (See the end of the proof of Theorem 2.1 in [CY21], p.11-p.12).

2. Clearly Kähler hyperbolicity implies Kähler $\widetilde{d}$-exactness. Note that these two notions depend only on the cohomology class of the Kähler form ω, in the following sense: if one representative of $\{\omega\} \in H^2_{DR}(X)$ is $\widetilde{d}$-exact (resp. $\widetilde{d}$-bounded), then all representatives are $\widetilde{d}$-exact (resp. $\widetilde{d}$-bounded) ([CY18], Lemma 2.4).

This leads to the following definition:

Definition 2.2.3. *[BKS24] A cohomology class $\{\alpha\} \in H^k_{DR}(X, \mathbb{R})$ is **hyperbolic** if α is $\widetilde{d}$-bounded. We denote by $V^k_{hyp}(X) \subset H^k_{DR}(X, \mathbb{R})$ the space of all hyperbolic classes.*

Remark 2.2.4. In the case $k = 2$, the subspace $V^2_{hyp}(X) \subset H^2_{DR}(X, \mathbb{R})$ depends only on the fundamental group of X (see Theorem 5.1 in [Ked09], Theorem 2.4 in [BKS24] or Proposition A.3 in [BDT25] for an alternative proof). This fact has been pointed out to the author by B. Claudon.

Let $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R}) := H^{1,1}_{BC}(X, \mathbb{C}) \cap H^2_{DR}(X, \mathbb{R})$ be the *Kähler cone* of X, namely, the set of all Kähler classes on X. Then:

1. X is Kähler $\widetilde{d}$-exact $\iff \left\{\{\omega\} \in \mathcal{K}_X \,:\, \widetilde{\omega} \in d\left(\mathscr{C}^\infty\left(\widetilde{X}, T^*_{\widetilde{X}}\right)\right)\right\} \neq \emptyset$.
2. X is Kähler hyperbolic $\iff \mathcal{K}_X \cap V^2_{hyp}(X) \neq \emptyset$.

The basic examples of Kähler hyperbolic manifolds are Kähler manifolds homotopic to negatively curved Riemannian manifolds [CY18] and submanifolds of Hermitian locally symmetric manifolds of non-compact type manifolds. Kähler $\widetilde{d}$-exact manifolds include other examples, such as Kähler non-elliptic manifolds; these are Kähler manifolds that admit Kähler metrics ω such that $\widetilde{\omega} = d\eta$ where η is a 1-form with **sublinear growth**, i.e.

$$|\eta|_{\widetilde{\omega}}(x) \leq C(1 + d_{\widetilde{\omega}}(x, x_0)) \ , \qquad \forall x \in \widetilde{X} \tag{2.2.1}$$

for some constant $C > 0$, where $d_{\widetilde{\omega}}$ is the distance induced by $\widetilde{\omega}$ and $x_0 \in \widetilde{X}$ is some fixed base point (see [JZ98, CX01]).

Definition 2.2.5. *Let X be a complex manifold (not necessarily compact), then it is said to be **Kobayashi hyperbolic** if the "Kobayashi pseudo-distance" (as defined in [Kob67a]) is a genuine distance on X.*

R. Brody gave a characterization of Kobayashi in the compact setting [Bro78], answering a couple of questions raised by S. Kobayashi in [Kob74] and [Kob05]. Namely:

Theorem 2.2.6. *Let X be a compact complex manifold. Then:*

$$X \text{ is Kobayashi hyperbolic} \iff \text{Every entire curve in } X \text{ is constant}$$

The latter property has come to be called **Brody hyperbolicity**.

Recently, generalized notions of hyperbolicity were introduced in the non-Kähler context. Here, we recall the definition of some special metrics and their associated hyperbolicity notions.

Definition 2.2.7. *Let X be a compact complex manifold of dimension $\dim_{\mathbb{C}} X = n$. A Hermitian metric ω on X is called:*

1. *[Str86, Bis89] **Strong Kähler with torsion** (**SKT**) or **pluriclosed** if $\partial\bar{\partial}\omega = 0$.*
2. *[Gau77b] **Gauduchon** if $\partial\bar{\partial}\omega^{n-1} = 0$*
3. *[Gau77a, Mic82] **Semi-Kähler** or **Balanced** if $d\omega^{n-1} = 0$.*
4. *[Pop13a] **Strongly Gauduchon** (**sG**) if $\partial\omega^{n-1}$ is $\bar{\partial}$-exact.*

Definition 2.2.8. *Let X be a compact complex manifold of dimension $\dim_{\mathbb{C}} X = n$. We say that X is:*

1. *([MP23],[MP22]) **Balanced hyperbolic** if there exists a balanced metric ω on X such that ω^{n-1} is $\widetilde{d}$-bounded.*
2. *([Mar23]) **SKT hyperbolic** if it admits an SKT metric ω which is $\widetilde{(\partial + \bar{\partial})}$**-bounded**, i.e. $\widetilde{\omega} = \partial\alpha + \bar{\partial}\beta$, where α and β are bounded $(0,1)$ and respectively $(1,0)$-forms*

on $\widetilde{X}$.

3. *([Mar23]* ***Gauduchon hyperbolic*** *if there exists a Gauduchon metric ω on X such that $\widetilde{\omega}^{n-1} = \partial\alpha + \bar{\partial}\beta$, where α and β are bounded $(n-2, n-1)$ and respectively $(n-1, n-2)$-forms on $\widetilde{X}$; that is to say that ω^{n-1} is $\widetilde{(\partial + \bar{\partial})}$-bounded.*

4. *([Ma24])* ***sG hyperbolic*** *if there exists an sG metric ω on X such that ω^{n-1} which is the $(n-1, n-1)$-component of a real d-closed $(2n-2)$-form Ω with $\pi^*\Omega = d\eta$, where η is a bounded $(2n-3)$-form on $\widetilde{X}$.*

M. Gromov mentioned in his paper [Gro91] that Kähler hyperbolic manifolds are Kobayashi hyperbolic. The same result holds for SKT hyperbolic manifolds ([Mar23], Theorem 3.5).

We will now recall some of the basic properties of Kähler $\widetilde{d}$-exact and Kähler hyperbolic manifolds.

Proposition 2.2.9. *Let X be a Kähler $\widetilde{d}$-exact manifold. Then there is no compact complex analytic subset Z of $\widetilde{X}$ of dimension $k = \dim_{\mathbb{C}} Z > 0$. In particular, X admits no simply connected compact complex submanifold Y (such as rational curves).*

Proof. Suppose that such a Z exists, then:

$$\mathrm{Vol}_{\widetilde{\omega}}(Z) = \frac{1}{k!}\int_Z \widetilde{\omega}^k = \frac{1}{k!}\int_Z d(\eta \wedge \widetilde{\omega}^{k-1}) \overset{(*)}{=} 0 \tag{2.2.2}$$

where $(*)$ is due to the compactness of Z. This is a contradiction since $\mathrm{Vol}_{\widetilde{\omega}}(Z) > 0$ by the positivity of the dimension. Now let $f : Y \hookrightarrow X$ be a simply connected submanifold of X, then f lifts to a map $\widetilde{f} : Y \hookrightarrow \widetilde{X}$. So similarly, one gets $f^*\omega = \widetilde{f}^*\widetilde{\omega}^* = 0$, hence f is constant, which is again a contradiction. $\square$

This implies by a result of S. Mori in [Mor79], that a *projective* Kähler $\widetilde{d}$-exact manifold has a nef canonical bundle, due to the absence of rational curves. A rather interesting question was asked by Gromov about the ampleness of the canonical bundle K_X of a Kähler hyperbolic manifold X. This question has been answered positively by B.-L. Chen and X. Yang:

Proposition 2.2.10 ([CY18], Theorem 2.11.)**.** *If X is a compact Kähler hyperbolic manifold, then its canonical bundle K_X is ample.*

This has been done through combining the fact that K_X is big ([Gro91], Corollary 0.4C) and cone theorems from [Mor82] and [Kaw91]. Hence, this gave a partial answer to a conjecture proposed by S. Kobayashi:

Conjecture 2.2.11. (***Kobayashi '70***) *Let X be a compact Kähler (or projective) manifold which is Kobayashi hyperbolic. Then K_X is ample.*

A result that can be deduced from Proposition 2.2.9 is the fact that the fundamental group of a compact Kähler $\widetilde{d}$-exact manifold is *large* in the sense of Kollár (Definition 4.6 in [Kol95]). Namely, we have:

Corollary 2.2.12. *Let X be a Kähler $\widetilde{d}$-exact manifold and $\iota : Z \hookrightarrow X$ an irreducible compact analytic subvariety of dimension $k = \dim_{\mathbb{C}} Z > 0$. Then the image of $\iota_* : \pi_1(Z) \longrightarrow \pi_1(X)$ is infinite. In particular, $\pi_1(X)$ is infinite.*

Proof. If the image of $\iota_* : \pi_1(Z) \longrightarrow \pi_1(X)$ were finite, passing to a finite cover $\nu : \hat{Z} \longrightarrow Z$ yields a map $\iota \circ \nu : \hat{Z} \longrightarrow X$ that lifts to the universal cover $\widetilde{X}$ of X, hence giving rise to a compact subvariety $\widetilde{Z}$ of $\widetilde{X}$ such that $\pi_{|\widetilde{Z}} : \widetilde{Z} \longrightarrow Z$ is finite. This implies that $\int_{\widetilde{Z}}(\pi^*\omega) > 0$, which is a contradiction according to Proposition 2.2.9. □

In particular, this means that the fundamental group of a Kähler hyperbolic manifold is infinite. This gives a partially positive answer to a question raised by S. Kobayashi in [Kob74], asking whether a compact (Kobayashi) hyperbolic complex manifold always has an infinite fundamental group, which turned out later to be false in general, since there are plenty of simply connected examples, mainly given by smooth projective general complete intersections of high degree.

A consequence that can be drawn from Corollary 2.2.12 is that if X is Kähler $\widetilde{d}$-exact, then the volume form on $\widetilde{X}$ admits a d-potential α with sublinear growth in the sense of (2.2.1) (Corollary 1.3, page 3 in [Sik01]). If X is Kähler hyperbolic, we have a little bit more.

Definition 2.2.13. *[Sul76, Gro81] A complete Riemannian manifold (M^m, g) is said to be* ***not closed*** *(or* ***open****)* ***at infinity*** *if it satisfies one of the following equivalent properties:*

1. *M^m admits a* ***bounded*** *volume form $\Omega > 0$ ($C^{-1}dV_g \leq \Omega \leq C dV_g$ for some constant $C > 0$) with bounded d-potential, i.e. $\Omega = d\Gamma$ such that Γ is a bounded $(m-1)$-form with respect to the Riemannian metric g.*
2. *The following* ***linear isoperimetric inequality*** *holds true: There exists a constant $C > 0$ such that for any bounded domain $D \subset M^m$ with $\mathscr{C}^1$-boundary ∂D we have:*

$$Vol_g(D) \leq C\, Area_g(\partial D) \tag{2.2.3}$$

It is easy to see that the universal cover $\widetilde{X}$ of a Kähler hyperbolic manifold (X, ω) is open at infinity, since:

$$\widetilde{\omega}^n = d(\eta \wedge \widetilde{\omega}^{n-1}) \tag{2.2.4}$$

and we have:

$$\|\eta \wedge \widetilde{\omega}^{n-1}\|_{L^\infty_{\widetilde{\omega}}(\widetilde{X})} \leq \|\eta\|_{L^\infty_{\widetilde{\omega}}(\widetilde{X})} \|\omega\|^{n-1}_{L^\infty_\omega(X)} < +\infty \tag{2.2.5}$$

One of the main purposes behind defining this notion of Kähler hyperbolicity was to give a partial affirmative answer to the Hopf-Chern conjecture:

Conjecture 2.2.14. (***Hopf-Chern***, *[Yau82], Problem 10.*) *Let M be a compact Riemannian manifold of real dimension $2n$ of negative sectional curvature. Then:*

$$(-1)^n \chi(M) > 0 \tag{2.2.6}$$

where $\chi(M)$ is the topological Euler characteristic of M.

This has been answered affirmatively in the case of Kähler hyperbolic manifolds (which includes compact Kähler manifolds with negative sectional curvature, see [Gro91]). The main tool to prove this was a lower bound on the spectrum of the Hodge-de Rham Laplace operator $\Delta = dd^* + d^*d$ acting on L^2-forms of degree $k \neq n$. Namely, we have the following spectral gap result:

Theorem 2.2.15. *[Gro91] Let X^n be a complete Kähler manifold equipped with a d-bounded Kähler form ω, i.e. $\omega = d\eta$, where η is a bounded 1-form. Then, there exists a constant $\lambda_0 = \lambda_0(\|\eta\|_{L^\infty_\omega(X)}) > 0$ such that for every L^2-form ψ of degree $k \neq n$ on X, we have:*

$$\langle \Delta\psi, \psi \rangle \geq \lambda_0^2 \|\psi\|^2_{L^2_\omega(X)} \tag{2.2.7}$$

Furthermore, the inequality (2.2.7) still holds for L^2-forms of degree n which are orthogonal to the harmonic n-forms.

In particular, this holds true on the universal cover of a compact Kähler hyperbolic manifold. In a similar fashion, S. Marouani proved this spectral gap when X is a complete Kähler manifold endowed with a $(\partial + \bar{\partial})$-bounded Kähler form, i.e. $\omega = \partial\alpha + \bar{\partial}\beta$, where α and β are bounded (see [Mar23], Theorem 3.19).

2.3 New hyperbolicity notions on compact Kähler manifolds

2.3.1 L^p-Kähler and L^p-SKT hyperbolicity

Now, we want to define a modified version of hyperbolicity on Kähler manifolds by relaxing the boundedness condition on the d-potential of the pullback of the Kähler form ω to the universal cover $\widetilde{X}$. We will also set a similar definition for the SKT case since it will be important for later purposes, specially in Section 4.

Definition 2.3.1. *Let X be a compact Riemannian manifold and let $\pi : \widetilde{X} \longrightarrow X$ be its universal cover. A **fundamental domain** of $\widetilde{X}$ is an open relatively compact subset $D \Subset \widetilde{X}$ that satisfies the following conditions:*

(i) The collection $\{\gamma\overline{D}\}_{\gamma\in\pi_1(X)}$ covers $\widetilde{X}$, i.e. $\bigcup\limits_{\gamma\in\pi_1(X)} \gamma\overline{D} = \widetilde{X}$.

(ii) $\partial D := \overline{D} \setminus D$ has zero measure and $\gamma D \cap D = \emptyset$, for all $\gamma \in \pi_1(X) \setminus \{Id\}$.

Remark 2.3.2. 1. ([MM07]-Section 3.6.1, [Ati76]-Section 3) A way to construct this type of domains is the following: Let $\{U_i\}_{1\leq i\leq N}$ be a finite cover of X by open balls, such that for each i, there exists an open set $\widetilde{U}_i \subset \widetilde{X}$ such that $\pi_{|\widetilde{U}_i} : \widetilde{U}_i \longrightarrow U_i$ is diffeomorphic with inverse $s_i : U_i \longrightarrow \widetilde{U}_i$. Set $V_i = U_i \setminus \left(\bigcup\limits_{j<i}\bar{U}_j \cap U_i\right)$. Then one can check that $D = \bigcup\limits_{1\leq i\leq N} s_i(V_i)$ is a fundamental domain of $\widetilde{X}$.

2. We have: $\pi(\overline{D}) = X$ and $\pi_{|D} : D \longrightarrow X$ is a diffeomorphism onto its image. More precisely, we have the following diagram:

$$\begin{tikzcd} \overline{D} \arrow[r, hook, "\iota"] \arrow[d, "\pi_{\overline{D}}"'] & \widetilde{X} \arrow[d, "\pi"] \\ F \arrow[r, "\simeq", "\phi"'] & X \end{tikzcd}$$

where $F = \overline{D}/\pi_1(X)$, ι is the inclusion map and ϕ is a diffeomorphism (see the proof of Theorem 9.2.4 in [Bea12]).

3. It can be shown that for any Riemannian metric g on X and for any two fundamental domains D and D' of $\widetilde{X}$, one has: $\mathrm{Vol}_{\widetilde{g}}(D) = \mathrm{Vol}_{\widetilde{g}}(D')$ ([Kat92], Theorem 3.1.1). In particular, we have: $\mathrm{Vol}_{\widetilde{g}}(\gamma D) \overset{(a)}{=} \mathrm{Vol}_{\widetilde{g}}(D) \overset{(b)}{=} \mathrm{Vol}_g(X)$, for any fundamental domain D of $\widetilde{X}$ and for any $\gamma \in \pi_1(X)$, where (a) comes from the fact $\pi_1(X)$ acts by isometries on $\widetilde{X}$ (Actually, one can easily check that if D is a fundamental domain

of $\widetilde{X}$, then so is γD, for any $\gamma \in \pi_1(X)$). As for (b), see [Kat92], p.75.

Definition 2.3.3. *Let (X, g) be a compact Riemannian manifold, and let D be a fundamental domain of $\widetilde{X}$. Let α be a k-form on $\widetilde{X}$, and let $1 \leq p \leq +\infty$. We say that α satisfies the **property** $(\boldsymbol{F}_p)$ if:*

$$(F_p)\text{:} \quad \exists C > 0, \quad \sup_{\gamma \in \pi_1(X)} \|\alpha\|_{L^p_{\widetilde{g}}(\gamma D)} \leq C$$

Remark 2.3.4.

1. The property (F_p) does not depend on the choice of the Riemannian metric g on X, since for any two Riemannian metrics g and g', the compactness of X implies the existence of a constant $A > 0$ such that:

$$\frac{1}{A} g' \leq g \leq A g' \Longrightarrow \frac{1}{A}\widetilde{g}' \leq \widetilde{g} \leq A\widetilde{g}'$$

2. The property (F_p) does not depend also on the choice of the fundamental domain. Namely, if α satisfies the property (F_p) relatively to some fundamental domain D of $\widetilde{X}$, then it satisies (F_p) for every other fundamental domain of $\widetilde{X}$. Indeed, let μ_g be the Riemannian volume measure induced by the metric g, and let D' be another fundamental domain of $\widetilde{X}$, then there exists a finite number of sets $\{\gamma D\}_{\gamma \in \pi_1(X)}$ (let us say $\{\gamma_1 D, \ldots, \gamma_k D\}$) such that $D' \underset{\text{a.e}}{\subset} \bigcup_{1 \leq j \leq k} \gamma_j D$ in the sense that:

$$\mu_g\left(\bigcup_{1 \leq j \leq k} \gamma_j D \setminus D'\right) = 0 \tag{2.3.1}$$

Hence, for any $\gamma_X \in \pi_1(X)$ we have:

$$\int_{\gamma_X D'} |\alpha|^p_{\widetilde{g}} \mu_g \leq \int_{\gamma_X\left(\bigcup\limits_{1 \leq j \leq k} \gamma_j D\right)} |\alpha|^p_{\widetilde{g}} \mu_g \leq k \sup_{1 \leq j \leq k} \int_{\gamma_{X,j} D} |\alpha|^p_{\widetilde{g}} \mu_g \tag{2.3.2}$$

where $\gamma_{X,j} = \gamma_X \cdot \gamma_j$. Thus (2.3.2) implies that

$$\sup_{\gamma \in \pi_1(X)} \|\alpha\|_{L^p_{\widetilde{g}}(\gamma D')} \leq C k^{\frac{1}{p}} =: C' > 0 \tag{2.3.3}$$

Which gives us the property (F_p) for α on D' as well.

Remark 2.3.5. Since the fundamental group of a compact manifold is countable (this is due to the fact that every compact topological manifold is homotopy equivalent to a finite CW-complex [Wes77], the latter has a finitely presented fundamental group), and a countable union of sets of measure zero has measure zero, together with the property

(i) of fundamental domains, leads to the following: If D is a fundamental domain of $\widetilde{X}$, then:

$$\sup_{\gamma \in \pi_1(X)} \|\alpha\|_{L^p_g(\gamma D)} = 0 \Longrightarrow \alpha = 0 \quad \text{almost everywhere} \tag{2.3.4}$$

Let us now define the notion of L^p-Kähler hyperbolicity.

Definition 2.3.6. *Let $1 \leq p \leq +\infty$, we say that a compact complex manifold X is:*

1. ***L^p-Kähler hyperbolic** if it admits a Kähler metric ω such that $\widetilde{\omega} = d\eta$, where η is a 1-form (not necessarily smooth) satisfying the property (F_p) on $\widetilde{X}$. Such a metric ω is called **L^p-Kähler hyperbolic**.*
2. ***L^p-SKT hyperbolic** if it admits an SKT metric ω such that $\widetilde{\omega} = \partial\alpha + \bar{\partial}\beta$, where α and β are $(0,1)$ and respectively $(1,0)$-forms (not necessarily smooth) that satisfy the property (F_p) on $\widetilde{X}$. Such a metric ω is called **L^p-SKT hyperbolic**.*

Remark 2.3.7.
1. Note that (when $p = +\infty$), L^∞-Kähler (resp. L^∞-SKT) hyperbolicity is equivalent to the classical notion of Kähler (resp. SKT) hyperbolicity for the reasons we mentioned in Remark 2.3.5.
2. If a manifold X is L^p-Kähler hyperbolic for some $1 \leq p \leq +\infty$, then X is $\widetilde{d}$-exact thanks to Remark 2.2.2.

Now we will try to investigate the relation between Kähler (resp. SKT) hyperbolicity and L^p-Kähler (resp. L^p-SKT) hyperbolicity for $1 \leq p < +\infty$.

Lemma 2.3.8. *Let ω be a Hermitian metric on X and let η be a $\widetilde{\omega}$-bounded k-form on $\widetilde{X}$. Then η has the property (F_p) for every $1 \leq p \leq +\infty$.*

Proof. The case $p = +\infty$ is due to Remark 2.3.7. Assume $p < +\infty$, and let D be a fundamental domain of $\widetilde{X}$ and $\gamma \in \pi_1(X)$. Then:

$$\|\eta\|_{L^p_{\widetilde{\omega}}(\gamma D)} = \left(\int_{\gamma D} |\eta|^p_{\widetilde{\omega}} \widetilde{\omega}_n\right)^{\frac{1}{p}} \leq \|\eta\|_{L^\infty_{\widetilde{\omega}}(\widetilde{X})} (\mathrm{Vol}_{\widetilde{\omega}}(\gamma D))^{\frac{1}{p}} \overset{(*)}{=} \|\eta\|_{L^\infty_{\widetilde{\omega}}(\widetilde{X})} (\mathrm{Vol}_{\omega}(X))^{\frac{1}{p}} =: C \tag{2.3.5}$$

where $C > 0$ is independent of γ and $(*)$ comes from the fact that $\pi_1(X)$ acts on $\widetilde{X}$ by isometries and Remark 2.3.2.(2). Hence η has the property (F_p). □

Corollary 2.3.9. *For any $1 \leq p \leq +\infty$, if α is a d-closed k-form on X such that $\pi^*\alpha$ has a d-potential with the (F_p) property, then for every k-form $\beta \in \{\alpha\}_{DR} \in H^k_{DR}(X, \mathbb{C})$, $\pi^*\beta$ also has a d-potential with the (F_p) property.*

Proposition 2.3.10.
1. *If X is a Kähler (resp. SKT) hyperbolic manifold, then X is L^p-Kähler (resp. L^p-SKT) hyperbolic, for any $1 \leq p \leq +\infty$. More generally, if*

X is L^p-Kähler (resp. L^p-SKT) hyperbolic, then X is L^r-Kähler (resp. L^r-SKT) hyperbolic for any $1 \leq r \leq p$.

2. *Let $1 \leq p \leq +\infty$. If X and Y are compact L^p-Kähler (resp. L^p-SKT) hyperbolic manifolds, then so is $X \times Y$.*

Proof. We are going to do the proof for the Kähler case, and the SKT case can be treated similarly.

1. If X is Kähler hyperbolic, apply Lemma 2.3.8. to get the fact that X is L^p-Kähler hyperbolic for any $1 \leq p < +\infty$. Now let ω be an L^p-Kähler hyperbolic metric on X, and let $q > 0$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then $\frac{p}{r}$ and $\frac{q}{r}$ are conjugate Hölder indices. Hence, Hölder's inequality yields for any $\gamma \in \pi_1(X)$:

$$\int_{\gamma D} |\eta|^r_{\widetilde{\omega}} \widetilde{\omega}_n \leq \left(\int_{\gamma D} |\eta|^p \widetilde{\omega}_n \right)^{\frac{r}{p}} \left(\int_{\gamma D} \widetilde{\omega}_n \right)^{\frac{r}{q}} \tag{2.3.6}$$

$$= \|\eta\|^r_{L^p_{\widetilde{\omega}}(\gamma D)} \mathrm{Vol}_{\omega}^{\frac{r}{q}}(X) \tag{2.3.7}$$

where (2.3.7) comes from the Remark 2.3.2.3. Hence we have:

$$\sup_{\gamma \in \pi_1(X)} \|\eta\|_{L^r_{\widetilde{\omega}}(\gamma D)} \leq \sup_{\gamma \in \pi_1(X)} \|\eta\|_{L^p_{\widetilde{\omega}}(\gamma D)} \mathrm{Vol}_{\omega}^{\frac{1}{r} - \frac{1}{p}}(X) =: C' \tag{2.3.8}$$

where $C' = C \mathrm{Vol}_{\omega}^{\frac{1}{r} - \frac{1}{p}}(X) > 0$ is obviously independent of γ.

2. Let ω_X (resp. ω_Y) be a Kähler metric on X (resp. Y) such that $\widetilde{\omega}_X = d\eta_X$ (resp. $\widetilde{\omega}_Y = d\eta_Y$), where η_X (resp. η_Y) have the (F_p) property with some constant $C_X > 0$ (resp. $C_Y > 0$). Let pr_X (resp. pr_Y) be the projection of $X \times Y$ on X (resp. on Y) and $\pi : \widetilde{X \times Y} \longrightarrow X \times Y$ be the universal cover of $X \times Y$. Then we have the following:

   (a) The projections pr_X and pr_Y lift to projections $\widetilde{pr_X}$ and $\widetilde{pr_Y}$ from $\widetilde{X \times Y}$ to $\widetilde{X}$ and respectively $\widetilde{Y}$ (since $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$ via the isomorphism given by: $[\gamma] \in \pi_1(X \times Y) \longmapsto ([pr_X \circ \gamma], [pr_Y \circ \gamma]) \in \pi_1(X) \times \pi_1(Y)$, and hence $\widetilde{X \times Y} \simeq \widetilde{X} \times \widetilde{Y}$ by unicity of the universal cover). In particular, we have:

$$pr_X \circ \pi = \pi_X \circ \widetilde{pr_X} \qquad \text{and} \qquad pr_Y \circ \pi = \pi_Y \circ \widetilde{pr_Y} \tag{2.3.9}$$

   (b) $\omega = pr_X^* \omega_X + pr_Y^* \omega_Y$ is a Kähler metric on $X \times Y$, and we have:

$$\widetilde{\omega} = \pi^* pr_X^* \omega_X + \pi^* pr_Y^* \omega_Y \overset{(*)}{=} \widetilde{pr_X}^* \widetilde{\omega}_X + \widetilde{pr_Y}^* \widetilde{\omega}_Y = d(\widetilde{pr_X}^* \eta_X + \widetilde{pr_Y}^* \eta_Y) \tag{2.3.10}$$

where $(*)$ comes from the identities in (2.3.9).

(c) If D_X and D_Y are fundamental domains of $\widetilde{X}$ and respectively $\widetilde{Y}$, then one can check that $D = D_X \times D_Y$ is a fundamental domain of $\widetilde{X} \times \widetilde{Y}$, and that for every $\gamma = (\gamma_X, \gamma_Y) \in \pi_1(X) \times \pi_1(Y)$ we have: $\gamma D = (\gamma_X D_X) \times (\gamma_Y D_Y)$. Hence, we get:

$$\|\widetilde{pr_X}^* \eta_X + \widetilde{pr_Y}^* \eta_Y\|_{L^p_{\widetilde{\omega}}(\gamma D)} \leq \|\widetilde{pr_X}^* \eta_X\|_{L^p_{\widetilde{\omega}}(\gamma D)} + \|\widetilde{pr_Y}^* \eta_Y\|_{L^p_{\widetilde{\omega}}(\gamma D)} \tag{2.3.11}$$

$$= \|\eta_X\|_{L^p_{\widetilde{\omega}_X}(\gamma_X D_X)} + \|\eta_Y\|_{L^p_{\widetilde{\omega}_Y}(\gamma_Y D_Y)} \tag{2.3.12}$$

Thus: $\sup_{\gamma \in \pi_1(X \times Y)} \|\widetilde{pr_X}^* \eta_X + \widetilde{pr_Y}^* \eta_Y\|_{L^p_{\widetilde{\omega}}(\gamma D)} \leq C_X + C_Y =: C > 0$, which means that the d-potential of $\widetilde{\omega}$ given in (2.3.10) satisfies the (F_p) property (which does not depend on the choice of the fundamental domain of $\widetilde{X} \times \widetilde{Y}$ by Remark 2.3.4). Hence $X \times Y$ is L^p-Kähler hyperbolic.

□

Note that for compact Riemann surfaces, this notion of L^p-Kähler hyperbolicity is not a new hyperbolicity notion. We already know that Kähler hyperbolicty implies Kobayashi/Brody hyperbolicity, which in turns implies negative curvature in dimension 1, which give us back Kähler hyperbolicity. Recall that we proved that Kähler hyperbolicty implies L^p-Kähler hyperbolicity in Proposition 2.3.10. Now, we show the following:

Proposition 2.3.11. *Let X be a compact Riemann surface, and let $1 \leq p \leq +\infty$. Then:*

$$X \text{ is } L^p\text{-Kähler hyperbolic} \implies X \text{ is Brody hyperbolic}$$

By Proposition 2.3.10, it is sufficient to show this result for $p = 1$. The proof uses the celebrated *uniformization theorem* established at the beginning of the 20th century by H. Poincaré [Poi08] and P. Koebe [Koe07] independently. It states that every simply connected Riemann surface is biholomorphic to the Riemann sphere $\mathbb{P}^1$, the complex plane $\mathbb{C}$, or the unit disc $\mathbb{D}$.

Proof. To show that X is Brody hyperbolic, we shall argue that the only possibility for the universal cover $\widetilde{X}$ of X is to be biholomorphic to $\mathbb{D}$.

First, we can easily rule out the case $\widetilde{X} \simeq \mathbb{P}^1$, since a Kähler form on $\mathbb{P}^1$ cannot be exact (because $\mathbb{P}^1$ is compact).

It remains to show that $\widetilde{X}$ cannot be biholomorphic to $\mathbb{C}$, i.e. X is not a complex torus. Assume that this is the case, then by the **Hodge isomorphism theorem**, one has: $H^{1,1}(X, \mathbb{R}) \simeq \mathcal{H}^{1,1}(X, \mathbb{R})$. On the other hand, note that a (1,1)-form α is of top degree (since $\dim_{\mathbb{C}} X = 1$), so it is proportional to the standard Kähler metric ω_0 on the torus X (the one induced by the euclidean metric on $\mathbb{C}$), i.e. $\alpha = f\omega_0$ for some $f \in \mathscr{C}^\infty(X)$. We consider Δ_0'' to be the $\bar{\partial}$-Laplacian induced by ω_0, then:

$$\Delta_0''\alpha = \Delta_0''(f\omega_0) = (\Delta_0''f)\omega_0$$

since the Lefschetz operator $(\cdot \longmapsto \omega_0 \wedge \cdot)$ commutes with the $\bar{\partial}$-Laplacian on Kähler manifolds. Hence α is harmonic if and only if f is harmonic. But since X is compact, f must be constant by the maximum principle. In particular, (1,1)-classes on X are in a one-to-one correspondence with constant hermitian forms on $\mathbb{C}$.

Thanks to Corollary 2.3.9, we only need to show that there is no Δ_0''-harmonic Kähler metric ω such that $\widetilde{\omega} = d\eta$, where η satisfies the (F_1) property. We may write:

$$\widetilde{\omega} = c\frac{i}{2}dz \wedge d\bar{z} = c\,dx \wedge dy \tag{2.3.13}$$

where $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$. We can assume (without loss of generality) that $c = 1$, and that the fundamental group $\pi_1(X) \simeq \mathbb{Z}^2$ is the standard lattice in $\mathbb{C}$ generated by $\gamma_1 = (1, 0)$ and $\gamma_2 = (0, 1)$. One can write $\widetilde{\omega}$ in polar coordinates (r, θ):

$$\widetilde{\omega} = d(r\cos\theta) \wedge d(r\sin\theta) = r dr \wedge d\theta = dr \wedge d\sigma \tag{2.3.14}$$

where $d\sigma = r d\theta$ is a 1-form on $\mathbb{C}$ that induces a length measure on the circles $\mathbb{S}_r \subset \mathbb{C}$. Namely, we have:

$$L_{\widetilde{\omega}}(\mathbb{S}_r) := \int_{\mathbb{S}_r} d\sigma_{|\mathbb{S}_r} > 0\ , \qquad \forall r > 0 \tag{2.3.15}$$

In particular, one can express η on circles as follows:

$$\eta_{|\mathbb{S}_r} = f_r(\theta) d\sigma_{|\mathbb{S}_r}\ , \qquad \forall r > 0\,, \quad \forall 0 \leq \theta < 2\pi \tag{2.3.16}$$

Simple computations yields that $|d\sigma|_{\widetilde{\omega}}(z) = 1$, $\forall z \in \mathbb{C}$. In particular, $|\eta_{|\mathbb{S}_r}|_{\widetilde{\omega}} = |f_r(\theta)|$, for all $r > 0$. Now, we measure the volume growth of the disks in $\mathbb{C}$. Using Stokes' theorem, we get for every $r > 0$:

$$A_{\widetilde{\omega}}(\mathbb{D}_r) = \int_{\mathbb{D}_r} \widetilde{\omega} = \int_{\mathbb{D}_r} d\eta = \int_{\mathbb{S}_r} \eta = \int_{\mathbb{S}_r} f_r(\theta) d\sigma_{|\mathbb{S}_r} \leq \int_{\mathbb{S}_r} |\eta|_{\widetilde{\omega}} d\sigma_{|\mathbb{S}_r} \tag{2.3.17}$$

This implies that for every $R > 0$:

$$F(R) := \int_0^R A_{\widetilde{\omega}}(\mathbb{D}_t) dt \leq \int_0^R \left(\int_{\mathbb{S}_t} |\eta|_{\widetilde{\omega}} d\sigma_{|\mathbb{S}_t} \right) dt = \int_{\mathbb{D}_R} |\eta|_{\widetilde{\omega}} \widetilde{\omega} \tag{2.3.18}$$

We denote by $C_{2r} := \{(x,y) \in \mathbb{R}^2 : \max\{|x|,|y|\} \leq r\}$ the square of side length $2r > 0$ with center the origin, and let D be the unit square passing through the origin (which is a fundamental domain of $\mathbb{C}$ for the prescribed action of $\pi_1(X)$). Then obviously $\mathbb{D}_r \subset C_{2r}$ for any $r > 0$, hence we get:

$$F(R) \leq \int_{\mathbb{D}_R} |\eta|_{\widetilde{\omega}} \widetilde{\omega} \leq \int_{C_{2R}} |\eta|_{\widetilde{\omega}} \widetilde{\omega} \leq N_R \sup_{\gamma \in \Gamma_R} \int_{D+\gamma} |\eta|_{\widetilde{\omega}} \widetilde{\omega} \ , \qquad \forall R > 0 \tag{2.3.19}$$

where $\Gamma_R = \{\gamma \in \pi_1(X) : (D+\gamma) \cap C_{2R} \neq \emptyset\}$, and $N_R = |\Gamma_R|$. Then $N_R = 4\lceil R \rceil^2$, $\forall R > 0$, where $\lceil \cdot \rceil$ is the *upper integer part* function. On the other hand $\sup_{\gamma \in \Gamma_R} \int_{D+\gamma} |\eta|_{\widetilde{\omega}} \widetilde{\omega} \leq C$, for some constant $C > 0$, since η satisies the (F_1) property. But, we know that $A_{\widetilde{\omega}}(\mathbb{D}_t) = \pi t^2$, for any $t \geq 0$, thus (2.3.19) becomes:

$$F(R) = \int_0^R \pi t^2 dt = \frac{\pi}{3} R^3 \leq 4C \lceil R \rceil^2 \ , \qquad \forall R > 0 \tag{2.3.20}$$

This is a contradiction. Hence, $\widetilde{X} \not\simeq \mathbb{C}$, so $\widetilde{X}$ is necessarily biholomorphic to $\mathbb{D}$, which means that X is Brody hyperbolic. □

Similar arguments can be used to show that complex tori $\mathbb{C}^n/\Lambda$ of any dimension n, are not L^p-Kähler hyperbolic for any $1 \leq p \leq +\infty$. However, it is not obvious for us whether L^p-Kähler hyperbolicity still implies Brody hyperbolicity in higher dimensions.

At this point, it is also difficult to find examples of L^p-Kähler hyperbolic manifolds for $p < +\infty$ other than the Kähler hyperbolic ones.

Problem 1. Let X be an L^p-Kähler hyperbolic manifold. Is X of general type ?

A potential way to answer this question is to prove the existence of a nonzero L^q-section of the pullback of K_X to the universal cover $\widetilde{X}$, for some $q > 0$ (Corollary 13.10, [Kol95]).

A positive answer to Problem 1 implies that K_X is in fact ample, since a Kähler $\widetilde{d}$-exact manifold of general type has ample canonical bundle thanks to Theorem 2.8, [Li19]. If one can show that L^p-Kähler hyperbolic manifolds are indeed Kobayashi/Brody hyperbolic, then answering Problem 1 means having a larger class of compact Kähler Kobayashi hyperbolic manifolds satisfying the Kobayashi conjecture 2.2.11.

2.3.2 Strong Kähler hyperbolicity

In this part, we will define a stronger hyperbolicity notion on compact Kähler manifolds which, despite the lack of examples, is interesting from the deformation theoretic point of view. We will start by recalling some basic definitions from [Gro87].

Let (M, d) be a metric space, and fix a reference point $x_0 \in M$. One defines the following *inner product* on M (also called the **Gromov product**):

$$(x, y)_{x_0} := \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y)) \ , \qquad \forall x, y \in M \tag{2.3.21}$$

Definition 2.3.12. *We say that a metric space (M, d) is δ-**hyperbolic** if there exists a base point $x_0 \in M$, and a constant $\delta \geq 0$ such that the following inequality holds for the inner product defined in (2.3.21):*

$$(x, y)_{x_0} \geq \min\{(x, z)_{x_0}, (y, z)_{x_0}\} - \delta \ , \qquad \forall x, y, z \in M \tag{2.3.22}$$

Remark 2.3.13. (Corollary 1.1.B, [Gro87]) If the inequality (2.3.22) is satisfied for some $x_0 \in M$ with constant δ, then a similar inequality holds for any base point $x \in M$ with constant 2δ. Hence, the hyperbolicity of (M, d) does not depend on the choice of the base point.

Now, let Γ be a finite-type group and let $S \subset \Gamma$ be a finite set generating Γ. We define the **length** of an element $\gamma \in \Gamma$ with respect to S as:

$$\|\gamma\|_S := \min\{k \in \mathbb{N} : \gamma = s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k}, s_i \in S, \varepsilon_i = \pm 1, \forall 1 \leq i \leq k\} \tag{2.3.23}$$

This gives a distance on Γ defined by:

$$d_S(\gamma_1, \gamma_2) := \|\gamma_2^{-1}\gamma_1\|_S \ , \qquad \forall \gamma_1, \gamma_2 \in \Gamma \tag{2.3.24}$$

Remark 2.3.14. If S and S' are two finite sets generating a finite-type group Γ, then the metric spaces (Γ, d_S) and $(\Gamma, d_{S'})$ are quasi-isometric (see [Ghy89], p.2 for instance).

Definition 2.3.15. *A finite-type group Γ is **hyperbolic** (or **Gromov hyperbolic**) if (Γ, d_S) is hyperbolic as a metric space in the sense of Definition 2.3.12 for some finite generating set S of Γ.*

More generally, one can associate to a finite-type group Γ with finite generating set S a **Cayley Graph** $\mathcal{G}(\Gamma, S)$. This is the graph whose vertices are the elements of Γ, and where an edge connects γ_1 to γ_2 if $d_S(\gamma_1, \gamma_2)$. This graph can be endowed with a natural metric, invariant under Γ, in which each edge is isometric to an interval of length 1 in $\mathbb{R}$, and such that the distance between two points is the shortest length of a path connecting them. Then the metric spaces (Γ, d_S) and $\mathcal{G}(\Gamma, S)$ are quasi-isometric by definition. In particular:

$$(\Gamma, d_S) \text{ is hyperbolic} \iff \mathcal{G}(\Gamma, S) \text{ is hyperbolic}$$

Remark 2.3.16. Definition 2.3.15 does not depend on the choice of the finite generating set of Γ by Remark 2.3.14, since hyperbolicity is a quasi-isometry invariant (see [Ghy89], Corollaire p.13 or [GdlH90], Théorème 12).

There are several definitions for hyperbolicity in the literature, that can be found in the previously mentioned sources. For instance, the inequality (2.3.22) ensures that on a *geodesic* metric space (M, d) (for example, the Cayley graph of finite type group), all the triangles are **δ-thin** for some constant $\delta \geq 0$. This means that for any points $x, y, z \in M$, and any choice of segments $[x, y]$, $[y, x]$ and $[z, x]$, we have the following property: Any point of $[x, y]$ is at most at a distance $\delta \geq 0$ from some point in $[y, z] \cup [z, x]$. The proof of the equivalence of definitions for hyperbolicity can be found in detail in [ABC+91], Chapter 2.

Definition 2.3.17. *We say that a compact Kähler manifold X is* ***strongly Kähler hyperbolic*** *(**SKH** for short) if it is Kähler $\widetilde{d}$-exact and $\pi_1(X)$ is hyperbolic.*

Remark 2.3.18. It is not hard to see that the Kähler $\widetilde{d}$-exactness condition is satisfied by X if $\pi_2(X)$ is torsion (Remark 1.8, [CY21]). Indeed, the universal covering map induces an isomorphism of homotopy groups $\pi_k(\widetilde{X}) \simeq \pi_k(X)$, $\forall k \geq 2$ since every map $f : \mathbb{S}^k \longrightarrow X$ with $k \geq 2$ lift to a map $\widetilde{f} : \mathbb{S}^k \longrightarrow \widetilde{X}$ by simple connectedness of $\mathbb{S}^k$ (which is not the case if $k = 1$). In particular, $\pi_2(\widetilde{X})$ is torsion. This implies that $H_2(\widetilde{X}, \mathbb{Z})$ is torsion by the Hurewicz theorem ($H_2(\widetilde{X}, \mathbb{Z}) \simeq \pi_2(\widetilde{X})$ since $\widetilde{X}$ is simply connected), which means that $H_2(\widetilde{X}, \mathbb{R}) = 0$. The universal coefficient theorem implies that $H^2(\widetilde{X}, \mathbb{R}) = 0$. By de Rham's theorem, we get that $H^2_{DR}(\widetilde{X}, \mathbb{R}) = 0$. This means that every closed 2-form on $\widetilde{X}$ is exact. In particular, any Kähler metric ω on X is $\widetilde{d}$-exact.

On the other hand, B-L. Chen and X. Yang proved that SKH manifolds are Kähler hyperbolic (Corollary 1.7, [CY21]). In particualr, the SKH property is pinched between *real hyperbolicity* (i.e. $\pi_1(X)$ is hyperbolic and $\pi_2(X) = 0$) for Kähler manifolds and Kähler hyperbolicity. Namely, we have:

Proposition 2.3.19. *Let X be a compact Kähler manifold. Then:*

$$X \textit{ is real hyperbolic} \Longrightarrow X \textit{ is SKH} \Longrightarrow X \textit{ is Kähler hyperbolic}$$

The following examples were kindly communicated to the author by P. Eyssidieux.

Examples 2.3.20. 1. The only known class of examples of Kähler groups (fundamental groups of compact Kähler manifolds) that are hyperbolic is the class of fundamental groups of negatively curved Kähler manifolds. These manifolds are known to be Kähler hyperbolic as mentioned above (hence Kähler $\widetilde{d}$-exact). In particular, these are SKH manifolds.

2. In complex dimension 1, Kähler hyperbolicity and the SKH property are clearly the same. However, this does not hold in higher dimensions (that is the left implication in Proposition 2.3.19 is strict). An example of a Kähler hyperbolic manifold which is not SKH is the following: Let $X = C_1 \times C_2$ be the product of two smooth curves of genus at least 2. It is obvious that X is Kähler hyperbolic since both C_1 and C_2 are Kähler hyperbolic. But $\pi_1(X)$ contains $\mathbb{Z}^2$, which means that for any $\delta \geq 0$, there are triangles that are not δ-**thin** in $\pi_1(X)$. Hence $\pi_1(X)$ is not Gromov hyperbolic and X is not SKH.

2.4 Deformations openness results

This section will be dedicated to the study of the deformations of some of the notions mentioned above. Let us first recall some fundamental results in deformation theory:

Definition 2.4.1. *A* ***holomorphic family of deformations*** *of a compact complex manifold X is a complex analytic family $(\mathscr{X}, B, \sigma)$, where $\mathscr{X}$ and B are complex manifolds and $\sigma : \mathscr{X} \longrightarrow B$ is a proper holomorphic submersion such that $X_0 := \sigma^{-1}(0) = X$, for a fixed base point $0 \in B$ (B is in general an open ball around the origin in some $\mathbb{C}^N$).*

The study of deformation openness for Kähler $\widetilde{d}$-exact and L^p-Kähler hyperbolic manifolds is motivated by two facts:

The first fact is that the Kähler property is open under holomorphic deformations:

Theorem 2.4.2. *([KS60], Theorem 15) If a fiber X_0 in a holomorphic family of compact complex manifolds is Kähler, then all the nearby fibers X_t are also Kähler. Moreover, any Kähler metric ω_0 on X_0 can be deformed to a $\mathscr{C}^\infty$ family of Kähler metrics ω_t on the nearby fibers X_t.*

And the second fact is that deformation openness results have been proved for some complex hyperbolicity notions, namely:

Theorem 2.4.3. *([Bro78], Theorem 3.1) The Kobayashi hyperbolicity property is open under holomorphic deformations.*

Theorem 2.4.4. *([Ma24], Theorem 2.4) The strongly Gauduchon hyperbolicity property is open under holomorphic deformations.*

Recall that by Ehresmann's Theorem [Ehr47], every holomorphic family of compact complex manifolds is locally $\mathscr{C}^\infty$ trivial, i.e. the neighboring fibers are diffeomorphic. In particular, for every degree k we have: $H^k_{DR}(X_t, \mathbb{C}) \simeq H^k_{DR}(X_0, \mathbb{C})$, for t close to 0 (and for all $t \in B$ if B is contractible). So it makes sense to define the following:

Definition 2.4.5. *Let (X, ω) be a compact Kähler manifold. We say that a deformation $X_t = \sigma^{-1}(t)$ of X is **polarized** by $\{\omega\}_{BC}$ if $\{\omega\}_{DR} \in H^2_{DR}(X)$ defines a Kähler class (hence remains of type $(1,1)$) for the complex structure of X_t.*

A direct application of Remark 2.2.2 and Corollary 2.3.9 yields our first deformation openness result regarding **polarized deformations**:

Proposition 2.4.6. *Let X be a compact Kähler manifold and $(\mathscr{X}, B, \sigma)$ be a holomorphic family of deformations of X.*

1. *If X is Kähler $\widetilde{d}$-exact, then so is every fiber X_t polarized by the fixed Kähler $\widetilde{d}$-exact class on X.*
2. *If X is L^p-Kähler hyperbolic for some $1 \leq p \leq +\infty$, then so is every fiber X_t polarized by the fixed L^p-Kähler hyperbolic class on X.*

Remark 2.4.7. If moreover X has a Gromov hyperbolic fundamental group in the first part of the previous proposition, i.e. X is SKH, then the polarized fibers X_t are also SKH since $X_t \underset{\mathscr{C}^\infty}{\simeq} X$, in particular $\pi_1(X_t) \simeq \pi_1(X)$.

2.4.1 Deformation of L^p-Kähler hyperbolic manifolds

Now we are going to deal with this deformation openness problem in its full generality. But before that, let us recall one important characterization of Kähler metrics. The following material can be found in [KS60] or [Sch07] (see also [Pop15] for some proofs).

Definition 2.4.8. *We define the 4^{th} order **Bott-Chern Laplacian** as follows:*

$$\Delta_{BC} = \partial^*\partial + \bar{\partial}^*\bar{\partial} + (\partial\bar{\partial})^*(\partial\bar{\partial}) + (\partial\bar{\partial})(\partial\bar{\partial})^* + (\partial^*\bar{\partial})^*(\partial^*\bar{\partial}) + (\partial^*\bar{\partial})(\partial^*\bar{\partial})^*$$

This Laplacian is elliptic and formally self-adjoint, so it induces the following orthogonal decomposition on any compact Hermitian manifold X for any $r, s \in \{0, \dots, n\}$:

$$\mathscr{C}^\infty_{r,s}(X) = \mathcal{H}^{r,s}_{BC}(X) \oplus \partial\bar{\partial}(\mathscr{C}^\infty_{r-1,s-1}(X)) \oplus (\partial^*(\mathscr{C}^\infty_{r+1,s}(X) + \bar{\partial}^*(\mathscr{C}^\infty_{r,s+1}(X)) \tag{2.4.1}$$

where $\mathcal{H}^{r,s}_{BC}(X) = \mathrm{Ker}\Delta_{BC} \cap \mathscr{C}^\infty_{r,s}(X)$ has the following decomposition:

$$\mathrm{Ker}\Delta_{BC} = \mathrm{Ker}\partial \cap \mathrm{Ker}\bar{\partial} \cap \mathrm{Ker}(\partial\bar{\partial})^* \tag{2.4.2}$$

Hence we have the following:

Lemma 2.4.9. *([Pop22], Lemma 2.6.5) Let ω be a Hermitian metric on a compact complex manifold X. Then:*

$$\omega \text{ is Kähler} \iff \Delta_{BC}\omega = 0 \tag{2.4.3}$$

Proof. ($\Longleftarrow$) If $\Delta_{BC}\omega = 0$, then (2.4.2) implies that $\partial\omega = \bar{\partial}\omega = 0$, hence ω is Kähler.

($\Longrightarrow$) Now assume that ω is Kähler, so we need to show that $\omega \in \mathrm{Ker}(\partial\bar{\partial})^*$ again by (2.4.2). By definition we have $*^2 = \mathrm{Id}$ on 2-forms, where $*$ is the Hodge star operator induced by the Kähler metric ω. This yields:

$$(\partial\bar{\partial})^* = \bar{\partial}^*\partial^* = - * \partial * (- * \bar{\partial} *) = - * \partial\bar{\partial} * \tag{2.4.4}$$

This implies that:

$$(\partial\bar{\partial})^*\omega = - * \partial\bar{\partial} * \omega = -\frac{1}{(n-1)!} * \partial\bar{\partial}\omega^{n-1} = 0 \tag{2.4.5}$$

since $\bar{\partial}\omega^{n-1} = (n-1)\omega^{n-2} \wedge \bar{\partial}\omega = 0$ by Kählerianity of ω. Hence $\Delta_{BC}\omega = 0$. $\square$

We have then the following deformation result:

Theorem 2.4.10. *Let $\sigma : \mathscr{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds $X_t = \sigma^{-1}(t)$, with $t \in B$. If X_0 is L^p-Kähler hyperbolic for some $1 \leq p \leq +\infty$, then the nearby fibers are L^p-SKT hyperbolic.*

Proof. Let U be a neighborhood of $0 \in B$ for which $\{X_t\}_{t\in U}$ are all Kähler manifolds, and let ω_0 be an L^p-Kähler hyperbolic metric on X_0, i.e. $\widetilde{\omega}_0 = d\eta$, where η is a 1-form satisfying the property (F_p) on some fundamental domain $D \Subset \widetilde{X}_0$. Let $\{\gamma_t\}_{t\in U}$ be a $\mathscr{C}^\infty$-family of Kähler metrics on $\{X_t\}_{t\in U}$ given by the Kodaira-Spencer theorem (Theorem 2.4.2) such that $\gamma_0 = \omega_0$. Then the induced complex structure $\mathcal{J}_t$ gives the following decomposition for ω_0 (seen as a 2-form) on the fiber X_t:

$$\omega_0 = (\omega_0)^{2,0}_{\mathcal{J}_t} + (\omega_0)^{1,1}_{\mathcal{J}_t} + (\omega_0)^{0,2}_{\mathcal{J}_t}\,, \qquad \forall t \in U \tag{2.4.6}$$

As a generalization of Proposition 8 in [KS60], one can prove that the Hodge numbers on every fiber X_t satisfy: $h^{r,s}(X_t) = h^{r,s}(X_0)$, for any $t \in U$ (see Theorem 2.6.4 in [Pop22] for the proof), then $\omega_t^{r,s} := F_t^{r,s}\left((\omega_0)^{r,s}_{\mathcal{J}_t}\right)$ form $\mathscr{C}^\infty$-families of forms of bidegree (r,s) on X_t (with $r+s=2$), where $F_t^{r,s} : \mathscr{C}^\infty_{r,s}(X) \longrightarrow \mathcal{H}^{r,s}_{BC}(X_t,\mathbb{C})$ are the $L^2_{\gamma_t}$-orthogonal projections (that vary smoothly with respect to $t \in U$ by Theorem 5 in [KS60]) onto the Bott-Chern harmonic spaces $\mathcal{H}^{r,s}_{BC}(X_t,\mathbb{C})$. Then $\{\omega_t^{1,1}\}_{t\in U}$ is a new $\mathscr{C}^\infty$-family of Kähler metrics on

X_t (up to shrinking U about 0 thanks to Lemma 2.4.9) such that $\omega_0^{1,1} = \omega_0$. On the other hand, the Hodge decomposition with respect to any fiber X_t (with $t \in U$) holds:

$$\{\omega_0\}_{DR} \in H^2(X_0, \mathbb{C}) \simeq H^2(X_t, \mathbb{C}) \simeq H^{2,0}_{BC}(X_t, \mathbb{C}) \oplus H^{1,1}_{BC}(X_t, \mathbb{C}) \oplus H^{0,2}_{BC}(X_t, \mathbb{C}) \quad (2.4.7)$$

Thus, for any $t \in U$ we have:

$$\omega_0 = \omega_t^{2,0} + \omega_t^{1,1} + \omega_t^{0,2} + u_t \,, \qquad u_t \in \mathscr{C}_2^\infty(X_t) \cap \operatorname{Im} d \quad (2.4.8)$$

Clearly, the family $\{u_t\}_{t \in U}$ is smooth, due to the smoothness of the other families of forms involved in this decomposition. So one can consider, for each $t \in U$, the L^2-minimal solution of the equation:

$$dv_t = u_t \;, \qquad t \in U \quad (2.4.9)$$

This is given by the formula: $v_t = d_t^* \Delta_t^{-1} u_t$, for all $t \in U$. Then, this family varies smoothly with respect to t again thanks to Theorem 5 in [KS60]. Hence, the decomposition (2.4.8) becomes:

$$\omega_0 = \omega_t^{2,0} + \omega_t^{1,1} + \omega_t^{0,2} + dv_t \;, \qquad v_t \in \mathscr{C}_1^\infty(X_t) \quad (2.4.10)$$

Pulling back to the universal cover, we get:

$$\widetilde{\omega}_0 := \pi^* \omega_0 = \pi^* \omega_t^{2,0} + \pi^* \omega_t^{1,1} + \pi^* \omega_t^{0,2} + d\pi^* v_t = d\eta \quad (2.4.11)$$

where π denotes the universal covering map $\widetilde{X_t} \longrightarrow X_t$ (by abuse of notation, since these fibers X_t are diffeomorphic, hence $\widetilde{X}_t \underset{\mathscr{C}^\infty}{\simeq} \widetilde{X}_0$). By taking the $(1,1)$-part in (2.4.11) we get:

$$\widetilde{\omega}_t^{1,1} := \pi_t^* \omega_t^{1,1} = \partial_t(\eta_t^{0,1} - \pi^* v_t^{0,1}) + \bar{\partial}_t(\eta_t^{1,0} - \pi^* v_t^{1,0}) \,, \qquad \forall t \in U \quad (2.4.12)$$

Put $\alpha_t := \eta_t^{0,1} - \pi^* v_t^{0,1}$ and $\beta_t := \eta_t^{1,0} - \pi^* v_t^{1,0}$, then

$$\widetilde{\omega}_t^{1,1} = \partial_t \alpha_t + \bar{\partial}_t \beta_t \quad (2.4.13)$$

If $p = +\infty$, then α_t and β_t satisfy the following inequalities for any $\gamma \in \pi_1(X_0)$:

$$\begin{cases} \|\alpha_t\|_{L^\infty_{\widetilde{\omega}_t^{1,1}}(\widetilde{X}_t)} \leq \|\eta_t^{0,1}\|_{L^\infty_{\widetilde{\omega}_t^{1,1}}(\widetilde{X}_t)} + \|\pi^* v_t^{0,1}\|_{L^\infty_{\widetilde{\omega}_t^{1,1}}(\widetilde{X}_t)} \leq C_t \|\eta\|_{L^\infty_{\widetilde{\omega}_0}(\widetilde{X}_0)} + \|v_t\|_{L^\infty_{\omega_t^{1,1}}(X)} < \infty \\ \|\beta_t\|_{L^\infty_{\widetilde{\omega}_t^{1,1}}(\widetilde{X}_t)} \leq \|\eta_t^{1,0}\|_{L^\infty_{\widetilde{\omega}_t^{1,1}}(\widetilde{X}_t)} + \|\pi^* v_t^{1,0}\|_{L^\infty_{\widetilde{\omega}_t^{1,1}}(\widetilde{X}_t)} \leq C_t \|\eta\|_{L^\infty_{\widetilde{\omega}_0}(\widetilde{X}_0)} + \|v_t\|_{L^\infty_{\omega_t^{1,1}}(X)} < \infty \end{cases}$$

where X is the differentiable manifold that underlies all the Kähler fibers X_t, and the constant $C_t > 0$ depends on the Kähler metric $\omega_t^{1,1}$ on the compact Kähler fiber X_t. Hence, the fibers X_t are SKT hyperbolic.

If $1 \leq p < +\infty$, then the following inequalities hold:

$$\begin{cases} \|\alpha_t\|_{L^p_{\tilde{\omega}_t^{1,1}}(\gamma D)} & \leq \|\eta_t^{0,1}\|_{L^p_{\tilde{\omega}_t^{1,1}}(\gamma D)} + \|\pi^* v_t^{0,1}\|_{L^p_{\tilde{\omega}_t^{1,1}}(\gamma D)} \\ & \leq C_t' \|\eta\|_{L^p_{\tilde{\omega}_0}(\gamma D)} + \|v_t\|_{L^\infty_{\omega_t^{1,1}}(X)} \mathrm{Vol}^{\frac{1}{p}}_{\tilde{\omega}_t^{1,1}}(X) \leq C_t'' \\ \|\beta_t\|_{L^p_{\tilde{\omega}_t^{1,1}}(\gamma D)} & \leq \|\eta_t^{1,0}\|_{L^p_{\tilde{\omega}_t^{1,1}}(\gamma D)} + \|\pi^* v_t^{1,0}\|_{L^p_{\tilde{\omega}_t^{1,1}}(\gamma D)} \\ & \leq C_t' \|\eta\|_{L^p_{\tilde{\omega}_0}(\gamma D)} + \|v_t\|_{L^\infty_{\omega_t^{1,1}}(X)} \mathrm{Vol}^{\frac{1}{p}}_{\tilde{\omega}_t^{1,1}}(X) \leq C_t'' \end{cases}$$

where $C_t' > 0$ depends on the Kähler metric $\omega_t^{1,1}$ on X_t, and hence $C_t'' > 0$ does not depend on γ. Hence, the fibers X_t are L^p-SKT hyperbolic. □

2.4.2 Deformation of Kähler hyperbolic manifolds

In this section, we will restrict ourselves to the study of deformations of Kähler hyperbolic manifolds. Based on Observation 4.4 in [Pop13a]), one may conjecture the following:

Conjecture 2.4.11. *Let X be a compact complex surface. Then:*

$$X \text{ is Kähler hyperbolic} \iff X \text{ is sG hyperbolic.}$$

If this can be proved, Theorem 2.4.4 would imply that Kähler hyperbolicity is **open** under holomorphic deformations in complex dimension 2. In higher dimensions, a case where we know that Kähler hyperbolicity is open under small deformations is the following:

Theorem 2.4.12. *Let $\sigma : \mathscr{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds $X_t = \sigma^{-1}(t)$, with $t \in B$. Fix an arbitrary reference point $0 \in B$. If X_0 is Kähler hyperbolic and the hodge number $h^{2,0}(X_0) = 0$, then the nearby fibers X_t are also Kähler hyperbolic.*

Proof. Since X_0 is Kähler, the fibers X_t are also Kähler for t close enough to 0 by Theorem 2.4.2 and $h^{2,0}(X_t) = h^{2,0}(X_0) = 0$, which in turns implies that $h^{0,2}(X_t) = h^{0,2}(X_0) = 0$ thanks to the Hodge symmetry. In particular, it follows from the Hodge decomposition that: $H^{1,1}(X_t, \mathbb{R}) \simeq H^{1,1}(X_0, \mathbb{R}) = H^2_{DR}(X, \mathbb{R})$, where X is the underlaying differentiable manifold. Which means that $\mathcal{K}_{X_t}$ is open in $H^2_{DR}(X, \mathbb{R})$, and since $\mathcal{K}_{X_0} \cap V^2_{hyp}(X_0) \neq \emptyset$, we also have $\mathcal{K}_{X_t} \cap V^2_{hyp}(X_t) \neq \emptyset$ for t close enough to 0 thanks to Remark 2.2.4. Hence the fibers X_t are also Kähler hyperbolic. □

Examples of Kähler hyperbolic manifolds that satisfy the condition $h^{2,0}(X) = 0$ are the *fake projective planes* (or **Mumford surfaces**), which are compact complex surfaces X with the same cohomology as $\mathbb{P}^2$ (see [Mum79]). B. Klingler and S.K. Yeung showed that these are arithmetic quotients of the hyperbolic ball $\mathbb{B}^2_{\mathbb{C}}$ ([Kli03], [YEU04]), hence they are Kähler hyperbolic and $h^{2,0}(X) = 0$. One can get a family of such examples in the following way: Let $X = S \times S$, where S is a fake projective plane (then again X is Kähler hyperbolic and $h^{2,0}(X) = 0$, embed X in some $\mathbb{P}^N$ and consider X_t to be a family of hyperplane sections of X. Then $\{X_t\}$ is a family of Kähler hyperbolic manifolds and $h^{2,0}(X_t) = 0$ by the Lefschetz hyperplane theorem. These examples and Theorem 2.4.12 were kindly communicated to the author by B. Claudon.

Now, we are going to try to tackle the following question:

Problem 2. Let X be a compact Kähler manifold. If X admits an SKT hyperbolic Kähler metric ω, then is ω L^2-Kähler hyperbolic ?

One key ingredient is the **spectral gap** for Hodge-de Rham Laplace operator Δ in Theorem 2.2.15 holds for the universal cover of Kähler manifolds carrying Kähler SKT hyperbolic metrics. More precisely, let (X, ω) be a complete Kähler manifold with $\dim_{\mathbb{C}} X = n$ such that $\omega = \partial\alpha + \bar\partial\beta$, where α and β are bounded $(0,1)$ and respectively $(1,0)$-forms on X. It is worth noting that the operators d, ∂, $\bar\partial$ and their associated Laplacians Δ, Δ', and Δ'' are **unbounded** operators when defined on $L^2_\bullet(X)$. For $T \in \{d, d^*, \partial, \partial^*, \bar\partial, \bar\partial^*\}$, we define: $\mathrm{Dom}(T) = \{u \in L^2_\bullet(X) : Tu \in L^2_\bullet(X)\} \subset L^2_\bullet(X)$. This domain is dense in $L^2_\bullet(X)$, and T is a closed operator. Consequently, the standard machinery of spectral theory for unbounded operators can be applied. Theorem 3.19 in [Mar23] provides a lower bound for the spectrum of the Laplace operator Δ acting on k-forms, for $k \neq n$. Here, Δ is considered as an unbounded operator on $L^2_\bullet(X)$ with domain:

$$\mathrm{Dom}(\Delta) = \left\{u \in L^2_\bullet(X) : u \in \mathrm{Dom}(d) \cap \mathrm{Dom}(d^*),\, du \in \mathrm{Dom}(d^*),\, d^*u \in \mathrm{Dom}(d)\right\}.$$

Namely, there exists a constant $\lambda = \lambda\left(\|\alpha\|_{L^\infty_\omega(X)}, \|\beta\|_{L^\infty_\omega(X)}\right) > 0$ such that:

$$\langle \Delta\psi, \psi\rangle \geq \lambda^2 \|\psi\|^2_{L^2_\omega(X)}, \qquad \forall \psi \in L^2_k(X) \cap \mathrm{Dom}(\Delta) \tag{2.4.14}$$

for any $k \neq n$. This implies in particular that the spectrum of Δ is included in $[\lambda^2, +\infty[$, when acting on L^2-forms of degree $k \neq n$. As a consequence, X has vanishing L^2-cohomology away from the middle degree, i.e. $\mathcal{H}^k_\Delta(X) = 0$, for any $k \neq n$ ([Mar23], Corollary 3.20). Moreover, the inequality (2.4.14) holds also true for L^2-forms of degree n which are orthogonal to the Δ-harmonic n-forms. The proof goes the same as for Theorem 1.4.A in [Gro91].

Remark 2.4.13. A spectral gap holds also for the $\bar\partial$-Laplacian $\Delta'' := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ and the ∂-Laplacian $\Delta' := \partial\partial^* + \partial^*\partial$ thanks to the classical **Bochner-Kodaira-Nakano** identity, or **BKN** for short (see [Dem97], Corollary 6.5 for instance):

$$\Delta = 2\Delta' = 2\Delta'' \tag{2.4.15}$$

with: $\mathrm{Spec}(\Delta') = \mathrm{Spec}(\Delta'') \subset \left[\frac{\lambda^2}{2}, +\infty\right[$ when acting on (p,q)-forms with $p+q \neq n$.

On a complete (Riemannian) manifold X, we have the following formula for the Laplacian acting on L^2-forms of degree k (Lemma p.7 in [Gro91]):

$$\langle \Delta\psi, \psi\rangle = \|d\psi\|^2_{L^2_k(X)} + \|d^*\psi\|^2_{L^2_k(X)}\,, \qquad \forall \psi \in L^2_k(X) \cap \mathrm{Dom}(\Delta) \tag{2.4.16}$$

This can be obtained by showing that d^* is indeed the adjoint of d when acting on the space of L^2-forms using **Gaffney's cutoff function** technique (Corollary p.6 in [Gaf54]) or *exhaustion functions* (Theorem 3.2, Chapter VII in [Dem97]). For simplicity, for each operator $T \in \{d, d^*, \partial, \partial^*, \bar\partial, \bar\partial^*\}$, we set: $T(L^2_k(X)) := \mathrm{Im}\big(T : L^2_k(X) \cap \mathrm{Dom}(T) \longrightarrow L^2_\bullet(X)\big)$. In particular, we have the following **weak Kodaira decompositon** (see [Kod49]):

$$L^2_k(X) = \mathcal{H}^k_\Delta(X) \oplus \overline{d(L^2_{k-1}(X))} \oplus \overline{d^*(L^2_{k+1}(X))} \quad , \qquad \forall k \in \{0, \ldots, 2n\} \tag{2.4.17}$$

Remark 2.4.14. One can prove that a similar formula to (2.4.16) holds for the ∂-Laplacian and the $\bar\partial$-Laplacian, as well as the decomposition (2.4.17).

Now, we will prove that the positive lower bound on the spectrum of the Laplacian in (2.4.14), that we have on the universal cover of a compact Kähler manifold carrying a Kähler metric which is $\widetilde{(\partial + \bar\partial)}$-bounded, implies that d and d^* have closed images. To show this, we need the following ingredient from functional analysis:

Lemma 2.4.15. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, and let $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a closed operator. Then the following are equivalent:*

1. *ImT is closed, i.e. $ImT = \overline{ImT}$.*
2. *ImT^* is closed.*
3. *There exists a constant $C > 0$ such that: $\|Tu\|_{\mathcal{H}_2} \geq C\|u\|_{\mathcal{H}_1}$, $\forall u \in DomT \cap (KerT)^\perp$.*

The proof of this Lemma can be found in almost any functional analysis textbook, or see for instance Proposition 2.3 and Theorem 2.4 in [Ohs15].

Proposition 2.4.16. *Let X be a complete Kähler manifold equipped with a Kähler metric ω such that: $\omega = \partial\alpha + \bar\partial\beta$, for some bounded forms α and β on X. Then:*

$$d : L^2_k(X) \longrightarrow L^2_{k+1}(X) \tag{2.4.18}$$
$$d^* : L^2_k(X) \longrightarrow L^2_{k-1}(X) \tag{2.4.19}$$

have closed images in every degree k.

Proof. First, we are going to prove the closedness of the images of d away from degree n. So let us take a sequence $\{\psi_\nu\}_{\nu\geq 1} \subset d(L^2_k(X))$ that converges to some $\psi \in L^2_{k+1}(X)$, so for every $\nu \geq 1$ we have $\psi_\nu = d\beta_\nu$ for some β_ν in $L^2_k(X)$. The β_ν's cannot be all closed unless $\psi = 0$, so we can choose the β_ν's such that: $\beta_\nu \in (\mathrm{Ker}d)^\perp \subset (\mathrm{Im}d)^\perp = \mathrm{Ker}d^*$. Hence $d^*\beta_\nu = 0$, so (2.4.14) together with (2.4.16) imply:

$$\|\psi_\nu\|_{L^2_{k+1}(X)} = \|d\beta_\nu\|_{L^2_{k+1}(X)} \geq \lambda\|\beta_\nu\|_{L^2_k(X)}$$

and this is equivalent to the closedness of $d(L^2_k(X))$ for every degree $k \neq n$ by Lemma 2.4.15. In a similar fashion, we can show that $d^*(L^2_k(X))$ is closed whenever $k \neq n$. When $k = n$, we can see that $d^* : L^2_n(X) \longrightarrow L^2_{n-1}(X)$ has a closed image due to Lemma 2.4.15, since it is the dual of $d : L^2_{n-1}(X) \longrightarrow L^2_n(X)$, which has a closed image as we proved previously. Similarly, $d : L^2_n(X) \longrightarrow L^2_{n+1}(X)$ has a closed image since it is the adjoint of the operator $d^* : L^2_{n+1}(X) \longrightarrow L^2_n(X)$, which has a closed image. $\square$

In particular, we have the following othogonal decompositions in every degree:

$$\begin{cases} L^2_k(X) &= d(L^2_{k-1}(X)) \oplus d^*(L^2_{k-1}(X))\,, \qquad \forall k \neq n \\ L^2_n(X) &= \mathcal{H}^n_\Delta(X) \oplus d(L^2_{n-1}(X)) \oplus d^*(L^2_{n-1}(X)) \end{cases} \tag{2.4.20}$$

Remark 2.4.17. Similarly, the images of ∂, ∂^*, $\bar\partial$ and $\bar\partial^*$ are also closed when acting on L^2-forms of any bidegree (p, q), due to Remark 2.4.13.

Now, we consider the following $\bar\partial$-type problem (in the sense of currents) on a complete Kähler manifold (X, ω) with $\omega = \partial\alpha + \bar\partial\beta$, for some bounded forms α and β on X:

$$\begin{cases} \bar\partial u = \bar\partial v\,, \qquad v \in L^2_{k,0}(X) \cap \mathrm{Dom}(\Delta_{BC}) \\ \partial u = 0 \end{cases} \tag{2.4.21}$$

Here, $\mathrm{Dom}(\Delta_{BC})$ denotes the intersection of the domains of each summand defining the Bott-Chern Laplacian Δ_{BC}. This kind of $\bar\partial$-problems has been treated in the compact Hermitian case by D. Popovici and S. Dinew ([DP21], Lemma 2.4) by the use of Δ_{BC}. The ellipticity of Δ_{BC} induces the following *weak* orthogonal Kodaira-type decomposition (which holds for any *complete* Hermitian manifold) for any bidegree (p, q):

$$L^2_{p,q}(X) = \mathcal{H}^{p,q}_{BC}(X) \oplus \overline{\partial\bar\partial(L^2_{p-1,q-1}(X))} \oplus \overline{\partial^*(L^2_{p+1,q}(X)) + \bar\partial^*(L^2_{p,q+1}(X))} \tag{2.4.22}$$

In order to get estimates on the solution of (2.4.21), we recall some of the fundamental identities for the Bott-Chern Laplacian Δ_{BC} on Kähler manifolds. For instance, one can show that Δ_{BC} can be expressed in the following way ([Sch07], Proposition 2.4):

$$\Delta_{BC} = (\Delta'')^2 + \partial^*\partial + \bar{\partial}^*\bar{\partial} \tag{2.4.23}$$

The formula (2.4.23) implies that $\mathrm{Spec}(\Delta_{BC}) \subset \left[\frac{\lambda^4}{4}, +\infty\right[$, when Δ_{BC} acts on L^2-forms of bidegree (p,q) with $p+q \neq n$. Moreover, when $p+q=n$ we have:

$$\langle \Delta_{BC}\psi, \psi\rangle \geq \frac{\lambda^4}{4}\|\psi\|^2_{L^2_{p,q}(X)}\ , \qquad \forall \psi \in (\mathcal{H}^{p,q}_{BC}(X))^{\perp} \cap \mathrm{Dom}(\Delta_{BC}) \tag{2.4.24}$$

In order to solve the $\bar{\partial}$-problem (2.4.21), it is crucial to have a *strong* Kodaira decomposition associated to the Bott-Chern Laplacian. Namely, we have:

Proposition 2.4.18. *Let X be a complete Kähler manifold equipped with a Kähler metric $\omega = \partial\alpha + \bar{\partial}\beta$, for some bounded forms α and β on X. Then for any $p, q \in \{0, \dots, n\}$, $\partial\bar{\partial}(L^2_{p,q}(X))$ and $\partial^*(L^2_{p+1,q}(X)) + \bar{\partial}^*(L^2_{p,q+1}(X))$ are closed.*

Proof. By Lemma 2.4.15, the closedness of $\partial\bar{\partial}(L^2_{p,q}(X))$ is equivalent to:

$$\exists C > 0\,, \quad \|\partial\bar{\partial}\psi\|_{L^2_{p+1,q+1}(X)} \geq C\|\psi\|_{L^2_{p,q}(X)}\ , \qquad \forall \psi \in L^2_{p,q}(X) \cap (\mathrm{Ker}\partial\bar{\partial})^{\perp} \cap \mathrm{Dom}(\partial\bar{\partial}) \tag{2.4.25}$$

where $\mathrm{Dom}(\partial\bar{\partial}) = \left\{\psi \in L^2_{p,q}(X) : \psi \in \mathrm{Dom}(\partial) \cap \mathrm{Dom}(\bar{\partial}),\, \partial\bar{\partial}\psi \in L^2_{p+1,q+1}(X)\right\}$.

Let $\psi = \psi_1 + \psi_2 \in \mathrm{Ker}\bar{\partial} + \mathrm{Ker}\partial$, then we have: $\partial\bar{\partial}\psi = \partial\bar{\partial}\psi_1 + \partial\bar{\partial}\psi_2 = \partial(\bar{\partial}\psi_1) - \bar{\partial}(\partial\psi_2) = 0$. Hence: $\mathrm{Ker}\bar{\partial} + \mathrm{Ker}\partial \subset \mathrm{Ker}\partial\bar{\partial}$, this implies that:

$$\begin{aligned}(\mathrm{Ker}\partial\bar{\partial})^{\perp} &\subset (\mathrm{Ker}\bar{\partial} + \mathrm{Ker}\partial)^{\perp} = (\mathrm{Ker}\bar{\partial})^{\perp} \cap (\mathrm{Ker}\partial)^{\perp} \\ &\subset (\mathrm{Im}\bar{\partial})^{\perp} \cap (\mathrm{Im}\partial)^{\perp} = \mathrm{Ker}\bar{\partial}^* \cap \mathrm{Ker}\partial^* \end{aligned} \tag{2.4.26}$$

So let $\psi \in L^2_{p,q}(X) \cap (\mathrm{Ker}\,\partial\bar{\partial})^{\perp} \cap \mathrm{Dom}(\partial\bar{\partial})$. Then $\partial^*\psi = \bar{\partial}^*\psi = 0$ by (2.4.26). Hence we have:

$$\|\partial\bar{\partial}\psi\|^2_{L^2_{p+1,q+1}(X)} \overset{(a)}{=} \|\partial\bar{\partial}\psi\|^2_{L^2_{p+1,q+1}(X)} + \|\partial^*\bar{\partial}\psi\|^2_{L^2_{p-1,q+1}(X)} \overset{(b)}{\geq} \frac{\lambda^2}{2}\|\bar{\partial}\psi\|^2_{L^2_{p,q+1}(X)} \tag{2.4.27}$$

$$\|\bar{\partial}\psi\|^2_{L^2_{p,q+1}(X)} \overset{(a')}{=} \|\bar{\partial}\psi\|^2_{L^2_{p,q+1}(X)} + \|\bar{\partial}^*\psi\|^2_{L^2_{p,q-1}(X)} \overset{(b')}{\geq} \frac{\lambda^2}{2}\|\psi\|^2_{L^2_{p,q}(X)} \tag{2.4.28}$$

where (a) follows from the identity $\partial^*\bar{\partial} = -\bar{\partial}\partial^*$ on a Kähler manifold together with the fact that $\partial^*\psi = 0$, and (b) follows from Remarks 2.4.14 and 2.4.13. Similarly, (a') follows from the fact that $\bar{\partial}^*\psi = 0$, while (b') follows from Remarks 2.4.14 and 2.4.13. Note that the inequality (b) remains valid when $p+q+1 = n$, since $\bar{\partial}\psi \notin \mathrm{Ker}\,\partial$. Indeed, this implies that $\bar{\partial}\psi \in \left(\mathcal{H}^{p,q+1}_{\Delta'}(X)\right)^{\perp}$. Likewise, the inequality (b') remains valid when $p+q=n$, since $\psi \notin \operatorname{Ker}\bar{\partial}$ (otherwise, we would have $\partial\bar{\partial}\psi = 0$), and therefore $\psi \in \left(\mathcal{H}^{p,q}_{\Delta''}(X)\right)^{\perp}$. Hence, (2.4.25) holds with constant $C = \dfrac{\lambda^2}{2} > 0$. It follows that $\partial\bar{\partial}(L^2_{p,q}(X))$ is closed.

As for the closedeness of $\partial^*(L^2_{p+1,q}(X)) + \bar{\partial}^*(L^2_{p,q+1}(X))$, consider $\widetilde{d^*}$, $\widetilde{\partial^*}$ and $\widetilde{\bar{\partial}^*}$ to be the restrictions of d^*, ∂^* and $\bar{\partial}^*$ respectively, to $\mathscr{D} = \operatorname{Dom}(\partial^*) \cap \operatorname{Dom}(\bar{\partial}^*)$, for $T \in \left\{\widetilde{d^*}, \widetilde{\partial^*}, \widetilde{\bar{\partial}^*}\right\}$, and denote by $\widetilde{\Delta_{BC}}$ the restriction of the Bott-Chern Laplacian Δ_{BC} when restricting ∂^* and $\bar{\partial}^*$ to $\mathscr{D}$. In a similar manner as before, one can show that $\widetilde{d^*}$, $\widetilde{\partial^*}$ and $\widetilde{\bar{\partial}^*}$ have closed images. Then for $p, q = 0, \ldots, n$, and for any $u \in L^2_{p+1,q}(X) \cap \mathscr{D}$, $v \in L^2_{p,q+1}(X) \cap \mathscr{D}$ we have: $\pi^{p,q}\left(\widetilde{d^*}(u+v)\right) = \widetilde{\partial^*}u + \widetilde{\bar{\partial}^*}v$, where $\pi^{p,q} : L^2_{p+q}(X) \longrightarrow L^2_{p,q}(X)$ is the orthogonal projection onto L^2-form of bidegree (p,q). In particular, we have:

$$\left(\pi^{p,q} \circ \widetilde{d^*}\right)\left(L^2_{p+1,q}(X) \oplus L^2_{p,q+1}(X)\right) = \widetilde{\partial^*}\left(L^2_{p+1,q}(X)\right) + \widetilde{\bar{\partial}^*}\left(L^2_{p,q+1}(X)\right)$$

Hence $\widetilde{\partial^*}\left(L^2_{p+1,q}(X)\right) + \widetilde{\bar{\partial}^*}\left(L^2_{p,q+1}(X)\right)$ is closed, for all $p, q = 0, \ldots, n$. On one hand, we clearly have that $\widetilde{\partial^*}\left(L^2_{p+1,q}(X)\right) + \widetilde{\bar{\partial}^*}\left(L^2_{p,q+1}(X)\right) \overset{(c)}{\subset} \partial^*\left(L^2_{p+1,q}(X)\right) + \bar{\partial}^*\left(L^2_{p,q+1}(X)\right)$, since $\mathscr{D} \subset \operatorname{Dom}(\partial^*)$ and $\mathscr{D} \subset \operatorname{Dom}(\bar{\partial}^*)$. On the other hand, the restricted Bott-Chern Laplacian $\widetilde{\Delta_{BC}}$ induces the following three-space orthogonal decomposition:

$$L^2_{p,q}(X) = \mathcal{H}^{p,q}_{BC}(X) \oplus \partial\bar{\partial}(L^2_{p-1,q-1}(X)) \oplus \left(\widetilde{\partial^*}\left(L^2_{p+1,q}(X)\right) + \widetilde{\bar{\partial}^*}\left(L^2_{p,q+1}(X)\right)\right) \tag{2.4.29}$$

Note that the Harmonic space $\mathcal{H}^{p,q}_{BC}(X)$ does not change when restricting the domain of Δ_{BC} since this is a *hypoelliptic* operator, which implies that harmonic forms are smooth. By uniqueness of the three-space decomposition, we get:

$$\overline{\partial^*(L^2_{p+1,q}(X)) + \bar{\partial}^*(L^2_{p,q+1}(X))} \overset{(d)}{=} \widetilde{\partial^*}\left(L^2_{p+1,q}(X)\right) + \widetilde{\bar{\partial}^*}\left(L^2_{p,q+1}(X)\right)$$

Combining (c) and (d) yields the equality:

$$\partial^*(L^2_{p+1,q}(X)) + \bar{\partial}^*(L^2_{p,q+1}(X)) = \widetilde{\partial^*}\left(L^2_{p+1,q}(X)\right) + \widetilde{\bar{\partial}^*}\left(L^2_{p,q+1}(X)\right)$$

Hence $\partial^*(L^2_{p+1,q}(X)) + \bar{\partial}^*(L^2_{p,q+1}(X))$ is closed for any $p, q = 0, \ldots, n$. $\square$

This implies that we have the following *strong* orthogonal decomposition in any bidegree (p,q) instead of the one in (2.4.22):

$$\begin{cases} L^2_{p,q}(X) = \mathcal{H}^{p,q}_{BC}(X) \oplus \partial\bar{\partial}(L^2_{p-1,q-1}(X)) \oplus \left(\partial^*(L^2_{p+1,q}(X)) + \bar{\partial}^*(L^2_{p,q+1}(X))\right) \\ \mathcal{H}^{p,q}_{BC}(X) = 0\ , \qquad \text{if } \ p+q \neq n \end{cases} \tag{2.4.30}$$

Lemma 2.4.19. *The L^2-minimal norm solution of (2.4.21) satisfies the following estimate:*

$$\|u\|^2_{L^2_{k,0}(X)} \leq 2\left(1+\frac{4}{\lambda^4}\right)\|v\|^2_{L^2_{k,0}(X)} \tag{2.4.31}$$

Proof. The solution of (2.4.21) is unique up to $\mathrm{Ker}\partial \cap \mathrm{Ker}\bar{\partial} = \mathrm{Ker}\Delta_{BC} \oplus \mathrm{Im}\partial\bar{\partial}$. Hence the decomposition (2.4.30) allows us to get the following formula for the minimal L^2-norm solution of (2.4.21) (see [DP21], proof of Lemma 2.4):

$$u = \Delta_{BC}^{-1}(\bar{\partial}^*\bar{\partial}v + \bar{\partial}^*\partial\partial^*\bar{\partial}v)$$

where Δ_{BC}^{-1} is the **Green operator** associated to Δ_{BC}, which is well-defined on $L^2_{p,0}(\widetilde{X})$; it is exactly the inverse of Δ_{BC} if $k \neq n$. If $k = n$, then Δ_{BC}^{-1} is the inverse of Δ_{BC} on $(\mathcal{H}^{n,0}_{BC}(X))^\perp$, which extend by 0 across $\mathcal{H}^{n,0}_{BC}(X)$. Hence we get:

$$\begin{aligned} u &= \Delta_{BC}^{-1}(\bar{\partial}^*\bar{\partial}v + \bar{\partial}^*\Delta'\bar{\partial}v) && (2.4.32)\\ &= \Delta_{BC}^{-1}(\bar{\partial}^*\bar{\partial}v + \Delta''\bar{\partial}^*\bar{\partial}v) && (2.4.33)\\ &= \Delta_{BC}^{-1}(\Delta''v + (\Delta'')^2v) && (2.4.34)\\ &= \Delta''\Delta_{BC}^{-1}v + (\Delta'')^2\Delta_{BC}^{-1}v && (2.4.35) \end{aligned}$$

where (2.4.32) comes from the fact that $\partial\bar{\partial}v = 0$, as this condition is imposed on v in order for (2.4.21) to be solvable. (2.4.33) is a consequence of the BKN identity in Remark 2.4.13 and $\Delta''\bar{\partial}^* = \bar{\partial}^*\Delta''$ on Kähler manifolds. (2.4.34) comes from the fact that $\bar{\partial}^*v = 0$ for bidegree reasons. As for (2.4.35), it holds true since it is easy to check that Δ_{BC} commutes with both $\bar{\partial}^*\bar{\partial}$ and $\bar{\partial}\bar{\partial}^*$, and hence it commutes with Δ'', which implies that Δ_{BC}^{-1} also commutes with Δ'' thanks to the Kählerianity of X. Hence we get the following inequality:

$$\|u\|^2_{L^2_{k,0}(X)} \leq 2\left(\|\Delta''\Delta_{BC}^{-1}v\|^2_{L^2_{k,0}(X)} + \|(\Delta'')^2\Delta_{BC}^{-1}v\|^2_{L^2_{k,0}(X)}\right) \tag{2.4.36}$$

Thanks to the formula (2.4.23), a similar proof of (2.4.16) yields for any $\psi \in L^2_{p,q}(X)$:

$$\langle\Delta_{BC}\psi, \psi\rangle = \|\partial\psi\|^2_{L^2_{p+1,q}(X)} + \|\bar{\partial}\psi\|^2_{L^2_{p,q+1}(X)} + \|\Delta''\psi\|^2_{L^2_{p,q}(X)} \tag{2.4.37}$$

in any bidegree (p, q). Hence, we obtain the following estimates:

$$\begin{aligned} \|\Delta''\Delta_{BC}^{-1}v\|^2_{L^2_{k,0}(X)} &\leq \langle\Delta_{BC}\Delta_{BC}^{-1}v, \Delta_{BC}^{-1}v\rangle = \langle v, \Delta_{BC}^{-1}v\rangle \leq \frac{4}{\lambda^4}\|v\|^2_{L^2_{k,0}(X)} && (2.4.38)\\ \|(\Delta'')^2\Delta_{BC}^{-1}v\|^2_{L^2_{k,0}(X)} &\leq \langle\Delta_{BC}\Delta''\Delta_{BC}^{-1}v, \Delta''\Delta_{BC}^{-1}v\rangle = \langle v, (\Delta'')^2\Delta_{BC}^{-1}v\rangle && (2.4.39)\\ &\overset{(*)}{\leq} \|v\|_{L^2_{k,0}(X)}\|(\Delta'')^2\Delta_{BC}^{-1}v\|_{L^2_{k,0}(X)} \leq \|v\|^2_{L^2_{k,0}(X)} && (2.4.40) \end{aligned}$$

where $(*)$ comes from the use of the Cauchy-Schwarz inequality. Thus, we get the desired estimate (2.4.31) for the L^2-norm of the L^2-minimal solution u of (2.4.21). $\square$

The same results and estimates can be obtained for the following "conjugate" problem of the $\bar{\partial}$-equation (2.4.21):

$$\begin{cases} \partial u = \partial v\,, & v \in L^2_{0,k}(X) \\ \bar{\partial} u = 0 \end{cases} \tag{2.4.41}$$

Namely, one can show that the L^2-minimal norm solution satisfies:

$$\begin{cases} u = \Delta_{BC}^{-1}(\partial^\star\partial v + \partial^\star\bar{\partial}\bar{\partial}^\star\partial v) = \Delta'\Delta_{BC}^{-1} v + (\Delta')^2\Delta_{BC}^{-1} v \\ \|u\|^2_{L^2_{0,k}(X)} \leq 2\left(1 + \dfrac{4}{\lambda^4}\right)\|v\|^2_{L^2_{0,k}(X)} \end{cases} \tag{2.4.42}$$

We now return to an attempt to address Problem 2: Let ω be a Kähler metric on X such that $\widetilde{\omega} = \partial\alpha + \bar{\partial}\beta$, where α and β are bounded $(0,1)$ and $(1,0)$-forms respectively on $\widetilde{X}$, in particular they satisfy the (F_2) property. That is, for any fundamental domain D of $\widetilde{X}$ we have:

$$\begin{cases} \exists C_\alpha > 0\,, & \sup\limits_{\gamma\in\pi_1(X)} \|\alpha\|_{L^2_{\widetilde{\omega}}(\gamma D)} \leq C_\alpha \\ \exists C_\beta > 0\,, & \sup\limits_{\gamma\in\pi_1(X)} \|\beta\|_{L^2_{\widetilde{\omega}}(\gamma D)} \leq C_\beta \end{cases} \tag{2.4.43}$$

Consider for any $\gamma \in \pi_1(X)$ the following:

$$\alpha_\gamma = \begin{cases} \alpha & \text{on } \gamma D \\ 0 & \text{elsewhere} \end{cases}\,, \qquad \beta_\gamma = \begin{cases} \beta & \text{on } \gamma D \\ 0 & \text{elsewhere} \end{cases} \tag{2.4.44}$$

We have: $\alpha = \sum\limits_{\gamma\in\pi_1(X)} \alpha_\gamma$ and $\beta = \sum\limits_{\gamma\in\pi_1(X)} \beta_\gamma$ almost everywhere. More precisely, these equalities hold true on $\widetilde{X} \setminus \bigcup\limits_{\gamma\in\pi_1(X)} \partial(\gamma D)$. Moreover, we have for any $\gamma \in \pi_1(X)$:

$$\|\alpha_\gamma\|_{L^2_{1,0}(\tilde{X})} = \|\alpha\|_{L^2_{1,0}(\gamma D)} \leq C_\alpha \qquad \text{and} \qquad \|\beta_\gamma\|_{L^2_{1,0}(\tilde{X})} = \|\beta\|_{L^2_{0,1}(\gamma D)} \leq C_\beta \tag{2.4.45}$$

Furthermore, α_γ and β_γ are smooth on $\widetilde{X}\setminus\partial(\gamma D)$. In particular, the question of whether they belong to $\mathrm{Dom}(\Delta_{BC})$ is entirely concentrated at the boundary $\partial(\gamma D)$. We consider for each $\gamma \in \pi_1(X)$, the L^2-minimal norm solutions α'_γ and β'_γ for the problems (2.4.21) and (2.4.41) on $\widetilde{X}\setminus\gamma D$ respectively with $v = \alpha_\gamma$ and $v = \beta_\gamma$. If we set $\theta := \sum\limits_{\gamma\in\pi_1(X)} \alpha'_\gamma + \beta'_\gamma$, then $\widetilde{\omega} = d\theta$ almost everywhere. This leads to the following intermediate question:

Problem 3. If X carries a SKT hyperbolic Kähler metric, then is X $\widetilde{d}$-exact ?

However, θ does not necessarily satisfy the (F_2) property. The issue here is that α'_γ and β'_γ are expressed in terms of the Green operator Δ_{BC}^{-1} associated with the Bott-Chern Laplacian, which is only a pseudodifferential operator rather than a differential one, and hence is nonlocal. Consequently, α'_γ and β'_γ are not necessarily supported in $\overline{\gamma D}$. In order to try to outcome this difficulty, we set $\alpha' = \sum\limits_{\gamma\in\pi_1(X)} \chi_{\gamma D}\alpha'_\gamma$ and $\beta' = \sum\limits_{\gamma\in\pi_1(X)} \chi_{\gamma D}\beta'_\gamma$. Then:

$$\begin{cases} \partial\alpha' = \partial\alpha + \sum\limits_{\gamma\in\pi_1(X)} \partial\chi_{\gamma D}\wedge\alpha'_\gamma \\ \bar\partial\alpha' = \sum\limits_{\gamma\in\pi_1(X)} \bar\partial\chi_{\gamma D}\wedge\alpha'_\gamma \end{cases} \quad \text{and} \quad \begin{cases} \bar\partial\beta' = \bar\partial\beta + \sum\limits_{\gamma\in\pi_1(X)} \partial\chi_{\gamma D}\wedge\beta'_\gamma \\ \partial\beta' = \sum\limits_{\gamma\in\pi_1(X)} \bar\partial\chi_{\gamma D}\wedge\beta'_\gamma \end{cases} \tag{2.4.46}$$

If we set the 1-form $\theta' = \alpha' + \beta'$, we get the follwoing identity on $\widetilde{X}\setminus \bigcup\limits_{\gamma\in\pi_1(X)} \partial(\gamma D)$:

$$\widetilde{\omega} = d\theta' - \sum_{\gamma\in\pi_1(X)} d\chi_{\gamma D}\wedge(\alpha'_\gamma + \beta'_\gamma) \tag{2.4.47}$$

However, $d\chi_{\gamma D}$ is supported in $\partial(\gamma D)$, hence: $\widetilde{\omega} = d\theta'$ on $\widetilde{X}\setminus \bigcup\limits_{\gamma\in\pi_1(X)} \partial(\gamma D)$. Moreover:

$$\begin{aligned} \|\theta'\|_{L^2_1(\gamma D)} &\leq \|\alpha'\|_{L^2_{1,0}(\gamma D)} + \|\beta'\|_{L^2_{0,1}(\gamma D)} && (2.4.48)\\ &\leq \sqrt{2\left(1+\frac{4}{\lambda^4}\right)}\left(\|\alpha\|_{L^2_{1,0}(\gamma D)} + \|\beta\|_{L^2_{0,1}(\gamma D)}\right) \qquad \text{(by (2.4.31) and (2.4.42))} && (2.4.49)\\ &\leq \sqrt{2\left(1+\frac{4}{\lambda^4}\right)}(C_\alpha + C_\beta) =: C_{\theta'} \quad , \qquad \forall\gamma\in\pi_1(X) && (2.4.50) \end{aligned}$$

where $C_{\theta'} > 0$ does not depend on γ. Which means that the Kähler metric ω is L^2-Kähler hyperbolic almost everywhere. An important consequence of solving Problem 2 or its weaker version Problem 3 is the deformation **openness** of the SKH property:

Corollary 2.4.20. *Let $\sigma : \mathscr{X} \longrightarrow B$ be a holomorphic family of compact complex manifolds $X_t = \sigma^{-1}(t)$, with $t\in B$. Fix an arbitrary reference point $0\in B$. If X_0 is SKH, then the nearby fibers are also SKH.*

Proof. Since X_0 is SKH, then X_0 is Kähler hyperbolic. Theorem 2.4.10 and Problem 3 imply that the fibers X_t are Kähler $\widetilde{d}$-exact. Since $\pi_1(X_0)$ is Gromov hyperbolic, then $\pi_1(X_t)$ is also Gromov hyperbolic by Remark 2.4.7. Thus, the fibers X_t are SKH. □

We hope to find a way to solve the $\bar\partial$-problem (2.4.21) and its conjugate problem (2.4.41) with L^∞-estimates. This will allow us to prove the following:

Conjecture 2.4.21. *Let X be a compact Kähler manifold. If X admits a Kähler metric ω which is SKT hyperbolic, then X is Kähler hyperbolic.*

If this is proved to be true, we would end up showing that the Kähler hyperbolicity property is open under small deformations.

One possible strategy for proving this conjecture is to assume that the universal cover $\widetilde{X}$ of a Kähler manifold carrying a Kähler SKT hyperbolic metric is *Stein*. When such manifolds arise as deformations of a Kähler hyperbolic manifold X_0, their universal cover has the same L^2-cohomology as $\widetilde{X}_0$. It follows that they are of general type, and therefore projective. Moreover, Proposition 2.2.9 and Corollary 2.2.12 remain valid for the universal cover $\widetilde{X}$ of a Kähler manifold carrying a Kähler SKT hyperbolic metric, so that its fundamental group is large in the sense of Kollár.

The **Shafarevich uniformization conjecture** predicts that the universal cover should be Stein. To show that the Shafarevich conjecture holds true for these manifolds, or at least ensure the existence of non-constant holomorphic maps on the universal cover, we do not have a general method except the following [Eys24]:

1. If the fundamental group does not satisfy **Kazhdan's property (T)** ([Kaz67]), N. Mok proved that the universal cover carries a non constant holomorphic function to some Hilbert space, equivariant for some affine isometric representation of the fundamental group [Mok95]. Consequently, the universal cover supports a non-constant holomorphic function with bounded differential, and thus, of at most linear growth (see [DG05], p.6). But there are some Kähler hyperbolic manifolds that are known to satisfy this property (compact quotients of irreducible bounded symmetric domains of rank 2).

2. Find a finite-dimensional linear representation of the fundamental group of infinite image [EKPR12]. We do not know of any fundamental group of a smooth complex algebraic variety that we can prove is infinite, and that does not have such linear representations.

This motivates a more thorough investigation of the fundamental groups of Kähler hyperbolic manifolds, with the goal of testing the validity of the Shafarevich conjecture for this class of projective manifolds. Indeed, the fundamental group has a significant influence on the growth of the primitive of pulled-back differential forms of degree ≥ 2. This consideration naturally leads to the following question, which is strictly stronger than the deformation openness of Kähler hyperbolicity:

Problem 4. Let X be a compact Kähler $\widetilde{d}$-exact manifold. Assume that X has a *Kähler* *hyperbolic* fundamental group, i.e. $\pi_1(X) \simeq \pi_1(Y)$ for some compact Kähler hyperbolic manifold Y. Is X Kähler hyperbolic ?

Chapter 3

Deformations of Non-Kähler Hyperbolicity Notions and Modifications of Degenerate Balanced Manifolds

We study deformation properties of balanced hyperbolicity, with a particular emphasis on degenerate balanced manifolds and their behavior under smooth modifications.

From a different perspective, we introduce two new notions of hyperbolicity for compact complex non-Kähler manifolds X of complex dimension $\dim_{\mathbb{C}} X = n$, in general degree $2p$ with $1 \leq p \leq n-1$. These notions are motivated by the work of D. Popovici and H. Kasuya on partial hyperbolicity in arbitrary degree and by the work of F. Haggui and S. Marouani on p-Kähler hyperbolicity. The first notion, called *p-SKT hyperbolicity*, extends SKT hyperbolicity and Gauduchon hyperbolicity to degree $2p$. Similarly, the second notion, called *p-HS hyperbolicity*, generalizes the notion of strongly Gauduchon hyperbolicity introduced by Y. Ma.

We then analyze the relationships between these analytic notions and geometric notions of hyperbolicity, namely Brody/Kobayashi hyperbolicity and *p-cyclic hyperbolicity* in degree $2p$ for $2 \leq p \leq n-1$. In addition, we study the behavior of p-HS hyperbolicity and p-Kähler hyperbolicity under holomorphic deformations, establishing openness results for these properties.

3.1 Introduction

In one of his seminal papers [Gro91], M. Gromov introduced a notion of hyperbolicity for compact Kähler manifolds, called *Kähler hyperbolicity*, defined in terms of a boundedness property relative to Kähler metrics. More precisely, let (X, ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$. A k-form α on X is said to be $\widetilde{d}$*(bounded)* if its pullback $\widetilde{\alpha} := \pi^*\alpha$ to the universal cover $\pi : \widetilde{X} \longrightarrow X$ satisfies $\widetilde{\alpha} = d\beta$ for some $\widetilde{\omega}$-bounded $(k-1)$-form β on $\widetilde{X}$. This notion clearly does not depend on the choice of the Hermitian metric on X, since all Hermtitian metrics on a compact complex manifold are comparable. The manifold X is called *Kähler hyperbolic* if it admits a Kähler metric ω that is $\widetilde{d}$(bounded).

Over the last few years, several authors have proposed extensions of this hyperbolicity notion beyond the Kähler setting in several ways. First, D. Popovici and S. Marouani introduced the notion of *balanced hyperbolicity* in [MP23] as a generalization of Gromov's Kähler hyperbolicity: a compact complex manifold X is said to be *balanced hyperbolic* if it admits a $\widetilde{d}$(bounded) balanced metric ω. This class includes, for example, Kähler hyperbolic manifolds and another special class called *degenerate balanced* manifolds). Further generalizations in the same spirit include *SKT hyperbolicity* and *Gauduchon hyperbolicity*, defined by S. Marouani in [Mar23], as well as *sG hyperbolicity*, introduced by Y. Ma in [Ma24].

The main goal of this paper is to study the deformation stability of such non-Kähler hyperbolicity notions, and to introduce new hyperbolicity concepts that fit naturally into this framework. But first, we obtain the following result concerning the behavior of degenerate balanced manifolds under modifications:

Theorem A. *Let $f : \widehat{X} \longrightarrow X$ be a smooth* ***surjective*** *modification of compact complex manifolds, with $\dim_{\mathbb{C}} X = n \geq 3$. If $\widehat{X}$ is degenerate balanced, then X is degenerate balanced.*

We then investigate the openness under holomorphic deformations of balanced hyperbolicity and of the degenerate balanced class of manifolds:

Theorem B. *Let $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ be an analytic family of compact complex n-dimensional manifolds such that X_0 is balanced. Assume that the fibers X_t satisfy the weak $(n-1, n)$-$\partial\bar{\partial}$-property for $t \neq 0$.*

1. *If X_0 is balanced hyperbolic, the fibers X_t carry balanced metrics ω_t that are Gauduchon hyperbolic for small t.*

2. *If moreover, X_0 is degenerate balanced and the fibers X_t satisy the mild $(n, n-2)$-*

$\partial\bar\partial$-property for $t \neq 0$, the fibers X_t are degenerate balanced for small t.

When the family $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ satisfies the $\partial\bar\partial$-property, we get the following stronger statement, which is new in the context of degenerate balanced manifolds:

Theorem C. *Let $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ be an analytic family of compact $\partial\bar\partial$-manifolds of complex dimension n such that X_0 is balanced.*

1. *Assume X_0 is balanced hyperbolic, and let ω be a balanced hyperbolic metric on X_0. Then ω extends to a $\mathscr{C}^\infty$-family of balanced metrics $\{\omega_t\}_{t\sim 0}$ that are Gauduchon hyperbolic on the respective fibers X_t for small t.*
2. *If moreover, ω is degenerate balanced, the $\mathscr{C}^\infty$-family of balanced metrics $\{\omega\}_{t\sim 0}$ on the fibers $\{X_t\}_{t\sim 0}$ are degenerate balanced with a $\mathscr{C}^\infty$-family of d-potentials $\{w_t\}_{t\sim 0}$ for small t, i.e. $\omega = dw_0$ and $\omega_t = dw_t$, $\forall t \sim 0$.*

Other notions of hyperbolicity were introduced by D. Popovici and H. Kasuya in [KP25], where they initiated the concept of *partial hyperbolicity*, which constitutes the first hyperbolicity notion formulated in intermediate degree $2p$ with $1 \leq p \leq n-1$. In a related direction, a natural way to generalize the previously mentioned *metric* notions of hyperbolicity is to impose analogous conditions in bidegree (p,p). This approach was developed by S. Marouani and F. Haggui in [HM23], where they introduced the notion of *p-Kähler hyperbolicity.*

In this paper, we introduce the notion of **Hermitian-symplectic** (or **HS**) **hyperbolicity**, in which the Kähler condition is replaced by the Hermitian-symplectic condition. This notion is motivated in part by a question raised by J. Streets and G. Tian on the existence of non-Kähler Hermitian–symplectic manifolds ([ST10], Question 1.7). We investigate the similarities and differences between hyperbolicity properties in the Hermitian–symplectic and Kähler settings. We also define degree $2p$ analogues of this notion and of SKT hyperbolicity, namely ***p*-SKT hyperbolicity** and ***p*-HS hyperbolicity**, and we prove the following results:

Theorem D. *Let X be a compact complex manifold of complex dimension $\dim_{\mathbb{C}} X = n \geq 2$, and let $1 \leq p \leq n-1$. Then:*

1. *If X is HS hyperbolic, then X is Kobayashi hyperbolic.*
2. *If X is p-HS hyperbolic or p-SKT hyperbolic, then X is p-cyclically hyperbolic (or p-hyperbolic in the sense of Definition 2.5, [KP25]).*

We also establish the following deformation results for p-HS hyperbolic manifolds and p-Kähler hyperbolic manifolds:

Theorem E. *Let $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ be an analytic family of compact complex n-dimensional* *manifolds, where $\mathbb{B}_\varepsilon(0) \subset \mathbb{C}^N$ is the open ball of radius $\varepsilon > 0$ centered at the origin, and fix $1 \leq p \leq n-1$. Then:*

1. *If X_0 is p-HS hyperbolic, then the fibers X_t are p-HS hyperbolic for all t sufficiently close to 0.*

2. *If X_0 is a p-Kähler hyperbolic $\partial\bar\partial$-manifold, then for sufficiently small t, the fibers X_t are p-SKT hyperbolic.*

3.2 Balanced hyperbolic manifolds

Let X be a compact connected complex manifold of complex dimension $\dim_{\mathbb{C}} X = n \geq 2$, and $\pi : \widetilde{X} \longrightarrow X$ its universal cover. Recall that a Hermitian metric ω on X is **semi-Kähler** (or **balanced**, or **co-Kähler**) if $d\omega^{n-1} = 0$ ([Gau77a],[Mic82]). Note that for $n = 2$, the semi-Kähler condition coincides with the classical Kähler condition.

Definition 3.2.1 (**Balanced hyperbolicity**, [MP23])**.** *We say that a compact complex manifold X is **balanced hyperbolic** if it admits a balanced metric ω such that ω^{n-1} is $\widetilde{d}$(bounded), i.e. there exists a $\widetilde{\omega}$-bounded $(2n-3)$-form α on $\widetilde{X}$ such that $\widetilde{\omega}^{n-1} := \pi^*\omega^{n-1} = d\alpha$.*

There are mainly four known classes of Balanced hyperbolic manifolds, namely:

1. Kähler hyperbolic manifolds (thanks to Lemma 2.2 in [MP23]).

2. **Degenerate balanced** manifolds, i.e. n-dimensional compact complex manifolds that carry a Hermitian metric ω such that ω^{n-1} is d-exact (see [Pop15], Proposition 5.4). Obviously, this class of manifolds does not contain any Kähler ones (if a compact complex manifold X admits a Kähler metric ω, then $\{\omega^{n-1}\} \neq 0 \in H^{2n-2}_{DR}(X, \mathbb{R})$). Examples of such manifolds appear starting from complex dimension 3. Two classes of examples of degenerate balanced manifolds are known:

   (a) The connected sums $X = \#_k(\mathbb{S}^3 \times \mathbb{S}^3)$ of $k \geq 2$ copies of $\mathbb{S}^3 \times \mathbb{S}^3$ endowed with the Friedman-Lu-Tian complex structure constructed via *conifold transitions* (see [Fri91], [LT96] and [FLY12]).

   (b) The **Yachou manifolds** $X = G/\Gamma$, where G is a *semi-simple* complex Lie group and $\Gamma \subset G$ a lattice ([Yac98]).

3. Products of (1) and (2) (thanks to Proposition 2.17, [MP23]), which need not be Kähler nor degenerate balanced.

4. Via smooth modifications of the previous examples. More precisely: Take a smooth modification $f : \widehat{X} \longrightarrow X$ of compact complex manifolds, with $\dim_{\mathbb{C}} X = n \geq 3$.

Assume that either $\pi_i(\widehat{X}) = 0$ for $2 \leq i \leq 2n-3$, or f is a *contraction to points*, i.e. f restricts to a biholomorphism outside of the preimage of a countable set of points in X. Then the following holds:

$$\widehat{X} \text{ is balanced hyperbolic} \Longrightarrow X \text{ is balanced hyperbolic}$$

This has been proved in [FWW24] (see Theorem 1.3.(1) and Theorem 1.4).

3.2.1 Modifications of Compact Degenerate Balanced Manifolds

It is natural to ask whether the degenerate balanced property is preserved under smooth modifications. Toward answering this question, we first make the following observation:

Observation 3.2.2. Let X be a compact balanced hyperbolic manifold with $\dim_{\mathbb{C}} X = n$.

1. If X is degenerate balanced, then there are no complex hypersurfaces in X.
2. If X contains a complex hypersurface, then $\pi_1(X)$ is infinite

Proof.

1. Assume X is degenerate balanced and ω is a degenerate balanced metric on X, that is $\omega^{n-1} = d\alpha$ for some smooth $(2n-3)$-form α on X. Assume there exists a complex hypersurface $H \subset X$, then:

$$\mathrm{Vol}_\omega(H) = \frac{1}{(n-1)!}\int_H \omega^{n-1} = \frac{1}{(n-1)!}\int_H d\alpha = \frac{1}{(n-1)!}\int_{\partial H} \alpha = 0 \quad (\text{since } \partial H = \emptyset)$$

   This is a contradiction, hence there are no complex hypersurfaces in X.

2. Let ω be a balanced metric on X such that $\widetilde{\omega}^{n-1} = d\alpha$ for some $\widetilde{\omega}$-bounded $(2n-3)$-form α on $\widetilde{X}$, and let $\iota : H \hookrightarrow X$ be a complex hypersurface in X. It suffices to show that the image of $\iota_* : \pi_1(H) \longrightarrow \pi_1(X)$ is infinite. We argue by contradiction. Suppose that the image of $\iota_* : \pi_1(H) \longrightarrow \pi_1(X)$ is finite. Passing to a finite cover, one then obtains a compact hypersurface $\widetilde{H} \subset \widetilde{X}$ such that the restriction $\pi_{|\widetilde{H}} : \widetilde{H} \longrightarrow H$ is finite (let us say of degree $k > 0$). This would imply that:

$$\mathrm{Vol}_{\widetilde{\omega}}(\widetilde{H}) = \frac{1}{(n-1)!}\int_{\widetilde{H}} \widetilde{\omega}^{n-1} = \frac{k}{(n-1)!}\int_H \omega^{n-1} = k\mathrm{Vol}_\omega(H) > 0$$

   On the other hand, we have:

$$\mathrm{Vol}_{\widetilde{\omega}}(\widetilde{H}) = \frac{1}{(n-1)!}\int_{\widetilde{H}} \widetilde{\omega}^{n-1} = \frac{1}{(n-1)!}\int_{\widetilde{H}} d\alpha = \frac{1}{(n-1)!}\int_{\partial\widetilde{H}} \alpha = 0$$

due to the compactness of $\widetilde{H}$, yielding a contradiction. Hence, $\pi_1(X)$ is infinite.

$\square$

Remark 3.2.3. 1. Note that Observation 3.2.2.(2) does not require the boundedness of α, it only uses the d-exactness of $\widetilde{\omega}^{n-1}$.

2. We can now infer that the degenerate balanced property is not stable under smooth modifications. Indeed, let X be a compact degenerate balanced manifold, and consider $\mu : \widehat{X} \longrightarrow X$ to be the blow-up of X at a point $x \in X$. Then the exceptional divisor $E := \mu^{-1}(x)$ is a complex hypersurface in $\widehat{X}$, hence $\widehat{X}$ is **not** degenerate balanced due to Observation 3.2.2.(1). Furthermore, $\widehat{X}$ is not even balanced hyperbolic since $E \simeq \mathbb{P}^{n-1}$ and $\mathbb{P}^{n-1}$ cannot be embedded in X. More precisely, we have:

   Proposition 3.2.4 ([MP23], Proposition 2.10)**.** *Let X be a compact balanced hyperbolic manifold. Then there is no holomorphic map $f : \mathbb{P}^{n-1} \longrightarrow X$ such that f is non-degenerate at some point $x \in \mathbb{P}^1$.*

It is natural to ask about the converse: if $f : \widehat{X} \longrightarrow X$ is a smooth modification of compact complex manifolds with $\widehat{X}$ is degenerate balanced, then is X degenerate balanced ?

When dealing with smooth modifications of compact degenerate balanced manifolds, we adopt the following characterization:

Proposition 3.2.5 ([Pop15], Proposition 5.4)**.** *A compact complex manifold X is degenerate balanced if and only if it admits no non-zero d-closed $(1,1)$-current $T \geq 0$.*

The first part of Observation 3.2.2 is actually a direct consequence of Proposition 3.2.5, since the existence of a complex hypersurface $H \subset X$ naturally gives rise to the *current of integration* over H:

$$[H] : \mathscr{C}^{\infty}_{n-1,n-1}(X) \longrightarrow \mathbb{C} \quad , \qquad u \longmapsto \int_H u$$

This defines a non-zero d-closed positive $(1,1)$-current.

Building on the ideas of [Pop13b], Theorem 2.2, we establish the following:

Proposition 3.2.6. *Let $f : \widehat{X} \longrightarrow X$ be a smooth* ***surjective*** *modification of compact complex manifolds, with $\dim_{\mathbb{C}} X = n \geq 3$. If $\widehat{X}$ is degenerate balanced, then X is again degenerate balanced.*

Proof. Let E be the exceptional divisor of f on $\widehat{X}$, and let $A \subset X$ be the analytic subset such that the restriction $f_{|\widehat{X}\setminus E} : \widehat{X}\setminus E \longrightarrow X\setminus A$ is a biholomorphism. The proof proceeds by contradiction. If X is not degenerate balanced, Proposition 3.2.5 yields the existence of a non-zero d-closed $(1,1)$-current $T \geq 0$. Then there exists an open cover $\{U_j\}_{j\in I}$ of X such that $T_{|U_j} = i\partial\bar{\partial}\varphi_j$ for some plurisubharmonic function $\varphi_j : U_j \longrightarrow \mathbb{R}\cup\{-\infty\}$. In particular, $f^*T_{|_{f^{-1}(u_j)}} = i\partial\bar{\partial}(\varphi_j \circ f)$, and these local pieces glue together into a globally defined d-closed $(1,1)$-current $f^*T \geq 0$ on $\widehat{X}$ thanks to the surjectivity of f (see [Meo96] for further details). It remains to check that f^*T is not the zero current on $\widehat{X}$ to conclude that $\widehat{X}$ is **not** degenerate balanced again thanks to Proposition 3.2.5. Assume, for the sake of contradiction, that $f^*T = 0$. This would mean that $\operatorname{Supp} T \subset A$. We distinguish the following two cases:

Case 1. If all the irreducible components A_j of A are such that $\operatorname{codim}_{\mathbb{C}} A_j \geq 2$ in X, the support theorem [Corollary 2.11 in [Dem97], Chapter III] forces $T = 0$, which is a contradiction.

Case 2. If A had certain global irreducible components A_j with $\operatorname{codim}_{\mathbb{C}} A_j = 1$ in X, then $T = \sum_j \lambda_j [A_j]$ where $\lambda_j \geq 0$ and $[A_j]$ is the current of integration on A_j thanks to another support theorem [Corollary 2.14 in [Dem97], Chapter III]. So $f^*T = \sum_j \lambda_j [f^{-1}(A_j)] = 0$ implies $\lambda_j = 0$ for all j, meaning that $T = 0$ on X, yielding again a contradiction. Hence the desired result follows. $\square$

3.2.2 Deformations of Balanced hyperbolic manifolds

The remainder of this section is devoted to the behavior of balanced hyperbolicity under small holomorphic deformations. Let $X_t : t \in \mathbb{B}_\varepsilon(0)$ be an analytic family of compact complex n-dimensional manifolds with X_0 balanced, where $\mathbb{B}_\varepsilon(0) \subset \mathbb{C}^N$ is the open ball of radius $\varepsilon > 0$ centered at 0. L. Alessandrini and G. Bassanelli observed that the balanced property is, in general, **not** preserved under small deformations; a classical counterexample is given by the small deformation of the Iwasawa manifold constructed by Nakamura (see [Nak72], [AB90]). A first extra condition ensuring stability is the $\partial\bar{\partial}$-property: C.-C. Wu proved in [Wu06] (Theorem 5.12) that satisfying the $\partial\bar{\partial}$-property is **open** under small deformations, and if X_0 is a balanced $\partial\bar{\partial}$-manifold, the nearby fibers X_t are balanced $\partial\bar{\partial}$-manifolds. However, D. Popovici conjectured in [Pop17] that every compact $\partial\bar{\partial}$-manifold carries a balanced metric (Conjecture 6.1). Nevertheless, the $\partial\bar{\partial}$ assumption can be weakened while still preserving the deformation stability of the balanced property (see [FY11], [AU17]). One such condition will be discussed later in this section.

Let us consider the *co-polarized* case (This is a generalization of the polarization of a fiber by the De Rham class of a Kähler metric, which is possible thanks to Ehresmann's Theorem [Ehr47]): Every holomorphic family of compact complex manifolds is locally $\mathscr{C}^\infty$ trivial. In particular, for every degree k we have: $H^k_{DR}(X_t,\mathbb{C}) \simeq H^k_{DR}(X_0,\mathbb{C})$). First, we recall the definition of *Bott-Chern cohomology* and *Aeppli cohomology*:

Definition 3.2.7. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.*

1. *The* ***Bott-Chern cohomology group*** *of bidegree (p,q) of X is defined as follows:*

$$H^{p,q}_{BC}(X,\mathbb{C}) = \frac{Ker\left(\partial : \mathscr{C}^\infty_{p,q}(X) \longrightarrow \mathscr{C}^\infty_{p+1,q}(X)\right) \cap Ker\left(\bar\partial : \mathscr{C}^\infty_{p,q}(X) \longrightarrow \mathscr{C}^\infty_{p,q+1}(X)\right)}{Im\left(\partial\bar\partial : \mathscr{C}^\infty_{p-1,q-1}(X) \longrightarrow \mathscr{C}^\infty_{p,q}(X)\right)}$$

   and the ***Bott-Chern numbers*** *are $h^{p,q}_{BC}(X) := \dim_{\mathbb{C}} H^{p,q}_{BC}(X,\mathbb{C})$.*

2. *The* ***Aeppli cohomology group*** *of bidegree (p,q) of X is defined as follows:*

$$H^{p,q}_{A}(X,\mathbb{C}) = \frac{Ker\left(\partial\bar\partial : \mathscr{C}^\infty_{p,q}(X) \longrightarrow \mathscr{C}^\infty_{p+1,q+1}(X)\right)}{Im\left(\partial : \mathscr{C}^\infty_{p-1,q}(X) \longrightarrow \mathscr{C}^\infty_{p,q}(X)\right) + Im\left(\bar\partial : \mathscr{C}^\infty_{p,q-1}(X) \longrightarrow \mathscr{C}^\infty_{p,q}(X)\right)}$$

   and the ***Aeppli numbers*** *are $h^{p,q}_{A}(X) := \dim_{\mathbb{C}} H^{p,q}_{A}(X,\mathbb{C})$.*

We recall some of the key characterizations of $\partial\bar\partial$-manifolds, highlighting their good Hodge-theoretic properties (which is why they are sometimes referred to as **cohomologically Kähler**), following ideas from [DGMS75], [Pop14] and [AT13] (see also the exposition in [Pop22], §1.3).

Proposition 3.2.8. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. The following statements are equivalent:*

1. *X is a $\partial\bar\partial$-manifold, i.e. $\forall p,q = 0,\dots,n$, $\forall u \in \mathscr{C}^\infty_{p,q}(X) \cap Ker\,\partial \cap Ker\,\bar\partial$, we have:*

$$u \in Im\ d \iff u \in Im\ \partial \iff u \in Im\ \bar\partial \iff u \in Im\ \partial\bar\partial$$

2. *X have the* ***Hodge Decomposition property****, i.e. $\forall p,q = 0,\dots,n$, every Dolbeault cohomology class $\{\alpha^{p,q}\}_{\bar\partial} \in H^{p,q}_{\bar\partial}(X,\mathbb{C})$ can be represented by a d-closed (p,q)-form and for every $k = 0,\dots,2n$, the linear map:*

$$\bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X,\mathbb{C}) \ni \sum_{p+q=k} \{\alpha^{p,q}\}_{\bar\partial} \longmapsto \left\{\sum_{p+q=k} \alpha^{p,q}\right\} \in H^{p,q}_{DR}(X,\mathbb{C})$$

   is ***well-defined*** *by means of d-closed* ***pure-type*** *representatives $\alpha^{p,q}$ of their respective Dolbeault cohomology classes (i.e. $\partial\alpha^{p,q} = \bar\partial\alpha^{p,q} = 0$) and* ***bijective****.*

3. $\forall p,q = 0,\dots,n$, *the canonical linear maps (induced by the identity):*

$$H^{p,q}_{BC}(X,\mathbb{C}) \longrightarrow H^{p,q}_{\bar\partial}(X,\mathbb{C}) \qquad \text{and} \qquad H^{p,q}_{\bar\partial}(X,\mathbb{C}) \longrightarrow H^{p,q}_{A}(X,\mathbb{C})$$

are isomorphisms.

Definition 3.2.9 ([Pop19], Definition 4.1)**.** *Let $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ be an analytic family of compact $\partial\bar\partial$-manifolds of complex dimension n. Assume X_0 is balanced, and fix a balanced class $\{\omega_0^{n-1}\}_{BC} \in H^{n-1,n-1}_{BC}(X_0,\mathbb{C}) \subset H^{2n-2}_{DR}(X_0,\mathbb{C})$. We say that a fiber X_t is **co-polarized** by the De Rham class $\{\omega_0^{n-1}\}_{BC}$ if it is of type $(n-1,n-1)$ for the complex structure on X_t.*

The deformation stability of balanced hyperbolicity on co-polarized fibers can be deduced from the following results:

Theorem 3.2.10 ([Pop19], Observation 7.2)**.** *Let (X_0,ω_0) be a balanced $\partial\bar\partial$-manifold. For $\varepsilon > 0$ sufficiently small, the De Rham class $\{\omega_0^{n-1}\}_{BC} \in H^{2n-2}_{DR}(X_0,\mathbb{C})$ defines a balanced class for the complex structure on the fibers X_t that are co-polarized by it.*

Lemma 3.2.11 ([CY18], Lemma 2.4)**.** *The $\widetilde{d}$-boundedness property is a **cohomological** property, i.e. if a closed k-form α on a compact Riemannian manifold X is $\widetilde{d}$(bounded), every k-form $\beta \in \{\alpha\} \in H^k_{DR}(X,\mathbb{R})$ is $\widetilde{d}$(bounded).*

Therefore, we arrive at the following conclusion:

Corollary 3.2.12. *Let $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ be an analytic family of compact $\partial\bar\partial$-manifolds with $\dim_{\mathbb{C}} X_t = n$, and assume (X_0,ω_0) balanced hyperbolic. Then the fibers X_t co-polarized by $\{\omega_0^{n-1}\} \in H^{2n-2}_{DR}(X_0,\mathbb{C})$ are also balanced hyperbolic (for t sufficiently close to 0).*

To establish a more general deformation result for balanced hyperbolic manifolds, we recall the following variants of the $\partial\bar\partial$-property in bidegree (p,q) (with $p,q \in \{0,\dots,n\}$) following Definition 5 in [FY11], Definition 2.1 in [AU17], Definition 3.1 and Theorem 1.1 in [RWZ19].

Definition 3.2.13. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$, and let $p,q \in \{0,\dots,n\}$. X is said to satisfy the :*

1. ***weak** (p,q)-$\partial\bar\partial$**-property** if for any form $\alpha \in \mathscr{C}^\infty_{p,q}(X)$ we have:*

$$\alpha \in Im\,\partial \cap Im\,\bar\partial \Longrightarrow \bar\partial\alpha \in Im\,\partial\bar\partial$$

2. ***mild* $(\boldsymbol{p},\boldsymbol{q})$-$\partial\bar{\partial}$-*property*** *if for any form $\alpha \in \mathscr{C}^{\infty}_{p,q}(X)$ we have:*

$$\alpha \in Ker\bar{\partial} \cap Im\partial \Longrightarrow \alpha \in Im\partial\bar{\partial}$$

3. ***strong* $(\boldsymbol{p},\boldsymbol{q})$-$\partial\bar{\partial}$-*property*** *if for any form $\alpha \in \mathscr{C}^{\infty}_{p,q}(X)$ we have:*

$$\alpha \in Kerd \cap (Im\partial + Im\bar{\partial}) \Longrightarrow \alpha \in Im\partial\bar{\partial}$$

Remark 3.2.14. 1. For any bidegree (p,q), we have :

$$\begin{aligned}\text{The strong}(p,q)\text{-}\partial\bar{\partial}\text{-property} &\Longrightarrow \text{The mild}(p,q)\text{-}\partial\bar{\partial}\text{-property}\\ &\Longrightarrow \text{The weak}(p,q)\text{-}\partial\bar{\partial}\text{-property}\end{aligned}$$

2. While the *strong* (p,q)-$\partial\bar{\partial}$-property are stable under small deformations of the complex structure (Proposition 4.8, [AU17]), the *weak* (p,q)-$\partial\bar{\partial}$-property and the *mild* (p,q)-$\partial\bar{\partial}$-property are **not** open under deformations (see [UV15], Example 3.7 for the case $(n-1,n)$).

If the weak $(n-1,n)$-$\partial\bar{\partial}$-property holds on the fibers X_t, for t sufficiently close to 0, then the balanced property is open under small deformations ([FY11], Theorem 6). Hence, a first result in the desired direction is the following:

Theorem 3.2.15. *Let $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ be an analytic family of compact complex n-dimensional manifolds such that X_0 is balanced. Assume that the fibers X_t satisfy the weak $(n-1,n)$-$\partial\bar{\partial}$-property for $t \neq 0$.*

1. *If X_0 is balanced hyperbolic, the fibers X_t carry balanced metrics ω_t that are **Gauduchon hyperbolic** for small t.*
2. *If moreover, X_0 is degenerate balanced and the fibers X_t satisy the mild $(n,n-2)$-$\partial\bar{\partial}$-property for $t \neq 0$, the fibers X_t are degenerate balanced for small t.*

Here, *Gauduchon hyperbolicity* is understood in the sense of Definition 3.8 of [Mar23] (its precise definition will be recalled in the next section; Definition 3.3.2).

Proof. Let us first revisit the beginning of the proof of Theorem 6 in [FY11]. Let X be the underlying differentiable manifold of the given analytic family, and let ω be a balanced metric on X_0, and let $\{h_t\}_{t\in\mathbb{B}_\varepsilon(0)}$ be a $\mathscr{C}^\infty$-family of Hermitian metrics on $(X_t)_{t\in\mathbb{B}_\varepsilon(0)}$ with $h_0 = \omega$. We decompose $\Omega := \omega^{n-1}$ with respect to the complex structure on X_t:

$$\Omega = \Omega_t^{n,n-2} + \Omega_t^{n-1,n-1} + \Omega_t^{n-2,n}, \qquad t \in \mathbb{B}_\varepsilon(0) \tag{3.2.1}$$

then $\Omega_t^{n-1,n-1} > 0$ for t close enough to 0 by continuity (since $\Omega_0^{n-1,n-1} = \omega^{n-1} > 0$). Since Ω is d-closed, we get: $\bar{\partial}_t\Omega_t^{n-1,n-1} + \partial_t\Omega_t^{n-2,n} = 0$, meaning that $\bar{\partial}_t\Omega_t^{n-1,n-1} \in \mathscr{C}^\infty_{n-1,n}(X_t)$ lies in $\operatorname{Im}\partial_t \cap \operatorname{Im}\bar{\partial}_t$. Applying the weak $(n-1,n)$-$\partial\bar{\partial}$-property yields: $\bar{\partial}_t\Omega_t^{n-1,n-1} = i\partial_t\bar{\partial}_t u_t$ for some $u_t \in \mathscr{C}^\infty_{n-2,n-1}(X_t) \cap (\operatorname{Ker}(\partial_t\bar{\partial}_t))^\perp$. Then:

$$\Omega_t := \Omega_t^{n-1,n-1} + i\partial_t u_t - i\bar{\partial}_t\bar{u}_t\,, \qquad \text{for } \ t \sim 0 \tag{3.2.2}$$

defines a continuous family of positive $(n-1,n-1)$-forms on X_t for t close enough to 0 (see the proof of Theorem 6, [FY11]). Hence, there exist a balanced metric ω_t on each respective fiber X_t for t sufficiently small, defined as the $(n-1)$-th root of Ω_t (Thanks to Michelson's trick (ii)', p.19 in [Mic82]), with $\omega_0 = \omega$.

1. Suppose that ω is balanced hyperbolic. Then $\widetilde{\Omega} = \widetilde{\omega}^{n-1} = d\Gamma$ for some smooth and $\widetilde{\omega}$-bounded $(2n-3)$-form Γ on $\widetilde{X}$. Then by identifying the bidegree $(n-1,n-1)$-component of this expression with respect to the complex structure on $\widetilde{X}_t$, we get:

$$\widetilde{\Omega_t}^{n-1,n-1} = \partial_t\Gamma_t^{n-2,n-1} + \bar{\partial}_t\Gamma_t^{n-1,n-2}\,, \qquad \forall t \in \mathbb{B}_\varepsilon(0) \tag{3.2.3}$$

   After lifting (3.2.2) to $\widetilde{X}_t$, (3.2.3) implies:

$$\widetilde{\Omega_t} = \partial_t(\Gamma_t^{n-2,n-1} + i\pi^\star u_t) + \bar{\partial}_t(\Gamma_t^{n-1,n-2} - i\pi^\star \bar{u}_t)\,, \qquad \forall t \in \mathbb{B}_\varepsilon(0) \tag{3.2.4}$$

   Put: $\alpha_t := \Gamma_t^{n-2,n-1} + i\pi^\star u_t$. Since $\widetilde{\Omega_t}^{n-1,n-1}$ is real, one can choose Γ such that $\Gamma_t^{n-1,n-2} = \overline{\Gamma_t^{n-2,n-1}}$. Then:

$$\widetilde{\Omega_t} = \widetilde{\omega_t}^{n-1} = \partial_t\alpha_t + \bar{\partial}_t\bar{\alpha}_t\,, \qquad \text{for } \ t \sim 0 \tag{3.2.5}$$

   with:

$$\|\alpha_t\|_{L^\infty_{\widetilde{h_t}}(\widetilde{X}_t)} \leq \|\Gamma_t^{n-2,n-1}\|_{L^\infty_{\widetilde{h_t}}(\widetilde{X}_t)} + \|\pi^\star u_t\|_{L^\infty_{\widetilde{h_t}}(\widetilde{X}_t)} \leq C_t\|\Gamma\|_{L^\infty_{\widetilde{\omega}}(\widetilde{X})} + \|u_t\|_{L^\infty_{h_t}(X_t)} < \infty$$

   where the constant $C_t > 0$ depends only on the metric h_t on the **compact** fiber X_t (the metrics ω and h_t are comparable on X, $\forall t \sim 0$). Hence, the balanced metrics ω_t are Gauduchon hyperbolic on the respective fibers X_t for $t \sim 0$.

2. Now let ω be a degenerate balanced metric on X_0, that is $\Omega := \omega^{n-1} = d\alpha$ for some $\alpha \in \mathscr{C}^\infty_{2n-3}(X)$ (in particular $n \geq 3$). Then the decomposition (3.2.1) yields:

$$\Omega_t^{n-1,n-1} = \bar{\partial}_t\alpha_t^{n-1,n-2} + \partial_t\alpha_t^{n-2,n-1}\,, \qquad t \in \mathbb{B}_\varepsilon(0) \tag{3.2.6}$$

   Since $\Omega_t^{n-1,n-1}$ is real, one can choose α such that $\alpha_t^{n-1,n-2} = \overline{\alpha_t^{n-2,n-1}}$. Hence, one gets:

$$\Omega_t = \partial_t(\alpha_t^{n-2,n-1} + iu_t) + \bar\partial_t(\alpha_t^{n-1,n-2} - i\bar u_t) = \bar v_t + v_t\,, \qquad t \in \mathbb{B}_\varepsilon(0) \tag{3.2.7}$$

where $v_t := \bar\partial_t(\alpha_t^{n-1,n-2} - i\bar u_t) \in \mathscr{C}^\infty_{n-1,n-1}(X_t) \overset{(*)}{\cap} \operatorname{Ker}\partial_t \cap \operatorname{Im}\bar\partial_t$ (here $(*)$ follows from the fact that $\partial_t\Omega_t = \bar\partial_t\Omega = 0$). It remains to check that the following system is solvable for:

$$\begin{cases} \bar\partial w_t = v_t \quad , \qquad t \neq 0 \\ \partial w_t = 0 \end{cases} \tag{3.2.8}$$

This would imply that $\omega_t^{n-1} = \Omega_t = \bar v_t + v_t = \partial_t\bar w_t + \bar\partial_t w_t = d(w_t + \bar w_t)$, hence the metrics ω_t are degenerate balanced.

Since $v_t \in \operatorname{Im}\bar\partial_t$, there exists $\beta_t \in \mathscr{C}^\infty_{n-1,n-2}(X_t)$, such that $v_t = \bar\partial_t\beta_t$, and since $\partial_t v_t = 0$, we have $\partial_t\beta_t \in \operatorname{Ker}\bar\partial_t \cap \operatorname{Im}\partial_t$. By applying the mild $(n,n-2)$-$\partial\bar\partial$-property on each fiber X_t, we get $\partial_t\beta_t = \partial_t\bar\partial_t\gamma_t$, for some $\gamma_t \in \mathscr{C}^\infty_{n-1,n-3}(X_t)$. Then $w_t := \beta_t - \bar\partial_t\gamma_t$ satisfies the system (3.2.8). Hence the desired result follows.

□

Remark 3.2.16. We propose the following "alternative version" of Theorem 3.2.15 in the case where the family satisfies the $\partial\bar\partial$-property.

Theorem 3.2.17. *Let $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ be an analytic family of compact $\partial\bar\partial$-manifolds of complex dimension n such that X_0 is balanced.*

1. *Assume X_0 is balanced hyperbolic, and let ω be a balanced hyperbolic metric on X_0. Then ω extends to a $\mathscr{C}^\infty$-family of balanced metrics $\{\omega_t\}_{t\sim 0}$ that are Gauduchon hyperbolic on the respective fibers X_t for small t.*

2. *If moreover, ω is degenerate balanced, the $\mathscr{C}^\infty$-family of balanced metrics $\{\omega\}_{t\sim 0}$ on the fibers $\{X_t\}_{t\sim 0}$ are degenerate balanced with a $\mathscr{C}^\infty$-family of d-potentials $\{w_t\}_{t\sim 0}$ for small t, i.e. $\omega = dw_0$ and $\omega_t = dw_t$, $\forall t \sim 0$.*

Proof. 1. As in Theorem 3.2.15, let ω be a balanced hyperbolic metric on X_0, and let $\{h_t\}_{t\in\mathbb{B}_\varepsilon(0)}$ be a $\mathscr{C}^\infty$-family of Hermitian metrics on $(X_t)_{t\in\mathbb{B}_\varepsilon(0)}$ with $h_0 = \omega$, and let $\Omega := \omega^{n-1}$. When decomposing Ω with respect to the complex structure of a given fiber X_t, we get thanks to the $\partial_t\bar\partial_t$-property that: $\bar\partial_t\Omega_t^{n-1,n-1}$ is $\partial_t\bar\partial_t$-exact, i.e. $\bar\partial_t\Omega_t^{n-1,n-1} = i\partial_t\bar\partial_t u_t$ for some $u_t \in \mathscr{C}^\infty_{n-2,n-1}(X_t) \cap (\operatorname{Ker}(\partial_t\bar\partial_t))^\perp$. The form u_t can be chosen as the minimal $L^2_{h_t}$-norm solution of the equation $\bar\partial_t\Omega_t^{n-1,n-1} = i\partial_t\bar\partial_t u_t$, which is given by the Neumann-type formula (See [Pop15], Theorem 4.1):

$$u_t = (\partial_t\bar\partial_t)^\star \mathbb{G}_{BC,t}(\bar\partial_t(i\Omega_t^{n-1,n-1})) \quad , \qquad t \sim 0$$

where $\mathbb{G}_{BC}$ is the **Green operator** associated with the **Bott-Chern Laplacian** Δ_{BC} defined as follows (see [KS60], Section 6):

$$\Delta_{BC} = \partial^\star\partial + \bar\partial^\star\bar\partial + (\partial\bar\partial)^\star(\partial\bar\partial) + (\partial\bar\partial)(\partial\bar\partial)^\star + (\partial^\star\bar\partial)^\star(\partial^\star\bar\partial) + (\partial^\star\bar\partial)(\partial^\star\bar\partial)^\star.$$

Then $\{u_t\}_{t\sim 0}$ defines a $\mathscr{C}^\infty$-family of $(n-2, n-1)$-forms on the fibers $(X_t)_{t\sim 0}$ thanks to Theorem 5 in [KS60]. Hence the family $\{\Omega_t\}_{t\in\mathbb{B}_\varepsilon(0)}$ defined by the formula (3.2.2) gives a $\mathscr{C}^\infty$-family of positive $(n-1, n-1)$-forms on the fibers $\{X_t\}_{t\sim 0}$ for t small enough, with $\Omega_0 = \Omega$. Thus, taking the $(n-1)$-root of each Ω_t defines a $\mathscr{C}^\infty$-family of balanced metrics $\{\omega_t\}_{t\sim 0}$ on the respective fibers X_t, that are Gauduchon hyperbolic according to Theorem 3.2.15.(1) with $\omega_0 = \omega$.

2. Moreover, if ω is degenerate balanced, the family of forms w_t defined by the system (3.2.8) (which is solvable thanks to the $\partial_t\bar\partial_t$-property) can be taken as the minimal $L^2_{h_t}$-norm solutions given by the following formula (Lemma 2.4, [DP21]):

$$w_t = \mathbb{G}_{BC,t}(\bar\partial_t^\star v_t + \bar\partial_t^\star\partial_t\partial_t^\star v_t)$$

Hence the family $\{w_t\}_{t\sim 0}$ varies in a smooth way with respect to t thanks to Theorem 5 in [KS60], and the $\mathscr{C}^\infty$-family of $(n-1, n-1)$ forms $\{\omega_t^{n-1}\}_{t\sim 0}$ admits a $\mathscr{C}^\infty$-family of d-potentials.

□

A consequence of Theorem 3.2.15 is the following:

Corollary 3.2.18. *If $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ is an analytic family of compact complex n-dimensional manifolds such that X_0 is balanced hyperbolic with $H^{2,0}_{\bar\partial}(X_0, \mathbb{C}) = 0$, the fibers X_t are Gauduchon hyperbolic for small t.*

Proof. Assume X_0 is a balanced hyperbolic manifold and $H^{2,0}_{\bar\partial}(X_0, \mathbb{C}) = 0$. Since the function $t \in \mathbb{B}_\varepsilon(0) \longmapsto h^{p,q}_{\bar\partial_t}(t) = \dim_{\mathbb{C}} H^{p,q}_{\bar\partial_t}(X_t, \mathbb{C})$ is upper semicontinuous in t (Theorem 6, [KS60]), we get $h^{2,0}_{\bar\partial_t}(X_t) \leq h^{2,0}_{\bar\partial}(X_0) = 0$, which implies that $H^{2,0}_{\bar\partial_t}(X_t, \mathbb{C}) = 0$ for small t. By Serre duality, we deduce that $H^{n-2,n}_{\bar\partial_t}(X_t, \mathbb{C}) = 0$. Now, let $\alpha_t \in \mathscr{C}^\infty_{n-1,n-1}(X_t)$ be such that $\bar\partial_t\alpha_t \in \text{Im}\partial_t$, i.e. $\bar\partial_t\alpha_t = \partial_t\beta_t$, for some $\beta_t \in \mathscr{C}^\infty_{n-2,n}(X_t)$. Since $\bar\partial_t\beta_t = 0$ (for bidegree reasons) and $H^{n-2,n}_{\bar\partial_t}(X_t, \mathbb{C}) = 0$, $\beta_t \in \text{Im}\bar\partial_t$. Hence $\bar\partial_t\alpha_t \in \text{Im}(\partial_t\bar\partial_t)$, which means that the $(n-1, n)$-th weak $\partial\bar\partial$-Lemma is satisfied on X_t. We conclude that the nearby fibers X_t are also balanced hyperbolic (for $t \sim 0$) thanks to Theorem 3.2.15. □

Examples 3.2.19. 1. Under mild hypotheses, R. Friedman proved that the connected sums $X = \#_k(S^3 \times S^3)$ for $k \geq 2$ are $\partial\bar{\partial}$-manifolds (viewed as small smoothings of **Clemens manifolds**) [Fri20]. This result was later extended to full generality by C. Li [Li24], thereby answering positively Question 5.6 in [Pop15]. It follows from Theorem 3.2.17.(2) that any degenerate balanced metric ω on X extends to a $\mathscr{C}^\infty$-family of degenerate balanced metrics $\{\omega\}_{t\sim 0}$ on any small family $\{X_t\}_{t\sim 0}$ of deformations of X.

2. Set $X = \#_k(\mathbb{S}^3 \times \mathbb{S}^3)$, and let Y be a *fake projective plane* (or a *Mumford surface*); this is a compact complex surface with the same cohomology as $\mathbb{P}^2$ (see [Mum79]). It can be viewed as an arithmetic quotient of the hyperbolic ball $\mathbb{B}^2_{\mathbb{C}}$. Since X is a degenerate balanced $\partial\bar{\partial}$-manifold and Y is Kähler hyperbolic, then $Z := X \times Y$ is a balanced hyperbolic $\partial\bar{\partial}$-manifold. Applying the Künneth formula and the Hodge decomposition yields :

$$h^{2,0}(Z) = h^{2,0}(X) + h^{1,0}(X)h^{1,0}(Y) + h^{2,0}(Y) = 0$$

Hence any small deformation of Z is balanced Gauduchon hyperbolic thanks to Corollary 3.2.18. Moreover, Theorem 3.2.17.(1) yields that given any balanced hyperbolic metric on Z, one construct a $\mathscr{C}^\infty$-family of balanced metrics $\{\omega_t\}_t$ that are Gauduchon hyperbolic on small deformations $\{Z_t\}_t$ of Z.

In the spirit of Conjecture 4.23 in [Khe25], we propose the following conjecture:

Conjecture 3.2.20. *Let X be a compact balanced manifold. If X admits a balanced metric ω which is Gauduchon hyperbolic, then X is balanced hyperbolic.*

In order to address the deformation openness of balanced hyperbolicity and answer positively Conjecture 3.2.20, one would need to solve the $\bar{\partial}$ and ∂-problems of the form 3.2.8 and 3.2.1 in bidegrees $(n-2, n-1)$ and $(n-1, n-2)$, respectively, with L^∞-estimates on a complete balanced manifold (equipped with a balanced metric ω such that ω^{n-1} is $(\partial + \bar{\partial})$(bounded) if needed). At present, this appears to be out of reach, even in the compact setting.

Even obtaining an analogue of Corollary 4.20 in [Khe25] appears to be highly nontrivial. Indeed, to the best of our knowledge, no satisfactory bounds are currently available for the spectra of the Laplacians in bidegrees $(n-1, n-2)$ and $(n-2, n-1)$, in contrast with the situation in bidegrees $(n, n-1)$ and $(n-1, n)$, where such estimates have been established in [MP22], Corollary 3.7. This difficulty is further compounded by the fact that, on a balanced manifold, the Bott-Chern Laplacian Δ_{BC} does not commute with the ∂ and $\bar{\partial}$ operators and their adjoints ∂^* and $\bar{\partial}^*$, while the commutation deficit remains hard to control.

3.3 Degree $2p$ Hyperbolicity Notions

Let us recall some of the basic definitions. Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$, and $\pi : \widetilde{X} \longrightarrow X$ be its universal cover. Recall that for any k-form α in X, we denote its pullback to the universal cover $\widetilde{X}$ by $\pi^*\alpha =: \widetilde{\alpha}$.

Definition 3.3.1 (**Special metrics**)**.** *A Hermitian metric ω on X is said to be:*

1. *[Str86, Bis89]* ***Strong Kähler with torsion*** *(**SKT** for short) or **pluriclosed** if $\partial\bar{\partial}\omega = 0$.*
2. *[Gau77b]* ***Gauduchon*** *(or **standard**) if $\partial\bar{\partial}\omega^{n-1} = 0$.*
3. *([Sul76],[HL83],[Gau84], [ST10])* ***Hermitian-symplectic*** *(**HS** for short) if $\exists \alpha \in \mathscr{C}^{\infty}_{2,0}(X)$ such that $d(\alpha + \omega + \bar{\alpha}) = 0$. Or equivalently, $\partial\omega$ is $\bar{\partial}$-exact.*

We also recall the definition of the hyperbolicity notions associated with SKT and Gauduchon metrics on a compact complex manifold X.

Definition 3.3.2 (**SKT and Gauduchon hyperbolicity**, [Mar23])**.**

1. *A (p, q)-form η on (X, ω) is said to be $\widetilde{(\partial + \bar{\partial})}$(**bounded**) if there exist a $\widetilde{\omega}$-bounded $(p-1, q)$-form α and a $\widetilde{\omega}$-bounded $(p, q-1)$-form β on $\widetilde{X}$ such that $\widetilde{\eta} = \partial\alpha + \bar{\partial}\beta$.*
2. *X is said to be **SKT hyperbolic** if it admits a $\widetilde{(\partial + \bar{\partial})}$(bounded) SKT metric.*
3. *X is said to be **Gauduchon hyperbolic** if it admits a Gauduchon metric ω such that ω^{n-1} is $\widetilde{(\partial + \bar{\partial})}$(bounded).*

3.3.1 Hermitian-symplectic Hyperbolicity

First, we recall the definition of sG metrics and sG *hyperbolicity*:

Definition 3.3.3 ([Pop13a], Definition 4.1 & Proposition 4.2)**.** *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. A Hermitian metric ω on X is **strongly Gauduchon** (or **sG** for short) if one of the following equivalent conditions holds:*

1. *The $(n, n-1)$-form $\partial\omega^{n-1}$ is $\bar{\partial}$-exact.*
2. *$\exists \alpha \in \mathscr{C}^{\infty}_{n,n-2}(X)$ such that $d(\alpha + \omega^{n-1} + \bar{\alpha}) = 0$.*

*If X carries such a metric, we say that X is a **strongly Gauduchon** (or **sG**) manifold.*

Definition 3.3.4 ([Ma24], Definition 2.2)**.** *We say that X is **sG hyperbolic** if there exists an sG metric ω on X such that ω^{n-1} is the $(n-1, n-1)$-component of a real $\widetilde{d}$(bounded), d-closed smooth $(2n-2)$-form Ω.*

Now, we define the notion of *Hermitian-symplectic hyperbolicity* (or *HS hyperbolicity* for short) on compact complex manifolds:

Definition 3.3.5. *We say that a compact complex manifold X with $\dim_{\mathbb{C}} X = n$ is* ***HS hyperbolic*** *if it admits a real smooth closed 2-form Ω such that:*

1. *$\omega := \Omega^{1,1}$ is positive definite, i.e. ω is a Hermitian metric.*
2. *Ω is $\widetilde{d}$(bounded), i.e. $\widetilde{\Omega} = d\eta$ for some* ***bounded*** *1-form η on $\widetilde{X}$.*

We also say that the metric ω is ***HS hyperbolic****.*

After all these definitions of hyperbolicity notions, we observe that the hierarchy that we have on special metrics remains preserved in the hyperbolic setting, that is to say:

Proposition 3.3.6. *Let (X, ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$. Then:*

$$\begin{tikzcd}
\omega \text{ is Kähler hyperbolic} \arrow[r, Rightarrow, "(1)"] \arrow[d, Rightarrow, "(3)"'] & \omega \text{ is HS hyperbolic} \arrow[r, Rightarrow, "(2)"] & \omega \text{ is SKT hyperbolic} \\
\omega \text{ is balanced hyperbolic} \arrow[r, Rightarrow, "(4)"'] & \omega \text{ is sG hyperbolic} \arrow[r, Rightarrow, "(5)"] & \omega \text{ is Gauduchon hyperbolic}
\end{tikzcd}$$

Almost all of the implications have been proved in the previously mentioned articles (where these hyperbolicity notions were defined). We provide proofs of the implications in this diagram to make the exposition self-contained.

Proof. (1) If ω is Kähler hyperbolic, $\Omega = \omega$ is a smooth real d-closed 2-form which is $\widetilde{d}$(bounded) and $\Omega^{1,1} = \Omega = \omega$.

(2) ω being HS hyperbolic means that $\omega > 0$ and there exists a smooth $(2,0)$-form α in X such that $\Omega = \alpha + \omega + \bar{\alpha}$ is a d-closed, $\widetilde{d}$(bounded) 2-form, i.e. $\widetilde{\Omega} = d\eta$, where η is a $\widetilde{\omega}$-bounded 1-form on $\widetilde{X}$. This implies that:

   (a) $\partial\bar{\partial}\omega = 0$ since $d\Omega = 0 \implies \omega$ is SKT.

   (b) $\widetilde{\omega} = \partial\eta^{0,1} + \bar{\partial}\eta^{1,0}$ (by taking the $(1,1)$-part in $\widetilde{\Omega} = d\eta$). Hence ω is SKT hyperbolic.

(3) If ω is Kähler hyperbolic, that is $\widetilde{\omega} = d\eta$ for some $\widetilde{\omega}$-bounded 1-form η on $\widetilde{X}$, then:

$$\widetilde{\omega}^{n-1} = \widetilde{\omega}^{n-2} \wedge \widetilde{\omega} = \widetilde{\omega}^{n-2} \wedge d\eta = d(\widetilde{\omega}^{n-2} \wedge \eta) \qquad \text{since } \widetilde{\omega} \text{ is closed}$$

Since $\widetilde{\omega}^{n-2} \wedge \eta$ is bounded, ω is balanced hyperbolic.

(4) This is obvious (the proof is similar to (1)).

(5) The proof is similar to (2).

□

S. Marouani proved in [Mar23] that SKT hyperbolicity implies Kobayashi hyperbolicity (Theorem 3.5). Thanks to (2) in Proposition 3.3.6, we deduce that:

Corollary 3.3.7. *Every HS hyperbolic compact complex manifold is Kobayashi hyperbolic.*

Based on the fact that every compact Hermitian-symplectic manifold is strongly Gauduchon (as shown in [YZZ23], Lemma 1, §.2, and further detailed in [DP21], Proposition 2.1), one can show that the following holds:

Proposition 3.3.8. *Every compact HS hyperbolic manifold X is sG hyperbolic.*

Proof. Let ω be a HS hyperbolic metric, this means that $\omega > 0$ and $\exists \alpha \in \mathscr{C}^\infty_{2,0}(X)$ such that $\Omega = \alpha + \omega + \bar{\alpha}$ is d-closed and $\widetilde{\Omega} = d\eta$ for some $\widetilde{\omega}$-bounded 1-form η on $\widetilde{X}$. Put $\Gamma := \Omega^{n-1}$, then:

1. $\widetilde{\Gamma} = \widetilde{\Omega}^{n-2} \wedge d\eta = d(\widetilde{\Omega}^{n-2} \wedge \eta)$ (since $\widetilde{\Omega}$ is closed) $\Longrightarrow \Gamma$ is $\widetilde{d}$(bounded).

2. $\Gamma = (\alpha + \omega + \bar{\alpha})^{n-1} = \displaystyle\sum_{k=0}^{n-1}\sum_{l=0}^{k} \binom{n-1}{k}\binom{k}{l} \alpha^l \wedge \bar{\alpha}^{k-l} \wedge \omega^{n-k-1}$. By taking the bidegree $(n-1, n-1)$ in this expression (which corresponds to $l = k - l$), we get:

$$\begin{aligned} \Gamma^{n-1,n-1} &= \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2l}\binom{2l}{l} \alpha^l \wedge \bar{\alpha}^l \wedge \omega^{n-2l-1} \\ &= \omega^{n-1} + 2\binom{n-1}{2} \alpha \wedge \bar{\alpha} \wedge \omega^{n-3} + 6\binom{n-1}{4} \alpha^2 \wedge \bar{\alpha}^2 \wedge \omega^{n-5} + \cdots \\ &\overset{(*)}{\geq} \omega^{n-1} > 0 \end{aligned}$$

where the inequality $(*)$ comes from the fact that $\alpha^l \wedge \bar{\alpha}^l \wedge \omega^{n-2l-1} \geq 0$ (see Chapter III in [Dem97], Example 1.2 and Proposition 1.11). We conclude thanks to Michelsohn's trick that $\omega' := \sqrt[n-1]{\Gamma^{n-1,n-1}}$ defines a sG hyperbolic metric on X.

□

3.3.2 p-SKT and p-HS Hyperbolicity

Now, we will generalize the (first-order) notions of SKT hyperbolicity and HS hyperbolicity to notions of hyperbolicity of degree $2p$ for $2 \leq p \leq n-1$, as was done in

[HM23], where Gromov's Kähler hyperbolicity was generalized to *p-Kähler hyperbolicity*. But first, let us recall the definition of a higher degree positivity notion (see Definition 1.1 in [HK72], and Definition 1.1, Chapter III in [Dem97]). This notion has been referred to as **transversality** in [Sul76], Definition I.3, and in [AA87], Definition 1.3.

Definition 3.3.9. *Let E be a $\mathbb{C}$-vector space with $\dim_{\mathbb{C}} E = n$, and E^* its dual. And let $1 \leq p, q \leq n-1$ be such that $p + q = n$.*

*(1) A (p,p)-form $u \in \Lambda^{p,p}(E^*)$ is **weakly strictly positive** if for any linearly independent $(1,0)$-forms $\alpha_1, \dots, \alpha_q \in E^*$ we have:*

$$u \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_q \wedge \bar{\alpha}_q$$

is a strictly positive volume form.

*(2) Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. A (p,p)-form $\alpha \in \mathscr{C}^{\infty}_{p,p}(X)$ is **weakly strictly positive** if $\forall x \in X$, $\alpha(x) \in \Lambda^{p,p} T^*_x X$ is weakly strictly positive in the sense of (1).*

Remark 3.3.10 (Corollary 1.5-Chapter III, [Dem97])**.** All weakly strictly positive (p,p)-forms u are **real**, i.e. $\bar{u} = u$.

Here, we recall the generalizations of special metrics in degree $2p$, with $1 \leq p \leq n-1$.

Definition 3.3.11. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$, and fix $1 \leq p \leq n-1$. Then X is said to be:*

1. ***p-Kähler** if it admits a d-closed weakly strictly positive (p,p)-form Ω. Here, Ω is called a **p-Kähler form** (Definition 1.11, [AA87]).*
2. ***p-Hermitian-symplectic** (or **p-HS** for short) if it admits a real d-closed $2p$-form Ω such that $\Omega^{p,p}$ is weakly strictly positive. Here, $\Omega^{p,p}$ is called a **p-HS form** (Definition 1.11, [AA87]).*
3. ***p-SKT** (or **p-pluriclosed**) if it admits a $\partial\bar{\partial}$-closed weakly strictly positive (p,p)-form Ω. Here, Ω is called a **p-SKT form** (Theorem 2.4, [Ale11]).*

Remark 3.3.12.

1. 1-Kähler=Kähler, 1-HS=HS and 1-SKT=SKT.
2. $(n-1)$-Kähler=Balanced, $(n-1)$-HS=sG, $(n-1)$-SKT=Gauduchon.
3. **Astheno-Kähler** $\Longrightarrow$ $(n-2)$-SKT (see §4 in [JY93]). More generally, the notion of p-SKT defined in [IP13] implies the notion of p-SKT used in this article, but they are not equivalent in general.
4. For every $1 \leq p \leq n-1$, we have: p-Kähler $\Longrightarrow$ p-HS $\Longrightarrow$ p-SKT.

Now, we define their associated hyperbolicity notions.

Definition 3.3.13. *Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$, and let $1 \leq p \leq n-1$. We say that X is:*

1. ***p-Kähler hyperbolic** if it admits a $\widetilde{d}$(bounded) p-Kähler form (Definition 3.1, [HM23]).*
2. ***p-HS hyperbolic** if it admits a $\widetilde{d}$(bounded) p-HS form (this was referred to as weakly p-Kähler hyperbolic in Definition 3.1, [Ma24]).*
3. ***p-SKT hyperbolic** if it admits a $\widetilde{(\partial+\bar{\partial})}$(bounded) p-SKT form.*

Remark 3.3.14.

1. The case $p = 1$ recovers the classical notions of Kähler hyperbolicity, HS hyperbolicity, and SKT hyperbolicity.
2. $(n-1)$-Kähler hyperbolic=Balanced hyperbolic, $(n-1)$-HS hyperbolic=sG hyperbolic and $(n-1)$-SKT hyperbolic=Gauduchon hyperbolic.
3. One can define *astheno-Kähler hyperbolicity* in a similar way: We say that a compact astheno-Kähler manifold X is **astheno-Kähler hyperbolic** if it admits an astheno-Kähler metric ω such that ω^{n-2} is $\widetilde{(\partial+\bar{\partial})}$(bounded). Then:
$$\text{Astheno-Kähler hyperbolic} \Longrightarrow (n-2)\text{-SKT hyperbolic}$$
4. The hierarchy between p-Kähler, p-HS and p-SKT forms still holds in the hyperbolic context. Namely, for every $1 \leq p \leq n-1$ we have:
$$p\text{-Kähler hyperbolic} \Longrightarrow p\text{-HS hyperbolic} \Longrightarrow p\text{-SKT hyperbolic}$$
5. By arguments similar to those used in Propositions 3.3.6.(3) and 3.3.8, one can show that a Kähler hyperbolic manifold is p-Kähler hyperbolic, and every HS hyperbolic manifold is p-HS hyperbolic, respectively, for any $1 \leq p \leq n-1$.

B.-L. Chen and X. Yang showed that a compact Kähler manifold homotopic to a compact negatively curved Riemannian manifold is Kähler hyperbolic (Theorem 1.1, [CY18]). This can be generalized to the p-Kähler hypebolic and p-HS hyperbolic cases for $1 \leq p \leq n-1$, thanks to the following property:

Proposition 3.3.15 (Proposition 7.1, [Pan93], Lemma 3.2, [CY18])**.** *Let M be a complete simply connected Riemannian manifold with Riemannian sectional curvature $K \leq -c^2 < 0$. Then every closed bounded k-form on M is d(bounded), for any $k \geq 2$.*

Note that such a manifold M cannot be compact since the Cartan–Hadamard theorem states that a complete simply connected Riemannian manifold with non-positive sectional curvature is diffeomorphic to $\mathbb{R}^{\dim M}$ via the exponential map at any point.

Theorem 3.3.16. *Let X be a compact p-Kähler (resp. p-HS) manifold with $\dim_{\mathbb{C}} X = n$, homotopic to a compact Riemannian manifold (Y, g_Y) with negative Riemannian sectional curvature, with $1 \leq p \leq n-1$. Then X is p-Kähler hyperbolic (resp. p-HS hyperbolic).*

Proof. We will spell out the proof in the p-HS case, the p-Kähler case can be handled in a similar manner. Let Ω be a real d-closed $2p$-form on X such that $\Omega^{p,p}$ is weakly strictly positive, i.e. $\Omega^{p,p}$ is a p-HS form. The goal is to show that Ω is $\widetilde{d}$(bounded). Let $\pi_X : \widetilde{X} \longrightarrow X$ and $\pi_Y : \widetilde{Y} \longrightarrow Y$ be the universal coverings of X and Y respectively. Since X and Y are homotopic, there exist two smooth maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that the induced map $(g \circ f)^*$ on $H^*_{DR}(X, \mathbb{R})$ is the identity. Thus for any $\Omega' \in [\Omega] \in H^{2p}_{DR}(X, \mathbb{R})$, there exists a $(2p-1)$-form u in X such that:

$$\Omega = \Omega' + du = (g \circ f)^*\Omega' + du = f^*(g^*\Omega') + du \tag{3.3.1}$$

Moreover, $g^*\Omega'$ is a d-closed $2p$-form on Y, hence $\widetilde{d}$(bounded) thanks to Proposition 3.3.15, that is $(\pi_Y^* \circ g^*)\Omega' = d\eta$ for some $\pi_Y^* g_Y$-bounded $(2p-1)$-form η on $\widetilde{Y}$. Let $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$ and $\widetilde{g} : \widetilde{Y} \longrightarrow \widetilde{X}$ be the liftings of f and g respectively such that the following diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_X"'] & \widetilde{Y} \arrow[r, "\widetilde{g}"] \arrow[d, "\pi_Y"'] & \widetilde{X} \arrow[d, "\pi_X"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & X
\end{tikzcd}$$

commutes. Hence, by lifting the formula (3.3.1) to the universal cover, we get:

$$\pi_X^*\Omega = (\pi_X^* \circ f^* \circ g^*)(\Omega') + \pi_X^*(du) = \widetilde{f}^* \circ (\pi_Y^* \circ g^*)(\Omega') + d(\pi_X^* u) \tag{3.3.2}$$

$$= \widetilde{f}^*(d\eta) + d(\pi_X^* u) = d(\widetilde{f}^*\eta + \pi_X^* u) \tag{3.3.3}$$

Fix a Riemannian metric g_X on X. Then by the triangle inequality we get:

$$\|\widetilde{f}^*\eta + \pi_X^* u\|_{L^\infty_{\pi_X^* g_X}(\widetilde{X})} \leq \|\widetilde{f}^*\eta\|_{L^\infty_{\pi_X^* g_X}(\widetilde{X})} + \|\pi_X^* u\|_{L^\infty_{\pi_X^* g_X}(\widetilde{X})} \tag{3.3.4}$$

On one hand, we have: $\|\pi_X^* u\|_{L^\infty_{\pi_X^* g_X}(\widetilde{X})} = \|u\|_{L^\infty_{g_X}(X)} < +\infty$ since π_X is a local isometry.. On the other hand, for any point $\widetilde{x} \in \widetilde{X}$ and tangent vectors $\widetilde{v}_1, \cdots, \widetilde{v}_{2p-1} \in T_{\widetilde{x}}\widetilde{X}$, consider

$x = \pi_X(\widetilde{x})$ and $v_i = (\pi_X)_* \widetilde{v}_i$ for $1 \leq i \leq 2p-1$. We have:

$$\begin{aligned}
|(\widetilde{f}^*\eta)_{\widetilde{x}}(\widetilde{v}_1, \ldots, \widetilde{v}_{2p-1})| &= \left|\eta_{\widetilde{f}(\widetilde{x})}\left(\widetilde{f}_*(\widetilde{v}_1), \ldots, \widetilde{f}_*(\widetilde{v}_{2p-1})\right)\right| \\
&\leq \|\eta\|_{L^\infty_{\pi_Y^* g_Y}} |\widetilde{f}_*(\widetilde{v}_1)|_{\pi_Y^* g_Y} \cdots |\widetilde{f}_*(\widetilde{v}_{2p-1})|_{\pi_Y^* g_Y} \\
&= \|\eta\|_{L^\infty_{\pi_Y^* g_Y}} |\widetilde{v}_1|_{\widetilde{f}^*(\pi_Y^* g_Y)} \cdots |\widetilde{v}_{2p-1}|_{\widetilde{f}^*(\pi_Y^* g_Y)} \\
&= \|\eta\|_{L^\infty_{\pi_Y^* g_Y}} |\widetilde{v}_1|_{\pi_X^*(f^* g_Y)} \cdots |\widetilde{v}_{2p-1}|_{\pi_X^*(f^* g_Y)} \quad \text{since } \pi_Y \circ \widetilde{f} = f \circ \pi_X \\
&= \|\eta\|_{L^\infty_{\pi_Y^* g_Y}} |f_* v_1|_{g_Y} \cdots |f_* v_{2p-1}|_{g_Y} \quad \text{since } \pi_X \text{ is a local isometry} \\
&\leq C\|\eta\|_{L^\infty_{\pi_Y^* g_Y}} \|f\|^{2p-1}_{\mathscr{C}^1_{g_X, g_Y}(X,Y)} |\widetilde{v}_1|_{\pi_X^* g_X} \cdots |\widetilde{v}_{2p-1}|_{\pi_X^* g_X}
\end{aligned}$$

where $C > 0$ depends only on p and X (Since X is compact, any two Riemannian metrics on X are uniformly equivalent). Moreover, in the last inequality we used the fact that π_X is a local isometry, so that $|v_i|_{g_X} = |\widetilde{v}_i|_{\pi_X^* g_X}$, for all $1 \leq i \leq 2p-1$. Hence, we get:

$$\|\widetilde{f}^*\eta + \pi_X^* u\|_{L^\infty_{\pi_X^* g_X}(\widetilde{X})} \leq C\|f\|^{2p-1}_{\mathscr{C}^1_{g_X, g_Y}(X,Y)} \|\eta\|_{L^\infty_{\pi_Y^* g_Y}(\widetilde{Y})} + \|u\|_{L^\infty_{g_X}(X)} < +\infty \tag{3.3.5}$$

Therefore, Ω is $\widetilde{d}$(bounded) and X is p-HS hyperbolic. $\square$

Remark 3.3.17. 1. Theorem 3.3.16 is not relevant when the manifold X itself admits a Hermitian metric ω with negative sectional curvature (for the **Chern connection**), since this implies the negativity of the first Chern-Ricci curvature form of ω. Then, the $(1,1)$-form:

$$\eta := -\mathrm{Ric}^{(1)}(\omega) = i\Theta_\omega(K_X) \underset{\mathrm{loc}}{=} i\partial\bar{\partial}\log(\omega^n).$$

defines a Kähler form on X. In particular, X is *Kähler hyperbolic* since (X, ω) is diffeomorphic to (X, η) and are therefore homotopic. Hence Theorem 3.3.16 applies.

2. The case $p = 1$ in Theorem 3.3.16 is already of particular interest. Indeed, constructing a compact non-Kähler Hermitian–symplectic manifold that is homotopic to a compact Riemannian manifold with $K < 0$ would provide an example of a non-Kähler Kobayashi hyperbolic manifold, assuming an affirmative answer to the question raised by J. Streets and G. Tian in Question 1.7 of [ST10] concerning the existence of compact non-Kähler Hermitian–symplectic manifolds. Such a scenario is, of course, widely regarded as unlikely in view of the Kobayashi conjecture, which predicts the ampleness of the canonical bundle K_X for any compact Kobayashi hyperbolic manifold X. Nevertheless, the existence of a compact non-Kähler Kobayashi hyperbolic manifold would simply indicate that the Kähler assumption in the Kobayashi conjecture must be made explicit.

Problem 5. Motivated by the search for non-Kähler Kobayashi hyperbolic manifolds, we ask whether Theorem 1.1 in [BS23] can be strengthened in the non-Kähler setting: that is, whether a compact complex manifold carrying an SKT metric with negative holomorphic sectional curvature (HSC < 0) must be SKT hyperbolic. Recall that any compact Hermitian manifold with HSC < 0 is Kobayashi hyperbolic by [Kob05], Theorem 4.1. As recently pointed out to the author by J. Streets, the negativity of the *real* bisectional curvature already forces a compact Hermitian manifold to be Kähler with ample canonical bundle (see [LS21], Theorem 1.1), indicating that even slight strengthening of the "HSC < 0" assumption may preclude genuinely non-Kähler examples. Note, for example, that a compact complex manifold with negative holomorphic bisectional curvature (HBC < 0) need not be Kähler SKT hyperbolic. Indeed, manifolds of the latter type have infinite fundamental group, since they have large fundamental group in the sense of Kollár (Definition 4.6 in [Kol95]). In contrast, there exist simply connected Kähler manifolds with HBC < 0, arising for instance as complete intersections in complex projective manifolds [Moh22].

In [KP25], Section §.5, it was observed that one does not need to assume negativity of the sectional curvature; instead, a weaker curvature-type condition introduced in the context of *pluriclosed star split* manifolds [Pop23] suffices to obtain hyperbolicity results in several settings. This suggests the possibility of exploring whether analogous conditions could imply other forms of hyperbolicity for compact complex manifolds, possibly in a partial sense. For instance, one may ask whether such curvature-type hypotheses could lead to a partial hyperbolicity of Hermitian–Symplectic, SKT, strongly Gauduchon, or Gauduchon type. Since partial hyperbolicity is weaker and more flexible than full Kobayashi hyperbolicity, it could potentially allow for a wider range of non-Kähler examples, opening the door to new constructions and examples in this direction.

Building upon the ideas of Theorem 5.5 in [Ale17] and Proposition 3.5 in [HM23], we derive the following results:

Proposition 3.3.18. *Let (X, ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$, and Y be compact complex manifold with $\dim_{\mathbb{C}} Y = m$. Fix $1 \leq q < m$.*

1. *If Y is p-HS (resp. p-HS hyperbolic) for all $q \leq p < m$, then $X \times Y$ is $(p+n)$-HS (resp. $(p+n)$-HS hyperbolic) for all $q \leq p < m$.*
2. *If Y is p-SKT (resp. p-SKT hyperbolic) for all $q \leq p < m$, then $X \times Y$ is $(p+n)$-SKT (resp. $(p+n)$-SKT hyperbolic) for all $q \leq p < m$.*

Proof. Let $\pi_X : \widetilde{X} \longrightarrow X$ and $\pi_Y : \widetilde{Y} \longrightarrow Y$ be the universal coverings of X and Y, respectively. Let $\mathrm{pr}_X : X \times Y \longrightarrow X$ and $\mathrm{pr}_Y : X \times Y \longrightarrow Y$ be the projections onto X and Y, respectively. The projections pr_X and pr_Y lift to projections $\widetilde{\mathrm{pr}_X} : \widetilde{X} \times \widetilde{Y} \longrightarrow \widetilde{X}$

and $\widetilde{\mathrm{pr}_Y} : \widetilde{X} \times \widetilde{Y} \longrightarrow \widetilde{Y}$. In particular, the following diagram commutes:

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi_X"'] & \widetilde{X} \times \widetilde{Y} \arrow[l, "\widetilde{\mathrm{pr}}_X"'] \arrow[r, "\widetilde{\mathrm{pr}}_Y"] \arrow[d, "\pi"] & \widetilde{Y} \arrow[d, "\pi_Y"] \\ X & X \times Y \arrow[l, "\mathrm{pr}_X"] \arrow[r, "\mathrm{pr}_Y"'] & Y \end{tikzcd}$$

1. For $q \leq p < n$, we let Ω_p be a real d-closed $2p$-form on Y such that $\Omega^{p,p}$ is weakly strictly positive, i.e. $\Omega^{p,p}$ is a p-HS form. Fix $q + n \leq j < m + n$ and let $h = \max\{0, j - m\}$. We set:

$$\Theta_j := \sum_{h \leq k \leq n} \mathrm{pr}_X^* \omega^k \wedge \mathrm{pr}_Y^* \Omega_{j-k} \tag{3.3.6}$$

Then Θ_j is obviously a real d-closed $2j$-form on $X \times Y$. Thanks to Propositions 1.11 and 1.12 in [Dem97], §.1, Chapter III, we infer that:

$$\Theta_j^{j,j} = \sum_{h \leq k \leq n} \mathrm{pr}_X^* \omega^k \wedge \mathrm{pr}_Y^* \Omega_{j-k}^{j-k,j-k} \tag{3.3.7}$$

is weakly strictly positive. Hence $\Theta_j^{j,j}$ defines a j-HS form on $X \times Y$. If moreover, we assume the p-forms Ω_p are $\widetilde{d}$(bounded), i.e. $\forall q \leq p < m$, there exists a bounded $(2p-1)$-form η_p in $\widetilde{Y}$ such that $\pi_Y^* \Omega_p = d\eta_p$. By pulling-back the expression (3.3.6) to $\widetilde{X} \times \widetilde{Y}$, we get:

$$\begin{aligned} \pi^* \Theta_j &= \sum_{h \leq k \leq n} (\pi^* \circ \mathrm{pr}_X^*) \omega^k \wedge (\pi^* \circ \mathrm{pr}_Y^*) \Omega_{j-k} \\ &= \sum_{h \leq k \leq n} \widetilde{\mathrm{pr}_X}^* \widetilde{\omega}^k \wedge \widetilde{\mathrm{pr}_Y}^* (\pi_Y^* \Omega_{j-k}) \\ &= \sum_{h \leq k \leq n} \widetilde{\mathrm{pr}_X}^* \widetilde{\omega}^k \wedge d(\widetilde{\mathrm{pr}_Y}^* \eta_{j-k}) \\ &= d \left(\sum_{h \leq k \leq n} \widetilde{\mathrm{pr}_X}^* \widetilde{\omega}^k \wedge \widetilde{\mathrm{pr}_Y}^* \eta_{j-k} \right) \qquad \text{since } \omega \text{ is } d\text{-closed} \end{aligned}$$

Hence for any $q + n \leq j < m + n$, Θ_j is $\widetilde{d}$(bounded).

2. For $q \leq p < n$, we let Ω_p be a p-SKT form on Y, and we fix $q + n \leq j < m + n$. Set $h = \max\{0, j - m\}$, then the (j, j)-form:

$$\Theta_j := \sum_{h \leq k \leq n} \mathrm{pr}_X^* \omega^k \wedge \mathrm{pr}_Y^* \Omega_{j-k} \tag{3.3.8}$$

defines a j-SKT form on $X \times Y$, again thanks to Propositions 1.11 and 1.12 in

[Dem97], §.1, Chapter III, and the fact ω is both ∂ and $\bar{\partial}$-closed. If moreover, the (p,p)-forms Ω_p are $\widetilde{(\partial + \bar{\partial})}$(bounded), i.e. $\forall q \leq p < m$, there exist a bounded $(p-1,p)$-form α_p and a bounded $(p,p-1)$-form β_p in $\widetilde{Y}$ such that $\pi_Y^* \Omega_p = \partial \alpha_p + \bar{\partial} \beta_p$, then:

$$
\begin{aligned}
\pi^* \Theta_j &= \sum_{h \leq k \leq n} (\pi^* \circ \mathrm{pr}_X^*) \omega^k \wedge (\pi^* \circ \mathrm{pr}_Y^*) \Omega_{j-k} \\
&= \sum_{h \leq k \leq n} \widetilde{\mathrm{pr}_X}^* \widetilde{\omega}^k \wedge \widetilde{\mathrm{pr}_Y}^* (\pi_Y^* \Omega_{j-k}) \\
&= \sum_{h \leq k \leq n} \widetilde{\mathrm{pr}_X}^* \widetilde{\omega}^k \wedge (\partial(\widetilde{\mathrm{pr}_Y}^* \alpha_{j-k}) + \bar{\partial}(\widetilde{\mathrm{pr}_Y}^* \beta_{j-k})) \\
&= \partial \left(\sum_{h \leq k \leq n} \widetilde{\mathrm{pr}_X}^* \widetilde{\omega}^k \wedge \widetilde{\mathrm{pr}_Y}^* \alpha_{j-k} \right) + \bar{\partial} \left(\sum_{h \leq k \leq n} \widetilde{\mathrm{pr}_X}^* \widetilde{\omega}^k \wedge \widetilde{\mathrm{pr}_Y}^* \beta_{j-k} \right)
\end{aligned}
$$

Hence Θ_j is a j-SKT hyperbolic form on $X \times Y$, for all $q + n \leq j < m + n$.

□

The analogue of Proposition 3.8 in [HM23] remains valid in the p-HS and p-SKT hyperbolic contexts. Namely:

Proposition 3.3.19. *Let X be a p-HS (resp. p-SKT) hyperbolic manifold with $\dim_{\mathbb{C}} X = n$, and S be submanifold of X with $\dim_{\mathbb{C}} S > p$. Then S is p-HS (resp. p-SKT) hyperbolic too.*

Proof. As the argument proceeds in the same manner as in the p-Kähler hyperbolic case, we shall restrict our attention to the p-SKT hyperbolic case, leaving the analogous proof for the p-HS hyperbolic case to the reader. Let $\iota : S \hookrightarrow X$ be the natural inclusion, and let $\pi_X : \widetilde{X} \longrightarrow X$ and $\pi_S : \widetilde{S} \longrightarrow S$ be the universal coverings of X and S respectively. Then ι lifts to a natural inclusion on universal covers:

$$
\begin{tikzcd}
\widetilde{S} \arrow[r, hook, "\widetilde{\iota}"] \arrow[d, "\pi_S"'] & \widetilde{X} \arrow[d, "\pi_X"] \\
S \arrow[r, hook, "\iota"'] & X
\end{tikzcd}
$$

Let Ω be a p-SKT hyperbolic form on X, that is: $\pi_X^* \Omega = \partial \alpha + \bar{\partial} \beta$, for some bounded $(p-1,p)$ form α and bounded $(p,p-1)$-form β on $\widetilde{X}$. Thanks to Proposition 1.12 in [Dem97], §.1, Chapter III, $\iota^* \Omega$ defines a p-SKT form on S and:

$$
\pi_S^* (\iota^* \Omega) = \widetilde{\iota}^* (\pi_X^* \Omega) = \partial (\widetilde{\iota}^* \alpha) + \bar{\partial} (\widetilde{\iota}^* \beta) \tag{3.3.9}
$$

Hence $\iota^*\Omega$ is a p-SKT hyperbolic form on S. $\square$

We now aim to highlight the relationship between the hyperbolicity notions defined in degree $2p$ and their corresponding geometric counterparts. Recall that in the case $p = 1$, these are the notions of Kähler hyperbolicity, HS hyperbolicity, and SKT hyperbolicity (Remark 3.3.14.1). They have been shown to imply Kobayashi hyperbolicity (see [CY18], Theorem 4.1, Corollary 3.3.7 and [Mar23], Theorem 3.5, respectively).

For $p = n - 1$, the corresponding notions are balanced hyperbolicity, sG hyperbolicity and Gauduchon hyperbolicity (Remark 3.3.14.2). These manifolds have been proven to be divisorially hyperbolic (see [MP23], Theorem 2.8; [Ma24], Theorem 2.6; and [Mar23], Theorem 3.10, respectively).

For general $2 \leq p \leq n-1$, we introduce a natural generalization of divisorial hyperbolicity (referred to as *p-dimensional hyperbolicity* in [HM23], Definition 3.14 and *p-hyperbolicity* in [KP25], Definition 2.5).

Let (X, ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n \geq 2$, and let $f : \mathbb{C}^p \longrightarrow X$ be a holomorphic map with $1 \leq p \leq n-1$, which is *non-degenerate* at some point $x_0 \in \mathbb{C}^p$, i.e. $d_{x_0} f : \mathbb{C}^p \longrightarrow T^{1,0}_{x_0} X$ is of maximal rank. Set: $\Sigma_f := \{x \in \mathbb{C}^p : f \text{ is degenerate at } x\}$; this is a proper analytic subset of $\mathbb{C}^p$. Then $f^*\omega$ defines a *degenerate metric* on $\mathbb{C}^p$ with degeneration set Σ_f ; that is $f^*\omega \geq 0$ on $\mathbb{C}^p$ and defines a genuine Hermitian metric on $\mathbb{C}^p \setminus \Sigma_f$. For any $r > 0$, we define the (ω, f)-**volume** of the ball $\mathbb{B}_r \subset \mathbb{C}^p$ of radius r centered at 0 to be:

$$\mathrm{Vol}_{\omega,f}(\mathbb{B}_r) := \int_{\mathbb{B}_r} f^*\omega_p > 0\,, \qquad \text{where } \omega_p = \frac{\omega^p}{p!} \tag{3.3.10}$$

For any $z \in \mathbb{C}^p$, let $\beta(z) = |z|^2$ be its squared Euclidean norm. By definition of the Hodge star operator induced by the Hermitian metric $f^*\omega$ on $\mathbb{C}^p \setminus \Sigma_f$, we have:

$$\frac{d\beta(z)}{|d\beta(z)|_{f^*\omega}} \wedge *_{f^*\omega}\left(\frac{d\beta(z)}{|d\beta(z)|_{f^*\omega}}\right) = f^*\omega_p(z)\,, \qquad \forall z \in \mathbb{C}^p \setminus \Sigma_f \tag{3.3.11}$$

This implies that the $(2p-1)$-form $d\sigma_{\omega,f} := *_{f^*\omega}\left(\frac{d\beta(z)}{|d\beta(z)|_{f^*\omega}}\right)$ defines an area measure on spheres $\mathbb{S}_r : \partial\mathbb{B}_r \subset \mathbb{C}^p$.

In particular, we define the (ω, f)-**area** of the sphere $\mathbb{S}_r$ of radius $r > 0$ by:

$$A_{\omega,f}(\mathbb{S}_r) := \int_{\mathbb{S}_r} d\sigma_{\omega,f,r} > 0\,, \qquad \text{where } d\sigma_{\omega,f,r} = (d\sigma_{\omega,f})_{|\mathbb{S}_r} \tag{3.3.12}$$

Definition 3.3.20 (Definition 2.3, [MP23] and Definition 2.3, [KP25]). *In the same* *setting, we say that f has **subexponential growth** if:*

(i) $\exists C_0, r_0 > 0$ *such that:* $\int_{\mathbb{S}_r} |d\beta|_{f^*\omega} d\sigma_{f,\omega,r} \leq C_0 r\, Vol_{\omega,f}(\mathbb{B}_r)$, $\forall r > r_0$.

(ii) $\forall \lambda > 0$, *we have:* $\limsup_{t \to +\infty} \left(\frac{t}{\lambda} - \log \left(\int_0^t Vol_{\omega,f}(\mathbb{B}_r) dr \right) \right) = +\infty$.

Remark 3.3.21. 1. Since X is compact, any two Hermitian metrics on X are equivalent. Therefore, the subexponential growth condition on the holomorphic maps from some $\mathbb{C}^p$ into X is independent of the choice of the Hermitian metric on X.

2. Let $F(t) := \int_0^t \mathrm{Vol}_{\omega,f}(B_r) dr$. Condition (ii) can be interpreted as a subexponential growth condition on F. Indeed, it states that

$$\forall \lambda > 0, \qquad \limsup_{t \to +\infty} \left(\frac{t}{\lambda} - \log F(t) \right) = +\infty. \tag{3.3.13}$$

This means that $\log F(t)$ grows strictly slower than any linear function of t. Equivalently, $F(t)$ grows strictly slower than any exponential function e^{ct}, $\forall c > 0$, which justifies the terminology *subexponential growth.* More precisely, for every $\lambda > 0$, the quantity $F(t)$ does not admit a uniform exponential upper bound of the form $e^{t/\lambda}$; instead, the difference $t/\lambda - \log F(t)$ becomes arbitrarily large along sequences $t \to +\infty$.

In particular, any polynomial growth satisfies condition (ii). Indeed, if

$$F(t) \leq C(1+t)^N \tag{3.3.14}$$

for some constants $C > 0$ and $N \geq 0$, then

$$\frac{t}{\lambda} - \log F(t) \geq \frac{t}{\lambda} - N \log(1+t) - \log C \longrightarrow +\infty \tag{3.3.15}$$

as $t \to +\infty$, for every $\lambda > 0$.

Definition 3.3.22 (Definition 2.5, [KP25] and Definition 3.14, [HM23])**.** *We say that X is p-**cyclically hyperbolic** for some $1 \leq p \leq n-1$ if there is no holomorphic map $f : \mathbb{C}^p \longrightarrow X$ that is non-degenerate at some point $x_0 \in \mathbb{C}^p$ and has subexponential growth in the sense of the previous Definition.*

Remark 3.3.23. 1. The case $p = n-1$ corresponds exactly to the definition of **divisorial hyperbolicity** (Definition 2.7, [MP23]).

2. When $p = 1$, it is clear that Kobayashi hyperbolicity implies 1-cyclic hyperbolicity. Conversly, assume X is a compact complex manifold which admits a non-constant holomorphic map $f : \mathbb{C} \longrightarrow X$. Thanks to **Brody's Renormalisation Lemma**

([Bro78], Lemma 2.1), we can assume f to be of order 2 (see [Lan87], Theorem 2.6-p.72), that is:

$$\mathrm{Vol}_{\omega,f}(\mathbb{B}_r) := \int_{\mathbb{B}_r} f^*\omega \underset{r\to\infty}{\sim} r^2$$

for some (hence any by compactness of X) Hermitian metric ω on X. Or equivalently:

$$\frac{1}{C_1}r^2 - C_2 \leq \mathrm{Vol}_{\omega,f}(\mathbb{B}_r) \leq C_1 r^2 + C_2\,, \qquad \forall r > r_0 \tag{3.3.16}$$

for some constants $C_1, C_2, r_0 > 0$. One readily checks that such an f satisfies condition (ii) in Definition 3.3.20. On the other hand, one can write $f^*\omega$ as:

$$f^*\omega = \frac{i}{2}\mu(z)dz \wedge d\bar{z}$$

where $\mu : \mathbb{C} \longrightarrow \mathbb{R}$ is a $\mathscr{C}^\infty$-function, positive on $\mathbb{C} \setminus \Sigma_f$ and vanishes along Σ_f (which is a countable set of points). For any $z = r(\cos\theta + i\sin\theta) \in \mathbb{C} \setminus \Sigma_f$, simple computations yield:

$$\beta(z) = |z|^2 \Longrightarrow d\beta(z) = 2|z|d|z| \Longrightarrow |d\beta(z)|_{f^*\omega} = \frac{2|z|}{\sqrt{\mu(z)}}$$

Hence, the length measure induced by f and ω is given in polar coordinates by:

$$\sigma_{\omega,f} = *_{f^*\omega}\left(\frac{d\beta}{|d\beta|_{f^*\omega}}\right) = r\sqrt{\mu}d\theta$$

Therefore, for any $r > 0$ we get:

$$\int_{\mathbb{S}_r} |d\beta|_{f^*\omega} d\sigma_{\omega,f,r} = \int_0^{2\pi} \frac{2r}{\sqrt{\mu}} r\sqrt{\mu}d\theta = 4\pi r^2$$

which is clearly dominated by $r\mathrm{Vol}_{\omega,f}(\mathbb{B}_r)$ for $r > r_0$ large enough. So f satisfies also condition (i) in Definition 3.3.20. Thus, 1-cyclic hyperbolicity coincides with Brody hyperbolicity (equivalently, with Kobayashi hyperbolicity).

3. Kobayashi hyperbolicity implies p-cyclic hyperbolicity, $\forall 1 \leq p \leq n-1$. Indeed, if $f : \mathbb{C}^p \longrightarrow X$ is a holomorphic map, non-degenerate at some point $x \in \mathbb{C}^p$, then f remains non-constant along every complex line passing through x.

Problem 6. Does p-cyclic hyperbolicity imply $(p+1)$-cyclic hyperbolicity ?

S. Marouani and F. Haggui established that every p-Kähler hyperbolic manifold is p-cyclically hyperbolic (Theorem 3.15 in [HM23]). Regarding p-HS hyperbolicity, Y. Ma showed that this condition also implies p-cyclic hyperbolicity in [Ma24], Theorem 3.2.

We now prove that the same holds for p-SKT hyperbolic manifolds:

Theorem 3.3.24. *Every compact p-SKT hyperbolic manifold is p-cyclically hyperbolic.*

Proof. Let (X, ω) be a compact Hermitian manifold with $\dim_{\mathbb{C}} X = n$ and $\pi : \widetilde{X} \longrightarrow X$ be the universal cover of X. Assume X equipped with a p-SKT hyperbolic form Ω, i.e. Ω is a smooth weakly strictly positive (p,p)-form such that $\partial\bar\partial\Omega = 0$ and $\widetilde{\Omega} = \partial\alpha + \bar\partial\gamma$, where α (resp. γ) is a smooth $\widetilde{\omega}$-bounded $(p-1,p)$-form (resp. $(p,p-1)$-form) on $\widetilde{X}$. Assume that a holomorphic map $f : \mathbb{C}^p \longrightarrow X$ such as in Definition 3.3.20 exists. Then $f^*\omega$ defines a degenerate Hermitian metric on X (with degeneracy locus Σ_f). Let us first make the following observation:

Observation 3.3.25. Let η be an $\widetilde{\omega}$-bounded k-form on $\widetilde{X}$, with $k \leq 2p$. Then $\widetilde{f}^*\eta$ is $f^*\omega$-bounded on $\mathbb{C}^p$.

Proof. Since $\mathbb{C}^p$ is simply connected, f lifts to a holomorphic map $\widetilde{f} : \mathbb{C}^p \longrightarrow \widetilde{X}$, that is: $\pi \circ \widetilde{f} = f$. In particular, the degeneracy locus of $\widetilde{f}$ is also Σ_f. Let η be a $\widetilde{\omega}$-bounded k-form on $\widetilde{X}$, with $k \leq 2p$. For any $x \in \mathbb{C}^p$, and any tangent vectors $\xi_1, \cdots, \xi_k \in \mathbb{C}^p$ at x, we have:

$$\begin{aligned}
|\widetilde{f}^*\eta(\xi_1, \cdots, \xi_k)|^2_{f^*\omega} &= |\eta(\widetilde{f}_*(\xi_1), \cdots, \widetilde{f}_*(\xi_k))|^2_{\widetilde{\omega}} \\
&\leq C|\widetilde{f}_*(\xi_1)|^2_{\widetilde{\omega}} \cdots |\widetilde{f}_*(\xi_k)|^2_{\widetilde{\omega}} && \text{since } \eta \text{ is bounded} \\
&= C|\xi_1|^2_{f^*\omega} \cdots |\xi_k|^2_{f^*\omega} && \text{since } \widetilde{f}^*\widetilde{\omega} = f^*\omega
\end{aligned}$$

$\square$

Now we can resume the proof of our theorem. $f^*\Omega$ defines a positive (p,p)-form (hence *strongly positive* by maximality of the dimension, see Corollary 1.9 in [Dem97], §.1, Chapter III) on $\mathbb{C}^p \setminus \Sigma_f$. Hence, there exists a smooth positive function $h : \mathbb{C}^p \setminus \Sigma_f \longrightarrow \mathbb{R}$ such that: $f^*\omega_p = hf^*\Omega$, where $\omega_p = \dfrac{\omega^p}{p!}$. But since both $f^*\omega_p$ and $f^*\Omega$ are smooth, h extends smoothly to the whole $\mathbb{C}^p$. In particular, we have for any $z \in \mathbb{C}^p$:

$$\sqrt{\binom{n}{p}} = |\omega_p(f(z))|_\omega = |f^*\omega_p(z)|_{f^*\omega} = h(z)|f^*\Omega(z)|_{f^*\omega} = h(z)|\Omega(f(z))|_\omega$$

Thus: $h(z) = \sqrt{\binom{n}{p}}|\Omega(f(z))|^{-1}_\omega$, $\forall z \in \mathbb{C}^p$. On the other hand, since $\Omega(x) \neq 0$ for any $x \in X$ by positivity, the compactness of X implies that there exists a constant $A > 0$ such that: $\dfrac{1}{A} \leq |\Omega(x)|_\omega \leq A$. Hence, we get for any $r > 0$:

$$\mathrm{Vol}_{\omega,f}(\mathbb{B}_r) = \int_{\mathbb{B}_r} f^*\omega_p = \int_{\mathbb{B}_r} hf^*\Omega \leq A\sqrt{\binom{n}{p}} \int_{\mathbb{B}_r} f^*\Omega \tag{3.3.17}$$

Moreover, we have:

$$f^*\Omega = \widetilde{f}^*\widetilde{\Omega} = \widetilde{f}(\partial\alpha + \bar{\partial}\gamma) \overset{(a)}{=} \widetilde{f}(d(\alpha+\gamma)) = d\widetilde{f}(\alpha+\gamma) \tag{3.3.18}$$

where (a) is a consequence of the bidegree constraints. Substituting (3.3.18) into (3.3.17) gives:

$$\mathrm{Vol}_{\omega,f}(\mathbb{B}_r) \leq A\sqrt{\binom{n}{p}} \int_{\mathbb{B}_r} d\widetilde{f}^*(\alpha+\gamma) = A\sqrt{\binom{n}{p}} \int_{\mathbb{S}_r} \widetilde{f}^*(\alpha+\gamma) \overset{(b)}{\leq} BA_{\omega,f}(\mathbb{S}_r)\,, \tag{3.3.19}$$

where (b) follows from Observation 3.3.25 (here $\eta = \alpha + \gamma$). On the other hand, the Cauchy-Schwarz inequality yields:

$$A_{\omega,f}(\mathbb{S}_r)^2 \leq \left(\int_{\mathbb{S}_r} \frac{1}{|d\beta|_{f^*\omega}} d\sigma_{\omega,f,r}\right)\left(\int_{\mathbb{S}_r} |d\beta|_{f^*\omega} d\sigma_{\omega,f,r}\right), \qquad \forall r > 0 \tag{3.3.20}$$

where $\beta(z) = |z|^2 = r^2$, hence $d\beta = 2rdr$. Therefore, by combining (3.3.19) and (3.3.20), we get for any $r > r_0$ sufficiently large:

$$\begin{aligned}
\mathrm{Vol}_{\omega,f}(\mathbb{B}_r) &= \int_0^r \left(\int_{\mathbb{S}_t} \frac{1}{|d\beta|_{f^*\omega}} d\sigma_{\omega,f,t}\right) d\beta \\
&\overset{(3.3.20)}{\geq} \int_0^r \frac{A_{\omega,f}(\mathbb{S}_t)^2}{\int_{\mathbb{S}_t} |d\beta|_{f^*\omega} d\sigma_{\omega,f,t}} 2tdt \\
&\overset{(3.3.19)}{\geq} \frac{2}{B^2} \int_0^r \frac{t\mathrm{Vol}_{\omega,f}(\mathbb{B}_t)}{\int_{\mathbb{S}_t} |d\beta|_{f^*\omega} d\sigma_{\omega,f,t}} \mathrm{Vol}_{\omega,f}(\mathbb{B}_t)dt \\
&\overset{(c)}{\geq} \frac{2}{C_0B^2} \int_0^r \mathrm{Vol}_{\omega,f}(\mathbb{B}_t)dt =: CF(r)
\end{aligned}$$

where $C := \dfrac{2}{C_0B^2} > 0$, $F(r) := \int_0^r \mathrm{Vol}_{\omega,f}(\mathbb{B}_t)dt$ and (c) follows from the condition (i) in Definition 3.3.20. Thus, the previous inequality takes the form:

$$F'(r) \geq CF(r) \Longrightarrow (\log F(r))' \geq C \overset{(*)}{\Longrightarrow} F(r) \geq F(r_0)\exp(C(r-r_0))\,, \qquad \forall r > r_0$$

where $(*)$ follows by integration over $[r_0, r]$. In this situation, condition (ii) in Definition 3.3.20 forces $F(r) = 0$, $\forall r > r_0$ which in turn yields: $\mathrm{Vol}_{\omega,f}(\mathbb{B}_r) = 0$, $\forall r > r_0$. Therefore $f^*\omega = 0$, contradicting our assumption. □

3.3.3 Deformation Stability Results

Finally, we will study the holomorphic deformations of p-HS hyperbolic and p-Kähler hyperbolic manifolds. Our first result in this direction is the deformation **openness** of the p-HS hyperbolicity property:

Theorem 3.3.26. *Let $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ be an analytic family of compact complex n-dimensional manifolds such that X_0 is p-HS hyperbolic for some $1 \leq p \leq n-1$. Then the fibers X_t are also p-HS hyperbolic for t sufficiently small.*

Proof. Fix a $\mathscr{C}^\infty$-family $\{\omega_t\}_{t\in\mathbb{B}_\varepsilon(0)}$ of Hermitian metrics on $(X_t)_{t\in\mathbb{B}_\varepsilon(0)}$. Let Ω be a d-closed $2p$-form on X_0 such that $\Omega^{p,p}$ is weakly strictly positive and $\widetilde{\Omega} = d\Gamma$, for some $\widetilde{\omega_0}$-bounded $(2p-1)$-form on $\widetilde{X_0}$. For $t \in \mathbb{B}_\varepsilon(0)$, let $\Omega_t^{p,p}$ be the (p,p)-component of Ω with respect to the complex structure on X_t (In particular: $\Omega_0^{p,p} = \Omega^{p,p}$, which is weakly strictly positive by hypothesis). By continuity, $\Omega_t^{p,p}$ is weakly strictly positive for t close enough to 0 (see for instance Lemma 4.2, [Bel20] or Proposition 4.12, [RWZ22] for the proof), meaning that $\Omega_t^{p,p}$ is a p-HS form on X_t, for $t \sim 0$. Moreover, $\widetilde{\Omega} = d\Gamma$ with: $\|\Gamma\|_{L^\infty_{\widetilde{\omega_t}}(\widetilde{X_t})} \leq C_t \|\Gamma\|_{L^\infty_{\widetilde{\omega_0}}(\widetilde{X_0})} < \infty$, for some $C_t > 0$ (since the metrics ω_0 and ω_t are comparable on the underlying **compact** manifold X). Hence X_t is also p-HS hyperbolic for t close enough to 0. $\square$

Since p-Kähler hyperbolicity implies p-HS hyperbolicity for any $1 \leq p \leq n-1$, it follows that small deformations of a compact p-Kähler hyperbolic manifold are p-HS hyperbolic. In particualr, in the case $p = 1$, we have:

Corollary 3.3.27. *Let $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ be an analytic family of compact complex manifolds of dimension n, such that X_0 is Kähler hyperbolic. Then the fibers X_t are HS-hyperbolic for t sufficiently small.*

When $2 \leq p \leq n-1$, the fibers are not necessarily p-Kähler. Sufficient conditions ensuring the stability of the p-Kähler property were given for instance in [RWZ19], Theorem 1.1 and [RWZ22], Theorem 1.8. In order to study the small deformations of p-Kähler hyperbolic manifolds, we will assume that the $\partial\bar{\partial}$-property is satisfied. Note that a p-Kähler $\partial\bar{\partial}$-manifold with $2 \leq p \leq n-1$ is not necessarily Kähler (take a proper modification of a compact Kähler manifold, see Theorem 4.8 in [AB93] and Theorem 5.22 in [DGMS75]). The following result extends part (1) of Theorem 3.2.15 as well as Theorem 4.10 of [Khe25] in the L^∞-setting, for arbitrary degree $1 \leq p \leq n-1$:

Theorem 3.3.28. *Let $\{X_t : t \in \mathbb{B}_\varepsilon(0)\}$ be an analytic family of compact complex n-dimensional manifolds such that X_0 is a p-Kähler hyperbolic $\partial\bar{\partial}$-manifold for some $1 \leq p \leq n-1$. Then there exists a $\mathscr{C}^\infty$-family of p-Kähler forms $\{\Omega_t\}_{t\sim 0}$ on the fibers X_t*

that are p-SKT hyperbolic for t sufficiently small.

Proof. Let $\{\omega_t\}_{t\in\mathbb{B}_\varepsilon(0)}$ be a $\mathscr{C}^\infty$-family of Hermitian metrics on the fibers X_t, and let Ω be a p-Kähler hyperbolic form on X_0, that is $\widetilde{\Omega}=d\eta$ for some $\widetilde{\omega_0}$-bounded $(2p-1)$-form η on $\widetilde{X_0}$. Recall that the Bott-Chern Laplacian Δ_{BC} (with respect to the metric ω_0), being elliptic and formally self-adjoint, induces the following $L^2_{\omega_0}$-orthogonal three-space decomposition in any bidegree (p,q) (see [Pop15], §2):

$$\mathscr{C}^\infty_{p,q}(X_0)=\mathcal{H}^{p,q}_{BC}(X_0)\oplus \operatorname{Im}\partial\bar\partial\oplus\left(\operatorname{Im}\partial^\star+\operatorname{Im}\bar\partial^\star\right)\quad , \tag{3.3.21}$$

and we have

$$\operatorname{Ker}\bar\partial\cap\operatorname{Ker}\partial=\mathcal{H}^{p,q}_{BC}(X_0)\oplus\operatorname{Im}\partial\bar\partial \tag{3.3.22}$$

yielding the Hodge isomorphism $\mathcal{H}^{p,q}_{BC}(X_0)\simeq H^{p,q}_{BC}(X_0)$. Now, since Ω is real (as Ω is weakly strongly positive), $d\Omega=0$ implies that $\partial\Omega=\bar\partial\Omega=0$. Hence $\Omega\in\mathcal{H}^{p,p}_{BC}(X_0)\oplus\operatorname{Im}\partial\bar\partial$ thanks to (3.3.22). Moreover, $\Omega=F^{p,p}_{BC}\Omega+\Delta_{BC}\mathbb{G}_{BC}\Omega$ by definition of the Green operator $\mathbb{G}_{BC}$, where $F^{p,p}_{BC}\Omega$ is the projection of Ω onto the Bott-Chern harmonic space $\mathcal{H}^{p,p}_{BC}(X_0)$.

The uniqueness of the decompostion of Ω with respect to (3.3.21) yields: $\Omega=F^{p,p}_{BC}\Omega+\partial\bar\partial\alpha$, for some $\alpha\in\mathscr{C}^\infty_{p-1,p-1}(X_0)$. We can take α to be the minimal $L^2_{\omega_0}$-norm solution of this equation, which is given by the following Neumann-type formula ([Pop15], Theorem 4.1):

$$\alpha=(\partial\bar\partial)^\star\mathbb{G}_{BC}(\Omega-F^{p,p}_{BC}\Omega)=(\partial\bar\partial)^\star\mathbb{G}_{BC}\Omega$$

In other words, one can write Ω as: $\Omega=(F^{p,p}_{BC}+\partial\bar\partial\bar\partial^\star\partial^\star\mathbb{G}_{BC})\Omega$. We decompose Ω as: $\Omega=\sum\limits_{j+k=2p}\Omega^{k,j}_t$. Since X_0 ia a $\partial\bar\partial$-manifold, the function $t\in\mathbb{B}_\varepsilon(0)\longmapsto h^{p,p}_{BC}(X_t)$ is constant (after shrinking ε if necessary, see [AT13], Corollary 3.7). Then one defines:

$$\Omega_t:=(F^{p,p}_{BC,t}+\partial_t\bar\partial_t\bar\partial_t^\star\partial_t^\star\mathbb{G}_{BC,t})\Omega^{p,p}_t\,,\qquad \forall t\in\mathbb{B}_\varepsilon(0) \tag{3.3.23}$$

where $\mathbb{G}_{BC,t}:\mathscr{C}^\infty_{p,p}(X_t)\longrightarrow\mathscr{C}^\infty_{p,p}(X_t)$ is the Green operator associated with the Bot-Chern Laplacian $\Delta_{BC,t}$, and $F^{p,p}_{BC,t}:\mathscr{C}^\infty_{p,p}(X_t)\longrightarrow\mathcal{H}^{p,p}_{BC}(X_t)$ is the projection onto the Bott-Chern harmonic (p,p)-forms. $\{\Omega_t\}_{t\in\mathbb{B}_\varepsilon(0)}$ defines a $\mathscr{C}^\infty$-family of d-closed (p,p)-forms (thanks to Theorem 5 in [KS60]) with $\Omega_0=\Omega$. In particular, $\{\Omega_t\}_{t\sim 0}$ is a $\mathscr{C}^\infty$-family of p-Kähler forms on the respective fibers X_t for $t\sim 0$, with $\Omega_0=\Omega$.

Now, $d\Omega=0$ implies: $\partial_t\bar\partial_t\Omega^{k,j}_t=0$, with $k+j=2p$, $\forall t\sim 0$. Using the fact that the canonical isomorphism given in Proposition 3.2.8.(3) is induced by the identity map (which exists thanks to the deformation openness of the $\partial\bar\partial$-property), we get:

$$[\Omega_t^{k,j}]_A = [F_{BC,t}^{k,j}\Omega_t^{k,j}]_A \quad , \qquad \text{with } k=j=2p \quad , \quad \forall t \sim 0.$$

as $[\Omega_t^{k,j}]_{BC} = [F_{BC,t}^{k,j}\Omega_t^{k,j}]_{BC}$. This implies that: $\Omega_t^{k,j} = F_{BC,t}^{k,j}\Omega_t^{k,j} + \partial_t u_t^{k-1,j} + \bar{\partial}_t u_t^{k,j-1}$, with $k+j=2p$, $t\sim 0$, and $u_t^{k-1,j} \in \mathscr{C}^\infty_{k-1,j}(X_t)$, $u_t^{k,j-1} \in \mathscr{C}^\infty_{k,j-1}(X_t)$.

Hence Ω decomposes with respect to the complex structure on X_t for each $t \in \mathbb{B}_\varepsilon(0)$ as:

$$\Omega = \sum_{j+k=2p} F_{BC,t}^{k,j}\Omega_t^{k,j} + \partial_t v_t + \bar{\partial}_t w_t \quad , \qquad v_t, w_t \in \mathscr{C}^\infty_{2p-1}(X_t) \quad , \quad \forall t\sim 0, \tag{3.3.24}$$

where $F_{BC,t}^{k,j} : \mathscr{C}^\infty_{k,j}(X_t) \longrightarrow \mathcal{H}_{BC}^{k,j}(X_t)$ is the projection onto the Bott-Chern harmonic (k,j)-forms. By lifting (3.3.24) to the universal cover and taking the (p,p)-component we get:

$$\widetilde{F_{BC,t}^{p,p}\Omega_t^{p,p}} = \partial_t(\eta_t^{p-1,p} - \pi^* v_t^{p-1,p}) + \bar{\partial}_t(\eta_t^{p,p-1} - \pi^* w_t^{p,p-1}) \,, \qquad t\sim 0. \tag{3.3.25}$$

Hence, lifting (3.3.23) yields for t sufficiently small:

$$\widetilde{\Omega_t} = \partial_t(\eta_t^{p-1,p} + \pi^*(\bar{\partial}_t\bar{\partial}_t^*\partial_t^*\mathbb{G}_{BC,t}\Omega_t^{p,p} - v_t^{p-1,p}) + \bar{\partial}_t(\eta_t^{p,p-1} - \pi^* w_t^{p,p-1}) = \partial_t\alpha_t + \bar{\partial}_t\beta_t \tag{3.3.26}$$

where $\alpha_t := \eta_t^{p-1,p} + \pi^*(\bar{\partial}_t\bar{\partial}_t^*\partial_t^*\mathbb{G}_{BC,t}\Omega_t^{p,p} - v_t^{p-1,p})$ and $\beta_t = \eta_t^{p,p-1} - \pi^* w_t^{p,p-1}$, with:

$$\begin{cases} \|\alpha_t\|_{L^\infty_{\tilde{\omega}_t}(\tilde{X}_t)} & \le \|\eta_t^{p-1,p}\|_{L^\infty_{\tilde{\omega}_t}(\tilde{X}_t)} + \|\pi^*(\bar{\partial}_t\bar{\partial}_t^*\partial_t^*\mathbb{G}_{BC,t}\Omega_t^{p,p} - v_t^{p-1,p})\|_{L^\infty_{\tilde{\omega}_t}(\tilde{X}_t)} \\ & \le C_t\|\eta\|_{L^\infty_{\tilde{\omega}_0}(\tilde{X})} + \|\bar{\partial}_t\bar{\partial}_t^*\partial_t^*\mathbb{G}_{BC,t}\Omega_t^{p,p}\|_{L^\infty_{\omega_t}(X_t)} + \|v_t\|_{L^\infty_{\omega_t}(X_t)} < \infty \\ \|\beta_t\|_{L^\infty_{\tilde{\omega}_t}(\tilde{X}_t)} & \le \|\eta_t^{p,p-1}\|_{L^\infty_{\tilde{\omega}_t}(\tilde{X}_t)} + \|\pi^* w_t^{p,p-1}\|_{L^\infty_{\tilde{\omega}_t}(\tilde{X}_t)} \le C_t\|\eta\|_{L^\infty_{\tilde{\omega}_0}(\tilde{X})} + \|w_t\|_{L^\infty_{\omega_t}(X)} < \infty \end{cases}$$

for some constant $C_t > 0$ depending on the metric ω_t. Hence Ω_t is p-SKT hyperbolic. □

Regarding the deformation behavior of p-SKT hyperbolicity, it is already known that in degree $p=1$, the SKT property is **not** open under small deformations of the complex structure (see, for instance, [TT17], Remark 4.6). However, under the $\partial\bar{\partial}$-assumption, the p-SKT property becomes stable under small deformations. Indeed, if X is a compact $\partial\bar{\partial}$-manifold, then (see [Ale11], Corollary 3.4, or [Bel20], Theorem 4.3):

$$X \text{ is } p\text{-SKT} \iff X \text{ is } p\text{-HS} \;, \qquad \text{where } 1 \le p \le n-1.$$

The $\partial\bar{\partial}$-assumption appears to be quite strong, since, as mentioned above, D. Popovici conjectured that every $\partial\bar{\partial}$-manifold is balanced, and the Fino-Vezzoni conjecture predicts that any balanced SKT manifold is in fact Kähler ([FV15], Problem 3). This suggests a deeper relationship between these structures. Motivated by this, we propose the following conjecture for any $1 \le p \le n-1$:

Conjecture 3.3.29. *Let X be a compact $\partial\bar{\partial}$-manifold. Then:*

$$X \text{ is } p\text{-SKT} \Longleftrightarrow X \text{ is } p\text{-Kähler} \ , \qquad \text{where } 1 \leq p \leq n-1.$$

A weaker criterion ensuring the stability of the p-SKT property under small deformations may be extracted from the arguments in [RWZ22], Proposition 4.1 and [Cav12], Theorem 8.11.

The preceding discussion, in conjunction with Conjecture 4.22 and Conjecture 4.25 in [Khe25], leads naturally to the following problem for degree $1 \leq p \leq n-1$:

Problem 7. Let X be a compact $\partial\bar{\partial}$-manifold and fix $1 \leq p \leq n-1$. We ask whether the following assertion holds:

$$X \text{ is } p\text{-SKT hyperbolic} \Longleftrightarrow X \text{ is } p\text{-HS hyperbolic} \Longleftrightarrow X \text{ is } p\text{-Kähler hyperbolic}$$

An affirmative answer to this question would imply the deformation openness of p-SKT hyperbolicity, provided the $\partial\bar{\partial}$-property holds.

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