2016 January **John Von Neumann Institute**

Stochastic calculus applied in Finance

This course contains seven chapters after some prerequisites, 18 hours plus exercises (12h).

0.1 Introduction, aim of the course, agenda

The purpose is to introduce some bases of stochastic calculus to get tools to be applied to Finance. Actually, it is supposed that the financial market proposes assets, the prices of them depending on time and hazard. Thus, they could be modelized by stochastic processes, assuming theses prices are known in continuous time. Moreover, we suppose that the possible states space, Ω, is infinite, that the information is continuously known, that the trading are continuous. Then, we consider that the model is indexed by time $t, t \in [0, T]$ or $\mathbb{R}^+$, and we will introduce some stochastic tools for these models.

Remark that actually the same tools could be useful in other areas, other than financial models.

(0) Prerequisites in Probability theory.
(i) Brownian motion: this stochastic process is characterized by the fact that little increments model the "noise", the physical measure error.... The existence of such a process is proved in the first chapter, Brownian motion is explicitly built, some of useful properties are shown.
(ii) Stochastic integral: actually Itô calculus allows to get more sophisticated processes by integration. This integral is defined in second chapter
(iii) Itô formula allows to differentiate functions of stochastic processes.
(iv) Stochastic differential equations: linear equation goes to Black-Scholes model and a first example of diffusion. Then Ornstein-Uhlenbeck equation models more complicate financial behaviors.
(v) Change of probability measures (Girsanov theorem) and martingale problems will be fifth chapter. Indeed, in these financial models, we try to set on a probability space where all the prices could be martingales, so with constant mean; in such a case, the prices are said to be "risk neutral". Thus we will get Girsanov theorem and martingale problem.
(vi) Representation of martingales, complete markets: we introduce the theorem of martingale representation, meaning that, under convenient hypotheses, any $\mathcal{F}_T$-measurable random variable is the value at time T of a martingale. In this chapter we also consider complete markets.
(vii) A conclusive chapter apply all these notions to financial markets : viable market, complete market, admissible portfolio, optimal portfolio and so on in case of a small investor. We also look (if time enough) at European options.

0.2 Prerequisites

Some definitions : on a set Ω a **σ-algebra** is a set $\mathcal{A}$ of subsets satisfying :
• $\emptyset \in \mathcal{A}$,
• if A and $B \in \mathcal{A}$, then $A \cup B$, $A \cap B$, $A^c = \Omega - A \in \mathcal{A}$,
• if $\forall n$, $A_n \in \Omega$ and $A_n \supset A_{n+1}$, $\cap_n A_n \in \mathcal{A}$.

A **probability** on $\mathcal{A}$ is an application $\mathbb{P} : \mathcal{A} \mapsto [0,1]$ such that $\mathbb{P}(\Omega) = 1$; $\mathbb{P}(A^c) = 1-\mathbb{P}(A)$; if A and $B \in \mathcal{A}$ and $A \cap B = \emptyset$, $\mathbb{P}(A \cup B) = \mathbb{P}(A)+\mathbb{P}(B)$; $\mathbb{P}(\cap_n A_n) = \lim_n \mathbb{P}(A_n)$.

A **probability space** is the triplet $(\Omega, \mathcal{A}, \mathbb{P})$.
Actually it a positive bounded measure on $(\Omega, \mathcal{A})$.

An important example of σ-algebra on R, R^d is the Borel σ-algebra generated by the open subset, meaning the smallest σ-algebra containing the open (or the closed) subsets.

A **filtration** is a set of increasing σ-algebras $(\mathcal{F}_t, t \in \mathbb{R}^+)$, and a filtered probability space is the set $(\Omega, \mathcal{A}, (\mathcal{F}_t, t \in \mathbb{R}^+), \mathbb{P})$, $\forall t, \mathcal{F}_t \subset \mathcal{A}$.

A **random variable** X on $(\Omega, \mathcal{A}, \mathbb{P})$ to a measurable space $(E, \mathcal{E})$ is an application from Ω to E such that $\forall A \in \mathcal{E}$, the reciprocal set $X^{-1}(A) \in \mathcal{A}$. It is said to be $\mathcal{A}$-measurable.
We denote the expectation $E_{\mathbb{P}}[X] = \int_\Omega X(\omega) d\mathbb{P}(\omega)$, and $E[X]$ if there is no ambiguity.

A **stochastic process** is an application X on $\Omega \times \mathbb{R}^+$. When ω is fixed, $t \mapsto X(\omega, t)$ is named a trajectory; this one could be continuous, right continuous (càd) left limited (làg), and so on.

On a filtered probability space, a process is said to be **adapted** to the filtration when $\forall t$, $X(., t)$ is $\mathcal{F}_t$-measurable.

0.3 Some convergences

Definition 0.1. *Let $\mathbb{P}_n$ series of probability measures on a metric space (E, d) endowed with Borel σ-algebra $\mathcal{B}$, and $\mathbb{P}$ measure on $\mathcal{B}$. The series $(\mathbb{P}_n)$ is said to* **weakly converge** *to $\mathbb{P}$ if $\forall \in \mathcal{C}_b(E)$, $\mathbb{P}_n(f) \to \mathbb{P}(f)$.*

Definition 0.2. *Let (X_n) a series of random variables on $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$ taking their values in a metric space $(E, d, \mathcal{B})$. The series (X_n) is said to* **converge in law** *to X if the series of probability measures $(\mathbb{P}_n X_n^{-1})$ weakly converges to $\mathbb{P}X^{-1}$, meaning: $\forall f \in \mathcal{C}_b(E)$, $\mathbb{P}_n(f(X_n)) \to \mathbb{P}(f(X))$.*

- L^p convergence: $E[|X_n - X|^p] \to 0$.

- convergence in probability: $\forall \varepsilon$, $\mathbb{P}\{\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\} \to 0$.

- almost sure convergence: $\mathbb{P}\{\omega : \lim_n X_n(\omega) = X(\omega)\} = 1$.

- limit sup and limit inf of sets: $\liminf_n A_n = \cup_n \cap_{k \geq n} A_k$, $\limsup_n A_n = \cap_n \cup_{k \geq n} A_k$.

We can express almost sure convergence:

$$\forall \varepsilon, \ \mathbb{P}(\liminf_n \{\omega : |X_n(\omega) - X(\omega)| \leq \varepsilon\} = 1.$$

And the following is now obvious:

Proposition 0.3. *Almost sure convergence yields probability convergence.*

Proposition 0.4. *L^p convergence yields probability convergence.*

- Lebesgue theorems: monotoneous, bounded convergence.

Theorem 0.5. *Fatou: For all series of events (A_n)*

$$\mathbb{P}(\liminf_n A_n) \leq \liminf_n \mathbb{P}(A_n) \leq \limsup_n \mathbb{P}(A_n) \leq \mathbb{P}(\limsup_n A_n).$$

Theorem 0.6. *Borel-Cantelli:*

$$\sum_n \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}(\limsup A_n) = 0.$$

When the events A_n are independent and $\sum_n \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(\limsup A_n) = 1$.

Proofs of these two theorem to be done as exercises: (I.0).

Definition 0.7. *A family of random variables $\{U_\alpha, \alpha \in A\}$ is* **uniformly integrable** *when*

$$\lim_{n\to\infty} \sup_\alpha \int_{\{|U_\alpha|\geq n\}} |U_\alpha| d\mathbb{P} = 0.$$

Theorem 0.8. *The following are equivalent:*

(i) Family $\{U_\alpha, \alpha \in A\}$ is uniformly integrable,

(ii) $\sup_\alpha E[|U_\alpha|] < \infty$ and $\forall \varepsilon, \exists \delta > 0 : A \in \mathcal{A}$ et $\mathbb{P}(A) \leq \delta \Rightarrow E[|U_\alpha| 1_A] \leq \varepsilon$.

RECALL: an almost surely convergent series which get a uniformly integrable family, moreover converges in L^1.
$X_n \to X$ in L^1 if and only if the family $(X_n, n \geq 0)$ is uniformly integrable and $X_n \to X$ in probability.

0.4 Conditional expectation

Definition 0.9. *Let X a random variable belonging to $L^1(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{B}$ a σ-algebra included in $\mathcal{A}$. $E_{\mathbb{P}}(X/\mathcal{B})$ is the unique random variable in $L^1(\mathcal{B})$ such that*

$$\forall B \in \mathcal{B}, \int_B X d\mathbb{P} = \int_B E_{\mathbb{P}}(X/\mathcal{B}) d\mathbb{P}.$$

Corollary 0.10. *If $X \in L^2(\mathcal{A})$, $\|X\|_2^2 = \|E_{\mathbb{P}}(X/\mathcal{B})\|_2^2 + \|X - E_{\mathbb{P}}(X/\mathcal{B})\|_2^2$.*

Exercises : Let $X \in L^1$ and a family of σ-algebras $\mathcal{F}^\alpha, \alpha \in A$. Then the family of conditional expectations $\{E[X/\mathcal{F}^\alpha], \alpha \in A\}$ is uniformly integrable.
Then Ex. 1.1 1.2 1.7.

0.5 Stopping time

This notion is related to a filtered probability space.

Definition 0.11. *A random variable* $T : (\Omega, \mathcal{A}, (\mathcal{F}_t), \mathbb{P}) \to (\mathbb{R}^+, \mathcal{B})$ *is a* **stopping time** *if* $\forall t \in \mathbb{R}^+$, *the event* $\{\omega / T(\omega) \leq t\} \in \mathcal{F}_t$.

Examples :

- a constant variable is a stopping time,

- let O be an open set in $\mathcal{A}$ and X a continuous process, then

$$T_O(\omega) = \inf\{t, X_t(\omega) \in O\}$$

is a stopping time, called 'hitting time'.

Definition 0.12. *Let* T *be a stopping time in filtration* $\mathcal{F}_t$. *The set* $\mathcal{F}_T = \{A \in \mathcal{A}, A \cap \{\omega / T \leq t\} \in \mathcal{F}_t\}$ *is called* **stopped σ-algebra** *at time* T.

Definition 0.13. *The process* $X_{.\wedge T}$ *is called "stopped process at* T*", denoted as* X^T.

Exercises I 3 to 8. The 1.6 is important, as a proposition: *A random variable* X *is* $\mathcal{F}^T-$*measurable if and only if* $\forall t \geq 0$, $X\mathbf{1}_{\{T \leq t\}}$ *is* $\mathcal{F}_t-$*measurable.*

0.6 Martingales

(cf. [30] pages 8-12 ; [20] pages 11-30.)

Definition 0.14. *An adapted real process* X *is a* **martingale** *(resp super/sub) if*
(i) $X_t \in L^1(\Omega, \mathcal{A}, \mathbb{P}), \forall t \in \mathbb{R}^+$,
(ii) $\forall s \leq t, E[X_t / \mathcal{F}_s] = X_s$. *(resp* $\leq, \geq$ *.)*

Lemma 0.15. *Let* X *be a martingale and* φ *a function such that* $\forall t\ \phi(X_t) \in L^1$.
If φ *is a convex function, then* $\varphi(X)$ *is a sub-martingale.*
If φ *is a concave function, then* $\varphi(X)$ *is an super-martingale.*

When X *is a sub-martingale and* ϕ *an increasing convex function (s.t.* $\forall t\ \phi(X_t) \in L^1$*), then* $\phi(X)$ *is a sub-martingale.*

Proof exercise II.1.

Definition 0.16. *The martingale* X *is said to be* **closed** *by* $Y \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ *if* $X_t = E[Y / \mathcal{F}_t]$.

Corollary 0.17. *A closed martingale is uniformly integrable.*

Proposition 0.18. *Any martingale admits a càdlàg modification (cf [30]).*

càdlàg means right continuous left limited, it is a french acronym

Theorem 0.19. *convergence of martingales: Let X be a càd super (or sub)-martingale such that* $\sup_t E[|X_t|] < \infty$. *Then* $\lim_{t\to\infty} X_t$ *exists almost surely and belongs to* $L^1(\Omega, \mathcal{A}, \mathbb{P})$. *If X is a martingale closed by Z, it is too by* $\lim_{t\to\infty} X_t$, *denoted as* X_∞, *equal to* $E[Z/\vee_{t\geq 0}\mathcal{F}_t]$.

The proof is really sophisticated and long enough, so we skip it.

Corollary 0.20. *A below bounded super-martingale converges almost surely to infinity.*

Proof. : Let X be an super-martingale, such that there exists $a \in \mathbb{R}$, $X_t(\omega) \geq a$ almost surely. So, $X_t - a$ is too a super-martingale satisfying $0 \leq X_t - a$ and $0 \leq E[X_t - a] = E(|X_t - a|]$. But the super-martingale property tells $E[X_t - a] \leq E[X_0 - a]$. So $\forall t$ $0 \leq E(|X_t - a|] \leq E[X_0 - a]$. The theorem property is satisfied and concludes the proof.•

Theorem 0.21. *In case of L^1 bounded martingale (meaning exactly* $\sup_t E[|X_t|] < \infty$*) there exists Y and Z positive martingales such that almost everywhere for all t,* $X_t = Y_t - Z_t$.

As a consequence, in many proofs, we could suppose that the martingale could be positive.

Theorem 0.22. *Let X be a martingale. The followings are equivalent :*
(i) X is uniformly integrable,
(ii) X_t converges almost surely to Y, Y belonging to L^1, when t goes to infinity, and $\{X_t, t \in \overline{\mathbb{R}^+}\}$ *is a martingale,*
(iii) (X_t) L^1 converges to Y when t goes to infinity.

Proof. = exercise II.3: $(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$.

The following is a corollary which stresses the point (ii) above : $\{X_t, t \in \overline{\mathbb{R}^+}\}$ is a martingale.

Corollary 0.23. *Let X be a uniformly integrable martingale; then the almost sure limit Y of X_t when t goes to infinity exists and belongs to L^1. Moreover* $X_t = E[Y/\mathcal{F}_t]$.

Actually the hypothesis "X be a uniformly integrable martingale" is the point (i) in previous theorem, so we get assertion (ii) which is exactly the corollary conclusion.

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Notation: let X be a stochastic process and T a stopping time on the filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t), \mathbb{P})$. Then $\mathbf{X}_T$ is the random variable $\omega \to X_{T(\omega)}(\omega)$.

Theorem 0.24. *Doob: Let* $(X_t, t \in \mathbb{R}^+)$ *be a càd* $\mathcal{F}$*-martingale,* S *and* T $\mathcal{F}$*-stopping times such that:*
(i) $E[|X_S|],\ E[|X_T|] < \infty$*,*
(ii) $\lim_{t\to+\infty} \int_{\{T>t\}} |X_t| d\mathbb{P} = \lim_{t\to+\infty} \int_{\{S>t\}} |X_t| d\mathbb{P} = 0$*,*
(iii) $S \leq T < \infty$ *almost surely.*
Then $E[X_T/\mathcal{F}_S] = X_S$ $\mathbb{P}-$ *almost surely.*

Let X *be a càd sub-martingale with terminal value* X_∞*, let two stopping times* S *and* T *satisfying (i)(ii)(iii). Then:*

$$X_S \leq E[X_T/\mathcal{F}_S]\ \mathbb{P}- \text{ almost surely.}$$

Proof: pages 19-20 [20]: to be detailed.

We provide the proof only in case of closed martingale: $\forall t,\ X_t = E[X_\infty/\mathcal{F}_t]$. Moreover, we restrain to the case of $X_t \geq 0$ without loss of generality, since a closed martingale can be written as following: $X_t = E[X^+_\infty/\mathcal{F}_t] - E[X^-_\infty/\mathcal{F}_t]$, difference between two non negative martingales.
(i) The first step will be to prove that in such a case, for all stopping time T:

$$X_T = E[X_\infty/\mathcal{F}_T]. \tag{1}$$

Then if $S \leq T$, $\mathcal{F}_S \subset \mathcal{F}_T$ and

$$E[X_T/\mathcal{F}_S] = E[E[X_\infty/\mathcal{F}_T]/\mathcal{F}_S] = E[X_\infty/\mathcal{F}_S] = X_S.$$

(ii) The second step is to consider T deterministic : then (1) is only the definition of a closed martingale.
(iii)We now consider that the stopping time T is such that $T(\Omega)$ is the discrete real subspace $\{t_1, \cdots, t_n, \cdots\}$. Then

$$X_T \mathbf{1}_{T=t_n} = X_{t_n} \mathbf{1}_{T=t_n} = E[X_\infty/\mathcal{F}_{t_n}] \mathbf{1}_{T=t_n}.$$

On another hand, X_T is both $\mathcal{F}_T-$measurable and integrable (assumption (i)). Let $A \in \mathcal{F}_T$ and compute

$$E[X_\infty \mathbf{1}_A] = \sum_n E[X_\infty \mathbf{1}_A \mathbf{1}_{T=t_n}] = \sum_n E[X_{t_n} \mathbf{1}_A \mathbf{1}_{T=t_n}]$$

since $A \cap \{T = t_n\} \in \mathcal{F}_{t_n}$ and positiveness allows the commutation between $\sum$ and E.

$$E[X_\infty \mathbf{1}_A] = \sum_n E[X_T \mathbf{1}_A \mathbf{1}_{T=t_n}] = E[X_T \mathbf{1}_A]$$

so (1) is satisfied.
(iv) Let T a general stopping time: there exists a decreasing sequence of stopping times T_n, $T_n \downarrow T$ and $\forall n$, T_n satisfied step (iii): so $\forall n$,

$$X_{T_n} = E[X_\infty/\mathcal{F}_{T_n}].$$

The right continuity tells us that the left hand above goes to X_T almost surely. Actually, (X_{T_n}) is a backward martingale, uniformly integrable, so we admit that this convergence is too a L^1 convergence.

Moreover $T_n \geq T$ implies $\mathcal{F}_T \subset \mathcal{F}_{T_n}$ and we get

$$E[X_\infty/\mathcal{F}_T] = E[E[X_\infty/\mathcal{F}_{T_n}]/\mathcal{F}_T] = E[X_{T_n}/\mathcal{F}_T].$$

Using the almost sure and L^1 convergence of (X_{T_n}) to X_T, the right hand above converges to X_T and (1) is proved.

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Corollary 0.25. *Under the same assumptions*

$$E[X_{t\wedge T}/\mathcal{F}_S] = X_{t\wedge S}.$$

Proof. : Doob theorem applied to stopping times $t\wedge T$, $t\wedge S$ yields $E[X_{t\wedge T}/\mathcal{F}_{t\wedge S}] = X_{t\wedge S}$. But actually, we can prove that $E[X_{t\wedge T}/\mathcal{F}_S]$ is $\mathcal{F}_{t\wedge S}$-measurable
(not so obvious, to detail...).
Then we can identify $E[X_{t\wedge T}/\mathcal{F}_S]$ as $E[X_{t\wedge T}/\mathcal{F}_{t\wedge S}]$. •

Definition 0.26. *The increasing process $\langle M\rangle$ ("bracket") is defined as:*

$$t \mapsto \langle M\rangle_t = \lim_{|\pi|\to 0} proba \sum_{t_i\in\pi}(M_{t_i} - M_{t_{i-1}})^2$$

π being partitions of $[0,t]$ and $|\pi| = \sup_i(t_{i+1} - t_i)$.

In next chapter, we will show that if $M = B$ is Brownian motion then $\langle B\rangle_t = t$.

Remark 0.27. *The squarred integrable martingales admit a bracket.*

Proposition 0.28. *$\langle M\rangle_t$ is the adapted increasing continuous unique processus such that $M_t^2 - \langle M\rangle_t$ is a martingale.*

This proposition is often used as the bracket definition and then Definition 0.26 is a consequence.

Proof: We can write $M_t^2 - \langle M\rangle_t$ as the limit in probability of

$$[\sum_{t_i\in\pi}(M_{t_i} - M_{t_{i-1}})]^2 - \sum_{t_i\in\pi}(M_{t_i} - M_{t_{i-1}})^2$$

and we developp the square

$$[\sum_{t_i\in\pi}(M_{t_i} - M_{t_{i-1}})]^2 - \sum_{t_i\in\pi}(M_{t_i} - M_{t_{i-1}})^2 = 2\sum_{i<j}(M_{t_i} - M_{t_{i-1}})(M_{t_j} - M_{t_{j-1}}).$$

We now take the $\mathcal{F}_s$ conditional expectation above,

$$E[M_t^2 - \sum_{t_i\in\pi}(M_{t_i} - M_{t_{i-1}})^2/\mathcal{F}_s] = M_s^2 - \sum_{t_i\leq s}(M_{t_i} - M_{t_{i-1}})^2 + 2E[\sum_{s\leq t_i<t_j}(M_{t_i} - M_{t_{i-1}})(M_{t_j} - M_{t_{j-1}})/\mathcal{F}_s].$$

But for any $s \leq t_i < t_j$,

$$E[(M_{t_i} - M_{t_{i-1}})(M_{t_j} - M_{t_{j-1}})/\mathcal{F}_s] = E[(M_{t_i} - M_{t_{i-1}})E[(M_{t_j} - M_{t_{j-1}})/\mathcal{F}_{t_i}]/\mathcal{F}_s] = 0.$$

Thus we can conclude getting $|\pi|$ to 0. •

Corollary 0.29. *For any pair $s \leq t$, $E[(M_t - M_s)^2/\mathcal{F}_s] = E[(\langle M\rangle_t - \langle M\rangle_s)/\mathcal{F}_s]$.*

Proof: We developp $(M_t - M_s)^2 = M_t^2 - 2M_sM_t + M_s^2$; since $E[M_t/\mathcal{F}_s] = M_s$, $E[M_t^2 - 2M_sM_s + M_s^2/\mathcal{F}_s] = E[M_t^2/\mathcal{F}_s] - M_s^2$. We now set the difference $E[(M_t - M_s)^2]/\mathcal{F}_s] - E[\langle M\rangle_t - \langle M\rangle_s/\mathcal{F}_s] = E[M_t^2 - \langle M\rangle_t/\mathcal{F}_s] - M_s^2 + \langle M\rangle_s$. •

Finally we admit some useful inequalities, (cf. [20] pp 13-14) namely **Doob's inequalities** (i) (ii) and Burkholder-Davis-Gundy inequality (iii).

Theorem 0.30. *Let X be a càd sub martingale and $0 \leq \sigma < \tau$, $\lambda > 0$. Then*

$$(i) \quad \lambda\mathbb{P}\{\sup_{\sigma \leq t \leq \tau} X_t \geq \lambda\} \leq E[X_\tau^+].$$

$$(ii) \quad \forall p > 1,\ E[\sup_{\sigma \leq t \leq \tau} |X_t|^p] \leq (\frac{p}{p-1})^p E[|X_\tau|^p]$$

(iii) If X is a local martingale, $X_0 = 0$, $\forall p \geq 1$, there exists $C_p > c_p > 0$ such that

$$\forall \text{ stopping time } \tau,\ c_p E[\langle X\rangle_\tau|^p] \leq E[(\sup_{t \leq \tau} |X_t|^{2p})] \leq C_p E[\langle X\rangle_\tau^p]$$

where $\langle X\rangle$ is the bracket.

Finally we provide a useful sub martingale decomposition:

Theorem 0.31. *Let be X a sub martingale of "class D" (meaning the family $\{X_S, S$ being $\mathcal{F}$ stopping times$\}$ is uniformly integrable). Then there exists a martingale M and an increasing process A such that almost surely $X_t = M_t + A_t$.*

Definition 0.32. *A process X is said to be "progressively measurable" for filtration $(\mathcal{F}_t, t \geq 0)$ if $\forall t \geq 0, \forall A \in \mathcal{B}(R) : \{(s,\omega)/0 \leq s \leq t\ ;\ X_s(\omega) \in A\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t$, meaning that the application on $([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t) : (s,\omega) \mapsto X_s(\omega)$ is measurable.*

Proposition 0.33. *(cf [20], 1.12) If X is a adapted measurable process, it admits a progressively measurable modification.*

Proof: cf. Meyer 1966, page 68.

0.7 Local martingales

To stop a process at a convenient stopping time allows to get some uniformly integrable martingales thus easy to manage with: we get results for all n, then n going to the infinity and using Lebesgue theorems (monotonous or bounded convergences). It is the reason of the introduction of stopping times and local martingales. The set of local martingales is denoted as $\mathcal{M}_{loc}$.

Definition 0.34. *(page 33 [30].) Let X an adapted càdlàg process. It is a* **local martingale** *if there exists a series of stopping times $(T_n)_n$, increasing to infinity, so that $\forall n$ the stopped process X^{T_n} is a martingale.*

Theorem 0.35. *(cf [30], th. 44, page 33) Let $M \in \mathcal{M}_{loc}$ and T stopping time such that M^T is uniformly integrable.*

(i) $S \leq T \Rightarrow M^S$ is uniformly integrable.

(ii) $\mathcal{M}_{loc}$ is a real vector space.

(iii) if M^S and M^T are uniformly integrable, then $M^{S\wedge T}$ is uniformly integrable.

Notation :

$$M_t^* = \sup_{0\leq s\leq t} |M_s| \ ; \ M^* = \sup_{0\leq s} |M_s|.$$

Theorem 0.36. *(cf [30], th. 47, page 35) If $M \in \mathcal{M}_{loc}$ is such that $E[M_t^*] < \infty \forall t$, then M is a "true" martingale.*
If moreover $E[M^*] < \infty$, then M is uniformly integrable.

Proof: to be admitted.
(i) $\forall s \leq t, |M_s| \leq M_t^*$ belongs to L^1. The sequence $T_n \wedge t$ is increasing to t and

$$E[M_{T_n\wedge t}/\mathcal{F}_s] = M_{T_n\wedge s}.$$

Taking almost sure limit in this equality and Lebesgue theorem allow the L^1 convergence.

(ii) Then M is a martingale and M^* is in L^1. Martingale convergence theorem shows the almost sure convergence of (M_t) to M_∞. Finally, the uniform integrability is to be proved (using equivalent definition of uniform integrability).

0.8

The following concerns general culture, but out of the agenda.

Definition 0.37. *Let X and Y two processes, X is said to be a* **modification** *of Y if:*

$$\forall t \geq 0, \mathbb{P}\{X_t = Y_t\} = 1.$$

X and Y are said to be **indistinguable** *if almost surely the trajectories coincide:*

$$\mathbb{P}\{X_t = Y_t, \forall t \geq 0\} = 1.$$

Remark 0.38. *This second notion is stronger than the first one.*

Proposition 0.39. *Let X be a $\mathcal{F}$-progressively measurable process and T be a $(\mathcal{F}_t)$ stopping time. Then*
(i) the application $\omega \mapsto X_{T(\omega)}(\omega)$ is $\mathcal{F}_T$-measurable
(ii) and the process $t \mapsto X_{t\wedge T}$ is $\mathcal{F}$-adapted.

Proof: (i) the fact that X is progressively measurable implies that for any Borel set A,

$$\forall t, \ \{(s,\omega), \ 0 \leq s \leq t, \ X_s(\omega) \in A\} \in \mathcal{B}_{[0,t]} \otimes \mathcal{F}_t.$$

Then $\{\omega : X_{T(\omega)}(\omega) \in A\} \cap \{\omega : T(\omega \leq t\} = \{\omega : X_{T(\omega)\wedge t}(\omega) \in A\} \cap \{T \leq t\}$.

T is a $\mathcal{F}$-stopping time, so the second event belongs to $\mathcal{F}_t$, and because of progressively measurability the first is too.

(ii) This second assertion moreover shows that X^T is too $\mathcal{F}$-adapted.

Proposition 0.40. *(cf [20], 1.13) If X is an adapted measurable process and admits càd or càg trajectories, it is progressively measurable.*

Proof: Define

$$X_s^{(n)}(\omega) = X_{(k+1)t2^{-n}}(\omega),\ s \in]\frac{kt}{2^n}, \frac{(k+1)t}{2^n}],\ X_0^{(n)}(\omega) = X_0(\omega)\ ;\ k = 0, \cdots, 2^n - 1.$$

Obviously the application $(s, \omega) \mapsto X_s^{(n)}(\omega)$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable. Using right continuity, the series $X_s^{(n)}(\omega)$ converges to $X_s(\omega)$ $\forall (s, \omega)$ then the limit is too $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable.

1 Introduction of Wiener process, Brownian motion

[20] pages 21-24 ; [30] pages 17-20.

Historically, this process first models the irregular motion of pollen particles suspended in water, observed by Robert Brown in 1828. This leads to dispersion of micro-particles in water, also called a "diffusion" of pollen in water. In fact, this movement is currently used in many other models of dynamic phenomena:

- Microscopic particles in suspension,
- Prices of shares on the stock exchange,
- Errors in physical measurements,
- Asymptotic behavior of queues,
- Any behavior from dynamic random (stochastic differential equations).

Definition 1.1. *The* **Brownian motion** *or* **Wiener process** *is a process B on a filtered space $(\Omega, \mathcal{A}, \mathcal{F}_t, \mathbb{P})$, adapted, continuous, taking its values in $\mathbb{R}^d$ such that:*

(i) $B_0 = 0$, $\mathbb{P}$-almost surely on Ω,

(ii) $\forall s \leq t, B_t - B_s$ is independent of $\mathcal{F}_s$, with centered Gaussian law with variance matrix $(t-s)I_d$.

Consequently, let a real sequence $0 = t_0 < t_1 < \cdots < t_n < \infty$, the sequence $(B_{t_i} - B_{t_{i-1}})_i$ follows a centered Gaussian law with variance matrix diagonal, diagonal $(t_i - t_{i-1})_i$. B is said to be a, **independent increments process**.

The first problem we solve is the existence of such a process. There are several classical constructions.

1.1 Existence based on vector construction, Kolmogorov lemma

([20] 2.2 ; [30] pages 17-20.) Very roughly, to get an idea without going into detailed proofs (long, delicate and technical), we proceed as follows. Let Ω be $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^d)$ and $B(t, \omega) = \omega(t)$ be the "coordinate applications" called **trajectories**. Space Ω is endowed with the smallest σ-algebra $\mathcal{A}$ which implies the variable $\{B_t, t \in \mathbb{R}^+\}$ measurable and with "natural" filtration generated by the process B : $\mathcal{F}_t = \sigma\{B_s, s \leq t\}$. On $(\Omega, \mathcal{A})$ the existence of a unique probability measure $\mathbb{P}$ is proved, satisfying $\forall n \in \mathbb{N}, t_1, \cdots, t_n \in \mathbb{R}^+, B_1, \cdots, B_n$ being Borel of $\mathbb{R}^d$:

$$\mathbb{P}\{\omega / \omega(t_i) \in B_i \forall i = 1, \cdots, n\} = \int_{B_1} \cdots \int_{B_n} p(t_1, 0, x_1) p(t_2 - t_1, x_1, x_2) \cdots p(t_n - t_{n-1}, x_{n-1}, x_n) dx_1..dx_n,$$

where $p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$.
Then the point is to show:

- This well defines a probability measure on the σ-algebra $\mathcal{A}$.

- Under this probability measure, the process $t \mapsto \omega(t)$ is a Brownian motion according to the original definition.

Definition 1.2. *This probability measure $\mathbb{P}$ is named* **the** *Wiener measure on Ω.*

In fact, this defines a probability measure on the Borel sets of the application space $\mathbb{A}(\mathbb{R}^+, \mathbb{R}^d)$, Ω not being one of its Borel sets. Instead of that, we choose $\Omega = \mathbb{A}(\mathbb{R}^+, \mathbb{R}^d)$ and Kolmogorov theorem (1933).

Definition 1.3. *A consistent family of finite dimensional distributions $(Q_t, t$ n-uple $\mathbb{R}^+)$ is a family of measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that*

- if $s = \sigma(t), s$ and $t \in (\mathbb{R}^+)^n$, σ a permutation of integers $\{1, \cdots, n\}$ $A_1, \cdots, A_n \in \mathcal{B}(\mathbb{R}^d)$, then $Q_t(A_1, \cdots, A_n) = Q_s(A_{\sigma(1)}, \cdots, A_{\sigma(n)})$,

- and if $u = (t_1, \cdots, t_{n-1}), t = (t_1, \cdots, t_{n-1}, t_n), \forall t_n, Q_t(A_1, \cdots, A_{n-1}, \mathbb{R}) = Q_u(A_1, \cdots, A_{n-1})$.

Theorem 1.4. *(cf [20] page 50 : Kolmogorov, 1933) Let $(Q_t, t \in (\mathbb{R}^+)^n)$ be a consistent family of finite dimensional distributions.*
Then there exists a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that for all $B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R}^d)$,

$$Q_t(B_1, \cdots, B_n) = \mathbb{P}\{\omega/\omega(t_i) \in B_i, i = 1, \cdots, n\}.$$

We apply this theorem to the family of measures

$$Q_t(A_1, \cdots, A_n) = \int_{\Pi_i A_i} p(t_1, 0, x_1) \cdots, p(t_n - t_{n-1}, x_{n-1}, x_n) dx.$$

Then we show the existence of a continuous modification of the process= coordinate applications of Ω (Kolmogorov-Centsov, 1956), to get to the existence of a continuous modification of the canonical process:

Theorem 1.5. *(Kolmogorov-Centsov, 1956,cf [20] page 53, [30] page 171) Consider X real random process on $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying:*

$$\exists \alpha, \beta, C > 0 : E|X_t - X_s|^\alpha \leq C|t - s|^{1+\beta},\ 0 \leq s, t \leq T,$$

then X admits a continuous modification $\tilde{X}$ which is locally $\gamma-$Hölder continuous:

$$\exists \gamma \in]0, \frac{\beta}{\alpha}[, \exists h \text{ random variable } > 0, \exists \delta > 0 \ :$$
$$\mathbb{P}\{\sup_{0<t-s<h; s,t \in [0,T]} |\tilde{X}_t - \tilde{X}_s| \leq \delta |t - s|^\gamma\} = 1.$$

Remark that this theorem is also true for $t \in \mathbb{R}^d$-indexed fields.

1.2 Second construction of Brownian motion, case $d = 1$

to skip in a first lecture
Once again we consider $\Omega = \mathcal{C}(\mathbb{R}^+, \mathbb{R})$, we define on it:

$$\rho(\omega_1, \omega_2) = \sum_{n\geq 1} 2^{-n} \sup_{0\leq t\leq n} (|\omega_1(t) - \omega_2(t)| \wedge 1)$$

meaning PROHOROV's distance.

Remark 1.6. *This metric implies a topology which is the uniform on any compact convergence in probability.* $\Omega = \mathcal{D}(\mathbb{R}^+, \mathbb{R})$ *is a complete space with respect to this norm (cf. [30], page 49.)*

On Ω, we call **finite dimensional cylindrical sets** subsets as $A = \{\omega/(\omega(t_1), \cdots, \omega(t_n)) \in B\}$ where B is a Borel set of $\mathbb{R}^n$ and t an n-uple of positive real numbers. Then Ω is endowed with the σ-algebra generated by these sets and we show:

Proposition 1.7. *(Exercise 4.2, [20] page 60) Let $\mathcal{G}_t$ be the σ-algebra generated by the cylindrical sets related to $n-$uples (t_i) such that $\forall i,\ t_i \leq t$.*

- *$\mathcal{G} = \vee_t \mathcal{G}_t$ coincides with (Ω, ρ) Borel sets.*
- *If*

$$\begin{array}{rcl}\varphi_t : \Omega & \rightarrow & \Omega \\ \omega & \mapsto & (s \mapsto \omega(s \wedge t))\end{array}$$

then $\mathcal{G}_t = \varphi_t^{-1}(\mathcal{G})$ meaning $\Omega^t = \mathcal{C}([0, t], \mathbb{R})$ Borel sets.

The construction is based on central limit theorem.

Theorem 1.8. *let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables, same law, centered,with variance σ^2. Then*

$$S_n = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} \xi_i \textit{ converges in distribution to } X \textit{ of law } \mathcal{N}(0, 1).$$

This tool will allow us to explicitly build the Brownian motion; the following theorem is called **Donsker's invariance theorem**.

Theorem 1.9. *On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ let be a sequence of independent random variables, same law, centered,with variance $\sigma^2 > 0$. Let be the family of continuous processes*

$$X_t^n = \frac{1}{\sigma\sqrt{n}} [\sum_{j=1}^{[nt]} \xi_j + (nt - [nt])\xi_{[nt]+1}].$$

Let $\mathbb{P}^n$ be the measure induced by X^n on $(\mathcal{C}(\mathbb{R}^+, \mathbb{R}), \mathcal{G})$. Then $\mathbb{P}^n$ weakly converges to $\mathbb{P}^*$, measure under which $B_t(\omega) = \omega(t)$ is a standard Brownian motion on $\mathcal{C}(\mathbb{R}^+, \mathbb{R})$.

The long proof (7 pages, cf. [20]) is based on the following topological tools:
- weak and distribution convergences,
- tight families and relative compacity,

which are the topic of the following sub-sections.

1.2.1 Tight families and relative compacity

Definition 1.10. *Let (S,ρ) be a metric space and Π a family of probability measures on $(S,\mathcal{B}(S))$; Π is said to be* **relatively compact** *if a weakly sub-sequence can be extracted from Π.*

The family Π is said to be **tight** *if*

$$\forall \varepsilon>0, \exists K \text{ compact } \subset S \text{ such that } \mathbb{P}(K)\geq 1-\varepsilon,\ \forall \mathbb{P}\in\Pi.$$

Similarly, a family of random variables $\{X_\alpha : (\Omega_\alpha,\mathcal{A}_\alpha)\ ;\ \alpha\in A\}$ is said to be **relatively compact or tight** *if the family of related probability measures on $(S,\mathcal{B}(S))$ is relatively compact or tight.*

We admit the following theorem.

Theorem 1.11. *(Prohorov theorem, 1956, [20] 4.7)*
Let Π be a family of probability measures on $(S,\mathcal{B}(S))$. Then Π is relatively compact if and only if it is tight.

This theorem is interesting since relative compacity allows to extract a weakly convergent sequence, but the tightness property is easier to check.

Definition 1.12. *On $\Omega=\mathcal{C}(\mathbb{R}^+)$, the* **continuity modulus** *on $[0,T]$ is the quantity*

$$m^T(\omega,\delta)=\max_{|s-t|\leq\delta, 0\leq s,t\leq T}|\omega(s)-\omega(t)|.$$

Exercise : we can show that this modulus is continuous on the metric space (Ω,ρ), ρ being Prohorov's distance, increasing with respect to δ, and that $\forall\omega, \lim_{\delta\to0} m^T(\omega,\delta)=0$.

The following theorem is a tightness criterion (thus of relative compacity) for a family of probability measures on $(\Omega,\mathcal{B}(\Omega))$.

Theorem 1.13. *([20] page 63, 4.10) A sequence of probability measures $(\mathbb{P}_n)$ is tight if and only if:*
(i)

$$\lim_{\lambda\to\infty}\sup_{n\geq1}\mathbb{P}_n\{\omega\ :\ |\omega(0)|>\lambda\}=0.$$

(ii)

$$\lim_{\delta\to0}\sup_{n\geq1}\mathbb{P}_n\{\omega\ :\ m^T(\omega,\delta)>\varepsilon\}=0, \forall T>0, \forall\varepsilon>0.$$

Proof It is based on the following lemma:

Lemma 1.14. *([20], 4.9 page 62: Arzelà-Ascoli theorem) Let be $A\subset\Omega$. Then $\bar{A}$ is compact if and only if*

$$\sup_{\{\omega\in A\}}|\omega(0)|<\infty \text{ and } \forall T>0, \lim_{\delta\to0}\sup_{\{\omega\in A\}} m^T(\omega,\delta)=0.$$

Proof : pages 62-63 de [20].

Then, to study the convergence of the processes (X^n) defined in Donsker theorem (1.9), we introduce notions of convergence related to processes. The convergence in law "process as a whole" is difficult to obtain. We introduce a concept easier to verify.

Definition 1.15. *The sequence of processes (X^n)* **converges in finite dimensional distribution** *to the process X if $\forall d\in\mathbb{N}$ and for any d-uple $(t_1,\cdots,t_d)$, $(X^n_{t_1},\cdots,X^n_{t_d})$ converges in distribution to $(X_{t_1},\cdots,X_{t_d})$*

To prove such a convergence, it is enough to use characteristic functions of such d-uples.

Proposition 1.16. *If the sequence of processes (X^n) converges in distribution to the process X, then it converges in finite dimensional distribution to the process X.*

Proof: indeed, $\forall d$ and for aand $\pi \circ X^n = (X^n_{t_1}, \cdots, X^n_{t_d})$ converges converges in distribution to $\pi \circ X$ since continuity keeps the convergence in distribution . •

Warning! the converse is not always true! It can be seen in the following example as an Exercise:

$$X_t^n = nt1_{[0,\frac{1}{2n}]}(t) + (1-nt)1_{[\frac{1}{2n},\frac{1}{n}]}(t)$$

converges in finite dimensional distribution to 0 but not in distribution.
But it is true in case of a tight sequence.

Theorem 1.17. *(4.15 [20]) Let (X^n) be a sequence of processes, constituting a tight family converging in finite dimensional distribution to a process X. Then, $\mathbb{P}_n$ law of X^n on $\mathcal{C}(\mathbb{R}^+)$ weakly converges to a measure $\mathbb{P}$ under which the process $B_t(\omega(= \omega(t)$ is limit in finite dimensional distribution of the sequence (X^n).*

Proof: based on Prohorov theorem. The family is tight thus relatively compact and there exists $\mathbb{P}$ weak limit of a subsequence of the family. Let Q be a weak limit of another subsequence and suppose $Q \neq \mathbb{P}$. The hypothesis yields $\forall d, \forall t_1, \cdots, t_d, \forall B$ Borel of $\mathbb{R}^d$:

$$\mathbb{P}\{\omega : (\omega(t_i)) \in B\} = Q\{\omega : (\omega(t_i)) \in B\}$$

since there is convergence in finite dimensional distribution. This means that $\mathbb{P}$ and Q coincide on cylindrical events, so on $\mathcal{B}$ which they generate. Thus any convergent subsequence weakly converges to this unique probability measure $\mathbb{P}$.

We now suppose that $(\mathbb{P}_n)$ doesn't weakly converge to $\mathbb{P}$. This means that there exists $f \in \mathcal{C}_b(\mathbb{R}^+)$ such that the real bounded sequence $(\mathbb{P}_n(f))$ doesn't converge to $\mathbb{P}(f)$.

Anyway, there exists at least a convergent subsequence $(\mathbb{P}_{n_k}(f))$, with limit which is not $\mathbb{P}(f)$. On the other hand, since the family is tight, a weakly convergent sequence can be extracted from family $\mathbb{P}_{n_k}$, still called $(\mathbb{P}_{n_k})$. But we saw that limit of $(\mathbb{P}_{n_k})$ is necessarily $\mathbb{P}(f)$, thus a contradiction and the proof is concluded. •

1.2.2 Donsker invariance principle and Wiener measure

In this section we prove the theorem building Brownian motion. We study the sequence of processes defined in principal theorem thanks to independent random variables $(\xi_j, j \geq 1)$. We need:

- to prove the convergence of sequence of processes $(X_n, n \geq 0)$,

- to prove the properties of the limit conveniently to the initial definition. Thus the scheme of the proof is:

1) this sequence converges in finite dimensional distribution to a process with Brownian motion properties,

2) this sequence is tight and Theorem 1.17 can be applied.

1)

Proposition 1.18. *(cf 4.17 [20]) Let be:*

$$X_t^n = \frac{1}{\sigma\sqrt{n}}(\sum_{j=1}^{[nt]} \xi_j + (nt - [nt])\xi_{[nt]+1}).$$

Then, $\forall d, \forall (t_1, \cdots, t_d) \in \mathbb{R}^+$, *we get the distribution convergence:*

$$(X^n_{t_1}, \cdots, X^n_{t_d}) \longrightarrow_{\mathcal{D}} (B_{t_1}, \cdots, B_{t_d})$$

where B *satisfies the properties defining the Brownian motion.*

Proof: a first simplification uses:

$$S^n_t = \frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{[nt]} \xi_j \text{ et } S^n_{\underline{t}} = (S^n_{t_1}, \cdots, S^n_{t_d}).$$

Remark:

$$X^n_t = S^n_t + \frac{nt - [nt]}{\sigma\sqrt{n}} \xi_{[nt]+1}.$$

Bienaymé-Tchebichev inequality yields:

$$\mathbb{P}\{\| X^n_{\underline{t}} - S^n_{\underline{t}} \| > \varepsilon\} \leq \frac{d}{n\sigma^2\varepsilon^2} \| \xi \|^2 \rightarrow 0$$

when n goes to infinity. Then it is enough to get the distribution convergence of $(S^n_{\underline{t}})$. *conclude the proof as an Exercise.*

Remark that $(S^n_{\underline{t}}, t \geq 0)$ is an independent increments process; if (t_i) are increasing ordered, the d random variables $(S^n_{t_1}, S^n_{t_2} - S^n_{t_1}, \cdots, S^n_{t_d} - S^n_{t_{d-1}})$ are independent. The application from $\mathbb{R}^d$ to $\mathbb{R}^d$: $x \mapsto (x_1, x_1 + x_2, \cdots, \sum_i x_i)$ is continuous and distribution convergence is maintained by the continuity. Then it is enough to look at the distribution convergence of the d-uple of increments, this is done using characteristic function:

$$\phi^n(u_1, \cdots, u_d) = E[e^{i\sum_j u_j (S^n_{t_j} - S^n_{t_{j-1}})}] = \Pi_j E[e^{\frac{iu_j}{\sigma\sqrt{n}} \sum_{[nt_{j-1}] < k \leq [nt_j]} \xi_k}]. \tag{2}$$

For any j, denoting $k_j = [nt_j]$, each factor is written as:

$$E\left[e^{iu_j\sqrt{\frac{k_j - k_{j-1}}{n}} \frac{\sum_k \xi_k}{\sigma\sqrt{k_j - k_{j-1}}}}\right].$$

But $\frac{k_j - k_{j-1}}{n} = \frac{[nt_j] - [nt_{j-1}]}{n}$ converges to $(t_j - t_{j-1})$ when n goes to infinity and the random variable $\frac{\sum_{[nt_{j-1}] < k \leq [nt_j]} \xi_k}{\sigma\sqrt{k_j - k_{j-1}}}$ converges in distribution to a standard Gaussian law (law of large numbers) thus its characteristic function goes to $e^{-t^2/2}$ and the jth factor goes to $e^{-\frac{1}{2}u_j^2(t_j - t_{j-1})}$. The limit law thus admits the characteristic function $\phi(u) = e^{-\frac{1}{2}\sum_j u_j^2(t_j - t_{j-1})}$ which is exactly this one of the d-uplet $(B_{t_1}, (B_{t_2} - B_{t_1}), \cdots, (B_{t_d} - B_{t_{d-1}}))$ coming from a Brownian motion.
Thus we get both law of limit process and property of independent increments.

2) We have now to prove that the family is tight, which will result of following lemmas:

Lemma 1.19. *(cf. [20], 4.18) Let* $(\xi_j, j \geq 1)$ *be a sequence of random variables, independent, same law, centered, variance* 1, *and let be* $S_j = \sum_{k=1}^j \xi_k$. *Then:*

$$\forall \varepsilon > 0, \lim_{\delta \to 0} \overline{\lim}_{n\to\infty} \frac{1}{\delta} \mathbb{P}\{max_{\{1 \leq j \leq [n\delta] > +1\}} |\xi_j| > \varepsilon\sigma\sqrt{n}\} = 0.$$

Lemma 1.20. *(cf. [20], 4.19) Under same hypotheses,*

$$\forall T > 0, \quad \lim_{\delta \to 0} \overline{\lim}_{n\to\infty} \mathbb{P}\{max_{\{1 \leq j \leq [n\delta] > +1\}} max_{\{1 \leq k \leq [nT] > +1\}} |S_{j+k} - S_j| > \varepsilon\sigma\sqrt{n}\} = 0.$$

Proof of Donsker invariance theorem :
Using Proposition 1.18 and Theorem 1.17, it is enough to show that the family is tight. Here we use the characterization given in Theorem 1.13. In this case $X_0^n = 0\ \forall n$, so it is enough to prove second criteron:

$$\lim_{\delta\to 0}\sup_n \mathbb{P}\{max_{|s-t|\leq\delta,0\leq s,t\leq T}|X_s^n - X_t^n| > \varepsilon\} = 0.$$

. $\overline{\lim}_n = \inf_m \sup_{n\geq m}$ could replace $\sup_n$ since for m bounded we can get empty events taking δ small enough: $(X^n, 0\leq n\leq m)$ is continuous on $[0,T]$ thus uniformly continuous.

$$\{max_{|s-t|\leq\delta,0\leq s,t\leq T}|X_s^n - X_t^n| > \varepsilon\} =$$
$$\{max_{|s-t|\leq\delta,0\leq s,t\leq T}|S_{j_s} - S_{j_t} + \frac{n_s - j_s}{\sigma\sqrt{n}}\xi_{j_s+1} - \frac{n_t - j_t}{\sigma\sqrt{n}}\xi_{j_t+1}| > \varepsilon\sigma\sqrt{n}\},$$

where $j_s = [ns]$, and if we denote $j_s = k$ and $j_t = k + j$, assuming $s\leq t$, this set is included in:

$$\{max_{|s-t|\leq\delta,0\leq s,t\leq T}|S_{j_s} - S_{j_t}| > \varepsilon\sigma\sqrt{n}\}$$

and Lemma 1.20 concludes. •

1.3 Properties of trajectories of Brownian motion

1.3.1 Gaussian process

Definition 1.21. *A process X is said to be* **Gaussian** *if $\forall d, \forall(t_1,\cdots,t_d)$ positive real numbers, the vector $(X_{t_1},\cdots,X_{t_d})$ admits a Gaussian law. If the law $(X_{t+t_i}; i=1,\cdots,d)$ doesn't depend on t, process X is said to be* **stationary**.

We call **covariance** *of vector X the matrix*

$$\rho(s,t) = E[(X_s - E(X_s))(X_t - E(X_t))^T],\ \ s,t\geq 0.$$

Proposition 1.22. *Brownian motion B is a centered continuous Gaussian process with covariance $\rho(s,t) = s\wedge t$.*

Reciprocally, any centered continuous Gaussian process with covariance $\rho(s,t) = s\wedge t$ is a Brownian motion.

Proposition 1.23. *The Brownian motion converges "in mean" to zero: $\frac{B_t}{t}\to 0$ almost surely when t goes to infinity.*

Proof Exercises. This last proposition is more or less a "law of large numbers". •

Other Brownian motions can be obtained by standard transformations, for instance changing the filtration.

(i) change of scaling: $(\frac{1}{\sqrt{c}}B_{ct}, \mathcal{F}_{ct})$.

(ii) inversion of time: $(Y_t, \mathcal{F}_t^Y)$, with $Y_t = tB_{\frac{1}{t}}$ si $t\neq 0$, $Y_0 = 0$ et $\mathcal{F}_t^Y = \sigma\{Y_s, s\leq t\}$.

(iii) reversing time: $(Z_t, \mathcal{F}_t^Z)_{0\leq t\leq T}$, with $Z_t = B_T - B_{T-t}$ et $\mathcal{F}_t^Z = \sigma\{Z_s, s\leq t\}$.

(iv) symmetry: $(-B_t, \mathcal{F}_t)$.

In each case we have to check that it is an adapted continuous process satisfying the characteristic property of Brownian motion or: that it is a centered continuous Gaussian process with covariance $\rho(s,t) = s \wedge t$.
The only difficult case is (ii) (Exercise).

1.3.2 Zeros set

This set is $\mathcal{X} = \{(t,\omega) \in \mathbb{R}^+ \times \Omega : B_t(\omega) = 0\}$. Let fixed a trajectory ω, denote $\mathcal{X}_\omega = \{t \in \mathbb{R}^+ : B_t(\omega) = 0\}$.

Theorem 1.24. *(cf. [20] 9.6, p. 105) $\mathbb{P}$-almost surely with respect to ω*

(i) Lebesgue measure of $\mathcal{X}_\omega$ is null,

(ii) $\mathcal{X}_\omega$ is closed no bounded,

(iii) $t = 0$ is an accumulation point of $\mathcal{X}_\omega$,

(iv) $\mathcal{X}_\omega$ is dense in itself.

Proof too difficult Exercise.... out of the agenda.

1.3.3 Variations of the trajectories

(cf. [20] pb 9.8 p. 106 et 125)
Notation: $\pi_n = (t_0 = 0, \cdots, t_n = t)$ is a "subdivision" of $[0,t]$, denote $\|\pi_n\| = \sup_i\{t_i - t_{i-1}\}$, called the "mesh" of π_n.

Theorem 1.25. *(cf. [30] 28 p. 18)*
Let π_n be a sequence of subdivisions of interval $[0,t]$ such that the mesh of π_n, $\|\pi_n\|$, goes to zero when n goes to infinity. Let be $\pi_n(B) = \sum_{t_i\in\pi_n}(B_{t_{i+1}} - B_{t_i})^2$.
Then, when n goes to infinity, $\pi_n(B)$ goes to t in $L^2(\Omega)$, and almost surely if moreover $\sum_n \| \pi_n \| < \infty$.

Proof : Let be $z_i = (B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)$; $\sum_i z_i = \pi_n(B) - t$. It is a centered independent random variables sequence since $B_{t_{i+1}} - B_{t_i}$ law is Gaussian law with null mean and variance $t_{i+1} - t_i$. Moreover we compute the expectation of z_i^2 :

$$E[z_i^2] = E[(B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)]^2 = E[(B_{t_{i+1}} - B_{t_i})^4 - 2(B_{t_{i+1}} - B_{t_i})^2(t_{i+1} - t_i) + (t_{i+1} - t_i)^2].$$

Knowing the moments of Gaussian law, we get:

$$E[z_i^2] = 2(t_{i+1} - t_i)^2.$$

The independence between the z_i shows that $E[(\sum_i z_i)^2] = \sum_i E[(z_i)^2]$ equal to $2\sum_i (t_{i+1} - t_i)^2 \leq 2\|\pi_n\|.t$, which goes to zero when n goes to infinity. This fact yields $L^2(\Omega)$ convergence (so probability convergence) of $\pi_n(B)$ to t.

If moreover $\sum_n \parallel \pi_n \parallel < \infty$, then $\mathbb{P}\{|\pi_n(B) - t| > \varepsilon\} \leq \frac{1}{\varepsilon^2} 2 \parallel \pi_n \parallel t$. Thus the series $\sum_n \mathbb{P}\{|\pi_n(B) - t| > \varepsilon\}$ converges and Borel-Cantelli lemma proves that

$$\mathbb{P}[\overline{\lim}_n \{|\pi_n(B) - t| > \varepsilon\}] = 0,$$

meaning:

$\mathbb{P}[\cap_n \cup_{m \geq n} \{|\pi_m(B) - t| > \varepsilon\}] = 0, \forall \varepsilon > 0$, almost surely $\cup_n \cap_{m \geq n} \{|\pi_m(B) - t| \leq \varepsilon\} = \Omega$,

this expresses almost sure convergence of $\pi_n(B)$ to t. •

Theorem 1.26. *(cf. [20] 9.9, p.106)*

$$\mathbb{P}\{\omega : t \mapsto B_t(\omega) \text{ is monotoneous on any interval}\} = 0.$$

Proof: let us denote $F = \{\omega :$ there exists an interval where $t \mapsto B_t(\omega)$ is monotonous$\}$. This could be expressed as:

$$F = \cup_{s,t \in Q, 0 \leq s < t} \{\omega : u \mapsto B_u(\omega) \text{ is monotonous on } (s,t)\}.$$

Let s and t be fixed in Q s.t. $0 \leq s < t$; we study the event

$$A = \{\omega : u \mapsto B_u(\omega) \text{ is increasing on} (s,t)\}.$$

Then, $A = \cap_n A_n$ où $A_n = \cap_{i=0}^{n-1} \{\omega : B_{t_{i+1}} - B_{t_i} \geq 0\}$ with $t_i = s + (t-s)\frac{i}{n}$. Using independence of increments, $\mathbb{P}(A_n) = \Pi_i \mathbb{P}\{\Delta_i B \geq 0\} = \frac{1}{2^n}$. For any n $\mathbb{P}(A) \leq \mathbb{P}(A_n)$ thus $\mathbb{P}(A) = 0$ for all s and t proving $\mathbb{P}(F) = 0$. •

Theorem 1.27. *(cf. [20] 9.18, p.110 : Paley-Wiener-Zygmund, 1933)*

$$\mathbb{P}\{\omega : \exists t_0 \; t \mapsto B_t(\omega) \text{ differentiable at point } t_0\} = 0.$$

More specifically, denoting $D^+ f(t) = \overline{\lim}_{h \to 0} \frac{f(t+h) - f(t)}{h}$; $D_+ f(t) = \underline{\lim}_{h \to 0} \frac{f(t+h) - f(t)}{h}$, *there exists an event F of probability measure 1 included in the set:*

$$\{\omega \; : \forall t, \; D^+ B_t(\omega) = +\infty \text{ ou } D_+ B_t(\omega) = -\infty\}.$$

Proof :

Let be ω such that there exists t such that $-\infty < D_+ B_t(\omega) \leq D^+ B_t(\omega) < +\infty$. Then,

$$\exists j, k \text{ such that } \forall h \leq 1/k, |B_{t+h} - B_t| \leq jh.$$

We can find n greater than $4k$ and $i, i = 1, \cdots, n$, such that :

$$\frac{i-1}{n} \leq t \leq \frac{i}{n}, \text{ and if } \nu = 1, 2, 3 : \frac{i+\nu}{n} - t \leq \frac{\nu+1}{n} \leq \frac{1}{k}.$$

These two remarks and triangle inequality $|B_{\frac{i+1}{n}} - B_{\frac{i}{n}}| \leq |B_{\frac{i+1}{n}} - B_t| + |B_t - B_{\frac{i}{n}}|$ induce the upper bound

$$|B_{\frac{i+1}{n}} - B_{\frac{i}{n}}| \leq \frac{3j}{n}.$$

We go on with $\nu = 2$ then 3 :

$$|B_{\frac{i+2}{n}} - B_{\frac{i+1}{n}}| \leq \frac{5j}{n},\ |B_{\frac{i+3}{n}} - B_{\frac{i+2}{n}}| \leq \frac{7j}{n}.$$

Thus the starting ω belongs to an event such that there exists $t \in [0,1]$, such that $\forall n \geq 4k$, $\exists i \in \{1, \cdots, n\}$ such that $t \in [\frac{i-1}{n}, \frac{i}{n}], \nu = 1,2,3 : |B_{\frac{i+\nu}{n}} - B_{\frac{i+\nu-1}{n}}| \leq \frac{(2\nu+1)j}{n}$.
These three events of B are independent; the probability measure of the event

$$\forall \nu = 1,2,3 : |B_{\frac{i+\nu}{n}} - B_{\frac{i+\nu-1}{n}}| \leq \frac{(2\nu+1)j}{n}$$

is bounded by $\frac{j^3.3.5.7}{n^{3/2}}$ and the one of the event

$$\forall n \geq 4k, \exists i = 1, \cdots, n, \nu = 1,2,3 : |B_{\frac{i+\nu}{n}} - B_{\frac{i+\nu-1}{n}}| \leq \frac{(2\nu+1)j}{n}$$

is bounded by $n\frac{j^3.3.5.7}{n^{3/2}}$ $\forall n \geq 4k$, thus goes to zero when k goes to infinity. •

Definition 1.28. *Let f be a function defined on interval $[a,b]$. We call* **variation of f on this interval** *:*

$$Var_{[a,b]}(f) = \sup_{\pi} \sum_{t_i \in \pi} |f(t_{i+1}) - f(t_i)|$$

where π belongs to the subdivisions of $[a,b]$ set.

Theorem 1.29. *(cf. [30] p.19-20 Let a and b be fixed in $\mathbb{R}^+$:*

$$\mathbb{P}\{\omega : Var_{[a,b]}(B) = +\infty\} = 1.$$

Proof :Let a and b be fixed in $\mathbb{R}^+$ and π a subdivision of $[a,b]$.

$$\sum_{t_i \in \pi} |B(t_{i+1}) - B(t_i)| \geq \frac{\sum_{t_i \in \pi} |B(t_{i+1}) - B(t_i)|^2}{sup_{t_i \in \pi}|B(t_{i+1}) - B(t_i)|}. \tag{3}$$

The numerator is the quadratic variation of B, known as converging to t. Then, $s \mapsto B_s(\omega)$ is continuous so uniformly continuous on interval $[a,b]$:

$$\forall \varepsilon, \exists \eta, \parallel \pi \parallel < \eta \Rightarrow sup_{t_i \in \pi}|B(t_{i+1}) - B(t_i)| < \varepsilon.$$

Thus the quotient (3) converges to infinity. •

1.3.4 Lévy Theorem

This theorem gives the magnitude of the modulus of continuity.

Theorem 1.30. *([20] th. 9.25 pp 114-115)*
Let be $g :]0,1] \to \mathbb{R}^+,\ \delta < 1,\ g(\delta) = \sqrt{-2\delta \log(\delta)}$. Then,

$$\mathbb{P}\{\omega : \overline{\lim}_{\delta \searrow 0} \frac{1}{g(\delta)} \sup_{0<s<t<1, t-s \leq \delta} |(B_t - B_s)(\omega)| = 1\} = 1.$$

This means that the magnitude of the modulus of continuity of B is $g(\delta)$.

Theorem 1.31. *(cf. [30] 31 p.22-23) Let be $\mathcal{F}_t = \sigma(B_s, s \leq t) \vee \mathcal{N}$. Then the filtration $\mathcal{F}$ is right continuous, meaning that $\mathcal{F}_{t^+} := \cap_{s>t}\mathcal{F}_s$ coincides with $\mathcal{F}_t$.*

Proof (Exercise) uses the fact that

$$\forall u_1, \forall u_2 \quad , \quad \forall z > v > t,$$
$$E[e^{i(u_1 B_z + u_2 B_v)}/\mathcal{F}_{t^+}] = \lim_{w \searrow t} E[e^{i(u_1 B_z + u_2 B_v)}/\mathcal{F}_w] =$$
$$E[e^{i(u_1 B_z + u_2 B_v)}/\mathcal{F}_t],$$

meaning that the $\mathcal{F}_{t^+}$ and $\mathcal{F}_t$ conditional laws are the same ones, so $\mathcal{F}_{t^+} = \mathcal{F}_t$

1.3.5 Markov and martingale properties

The Brownian motion is a Markov process, meaning that:

$$\forall x \in \mathbb{R},\ \forall f \text{ bounded Borel, } E_x[f(B_{t+s})/\mathcal{F}_s] = E_{B_s}[f(B_t)].$$

The proof is easy, possibly "handmade" : under $\mathbb{P}_x$, $B_{t+s} = x + W_{t+s}$ and

$$f(B_{t+s}) = f(x + W_{t+s} - W_s + W_s),$$

we conclude using independence of $x + W_s$ and $W_{t+s} - W_s$.
As a corollary, we get that B is a martingale for its own filtration.

1.4 Computation of $2\int_0^t B_s dB_s$ (Exercise)

The trajectories of B aren't differentiable, anyway we look for a meaning to this integral. The intuition could say that it is B_t^2, but it is not. To stress the difference between both, we decompose B_t^2 as a sum of differences along a subdivision of interval $[0, t]$, denoted as $t_i = it/n$, then developed using Taylor formula:

$$B_t^2 = \sum_i (B_{t_{i+1}}^2 - B_{t_i}^2) = \sum_i 2B_{t_i}[B_{t_{i+1}} - B_{t_i}] + \sum_i [B_{t_{i+1}} - B_{t_i}]^2.$$

The first term "naturally" converges to the expected formula: $2\int_0^t B_s dB_s$ (we will justify this convergence in Chapter 2). We could think that the second term converges to 0, here is the paradox. We have to remark that, by definition of Brownian motion, this second term is sum of the squared of n centered independent Gaussian variables with variance t/n; thus it is a random variable with $\frac{t}{n}\chi_n^2$ law. Its expectation is t and its variance is $t^2/nVar\chi_1^2$: thus this term L^2-converges (thus probability convergence) to its expectation t. Later, we will more specifically prove

$$B_t^2 = 2\int_0^t B_s dB_s + t.$$

2 Stochastic integral

The main purpose of this chapter is to give meaning to notion of integral of some processes with respect to Brownian motion or, more generally, with respect to a martingale. Guided by the "pretext" of this course (stochastic calculus applied to Finance), we can motivate the stochastic integral as following: for a moment study a model where the price of a share would be given by a martingale M_t at time t. If we have $X(t)$ of such shares at time t and if we conduct transactions at times t_k wealth is finally increased:

$$\sum_k X(t_{k-1})(M_{t_k} - M_{t_{k-1}}).$$

But if we want to trade in continuous time, at any time t we must be able to define a mathematical tool to move to limit in the above expression with the problem, especially if $M = B$, the derivative B' doesn't exist! this expression is a sum which is intended to converge to a Stieljes integral, but since the variation $V(B)$ is infinite, this can not converge in a " deterministic" sense: the stochastic integral "naive" is impossible (cf. Protter page 40) as the following result shows it.

Theorem 2.1. *Let $\pi = (t_k)$ be a subdivision of $[0, T]$. If $\lim_{\|\pi\| \to 0} \sum_k x(t_{k-1})(f(t_k) - f(t_{k-1}))$ exists, then f is finite variation. (cf. Protter, th. 52, page 40)*

The proof uses Banach Steinhaus theorem, id est: if X is a Banach space and Y normed vector space, (T_α) a sequence of bounded operators from X to Y such that $\forall x \in X, (T_\alpha(x))$ is bounded, then the sequence $(\| T_\alpha \|)$ is bounded in $\mathbb{R}$.

Reciprocally, we get: $V(f) = +\infty$ yields the limit doesn't exist, this the case if $f : t \mapsto B_t$ is Brownian motion.
We thus must find other tools. The idea of Itô was to restrict integrands to be processes that can not "see" the increments in the future, that is adapted processes, so that, at least for the Brownian motion, $x(t_{k-1})$ and $(B_{t_k} - B_{t_{k-1}})$ are independent, so trajectory by trajectory nothing can be done. But we will work in probability, in expectation.

The plan is as follows: after introducing the problem and some notations (2.1.1), we first define (2.1.2) the integral on the "simple processes" ($\mathcal{S}$ denotes the set of simple processes, which will be defined below). Then 2.1.3 will give the properties of this integral over $\mathcal{S}$ thereby operator extended by continuity on the closure of $\mathcal{S}$ for a well chosen topology, so to have a reasonable amount of integrands.

2.1 Stochastic integral

2.1.1 Introduction and notations

Let M be a square integrable continuous martingale on the filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ where $\mathcal{F}_t$ is for instance the natural filtration generated by the Brownian motion, completed by negligible events. For any measurable process X, $\forall n \in \mathbb{N}$ and $\forall t \in \mathbb{R}^+$ let us define:

$$I_n(X,t) = \sum_j X(\frac{j-1}{2^n} \wedge t)(M_{\frac{j}{2^n}\wedge t} - M_{\frac{j-1}{2^n}\wedge t}).$$

This quantity doesn't necessarily have a limit. We have to restrict to a class of almost surely square integrable (with respect to increasing process $\langle M \rangle$ defined below), adapted, measurable processes X.

Definition 2.2. *The increasing process $\langle M \rangle$ is defined as:*

$$t \mapsto \langle M \rangle_t = \lim_{\|\pi\| \to 0} \textit{probability} \sum_{t_i \in \pi} (M_{t_i} - M_{t_{i-1}})^2$$

where π describe the subdivisions of $[0,t]$. It is named "bracket".

The construction of $I(X,t)$ is due to Ito (1942) in case of M Brownian motion, and Kunita and Watanabe (1967) for square integrable martingales. An exercise in Chapter 1 with $M = B$ proves $\langle B \rangle_t = t$.

Remark 2.3. *The square integrable continuous martingales admit a bracket.*

Recall:

Proposition 2.4. *$\langle M \rangle_t$ is the unique adapted increasing continuous process such that $M_t^2 - \langle M \rangle_t$ is a martingale.*

Very often, this proposition is bracket definition, and then Definition 0.26 is a consequence.
Notation: let us define a measure on σ-algebra $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$ as

$$\mu_M(A) = E[\int_0^\infty 1_A(t,\omega) d\langle M \rangle_t(\omega)].$$

X and Y are said to be equivalent if $X = Y \mu_M$ *a.s.*
Notation: for any adapted process X, we note $[X]_T^2 = E[\int_0^T X_t^2 d\langle M \rangle_t]$.

Remark that X et Y are equivalent if and only if $[X - Y]_T^2 = 0 \ \forall T \geq 0$.

Let us introduce the following set of processes:

(4) $\quad \mathcal{L}(M) = \{$ classes of measurable $\mathcal{F}$-adapted processes $X s.t.\ \forall T \ [X]_T < +\infty\}$

endowed with the metric:

$$d(X,Y) = \sum_{n \geq 1} \frac{1 \wedge [X-Y]_n}{2^n}, \tag{5}$$

then the subset of the previous:

$$\mathcal{L}^*(M) = \{X \in \mathcal{L} \text{ progressively measurable}\}.$$

When the martingale M is such that $\langle M \rangle$ is absolutely continuous with respect to Lebesgue measure, since any element of $\mathcal{L}$ admits a modification in $\mathcal{L}^*(M)$, in such a case, we manage in $\mathcal{L}$, but generally, we will restrict to $\mathcal{L}^*(M)$.

Proposition 2.5. *Let $\mathcal{L}_T(M)$ be the set of adapted measurable processes X on $[0,T]$ such that:*

$$[X]_T^2 = E[\int_0^T X_s^2 d\langle M\rangle_s] < +\infty.$$

$\mathcal{L}_T^*(M)$, set of progressively measurable processes of $\mathcal{L}_T(M)$, is closed in $\mathcal{L}_T(M)$. In particular, it is complete for the norm $[.]_T$.

Proof: Let (X^n) be a sequence in $\mathcal{L}_T^*(M)$, converging to X: $[X - X^n]_T \to 0$. It is a sequence in L^2 space, thus complete and $X \in \mathcal{L}_T(M)$, convergence L^2 yields the existence of an almost surely convergent subsequence. Let Y be the almost sure limit on $\Omega\times[0,T]$, meaning that $A = \{(\omega,t) : \lim_n X_t^n(\omega,t) \text{ exists }\}$ has probability equal to 1 and $Y(\omega,t) = X(\omega,t)$ if $(\omega,t) \in A$, and if not is equal to 0. The fact that $\forall n,\ X^n \in \mathcal{L}_T^*(M)$ shows that $Y \in \mathcal{L}_T^*(M)$ and Y is equivalent to X. •

2.1.2 Integral of simple processes and extension

Definition 2.6. *Process X is said to be* **simple** *if there exists a sequence of real numbers (t_i) increasing to infinity and a bounded family (ξ_i) of $\mathcal{F}_{t_i}-$measurable random variables such that:*

$$X_t = \xi_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=1}^{\infty} \xi_i \mathbf{1}_{]t_i t_{i+1}]}(t).$$

Denote $\mathcal{S}$ their set, note the inclusions $\mathcal{S} \subset \mathcal{L}^*(M) \subset \mathcal{L}$. (to check as Exercise)
Exercise: compute $[X]_T^2$ when $X \in \mathcal{S}$.

Definition 2.7. *Let be $X \in \mathcal{S}$. The stochastic integral of X with respect to M is*

$$I_t(X) = \sum_{i=1}^{\infty} \xi_i (M_{t\wedge t_{i+1}} - M_{t\wedge t_i}).$$

Notation: $(\mathbf{X}.\mathbf{M})_\mathbf{t} := \mathbf{I}_\mathbf{t}(\mathbf{X})$ *to express it is a stochastic integral w.r.t. martingale M.*

We have now to extend this definition to a larger class of integrands, at least in case of M is Brownian motion, meaning $\langle M\rangle_t = t$.

Lemma 2.8. *For any* **bounded** *process $X \in \mathcal{L}(B)$ there exists a sequence of processes $X_n \in \mathcal{S}$ such that* $\sup_{T\geq 0} \lim_n E[\int_0^T (X_n - X)^2 dt] = 0$.

Proof

(a) Case when X is continuous: set $X_t^n = X_{\frac{j-1}{2^n}}$ on the interval $]\frac{j-1}{2^n} \frac{j}{2^n}]$. By continuity, obviously $X_t^n \to X_t$ almost surely. Moreover by hypothesis X is bounded; dominated convergence theorem allows to conclude.

(b) Case when $X \in \mathcal{L}^*(M)$: set $X_t^m = m\int_{(t-1/m)^+}^t X_s ds$, this one is continuous and stay measurable adapted bounded in $\mathcal{L}$. Using step (a) $\forall m$, there exists a sequence $X^{m,n}$ of simple processes converging to X^m in $L^2([0,T]\times\Omega, d\mathbb{P}\times dt)$ meaning that:

$$\forall m\ \forall T\ \lim_{n\to\infty} E[\int_0^T (X_t^{m,n} - X_t^m)^2 dt] = 0. \tag{6}$$

Let be $A = \{(t,\omega)\in[0,T]\times\Omega\ :\ \lim_{m\to\infty} X_t^m(\omega) = X_t(\omega)\}^c$ and its $\omega-$section A_ω, $\forall\omega$. Since X is progressively measurable, $A_\omega \in \mathcal{B}([0,T])$. Using **Lebesgue fundamental theorem** (cf. for instance STEIN: "Singular Integrals and Differentiability Properties of Functions") X is integrable yields:

$$X_t^m - X_t = m\int_{(t-1/m)^+}^t (X_s - X_t)ds \to 0$$

for almost any t and Lebesgue measure of A_ω is null. On another hand, X and X^m are uniformly bounded; bounded convergence theorem in $[0,T]$ proves that
$\forall\omega\ \int_0^T (X_s - X_x^m)^2 ds \to 0$.

Once again we apply bounded convergence theorem but in Ω so that $E[\int_0^T (X_s - X_x^m)^2 ds] \to 0$. This fact added to (6) concludes (b).

(c) Case when X is **bounded adapted measurable**: we go to case (b) recalling that any adapted measurable process admits a progressively measurable modification, named Y. Then there exists a sequence (Y^n) of simple processes converging to Y in $L^2([0,T]\times\Omega, d\mathbb{P}\times dt)$:

$$E[\int_0^T (Y_s - Y_s^m)^2 ds] \to 0 \text{ et } \forall t\ \mathbb{P}(X_t = Y_t) = 1.$$

Set $\eta_t = \mathbf{1}_{\{X_t \neq Y_t\}}$. Using Fubini theorem we get:

$$E[\int_0^T \eta_t dt] = \int_0^T \mathbb{P}(X_t \neq Y_t)dt = 0$$

thus $\int_0^T \eta_t dt = 0$ almost surely.

$$\eta_t + \mathbf{1}_{\{X_t=Y_t\}} = 1 \Rightarrow E[\int_0^T \mathbf{1}_{\{X_t=Y_t\}} dt] = T \text{ and } \mathbf{1}_{\{X_t=Y_t\}} = 1\ dt\times d\mathbb{P} \text{ almost surely}$$

Finally:

$$E[\int_0^T (Y_s - Y_s^m)^2 ds] = E[\int_0^T \mathbf{1}_{\{X_s=Y_s\}}(Y_s - Y_s^m)^2 ds] = E[\int_0^T (X_s - Y_s^m)^2 ds]$$

which gives the conclusion. •

Proposition 2.9. *If the increasing process $t \mapsto \langle M\rangle_t$ is $\mathbb{P}$-almost surely absolutely continuous with respect to Lebesgue measure dt, then the set $\mathcal{S}$ is dense in the metric space $(\mathcal{L}, d)$ with metric d defined in (5).*

Proof

(i) Let be $X \in \mathcal{L}$ and bounded: the previous lemma proves the existence of a sequence of simple processes (X^n) converging to X in $L^2(\Omega\times[0,T], d\mathbb{P}\otimes dt), \forall T$. Thus there exists an almost surely converging subsequence. Bounded convergence theorem and $d\langle M\rangle_t = f(t)dt$ get the conclusion.

(ii) Let be $X \in \mathcal{L}$ no bounded: set $X_t^n(\omega) = X_t(\omega)\mathbf{1}_{\{|X_t(\omega)|\leq n\}}$. The distance

$$d(X^n, X) = E[\int_0^T X_s^2 \mathbf{1}_{\{|X_t(\omega)|\geq n\}} d\langle M\rangle_s] \to 0$$

since the integrand converges almost surely to 0, is bounded by X^2 which is integrable (bounded convergence theorem). But $\forall n$ $X^n \in \mathcal{L}$ and are bounded: their set is dense in $\mathcal{L}$.

(iii) The set of simple processes is dense in the subset of bounded processes of $\mathcal{L}$; (i) and (ii) yields the conclusion. •

This proposition therefore provides the density of simple processes set in $\mathcal{L}$ in the case of increasing process $\langle M\rangle_t$ is absolutely continuous with respect to dt. If not, there exists the density of simple processes only in $\mathcal{L}^*(M)$ with the following proposition.

Proposition 2.10. *$\mathcal{S}$ is dense in the metric space $(\mathcal{L}^*(M), d)$ with metric d defined in (5).*

Proof: Cf. Proposition 2.8 and Lemma 2.7. in [20], pages 135-137.

Remark 2.11. *useful: the metric d defined in (5) induces the following equivalent topology* $\lim_{n\to\infty} d(X_n, X) = 0$ *if and only if*

$$\forall T > 0, \lim_{n\to\infty} E[\int_0^T |X_n(t) - X(t)|^2 d\langle M\rangle_t] = 0.$$

2.1.3 Construction of the stochastic integral, elementary properties

Remember the stochastic integral of a simple process X:

$$I_t(X) = \sum_{i=1}^{\infty} \xi_i(M_{t\wedge t_{i+1}} - M_{t\wedge t_i}).$$

Let us denote $I_t(X) = \int_0^t X_s dM_s$ or $(X.M)_t$ in case of integrator M. This simple stochastic integral admits the following properties (Exercise):

Exercise. Let $\mathcal{S}$ be the set of simple processes on which the stochastic integral with respect to M is defined:

$$I_t(X) = \sum_j \xi_j(M_{t_{j+1}\wedge t} - M_{t_j\wedge t}).$$

Prove that I_t satisfies the following properties

(i) I_t is a linear application.

(ii) $I_t(X)$ is square integrable.

(iii) Expectation of $I_t(X)$ is null.

(iv) $t \mapsto I_t(X)$ is a continuous martingale.

(v) $E[I_t(X)]^2 = E[\int_0^t X_s^2 d\langle M\rangle_s]$.

(vi) $E[(I_t(X) - I_s(X))^2/\mathcal{F}_s] = E[\int_s^t X_u^2 d\langle M\rangle_u/\mathcal{F}_s]$.

(vii) $\langle I_.(X)\rangle_t = \int_0^t X_s^2 d\langle M\rangle_s$.

Remark that (v) proves that I_t is an isometry.

We now extend the set of integrands over simple processes thanks to above density results then we check that this new operator satisfies the same properties.

Proposition 2.12. *Let be $X \in \mathcal{L}^*(M)$ and a sequence of simple processes (X^n) converging to X. Then the sequence $(I_t(X^n))$ is a Cauchy sequence in $L^2(\Omega)$. The limit doesn't depend of the chosen sequence so it defines the stochastic integral of X with respect to the martingale M, denoted as $I_t(X)$ or $\int_0^t X_s dM_s$ or $(X.M)_t$.*

Proof: using property (v) above we compute the norm L^2 of $I_t(X^n)$:

$$\| I_t(X^n) - I_t(X^p) \|_2^2 = E[\int_0^t |X_s^n - X_s^p|^2 d\langle M\rangle_s] \to 0$$

$\forall t > 0$ since $d(X^n, X^p) \to 0$. Clearly the same kind of argument proves that changing sequence approaching X does not change this limit:

$$\| I_t(X^n) - I_t(Y^n) \|_2 \to 0$$

along with $d(X^n, Y^n) \leq d(X^n, X) + d(X, Y^n)$. •

We now prove the properties:

Proposition 2.13. *let be $X \in \mathcal{L}^*(M)$, then:*
i) I_t is a linear application.
(ii) $I_t(X)$ is square integrable.
(iii) Expectation of $I_t(X)$ is null.
(iv) $t \mapsto I_t(X)$ is a continuous martingale.
(v) $E[I_t(X)]^2 = E[\int_0^t X_s^2 d\langle M\rangle_s]$.
(vi) $E[(I_t(X) - I_s(X))^2/\mathcal{F}_s] = E[\int_s^t X_u^2 d\langle M\rangle_u/\mathcal{F}_s]$.

(vii) $\langle I_.(X)\rangle_t = \int_0^t X_s^2 d\langle M\rangle_s$.
(vi') $E[(I_t(X))^2/\mathcal{F}_s] = I_s^2(X) + E[\int_s^t X_u^2 d\langle M\rangle_u/\mathcal{F}_s]$.
(vii) $\langle I_.(X)\rangle_t = \int_0^t X_s^2 d\langle M\rangle_s$.

CONCLUSION: $X \in \mathcal{L}^*(M) \Rightarrow X.M$ is a square integrable martingale.

Proof: most of these properties are obtained passing to the L^2 limit of properties satisfied by $I_t(X^n)$ $\forall n$, for instance (i) (ii) (iii) (iv); (concerning (iv) note that the set of square integrable continuous martingales is complete in L^2).

(v) is a consequence of (vi) with $s = 0$.

(vi) Set $s < t$ and $A \in \mathcal{F}_s$, and compute:

$$E[\mathbf{1}_A(I_t(X) - I_s(X))^2] = \lim_n E[\mathbf{1}_A(I_t(X^n) - I_s(X^n))^2] =$$

$$\lim_n E[\mathbf{1}_A \int_s^t (X_u^n)^2 d\langle M\rangle_u] = E[\mathbf{1}_A \int_s^t X_u^2 d\langle M\rangle_u]$$

since $d(X^n, 0) \to d(X, 0)$.
(vii) is a consequence of (vi') and second characterization of bracket (0.28). •

Proposition 2.14. *For any stopping times S and T, $S \le T$, satisfying Doob Theorem hypotheses, we get:*

$$E[I_{t\wedge T}(X)/\mathcal{F}_S] = I_{t\wedge S}(X).$$

If X and $Y \in \mathcal{L}^*$, almost surely,

$$E[(I_{t\wedge T}(X) - I_{t\wedge S}(X))(I_{t\wedge T}(Y) - I_{t\wedge S}(Y))/\mathcal{F}_S] = E[\int_{t\wedge S}^{t\wedge T} X_u Y_u d\langle M\rangle_u/\mathcal{F}_S].$$

Proof: $t \mapsto I_t(X)$ is a martingale, we apply Doob Corollary 0.25 concerning the two bounded stopping times $t \wedge S$ and $t \wedge T$: so $E[I_{t\wedge T}(X)/\mathcal{F}_S] = I_{t\wedge S}(X)$.

Let be $t \ge 0$ and the bracket of $I_.(X)$ is $\int_0^t X_s^2 d\langle M\rangle_s$, so $I_t(X)^2 - \int_0^t X_s^2 d\langle M\rangle_s$ is a martingale; once again we apply Doob theorem concerning the stopping between two bounded stopping times $S \wedge t$ et $T \wedge t$, meaning

$$E[I_{T\wedge t}(X)^2 - I_{S\wedge t}(X)^2/\mathcal{F}_{S\wedge t}] = E[\int_{S\wedge t}^{T\wedge t} X_u^2 d\langle M\rangle_u/\mathcal{F}_{S\wedge t}].$$

This implies the second point using Corollary 0.25 once again, finally we conclude using polarization argument. •

2.2 Quadratic co-variation

(cf. [20], pages 141-145 ; [30], pages 58-60) Similarly the definition of $\langle M\rangle_t$ as probability limit of quadratic increments sums of M, the quadratic co-variation of two square integrable continuous martingales M and N, if π are subdivisions of $[0, t]$, is defined as

$$\langle M, N\rangle_t = \lim_{|\pi|\to 0} \text{proba} \sum_{t_i\in\pi} (M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}),$$

or equivalently

$$4\langle M,N\rangle_t := \langle M+N\rangle_t - \langle M-N\rangle_t.$$

Example : if B is a vector Brownian motion, then $\langle B^i, B^j\rangle_t = t$ if $i=j$ and $=0$ if $i\neq j$.

So, in case of X and $Y \in \mathcal{L}^*(M)$, we now can study the "bracket" $\langle I(X), I(Y)\rangle$. But previously we recall some useful results on the brackets of square integrable continuous martingales.

Proposition 2.15. *Let M and N be two square integrable continuous martingales, then:*

(i) $|\langle M,N\rangle_t|^2 \le \langle M\rangle_t\langle N\rangle_t$ *;*

(ii) $M_tN_t - \langle M,N\rangle_t$ *is a martingale.*

Proof: (i) is proved as any Cauchy inequality. Since $M+N$ is a square integrable continuous martingale, the difference $(M+N)^2 - \langle M+N\rangle_t$ is a martingale and (ii) is a consequence. •

Proposition 2.16. *Let T be a stopping time, M and N be two square integrable continuous martingales. Then:* $\langle M^T, N\rangle = \langle M, N^T\rangle = \langle M,N\rangle^T$.

Proof: cf. Protter [30] th.25, page 61.

Let π be a subdivision of $[0,t]$.

$$\langle M^T, N\rangle_t = \lim_{|\pi|\to 0}\sum_i (M^T_{t_{i+1}} - M^T_{t_i})(N_{t_{i+1}} - N_{t_i}).$$

The family $(t_i \wedge T)$ is a subdivision of $[0, t\wedge T]$.

$$\langle M,N\rangle_{t\wedge T} = \lim_{|\pi|\to 0}\sum_i (M_{T\wedge t_{i+1}} - M_{T\wedge t_i})(N_{T\wedge t_{i+1}} - N_{T\wedge t_i}).$$

The difference between these two sums is null on the event $\{T\ge t\}$ and on the complement $\{T<t\}$, it is

$$(M_T - M_{t_i})(N_{t\wedge t_{i+1}} - N_{T\wedge t_{i+1}}),$$

the index i being such that $T\in[t_i, t_{t_{i+1}}]$. All these processes are continuous, so the limit is almost surely null, thus too in probability. •

Theorem 2.17. *(Kunita-Watanabe inequality) Let M and N be two square integrable continuous martingales, $X \in \mathcal{L}^*(M)$ et $Y\in\mathcal{L}^*(N)$. Then almost surely:*

$$(7)\qquad (\int_0^t |X_sY_s|d\langle M,N\rangle_s)^2 \le \int_0^t |X_s|^2 d\langle M\rangle_s \int_0^t |Y_s|^2 d\langle N\rangle_s.$$

Proof:

(i) first remark the almost sure inequality:

$$\langle M,N\rangle_t - \langle M,N\rangle_s \le \frac{1}{2}(\int_s^t d\langle M\rangle_u + \int_s^t d\langle N\rangle_u)$$

consequence of inequality :

$$2\sum_i (M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}) \leq \sum_i (M_{t_{i+1}} - M_{t_i})^2 + \sum_i (N_{t_{i+1}} - N_{t_i})^2$$

where we pass to probability limit thus almost sure for a subsequence.

Let A be the increasing process $\langle M \rangle + \langle N \rangle$. All the finite variation processes $\langle M \rangle, \langle N \rangle, \langle M, N \rangle$ are absolutely continuous with respect to A. Thus it could be set

$$d\langle M, N \rangle_t = f(t)dA_t,\ d\langle M \rangle_t = g(t)dA_t,\ d\langle N \rangle_t = h(t)dA_t.$$

(ii) For any a and b:

$$\int_0^t (aX_s\sqrt{g(s)} + bY_s\sqrt{h(s)})^2 ds \geq 0.$$

Using classic method in case of Cauchy inequalities, yields:

$$(\int_0^t |X_s Y_s|\sqrt{g(s)h(s)}ds)^2 \leq \int_0^t |X_s|^2 d\langle M \rangle_s \int_0^t |Y_s|^2 d\langle N \rangle_s. \tag{8}$$

(iii) For any a the process $\langle aX.M + Y.N \rangle$ is increasing, so:

$$\int_s^t (a^2 g(u) + 2af(u) + h(u))dA_u \geq 0, \forall s \leq t.$$

Since A is increasing, this implies that the integrand is positive: $a^2 g(s) + 2af(s) + h(s) \geq 0\ \forall a \in \mathbb{R}$, meaning $f(s) \leq \sqrt{g(s)h(s)}$.
This and (8) go to the conclusion. •

Proposition 2.18. *Let M and N be two square integrable continuous martingales, $X \in \mathcal{L}^*(M)$ and $Y \in \mathcal{L}^*(N)$. Then:*

$$\langle X.M, Y.N \rangle_t = \int_0^t X_u Y_u d\langle M, N \rangle_u,\ \forall t \in \mathbb{R},\ \mathbb{P}\ a.s. \tag{9}$$

and

$$E[\int_s^t X_u dM_u \int_s^t Y_u dN_u / \mathcal{F}_s] = E[\int_s^t X_u Y_u d\langle M, N \rangle_u / \mathcal{F}_s],\ \forall s \leq t,\ \mathbb{P}\ a.s. \tag{10}$$

Proof: needs some preliminary lemmas

Lemma 2.19. *Let M and N be two square integrable continuous martingales, and $\forall n$ $X^n, X \in \mathcal{L}^*(M)$ such that $\forall t$:*

$$\lim_n \int_0^t |X_u^n - X_u|^2 d\langle M \rangle_u = 0,\ \mathbb{P}\ a.s.$$

Then:

$$\langle I^M(X^n), N \rangle_t \to_{n\to\infty} \langle I^M(X), N \rangle_t,\ \mathbb{P}\ a.s.$$

Proof: We look for evaluating Cauchy rest.

$$|\langle I^M(X^n), N\rangle_t - \langle I^M(X^p), N\rangle_t|^2 = |\langle I^M(X^n - X^p), N\rangle_t|^2$$
$$\leq \langle I^M(X^n - X^p)\rangle_t \langle N\rangle_t = \int_0^t |X_u^n - X_u^p|^2 d\langle M\rangle_u \langle N\rangle_t$$

the inequality coming from Cauchy-Schwartz inequality concerning brackets (cf. Proposition 2.15 (i)). Thus the convergence is an immediate consequence of the hypothesis. •

Lemma 2.20. *Let M and N be two square integrable continuous martingales and $X \in \mathcal{L}^*(M)$. Then for almost any t:*

$$\langle I^M(X), N\rangle_t = \int_0^t X_u d\langle M, N\rangle_u \mathbb{P}\ a.s.$$

Proof: let (X^n) be a sequence of simple processes going to X :

$$\lim_n E[\int_0^\infty |X_u^n - X_u|^2 d\langle M\rangle_u = 0.$$

Let t be fixed, and a subsequence, converging $\mathbb{P}\ a.s.$: $\int_0^t |X_u^n - X_u|^2 d\langle M\rangle_u \to 0$. Lemma 2.19 proves:

$$\langle I^M(X^n), N\rangle_t \to \langle I^M(X), N\rangle_t \mathbb{P}\ a.s. \tag{11}$$

For simple processes:

$$\langle I^M(X^n), N\rangle_t = \sum_i X_{t_i}^n \sum_{s_k \in [t_i t_{i+1}]} (M_{s_{k+1}} - M_{s_k})(N_{s_{k+1}} - N_{s_k})$$

which goes to $\int_0^t X_u^n d\langle M, N\rangle_u$ when $\sup_k |s_{k+1} - s_k| \to 0$. Finally

$$|\int_0^t X_u^n d\langle M, N\rangle_u - \int_0^t X_u d\langle M, N\rangle_u|^2 = \tag{12}$$
$$|\int_0^t (X_u^n - X_u) d\langle M, N\rangle_u|^2 \leq \int_0^t |X_u^n - X_u|^2 d\langle M\rangle_u \langle N\rangle_t$$

using Kunita-Watanabé inequality (7), then we take almost sure right limit by construction of X^n. Then (12) goes to zero; this limit and the previous (11) prove the result. •

Proof of Proposition 2.18

(i) Set $N_1 = Y.N$, Lemma 2.20 yields:

$$\langle X.M, N_1\rangle_t = \int_0^t X_u d\langle M, N_1\rangle_u \text{ and } \langle M, Y.N\rangle_t = \int_0^t Y_u d\langle M, N\rangle_u$$

We compose finite variation integrals to conclude.

(ii) The property is true for any simple process; then take the probability limit. *Exercise*.

Proposition 2.21. *Let M be a square integrable continuous martingale and $X \in \mathcal{L}^*(M)$. Then $X.M$ is the unique square integrable continuous martingale Φ null at $t = 0$ such that, for any square integrable continuous martingale N:*

$$\langle \Phi, N \rangle_t = \int_0^t X_u d\langle M, N \rangle_u \mathbb{P} \ a.s.$$

Proof: actually $X.M$ satisfies this relation according to Lemma 2.20. Then let Φ satisfying hypotheses of the proposition; for any square integrable continuous martingale N:

$$\langle \Phi - X.M, N \rangle_t = 0, \ \mathbb{P} \ a.s.$$

As a particular case, if we choose $N = \Phi - X.M$, we get $\langle N \rangle_t = 0 \ \mathbb{P} \ a.s.$ that is

$$\Phi - X.M = 0, \ \mathbb{P} \ a.s.$$

•

Corollary 2.22. *Let M and N be two square integrable continuous martingales, $X \in \mathcal{L}^*(M)$, $Y \in \mathcal{L}^*(N)$, T a stopping time such that $\mathbb{P}$ a.s. :*

$$X_{t\wedge T} = Y_{t\wedge T} \ et \ M_{t\wedge T} = N_{t\wedge T}.$$

Then:

$$(X.M)_{t\wedge T} = (Y.N)_{t\wedge T}.$$

Proof: let H be a square integrable continuous martingale; using Proposition 2.16:

$$\langle M - N, H \rangle^T = \langle M^T - N^T, H \rangle = 0, \ \mathbb{P} \ a.s.$$

On one hand:

$$\forall H, \langle X.M - Y.N, H \rangle_{t\wedge T} = \int_0^{t\wedge T} X_u d\langle M, H \rangle_u - \int_0^{t\wedge T} Y_u d\langle N, H \rangle_u,$$

on the other hand hypothesis $X_{t\wedge T} = Y_{t\wedge T}$, Proposition 2.16 and Lemma 2.20 imply:

$$\langle (X.M)^T, H \rangle = \langle X.M, H \rangle^T = \int_0^{t\wedge T} X_u d\langle M, H \rangle_u = \int_0^{t\wedge T} Y_u d\langle N, H \rangle_u$$

Thus we can deduce with 2.21

$$\langle X.M - Y.N, H \rangle^T = 0, \ \mathbb{P} \ a.s. \tag{13}$$

So $(X.M - Y.N)^T$ is a martingale, orthogonal to any square integrable continuous martingale, and in particular to itself, so it is null. •

Proposition 2.23. *The stochastic integral has associative property: if $H \in \mathcal{L}^*(M)$ and $G \in \mathcal{L}^*(H.M)$, then $GH \in \mathcal{L}^*(M)$ and:*

$$G.(H.M) = GH.M$$

Proof: Exercise, cf. Protter th. 19 page 55 or K.S. corollary 2.20, page 145. •

2.3 Integration with respect to local martingales

Corollary 2.22 allows the extension of integrators set and integrands set. In this subsection, M is a continuous local martingale.

Definition 2.24. *Let $\mathcal{P}^*(M)$ be the set of progressively measurable processes such that*

$$\forall t, \ \int_0^t X_s^2 d\langle M\rangle_s < \infty, \ \mathbb{P} \ a.s.$$

Definition 2.25. *Let be $X \in \mathcal{P}^*(M)$ and M a local martingale, with sequence of stopping times S_n. Let be $R_n(\omega) = \inf\{t/ \int_0^t X_s^2 d\langle M\rangle_s \geq n\}$ and $T_n = R_n \wedge S_n$ We now define the stochastic integral of X with respect to M :*

$$X.M = X^{T_n}.M^{T_n} \ on \ \{t \leq T_n(\omega)\}.$$

Proposition 2.26. *This is a "robust" definition since if $n \leq m$, $X^{T_n}.M^{T_n} = X^{T_m}.M^{T_m}$ on $\{t \leq T_m(\omega)\}$ and the process $X.M$ so defined is a local martingale.*

Proof: Corollary 2.22 says that if $t \leq T_m$

$$(X^{T_m}.M^{T_m})_t^{T_n} = (X^{T_m\wedge T_n}.M^{T_m\wedge T_n})_t = (X^{T_m}.M^{T_m})_t.$$

Moreover thanks to this corollary, this definition doesn't depend on the chosen sequence.
Finally by construction, $\forall n, \ (X.M)^{T_n}$ is a martingale, and this exactly means that $X.M$ is a local martingale. •

This stochastic integral doesn't keep all the previous "good" properties. For instance the ones concerning expectations are lost (generally $X.M$ is not integrable), as are the ones concerning conditional expectations. But we have:

Proposition 2.27. *Let M be a continuous local martingale and $X \in \mathcal{P}(M)$. Then $X.M$ is the unique local martingale Φ such that for any square integrable continuous martingale N:*

$$\langle \Phi, N\rangle_t = \int_0^t X_u d\langle M, N\rangle_u.$$

Proof: this is the "local" version of Proposition 2.21. On the event $\{t \leq T_n\}, X.M = X^{T_n}.M^{T_n}$ and satisfies $\forall t$, $\forall n$ and any martingale N,

$$\langle X^{T_n}.M^{T_n}, N\rangle_t = \int_0^t X_{T_n\wedge s} d\langle M^{T_n}, N\rangle_s$$

meaning $\int_0^{T_n\wedge t} X_u d\langle M, N\rangle_u$ which converges almost surely to $\int_0^t X_u d\langle M, N\rangle_u$ when n goes to infinity.

Reciprocally, for any martingale N we get the almost sure equality $\langle \Phi - X.M, N\rangle_t = 0$, particularly for $N = (\Phi - X.M)^{T_n}$. Thus for any localising sequence (T_n), the martingale $(\Phi - X.M)^{T_n}$ bracket is null; so $(\Phi - X.M)^{T_n} = 0$ and almost surely $\Phi = X.M$.
We implicitly used $X^T.M = (X.M)^T$ and the result concerning brackets 2.16. •

3 Itô formula

(cf. [20], pages 149-156, [30], pages 70-83)

This tool allows integro differential calculus, usually called “Itô calculus”, calculus on trajectories of processes, thus the knowledge of what happens to a realization $\omega \in \Omega$.

First recall the standard integration with respect to finite variation processes.

Definition 3.1. *Let A be a continuous process. It is said to be* **finite variation** *if $\forall t$, given the subdivisions π of $[0,t]$ we get:*

$$\lim_{|\pi|\to 0} \sum_{t_i \in \pi} |A_{t_{i+1}} - A_{t_i}| < \infty \ \mathbb{P} \ a.s.$$

Example: $A_t = \int_0^t Y_s ds$ when Y is a continuous process.

Such processes, ω being fixed, give rise to Stieltjes integral.

Theorem 3.2. *(cf. Protter, th. 31 page 71). Let A a continuous finite variation process, f of class C^1. Then, $f(A_.)$ is a continuous finite variation process:*

$$f(A_t) = f(A_0) + \int_0^t f'(A_s) dA_s.$$

This is the order 1 Taylor formula.

These processes joined to continuous local martingales generate a large enough space of integrators, defined below.

Definition 3.3. *A* **continuous semi-martingale** *is a process X on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ $\mathbb{P}$ a.s. defined:*

$$X_t = X_0 + M_t + A_t, \ \forall t \geq 0,$$

where X_0 is $\mathcal{F}_0$-measurable, M is a continuous local martingale and $A = A^+ - A^-$, A^+ et A^- adapted finite variation increasing processes.

Recall: under AOA hypothesis, the prices are semi-martingales, cf. [7].
Important: $\forall A$ finite variation process and $\forall Y$ continuous semi-martingale, the bracket $\langle A, Y \rangle_t = 0$.

3.1 Itô formula

Theorem 3.4. *(Itô 1944, Kunita-Watanabé 1967) Let be $f \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ and X a continuous semi-martingale. Then, $\mathbb{P}$ a.s. and $\forall t \geq 0$:*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dM_s + \int_0^t f'(X_s) dA_s + \frac{1}{2} \int_0^t f"(X_s) d\langle M \rangle_s,$$

the first integral is a stochastic integral, the two others are Stieltjes integrals.

Differential notation : sometimes, we say that the "stochastic differential" of $f(X_t)$ is:

$$df(X_s) = f'(X_s)dX_s + \frac{1}{2}f"(X_s)d\langle X\rangle_s,$$

from where we deduce a stochastic differential calculus. This formula could be summarized as an order 2 Taylor formula.
Proof: four steps.

we "localize" to go to a bounded case,

we get the Taylor development of function f up to order 2,

we study the term inducing stochastic integral,

finally the quadratic variation term.

(1) Let be the stopping time

$$\begin{aligned} T_n &= 0 \text{ si } |X_0| \geq n, \\ & \inf\{t \geq 0; |M_t| \geq n \text{ or } |A_t| \geq n \text{ or } \langle M\rangle_t \geq n\} \\ & \text{ and infinity if above set is empty.} \end{aligned}$$

Obviously this sequence of stopping times is almost surely increasing to infinity. The property to prove is trajectorial, it is enough to show it for the process stopped at time T_n (then n goes to infinity). We thus can assume that the processes $M, A, \langle M\rangle$ and random variable X_0 are bounded. The process X is too bounded and we can consider function f admitting a compact support: $f, f', f"$ are bounded.

(2) To get this formula, and particularly the stochastic integral term, we cut the interval $[0,t]$ as a subdivision $\pi = (t_i, i = 1, ..., n)$ and we study the increments of $f(X_t)$ on this subdivision:

$$(14) \qquad f(X_t) - f(X_0) = \sum_{i=0}^{n-1}(f(X_{t_{i+1}}) - f(X_{t_i})) =$$

$$\sum_{i=0}^{n-1} f'(X_{t_i})(M_{t_{i+1}} - M_{t_i}) + \sum_{i=0}^{n-1} f'(X_{t_i})(A_{t_{i+1}} - A_{t_i}) + \frac{1}{2}\sum_{i=0}^{n-1} f"(\eta_i)(X_{t_{i+1}} - X_{t_i})^2,$$

where $\eta_i \in [X_{t_i}, X_{t_{i+1}}]$.
Obviously the second term converges to Stieltjes integral of $f'(X_s)$ with respect to A. Here, nothing is stochastic.

(3) Concerning the first term, we consider the simple process associated to the subdivision π :

$$Y_s^\pi = f'(X_{t_i}) \text{ si } s \in]t_i, t_{i+1}].$$

Then this first term, by definition, is equal to $\int_0^t Y_s^\pi dM_s$. But

$$\int_0^t |Y_s^\pi - f'(X_s)|^2 d\langle M\rangle_s = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |f'(X_{t_i}) - f'(X_s)|^2 d\langle M\rangle_s.$$

The application $s \mapsto f'(X_s)$ being continuous, the integrand above converges almost surely to zero. The fact that f' is bounded and bounded convergence Theorem prove that Y_s^π converges to $f'(X_s)$ in $L^2(d\mathbb{P} \times d\langle M\rangle)$: by definition, the first term converges in L^2 to the stochastic integral

$$\int_0^t f'(X_s)dM_s.$$

(4) **Quadratic variation term**: we decompose it in three terms:

$$\sum_{i=0}^{n-1} f"(\eta_i)(X_{t_{i+1}} - X_{t_i})^2 = \sum_{i=0}^{n-1} f"(\eta_i)(M_{t_{i+1}} - M_{t_i})^2 \tag{15}$$
$$+2\sum_{i=0}^{n-1} f"(\eta_i)(M_{t_{i+1}} - M_{t_i})(A_{t_{i+1}} - A_{t_i}) + \sum_{i=0}^{n-1} f"(\eta_i)(A_{t_{i+1}} - A_{t_i})^2$$

The last term is bounded by $\|f"\| \sup_i |\Delta_i A| \sum_{i=0}^{n-1} |\Delta_i A|$, by hypothesis $\|f"\|$ and $\sum_{i=0}^{n-1} |\Delta_i A|$ are bounded; $\sup_i |\Delta_i A|$ goes to zero almost surely since A is continuous.

The second term is bounded by $2\|f"\| \sup_i |\Delta_i M| \sum_{i=0}^{n-1} |\Delta_i A|$ which similarly converges almost surely to zero since M is continuous.

The first term of (15) is near to be

$$\sum_{i=0}^{n-1} f"(X_{t_i})(M_{t_{i+1}} - M_{t_i})^2.$$

Indeed:

$$\sum_{i=0}^{n-1}(f"(\eta_i) - f"(X_{t_i}))(\Delta_i M)^2 \leq \sup_i |f"(\eta_i) - f"(X_{t_i})| \sum_{i=0}^{n-1}(\Delta_i M)^2$$

where $\sup_i |f"(\eta_i) - f"(X_{t_i})|$ goes almost surely to zero using $f"$ continuity, and $\sum_{i=0}^{n-1}(\Delta_i M)^2$ goes to $\langle M\rangle_t$, by definition, in probability so there exists a subsequence which converges almost surely. Thus the product goes to zero in L^2 using the bounded convergence Theorem. It remains to study

$$\sum_{i=0}^{n-1} f"(X_{t_i})(M_{t_{i+1}} - M_{t_i})^2$$

to be compared to $\sum_{i=0}^{n-1} f"(X_{t_i})(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i})$. Its limit in L^2 is $\int_0^t f"(X_s)d\langle M\rangle_s$ since
- by continuity the simple process $t \mapsto f"(X_{t_i})$ if $t \in]t_i, t_{i+1}]$ converges almost surely to $f"(X_s)$;
- the bounded convergence Theorem concludes.

Let be the difference:

$$\sum_{i=0}^{n-1} f"(X_{t_i})[(M_{t_{i+1}} - M_{t_i})^2 - (\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i})],$$

we study its limit in L^2; look at the expectation of rectangular terms:

$$i < k \; : \; E[f"(X_{t_i})f"(X_{t_k})(\Delta_i M^2 - \langle M\rangle_{t_i}^{t_{i+1}})(\Delta_k M^2 - \langle M\rangle_{t_k}^{t_{k+1}})].$$

Applying $\mathcal{F}_{t_k}$ conditional expectation, $f"(X_{t_i})f"(X_{t_k})(\Delta_i M^2 - \langle M\rangle_{t_i}^{t_{i+1}})$ get out the conditional expectation, since $M^2 - \langle M\rangle$ is a martingale we get $E[(\Delta_k M^2 - \langle M\rangle_{t_k}^{t_{k+1}})/\mathcal{F}_{t_k}] = 0$ and we conclude that these terms are null.
Look at the squared terms:

$$\begin{aligned}\sum_i E[(f"(X_{t_i}))^2(\Delta_i M^2 - \langle M\rangle_{t_i}^{t_{i+1}})^2] &\leq 2\|f"\|_\infty^2 \sum_i [E(\Delta_i M^4) + E((\langle M\rangle_{t_i}^{t_{i+1}})^2)] \\ &\leq 2\|f"\|_\infty^2 E[(\sup_i \Delta_i M^2 \sum_i \Delta_i M^2) + \sup_i(\langle M\rangle_{t_i}^{t_{i+1}})\langle M\rangle_t].\end{aligned}$$

In the bound, $\sup_i \Delta_i M^2$ and $\sup_i(\langle M\rangle_{t_i}^{t_{i+1}})$ are bounded and converge almost surely to zero by continuity; by definition , $\sum_i \Delta_i M^2$ converges to $\langle M\rangle_t$ in probability; using bounded convergence Theorem, globally it converges to zero L^1, at least for a subsequence.

As a conclusion, the sequence of sums (14) converges in probability to the result of Theorem; we conclude thanks to the almost sure convergence of a subsequence. •

3.1.1 Extension and applications

We can extend this result to functions of vector semi-martingales depending also on time.

Proposition 3.5. *Let M be a d-dimensional vector of continuous local martingales, A a d-dimensional vector of continuous adapted processes with finite variation, X_0 a random variable, $\mathcal{F}_0$-measurable. Let be $f \in C^{1,2}(\mathbb{R}^+, \mathbb{R}^d)$. Set $X_t = X_0 + M_t + A_t$. Then, $\mathbb{P}$ almost surely:*

$$\begin{aligned}f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial}{\partial t} f(s, X_s)ds + \int_0^t \sum_i \frac{\partial}{\partial x_i} f(s, X_s)dM_s^i + \int_0^t \sum_i \frac{\partial}{\partial x_i} f(s, X_s)dA_s^i \\ &\quad + \frac{1}{2}\int_0^t \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(s, X_s)d\langle M^i, M^j\rangle_s\end{aligned}$$

Proof: to write it as a problem.

When f and its derivatives are bounded and M is a square integrable martingale, the stochastic integral term above is a "true" martingale, null in $t = 0$ and yields:

$$f(t, X_t) - f(0, X_0) - \int_0^t \frac{\partial}{\partial t} f(s, X_s)ds - \int_0^t \frac{\partial}{\partial x_i} f(s, X_s)dA_s^i - \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(s, X_s)d\langle M^i, M^j\rangle_s \in \mathcal{M}$$

For instance, if $A = 0$ and $X = M$ is Brownian motion, yields:

$$f(t, X_t) - f(0, X_0) - \int_0^t \mathcal{L}f(s, X_s)ds \text{ is a martingale}$$

where the differential operator $\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i \partial x_i}$.

From Itô formula we can deduce the solution of the so-called "heat equation", meaning the partial differential equation (PDE):

$$f \in C^{1,2}(\mathbb{R}^+, \mathbb{R}^d),\ \frac{\partial}{\partial t} f = \sum_i \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_i} f \text{ and } f(0,x) = \varphi(x)$$

where $\varphi \in C_b^2(\mathbb{R}^d)$ and the unique solution is

$$f(t,x) = E[\varphi(x + B_t)].$$

We easily check that this function is actually solution applying Itô formula; the uniqueness is a little bit more difficult to check.

For the following corollary, we set the following notation-definition:

Definition 3.6. *If X is the continuous real semi-martingale $X_0 + M + A$, denote $\langle X \rangle$ (which is actually $\langle M \rangle$). Similarly for two continuous semi-martingales X and Y, denote $\langle X, Y \rangle$ the bracket of their martingale part.*

Corollary 3.7. *Let be two continuous real semi-martingales X and Y; then:*

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \langle X, Y \rangle_t.$$

This is the important formula, named **integration by part formula**.

Proof: Exercise, as a simple application of Itô formula.

4 Examples of stochastic differential equations (SDE)

Here are other applications of Itô formula: a great use of Brownian motion is to model additive noises, measurement error in ordinary differential equations. For instance let us assume dynamics given by:

$$\dot{x}(t) = a(t)x(t),\ t \in [0,T],\ x(0) = x.$$

But it is not exactly this, in addition to the speed there is a little noise, and we model the dynamics as following:

$$dX_t = a(t)X_t dt + b(t)dB_t,\ t \in [0,T],\ X_0 = x,$$

called **stochastic differential equation**. We do not discuss the theory in this course, but we give another example below.

4.1 Black and Scholes model

This model is the one of a stochastic exponential with constant coefficients. We assume that the risky assets is solution to the SDE

$$dS_t = S_t b dt + S_t \sigma dW_t, S_o = s, \tag{16}$$

coefficient b is called "trend" and σ "volatility". Using the previous, it admits the explicit unique solution:

$$S_t = s \exp[\sigma W_t + (b - \frac{1}{2}\sigma^2)t].$$

Let us remark that $\log S_t$ has a Gaussian law.

Exercise: prove the uniqueness of the solution of (16); you could use Itô formula and apply it to the quotient of two solutions.

The following definitions will be seen with more details in Chapter 8.

Definition 4.1. *A strategy $\theta = (a, d)$ is said to be* **self-financing** *if $V_t(\theta) = a_t S_t^0 + d_t S_t = \langle \theta_0, p_0 \rangle + \int_0^t a_s dS_s^0 + \int_0^t d_s dS_s$.*

Moreover it is said to be **admissible** *if it is self-financing and if its value*

$$V_t(\theta) = V_0 + \int_0^t \theta_s . dS_s$$

is almost surely bounded below by a real constant.

An **arbitrage opportunity** *is an admissible strategy θ such that the value $V_.(\theta)$ satisfies $V_0(\theta) = 0$, $V_T(\theta) \geq 0$ and $\mathbb{P}(V_T(\theta) > 0) > 0$.*
AOA hypothesis *is the non existence of such a strategy.*

We call **risk neutral probability measure** *any probability measure Q which is equivalent to $\mathbb{P}$ and so that any discounted prices (id est $e^{-rt}S_t$ where r is a discount coefficient,*

for instance inflatio rate) are $(\mathcal{F}, Q)-$martingales. *Needs Section 5.*
A market is **viable** *is AOA hypothesis is satisfied. A sufficient condition is there exists at least one risk neutral probability measure.* *Needs Section 6.3.*
A market is **complete** *as soon as* $\forall X \in L^1(\Omega, \mathcal{F}_T, \mathbb{P})$ *there exists a strategy* θ *which is stochastically integrable with respect to the prices vector and such that* $X = E(X) + \int_0^T \theta_t dS_t$. *Needs Sections 6.1 and 6.2.*

The market under Black and Scholes model is viable, complete, with the unique risk neutral probability measure

$$Q = L_T\mathbb{P}, dL_t = -L_t\sigma^{-1}(b-r)dW_t, t \in [0,T], L_0 = 1.$$

Definition 4.2. *We call "call option" the following contract: at time $t = 0$ the buyer pays a sum q which gives the possibility to buy at time $t = T$ a share to price K but without obligation. If at time $T, S_T > K$, he exercises his right and wins $(S_T - K)^+ - q$. Otherwise, and if he does not exercise, it will have lost q. Overall, he earns $(S_T - K)^+ - q$.*

We call "put option" the following contract: at time $t = 0$ the buyer pays a sum q which gives the possibility to sell at time $t = T$ a share to price K but without obligation. If at time $T, S_T < K$, he exercises his right and wins $K - S_T - q$. Otherwise, and if he does not exercise, it will have lost q. Overall, he earns $(K - S_T)^+ - q$.

The problem is then to find a "fair price" q, between seller and buyer of this contract. This is the aim of the so called **Black and Scholes formula**.
To do this, we assume that the hedging portfolio θ, is such that there exists a class $(1, 2)$ function C such that the value is:

$$V_t(\theta) = C(t, S_t). \tag{17}$$

On another hand, θ is the pair (a, d) and

$$V_t(\theta) = a_t S_t^0 + d_t S_t = \langle \theta_0, p_0 \rangle + \int_0^t a_s dS_s^0 + \int_0^t d_s dS_s. \tag{18}$$

With this self financing strategy θ the seller of the option (for instance $(S_T - K)^+$) could "hedge the option using initial price $q = V_0$ to finally have $V_T(\theta) = C(T, S_T)$.

The key is the two ways of computing the stochastic differential of this value and their identification:

$$dV_t(\theta) = \frac{\partial}{\partial t}C(t, S_t)dt + \frac{\partial C}{\partial x}(t, S_t)dS_t + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}C(t, S_t)S_t^2\sigma^2 dt,$$

using (17), then using (18):

$$dV_t(\theta) = ra_t S_t^0 dt + d_t S_t(bdt + \sigma dW_t).$$

The identification gives two equations, and recall (18) which is $C(t, S_t)$:

$$\begin{aligned}(19)\qquad & \frac{\partial C}{\partial t}(t,S_t) + bS_t\frac{\partial C}{\partial x}(t,S_t) + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}S_t^2\sigma^2 = ra_tS_t^0 + d_tS_tb \\ & \frac{\partial C}{\partial x}(t,S_t)S_t\sigma = d_tS_t\sigma.\end{aligned}$$

Thus we get the hedging portfolio:

$$(20)\qquad d_t = \frac{\partial C}{\partial x}(t,S_t)\ ;\ a_t = \frac{C(t,S_t) - S_t\frac{\partial C}{\partial x}(t,S_t)}{S_t^0}.$$

Plugging (20) in (19) we get an almost sure equality

$$\frac{\partial C}{\partial t}(t,S_t) + rS_t\frac{\partial C}{\partial x}(t,S_t) + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}(t,S_t)S_t^2\sigma^2 = rC(t,S_t),$$

and when $t = T$ we need $C(T, S_T) = (S_T - K)^+$. But actually because $\log S_t$ admits a Gaussian law, we get that $S_t(\Omega) = \mathbb{R}^+ - \{0\}$, so we can replace above all S_t by an $x > 0$, and we get the PDE with boundary condition:

$$\frac{\partial C}{\partial t}(t,x) + rx\frac{\partial C}{\partial x}(t,x) + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}(t,x)x^2\sigma^2 = rC(t,x),$$

$$C(T,x) = (x-K)^+, x \in \mathbb{R}^+.$$

We solve this problem using **Feynman-Kac formula**. Set

$$dY_s = Y_s(rds + \sigma dW_s), Y_t = x.$$

Then $Y_s = x\exp[\sigma(W_s - W_t) - (s-t)(\frac{1}{2}\sigma^2 + r)]$ denoted as $Y_s^{(t,x)}$ and

$$C(t,x) = E_x[e^{-r(T-t)}(Y_T^{t,x} - K)^+]$$

is the expected solution, the portfolio being given by equations (20). The so famous Black-Scholes formula allows an explicit computation of this function, setting φ the distribution function of standard Gaussian law:

$$\begin{aligned}C(t,x) \quad = \quad & x\varphi\left(\frac{\log(x/K) + (T-t)(r + \frac{1}{2}\sigma^2)}{\sigma\sqrt{T-t}}\right) \\ & -Ke^{-r(T-t)}\varphi\left(\frac{\log(x/K) + (T-t)(r - \frac{1}{2}\sigma^2)}{\sigma\sqrt{T-t}}\right).\end{aligned}$$

The initial price q of this option is then $C(0,x)$.

Actually, the way is to solve after a change of (variable,function):

$$x = e^y, y \in \mathbb{R}\ ;\ D(t,y) = C(t,e^y)$$

which allows to go to Dirichlet problem:

$$\frac{\partial}{\partial t}D(t,y) + r\partial_y D(t,y) + \frac{1}{2}\partial^2_{y^2}D(t,y)\sigma^2 = rD(t,y), y \in \mathbb{R},$$

$$D(T,y) = (e^y - K)^+, y \in \mathbb{R},$$

associated to the stochastic differential equation:

$$dX_s = rds + \sigma dW_s, s \in [t,T], X_t = y.$$

This is exactly what we saw in Proposition 4.11, with $g = 0,\ f(x) = (e^x - k)^+,\ k(x) = r$. *Thus*

$$D(t,y) = E_y[e^{-r(T-t)}(e^{X_T} - K)^+],$$

and the explicit formula since X_T *admits a Gaussian law.*

The price at time t is $C(t,S_t) = E_Q e[^{-r(T-t)}(e^{X_T} - K)^+/\mathcal{F}_t]$; this is easy to compute: the law of X_T given $\mathcal{F}_t$ is a Gaussian law, with mean $S_t + r(T-t)$ and variance $\sigma^2(T-t)$.

4.2 Stochastic exponential

Let us consider the function C^∞, $f : x \mapsto e^x$, and a continuous semi-martingale X, $X_0 = 0$, let us apply Itô formula to the process $Z_t = \exp(X_t - \frac{1}{2}\langle X\rangle_t)$. Yields:

$$Z_t = 1 + \int_0^t [\exp(X_s - \frac{1}{2}\langle X\rangle_s)(dX_s - \frac{1}{2}d\langle X\rangle_s) + \frac{1}{2}\exp(X_s - \frac{1}{2}\langle X\rangle_s)d\langle X\rangle_s].$$

So, after some cancellation:

$$Z_t = 1 + \int_0^t \exp(X_s - \frac{1}{2}\langle X\rangle_s)dX_s,$$

or using differential notation:

$$dZ_s = Z_s dX_s.$$

This is an example of (stochastic) differential equation. Then there is the following result:

Theorem 4.3. *Let* X *be a continuous semi martingale,* $X_0 = 0$. *Then there exists a unique continuous semi martingale which is solution of the stochastic differential equation:*

$$Z_t = 1 + \int_0^t Z_s dX_s \tag{21}$$

which is explicitly:

$$Z_t(X) = \exp(X_t - \frac{1}{2}\langle X\rangle_t).$$

Itô formula shows that this process is actually solution of the required equation. *Exercise: show the uniqueness assuming that there exists two solutions Z and Z', then apply Itô formula to the quotient $Y_t = \frac{Z_t}{Z'_t}$.*

Definition 4.4. *Let X be a continuous semi martingale, $X_0 = 0$. The* **stochastic exponential** *of X, denoted as $\mathcal{E}(X)$, is the unique solution of the stochastic differential equation (21).*

Example: Let be $X = aB$ where a is a real number and B the Brownian motion; then $\mathcal{E}_t(aB) = \exp(aB_t - \frac{1}{2}a^2 t)$, sometimes called "geometric Brownian motion".

Here are some results on these stochastic exponentials.

Theorem 4.5. *(cf. [30], Th. 37) Let X and Y be two continuous semimartingales, $X_0 = Y_0 = 0$. Then*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + \langle X, Y\rangle).$$

Preuve: set $U_t = \mathcal{E}_t(X)$ et $V_t = \mathcal{E}_t(Y)$ and apply integration by part formula (3.7):

$$U_t V_t - 1 = \int_0^t U_s dV_s + V_s dU_s + d\langle U, V\rangle_s$$

Setting $W = UV$ and using the differential definition of the stochastic exponential we get the result. •

Corollary 4.6. *Let X be a continuous semi martingale, $X_0 = 0$. Then the inverse $\mathcal{E}_t^{-1}(X) = \mathcal{E}_t(-X + \langle X\rangle)$*

Proof as an Exercise.

Let us now consider more general linear stochastic differential equations.

Theorem 4.7. *(cf. [30], th. 52, page 266.) Let Z and H two real continuous semi martingales, $Z_0 = 0$. Then the stochastic differential equation:*

$$X_t = H_t + \int_0^t X_s dZ_s$$

admits the unique solution

$$\mathcal{E}_H(Z)_t = \mathcal{E}_t(Z)(H_0 + \int_0^t \mathcal{E}_s^{-1}(Z)(dH_s - d\langle H, Z\rangle)_s).$$

Preuve: we use the method of constant variation. Let us assume that the solution admits the form:

$$X_t = \mathcal{E}_t(Z) C_t$$

and apply Itô formula:

$$dX_t = C_t d\mathcal{E}_t(Z) + \mathcal{E}_t(Z) dC_t + d\langle \mathcal{E}(Z), C\rangle_t,$$

so, replacing $d\mathcal{E}_t(Z)$ by its value and using the particular form of X:

$$dX_t = X_t dZ_t + \mathcal{E}_t(Z)[dC_t + d\langle Z, C\rangle_t].$$

If X is solution of the required equation, by identification we get two different expressions for dX_t and by identification we get:

$$dH_t = \mathcal{E}_t(Z)[dC_t + d\langle Z, C\rangle_t].$$

But since $\mathcal{E}_t(Z)$ is an exponential and since $(Z_t - \frac{1}{2}\langle Z\rangle_t)$ is finite, $\mathcal{E}_t^{-1}(Z)$ exists and

$$dC_t = \mathcal{E}_t^{-1}(Z)dH_t - d\langle Z, C\rangle_t$$

so yields:

$$d\langle Z, C\rangle_t = \mathcal{E}_t^{-1}(Z)d\langle H, Z\rangle_t,$$

and finally:

$$dC_t = \mathcal{E}_t^{-1}(Z)[dH_t - d\langle H, Z\rangle_t].$$

We used the co-variation of C and Z is the same as the one of $\mathcal{E}_t(Z)^{-1}.H$ and Z. •

4.3 Ornstein-Uhlenbeck equation

Another important example used in Finance (for instance to model the dynamics of rate) is the one of **Ornstein-Uhlenbeck** equation (cf. [20], page 358):

$$dX_t = a(t)X_t dt + b(t)dB_t,\ t \in [0, T],\ X_0 = x$$

where a and b are $\mathcal{F}-$adapted processes, a almost surely integrable with respect to to time, $b \in L^2(\Omega \times [0, T], d\mathbb{P} \otimes dt)$. When a and b are constant $a(t) = -\alpha$ and $b(t) = \sigma$, we get the solution:

$$X_t = e^{-\alpha t}(x + \int_0^t \sigma e^{\alpha s} dB_s).$$

Morever it can be shown:

$$\begin{aligned} m(t) &= E(X_t) = m(0)e^{-\alpha t} \\ V(t) &= Var(X_t) = \frac{\sigma^2}{2\alpha} + (V(0) - \frac{\sigma^2}{2\alpha})e^{-2\alpha t} \\ \rho(s, t) &= cov(X_s, X_t) = [V(0) + \frac{\sigma^2}{2\alpha}(e^{2\alpha(t\wedge s)} - 1)]e^{-\alpha(t+s)} \end{aligned}$$

Finally one more example, "Mean reversion" model, is the **Cox Ingersoll Ross model**:

$$dY_t = \lambda(\eta - Y_t)dt + \theta\sqrt{Y_t}dB_t, Y_t = y.$$

With the hypothesis

$$2\lambda\eta \geq \theta^2$$

we get $Y_t > 0$. This is convenient to model stochastic volatility or interest rates.

4.4 Insight into more general stochastic differential equations

Generally, there is existence (and uniqueness) sufficient conditions for solution of the equation with initial condition $X_t = x$:

$$X_s^{t,x} = x + \int_t^s b(u, X_u)du + \sigma(u, X_u)dW_u, \tag{22}$$

for instance hypotheses on coefficients could be:
(i) continuous, with sub linear increase with respect to space,
(ii) such that there exists a solution to the equation unique in law, meaning **weak solution**:there exists a probability $\mathbb{P}_x$ on Wiener space $(\Omega, \mathcal{F})$ under which

. X is $\mathcal{F}-$adapted continuous, taking its value in $\overline{\mathbb{R}}$,

. if $S_n = \inf\{t : |X_t| > n\}$, X^{S_n} satisfies the existence conditions of **strong solutions** (meaning trajectorial solutions).
The increasing limit of times S_n is called explosion time. Then $\mathbb{P}_x$-almost surely for all n

$$X_{t\wedge S_n} = x + \int_t^{t\wedge S_n} b(u, X_u)du + \int_t^{t\wedge S_n} \sigma(u, X_u)dW_u.$$

For clarification, let us quote the existence Theorem 6 page 194 in [30].

Theorem 4.8. *Let Z be a semi martingale with $Z_0 = 0$ and let $f : \mathbb{R}^+ \times \mathbb{R} \times \Omega$ be such that*

(i) for fixed x, $(t, \omega) \mapsto f(t, x, \omega)$ is adapted right continuous-left limited,

(ii) for each (t, ω), $|f(t, x, \omega) - f(t, y, \omega)| \leq K(\omega)|x - y|$ for some finite random variable K.
Let X_0 be finite and $\mathcal{F}_0$-measurable. Then the equation

$$X_t = X_0 + \int_0^t f(s, ., X_{s-})dZ_s$$

admits a solution. This solution is unique and it is a semi martingale.

Or Theorem 2.5 page 287 in [20].

Theorem 4.9. *Let the EDS*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

such that the coefficient b and σ are locally Lipschitz continuous in the space variable; i.e. for every integer $n \geq 1$ there exists a constant K_n such that for every $t \geq 0$, $\|x\| \leq n$, and $\|y\| \leq n$

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K_n\|x - y\|.$$

Then strong uniqueness holds.

4.5 Link with partial differential equations, Dirichlet Problem

(cf. [20] 5.7 pages 363 et sq.)

Definition 4.10. *Let D be an open subset of $\mathbb{R}^d$. An order 2 differential operator $\mathcal{A} = \sum_{i,j} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}$ is said to be elliptic for x if*

$$\forall \xi \in \mathbb{R}^d_*, \ \sum_{i,j} a_{i,j}(x)\xi_i\xi_j > 0.$$

If $\mathcal{A}$ is elliptic for any point $x \in D$, it is said to be elliptic in D.
If there exists $\delta > 0$ such that

$$\forall \xi \in \mathbb{R}^d, \ \sum_{i,j} a_{i,j}(x)\xi_i\xi_j \geq \delta\|\xi\|^2,$$

it is said to be uniformly elliptic.

Dirichlet problem is the one to find a C^2 class function u on bounded open subset D, $u(x) = f(x)$ $\forall x \in \partial D$, and satisfying in D:

$$\mathcal{A}u - ku = -g$$

with $\mathcal{A}$ elliptic, $k \in \mathcal{C}(\bar{D}, \mathbb{R}^+)$, $g \in \mathcal{C}(\bar{D}, \mathbb{R})$, $f \in \mathcal{C}(\partial D, \mathbb{R})$.

Proposition 4.11. *(Proposition 7.2, page 364 [20])*
Let u be solution of Dirichlet problem $(\mathcal{A}, D)$ and X solution of (22) with operator $\mathcal{A} = \frac{1}{2}\sum_{i,j,l} \sigma_l^i \sigma_l^j(x) \frac{\partial^2}{\partial x_i \partial x_j} + \nabla.b(x)$; T_D the exit time of D by X. If $\forall x \in D$,

$$E_x(T_D) < \infty \tag{23}$$

then $\forall x \in \bar{D}$,

$$u(x) = E_x[f(X_{T_D}) \exp(-\int_0^{T_D} k(X_s)ds) + \int_0^{T_D} g(X_t) \exp(-\int_0^t k(X_s)ds)dt].$$

Proof Exercise (problem 7.3 in [20], correction page 393).
First let us remark that the continuity of X implies $X_{T_D} \in \partial D$.
Indication: prove

$$M : t \mapsto u(X_{t\wedge T_D}) \exp\left(-\int_0^{t\wedge T_D} k(X_s)ds\right) + \int_0^{t\wedge T_D} g(X_s) \exp\left(-\int_0^s k(X_u)du\right) ds, t \geq 0$$

is a uniformly integrable martingale with respect to $\mathbb{P}_x$: compute $E_x(M_0) = E_x(M_\infty)$; on $\{t < T_D\}$, do the Itô differential of M and use on $D, \mathcal{A}u - ku + g = 0$. $M_0 = u(x)$ since $X_0 = x$ under $\mathbb{P}_x$,

$$dM_t = \exp(-\int_0^{t\wedge T_D} k(X_s)ds) \times [\mathcal{A}u(X_{t\wedge T_D})dt + \nabla u(X_{t\wedge T_D})\sigma(t, X_{t\wedge T_D})dW_t + g(X_{t\wedge T_D}) - (k.u)(X_{t\wedge T_D})dt],$$

functions ∇u and σ are continuous thus bounded on compact $\bar{D}$, so the second term above is a martingale, moreover the other terms cancel since $\mathcal{A}u - ku + g = 0$ and for any $t, E_x[M_t] = u(x)$.

This martingale is bounded in L^2 so uniformly integrable and we could do t going to infinity and apply stopping Theorem since $E_x[T_D] < \infty$. •

Remark 4.12. *(Friedman, 1975) A sufficient condition for hypothesis (23) is: $\exists l, \exists \alpha \ : \ a_{l,l}(x) \geq \alpha > 0$. This condition is stronger than ellipticity, but weaker than uniform ellipticity in D.*

Set:

$$b^* = \max\{|b_l(x)|, x \in \bar{D}\}, q = \min\{x_l, x \in \bar{D}\},$$

and choose $\nu > 4b^*/\alpha, h(x) = -\mu \exp(\nu x_l), x \in D$, μ will be chosen later. Then h is C^∞ class and $-\mathcal{A}h(x)$ is computed and bounded:

$$-\mathcal{A}h(x) = (\frac{1}{2}\nu^2 a_{ll} + \nu b_l(x))\mu e^{\nu x_l} \geq (\frac{8(b^*)^2}{\alpha} - \frac{4b^*}{\alpha}b^*)\mu e^{\nu x_l} \geq \frac{4(b^*)^2}{\alpha}\mu e^{\nu q} \geq 1.$$

Then we choose μ great enough so that $-\mathcal{A}h(x) \geq 1$; $x \in D$, h and its derivatives are bounded in D, and we apply Itô formula to h

$$h(X_t^{T_D}) = h(x) + \int_0^{t\wedge T_D} \mathcal{A}h(X_s)ds + \int_0^{t\wedge T_D} \nabla h(X_s)\sigma(X_s)dW_s.$$

Thus yields

$$t \wedge T_D \leq h(x) - h(X_t^{T_D}) = -\int_0^{t\wedge T_D} \mathcal{A}h(X_s)ds$$

plus a uniformly integrable martingale. Thus $E_x[t \wedge T_D] \leq 2\|h\|_\infty$ and finally let t goes to infinity.

5 Change of probability, Girsanov theorem

The motivation of this chapter is: martingales and local martingales are powerful tools, and it is therefore worthwhile to model reality so that the processes involved are martingales, at least locally. Thus, for the application of stochastic calculus to Finance, the data are a set of processes that model the evolution over time of share price on the financial market, and one can legitimately ask the question:

Is there a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ on which the price process are all martingales (at least locally)?

Specifically, does it exist a probability $\mathbb{P}$ which satisfies the property? Hence the two problems discussed are the following:

- How to move from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\Omega, \mathcal{F}, Q)$ in a simple way? does it exist a density $\frac{d\mathbb{P}}{dQ}$? How then are transformed Brownian motion and martingales? This is Girsanov theorem, Section 5.1. Section 5.2 gives a sufficient condition to apply Girsanov theorem.

- Finally, given a family of semi-martingales on filtered probability space $(\Omega, (\mathcal{F}_t))$, does it exist a probability $\mathbb{P}$ such that all these processes are martingales on filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$? This is a "martingale problem" that we will see in Chapter 6.

We a priori consider a filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$ which is defined linked to a d-dimensional Brownian motion B, $B_0 = 0$. The filtration is generated by the Brownian motion and we note $\mathcal{M}(\mathbb{P})$ the set of martingales on $(\Omega, (\mathcal{F}_t), \mathbb{P})$.

Recall the notion of **local martingales**, their set is denoted as $\mathcal{M}_{loc}(\mathbb{P})$ meaning adapted process M such that there exists a sequence of stopping times (T_n) increasing to infinity and such that $\forall n$ the T_n stopped process M^{T_n} is a true martingale.

5.1 Girsanov theorem

([20] 3.5, p 190-196; [30] 3.6, p 108-114)

Let X be an adapted measurable process in $\mathcal{P}(B)$:
$\mathcal{P}(B) := \{X \text{adapted measurable process:} \forall T,\ \int_0^T \| X_s \|^2 \, ds < +\infty \ \mathbb{P}\ a.s.\}$

This set is larger than $\mathcal{L}(B) = L^2(\Omega \times \mathbb{R}^+, d\mathbb{P} \otimes dt)$.
Generally we define for any martingale M the set $\mathcal{P}(M)$ which contains $\mathcal{L}(M) = L^2(\Omega \times \mathbb{R}^+, d\mathbb{P} \otimes d\langle M \rangle)$:

$$\mathcal{P}(M) := \{X \textit{ adapted measurable process: } \forall T, \int_0^T \| X_s \|^2 \, d\langle M \rangle_s < +\infty \ \mathbb{P}\ a.s.\}$$

For such process X, $X.M$ is only a "local" martingale.

Think of $d\langle M \rangle_s$ as $f(s, \omega) ds$.

Thus we can define the local martingale $X.B$ and its Doléans exponential (stochastic exponential) as soon as $\forall t \int_0^t \| X_s \|^2 ds < +\infty \ \mathbb{P}\ a.s.$:

$$\mathcal{E}_t(X.B) = \exp[\int_0^t (\sum_i X_s^i dB_s^i - \frac{1}{2} \| X_s \|^2 ds)],$$

solution of the SDE

$$dZ_t = Z_t \sum_i X_t^i dB_t^i \ ; \ Z_0 = 1, \tag{24}$$

which is too a local martingale since $\int_0^t Z_s^2 \| X_s \|^2 ds < +\infty \ \mathbb{P}\ a.s.$ by continuity of the integrand on $[0,t]$.

Under some conditions, $\mathcal{E}_.(X.B)$ is a "true" martingale, then $\forall t,\ E[Z_t] = 1$, this allows a change of probability measure on the σ-algebra $\mathcal{F}_t$:

$$Q = Z_t.\mathbb{P} \text{ meaning if } A \in \mathcal{F}_t,\ Q(A) = E[1_A Z_t].$$

Note that E is the expectation with respect to probability measure $\mathbb{P}$. Since $Z_t > 0$, both probability measures are equivalent and $\mathbb{P}(A) = E_Q[Z_t^{-1}\mathbf{1}_A]$.

Theorem 5.1. *(Girsanov, 1960 ; Cameron-Martin, 1944)*
If the process $Z = \mathcal{E}(X.B)$ solution of (24) belongs to $\mathcal{M}(\mathbb{P})$, and if Q is the probability measure defined on $\mathcal{F}_T$ by $Z_T.\mathbb{P}$ then:

$$\tilde{B}_t = B_t - \int_0^t X_s ds,\ t \leq T$$

is a Brownian motion on $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, Q)$.

The proof needs a preliminary lemma. Below E_Q notes the Q-expectation.

Lemma 5.2. *Let be $T \geq 0$, $Z \in \mathcal{M}(\mathbb{P})$, $Q = Z_T\mathbb{P}$. Let be $0 \leq s \leq t \leq T$ and a random variable Y, in $L^1(Q, \mathcal{F}_t)$, then $E_Q(Y/\mathcal{F}_s) = \frac{E(YZ_t/\mathcal{F}_s)}{Z_s}$.*

This is, more or less, a Bayes formula.
Proof (Exercise): let be $A \in \mathcal{F}_s$:

$$E_Q(1_A \frac{E(YZ_t/\mathcal{F}_s)}{Z_s}) = E(1_A E(YZ_t/\mathcal{F}_s))$$

since on $\mathcal{F}_s$, $Q = Z_s\mathbb{P}$. Then: $E[1_A E(YZ_t/\mathcal{F}_s)] = E(1_A Y Z_t)$
by definition of conditional expectation, and finally using definition of Q, and since $1_A Y$ is $\mathcal{F}_T$-measurable

$$E(1_A Y Z_t) = E_Q(1_A Y).$$

This is true $\forall A \in \mathcal{F}_s$, so we can identify $\frac{E(YZ_t/\mathcal{F}_s)}{Z_s}$ as the expected conditional expectation.

•

Proposition 5.3. *Under hypotheses of Girsanov theorem, for any continuous local $\mathbb{P}$-martingale M, the process N below is a $Q-$local martingale:*

$$N = M - \int_0 \sum_i X_s^i d\langle M, B^i\rangle_s.$$

Proof: (Exercise)

•

It yields as a corollary that $\tilde{B}$ is a Q-martingale with bracket t. To prove it is a Q-Brownian motion, it is enough to show that it is an independent increments process with Gaussian law (or that it is a Gaussian process).

Now we look things in "reverse" order, that is, if there exists equivalent probability measures, to look for a link between martingales related to the one or the other probability, and related to the same filtration.

Proposition 5.4. *Let $\mathbb{P}$ and Q be two equivalent probability measures on $(\Omega, \mathcal{F})$ and the uniformly integrable continuous martingale $Z_t = E[\frac{dQ}{d\mathbb{P}}/\mathcal{F}_t]$. Then $M \in \mathcal{M}^c_{loc}(Q) \Leftrightarrow MZ \in \mathcal{M}^c_{loc}(\mathbb{P})$.*

Proof: Let (T_n) be a sequence of stopping times, localizing for M and recall Doob Corollary 0.25

$$E_Q[M_{t\wedge T_n}/\mathcal{F}_s] = E_Q[M_{t\wedge T_n}/\mathcal{F}_{s\wedge T_n}] = M_{s\wedge T_n}.$$

We now apply Lemma 5.2, for $s \leq t$ it yields both:

$$E_Q[M_{t\wedge T_n}/\mathcal{F}_s] = \frac{E_{\mathbb{P}}[Z_t M_{t\wedge T_n}/\mathcal{F}_s]}{Z_s} \tag{25}$$

and

$$E_Q[M_{t\wedge T_n}/\mathcal{F}_{s\wedge T_n}] = \frac{E_{\mathbb{P}}[Z_{t\wedge T_n} M_{t\wedge T_n}/\mathcal{F}_{s\wedge T_n}]}{Z_{s\wedge T_n}}$$

so

$$E_Q[M_{t\wedge T_n}/\mathcal{F}_s] = \frac{E_{\mathbb{P}}[Z_{t\wedge T_n} M_{t\wedge T_n}/\mathcal{F}_{s\wedge T_n}]}{Z_{s\wedge T_n}}.$$

Then the fact that $M^{T_n} \in \mathcal{M}(Q)$ is equivalent to $(MZ)^{T_n} \in \mathcal{M}(\mathbb{P})$.

•

Theorem 5.5. *Girsanov-Meyer: Let be $\mathbb{P}$ and Q two equivalent probability measures, $Z_t = E[\frac{dQ}{d\mathbb{P}}/\mathcal{F}_t]$ and X a semi-martingale on $(\Omega, \mathcal{F}, \mathbb{P})$ decomposed as $X = M + A$. Then, X is too semi-martingale on $(\Omega, \mathcal{F}, Q)$ decomposed as $X = N + C$, where*

$$N = M - \int_0^t Z_s^{-1} d\langle Z, M\rangle_s \; ; \; C = A + \int_0^t Z_s^{-1} d\langle Z, M\rangle_s.$$

Proof: (i) C is a finite variation process as sum of two finite variation processes.

(ii) Compute the product NZ using Itô formula under $\mathbb{P}$:

$$d(NZ)_t = N_t dZ_t + Z_t dM_t - Z_t Z_t^{-1} d\langle Z, M\rangle_t + d\langle Z, N\rangle_t.$$

But N is a $\mathbb{P}$-semi martingale with martingale part M: the bracket $\langle Z, N\rangle$ is the one of M with Z, so a cancellation proves that NZ is a $\mathbb{P}$-martingale so (using Proposition 5.4) N is a Q-martingale. •

5.2 Novikov condition

(cf. [20] pages 198-201).
The previous subsection is based on the hypothesis that the process $\mathcal{E}(X.B)$ is a true martingale. We now look for sufficient conditions on X so that this hypothesis will be satisfied. Generally $\mathcal{E}(X.B)$ is at least a local martingale with localising sequence

$$T_n = \inf\{t \geq 0, \ \int_0^t \| \mathcal{E}_s(X.B)X_s \|^2 ds > n\}.$$

Lemma 5.6. *$\mathcal{E}(X.B)$ is an super martingale; it is a martingale if and only if:*

$\forall t \geq 0 \ E[\mathcal{E}_t(X.B)] = 1.$

Proof: there exists an increasing sequence of stopping times T_n such that $\forall n, \mathcal{E}(X.B)^{T_n} \in \mathcal{M}(\mathbb{P})$ thus for any $s \leq t$ we get

$$E[\mathcal{E}_{T_n \wedge t}(X.B)/\mathcal{F}_s] = \mathcal{E}_{T_n \wedge s}(X.B).$$

Using Fatou lemma, we deduce from this equality going to the limit that actually $\mathcal{E}(X.B)$ is a super martingale (remember that any positive local martingale is a supermartingale). Since $E[\mathcal{E}_0(X.B)] = 1$, it is enough that, $\forall t \geq 0$, we could have $E[\mathcal{E}_t(X.B)] = 1$ to check that $\mathcal{E}(X.B)$ is a martingale. •

Proposition 5.7. *([20] pp. 198-199)*
Let M be a continuous local martingale with respect to $\mathbb{P}$ and $Z = \mathcal{E}(M)$ such that $E[\exp \frac{1}{2}\langle M\rangle_t] < \infty \ \forall t \geq 0$. Then $\forall t \geq 0, E[Z_t] = 1$.

Corollary 5.8. *(Novikov, 1971) : Let X be an adapted vectorial measurable process such that:*

$$E[\exp \frac{1}{2} \int_0^t \| X_s \|^2 ds] < \infty \text{ pour tout } t \geq 0$$

(where $\|x\|^2 = \sum_i x_i^2$,) then $\mathcal{E}(X.B) \in \mathcal{M}(\mathbb{P})$.

To close this subsection, here is an example of process $X \in \mathcal{P}(B)$ which doesn't satisfy Novikov condition, such that $\mathcal{E}(X.B) \in \mathcal{M}^c_{loc}(\mathbb{P})$ but it is not a "true" martingale

(Exercise):
Let be the stopping time $T = \inf\{1 \geq t \geq 0, t + B_t^2 = 1\}$ *and*

$$X_t = -\frac{2}{(1-t)^2} B_t 1_{\{t \leq T\}} \ ; \ 0 \leq t < 1, \ X_1 = 0.$$

(i) Prove that $T < 1$ *almost surely and thus* $\int_0^1 X_t^2 dt < \infty$ *almost surely.*

(ii) Apply Itô formula to the process $t \to \frac{B_t^2}{(1-t)^2}$; $0 \leq t < 1$ *to prove:*

$$\int_0^1 X_t dB_t - \frac{1}{2}\int_0^1 X_t^2 dt = -1 - 2\int_0^T [\frac{1}{(1-t)^4} - \frac{1}{(1-t)^3}] B_t^2 dt < -1.$$

(iii) The local martingale $\mathcal{E}(X.B)$ *is not a martingale (not up to* 1 *anyway!): we deduce from (ii) that its expectation is bounded by* $\exp(-1) < 1$ *and this contradicts Lemma 5.6. Anyway, we can prove that* $\forall n \geq 1$ *and* $\sigma_n = 1 - (1/\sqrt{n})$*, the process* $\mathcal{E}(X.B)^{\sigma_n}$ *is a martingale.*

6 Martingale representation theorem, martingale problem

(cf. Protter [30], pages 147-157.)
The motivation of this chapter is to show that a large enough class of martingales could be identified as a stochastic integral $X.B$. This will allow us to find a common probability measure $\mathbb{P}$ for all the price processes such that these ones are all $\mathbb{P}$-martingales, at least local martingales.

6.1 Representation property

We here consider martingales in $\mathcal{M}^{2,c}$, null at time $t = 0$, and satisfying $\langle M\rangle_\infty \in L^1$. Then, $E[\sup_t M_t^2] \leq C_2 \sup_t E[\langle M\rangle_t] = E[\langle M\rangle_\infty] < \infty$. These martingales are uniformly integrable, there exists M_∞ such that $\forall t \geq 0,\ M_t = E[M_\infty/\mathcal{F}_t]$. Let us denote their set as $\mathcal{H}_0^2$.

$$\mathcal{H}_0^2 = \{M \in \mathcal{M}^{2,c}, M_0 = 0, \langle M\rangle_\infty \in L^1\}.$$

Recall following notations:

$$\mathcal{L}(M) = \{X \text{ adapted } \in L^2(\Omega\times\mathbb{R}^+, \mathbb{P}\otimes d\langle M\rangle)\}\ ;\ \mathcal{L}^*(M) = \{X \text{ progressive } \mathbb{P}\, a.s.\ \in L^2(\mathbb{R}^+, d\langle M\rangle)\},$$

and if X is càd or càg, then adapted is equivalent to progressive. For now on, we only consider such a case.
We have to look at what happens after a change of probability measure.

Definition 6.1. *Let be $\mathcal{A} \subset \mathcal{H}_0^2(\mathbb{P})$ and denote $\mathcal{M}(\mathcal{A})$ the set of probability measures Q on $\mathcal{F}_\infty$, absolutely continuous with respect to $\mathbb{P}$, equal to $\mathbb{P}$ on $\mathcal{F}_0$, such that $\mathcal{A} \subset \mathcal{H}_0^2(Q)$.*

Lemma 6.2. *$\mathcal{M}(\mathcal{A})$ is convex.*

Proof: exercise.

Definition 6.3. *Let be $\mathcal{A} \subset \mathcal{H}_0^2$, $\mathcal{A}$ is said to have the* **predictable representation property** *if:*

$$\mathcal{I} = \{X = \sum_{i=1}^n H^i M^i,\ M^i \in \mathcal{A}, H^i \in \mathcal{L}^*(M^i) \cap L^2(d\mathbb{P} \otimes d\langle M^i\rangle)\} = \mathcal{H}_0^2.$$

Below, we will see an important example of such an $\mathcal{A}$, Theorem 6.14:
Let be $\mathcal{A} = (M^1, \cdots, M^n) \subset \mathcal{H}_0^2(\mathbb{P})$ satisfying $M^i \dagger M^j, i \neq j$. $\mathbb{P}$ is extremal in $\mathcal{M}(\mathcal{A})$ yields that $\mathcal{A}$ has the predictable representation property.

And really important is Theorem 6.15:
Let B be a n-dimensional Brownian motion on $(\Omega, \mathcal{F}_t^B, \mathbb{P})$. Then $\forall M \in \mathcal{M}_{loc}^{c,2}$, there exists $H^i \in \mathcal{L}(B^i), i = 1, \cdots, n$, such that:

$$M_t = M_0 + \sum_{i=1}^n (H^i.B^i)_t.$$

Theorem 6.4. *Let F be a closed vector subspace of $\mathcal{H}_0^2$. Then the followings are equivalent definitions of a* **stable subspace***:*

(i) if $M \in F$ and $A \in \mathcal{F}_t$, $(M - M^t)1_A \in F$, $\forall t \geq 0$.

(ii) F satisfies: if $\forall M \in F$ and for any stopping time T then $M^T \in F$.

(iii) if $M \in F$ and H bounded $\in \mathcal{L}^*(M)$ then $H.M \in F$.

(iv) if $M \in F$ and $H \in \mathcal{L}^*(M) \cap L^2(d\mathbb{P} \otimes d\langle M \rangle)$, then $H.M \in F$.

More or less, "stability" means stability with respect to stochastic integration.

Proof: Since $\mathcal{L}_b^*(M) \subset \mathcal{L}^*(M) \cap L^2(d\mathbb{P} \otimes d\langle M \rangle)$, the implication (iv) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii): it is enough to consider any stopping time T and the process $H_t = 1_{[0,T]}(t)$. Then

$$(H.M)_t = \int_0^t 1_{[0,T]}(s)dM_s = M_{t \wedge T} \in F,$$

meaning M^T is an element of F.

(ii) $\Rightarrow$ (i): let t be fixed, $A \in \mathcal{F}_t$ and $M \in F$. We build the stopping time $T(\omega) = t$ if $\omega \in A$ and infinity if not. This is actually a stopping time since $A \in \mathcal{F}_t$. Otherwise, on one hand:

$$\begin{aligned} (M - M^t)1_A &= (M - M^t) \text{ if } \omega \in A, \text{ which is equivalent to } T(\omega) = t \\ &= 0 \text{ if not }, \end{aligned}$$

on the other hand:

$$\begin{aligned} M - M^T &= (M - M^t) \text{ if } \omega \in A, \\ &= 0 \text{ if not }, \end{aligned}$$

this means that $(M - M^t)1_A = M - M^T$. But F is stable, thus M and $M^T \in F$, so $(M - M^t)1_A \in F$ for any $t \geq 0$: this is property (i).

(i) $\Rightarrow$ (iv): let be $H \in \mathcal{P}$ which could be written as:

$$H = H_0 + \sum_i H_i 1_{]t_i, t_{i+1}]}$$

where $H_i = 1_{A_i}, A_i \in \mathcal{F}_{t_i}$. Then

$$H.M = \sum_i 1_{A_i}(M_{t_{i+1}} - M_{t_i}) = \sum_i 1_{A_i}(M - M^{t_i})_{t_{i+1}}$$

which belongs to F using (i). Any simple process is limit of linear combinations of processes as H above; the stochastic integral being linear we get for any simple process X that $X.M \in F$, vector space. To conclude we take the limits of simple processes since $\mathcal{P}$ is dense in $\mathcal{L}^*(M) \cap L^2(d\mathbb{P} \otimes d\langle M \rangle)$ (cf. Proposition 2.10) •

Definition 6.5. *Let $\mathcal{A}$ be a subset of $\mathcal{H}_0^2$. We denote $\mathbf{S}(\mathcal{A})$ the smallest stable closed vectorial subspace which contains $\mathcal{A}$.*

Definition 6.6. *Let be M and $N \in \mathcal{H}_0^2$, M and N are said to be* **orthogonal** *if $E[M_\infty N_\infty] = 0$,* **strongly orthogonal** *if MN is a martingale.*

By definition $MN - \langle M, N\rangle$ is a martingale, thus the strong orthogonality is equivalent to $\langle M, N\rangle = 0$. This is a simple way to prove that strong orthogonality implies orthogonality; the converse is false: let us consider $M \in \mathcal{H}_0^2$ and Y a Bernoulli random variable (values ± 1 with probability $\frac{1}{2}$), independent of M. Let be $N = YM$.
Exercise: prove that M and N are orthogonal but no strongly orthogonal.

Let $\mathcal{A}$ be a subset of $\mathcal{H}_0^2$, denote $\mathcal{A}^\perp$ its orthogonal space, $\mathcal{A}^\dagger$ its strong orthogonal space.

Lemma 6.7. *Let $\mathcal{A}$ be a subset of $\mathcal{H}_0^2$, then $\mathcal{A}^\dagger$ is stable closed vector subspace.*

Proof: let M^n be a sequence in $\mathcal{A}^\dagger$, converging to M in $\mathcal{H}_0^2$, and let be $N \in \mathcal{A}$: $\forall n,\ M^nN$ is a uniformly integrable martingale. On another hand, $\forall t \geq 0$, using Cauchy-Schwartz inequality

$$E[|\langle M^n - M, N\rangle_t|^2] \leq E[\langle M^n - M\rangle_t]E[\langle N\rangle_t]$$

which goes to zero. Thus $\langle M^n, N\rangle_t \to \langle M, N\rangle_t$ in L^2. But $\forall n$ and $\forall t,\ \langle M^n, N\rangle_t = 0$, thus $\langle M, N\rangle_t = 0$ and M is orthogonal to N. •

Lemma 6.8. *Let M and N be two martingales in $\mathcal{H}_0^2$, the following are equivalent:*

(i) M and N strongly orthogonal, denoted as $M \dagger N$,
(ii) $S(M) \dagger N$
(iii) $S(M) \dagger S(N)$
(iv) $S(M) \perp N$
(v) $S(M) \perp S(N)$

Proof: exercise.

Theorem 6.9. *Let be $M^1, \cdots, M^n \in \mathcal{H}_0^2$ such that for $i \neq j,\ M^i \dagger M^j$. Then,*

$$S(M^1, \cdots, M^n) = \{\sum_{i=1}^n H^i M^i\ ; H^i \in \mathcal{L}^*(M^i) \cap L^2(d\mathbb{P} \otimes d\langle M^i\rangle)\}.$$

It means that, in this case, actually, the right hand is a closed vectorial subspace.
Proof: let us denote $\mathcal{I}$ the right hand. By construction and property (iv) , $\mathcal{I}$ is a stable space. Consider now the application:

$$\begin{array}{rcl} \oplus_i \mathcal{L}^*(M^i) \cap L^2(d\mathbb{P} \otimes d\langle M^i\rangle) & \longrightarrow & \mathcal{H}_0^2 \\ (H^i) & \longmapsto & \sum_{i=1}^n H^i.M^i \end{array}$$

We easily check that this is an isometry, using that for $i \neq j,\ M^i \dagger M^j$:

$$\| \sum_{i=1}^n H^i M^i \|_2^2 = \sum_{i=1}^n \| H^i M^i \|_2^2 = \sum_{i=1}^n E[\int_0^\infty |H_s^i|^2 d\langle M^i\rangle_s].$$

Thus the set $\mathcal{I}$, image of a closed set by an isometric application is a closed set so contains $S(M^1, \cdots, M^n)$.

Conversely, using Theorem 6.4 (iv), any stable closed set F which contains M^i contains too $H^i.M^i$ so $\mathcal{I} \subset \mathcal{F}$. •

Here, too, vector subspace is closed.

Proposition 6.10. *Let be $\mathcal{A} = (M^1, \cdots, M^n) \subset \mathcal{H}_0^2$ satisfying $M^i \dagger M^j, i \neq j$. If for any $N \in \mathcal{H}_0^2$ strongly orthogonal to $\mathcal{A}$ is null, then $\mathcal{A}$ has the predictable representation property.*

Proof: Theorem 6.9 proves that $S(\mathcal{A})$ is the set $\mathcal{I}$, defined above. Then let be $N \in \mathcal{A}^\dagger$. Using Lemma 6.8(ii),

$$N \in S(\mathcal{A})^\dagger = \mathcal{I}^\dagger.$$

Hypothesis theorem tells us that N is null, meaning $\mathcal{I}^\dagger = \{0\}$, thus $\mathcal{I} = \mathcal{H}_0^2$. •

These orthogonality and representation properties are related to underlying probability measure.

Definition 6.11. *$Q \in \mathcal{M}(\mathcal{A})$ is said to be* **extremal** *if*

$$Q = aQ_1 + (1-a)Q_2, a \in [0,1], Q_i \in \mathcal{M}(\mathcal{A}) \Rightarrow a = 0 \text{ ou } 1.$$

Next theorem is a necessary condition for PRP (predictable representation property).

Theorem 6.12. *Let be $\mathcal{A} \subset \mathcal{H}_0^2(\mathbb{P})$. $S(\mathcal{A}) = \mathcal{H}_0^2(\mathbb{P})$ yields that $\mathbb{P}$ is extremal in $\mathcal{M}(\mathcal{A})$.*

Proof : cf. Th. 37 page 152 [30].
We assume that $\mathbb{P}$ is not extremal so could be written as $aQ_1 + (1-a)Q_2$ with $Q_i \in \mathcal{M}(\mathcal{A})$. Probability measure $Q_1 \leq \frac{1}{a}\mathbb{P}$, so admits a density Z with respect to $\mathbb{P}$, such that $Z_t \leq \frac{1}{a}$ and $Z_. - Z_0 \in \mathcal{H}_0^2(\mathbb{P})$. Remark that $\mathbb{P}$ and Q_1 coincide on $\mathcal{F}_0$ implies $Z_0 = 1$. Let be $X \in \mathcal{A}$: so it is a $\mathbb{P}$ and Q_1-martingale thus ZX is a $\mathbb{P}$-martingale and also $(Z - Z_0)X = (Z-1)X$ is a $\mathbb{P}$-martingale; this proves that $Z - Z_0$ is orthogonal to any X, so to $\mathcal{A}$, so to $S(\mathcal{A})$. This set being $\mathcal{H}_0^2(\mathbb{P})$, $Z - 1 = 0$ and $P = Q_1$ is extremal. •

Proposition 6.13. *Let be $\mathcal{A} \subset \mathcal{H}_0^2(\mathbb{P})$ and $\mathbb{P}$ extremal in $\mathcal{M}(\mathcal{A})$. If $M \in \mathcal{M}_b^c(\mathbb{P}) \cap \mathcal{A}^\dagger$ then M is null.*

Proof: Let c be a bound of the bounded martingale M and we assume M is not identically null. Thus we can define

$$dQ = (1 - \frac{M_\infty}{2c})d\mathbb{P} \text{ et } dR = (1 + \frac{M_\infty}{2c})d\mathbb{P}.$$

Then $\mathbb{P} = \frac{1}{2}(Q+R)$, Q and R are absolutely continuous with respect to $\mathbb{P}$ and equal $\mathbb{P}$ on $\mathcal{F}_0$ since $M_0 = 0$. Let be $X \in \mathcal{A} \subset \mathcal{H}_0^2(\mathbb{P})$: using Proposition 5.4, $X \in \mathcal{H}_0^2(Q)$ if and only if $(1 - \frac{M_t}{2c})X_t \in \mathcal{H}_0^2(\mathbb{P})$. But $X \dagger M$ so actually this property is true and as well $X \in \mathcal{H}_0^2(Q)$. Thus Q and $R, \in \mathcal{M}(\mathcal{A})$.
So it could exist a decomposition of $\mathbb{P}$, and this contradicts the hypothesis: M is necessarily null. •

The following is now a sufficient condition for PRP (predictable representation property).

Theorem 6.14. *Let be $\mathcal{A} = (M^1, \cdots, M^n) \subset \mathcal{H}_0^2(\mathbb{P})$ satisfying $M^i \dagger M^j, i \neq j$. $\mathbb{P}$ is extremal in $\mathcal{M}(\mathcal{A})$ yields that $\mathcal{A}$ has the predictable representation property.*

Proof: Proposition 6.10 proves that it is enough to show that any $N \in \mathcal{H}_0^2(\mathbb{P}) \cap \mathcal{A}^\dagger$ is null. Let N be such a martingale and a sequence of stopping times $T_n = \inf\{t \leq 0\ ;\ |N_t| \geq n\}$. The stopped martingale N^{T_n} is bounded and belongs to $\mathcal{A}^\dagger$; $\mathbb{P}$ is extremal. Theorem 6.13 shows that N^{T_n} is null $\forall n$, so $N = 0$. •

6.2 Fondamental theorem

Theorem 6.15. *Let B be a n-dimensional Brownian motion on $(\Omega, \mathcal{F}_t^B, \mathbb{P})$. Then $\forall M \in \mathcal{M}_{loc}^{c,2}$, there exists $H^i \in \mathcal{L}(B^i), i = 1, \cdots, n$, such that:*

$$M_t = M_0 + \sum_{i=1}^{n} (H^i.B^i)_t.$$

Proof: exercise.
This is an application Theorem 6.14 to the component of Brownian motion, we prove that $\mathbb{P}$ is the unique element of $\mathcal{M}(B)$. We do as following: let be $Q \in \mathcal{M}(B)$ and the martingale $Z = E[\frac{dQ}{dP}/\mathcal{F}_.]$ which is a function g of B_t^i since B is a Markov process; B is both $\mathbb{P}$ and Q-martingale; Girsanov theorem implies that ZB is a $\mathbb{P}$ martingale, so the bracket $\langle Z, B\rangle = 0$ and Itô formula proves $g = 1$, meaning $\mathbb{P} = Q$.
Use that $Z_t = E[\frac{dQ}{d\mathbb{P}}/\mathcal{F}_t^B]$ is a measurable function of vector $(B_t^1, \cdots, B_t^n)$.

Then we localize martingale M.

Corollary 6.16. *Under the same hypotheses, let be $Z \in L^1(\mathcal{F}_\infty, \mathbb{P})$, then there exists $H^i \in \mathcal{L}(B^i), i = 1, \cdots, n$, such that:*

$$Z = E[Z] + \sum_{i=1}^{n} (H^i.B^i)_\infty.$$

Proof: apply Theorem 6.15 to the martingale $M_t = E[Z/\mathcal{F}_t]$ and do t going to infinity.
•

Let be $\mathbb{P}$ and Q two equivalent probability measures and denote Z the $\mathbb{P}-$integrable variable $\frac{dQ}{d\mathbb{P}} > 0, E_\mathbb{P}[Z] = 1$. The martingale $Z_t = E_\mathbb{P}[Z/\mathcal{F}_t] > 0$ could be "represented" as a Brownian martingale: there exists $\psi \in \mathcal{L}(B)$ such that $dZ_t = \psi_t dB_t$.
This is an exponential martingale: indeed, since $Z_t > 0$, there exists a process $\phi = Z^{-1}\psi$ such that $dZ_t = Z_t\phi_t dB_t$.

This is important in case of Ito formula use, computation of bracket, etc.

Warning! in case of a vector martingale M, its components not being strongly orthogonal, the set $\mathcal{L}(M)$ contains the set $\{H = (H^i),\ \forall i\ \ H^i \in \mathcal{L}(M^i)\}$ but they aren't equal: $H \in \mathcal{L}(M) \Leftrightarrow \forall t,\ \int_0^t \sum_{i,j} H_s^i H_s^j d\langle M^i, M^j\rangle_s < \infty$.

6.3 Martingale problem

(cf. Jacod [19], pages 337-340).
In case of Finance, it is the following problem: let be a set of price processes with dynamics modeled by a family of adapted continuous processes on the filtered probability space $(\Omega, \mathcal{B}, \mathcal{F}_t, \mathbb{P})$, actually semi martingales. Does it exist a probability measure Q such that this family could be a subset of $\mathcal{M}^c_{loc}(Q)$? This is a martingale problem. We assume that $\mathcal{B} = \mathcal{F}_\infty$.

In this subsection we consider a larger set of martingales:

$$\mathcal{H}^1(\mathbb{P}) = \{M \in \mathcal{M}^c_{loc}(\mathbb{P}) \ ; \ \sup_t |M_t| \in L^1\}.$$

This definition is equivalent to:

$$\mathcal{H}^1(\mathbb{P}) = \{M \in \mathcal{M}^c_{loc}(\mathbb{P}) \ ; \ \langle M\rangle_\infty^{\frac{1}{2}} \in L^1\}$$

using Burkholder inequality:

$$\| \sup_t |M_t| \|_q \leq c_q \| \langle M \rangle^{\frac{1}{2}} \|_q \leq C_q \| \sup_t |M_t| \|_q.$$

Definition 6.17. *Let $\mathcal{X}$ be a family of adapted continuous processes on $(\Omega, \mathcal{B}, \mathcal{F}_t)$. We call solution of the* **martingale problem related to** $\mathcal{X}$ *any probability $\mathbb{P}$ such that $\mathcal{X} \subset \mathcal{M}^c_{loc}(\mathbb{P})$. We note $M(\mathcal{X})$ this set of probability measures and we recall that $S(\mathcal{X})$ is the smallest stable subset of $\mathcal{H}^1(\mathbb{P})$ containing $\{H.M, H \in \mathcal{L}^*(M), M \in \mathcal{X}\}$.*

Proposition 6.18. *$M(\mathcal{X})$ is convex.*

Proof: exercise.

We note $\mathbf{M_e}(\mathcal{X})$ the extremal elements of this set.

Theorem 6.19. *(cf. th. 11.2 [19] page 338.)*
Let be $\mathbb{P} \in M(\mathcal{X})$; the followings are equivalent:

(i) $\mathbb{P} \in M_e(\mathcal{X})$
(ii) $\mathcal{H}^1(\mathbb{P}) = S(\mathcal{X} \cup \{1\})$ *and* $\mathcal{F}_0 = (\emptyset, \Omega)$
(iii) $\forall N \in \mathcal{M}_b(\mathbb{P}) \cap \mathcal{X}^\dagger$ *such that* $\langle N \rangle$ *is bounded* $, N = 0$ *and* $\mathcal{F}_0 = (\emptyset, \Omega)$.

Remark 6.20. *Property (ii) exactly means that a market generated by a set of prices processes $\mathcal{X}$ is complete. It has the representation property.*

Corollary 6.21. *If moreover $\mathcal{X}$ is finite, or containing uniquely almost sure continuous processes, (i) (ii) (iii) are equivalent to*

$$(iv) \{Q \in M(\mathcal{X}), \ Q \sim \mathbb{P}\} = \{\mathbb{P}\}.$$

Proof:
(ii)⇒(iii) let M be a bounded, $M_0 = 0$, strongly orthogonal to any element of $\mathcal{X}$ meaning $\langle M, X\rangle = 0,\ \forall X \in \mathcal{X}$.

Since by hypothesis $\mathcal{X} \cup \{1\}$ generate the set $\mathcal{H}^1(\mathbb{P})$, any $N \in \mathcal{H}^1(\mathbb{P})$ is limit of a sequence of processes as $N_0 + \sum_i H_i.X_i$. Thus,

$$\langle M, N\rangle_t = \lim M_0 N_0 + \sum_i \langle M, H_i.X_i\rangle_t = \sum_i \int_0^t H_i d\langle M, X_i\rangle_s$$

which is null v on M, which so is orthogonal to any element of $\mathcal{H}^1(\mathbb{P})$. Moreover, M is bounded so belongs to $\mathcal{H}^1(\mathbb{P})$, thus it orthogonal to itself thus it is null.

(iii)⇒(ii) By the definition we get the inclusion $S(\mathcal{X} \cup \{1\}) \subset \mathcal{H}^1(\mathbb{P})$. But let us suppose that this inclusion is strict. Since $S(\mathcal{X} \cup \{1\})$ is a closed convex subset of $\mathcal{H}^1(\mathbb{P})$, there exists $M \in \mathcal{H}^1(\mathbb{P})$ orthogonal to $S(\mathcal{X} \cup \{1\})$. Particularly M is orthogonal to 1, thus $M_0 = 0$. Let $T_n = \inf\{t/|M_t| \geq n\}$ be the sequence of stopping times such that M^{T_n} is a bounded martingale, null in 0, orthogonal to $\mathcal{X}$: Hypothesis (iii) implies M is null and the equality of both sets is satisfied.

(i)⇒(iii) $\mathbb{P}$ is extremal in $M(\mathcal{X})$. Let Y be a bounded $\mathcal{F}_0$-measurable random variable and N' a bounded martingale, null in zero, orthogonal to $\mathcal{X}$. Set $N = Y - E[Y] + N'$ and remark that $\forall t \geq 0,\ E_{\mathbb{P}}(N_t) = 0$. Then set

$$a = \|N\|_\infty \ ;\ Z_1 = 1 + \frac{N}{2a} \ ;\ Z_2 = 1 - \frac{N}{2a}.$$

Obviously $E(Z_i) = 1, Z_i \geq \frac{1}{2} > 0$, so the measures $Q_i = Z_i\mathbb{P}$ are equivalent to $\mathbb{P}$ probability measures, their half-sum is $\mathbb{P}$.

Since Y is $\mathcal{F}_0$-measurable and N' is orthogonal to X $\forall X \in \mathcal{X}$, and NX is a $\mathbb{P}$- martingale. Thus $Z_i X = X \pm \frac{NX}{2a}$ is too a $\mathbb{P}$- martingale. Using Proposition 5.4, $X \in \mathcal{M}^c_{loc}(Q_i)$ and $Q_i \in M(\mathcal{X})$; this contradicts that $\mathbb{P}$ is extremal unless $N_t = 0, \forall t \geq 0$ meaning bot $Y = E[Y]$ and $N' = 0$. This concludes (iii).

(iii)⇒(i) Let us assume that $\mathbb{P}$ admits the decomposition in $M(\mathcal{X}):\ \mathbb{P} = aQ_1 + (1-a)Q_2$. So Q_1 is absolutely continuous with respect to $\mathbb{P}$ and the density Z exists, bounded by $\frac{1}{a}$, $E[Z] = 1$ and since $\mathcal{F}_0 = (0, \Omega)$, $Z_0 = 1$ almost surely: $Z - 1$ is a bounded null in zero martingale.

On another hand, $\forall X \in \mathcal{X}, X \in \mathcal{M}^c_{loc}(\mathbb{P}) \cap \mathcal{M}^c_{loc}(Q_1)$ since $\mathbb{P}$ and $Q_1 \in M(\mathcal{X})$. Once again, Proposition 5.4 proves that $ZX \in \mathcal{M}^c_{loc}(\mathbb{P})$ and $(Z-1)X \in \mathcal{M}^c_{loc}(\mathbb{P})$ meaning $Z-1$ is orthogonal to any X and Hypothesis (iii) proves $Z - 1 = 0$, meaning $Q_1 = \mathbb{P}$ which, so, is extremal.

(iv)⇒(iii) is proved as (i)⇒(iii), this proof doesn't need any property to $\mathcal{X}$.

(ii)$\Rightarrow$(iv) Let us assume that there exists $\mathbb{P}' \neq \mathbb{P}$ in $M(\mathcal{X})$, equivalent to $\mathbb{P}$. In case of finite $\mathcal{X}$, (ii) means (cf. Theorem 6.9):

$$\mathcal{H}^2(\mathbb{P}) = \{a + \sum_{i=1}^{n} H^i X^i \ ; \ a \in \mathbb{R}, \ H^i \in \mathcal{L}^*(X^i) \cap L^2(d\mathbb{P} \otimes d\langle X^i \rangle), \ X^i \in \mathcal{X}\}.$$

Let Z be the martingale density of $\mathbb{P}'$ with respect to $\mathbb{P}$: $\mathbb{P}' = Z\mathbb{P}$ where Z is a $\mathbb{P}$-martingale, expectation 1, equal to 1 at zero. Any X of $\mathcal{X}$ belongs to $\mathcal{M}^c_{loc}(\mathbb{P}) \cap \mathcal{M}^c_{loc}(\mathbb{P}')$, but Proposition 5.4 says that $ZX \in \mathcal{M}^c_{loc}(\mathbb{P})$, thus $(Z-1)X \in \mathcal{M}^c_{loc}(\mathbb{P})$, meaning that $Z-1$ is orthogonal to X so to $S(\mathcal{X} \cup \{1\}) = \mathcal{H}^1(\mathbb{P})$. Localizing, we bound this martingale, the stopped martingale is orthogonal to itself, thus null.

7 Finance application

The application is twofold: if there exists a probability Q, equivalent to the natural probability such that any price process is a Q-martingale, Q is said *risk neutral probability* (or "martingale measure"), then the market is said VIABLE, meaning that there exists no arbitrage (arbitrage is to win with a strictly positive probability starting with a null initial wealth meaning $V_T(\theta) \geq 0$ and $\mathbb{P}\{V_T(\theta) > 0\} > 0$.
RECIPROCAL is false, contrarily to what it is too often said or written.

When the set of these price processes, Q-martingales, has the representation property for Q-martingales, the market is said to be COMPLETE.

7.1 Research of a risk neutral probability measure

We assume that the share prices are $S^i, i = 1, ...n$, strictly positive semi martingales:

$$dS_t^i = S_t^i b_t^i dt + S_t^i \sum_j \sigma_j^i(t) dB_t^j.$$

Otherwise look at the equivalent probability $Q = \mathcal{E}(X.B)\mathbb{P} = Z\mathbb{P}$. Using Girsanov Theorem, $\forall j$:

$$\tilde{B}_t^j = B_t^j - \int_0^t X_s^j ds$$

is a Q-Brownian motion. So actually processes S^i are too Q-semi martingales as following:

$$dS_t^i = S_t^i (b_t^i + \sum_j \sigma_j^i(t) X_t^j) dt + S_t^i \sum_j \sigma_j^i(t) d\tilde{B}_t^j.$$

Thus the problem is now to find a vector X in $\mathcal{L}(B)$ satisfying (for instance) Novikov condition such that $\forall i = 1, ...n$ we get the system with n equations and d unknown:

$$b_t^i + \sum_j \sigma_j^i(t) X_t^j = 0.$$

Exercise: solve this system when $n = d = 1$, then $n = d$. What to do if $n \neq d$?

7.2 Application: to hedge an option

In case of a complete market, using representation Theorem, we can "hedge" an option.

Remember that an option is a financial asset based on a share price p but it is a right that can carry forward in two ways :

- call option with terminal value $(S_T - K)^+$,

- put option with terminal value $(K - S_T)^+$,

K being the exercise price of the option and T the maturity.
Concretely, at time 0 we buy

- the right to buy at price K even if the price S_T is over (call)

- or the right to sell at price K even if the price S_T is under (put).

But to find the "fair price" of this contract, the seller of the option could honor the contract, thus placing the sum obtained by selling the contract so he can (at least in average) pay the buyer at time T.

Definition 7.1. *We call the fair price of a contingent claim H the smallest $x \geq 0$ such that*
there exists a self-financing admissible strategy π which realizes at time T the value $g(S_T) = V_T(\pi)$, the discounted price being $e^{-rT}V_T(\pi) = H$, initial value being $V_0(\pi) = x$.

Recall: A self-financing strategy π is said to be **admissible** if its value

$$V_t(\pi) = V_0 + \int_0^t \pi_s.dS_s$$

is almost surely bounded below by a real constant.

For instance for the "call option", the claim is $H = g(S_T) = (S_T - K)^+$, and the seller of the contract looks for "hedging". Here are useful the "martingale representation" Theorems. If r is the discount (e.g. savings rate), $e^{-rT}g(S_T)$ is the discounted claim.

Let us assume that we are in 7.1 scheme with $n = d$, σ invertible and the market admitting a risk neutral probability measure on $\mathcal{F}_T$: $Q = \mathcal{E}_T(X.B)\mathbb{P}$. Using fundamental Theorem there exists a vector θ such that

$$e^{-rT}g(S_T) = E_Q[e^{-rT}g(S_T)] + \int_0^T \sum_j \theta_t^j d\tilde{B}_t^j. \tag{26}$$

But using Q-Brownian motion $\tilde{B}$ above, yields:

$$dS_t^i = S_t^i \sum_j \sigma_j^i(t) d\tilde{B}_t^j$$

so $\forall j$

$$d\tilde{B}_t^j = \sum_i (\sigma^{-1})_i^j(t)(S_t^i)^{-1} dS_t^i$$

to be replaced in (26):

$$e^{-rT}g(S_T) = E_Q[e^{-rT}g(S_T)] + \int_0^T \sum_{i,j} \theta_t^j (S_t^i)^{-1}(\sigma^{-1})_i^j(t) dS_t^i$$

which allows us to identify the hedging portfolio

$$\pi_t^i = (S_t^i)^{-1} \sum_j \theta_t^j (\sigma^{-1})_i^j(t)$$

and finally the fair price is:

$$q = E_Q[e^{-rT}g(S_T)].$$

8 Financial model, continuous time, continuous prices

(Cf. [9] chap 12.1 to 12.5, [20] Section 5.8, pages 371 et sq.)
Here are assumed AOA hypothesis (cf. Definition 4.1): thus the price processes are semi martingales.

8.1 Model

We consider finite horizon $t \in [0, T]$, the market is denoted as S with $n+1$ assets, the prices of which being continuous semi martingales. Real quantities of these assets could be bought or sold, there is neither trade nor transaction costs. The semi martingales are continuous, build on Wiener space, filtered probability space: $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F}_t)$, on which is defined a n-dimensional Brownian motion, B. Moreover we assume $\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_T = \mathcal{A}$.

Hypothesis on market S: the first assets is risk less, constant rate, namely the "bond", $S_t^0 = e^{rt}$ thus:

$$dS_t^0 = S_t^0 r dt,\ r > 0,\ S_0^0 = 1.$$

The n risky assets on the market are supposed to be strictly positive semi martingales satisfying: $\forall i = 1, ..., n$, there exists a semi martingale X^i such that :

$$S_t^i = \mathcal{E}_t(X^i),\ t \in [0, T].$$

Concretely,

$$dX_t^i = \sum_j (\sigma_j^i(t) dB_t^j + b^i(t)dt), i = 1, \cdots, n; dX_t^0 = rdt. \tag{27}$$

There is a perishable consumption good and there are I economic agents with access to information $\mathcal{F}_t$ on time t. For any $k = 1, \cdots, I$, the $k-$th agent has resources (endowments) $e_0^k \in \mathbb{R}^+$ on the beginning and $e_T^k \in L^1(\Omega, \mathcal{F}_T, \mathbb{P})$ at the end, he consumes $c_0^k \in \mathbb{R}$ on the beginning and $c_T^k \in L^1(\Omega, \mathcal{F}_T, \mathbb{P})$ at the end. He has no intermediary resources or consumption.

We denote X a subset of $\mathbb{R} \times \mathcal{A}(\Omega, \mathcal{F}_T, \mathbb{P})$, set of claims to reach, equipped with a complete, continuous, increasing, convex preference relation (that will be built later and is different from an order relation, it lacks the antisymmetry and transitivity).

Definition 8.1. *A preference relation (denoted as $\prec$) is said to be* **complete** *if for any c_1 and c_2 in X, it is either $c_1 \prec c_2$ or $c_2 \prec c_1$*
It is said to be **continuous** *if $\forall c \in X$, $\{c' \in X,\ c' \prec c\}$ and $\{c' \in X,\ c \prec c'\}$ are closed sets.*
It is said to be **increasing** *if all the coordinates of c' are greater or equal to those of c implies $c \prec c'$.*
It is said to be **convex** *if c' and $c" \prec c$ then $\forall \alpha \in [0, 1]$, $\alpha c' + (1-\alpha)c" \prec c$.*

8.2 Equilibrium price measure, or risk neutral probability measure

Definition 8.2. *Let $(S^0, \cdots, S^n)$ be a price system, an* **equilibrium price measure** *or risk neutral probability measure on $(\Omega, \mathcal{F}_t)$ is a probability Q, equivalent to $\mathbb{P}$, such the discounted prices $e^{-rt}S^i$, denoted $\tilde{S}^i$, are local Q-martingales.*

We note $\mathcal{Q}_S$ the set of such probability measures.
Remark that $\mathcal{Q}_S$ is included in the set $M(S)$, cf. Definition 6.17.
We now assume that $\mathcal{Q}_S$ **is non empty**, we choose $Q \in \mathcal{Q}_S$; it is not necessarily unique, but most of the results don't depend on the chosen element in $\mathcal{Q}_S$.
This hypothesis implies the absence of arbitrage opportunity (Definition 8.7 and Theorem 8.9 below). Once again, contrary to what we read too often it is not equivalent to it. This is a sufficient condition but not necessary for the absence of arbitrage. Instead, it is equivalent to a condition called NFLVR(cf. [7]).

Exercise: In this context, express the major hypothesis of the model (27), namely the existence of a equilibrium price measure Q , i.e. the discounted price processes $\tilde{S}^n$ are Q-martingales. Itô formula is a good tool to solve it.

$$(28) \qquad d\tilde{S}^i_t = e^{-rt}dS^i_t - rS^i_t e^{-rt}dt = \tilde{S}^i_t(dX^i_t - rdt) = \tilde{S}^i_t[\sum_j \sigma^i_j(t)dB^j_t + (b^i(t) - r)dt].$$

So the problem is to find Q, equivalent to $\mathbb{P}$, and a $Q-$Brownian motion $\tilde{B}$ such that $dX^i_t - rdt = \sigma_t d\tilde{B}_t$. Here we use Girsanov theorem denoting $Z_t = E_{\mathbb{P}}[\frac{d\mathbb{P}}{dQ}/\mathcal{F}_t]$ which could be expressed as a martingale, stochastic integral with respect to the d-dimensional Brownian motion B: there exists a vector process $X \in \mathcal{P}(B)$ such that $dZ_t = Z_t \sum_{j=1}^d X^j dB^j_t$.
To find risk neutral Q is equivalent to find X.
End the exercise by assuming for example that the matrix ${}^t\sigma.\sigma$ has rank d thus is invertible and there is a Novikov-type condition on the vector $v_. = ({}^t\sigma_.\sigma_.)^{-1} \times {}^t\sigma_.(b_. - r.\mathbf{1})$ where $\mathbf{1} = (1, \cdots, 1)$.
More generally, discuss the existence of risk-neutral probabilities depending on
whether $d = n, d < n, d > n$.

8.3 Trading strategies

Notation: below, $\langle x, y\rangle$ notes the scalar product between both vectors x and y, not to be confused with the stochastic bracket between two martingales or semi martingales!

A strategy is a portfolio θ, $\mathcal{F}$-adapted process taking its values in $\mathbb{R}^{n+1}$, θ^i representing the portion of the portfolio invested in the ith financial assets. The conditions to assume are those allowing the real process $\int\langle\theta_s, dS_s\rangle$ to be defined: θ has to be integrable on $[0, t]$, $\forall t$ respectively with respect to the martingale part and the finite variation part of the semi-martingale, discounted price process $\tilde{S}^i$. This quantity $\int_0^t\langle\theta_s, dS_s\rangle$ represents the gain from the exchange between 0 and t and $\int_0^t\langle\theta_s, d\tilde{S}_s\rangle$ represents the discounted gain from the exchange between 0 and t.

Definition 8.3. *An* **admissible strategy** *is an adapted process taking its values in* $\mathbb{R}^{N+1}$ *on* $(\Omega, \mathcal{F}_t, Q)$*, stochastically integrable (cf. Section 2) with respect to the price vector* S*.*

Definition 8.4. *A strategy is* **self-financing** *if moreover* $\forall t \in \mathbb{R}^+$ *the portfolio value satisfies:*

$$V_t(\theta) = \langle \theta_t, S_t \rangle = \langle \theta_0, S_0 \rangle + \int_0^t \langle \theta_s, dS_s \rangle.$$

Remark: This is interpreted as follows: there are no external resources, only the change of the portfolio is changing wealth.
This may be clearer in discrete time:

$$\begin{aligned} V_{t+1} - V_t &= \langle \theta_{t+1}, S_{t+1} \rangle - \langle \theta_t, S_t \rangle = \langle \theta_{t+1}, S_{t+1} - S_t \rangle \\ \text{is equivalent to} \quad & \langle \theta_{t+1}, S_t \rangle = \langle \theta_t, S_t \rangle. \end{aligned} \tag{29}$$

The portfolio is change between t and $t+1$ by internal reorganization between the assets.

This not an obligation but here we assumed that the price processes are stochastic exponentials, so that they are strictly positive.

Theorem 8.5. *Let* θ *be an admissible strategy. It is self-financing if and only if the discounted value of the portfolio* $\tilde{V}_t(\theta) = e^{-rt} V_t(\theta)$ *satisfies:*

$$\tilde{V}_t(\theta) = V_0(\theta) + \int_0^t \langle \theta_s, d\tilde{S}_s \rangle$$

where the scalar product is in $\mathbb{R}^n$ *instead of* $\mathbb{R}^{n+1}$ *since* $d\tilde{S}_s^0 = 0$.

Proof. : exercise, using Ito formula on the product $e^{-rt} \times V_t(\theta)$, then using (28).

Corollary 8.6. *let* Q *be an equilibrium price measure. For any* θ *self-financing strategy, element of* $\mathcal{P}(\tilde{S})$*, the discounted value of the portfolio is a local* $Q-$*martingale.*

Proof. : Exercise

Definition 8.7. θ *is said to be an* **arbitrage strategy** *if it is admissible, self-financing and satisfies one of these three properties:*

$$\begin{aligned} &\langle \theta_0, S_0 \rangle \leq 0 \text{ and } \langle \theta_T, S_T \rangle \geq 0 \text{ almost surely and } \neq 0 \text{ with probability } > 0, \\ &\langle \theta_0, S_0 \rangle < 0 \text{ and } \langle \theta_T, S_T \rangle \geq 0 \text{ almost surely,} \\ &\langle \theta_0, S_0 \rangle = 0 \text{ and } \langle \theta_T, S_T \rangle \geq 0 \text{ almost surely and } \neq 0 \text{ with probability } > 0. \end{aligned} \tag{30}$$

Proof: exercise, prove the equivalence of these three definitions.
For instance, $2 \Rightarrow 3$, if $\langle \theta_0, S_0 \rangle = a < 0$, we define a new strategy which satisfies the third property:

$$\theta'^i = \theta^i, i = 1, \cdots, n \; ; \; \theta'^0(t) = \theta^0(t) - ae^{-rt}, \forall t \in [0, T].$$

Then

$$\langle \theta'_0, S_0 \rangle = \theta'^0_0, S_0^0 + \sum_1^n \langle \theta_0^i, S_0^i \rangle = \langle \theta_0, S_0 \rangle - a = 0$$

and $\langle \theta'_T, S_T \rangle = \langle \theta_T, S_T \rangle - ae^{-rT} e^{rT} > \langle \theta_T, S_T \rangle \geq 0$. Thus, $\langle \theta'_T, S_T \rangle$ is positive, non null. •

Definition 8.8. *A market where there is no arbitrage strategy is said to be* **viable**. *We say that it satisfies the AOA hypothesis arbitrage opportunity absence).*

We now give some sufficient conditions to make a market S viable.

Theorem 8.9. *(cf. [9], 12.2 et sq.) If the set $\mathcal{Q}_S$ is non empty, then the market is viable.*

Proof. : Exercise with the following steps. *Let be $Q \in \mathcal{Q}_S$:*
1. If for any self-financing strategy θ, $\tilde{V}_t(\theta)$ is a $Q-$super martingale, then the market is viable.
2. If any self-financing strategy of $\mathcal{P}(\tilde{S})$ is such that $\tilde{V}_t(\theta) \geq 0$, then the market is viable.

1. The fact that $\tilde{V}_t(\theta)$ is a $Q-$super martingale could be written as:

$$\forall s \leq t,\ E_Q[\tilde{V}_t(\theta)/\mathcal{F}_s] \leq \tilde{V}_s(\theta).$$

Particularly, since the initial $\sigma-$algebra $\mathcal{F}_0$ is trivial, for $s = 0$,

$$E_Q[\tilde{V}_T(\theta)] \leq \tilde{V}_0(\theta) \text{ meaning } \langle \theta_0, S_0 \rangle.$$

Thus let us assume that there exists an arbitrage strategy: $\langle \theta_0, S_0 \rangle = 0, \langle \theta_T, S_T \rangle \geq 0$. Thus $E_Q[\tilde{V}_T(\theta)] \leq 0$ and since $\tilde{V}_T(\theta) = e^{-rT}\langle \theta_T, S_T \rangle \geq 0,\ \tilde{V}_T(\theta) = 0$, so strategy θ cannot be arbitrage strategy.

2. Since the strategy θ is self-financing,

$$\tilde{V}_t(\theta) = \langle \theta_0, S_0 \rangle + \int_0^t \langle \theta_s, d\tilde{S}_s \rangle.$$

Corollary 8.6 shows that $\tilde{V}_t(\theta)$ is a local $Q-$martingale moreover positive, thus it is a super martingale (cf. proof of Lemma 5.6) and we go back to (1) to conclude. •

As a conclusion, to avoid arbitrage, we add in the definition of admissible strategy θ the obligation to check

$$V_t(\theta) \geq 0,\ dt \otimes d\mathbb{P} \text{ almost surely }.$$

Remark 8.10. *We stress the sequence of implications: $\mathcal{Q}_S$ is non empty $\Rightarrow$ no arbitrage $\Rightarrow$ price processes are semi-martingale*
without however, having the reciprocal.....

8.4 Complete market

Here we use the tools introduced in Subsection 6.1. Let be $Q \in \mathcal{Q}_S$.

Definition 8.11. *A contingent claim $X \in L^1(\Omega, \mathcal{F}_T, Q)$ is* **simulable** *or* **attainable** *under probability measure Q if there exists a self-financing admissible strategy θ and a real number x such that*

$$X = \langle \theta_T, S_T \rangle = x + \int_0^T \theta_s . dS_s.$$

A market is said to be **complete** *under probability measure Q for the price system S is any $X \in L^1(\Omega, \mathcal{F}_T, Q)$ is simulable.*

In this subsection we look for a characterization of complete market, at least to exhibit some sufficient conditions for completeness.

Theorem 8.12. *A claim X is simulable if and only if there exists a vector process $\alpha \in \mathcal{P}(\tilde{S})$, N-dimensional such that:*

$$E_Q[X/\mathcal{F}_t] = e^{-rT} E_Q[X] + \int_0^t \langle \alpha_s, d\tilde{S}_s \rangle.$$

Proof:

If X is simulable, this means there exists a self-financing admissible strategy θ and a real number x such that $X = V_T(\theta) = \langle \theta_T, S_T \rangle = x + \int_0^T \langle \theta_s, dS_s \rangle$.
Since θ is admissible, by definition, it is stochasticaly integrable with respect to S so to $\tilde{S}$; it is self-financing meaning (cf. Theorem 8.5) $d\tilde{V}_t(\theta) = \langle \theta_t, d\tilde{S}_t \rangle$. But $X = \langle \theta_T, S_T \rangle$ or $\tilde{V}_T(\theta) = e^{-rT} X$ and finally $\tilde{V}(\theta)$ is a martingale:

$$\tilde{V}_t(\theta) = E_Q[\tilde{V}_T(\theta)/\mathcal{F}_t] = E_Q[\tilde{V}_T(\theta)] + \int_0^t \langle \theta_s, d\tilde{S}_s \rangle.$$

The first term is $e^{-rT} E_Q[X]$ and the process α is identified as the required process, the strategy θ on coordinates $1, \cdots, N$.

Conversely, if α exists, let us define the strategy

$$\theta^i = \alpha^i,\ i = 1, \cdots, n\ ;\ \theta^0 = e^{-rT} E_Q[c(T)] + \int_0^T \langle \alpha_s, d\tilde{S}_s \rangle - \sum_1^n \langle \theta_s^i, \tilde{S}_s^i \rangle.$$

We check that this strategy actually hedges the claim X, thus simulable, then that this strategy θ is actually self-financing. •

Let us admit the theorem:

Theorem 8.13. *Let Q be a risk neutral probability measure. If $\mathcal{F}_0 = \{\Omega, \emptyset\}$, the following are equivalent:*

(i) The market is complete with respect to price system $\{S\}$.

(ii) $\mathcal{Q}_S = \{Q\}$

Proof: Exercise, in case of N assets, semi martingales driven by a $d-$Brownian motion B:

$$dS_t^i = S_t^i b_t^i dt + S_t^i \sum_{j=1}^d \sigma_t^{ij} dB_t^j,\ i = 1, \cdots, n.$$

9 EXERCISES

We consider that we are on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t), \mathbb{P})$.
i(*) means exo i is difficult to solve but its result is useful.

9.1 Prerequisites: conditional expectation, stopping time

0. Recall Borel-Cantelli and Fatou lemmas.

1. Let $\mathcal{G}$ be a sub-σ algebra of $\mathcal{A}$ and an almost surely positive random variable X. Prove that the conditional expectation $E[X/\mathcal{G}]$ is also strictly positive.
Prove that the reciprocal is false given a contra-example (for instance use the trivial σ-algebra $\mathcal{G}$).

2. Let $\mathcal{G} \subset \mathcal{H} \subset \mathcal{A}$ and $X \in L^2(\Omega, \mathcal{A}, \mathbb{P})$. Prove (Pythagore Theorem):

$$E[(X - E[X/\mathcal{G}])^2] = E[(X - E[X/\mathcal{H}])^2] + E[(E[X/\mathcal{H}] - E[X/\mathcal{G}])^2].$$

3. Let O be an open sand in $\mathcal{A}$ and a $\mathcal{F}-$adapted continuous process X. One notes

$$T_0 = \inf\{t \ : \ X_t \in O\}.$$

Prove that T_O is a stopping time.

4. Let be stopping times S and T.
(i) Prove that $S \wedge T$ is a stopping time.
(ii) Prove

$$\mathcal{F}_{S \wedge T} = \mathcal{F}_S \cap \mathcal{F}_T.$$

5. Let be T a stopping time and $A \in \mathcal{A}$. Prove that

$$\begin{aligned} T_A &= T \quad \text{sur} \quad A, \\ &= +\infty \quad \text{sur} \quad A^c, \end{aligned}$$

is a stopping time if and only if $A \in \mathcal{F}_T$.

6. A real random variable X is $\mathcal{F}_T$ measurable if and only if $\forall t \geq 0$, $X\mathbf{1}_{T \leq t}$ is $\mathcal{F}_t$ measurable.

7. Let $X \in L^1$ and a family of σ-algebras $\mathcal{F}^\alpha, \alpha \in A$. Then the family of conditional expectations $\{E[X/\mathcal{F}^\alpha], \alpha \in A\}$ is uniformly integrable.

8. let X be a $\mathcal{F}$-progressively measurable process and T a $(\mathcal{F}_t)$ stopping time. Then
(i) the application $\omega \mapsto X_{T(\omega)}(\omega)$ is $\mathcal{F}_T$-measurable
(ii) the process $t \mapsto X_{t \wedge T}$ is $\mathcal{F}$-adapted.

9. If X is an adapted measurable process admitting càd or càg trajectories, it is progressively measurable.

9.2 Martingales

1. Let X be a martingale, φ a function such that $\forall t\ \varphi(X_t) \in L^1$.
(i) if φ is a convex function, then $\varphi(X)$ is a sub-martingale ; if φ is a concave function $\varphi(X)$is a super-martingale.
(ii) When X is a sub-martingale and φ an increasing convex function such that $\forall t\ \varphi(X_t) \in L^1$, then $\varphi(X)$ is a sub-martingale.

2. Martingale convergence: admit the following: let X be a càd super (or sub)-martingale such that $\sup_t E[|X_t|] < \infty$. Then $\lim_{t\to\infty} X_t$ exists almost surely and belongs to $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

And deduce the Corollary : if X is a càd bounded from below super-martingale, then $\lim_{t\to\infty} X_t$ exists almost surely and belongs to $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

3. let X be a martingale. Prove the following are equivalent:
(i) X is uniformly integrable.
(ii) X_t converges almost surely to Y (which belongs to L^1) when t goes to infinity and $\{X_t, t \in \overline{\mathbb{R}^+}\}$ is a martingale.
(iii) X_t converges to Y in L^1 when t goes to infinity.

Indication: (i) → (iii) → (ii) → (i)

4. let be $(X_t)_{t\geq 0}$ a positive right continuous upper-martingale and

$$T = \inf\{t > 0\ : X_t = 0\}.$$

(i) Prove that almost surely $\forall t \geq T,\ X_t = 0$. (First prove $\mathbf{E}(X_t \mathbf{1}_{T\leq t}) = 0$.)
(ii) Prove that almost surely $X_\infty = \lim_{t\to\infty} X_t$ exists. Deduce:

$$\{X_\infty > 0\} \subset \{\forall t, \quad X_t > 0\} = \{T = +\infty\}.$$

Give a contra-example using

$$\{X_\infty > 0\} \neq \{T = +\infty\}.$$

5. If $M \in \mathcal{M}_{loc}$ is such that $E[M_t^*] < \infty \forall t$, then M is a 'true' martingale.
Moreover suppose $E[M^*] < \infty$, then M is uniformly integrable.

6. If X is a closed martingale with Z, meaning Z is integrable and $\forall t,\ X_t = E[Z/\mathcal{F}_t]$, prove that it also closed with $\lim_{t\to\infty} X_t$ denoted as X_∞ equal to $E[Z/\vee_{t\geq 0}\mathcal{F}_t]$.

9.3 Brownian motion

1. Prove that the real Brownian motion is a centered continuous Gaussian process with covariance function $\rho(s,t) = s \wedge t$.
Conversely a centered continuous Gaussian process with covariance function $\rho(s,t) = s \wedge t$ is a real Brownian motion.

2. Prove that the Brownian motion is a martingale with respect to its proper filtration, i.e. $\mathcal{F}_t = \sigma(B_s, s \leq t)$.
Prove that it is also a Markov process.

3. Let be $\mathcal{G}_t = \sigma(B_s, s \leq t) \vee \mathcal{N}, t \geq 0$. Prove this filtration is càd, meaning $\mathcal{G}_{t^+} = \cap_{s>t}\mathcal{G}_s$.
Indication: use
1. the $\mathcal{G}_{t^+}$-conditional characteristic of the vector (B_u, B_z), $z, u > t$ is the limit of $\mathcal{G}_w$-conditional characteristic function of the vector (B_u, B_z), when w decreases to t,
2. this limit is equal to the $\mathcal{G}_t$-conditional characteristic of the vector (B_u, B_z), $z, u > t$,
3. thus for any integrable Y $E[Y/\mathcal{G}_{t^+}] = E[Y/\mathcal{G}_t]$. So any $\mathcal{G}_{t^+}$-measurable is $\mathcal{G}_t$-measurable and conclude.

4(*). Paley-Wiener-Zygmund' Theorem, 1933, *cf.pp. 110-111, Karatzas-Schreve.* For almost all $\omega \in \Omega$, the application $t \mapsto B_t(\omega)$ is not differentiable. More specifically, we have

$$\mathbb{P}\{\omega \in \Omega : \forall t, \ \overline{lim}_{h\to 0^+} \frac{(B_{t+h} - B_t)(\omega)}{h} = +\infty \text{ and } \underline{lim}_{h\to 0^+} \frac{(B_{t+h} - B_t)(\omega)}{h} = -\infty\} = 1.$$

6. Let be (B_t) a real Brownian motion.
a) Prove that the sequence $\frac{B_n}{n}$ goes to 0 almost surely.
b) Use that B is a martingale and a Doob inequality (cf. Theorem 0.30 page 8 Lecture Notes) to deduce the majoration

$$E[\sup_{\sigma \leq t \leq \tau} (\frac{B_t}{t})^2] \leq \frac{4\tau}{\sigma^2}.$$

c) Let be $\tau = 2\sigma = 2^{n+1}$, give a bound for $\mathbb{P}\{\sup_{2^n \leq t \leq 2^{n+1}} |\frac{B_t}{t}| > \varepsilon\}$ that proves the convergence of this sequence, then apply Borel Cantelli lemma.
d) Deduce $\lim_{t\to\infty} \frac{B_t}{t} = 0$ almost surely. (meaning the large numbers law, cf. problem 9.3, correction pages 124-125, in Karatzas-Schreve.)

7. Let be $Y_t = t.B_{1/t}$; $Y_0 = 0$ and $\mathcal{F}_t^Y$ the natural filtration associated to the process Y. Prove that $(Y_t, \mathcal{F}_t^Y)$ is a Brownian motion (use the criterium in 1 and exercise 6 above).

9.4 Stochastic integral

In this section and the following let be M square integrable martingale on the filtered probability space $(\Omega, \mathcal{F}_t, P)$ such that $d\langle M\rangle_t$ is absolutely continuous w.r.t. Lebesgue measure dt: $\exists\, f$ measurable positive function on [0,T] s.t. $d\langle M\rangle_t = f(t)dt$.
1. Let be $\mathcal{L}_T(M)$ the set of adapted processes X on $[0,T]$ such that:

$$[X]_T^2 = E[\int_0^T X_s^2 d < M >_s] < +\infty.$$

Prove that $\mathcal{L}_T(M)$ is a metric space w.r.t. the distance d: $d(X,Y) = \sqrt{[X-Y]_T^2}$.
Actually it is a semi-norm which defines an equivalence relation $X \sim Y$ *if* $d(X,Y) = 0$.
2. Prove the equivalence

$$\sum_{j\geq 1} 2^{-j} \inf(1, [X - X_n]_j) \to 0 \iff \forall T,\ [X - X_n]_T \to 0.$$

3. Let be $\mathcal{S}$ the set of simple processes for which is defined the stochastic integral w.r.t. M :

$$I_t(X) = \sum_{j=0}^{J-1} X_j(M_{t_{j+1}} - M_{t_j}) + X_J(M_t - M_{t_J}) \text{ on the event } \{t_J \leq t \leq t_{J+1}\}.$$

Prove that I_t satisfies the following:

(i) I_t is a linear application on $\mathcal{S}$.

(ii) $I_t(X)$ is $\mathcal{F}_t$-measurable and square integrable.

(iii) $E[I_t(X)] = 0$.

(iv) $I_t(X)$ is a continuous martingale.

(v) $E[(I_t(X) - I_s(X))^2/\mathcal{F}_s] = E[I_t^2(X) - I_s^2(X)/\mathcal{F}_s] = E[\int_s^t X_u^2 d < M >_u /\mathcal{F}_s]$.

(vi) $E[I_t(X)]^2 = E[\int_0^t X_s^2 d < M >_s] = [X]_t^2$.

(vii) $< I_.(X) >_t = \int_0^t X_s^2 d < M >_s$.

Indication: actually, (vi) and (vii) are consequence of (v).

4. Prove Proposition 2.18: Let M and N be two square integrable continuous martingales, $X \in \mathcal{L}^*(M)$ and $Y \in \mathcal{L}^*(N)$. Then:

$$\langle X.M, Y.N\rangle_t = \int_0^t X_u Y_u d\langle M, N\rangle_u,\ \forall t \in \mathbb{R},\ \mathbb{P}\ a.s. \tag{31}$$

and

$$E[\int_s^t X_u dM_u \int_s^t Y_u dN_u/\mathcal{F}_s] = E[\int_s^t X_u Y_u d\langle M, N\rangle_u/\mathcal{F}_s],\ \forall s \leq t,\ \mathbb{P}\ a.s. \tag{32}$$

5. Prove that stochastic integral is associative, meaning: if H is stochastically integrable w.r.t. the martingale M, giving the integral $H.M$, and if G is stochastically

integrable w.r.t. the martingale $H.M$, then $G.H$ is stochastically integrable w.r.t. the martingale M and:

$$G.(H.M) = (G.H).M.$$

6. Let be M a continuous martingale and $X \in \mathcal{L}(M)$. let be $s < t$ and Z a $\mathcal{F}_s$-measurable bounded random variable. Compute $E[\int_s^t ZX_u dM_u - Z\int_s^t X_u dM_u]^2$ and prove:

$$\int_s^t ZX_u dM_u = Z\int_s^t X_u dM_u.$$

6. Let be T a stopping time, two processes X and Y such that $X^T = Y^T$, two martingales M and N such that $M^T = N^T$. Suppose $X \in \mathcal{L}(M)$ and $Y \in \mathcal{L}(N)$. Prove that $I_M(X)^T = I_N(Y)^T$.
(Use that for any square integrable martingale: $M_t = 0\ a.s. \Longleftrightarrow < M >_t = 0\ a.s.$)

7. Let M and N square integrable continuous martingales, and processes $X \in \mathcal{L}_\infty(M)$, $Y \in \mathcal{L}_\infty(N)$. Prove that
(i) $X.M$ and $Y.N$ are uniformly integrable, with terminal value $\int_0^\infty X_s dM_s$ and $\int_0^\infty Y_s dN_s$.
(ii) $\lim_{t\to\infty}\langle X.M, Y.N\rangle_t$ exists almost surely.
This is a direct application of Kunita-Watanabe's inequality.
(iii)$E[X.M_\infty Y.N_\infty] = E[\int_0^\infty X_s Y_s d\langle M, N\rangle_s]$.

Use the following theorem: if M is a continuous local martingale such that $E[\langle M\rangle_\infty] < \infty$, then it is uniformly integrable and converges almost surely when $t \to \infty$. Moreover $E[\langle M\rangle_\infty] = E[M_\infty^2]$.

8. Let be M and N two local continuous martingales and real numbers a and b, $X \in \mathcal{L}_\infty(M) \cap \mathcal{L}_\infty(N)$. Prove that the stochastic integration with respect to the local continuous martingales is a linear application, meaning $X.(aM + bN) = aX.M + bX.N$

9. Stratonovitch integral is defined as:

$$\int_0^t Y_s \circ dX_s = \int_0^t Y_s \circ dX_s + \frac{1}{2}\langle Y, X\rangle_t.$$

Let be $\varepsilon = \frac{1}{2}$. Prove that:

$$\lim_{\|\pi\|\to 0} S_\varepsilon(\Pi) = \sum_{i=0}^{m-1}[(1-\varepsilon)W_{t_i} + \varepsilon W_{t_{i+1}}](W_{t_{i+1}} - W_{t_i}) = \int_0^t W_s \circ dW_s = \frac{1}{2}W_t^2$$

where $\|\pi\| = \sup_i(t_{i+1} - t_i)$.

Let be X and Y two continuous semi-martingales,and π a partition [0,t]. Prove that

$$\lim_{\|\pi\|\to 0}\sum_{i=0}^{m-1}\frac{1}{2}(Y_{t_{i+1}} + Y_{t_i})(X_{t_{i+1}} - X_{t_i}) = \int_0^t Y_s \circ dX_s.$$

Let be X a d-dimensional vector of continuous semi-martingales, and f a C^2 function. Prove that:

$$f(X_t) - f(X_0) = \int_0^t \frac{\partial f}{\partial x_i}(X_s) \circ dX_s^i.$$

9.5 Itô formula

1. The quadratic co-variation of two continuous square integrable semi martingales X and Y is the limit in probability, when $\sup_i |t_{i+1} - t_i| \to 0$ of:

$$\langle X, Y \rangle_t = \lim_{proba} \sum_{i=1}^{n} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}).$$

Prove this co-variation is null when X is a continuous semi-martingale and Y a finite variation process.
2. **Lévy Theorem** : Let be X a continuous (semi-)martingale, $X_0 = 0$ almost surely. X is a real Brownian motion if and only if X is a continuous local martingale s.t. $\langle X \rangle_t = t$. First step: compute the $\mathcal{F}_s$-conditional characteristic function of $X_t - X_s$ using Itô formula, $\forall\ s \le t$.
3. Prove that the unique solution in $\mathcal{C}_b^{1,2}(R^+, R^d)$ of the partial differential equation (heat equation)

$$\frac{\partial f}{\partial t} = \frac{1}{2}\Delta f, f(0, x) = \varphi(x),\ \forall x \in R^d$$

where $\varphi \in \mathcal{C}_b^2(R^d)$ is $f(t, x) = E[\varphi(x + B_t)]$, B $d-$dimensional Brownian motion.
could we avoid boundedness of f and ϕ ?

4. *Long and tedious proof...* Let be M a d-dimensional vector of continuous martingales, A an adapted continuous d-dimensional vector with with finite variation, X_0 a $\mathcal{F}_0$-measurable random variable; let be $f \in C^{1,2}(\mathbb{R}^+, \mathbb{R}^d)$ and $X_t := X_0 + M_t + A_t$. Prove that $\mathbb{P}$ almost surely:

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)ds + \int_0^t \sum_i \frac{\partial f}{\partial x_i}(s, X_s)dM_s^i + \int_0^t \sum_i \frac{\partial f}{\partial x_i}(s, X_s)dA_s^i \\ &\quad + \frac{1}{2}\int_0^t \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s)d\langle M^i, M^j \rangle_s \end{aligned}$$

5. a)Use exercise 4 with two semi-martingales $X = X_0 + M + A$ and $Y = Y_0 + N + C$. Prove that $\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \langle X, Y \rangle_t$.
This the **integral by part formula**.

b) Use Ito formula to get the stochastic differential of the processes

$$t \mapsto X_t^{-1}\ ;\ t \mapsto \exp(X_t)\ ;\ t \mapsto X_t.Y_t^{-1}.$$

6. Prove that

$$(\exp \int_0^t a_s ds)(x + \int_0^t b_s \exp(-\int_0^t a_u du)dB_s)$$

is solution to the SDE

$$dX_t = a(t)X_t dt + b(t)dB_t,\ t \in [0, T],\ X_0 = x,$$

after justification of any integral in the formula.

9.6 Stochastic differential equations

1. Prove that the process $t \mapsto (\exp \int_0^t a_s ds)(x + \int_0^t b_s \exp(-\int_0^s a_u du) dB_s)$ is solution to the SDE $dX_t = a(t)X_t dt + b(t)dB_t,\ t \in [0,T],\ X_0 = x$, after justification of any integral in the formula. (meaning specify useful hypotheses on parameters a and b.

2. Let be B a real Brownian motion. Prove that $B_t^2 = 2\int_0^t B_s dB_s + t$.
If $\forall t\ X \in \mathcal{L}_t(B)$, then:

$$(X.B)_t^2 = 2\int_0^t (X.B)_s X_s dB_s + \int_0^t X_s^2 ds.$$

Let be $Z_t = \exp((X.B)_t - \frac{1}{2}\int_0^t X_s^2 ds)$. Prove that Z is solution to the SDE:

$$Z_t = 1 + \int_0^t Z_s X_s dB_s.$$

Prove that $Y = Z^{-1}$ is solution to the SDE:

$$dY_t = Y_t(X_t^2 dt - X_t dB_t).$$

Prove that there exists a unique solution to the SDE $dX_t = X_t b_t dt + X_t \sigma_t dB_t,\ X_t = x \in \mathbb{R}$ when $b, \sigma^2 \in L^1(\mathbb{R}^+)$, computing the stochastic differential of two solutions ratio.
3. Let be Ornstein Uhlenbeck stochastic differential equation:

$$dX_t = -\alpha X_t dt + \sigma dB_t,\ X_0 = x,$$

where $x \in L^1(\mathcal{F}_0)$.
(i) Prove that the following is the solution of this SDE:

$$X_t = e^{-\alpha t}(x + \int_0^t \sigma e^{\alpha s} dB_s).$$

(ii) Prove that the expectation $m(t) = E[X_t]$ is solution of an ordinary differential equation which is obtained by integration of $X_t = x - \alpha \int_0^t X_s ds + \sigma B_t$. Deduce $m(t) = m(0)e^{-\alpha t}$.
(iii) Prove the covariance

$$V(t) = Var[X_t] = \frac{\sigma^2}{2\alpha} + (V(0) - \frac{\sigma^2}{2\alpha})e^{-2\alpha t}.$$

(iv) Let be x a $\mathcal{F}_0$-measurable variable, with law $\mathcal{N}(0, \frac{\sigma^2}{2\alpha})$, Prove that X is a Gaussian process with co-variance function $\rho(s,t) = \frac{\sigma^2}{2\alpha} e^{-\alpha|t-s|}$.

9.7 Black-Scholes Model

to do with Man next week to prepare the terminal term test!

1.Assume that a risky asset price process is solution to the SDE

$$dS_t = S_t b dt + S_t \sigma dW_t, S_o = s, \tag{33}$$

b is named "trend' and σ "volatility". Prove that (33) admits a unique solution, using Ito formula to compute the ratio $\frac{S^1}{S^2}$ with $S^i, i = 1, 2$ two solutions to the SDE.

2. Assume that the portfolio θ value $V_t(\theta)$ is such that there exists a $C^{1,2}$ regular function C satisfying

$$V_t(\theta) = C(t, S_t). \tag{34}$$

Otherwise, θ is the pair (a, d) and

$$V_t(\theta) = a_t S_t^0 + d_t S_t = V_0(\theta) + \langle \theta_0, p_0 \rangle + \int_0^t a_s dS_s^0 + \int_0^t d_s dS_s. \tag{35}$$

With this "self-financing" strategy θ the option seller (for instance option $(S_T - K)^+$) could "hedge" the option with the initial price $q = V_0$: $V_T(\theta) = C(T, S_T)$.
Use the two different expressions of stochastic differential of $V_t(\theta)$, meaning starting with $\langle \theta_0, p_0 \rangle + \int_0^t a_s dS_s^0 + \int_0^t d_s dS_s$ or with $V_T(\theta) = C(T, S_T)$, to get a PDE (partial differential equation) the solution of which will be the researched function C.

3. Actually this PDE is solved using the change of (variable,function) :

$$x = e^y, y \in \mathbb{R} \; ; \; D(t, y) = C(t, e^y).$$

Thus, prove that we turn to the Dirichlet problem

$$\begin{aligned}
&\frac{\partial D}{\partial t}(t, y) + r\frac{\partial D}{\partial y}(t, y) + \frac{1}{2}\frac{\partial^2 D}{\partial y^2}(t, y)\sigma^2 = rD(t, y), y \in \mathbb{R}, \\
&D(T, y) = (e^y - K)^+, y \in \mathbb{R}.
\end{aligned}$$

Now let be the SDE:

$$dX_s = rds + \sigma dW_s, s \in [t, T], X_t = y.$$

Deduce the solution

$$D(t, y) = E_y[e^{-r(T-t)}(e^{X_T} - K)^+],$$

and the explicit formula, "Black-Scholes" formula, which uses the fact that the law of X_T is a Gaussian law.

9.8 Change of probability measures, Girsanov theorem

1. Let be the probability measure Q equivalent to $\mathbb{P}$ defined as $Q = Z.P$, $Z \in L^1(\Omega, \mathcal{F}_T, \mathbb{P})$ meaning $Q_{|}\mathcal{F}_t = Z_t.P,\ Z_t = E_P[Z/\mathcal{F}_t]$.
Prove that $\forall t$ and $\forall Y \in L^\infty(\Omega, \mathcal{F}_t, P), E_P[YZ_t/\mathcal{F}_s] = Z_s E_Q[Y/\mathcal{F}_s]$.
Indication: compute $\forall A \in \mathcal{F}_s$, the expectations $E_P[1_A Y Z_t]$ and $E_P[1_A Z_s E_Q[Y/\mathcal{F}_s]]$.
2. Let be $T \geq 0$, $Z \in \mathcal{M}(\mathbb{P})$ and $Q = Z_T \mathbb{P}$, $0 \leq s \leq t \leq T$ and a $\mathcal{F}_t-$measurable random variable $Y \in L^1(Q)$. Prove (Bayes formula)

$$E_Q(Y/\mathcal{F}_s) = \frac{E_{\mathbb{P}}(YZ_t/\mathcal{F}_s)}{Z_s}.$$

3. Let be M a $\mathbb{P}$-martingale, $X \in \mathcal{L}(B)$ such that $Z = \mathcal{E}(X.B)$ is a $\mathbb{P}$-martingale (remember: $dZ_t = Z_t X_t dB_t,\ Z_0 = 1$). Let be $Q := Z_T \mathbb{P}$ an equivalent probability measure to $\mathbb{P}$ on σ-algebra $\mathcal{F}_T$.

(i) Prove that $d\langle M, Z\rangle = ZXd\langle M, B\rangle$.

(ii) Use Itô formula to develop $M_t Z_t - M_s Z_s$, compute $E_{\mathbb{P}}[M_t Z_t/\mathcal{F}_s]$.

(iii) Use Itô formula between s and t to process $Z_. \int_0^. X_u d\langle M, B\rangle_u$.

(iv) Deduce $M_. - \int_0^. X_u d\langle M, B\rangle_u$ is a $Q-$martingale.

4. The following is a contra-example when Novikov condition is not satisfied: let be the stopping time $T = \inf\{1 \geq t \geq 0, t + B_t^2 = 1\}$ and

$$X_t = -\frac{2}{(1-t)^2} B_t 1_{\{t \leq T\}}\ ;\ 0 \leq t < 1,\ X_1 = 0.$$

(i) Prove that $T < 1$ almost surely, so $\int_0^1 X_t^2 dt < \infty$ almost surely.

(ii) Apply Itô formula to the process$t \to \frac{B_t^2}{(1-t)^2}\ ;\ 0 \leq t < 1$ to prove:

$$\int_0^1 X_t dB_t - \frac{1}{2}\int_0^1 X_t^2 dt = -1 - 2\int_0^T \frac{t}{(1-t)^4} B_t^2 dt < -1.$$

(iii) The local martingale $\mathcal{E}(X.B)$ is not a martingale: we deduce from (ii) that $E[\mathcal{E}_t(X.B)] \leq \exp(-1) < 1$ and this fact contradicts that for any martingale $E(M_t) = M_0$, here it could be 1.... Anyway, prove that $\forall n \geq 1$ and $\sigma_n = 1 - (1/\sqrt{n})$, the stopped process $E(X.B)^{\sigma_n}$ is a martingale.

9.9 Representation theorems, martingale problem

Recall:
$\mathcal{H}_0^2 = \{M \in \mathcal{M}^{2,c}, M_0 = 0, \langle M \rangle_\infty \in L^1\}$,
M and N are said to be orthogonal if $E[M_\infty N_\infty] = 0$, noted $M \perp N$,
and strongly orthogonal if MN is a martingale, noted as $M \dagger N$.
Let be $\mathcal{A} \subset \mathcal{H}_0^2$: denote $S(\mathcal{A})$ the smallest stable closed vector subspace which contains $\mathcal{A}$.

1. Let be $M \in \mathcal{H}_0^2$ and Y a centered Bernoulli random variable independent on M. Let be $N := YM$. Prove $M \perp N$ but no $M \dagger N$.
2. Let be $\mathcal{M}(\mathcal{A})$ the set of probability measures Q on $\mathcal{F}_\infty$, $Q << \mathbb{P}$, $\mathbb{P}_{|\mathcal{F}_0} = Q_{|\mathcal{F}_0}$, and such that $\mathcal{A} \subset \mathcal{H}_0^2(Q)$. Prove that $\mathcal{M}(\mathcal{A})$ is convex.
Study carefully the difference between $\mathcal{M}(\mathcal{A})$ and $M(\mathcal{A})$ (cf. Def 6.1 and 6.17 in Lecture Notes).
3. Let be B a $n-$dimensional Brownian motion on $(\Omega, \mathcal{F}_t, \mathbb{P})$. Prove that $\forall M \in \mathcal{M}^{c,2}$, $\exists H^i \in \mathcal{P}(B^i), i = 1, \cdots, n$, such that:

$$M_t = M_0 + \sum_{i=1}^{n} (H^i.B^i)_t.$$

Indication: apply extremal probability measure theorem (th 6.14) to the set $M(\mathcal{B})$ (actually the singleton $\{\mathbb{P}\}$) when $\mathcal{B}$ is the set of Brownian motion.
4. Prove that the above vector process H is unique, meaning $\forall H'$ satisfying $M_t = M_0 + \sum_{i=1}^{n} (H'^i.B^i)_t$ is such that :

$$\int_0^t \sum_{i=1}^{n} |H_s'^i - H_s^i|^2 ds = 0 \text{ almost surely.}$$

9.10 Example: optimal strategy for a small investor

To do later to prepare the terminal term test....

Let be a set of price processes: $S_t^n = \mathcal{E}_t(X^n), t \in [0,T]$, with:

$$dX_t^n = \sum_{j=1}^{d} \sigma_j^n(t) dW_t^j + b^n(t)dt, n = 1, \cdots, N; dX_t^0 = r_t dt.$$

Suppose the matrix σ satisfies $dt \otimes d\mathbb{P}$ almost surely : $\sigma\sigma^* \geq \alpha I$, σ^* is the transpose matrix of σ and $\alpha > 0$. The coefficients b, σ, r are$\mathcal{F}-$adapted bounded $[0,T] \times \Omega$ processes.
1. Look for a condition so that the market is viable, meaning a condition such that there is no arbitrage opportunity.

(i) Prove that a market is viable as soon as there exists a risk neutral probability measure Q.

(ii) Propose some hypotheses on the above model, sufficient for the existence of Q.

(iii) Propose some hypotheses on the above model, sufficient for the market be complete, meaning any contingent claim is "attainable" (hedgible).

Start with case $N = d = 1$, then $N = d > 1$.

Remark: If $d < N$ and σ surjective, there is no uniqueness of vector u so that $\sigma dW + (b-r)dt = \sigma d\tilde{W}$. In this case, the market is not complete and the set $\mathcal{Q}_S$ is bijective with $\sigma^{-1}(r-b)$.

Recall: let be a set of price processes S, a **risk neutral probability measure** on$(\Omega, (\mathcal{F}_t))$ is a probability measure Q equivalent to $\mathbb{P}$ such that the discounted prices $e^{-rt}S^n$, denoted as $\tilde{S}^n$, are local Q-martingales; denote their set $\mathcal{Q}_S$.

2. Let be θ an admissible strategy. Prove it is self-financing if and only if the discounted portfolio value $\tilde{V}_t(S) = e^{-rt}V_t(S)$ satisfies:

$$\tilde{V}_t(S) = V_0(S) + \int_0^t < \theta_s, d\tilde{S}_s > .$$

Use Ito formula; then deduce that $(\tilde{V}_t(S))$ is a local Q-martingale $\forall Q \in \mathcal{Q}_S$.

3. Prove the equivalence between the three properties defining the self-financing admissible strategy θ as an **arbitrage strategy** :

$$\langle \theta_0, S_0 \rangle \leq 0 \text{ and } \langle \theta_T, S_T \rangle \geq 0 \text{ almost surely and } \neq 0 \text{ with probability } > 0,$$
$$\langle \theta_0, S_0 \rangle < 0 \text{ and } \langle \theta_T, S_T \rangle \geq 0 \text{ almost surely},$$
$$\langle \theta_0, S_0 \rangle = 0 \text{ and } \langle \theta_T, S_T \rangle \geq 0 \text{ almost surely and } \neq 0 \text{ with probability } > 0.$$

4. Prove Theorem 8.9: If the set $\mathcal{Q}_S$ is non empty, then the market is viable.

5. A sufficient and necessary condition for a strategy (π, c) to be admissible: let be fixed the discounted "objective" consumption $\int_0^T e^{-rs}c_s ds$. Prove that

$$(*) \qquad E_Q[\int_0^T e^{-rs}c_s ds] \leq x$$

is equivalent to the existence of an admissible strategy π such that $X_T = x + \int_0^T \pi_s.d\tilde{S}_s$.

6. Optimal strategies: Prove that actually the problem is as following: the small investor

evaluates the quality of his investment with an "utility function" (U_i is positive, concave, strictly increasing, C^1 class); he look for the maximization:

$$(c, X_T) \to E_{\mathbb{P}}[\int_0^T U_1(c_s)ds + U_2(X_T)]$$

under the above constraint 5 (*). Solve this constrained optimization problem using Lagrange method and Kuhn and Tucker Theorem.

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