Much ado about Derrida's CREM

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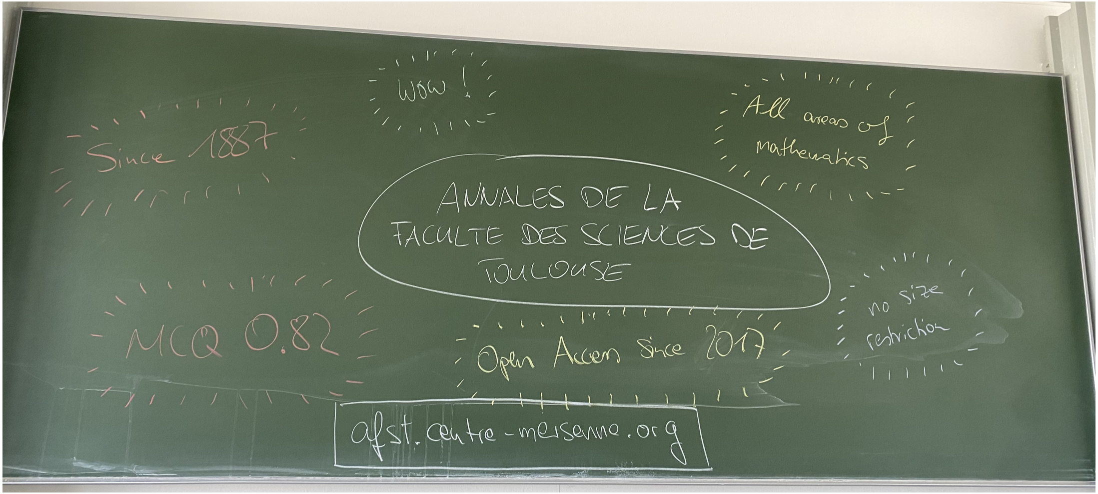

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“Much ado about Derrida’s GREM” Bovier–Kurkova 2007

3 papers by A. Bovier and I. Kurkova (2004–07):

- Initiated mathematical study of the *generalized random energy model (GREM)* Derrida 1985, a toy model for *spin glasses*
- Defined the *continuous random energy model (CREM)* and studied it through approximation by the GREM.

Much Ado about Derrida’s GREM

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Summary. We provide a detailed analysis of Derrida’s generalised random energy model (GREM). In particular, we describe its limiting Gibbs measure in terms of Ruelle’s Poisson cascades. Next we introduce and analyse a more general class of continuous random energy models (CREMs) which differs from the well-known class of Sherrington–Kirkpatrick models only in the choice of distance on the space of spin configurations: the Hamming distance defines the later class while the ultrametric distance corresponds to the former one. We express explicitly the geometry of its limiting Gibbs measure in terms of genealogies of Neveu’s continuous state branching process via an appropriate time change. We also identify the distances between replicas under the limiting CREM’s Gibbs measure with those between integers of Bolthausen–Sznitman coalescent under the same time change.

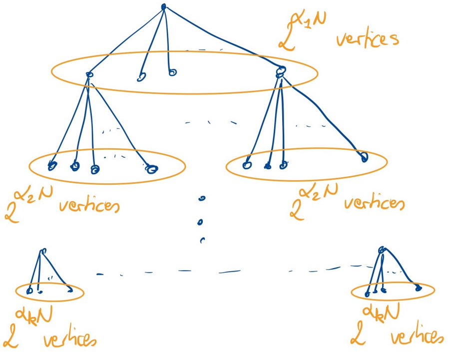
Key words: Gaussian processes, spin-glasses, generalised random energy model, Poisson point processes, branching processes, coalescence.

Generalized random energy model (GREM)

$k \geq 1, \alpha_1, \ldots, \alpha_k > 0, \sigma_1, \ldots, \sigma_k \geq 0.$
Gaussian field on rooted tree with k levels, vertex at level $k-1$ has $2^{\alpha_k N}$ children.
Leaf $v = (v_1, \ldots, v_k)$,

$$X_v = \sum_{i=1}^{k} X_{v_1,\ldots,v_i},$$

$(X_{v_1,\ldots,v_i})_{i \leq k; v_1 \leq 2^{\alpha_1 N}, \ldots, v_i \leq 2^{\alpha_i N}}$ are independent r.v. with respective laws $\mathcal{N}(0, \sigma_i^2 N)$.

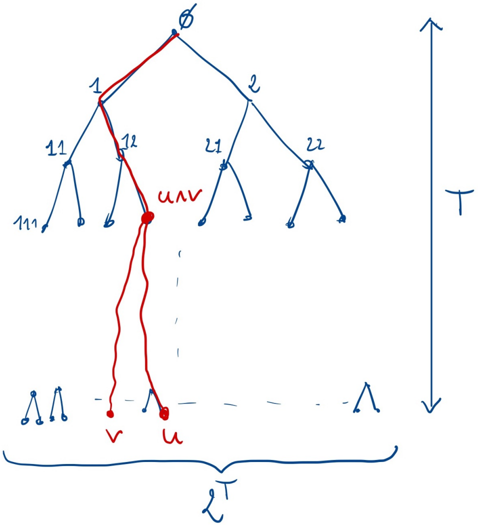


Continuous random energy model (CREM)

Rooted binary tree $\mathbb{T}_T$ of depth T
$(X_u)_{u\in\mathbb{T}_T}$: centered Gaussian field
$A : [0,1] \to [0,1]$ non-decreasing
$|u| = \mathrm{dist}(\emptyset, u)$

$$\mathrm{Cov}(X_u, X_v) = T \cdot A\left(\frac{|u \wedge v|}{T}\right)$$

Examples
A piecewise constant: GREM
$A(x) = x$: Gaussian branching random walk



Bovier-Kurkova 2004-07: key results

A non-decreasing, $A(0)=0$, $A(1)=1$, $\hat{a}(t)$: left-derivative of concave hull of A.

Free energy: $F(\beta)=\lim_{T\to\infty}\frac{1}{\beta T}\mathbb{E}\left[\log Z_T(\beta)\right]=\frac{1}{\beta}\int_0^1 f(\beta\sqrt{\hat{a}(t)})\,dt$,

where $Z_T(\beta)=\sum_{u\in\partial\mathbb{T}_T}e^{\beta X_u}$, $\quad f(\beta)=\min\left(\log 2+\frac{\beta^2}{2},\sqrt{2\log 2}\,\beta\right)$.

Mean overlap: For all $t\in(0,1)$, s.t. $\hat{a}$ is continuous at t,

$$\mathbb{E}\left[\sum_{u,v\in\partial\mathbb{T}_T}\mathbf{1}_{\left(\frac{|u\wedge v|}{T}\le t\right)}\mu_{\beta,T}^{\otimes 2}(u,v)\right]\to\min\left(\frac{\sqrt{2\log 2}}{\beta\sqrt{\hat{a}(t)}},1\right),\text{ where }\mu_{\beta,T}(u)=\frac{1}{Z_T(\beta)}e^{\beta X_u}.$$

Full overlap distribution: Limiting distribution of overlaps under $\mu_{\beta,T}^{\otimes k}$ obtained through a time-changed Bolthausen-Sznitman coalescent.

Extremal point process (GREM only): Convergence of extremal point process to a certain *cluster Poisson process* (see below).

Extremal point process (GREM)

Branching convolution operation: $\mathcal{X}, \mathcal{Y}$ laws of point processes on $\mathbb{R}$. X : point process of law $\mathcal{X}$, $(Y^{(i)})_{i\geq 1}$: iid point processes of law $\mathcal{Y}$, indep. of X. Δ_x: operator of translation by x. Then $\mathcal{X} \circledast \mathcal{Y}$ is the law of the process

$$\sum_{x_i \in X} \Delta_{x_i} Y^{(i)}.$$

Define Π_α to be the law of the Poisson point process with intensity measure $e^{-\alpha x}\, dx$.

Theorem (Bovier-Kurkova 2004)

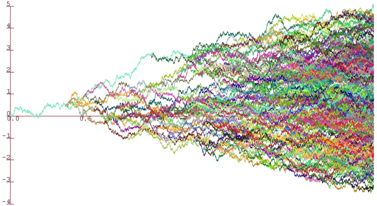
Let A be piecewise constant and let $0 = t_0 < t_1 < \ldots < t_m \leq 1$ the extremal points of its concave hull. Then there exists a deterministic recentering term m_T, such that as $T \to \infty$,

$$\sum_\sigma \delta_{X_\sigma - m_T} \Longrightarrow \Pi_{\sqrt{2\log 2}/\sqrt{\hat{a}(t_1)}} \circledast \cdots \circledast \Pi_{\sqrt{2\log 2}/\sqrt{\hat{a}(t_m)}}.$$

Branching Brownian motion (BBM)

Definition

- A particle performs standard Brownian motion started at a point $x \in \mathbb{R}$.
- With rate $1/2$, it branches into 2 offspring (can be generalized)
- Each offspring repeats this process independently of the others.
- $\mathcal{N}_t$: particles at time t
  $X_u(t)$: position of particle u at time t.



Picture by Matt Roberts

Extreme values of BBM/BRW

- Convergence of extremal point processes Arguin–Bovier–Kistler 2011-13 Aïdékon–Berestycki–Brunet–Shi 2013 Madaule 2017
- Characterization of limiting extremal point process Aizenman–Ruzmaikina 2005 M. 2013 Biskup–Louidor 2016 Subag–Zeitouni 2015 M.–Mallein 2021 Chen–Garban–Shekhar 2022
- Fine properties of limiting extremal point process Cortines–Hartung–Louidor 2019

Remark 1: And previous works by Brunet, Derrida and coauthors (1997-2007), mostly using the FKPP equation.
Remark 2: Ramifications to other models (log-correlated Gaussian fields, characterisic polynomials of random matrices, zeta function)

Shifted decorated point processes

$\mathcal{X} \circledast \mathcal{Y}$: law of the process $\sum_{x_i \in X} \Delta_{x_i} Y^{(i)}$, with Δ_x the operator of translation by x.
Π_α: law of the Poisson point process with intensity measure $e^{-\alpha x}\, dx$.

Shifted decorated point processes Subag-Zeitouni 2015 M.-Mallein 2021

S: positive random variable, $\alpha > 0$, $\mathcal{D}$: law of a point process on $\mathbb{R}$. Then

$$\mathrm{SDPPP}(S, \alpha, \mathcal{D}) := \mathscr{L}\left(\delta_{\alpha^{-1}\log S}\right) \circledast \Pi_\alpha \circledast \mathcal{D} = \mathrm{PPP}(S\, e^{-\alpha x}\, dx) \circledast \mathcal{D}.$$

Remark: Under mild assumptions, $\mathrm{SDPPP}(S, \alpha, \mathcal{D})$ are unique solutions to the equation $\mathcal{E} = \mathcal{Z} \circledast \mathcal{E}$ M.-Mallein 2021

Convergence of extremal point process of BBM

$$m_T = T - \frac{3}{2}\log T \qquad E_T = \sum_{u \in \mathcal{N}_T} \delta_{X_u(T) - m_T}.$$

Derivative martingale limit Lalley–Sellke 1987

$$D = \lim_{T\to\infty} \sum_{u\in\mathcal{N}_T} (T - X_u(T)) e^{X_u(T)-T} > 0$$

Theorem (Arguin–Bovier–Kistler 2011-13 Aïdékon–Berestycki–Brunet–Shi 2013)

As $T \to \infty$, E_T converges in law to $\mathrm{SDPPP}(cD, 1, \mathcal{D}^1)$, for some $c > 0$ and where

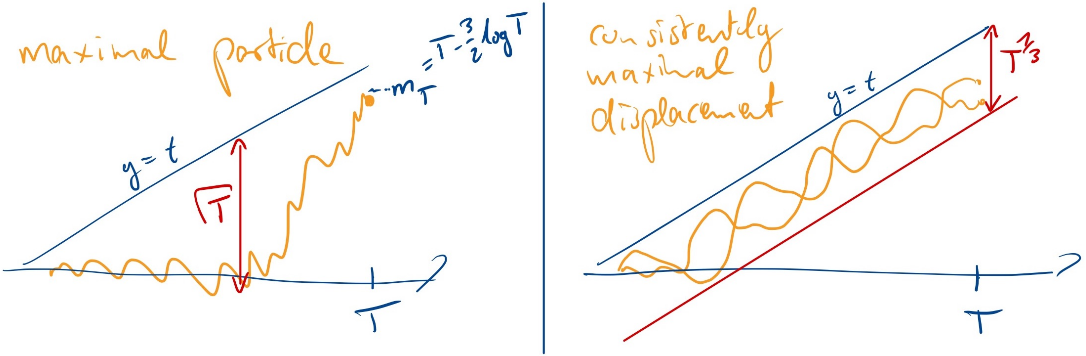
$$\mathcal{D}^1 = \lim_{T\to\infty} \mathscr{L}\left(\Delta_{(-\max E_T)} E_T \mid \max E_T > T\right).$$

Consistently maximal particles

- Consistent(ly) maximal displacement Hu–Shi 2007 Fang–Zeitouni 2010 Faraud–Hu–Shi 2012 Jaffuel 2012 Roberts 2015
- BBM/BRW with absorption Kesten1978 Aldous 1998 Pemantle2009 Bérard–Gouéré 2011 Gantert–Hu–Shi 2011 Berestycki–Berestycki–Schweinsberg 2011-13 M.–Schweinsberg 2022
- N-particle BBM/BRW Bérard–Gouéré 2010 M. 2016 Mallein 2018

And previous works by Brunet, Derrida and coauthors (1997-2007) using the FKPP equation.

Maximum *vs.* consistently maximal displacement



Example: BBM with absorption, critical drift

BBM with (critical) drift -1, kill particles at 0. Gets extinct almost surely Kesten1978.
ζ: time of extinction. $\mathbb{P}_y$: law of process started at y.

Theorem (M.-Schweinsberg 2022)

Let $c = (3\pi^2/2)^{1/3}$. There exists $C > 0$, such that, as $T \to \infty$,

$$\mathbb{P}_{cT^{1/3}+x}(\zeta \le T) \to \phi(x), \quad \phi(x) = \mathbb{E}[\exp(-CDe^x)].$$

Remark: Also obtain Yaglom-type theorem and other results.

BBM with absorption, critical drift (contd.)

- $L_T(t) = c(T-t)^{1/3}$, $c = (3\pi^2/2)^{1/3}$.
- $Z_T(t) = \sum_{u\in\mathcal{N}_t} L_T(t) \sin\left(\pi X_u(t)/L_T(t)\right) e^{X_u(t)-L_T(t)}$.

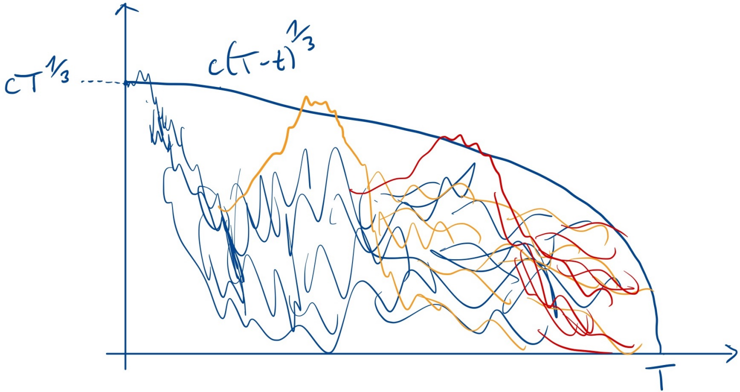
Theorem (M.-Schweinsberg 2022)

Let $\tau(t) = -\frac{2}{3}\log(1-t)$. *There are* $C > 0$, $a \in \mathbb{R}$, *such that we have the following convergence in finite-dimensional distributions as* $T \to \infty$ *under* $\mathbb{P}_{cT^{1/3}+x}$*:*

$$(Z_T(tT); t \in (0,1)) \Longrightarrow (\Xi(\tau(t)); t \in (0,1)),$$

where $(\Xi(s); s \geq 0)$ *is a continuous-state branching process with branching mechanism* $u \mapsto au + u\log u$ *(Neveu's CSBP) started at* $Ce^x D$.

Basic proof idea (originating in BBS 2013)



1) Stop particles when they come to distance $O(1)$ of curve $c(T-t)^{1/3}$. 2) Estimate their descendence. 3) Treat remaining particles with 1st and 2nd moment methods.

Back to CREM/variable-speed BBM

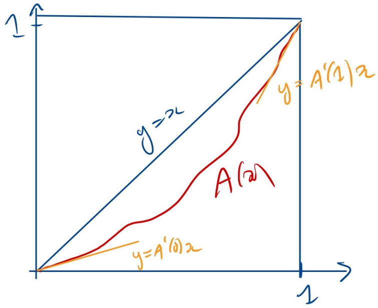
CREM $\approx$ variable-speed BBM (but $\sqrt{2\log 2}$ disappears)
Weak correlation regime: $A(x) < x$, $A'(0) = \sigma_0^2 < 1$, $A'(1) = \sigma_1^2 > 1$, $A(0) = 0$, $A(1) = 1$.

Theorem (Bovier-Hartung 2015)

There is $c > 0$, such that as $T \to \infty$

$$\sum_{u \in \mathcal{N}_T} \delta_{X_u(T) - (T - \frac{1}{2}\log T)} \Longrightarrow \mathrm{SDPPP}(cW^{\sigma_0}, 1/\sigma_1, \mathcal{D}^{\sigma_1}).$$

Here, W^σ is an additive martingale of standard BBM with parameter σ and $\mathcal{D}^\sigma$ is the limiting extremal point process of standard BBM conditioned on the maximum being greater than σT.



BBM with strictly decreasing variance

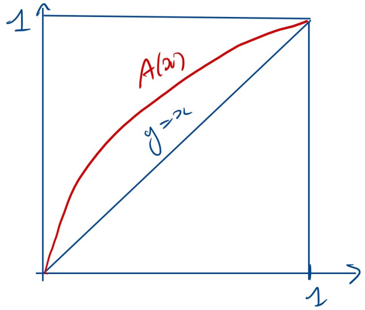
A *strictly concave*, smooth. Set $\sigma(t) = \sqrt{A'(t)}$ (σ decreasing and $\sigma(t) = \sqrt{\hat{a}(t)}$).

Theorem (Hartung-Legrand-M.-Pain, in prep.)

There exists constants v_1, v_2, such that, with $m_T = v_1 T - v_2 T^{1/3} - \sigma(1)\log T$, the following convergence in law holds as $T \to \infty$:

$$\sum_{u \in \mathcal{N}_T} \delta_{X_u(T) - m_T} \Longrightarrow \mathrm{SDPPP}\left(S, 1/\sigma(1), \mathcal{D}^{\sigma(1)}\right),$$

where $S = \Xi\left(\log(\sigma(0)) - \log(\sigma(1))\right)$ for $(\Xi(s))_{s \geq 0}$ a Neveu's CSBP started at cD, for some $c > 0$.



Remarks: Also obtain stronger result incorporating the genealogy. Proof uses techniques from M.-Zeitouni 2015 and M.-Schweinsberg 2022.

N-particle variable-speed BBM

A general but smooth. At each time step, keep N maximal particles, kill others.

Homogeneous case $A(x) = x$: N-BRW Bérard-Gouéré 2010

Brunet-Derrida correction: speed $= 1 - \frac{\pi^2}{2(\log N)^2} + \cdots$

Let $N = \exp(L(T))$, $1 \ll L(T) \ll T$. M_T: maximum at time T. $v = \int_0^1 \sqrt{A'(t)}\, dt$.

Theorem (Legrand-M., in prep.)

1. *If $L(T) \ll T^{1/3}$, then $M_T = vT\left(1 - \frac{\pi^2}{2L(T)^2}\right) + \cdots$.*
2. *If $L(T) \ll T^{1/3}$, then $M_T = vT - (\sigma(1) - \sigma(0))L(T) + \cdots$.*
3. *If $L(T) \sim \alpha T^{1/3}$ for some $\alpha > 0$, then $M_T = vT + \Phi(\alpha, A)T^{1/3} + \cdots$, for some explicit functional $\Phi(\alpha, A)$.*

Algorithmic hardness for CREM

The previous result is related to *algorithmic hardness thresholds* for optimization algorithms on CREM. Algorithm: Explore vertices one at a time, choice of next vertex depending on vertices discovered so far, plus additional randomness.
Define $x_* = \sqrt{2\log 2}\int_0^1 \sqrt{A'(t)}\,dt$.

Theorem (Addario-Berry–M. 2021)

1. *For $x < x_*$, there exists an algorithm with runtime $O(T)$, which finds a vertex u with $X_u \geq xT$ with high probability.*
2. *For $x > x_*$ every algorithm, which finds a vertex u with $X_u \geq xT$, has runtime $\geq e^{\gamma T}$ with high probability, for some $\gamma = \gamma(x) > 0$.*

Remark: For efficiency of *sampling algorithms*, see Ho–M. 2022, Ho, in prep.

Thank you for your attention!