FLUCTUATIONS OF THE BBM GIBBS MEASURE AT CRITICAL TEMPERATURE

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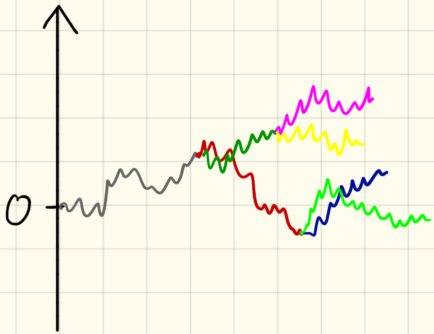
BRANCHING BROWNIAN MOTION (BBM)



- Particles move according to Brownian motions with drift μ and variance σ^2
- Particles branch at rate β into random number of offspring distributed as r.v. L
- $E[L] > 1$ (supercritical branching)
- $\mathcal{N}_t$: set of particles at time t
  $X_u(t)$: position of particle u at time t

Fix parameters: $\mu = 1$ $\sigma^2 = 1$ $\beta = \frac{1}{2(E[L]-1)}$

Then for all $t \geq 0$:

$$E\left[\sum_{u \in \mathcal{N}_t} e^{-X_u(t)}\right] = 1, \quad E\left[\sum_{u \in \mathcal{N}_t} X_u(t)\, e^{-X_u(t)}\right] = 0$$

$$E\Big[\sum_{u\in\mathcal{N}_t} e^{-X_u(t)}\Big] = 1, \quad E\Big[\sum_{u\in\mathcal{N}_t} X_u(t)\, e^{-X_u(t)}\Big] = 0$$

Martingales:

$$W_t = \sum_{u\in\mathcal{N}_t} e^{-X_u(t)} \qquad Z_t = \sum_{u\in\mathcal{N}_t} X_u(t)\, e^{-X_u(t)}$$

(critical additive martingale) (derivative martingale)

$$W_t^{(\theta)} = \sum_{u\in\mathcal{N}_t} e^{-\theta X_u(t) - \varphi(\theta) t}, \quad \varphi(\theta) = \frac{1}{2}(1-\theta)^2$$

(additive martingales)

$$W_t = W_t^{(1)} \qquad Z_t = -\frac{d}{d\theta} W_t^{(\theta)}$$

LIMIT THEOREMS

$$W_t^{(\theta)} = \sum_{u \in \mathcal{N}_t} e^{-\theta X_u(t) - \varphi(\theta) t}, \qquad \varphi(\theta) = \frac{1}{2}(1-\theta)^2$$

Thm (Biggins '77, Lyons '97):
$(W_t^{(\theta)})_{t \geq 0}$ is uniformly integrable iff
$|\theta| < 1$ and $E[L \log L] < \infty$.
(if not, $W_t^{(\theta)} \longrightarrow 0$ a.s.)

Thm (Biggins '78): $|\theta| < 1$, $a < b$, $E[L (\log L)^{5/2}] < \infty$

$$\frac{\#\{u \in \mathcal{N}_t : X_u(t) \in (1-\theta)t + [a,b]\}}{E[\#\{u \in \mathcal{N}_t : X_u(t) \in (1-\theta)t + [a,b]\}]} \xrightarrow[t \to \infty]{a.s.} W_\infty^{(\theta)}$$

LIMIT THEOREMS (CRITICAL CASE)

Thm (Biggins '77, Lyons '97): $W_t \xrightarrow{t\to\infty} 0$ a.s.

Thm (Lalley-Sellke '87, Kyprianou '04, Yang-Ren '11)

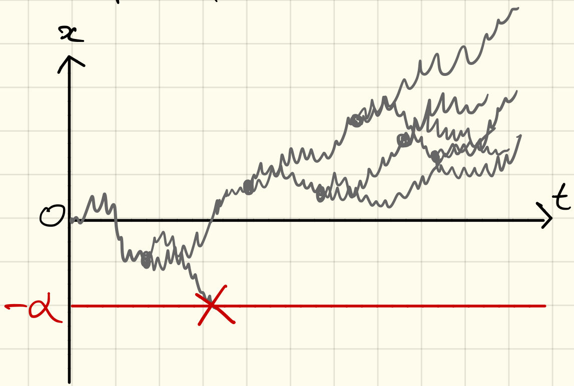
$Z_t \longrightarrow Z_\infty$ a.s. If $E[L(\log L)^2] < \infty$, then

$Z_\infty > 0$ a.s. on event of survival, else $Z_\infty = 0$ a.s.

Key proof steps: • Introduce absorbing barrier at $-\alpha$:

- $Z_t^\alpha = Z_t + \alpha W_t$ (positive) martingale
- $\lim_{t\to\infty} Z_t = \lim_{t\to\infty} Z_t^\alpha$ if no absorption
- $(Z_t^\alpha)_{t\geq 0}$ is u.i. if $E[L(\log L)^2] < \infty$

  (show for example by spine techniques)
- $\alpha \to \infty$



IMPORTANCE OF DERIVATIVE MARTINGALE

- Minimal position (Bramson '83 + Lalley-Sellke '87, Aïdékon '11)

$$\min_{u\in\mathcal{N}_t} X_u(t) - \frac{3}{2}\log t \Longrightarrow -\log(C Z_\infty) - G$$

Gumbel, $\perp\!\!\!\perp$ of Z_∞

- Seneta-Heyde norming (Aïdékon-Shi '12)

$$\sqrt{t}\, W_t \xrightarrow{\mathbb{P}} \sqrt{\frac{2}{\pi}}\, Z_\infty$$

- Gibbs measure (Madaule '15): if $f \in C((0,\infty))$, s.t. $x f(x) \le C e^{Cx}$ for some $C>0$,

$$Z_t(f) := \sum_{u\in\mathcal{N}_t} X_u(t)\, e^{-X_u(t)} f\left(\frac{X_u(t)}{\sqrt{t}}\right) \xrightarrow{\mathbb{P}} Z_\infty\, \mu(f)$$

$$\mu(f) = \int f(x)\,\mu(dx)$$

$$\mu(dx) = \sqrt{\frac{2}{\pi}}\, x^2 e^{-x^2/2}\, dx$$

- $Z_t \xrightarrow{a.s.} Z_\infty$
- $\sqrt{t}\, W_t \xrightarrow{\mathbb{P}} W_\infty$
- $Z_t(y) \xrightarrow{\mathbb{P}} Z_\infty \mu(y)$

Q : 1) RATE OF CONVERGENCE ?

2) (NON-STANDARD) CENTRAL LIMIT THEOREMS ?

A : 1) $1/\sqrt{t}$, with possible $O\left(\frac{\log t}{\sqrt{t}}\right)$ corrections

2) Yes, involving mixtures of 1-stable distributions

(M., Pain, in preparation)

OUR RESULTS

<u>Thm 1</u> (M., Pain, in prep.): Assume $E\left[L(\log L)^3\right] < \infty$

There exists a spectrally positive 1-stable Lévy-process $(S_t)_{t\geq 0}$, independent of Z_∞, such that

$$\sqrt{t}\left(\left(1+\frac{\log t}{\sqrt{2\pi a t}}\right)Z_\infty - Z_{at}\right)_{a\geq 1} \xrightarrow{\text{fidis}} \left(S_{Z_\infty/\sqrt{a}}\right)_{a\geq 1}.$$

<u>Extension:</u> Conditionally on $\mathcal{F}_t$, the fidis of $\left(Z_\infty - Z_{at} + \frac{\log t}{\sqrt{2\pi a t}} Z_\infty\right)_{a\geq 1}$ converge weakly in probability to $\left(S_{Z_\infty/\sqrt{a}}\right)_{a\geq 1}$ conditioned on Z_∞.

<u>Remark:</u> Hypothesis $E\left[L(\log L)^3\right] < \infty$ probably optimal

OUR RESULTS (contd.)

Thm 2 (M., Pain, in preparation) Assume $E[L^2] < \infty$.

Let $f \in C^2((0,\infty))$, such that $(xf)''(x) \le Ce^{Cx}$. Then,

$$\sqrt{t}\left(\left(\mu(f) + c_1(f)\frac{\log t}{\sqrt{2\pi t}}\right) Z_\infty - Z_t(f)\right) \xrightarrow{\mathcal{L}} S_{Z_\infty},$$

where $(S_t)_{t\geq 0}$ is a 1-stable Lévy process with asymmetry parameter $\beta = \dfrac{c_1(f)}{c_2(f)}$

$$c_1(f) = \int_0^\infty \left(\mu(f) - \mu\left(f\left((1-u)^{\frac{1}{2}}\,\cdot\right)\right) \mathbb{1}_{u<1}\right)\frac{du}{2u^{3/2}} = \sqrt{\frac{\pi}{2}}\,\mu\left(\frac{(xf)'}{x}\right)$$

$$c_2(f) = \int_0^\infty \left|\mu(f) - \mu\left(f\left((1-u)^{\frac{1}{2}}\,\cdot\right)\right) \mathbb{1}_{u<1}\right|\frac{du}{2u^{3/2}}$$

In particular $\left(f(x) = \frac{1}{x}\right)$: $\sqrt{t}\left(\sqrt{\frac{2}{\pi}}\, Z_\infty - \sqrt{t}\, W_t\right) \xrightarrow{\mathcal{L}} S_{Z_\infty}$,

with $(S_t)_{t\geq 0}$ a Cauchy process.

RELATED WORKS

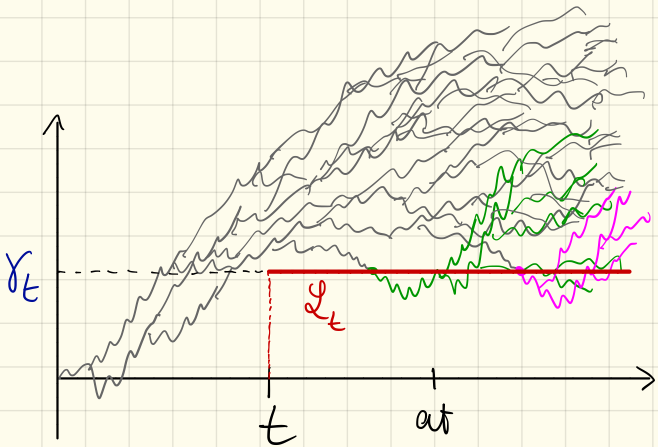
convergence exponentially fast

- $\theta < \frac{1}{2}$: (standard) CLT w/ random variance (Iksanov-Kabluchko '16)
  (subordinated by Z_∞)
- $\theta \in [\frac{1}{2}, 1)$: $\frac{1}{\theta}$-stable distributions (Iksanov-Kolesko-Meiners, in prep.)
  ↳ key tool: fixed-point equation (FPE) $W_\infty^{(\theta)} = \sum_{u \in \mathcal{N}_t} e^{-\theta X_u(t) - \varphi(\theta) t} \cdot W_\infty^{(\theta),u}$
  ↳ can also be used for Z_t, but only for one-dimensional marginals
- complex θ: CLT (w/ random variance) in certain phases (Hartung-Klimovsky '15, '17)
- $\theta > 1$: construction of solutions to FPE by subordination of $1/\theta$-stable process (Durrett-Liggett '83, Guivarc'h '90, Barral-Jin-Rhodes-Vargas '13)
- Fluctuations of partition function in other stat. mech. models:
  - SK (Comets-Neveu '95) } CLT
  - p-spin (Bovier-Kurkova-Löwe '02) } CLT
  - spherical SK + Curie-Weiss (Baik-Lee '16, '17, Baik-Lee-Wu '18+) → CLT, → Tracy-Widom, → ...
  - ...

PROOF IDEAS (THM 1)

- Prove simple version: (Lemma: $\left(\sqrt{t}\, W_t - \sqrt{\tfrac{2}{\pi}}\, Z_\infty\right) = O(t^{-\eta}),\ \eta > 0$)

$$\sqrt{t}\left(Z_\infty - \left(Z_{at} - \frac{\log t}{2} W_{at}\right)\right)_{a \geq 1} \xrightarrow{\text{fidis}} \left(S_{Z_\infty/\sqrt{a}}\right)_{a \geq 1}.$$



BASIC IDEA:

From time t on, introduce barrier $\mathcal{L}_t$ at suitably chosen γ_t

GOOD CHOICE:

$$\boxed{\gamma_t = \frac{1}{2}\log t + \beta_t, \quad \beta_t \to \infty \text{ slowly}}$$

Why $\frac{1}{2}\log t$? $\lim_{t\to\infty} \dfrac{\min_{u \in \mathcal{N}_t} X_u(t)}{\log t} = \dfrac{1}{2}$ a.s. (Hu-Shi '09)

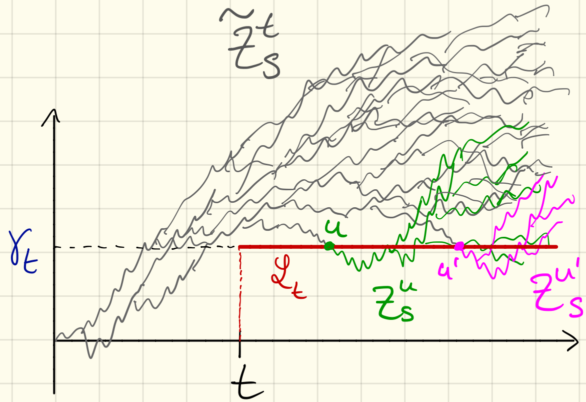
$$\boxed{\gamma_t = \tfrac{1}{2}\log t + \beta_t,\quad \beta_t \to \infty \text{ slowly}}$$

$s \geq t$: $\quad Z_s^t := Z_s - \gamma_t W_s$

$$= \sum_{u \in \mathcal{N}_s} (X_u(s) - \gamma_t)\, e^{-X_u(s)}$$

$$= \tilde{Z}_s^t + \sum_{u \in \mathcal{L}_t} Z_s^u$$

positive martingale



$$\tilde{Z}_s^t = \sum_{u \in \mathcal{N}_s} (X_u(s) - \gamma_t)\, e^{-X_u(s)}\, \mathbb{1}(u < \mathcal{L}_t)$$

$$Z_s^u = \sum_{v \in \mathcal{N}_s} (X_v(s) - \gamma_t)\, e^{-X_v(s)}\, \mathbb{1}(v \succeq u)$$

Take limit $s \to \infty$:

$$\boxed{Z_\infty = \tilde{Z}_\infty^t + \sum_{u \in \mathcal{L}_t} Z_\infty^u}$$

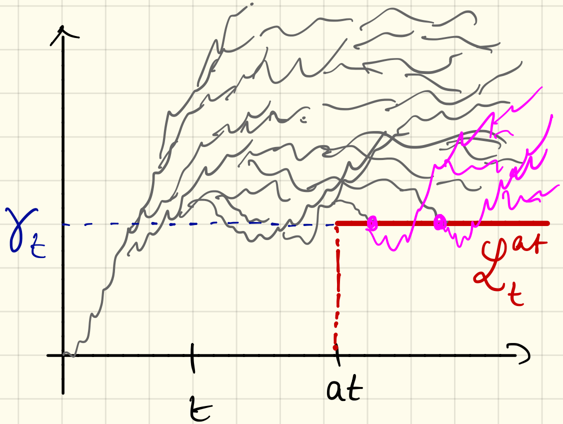
First moment and (truncated) second moment computations yield:

$$\tilde{Z}^t_\infty = Z^t_t + o\left(1/\sqrt{t}\right)$$

$$\Longrightarrow \quad Z_\infty = Z_t - \frac{\log t}{2} W_t + \sum_{u \in \mathcal{L}_t} Z^u_\infty - \beta_t W_t + o\left(\frac{1}{\sqrt{t}}\right)$$

Can do the same for barrier $\mathcal{L}^{at}_t$ starting at at:

$$\boxed{\begin{aligned} &\sqrt{t}\left(Z_\infty - \left(Z_{at} - \frac{\log t}{2} W_{at}\right)\right)_{a \geq 1} \\ &\approx \sqrt{t}\left(\sum_{u \in \mathcal{L}^{at}_t} Z^u_\infty - \beta_t W_{at}\right)_{a \geq 1} \end{aligned}}$$



Strategy similar to Berestycki - Berestycki - Schweinsberg '13

REMAINS TO PROVE:

$$\sqrt{t}\left(\sum_{u \in \mathcal{N}_t^{at}} Z_\infty^u - \beta_t W_{at}\right)_{a \geq 1} \xrightarrow{\mathcal{L}} \left(S_{Z_\infty / \sqrt{a}}\right)_{a \geq 1}$$

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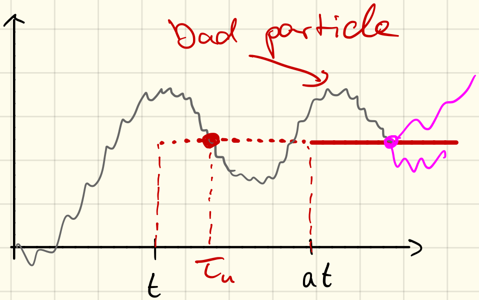
REMAINS TO PROVE:

$$\sqrt{t}\left(\sum_{u \in \mathcal{Y}_t^{at}} Z_\infty^u - \beta_t W_{at}\right)_{a \geq 1} \xrightarrow{\mathcal{L}} \left(S_{Z_\infty/\sqrt{a}}\right)_{a \geq 1}$$

1) $\displaystyle\sum_{u \in \mathcal{Y}_t^{at}} Z_\infty^u = \sum_{\underline{\underline{u \in \mathcal{L}_t}},\ \underline{\underline{\tau_u \geq at}}} Z_\infty^u + o\left(\frac{1}{\sqrt{t}}\right)$

bad particles

↪ sum over a decreasing set



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REMAINS TO PROVE:

$$\sqrt{t}\Big(\sum_{u\in\mathcal{A}_t^{at}} Z^u_\infty - \beta_t W_{at}\Big)_{a\geq 1} \xrightarrow{\mathcal{L}} \big(S_{Z_\infty/\sqrt{a}}\big)_{a\geq 1}$$

1) $$\sum_{u\in\mathcal{A}_t^{at}} Z^u_\infty = \sum_{u\in\underline{\underline{\mathcal{L}_t}},\ \underline{\tau_u\geq at}} Z^u_\infty + o\Big(\frac{1}{\sqrt{t}}\Big)$$

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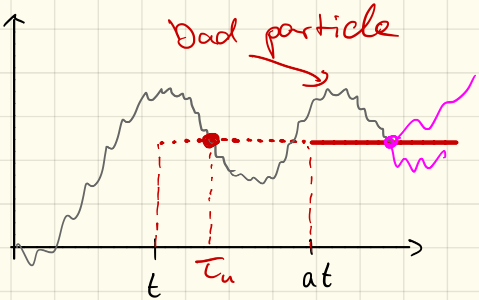

2) particles hit barrier at rate

$$\overset{s\geq t}{\approx} \sqrt{\frac{2}{\pi}}\, Z_t\, e^{\gamma_t} \frac{ds}{2s^{3/2}} \overset{s=at}{\approx} \sqrt{\frac{2}{\pi}}\, Z_\infty\, e^{\beta_t} \Big|d\frac{1}{\sqrt{a}}\Big|$$

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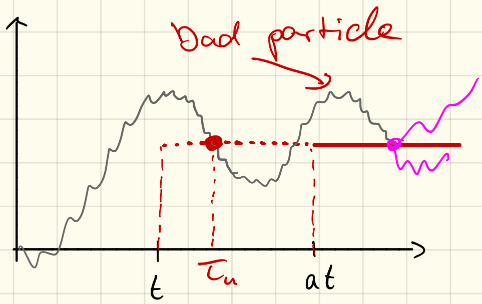

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3) Contribution of each particle: $Z_\infty^u \overset{law}{=} e^{-\gamma_t} Z_\infty = \frac{1}{\sqrt{t}}\cdot e^{-\beta_t} Z_\infty$

Known: $P(Z_\infty > x) \overset{x\to\infty}{\sim} \frac{1}{x}$, $E[Z_\infty \mathbb{1}_{Z_\infty\leq x}] = \log x + c + o(1)$
(BBS, M.)

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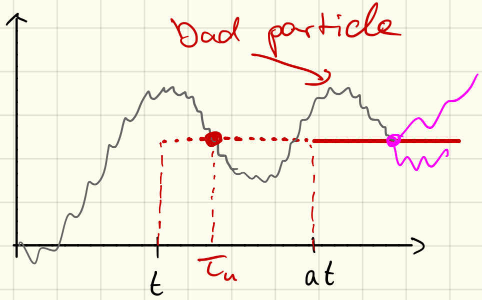

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$$\overset{s\geq t}{\approx} \sqrt{\frac{2}{\pi}}\, Z_t\, e^{\gamma_t} \frac{ds}{2s^{3/2}} \overset{s=at}{\approx} \sqrt{\frac{2}{\pi}}\, Z_\infty\, e^{\beta_t} \left|d\frac{1}{\sqrt{a}}\right|$$

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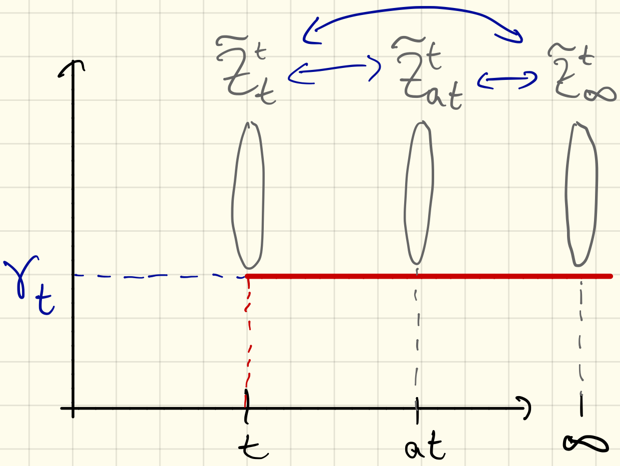
4) $\displaystyle E\left[\sum_{u\in\mathcal{L}_t,\ \tau_u\geq at} Z_\infty^u \mathbb{1}\left(Z_\infty^u \leq \frac{1}{\sqrt{t}}\right)\right] \approx \cancel{\sqrt{at}}\, W_{at}\, \cancel{e^{\beta_t}} \cdot \frac{1}{\cancel{\sqrt{a}}} \times \frac{1}{\cancel{\sqrt{t}}} \cancel{e^{-\beta_t}} \left(\beta_t + c + o(1)\right)$

$= (\beta_t + c + o(1))\, W_{at}$

(≙ linear compensator of jumps)

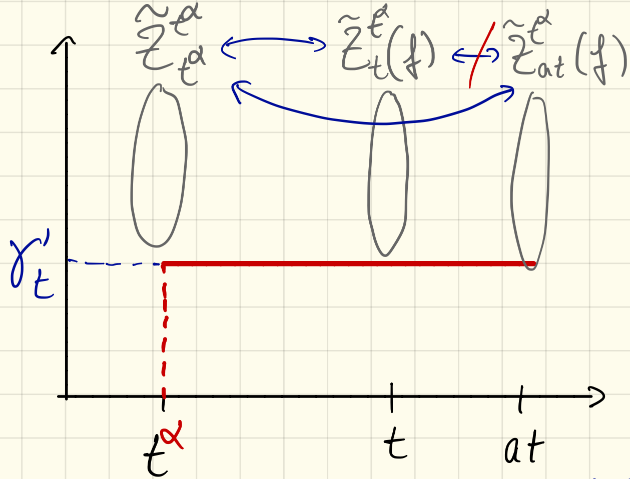
$\blacksquare$

THEOREM 1



Can compare Z_{at}, $a>1$ with Z_t because in the process with absorption at γ_t, $(\tilde{Z}^t_s)_{s\geq t}$ is a martingale

THEOREM 2



Can not compare directly $Z_{at}(f)$ and $Z_t(f)$, only through Z_s for some $s=t^\alpha$, $\alpha\in(0,1)$

α large $\Rightarrow$ Errors in $\tilde{Z}^{t^\alpha}_t(f)$ too large

α small $\Rightarrow$ Many particles hitting barrier

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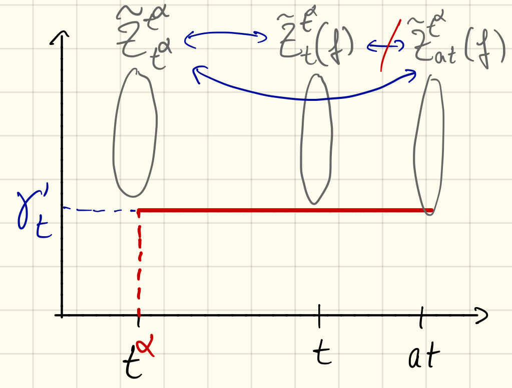

OPTIMAL CHOICE:

$\rightarrow \alpha = \frac{1}{2} - \eta$ (η small)

$\rightarrow \gamma_t' = \frac{1}{2}\log t + \beta_t'$, with β_t' growing fast enough $(\beta_t' - \log\log t \longrightarrow \infty)$

Then:

- $\left| \tilde{Z}_t^{t^\alpha}(f) - \mu(f)\tilde{Z}_t^{t^\alpha} \right| = o(t^{-\frac{1}{2}})$
- handle particles hitting barrier by a (actually, two-scale) bootstrap/multiscale argument involving several barriers

SUMMARY

- Fluctuations of Z_t and $Z_t(f)$ are $O\left(\frac{1}{\sqrt{t}}\right)$ and caused by particles going down to $\approx \frac{1}{2}\log t$.

- Limit laws are mixtures of 1-stable laws with asymmetry parameter
$$\beta = \begin{cases} 1, & Z_t \\ 0, & \sqrt{t}W_t \\ \text{anything}, & Z_t(f) \end{cases}$$
(reason: particles reaching $\frac{1}{2}\log t$ at different times contribute with possibly different signs)

- 1-stable laws arise in other (but related) contexts:
  - BBM with absorption (Brunet-Derrida-Mueller-Munier '06, Berestycki-Berestycki-Schweinsberg '11-'13, Berestycki-M.-Schweinsberg, in prep.)
  - BBM with variance decreasing in time (see open problems)

FURTHER OUTLOOK

- "Moderate" deviations:

$$\beta \in \left[0, \tfrac{1}{2}\right), \qquad P\left(Z_t(f) - \mu(f) Z_\infty \approx t^{-\beta}\right) = ?$$

- Conjecture (minimal particle):

Cauchy process

$$\min_{u \in \mathcal{N}_t} X_u(t) = \frac{3}{2} \log t + \log(C Z_\infty) + G + \frac{S_{Z_\infty}}{\sqrt{t}} + o\left(\frac{1}{\sqrt{t}}\right)$$

Gumbel

(formalize in terms of mod-ϕ convergence)

- Ebert-van Saarloos correction:

$$\text{median of } \min_{u \in \mathcal{N}_t} X_u(t) = \frac{3}{2} \log t + C + \frac{3\sqrt{2\pi}}{\sqrt{t}} + o\left(\frac{1}{\sqrt{t}}\right)$$

(Ebert-van Saarloos '00, Nolen-Roquejoffre-Ryzhik '16+)

$\rightarrow$ related, but not equivalent problem.

$$Z_t(F) = \mu_t(xF) = \mu_t(G)$$

$$E[F(R_1)] = \sqrt{\frac{2}{\pi}}\,\mu(G)$$

$$\mu_\infty(x) = \int_0^\infty x^2 e^{-x^2/2}\,dx \overset{IPP}{=} \int_0^\infty e^{-x^2/2}\,dx$$

$$\Rightarrow \mu_\infty(x) = \sqrt{\frac{\pi}{2}}$$

$$\left(\mu_t(G) - W_\infty\,\mu(G) - c_1(F)\,\frac{\log t}{2\sqrt{t}}\,Z_\infty\right)$$

$$\longrightarrow S_{Z_\infty}, \quad \text{asymmetry } \frac{\pi}{2}\cdot\frac{c_1(F)}{c_2(F)}$$

$$c_1(F) = c_1\left(\frac{G(x)}{x}\right) = E\left[\frac{G'(R_1)}{R_1}\right] = \sqrt{\frac{2}{\pi}}\,\mu(G')$$

$$F_a(x) = F(ax)$$

$$G_a(x) = xF_a(x) = xF(ax) = \frac{1}{a}G(ax)$$

$$\frac{c_1(F)}{c_2(F)} = \frac{2}{\pi}\cdot\frac{}{\int_0^\infty \left|\frac{1}{\sqrt{1-u}}\mu\left(G(\sqrt{1-u}\,\cdot)\right)\mathbb{1}_{u<1} - \mu(G)\right|\frac{du}{2u^{3/2}}}$$