Fixed Points of the Smoothing Transform: An Analysis Through the Branching Random Walk

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Chapter I

Introduction

Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of non-negative random variables with no particular dependence assumed between them. Let Y, Y_i, $i \in \mathbb{N}$ be i.i.d. non-negative random variables, independent of $(A_i)_{i\in\mathbb{N}}$. We look at the equation

$$\text{(FPE)} \quad Y \overset{(d)}{=} \sum_{i\in\mathbb{N}} A_i Y_i,$$

where $\overset{(d)}{=}$ means equivalence in law. Whether Y satisfies (FPE) or not obviously depends only on its law $\mathscr{L}(Y)$, so we will use the phrases "Y satisfies (FPE)" and "$\mathscr{L}(Y)$ satisfies (FPE)" interchangeably.

The above equation can be seen as the *fixed point equation* of a transform T on the space of probability measures, defined by

$$T\mathscr{L}(Y) = \mathscr{L}\left(\sum_{i\in\mathbb{N}} A_i Y_i\right),$$

where the A_i and Y_i are defined as above. This transformation was called the *smoothing transformation* in [HL81], where it appeared in the context of infinite particle systems.

Since we are working with non-negative random variables, we can make use of Laplace transforms. The above equation and the smoothing transform then become the *functional equation*

$$\text{(FE)} \quad \phi(x) = T\phi(x) = E[\prod_{i\in\mathbb{N}} \phi(xA_i)].$$

If a Laplace transform of a probability measure ϕ satisfies (FE) and is not equal to 1, then we say that ϕ is a *non-trivial solution* to (FE) and define $\mathcal{L}$ to be the set of non-trivial solutions to (FE). This set is of course in one-to-one correspondence to the set of solutions to (FPE) that are not concentrated on 0. Note that we could also consider more general solutions to (FE), i.e. which are not Laplace transforms, but we won't do this here, see for example [Kyp98] for some results in this direction.

The equations (FPE) and (FE) have been extensively studied during the last four decades in various fields and under various disguises. The first example has probably been [KP76], where the equation arises in multiplicative cascades. There, the A_i are i.i.d. up to a (deterministic) index N and 0 from $N + 1$ on. The next example is the already cited paper [HL81], here the A_i are constant multiples of the same variable for $i \leq N$ and 0 from $N + 1$ on, too. The first paper to treat the equations with general A_i, but still with a finite number of them, was [DL83]. Not only did they provide necessary and sufficient conditions for $\mathcal{L} \neq \emptyset$, they also characterised the solutions to (FPE) and examined their tail behaviour, i.e. the behaviour of the Laplace transform ϕ at the origin. These questions were then considered to be settled, and so for over one decade it was quiet around the equation (FPE).

The usual way to investigate the fixed points of a transformation T is to repeatedly apply T, i.e. look at the sequence $T\phi$, $T^2\phi$, $T^3\phi, \ldots$ and to hope that this sequence converges to some limit satisfying the fixed point equation. The "right" way to do this in our setting is via *branching random walks*, which transfer the equation from the multiplicative to the easier to handle additive scale. Although [DL83] already used some elements of the theory, it was not until [BK97] that the full dependence between the two settings was exploited, with the help of new developments, notably those in the field of branching Brownian motion. Many papers of different authors followed, the results of which will come up one by one in this paper, so that we don't cite them at this place.

The object of this paper is to provide a homogenised view of the analysis of the fixed point equation (FPE) via branching random walks. The starting point was the article [BK05], which already gives an overview but leaves out the proofs. Here, we develop all the results presented in [BK05] from the ground up, so that every probabilist can delve into the subject without having to consult a pile of articles. The paper strives to be essentially self-contained, with a few exceptions when the material is too technical or requires too many results of an external field. What the paper does *not* try to do, however, is to give a broad overview of the literature treating this or related subjects, hence it is not a survey. Although some historical remarks are given, most sources are only cited if they have a direct influence on the development presented in this paper.

The content is organised as follows. Chapter II presents the branching random walk and some basic results about it, notably the law of large numbers of the left-most individual. These results are used to derive a necessary condition for $\mathcal{L}$ to be non-empty. Some results concerning this condition are new, to the knowledge of the author. They can be identified as those not being associated to any source. Chapter III establishes sufficient conditions for $\mathcal{L}$ to be non-empty. It presents the measure change of the BRW introduced by [Lyo97] and his proof of Biggins' martingale convergence result and shows how to treat the cases not covered by this method. In Chapter IV we consider a general Markovian branching process in the setting of [BK04] and the theory of optional lines. Chapter V is devoted to the study of the behaviour of $\phi \in \mathcal{L}$ at the origin and the consequences of these results. In Chapter VI these results are greatly refined in a boundary case, with the help of results about branching random walks with absorption, which are established in this chapter as well.

A few notes about notation are in order. In most cases we are working on a canonical or an undefined probability space. If the law/probability measure is not specified, then probability and expectation are written as $P()$ and $E[]$, respectively. However, if the law/probability measure is specified and called $\mathbb{P}$, say, then $\mathbb{P}()$ and $\mathbb{P}[]$ specify probability and expectation, respectively. The indicator function is denoted by $\mathbb{1}$. If f

is a function, then $f(a\bullet)$ denotes the function $x \mapsto f(ax)$.

I am indebted to my advisor Zhan Shi for offering me to work on this subject and for his kind help, which I greatly appreciated.

Chapter II

The branching random walk

In this chapter we introduce the branching random walk and present some basic results about it. They will have as a corollary a necessary condition for the existence of non-trivial solutions to (FE).

II.1 Basic definition

The *branching random walk* (*BRW* for short) can be seen as a system of individuals positioned on the real line and reproducing according to the following process: Let x be the position of an individual. When she dies, she gives birth to a finite or countable number of children with positions $(x + x_i)_{i=1,2,3,\ldots}$, the tuple/sequence $(x_i)_{i=1,2,3,\ldots}$ following some fixed law. Each child then reproduces in the same way and independently of the other past and present individuals.

We are going to make this more precise using the *Ulam-Harris labelling*. We start with a single ancestor, who forms the 0-th generation and is labelled with the symbol $\emptyset$ (empty set). Now, every individual of the n-th generation is labelled with an n-tuple $u \in \mathbb{N}^n$ describing its ancestry. For example, the sixth child of the second child of the third child of the ancestor has the label $(3, 2, 6)$ or 326 for short. This labelling is obviously injective, thus enabling us to identify an individual with its label and speak of "the individuals" $(1, 5)$, $(3, 17, 2)$, or $\emptyset$, say. Consequently, we define the *space of individuals* or *universe* as the (countable) set

$$U = \bigcup_{n \in \mathbb{N}_0} \mathbb{N}^n$$

and the following operations on U:

- Two individuals $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$ can be *concatenated*, forming the $(n + m)$-tuple
$$uv = (u_1, \ldots, u_n, v_1, \ldots, v_m).$$
In particular, if u is any individual and $i \in \mathbb{N}$, then $v = ui$ is u's i-th child and u is v's *mother*, also denoted by
$$m(v) = m(ui) = u.$$

- We define the relations $\le$ and $<$ on U by

  $$u \le v \iff \exists w \in U : uw = v$$

  and

  $$u < v \iff u \le v \text{ and } u \ne v.$$

  If $u \le v$ or $u < v$ we say that u is an *ancestor* or a *proper ancestor* of v, respectively. This obviously renders $(U, \le)$ a partially ordered set and indeed a semi-lattice, i.e. each finite subset $u_1, \ldots, u_n$ has a greatest lower bound, the last common ancestor, denoted by $u_1 \wedge \ldots \wedge u_n$.

- If $n \in \mathbb{N}_0$ and $u \in \mathbb{N}^n$, then we define $|u| := n$ and say that u belongs to the *n-th generation*.

We can now define the space the branching random walk is going to live in. First let $\partial \notin \mathbb{R}$ be some cemetery symbol and set $\overline{\mathbb{R}} = \mathbb{R} \cup \{\partial\}$. Now set

$$\mathbb{T} := \overline{\mathbb{R}}^U = \{z : U \to \overline{\mathbb{R}}\}$$

and call $\mathbb{T}$ the *space of labelled trees*, endowed with the product σ-field $\mathcal{F}$. If $z \in \mathbb{T}$ and $u \in U$, then the interpretation of $z_u := z(u)$ is:

- $z_u = \partial$: the individual u does not exist

- $z_u \in \mathbb{R}$: the individual u exists and z_u denotes its position on the real line

If ∂ appears in arithmetic expressions, it should be interpreted as the symbol ∞, e.g. $\partial + r = \partial$ for all $r \in \overline{\mathbb{R}}$, $\mathrm{e}^{-\partial} = 0$, etc.

In order to describe dynamics we define the filtration $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$ on $\mathbb{T}$ by

$$\mathcal{F}_n = \sigma\left(\pi_u; |u| \le n\right),$$

where $\pi_u : \mathbb{T} \to \overline{\mathbb{R}}$ denotes the projection $z \mapsto z_u$. We now have the tools we need to define the branching random walk:

Definition II.1. Let $Q(d\vec{x})$ be a probability measure on $\overline{\mathbb{R}}^{\mathbb{N}}$, $\mathbb{P}$ a probability measure on $\mathbb{T}$ and $(z_u)_{u \in U}$ the canonical process. For $x \in \overline{\mathbb{R}}$ let $Q_x(d\vec{x}) = Q(d(\vec{x} + x))$, where $\vec{x} + x = (x_1 + x, x_2 + x, \ldots)$. $\mathbb{P}$ is called *the law of a branching random walk with reproduction* Q iff the following holds:

1. $z_\varnothing = 0$ $\mathbb{P}$-a.s.

2. If $f_u : \overline{\mathbb{R}}^{\mathbb{N}} \to \mathbb{R}_+$, $u \in U$, are measurable functions then

$$\mathbb{P}[\prod_{|u|=n} f_u\left((z_{ui})_{i \in \mathbb{N}}\right) \mid \mathcal{F}_n] = \prod_{|u|=n} Q_{z_u}[f_u]$$

The law $\mathbb{P}$ in the above definition is clearly unique and we assign it the symbol $\mathbb{B}$. A random variable z with values in $\mathbb{T}$ following the law $\mathbb{B}$ is called a *branching random walk with reproduction* Q or simply a *branching random walk*. If z is a BRW with reproduction Q and $x \in \overline{\mathbb{R}}$, then $z + x = (z_u + x)_{u\in U}$ is called a branching random walk with reproduction Q *starting at* x and its law will be denoted by $\mathbb{B}_x$.

The existence of the law $\mathbb{B}$ follows by standard methods involving Ionescu Tulcea's theorem and the fact that we can count the individuals in such a way that each mother precedes her children. To see this, note that for each $n \in \mathbb{N}_0$ the finite set

$$U_n = \{u \in U : |u| \le n \text{ and } \max_{1\le i\le |u|} u_i \le n\}$$

can be counted generation-wise and that no $u \in U_{n+1}\backslash U_n$ is an ancestor of any $v \in U_n$. Hence, if we count first U_0, then $U_1\backslash U_0$, then $U_2\backslash U_1$ and so on, we obtain the desired counting.

With the monotone class theorem one can show that the law $\mathbb{B}$ satisfies the *branching property*:

Proposition II.2. *If* $f_u : \mathbb{T} \to \mathbb{R}_+$, $u \in U$ *are measurable functions then*

$$\mathbb{B}[\prod_{|u|=n} f_u\left((z_{u\bullet})\right) \mid \mathcal{F}_n] = \prod_{|u|=n} \mathbb{B}_{z_u}[f_u],$$

i.e. conditionally on $\mathcal{F}_n$ *the processes* $(z_{uv})_{v\in U}$ *for each* $u \in U$ *are independent BRW with reproduction* Q *starting at* z_u *and depend on* $\mathcal{F}_n$ *only through* z_u, $u \in U$.

In what follows, we will often have to do with sums, products, infima, etc. over the positions of the individuals. In order not to be in trouble because of the individuals with label ∂, we define the following

Notation: The symbols

$$\sum_{|u|=n}, \prod_{|u|=n}, \inf_{|u|=n}, \sup_{|u|=n}$$

mean sum, product, infimum and supremum over all individuals in the set $\{|u| = n; z_u \neq \partial\}$. If this set is empty, the expressions take the values $0,\ 1,\ \infty,\ -\infty$, respectively. Similar rules may apply in obvious manners at other places, e.g. $\sum_{i\in\mathbb{N}} f(z_i)$, for some function $f : \mathbb{R} \to \mathbb{R}$.

II.2 The relation with the functional equation

Let us turn back to our primary object of study and see how the functional equation and the branching random walk are intertwined. We define the random variables $x_i := -\log A_i \in \overline{\mathbb{R}}$, $i \in \mathbb{N}$ (remember ∂ should be interpreted as ∞) and set $Q := \mathscr{L}((x_i)_{i\in\mathbb{N}})$. Let z be a branching random walk with reproduction Q.

Having thus changed from the multiplicative scale to the additive scale, we obtain the following important

Proposition II.3. *Let ϕ be a solution to (FE). Fix $x \in \mathbb{R}_+$. Then the process*

$$M_n(x) = \prod_{|u|=n} \phi(x\mathrm{e}^{-z_u})$$

is an $(\mathcal{F}_n)$-martingale.

Proof. Let $n \in \mathbb{N}_0$.

$$\begin{aligned}
\mathbb{B}[M_n(x) \mid \mathcal{F}_n] &= \mathbb{B}\left[\prod_{|u|=n} \phi(x\mathrm{e}^{-z_u}) \mid \mathcal{F}_n\right] \\
&= \prod_{|u|=n} Q_{z_u}\left[\prod_{i\in\mathbb{N}} \phi(x\mathrm{e}^{-x_i})\right] && \text{by definition II.1} \\
&= \prod_{|u|=n} Q\left[\prod_{i\in\mathbb{N}} \phi(x\mathrm{e}^{-z_u}\mathrm{e}^{-x_i})\right] \\
&= \prod_{|u|=n} E\left[\prod_{i\in\mathbb{N}} \phi(x\mathrm{e}^{-z_u}A_i)\right] \\
&= \prod_{|u|=n} \phi(x\mathrm{e}^{-z_u}) && \text{by equation (FE)} \\
&= M_n(x)
\end{aligned}$$

□

This martingale is called the *multiplicative martingale* associated to the BRW z and was first discovered by [Nev88] for branching Brownian motion, an analogue to the branching random walk in continuous time, then by [BK97] for the branching random walk. It gives us immediately a necessary condition for the existence of non-trivial solutions to (FE).

Definition II.4. For a branching random walk z let

$$Z_n(z) = \#\{u \in U : |u| = n,\ z_u \in \mathbb{R}\}$$

denote the number of individuals in the n-th generation and let

$$L_n(z) = \inf_{|u|=n} z_u$$

denote the position of the *left-most individual* ($L_n \in [-\infty, \infty]$). Finally, assuming that we are working on the probability space Ω, define the event of ultimate survival, called the *survival set*, by

$$S := \{\omega \in \Omega : \forall n \in \mathbb{N} : Z_n(z(\omega)) > 0\}.$$

Theorem II.5. *Let S be the survival set of the branching random walk z. If*

$$\liminf_{n\to\infty} L_n = -\infty \quad \textit{a.s. on } S,$$

then $\mathcal{L} = \emptyset$. Note that this is trivially true if $P(S) = 0$.

Proof. Assume that $\mathcal{L} \neq \emptyset$ and let $\phi \in \mathcal{L}$. Fix $x \in \mathbb{R}_+$. The martingale $(M_n(x))$ is clearly bounded by 1 and positive, therefore it converges a.s. and in L^1 to a r.v. $M \in [0,1]$. If $P(S) = 0$, then we have

$$M(x) = \lim_{n\to\infty} \prod_{|u|=n} \phi(x\mathrm{e}^{-z_u}) = 1$$

and therefore $\phi(x) = M_0(x) = E[M(x)] = 1$, which is a contradiction.

Now assume $P(S) > 0$ and $\phi \not\equiv 1$, s.t. $\phi(\infty) = \lim_{y\to\infty} \phi(y) < 1$. We then have a.s.

$$\begin{aligned} M(x) &= \liminf_{n\to\infty} \prod_{|u|=n} \phi(x\mathrm{e}^{-z_u}) \\ &\le \liminf_{n\to\infty} \mathbb{1}_S \cdot \phi(x\mathrm{e}^{-L_n}) + \mathbb{1}_{\Omega\setminus S} \\ &= \mathbb{1}_S \cdot \phi(\infty) + \mathbb{1}_{\Omega\setminus S}, \end{aligned}$$

so that $\phi(x) = M_0(x) = E[M(x)] = P(S)\phi(\infty) + 1 - P(S) < 1$ for all x. Hence, ϕ cannot be a Laplace transform of a probability measure, which is a contradiction. □

Remark II.6. We have for $n \in \mathbb{N}_0$

$$Z_{n+1} = \sum_{|u|=n+1} 1 = \sum_{\substack{|u|=n \\ z_u \neq \partial}} \sum_{i\in\mathbb{N}} \mathbb{1}\{z_{ui} - z_u \neq \partial\} \overset{(\mathrm{d})}{=} \sum_{i=1}^{Z_n} X_i,$$

where the X_i are i.i.d., independent of Z_n and

$$X_i \overset{(\mathrm{d})}{=} \sum_{i\in\mathbb{N}} \mathbb{1}\{z_i \neq \partial\} = Z_1.$$

Therefore, (Z_n) is a Galton-Watson process with reproduction $\mathscr{L}(Z_1)$. In particular, we have

$$P(S > 0) \iff E[Z_1] > 1$$

(see any standard text on branching processes for this result, for example [AH83], Proposition II.1.2).

The next step is to determine conditions under which Theorem II.5 can be applied. This leads to a study of the behaviour of the left-most individual in the BRW, which we will develop next.

II.3 The left-most individual

[Big77a] showed that the left-most individual L_n in the branching random walk follows a law of large numbers, i.e. that $\frac{L_n}{n}$ converges a.s. to a constant on the survival set. We are going to present his proof but skip some technical details of it.

Before stating the theorem we need some definitions, which will also be important throughout the rest of this paper. Let μ denote the intensity measure of the (not necessarily locally finite) point process $\sum_{i\in\mathbb{N}} \delta_{z_i}$, i.e.

$$\mu(A) := E[\sum_{i\in\mathbb{N}} \mathbb{1}\{z_i \in A\}] = E[\#\{i \in \mathbb{N} : z_i \in A\}]$$

for any Borel subset A of $\mathbb{R}$. Now define

$$v(\theta) := \log E[\sum_{i\in\mathbb{N}} \mathrm{e}^{-\theta z_i}] = \log \int_{\mathbb{R}} \mathrm{e}^{-\theta x}\mu(dx)$$

for $\theta \geq 0$ and $v(\theta) := +\infty$ for $\theta < 0$. v is the logarithm of the Laplace transform of a positive measure and therefore convex, lower semi-continuous, continuous on $\{\theta : v(\theta) < \infty\}$ and differentiable on $\mathrm{int}\{\theta : v(\theta) < \infty\}$.

In what follows, we suppose that there exists $\theta > 0$, s.t. $v(\theta) < \infty$. Define

$$I(a) := \sup_{\theta\in\mathbb{R}}\{-a\theta - v(\theta)\} \in (-\infty, \infty],$$

so that $I(-a)$ is the Legendre transform of v. It follows that I is convex and therefore continuous on $\{a : I(a) < \infty\}$. Since $v(\theta) = \infty$ for $\theta < 0$, it follows that I is non-increasing and thus by convexity strictly decreasing until it reaches its minimum $-v(0)$. We further define

$$\gamma := \inf\{a : I(a) < 0\}.$$

We may now state the

Theorem II.7. *Let $v(\theta) < \infty$ for some θ. Let S be the survival set of the branching random walk and L_n the left-most individual in the n-th generation. Then:*

$$\frac{L_n}{n} \xrightarrow{n\to\infty} \gamma \quad \text{a.s. on } S.$$

Before giving the proof a

Remark II.8. The interesting case for the theorem is when $P(S) > 0$. Since we have

$$v(0) = \log E[\sum_{i\in\mathbb{N}} \mathbb{1}\{z_i \neq \partial\}] = \log E[Z_1],$$

this is exactly the case when $v(0) > 0$ (see Remark II.6). In this situation, $I(a) = 0 \iff a = \gamma$ and $\gamma < 0$, $= 0$ or > 0 according to whether $\inf_\theta v(\theta) < 0$, $= 0$ or > 0, respectively.

Proof of Theorem II.7. Define

$$Z_n(t) := E[\sum_{|u|=n} \mathbb{1}\{z_u \leq t\}]$$

and

$$F(t) := E[Z_1(t)] = \mu((-\infty, t]).$$

The assumption $v(\theta) < \infty$ for some $\theta > 0$ entails that $F(t) < \infty$ for all $t \in \mathbb{R}$, so that $v(\theta) = \int_{-\infty}^{\infty} \mathrm{e}^{-\theta x} dF(x)$. We claim that $\forall n \in \mathbb{N} : F^{n*}(t) = E[Z_n(t)]$, where F^{n*} denotes the n-fold Stieltjes convolution of F ($F^{0*} = \delta_0$). This is obviously true for $n = 0$. Assume that it holds for some n, then

$$\begin{aligned}
E[Z_{n+1}(t)] &= E[\sum_{|u|=n}\sum_{i\in\mathbb{N}} \mathbb{1}\{z_{ui} \le t\}] \\
&= E[\sum_{|u|=n} E[\sum_{i\in\mathbb{N}} \mathbb{1}\{z_{ui} - z_u \le t - z_u\} \mid \mathcal{F}_n]] \\
&= E[\sum_{|u|=n} F(t - z_u)] && \text{by definition II.1} \\
&= \int_{-\infty}^{\infty} F(t-x)dF^{n*}(x) && \text{by ind. hypothesis} \\
&= F^{n*}(t)
\end{aligned}$$

In his paper, Biggins obtained a large deviations result for the family $(F^{n*})_{n\in\mathbb{N}}$, similar to Cramér's famous theorem.

Lemma II.9. *Let $a \in \mathbb{R}$.*

a) $\forall n \in \mathbb{N} : \frac{1}{n}\log F^{n*}(na) \le -I(a)$

b) $\frac{1}{n}\log F^{n*}(na) \overset{n\to\infty}{\longrightarrow} -I(a)$

We skip the proof of this result, since it is similar to the standard one with some additional technical difficulties.

By Remark II.8, $I(a)$ is strictly decreasing at γ if $P(S) > 0$. Theorem II.7 is therefore a corollary of the following lemma, which is the stochastic equivalent of Lemma II.9.

Lemma II.10. *Let $a \in \mathbb{R}$.*

a) If $I(a) > 0$, then a.s. $Z_n(na) = 0$ for almost all n.

b) If $I(a) < 0$, then

$$\frac{1}{n}\log Z_n(na) \overset{n\to\infty}{\longrightarrow} -I(a) \quad \text{a.s. on } S.$$

Proof. For $c < I(a)$ we have

$$\begin{aligned}
P(\frac{1}{n}\log Z_1(na) \ge -c) &= P(Z_1(na) \ge \mathrm{e}^{-cn}) \\
&\le \mathrm{e}^{cn}E[Z_1(na)] && \text{by the Markov inequality} \\
&\le \mathrm{e}^{(c-I(a))n} && \text{by Lemma II.9 a)}
\end{aligned}$$

Hence, $\limsup_{n\to\infty}\frac{1}{n}\log Z_n(na) \le -c$ by the virtue of the Borel-Cantelli lemma. Letting c tend to $-I(a)$, this gives

$$\limsup_{n\to\infty}\frac{1}{n}\log Z_n(na) \le -I(a).$$

Since Z_n is integer-valued, this means that $Z_n(na)$ must be ultimately 0 if $I(a) < 0$. We have thus proved part a) and the first half of part b).

The second half is more delicate and will be tackled by reduction to a known convergence result about Galton-Watson processes. Assume w.l.o.g. that $v(0) > 0$ (otherwise $P(S) = 0$). Then, due to Remark II.6, $\{x : I(x) < 0\}$ is an open subset of $\mathbb{R}$ and thus there exists $b < a$ s.t. $I(b) < 0$. Thanks to Lemma II.9, there exists a $k_0 \in \mathbb{N}$, s.t. $\forall k \geq k_0 : F^{k*}(kb) > 1$.

Fix $k \geq k_0$ and let $j : \mathbb{N} \to \mathbb{N}^k$ be some bijective function. We define a new process (ζ_u): For $u_1, \dots, u_n \in \mathbb{N}$ let

$$\zeta_{u_1 \cdots u_n} := \begin{cases} z_{j(u_1)\cdots j(u_n)} & \text{if } z_{j(u_1)\cdots j(u_n)} \neq \partial \text{ and} \\ & z_{j(u_1)\cdots j(u_i)} - z_{j(u_1)\cdots j(u_{i-1})} \leq kb,\ i = 1, \dots, n \\ \partial & \text{otherwise} \end{cases}$$

Intuitively, this means that the n-th generation of ζ consists of the (nk)-th generation of z, where we keep only those individuals whose ancestor in the (ik)-th generation is at most (kb) units to the right of the ancestor in the $((i-1)k)$-th generation. Call this process the $(b, k, \emptyset)$-extreme process. Using the branching property (Proposition II.2) we can repeat the above definition on the set $\{z_u \neq \partial\}$ for the BRW $(z_{u\bullet} - z_u)$ initiated by u, yielding the *(b, k, u)-extreme process*.

One can show that conditioned on $z_u \neq \partial$ the (b, k, u)-extreme process is again a BRW, so that N_n, the number of individuals in its n-th generation, is a Galton-Watson process (see Remark II.6. Clearly, $E[N_1] = F^{k*}(kb) > 1$. The known result about Galton-Watson processes mentioned above is

Lemma II.11.
$$\frac{1}{n} \log N_n \overset{n\to\infty}{\longrightarrow} \log F^{k*}(kb)$$
a.s. on the survival set of (N_n).

This result can be deduced easily from Corollary II.1.6, Lemma II.5.4 or Corollary III.5.3 of [AH83].

Now fix $r \in \mathbb{N}$ and $s \in \mathbb{N}$ with $0 \leq s < k$. If u is an individual in the $(rk + s)$-th generation then we can construct the (b, k, u)-extreme process if $z_u \neq \partial$. Let $N_n(u)$ be the number of individuals in its n-th generation. The people in the n-th generation of the (b, k, u)-extreme process lie to the left of $z_u + nkb$ and therefore also to the left of $((n + r)k + s)a$ for large n. Since they are part of the $((n + r)k + s)$-th generation of the original process z, we have

$$\begin{aligned} &\liminf_{m\to\infty} \frac{1}{mk+s} \log Z_{mk+s}\left((mk+s)a\right) \\ =&\liminf_{n\to\infty} \frac{1}{nk} \log Z_{(n+r)k+s}(((n+r)k+s)a) \\ \geq&\liminf_{n\to\infty} \frac{1}{nk} \log N_n(u) \\ =&\frac{1}{k} \log F^{k*}(kb) \quad \text{by Lemma II.11} \end{aligned} \tag{II.1}$$

a.s. on $S_{r,u} = \{\omega : z_u \neq \partial$ and the (b, k, u)-extreme process survives$\}$. This inequality therefore holds a.s. on $S_r = \bigcup_{|u|=rk+s} S_{r,u}$, which is the event that (z_u) has survived until the $(rk+s)$-th generation and that at least one of the extreme processes survived forever. It is clear that $S_r \subset S_{r+1} \subset S\ \forall r \in \mathbb{N}$ and one can show that in fact $S = \bigcup_r S_r$ a.s., therefore II.1 holds on S. It follows that

$$\liminf_{n\in\infty} \frac{1}{n} \log Z_n(na) \geq \frac{1}{k} \log F^{k*}(kb) \quad \text{a.s. on } S$$

and letting k go to infinity, Lemma II.9 gives us $\liminf_{n\to\infty} \frac{1}{n} \log Z_n(na) \geq -I(b)$ a.s. on S. Since I is continuous at a we can let b tend up to a to see that

$$\liminf_{n\to\infty} \frac{1}{n} \log Z_n(na) \geq -I(a) \quad \text{a.s. on } S.$$

This completes the proof. □

We can now put the pieces together to formulate

Theorem II.12. *If $\mathcal{L} \neq \emptyset$, then $v(0) > 0$ and $\exists\theta : v(\theta) \leq 0$.*

Proof. By contradiction. The case $v(0) \leq 0$ follows from Theorem II.7, since in this case $P(S) = 0$ (see Remark II.8). Assume from now on that $v(\theta) > 0$ for all $\theta \in \mathbb{R}$, thus $\inf_\theta v(\theta) \geq 0$.

Let us first look at the case $\inf_\theta v(\theta) = 0$. Since v is convex and lower semi-continuous, this can only happen if $v(\theta) \downarrow 0$ for $\theta \to \infty$. This implies that $\mu = \delta_0 + \nu$, where v is a measure concentrated on $(0, \infty)$ of strictly positive mass. Comparing this with (FE) this amounts to saying that for $\phi \in \mathcal{L}$

$$\phi(x) = E[\phi(x)^N \cdot \prod_{\substack{i\in\mathbb{N} \\ z_i>0}} \phi(x\mathrm{e}^{-z_i})],$$

where $N = \sum_i \mathbb{1}\{z_i = 0\}$ and $E[N] = 1$. Since $\nu \neq 0$, there exists a $c \in (0, 1)$, s.t. the event $A := \{\omega : \exists i \in \mathbb{N} : \mathrm{e}^{-z_i} \in (c, 1)\}$ has positive probability. We thus have

$$\phi(x) \leq E[\phi(x)^N \cdot \phi(xc)^{\mathbb{1}_A}] \leq E[\phi(x)^N \cdot \phi(x)^{c\mathbb{1}_A}] = E[\phi(x)^{N+c\mathbb{1}_A}],$$

where the second inequality stems from the fact that

$$\phi(xc) = E[\mathrm{e}^{xcY}] = E[(\mathrm{e}^{xY})^c] \leq \left(E[\mathrm{e}^{xY}]\right)^c = \phi(x)^c,$$

because the function $x \mapsto x^c$ is concave for $c < 1$. This entails that $\varphi(s) := E[s^{N+c\mathbb{1}_A}] \geq s$ for s in $(\phi(\infty), 1]$, whence

$$1 \geq \varphi'(1) = E[N + c\mathbb{1}_A] > 1,$$

which is a contradiction.

Assume now that $\inf_\theta v(\theta) > 0$ and further that $v(\theta) < \infty$ for some $\theta \geq 0$. Then we have $\gamma < 0$ by Remark II.8 and so $L_n \to -\infty$ by Theorem II.7. Theorem II.5 enables us to conclude that $\mathcal{L} = \emptyset$.

The remaining case $v \equiv \infty$ will be tackled by truncation. We first need the following

Lemma II.13. *Assume that $\mathcal{L} \neq \emptyset$. Then $E[Z_1((-\infty, a])] \leq \mathrm{e}^a$ for every $a \geq 0$.*

Proof. Let $a \geq 0$. Let f be the generating function of the r.v. $Z_1((-\infty, a])$, i.e. $f(s) = E[s^{Z_1((-\infty,a])}]$. Let $\phi \in \mathcal{L}$. We have for $x \in \mathbb{R}_+$

$$\phi(x) = E[\prod_{i \in \mathbb{N}} \phi(x\mathrm{e}^{-z_i})] \leq E[\phi(x\mathrm{e}^{-a})^{Z_1((-\infty,a])}] = f(\phi(x\mathrm{e}^{-a})).$$

It follows that

$$1 - f(\phi(x\mathrm{e}^{-a})) \leq 1 - \phi(x) \leq \mathrm{e}^a(1 - \phi(x\mathrm{e}^{-a}))$$

because of the convexity of ϕ. Hence, $1 - f(s) \leq \mathrm{e}^a(1-s)$ on $(\phi(\infty), 1]$ and thus $\lim_{s\to 1} f(s) = 1$ and $f'(1) \leq \mathrm{e}^a$. This proves the lemma. □

We now proceed with the proof of Theorem II.12 for the case $v \equiv +\infty$. Assume $\mathcal{L} \neq \emptyset$ and let $\phi \in \mathcal{L}$. Lemma II.13 tells us in particular that there exists a $\beta > 0$, s.t. $\int_{[0,\infty)} \mathrm{e}^{-\beta x} \mu(dx) < \infty$. Therefore, $\mu(-\infty, 0) > 0$.

Let $c < 0$. We define the truncated reproduction $\mathscr{L}((z_i^{(c)})_{i\in\mathbb{N}})$ by:

$$z_i^{(c)} := \begin{cases} z_i & \text{if } z_i \geq c \\ \partial & \text{otherwise} \end{cases}$$

Let $z^{(c)}$ be a BRW with this reproduction and define $v^{(c)}(\theta) := E[\sum_{i\in\mathbb{N}} \mathrm{e}^{-\theta z_i}]$, which satisfies $v^{(c)}(\beta) < \infty$, since $\mu((-\infty, 0]) \leq 1$ by Lemma II.13. Then $\forall \theta \in \mathbb{R}$:

$$v^{(c)}(\theta) \overset{c\to\infty}{\longrightarrow} v(\theta) = +\infty.$$

For c large enough we have $v^{(c)}(\theta) \overset{\theta\to\infty}{\longrightarrow} +\infty$ and so, by convexity,

$$\inf_{\theta\in\mathbb{R}} v^{(c)}(\theta) \overset{c\to\infty}{\longrightarrow} +\infty.$$

There exists thus $c < 0$, s.t. $\inf_{\theta\in\mathbb{R}} v^{(c)}(\theta) > 0$. Since $v^{(c)}(\beta) < \infty$, we can apply Theorem II.7 to conclude that $L_n^{(c)}$, the left-most individual in the n-th generation of the BRW $z^{(c)}$, satisfies $L_n^{(c)} \to -\infty$ a.s., which entails $L_n \to -\infty$ a.s. We can thus use Theorem II.5 to conclude. □

It is astonishing that the function v, although it only depends on the one-dimensional marginals of $(A_i)_{i\in\mathbb{N}}$, contains enough information about the non-existence of non-trivial solutions to (FE). Of course, we don't know if we can further refine the result, and indeed we are going to show later that v actually has to satisfy $\inf_{\theta\in(0,1]} v(\theta) \leq 0$. But this is the best one can do: As we are going to show in the next chapter, the conditions $v(0) > 0$ and $\inf_{\theta\in(0,1]} v(\theta) \leq 0$ (plus some other mild assumptions) are also *sufficient* for $\mathcal{L} \neq \emptyset$, a result that is very satisfying.

Chapter III

Existence of solutions to the functional equation

Our goal in this chapter is to establish the existence of solutions to the functional equation unter the conditions $v(0) > 0$ and $\inf_{\theta\in(0,1]} v(\theta) \le 0$ plus some additional assumptions that are to be determined.

III.1 The additive martingale

Let us recall the definition of v in terms of $(A_i)_{i\in\mathbb{N}}$: If $\theta < 0$, then $v(\theta) = +\infty$ and if $\theta \ge 0$, then

$$v(\theta) = E[\sum_{i\in\mathbb{N}} \mathrm{e}^{-\theta z_i}] = E[\sum_{i\in\mathbb{N}} A_i^\theta] \quad \text{(with the rule } 0^0 = 0\text{)}.$$

In particular, $v(0) = E[\sum_{i\in\mathbb{N}} \mathbb{1}\{A_i > 0\}]$ and $v(1) = E[\sum_{i\in\mathbb{N}} A_i]$. We define

$$N := \sum_{i\in\mathbb{N}} \mathbb{1}\{A_i > 0\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Then, $E[N] = v(0)$.

If $v(\theta) < \infty$ and $v(\theta_0) < \infty$ for some $\theta_0 < \theta$, then the left derivative of v exists and equals

$$v'(\theta) = -E[\sum_{i\in\mathbb{N}} z_i \mathrm{e}^{-\theta z_i}] = -E[\sum_{i\in\mathbb{N}} A_i^\theta \log A_i] \quad \text{(where } 0\log 0 = 0\text{)}.$$

If there is no such θ_0 or if $v(\theta) = \infty$ then $E[\sum_{i\in\mathbb{N}} z_i \mathrm{e}^{-\theta z_i}]$ may be well defined in spite of this, i.e. either $E[\big(\sum_{i\in\mathbb{N}} z_i \mathrm{e}^{-\theta z_i}\big)^+]$ or $E[\big(\sum_{i\in\mathbb{N}} z_i \mathrm{e}^{-\theta z_i}\big)^-]$ may be finite. In any of there cases, we say that $v'(\theta)$ *exists* and set

$$v'(\theta) = -E[\sum_{i\in\mathbb{N}} z_i \mathrm{e}^{-\theta z_i}].$$

As a first step, we shall look for conditions for (FE) to have non-trivial solutions of *finite mean*. Since $E[Y] = E[Y] \cdot E[\sum_i A_i] = E[Y] \cdot \mathrm{e}^{v(1)}$ a necessary condition is obviously

$$v(1) = 0,$$

so let us assume this for the sequel.

Define

$$W_n := \sum_{|u|=n} \mathrm{e}^{-z_u}.$$

Then we have

$$\begin{aligned}
E[W_{n+1} \mid \mathcal{F}_n] &= E[\sum_{|u|=n} \sum_{i\in\mathbb{N}} \mathrm{e}^{-z_{ui}+z_u-z_u} \mid \mathcal{F}_n] \\
&= \sum_{|u|=n} \mathrm{e}^{-z_u} E[\sum_{i\in\mathbb{N}} \mathrm{e}^{-(z_{ui}-z_u)} \mid \mathcal{F}_n] \\
&= \sum_{|u|=n} \mathrm{e}^{-z_u} \cdot \mathrm{e}^{v(1)} = W_n,
\end{aligned}$$

so that W_n is an $\mathcal{F}_n$-martingale. Since it is non-negative, it converges a.s. to a limit $W = \lim_{n\to\infty} W_n$.

W_n is called the *additive martingale* associated with the BRW and has come up much earlier than the multiplicative martingale introduced in the previous chapter. Its study goes back at least to [Big77b], where the author gave an *L log L-type condition* for this martingale to be uniformly integrable.

Let us see how the additive martingale is related to solutions to (FE). Define

$$\overline{W}_n^{(i)} := \begin{cases} \sum_{|u|=n} \mathrm{e}^{-(z_{iu}-z_i)} & \text{if } z_u \in \mathbb{R} \\ 0 & \text{if } z_u = \partial \end{cases}$$

Then

$$W_{n+1} = \sum_{i\in\mathbb{N}} \sum_{|u|=n} \mathrm{e}^{-(z_{iu}-z_i)-z_i} = \sum_{i\in\mathbb{N}} \mathrm{e}^{-z_i} \cdot \overline{W}_n^{(i)} = \sum_{i\in\mathbb{N}} A_i \cdot \overline{W}_n^{(i)},$$

and making use of the branching property and the fact that $A_i = 0 \iff z_i = \partial$ we obtain

$$W_{n+1} \stackrel{(\mathrm{d})}{=} \sum_{i\in\mathbb{N}} A_i W_n^{(i)},$$

where the $(W_n^{(i)})$ are independent copies of (W_n) and independent of $(A_i)_{i\in\mathbb{N}}$.

If W_n converges in L^1 to W, then $E[\sum_{i\in\mathbb{N}} A_i] < \infty$ implies that the right-hand side of the equation converges in L^1 as well and has the limit $\sum_{i\in\mathbb{N}} A_i W^{(i)}$, with $W^{(i)} = \lim_{n\to\infty} W_n^{(i)}$. It follows that $\mathscr{L}(W)$ satisfies (FPE) and moreover that $E[W] = 1$. We have thus obtained the following

Proposition III.1. *Assume $v(1) = 0$. If the martingale $(W_n)_{n\in\mathbb{N}}$ is uniformly integrable, then there is a non-trivial solution to (FE) of finite mean.*

Remark III.2. [Iks04] showed that the L^1-convergence is also necessary for (FE) to have a non-trivial solution to (FE) of finite mean and further generalised this to "α-elementary fixed points". The reader should be careful while consulting his article, however, since it contains several errors (in particular, Propositions 1b and 3c are erroneous).

III.2 Lyons' measure change argument

In view of Proposition III.1, the next step is to determine conditions for the martingale (W_n) to be uniformly integrable. As said before, the definite criterion is an L log L-type condition found by [Big77b]. In our proof, we will follow [Lyo97], who presented a simple proof based on an *measure change* argument originally introduced in [LPP95] for the Galton-Watson process. This paper saw a lot of interest, although their argument was already used before in similar settings, for example in [CR88] for branching Brownian motion. See [Lyo97] and [LPP95] for further references.

Let z be a BRW with reproduction Q and let $\mathbb{B}$ be its law on the space $\mathbb{T}$ of labelled trees (see section II.1). Assume $E[\sum_{i\in\mathbb{N}} \mathrm{e}^{-z_i}] = \mathrm{e}^{v(1)} = 1$. If $\xi_0 = \varnothing, \xi_1, \xi_2, \ldots$ is a sequence of individuals with $\xi_i = m(\xi_{i+1})$ $\forall i \in \mathbb{N}_0$, then we call $\xi = (\xi_i)_{i\in\mathbb{N}_0}$ a *trunk*. Note that a trunk can also be viewed as a sequence $(n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, hence we define $\mathbb{N}^{\mathbb{N}}$ as the *space of trunks*. Then $\mathbb{T}^* = \mathbb{T} \times \mathbb{N}^{\mathbb{N}}$ is called the *space of labelled trees with trunk*.

We define two filtrations $(\mathcal{F}_n^*)$ and $(\mathcal{G}_n^*)$ over $\mathbb{T}^*$ by

$$\mathcal{F}_n^* = \mathcal{F}_n \otimes \sigma\{\xi_i; 0 \le i \le n\}$$

and

$$\mathcal{G}_n^* = \mathcal{F}_{n+1} \otimes \sigma\{\xi_i; 0 \le i \le n\}.$$

Then any $\mathcal{F}_n^*$-measurable function f can be written as

$$f(z,\xi) = \sum_{|u|=n} f_u(z) \mathbb{1}\{\xi_n = u\},$$

where the f_u are $\mathcal{F}_n$-measurable.

We define a branching process (z,ξ) on $\mathbb{T}^*$ in the following way: Start with $z_\varnothing = 0$, $\xi_0 = \varnothing$. Now given $\mathcal{F}_n^*$ generate the next generation of the tree as for the BRW except the children of ξ_n: $(z_{\xi_n i} - z_{\xi_n})_{i\in\mathbb{N}}$ follows the law $\hat{Q}$ which is defined by its Radon-Nikodym derivative w.r.t. Q:

$$\frac{d\hat{Q}}{dQ}((z_i)_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} \mathrm{e}^{-z_i}.$$

Note that $\hat{Q}(\partial,\partial,\ldots) = 0$. In the second step, conditionally on $\mathcal{G}_n^*$, choose $\xi_{n+1} = u$ with probability $\mathrm{e}^{-(z_{ui}-z_u)} / \sum_{i\in\mathbb{N}} \mathrm{e}^{-(z_{ui}-z_u)}$. We denote the law of the branching process thus defined by $\hat{\mathbb{B}}^*$.

Another way of seeing this is the following: For $n \in \mathbb{N}_0$ let $\mathbb{B}_n^*$ be the (non-probability) measure over $\mathbb{T}^*$ satisfying

$$\int f(z,\xi) d\mathbb{B}_n^* = \int \sum_{|u|=n} f_u(z) d\mathbb{B}_n. \tag{III.1}$$

for every measurable function $f : \mathbb{T} \to \mathbb{R}_+$. Let $\hat{\mathbb{B}}_n^*$ be the restriction of $\hat{\mathbb{B}}^*$ to $\mathcal{F}_n^*$. Then the above definition translates to:

$$\begin{aligned}\frac{d\hat{\mathbb{B}}_{n+1}^*}{d\mathbb{B}_{n+1}^*}(z,\xi) &= \frac{d\hat{\mathbb{B}}_n^*}{d\mathbb{B}_n^*}(z,\xi) \cdot \frac{d\hat{Q}}{dQ}((z_{\xi_n i} - z_{\xi_n})_{i\in\mathbb{N}}) \cdot \frac{\mathrm{e}^{-(z_{\xi_{n+1}} - z_{\xi_n})}}{\sum_{i\in\mathbb{N}} \mathrm{e}^{-(z_{\xi_n i} - z_{\xi_n})}} \\ &= \frac{d\hat{\mathbb{B}}_n^*}{d\mathbb{B}_n^*}(z,\xi) \cdot \mathrm{e}^{-(z_{\xi_{n+1}} - z_{\xi_n})}\end{aligned}$$

Hence,

$$\frac{d\hat{\mathbb{B}}_n^*}{d\mathbb{B}_n^*}(z,\xi) = \mathrm{e}^{-z_{\xi_n}} \tag{III.2}$$

and if $\hat{\mathbb{B}}$ denotes the projection of $\hat{\mathbb{B}}^*$ onto $\mathbb{T}$, it follows for any measurable function $f : \mathbb{T} \to \mathbb{R}_+$:

$$\int f(z)d\hat{\mathbb{B}}_n = \int f(z)d\hat{\mathbb{B}}_n^* = \int f(z)\mathrm{e}^{-z_{\xi_n}}d\mathbb{B}_n^* = \int \sum_{|u|=n} \mathrm{e}^{-z_u} f(z)d\mathbb{B}_n,$$

and therefore

$$\frac{d\hat{\mathbb{B}}_n}{d\mathbb{B}_n}(z) = \sum_{|u|=n} \mathrm{e}^{-z_u} = W_n(z) \tag{III.3}$$

The following result, taken from [Dur91], page 210, Exercise 3.6, will enable us to study the convergence of W_n with the help of the measure $\hat{\mathbb{B}}$:

Lemma III.3. *On some measurable space with a filtration $(\mathcal{F}_n)$, let μ and ν be probability measures and let μ_n and ν_n be the restrictions of μ and ν to $\mathcal{F}_n$, respectively. Assume $\mu_n \ll \nu_n$ for every n and define $X := \limsup_{n\to\infty} \frac{d\mu_n}{d\nu_n}$. Then*

a) $\mu \ll \nu \iff \mu(X < \infty) = 1 \iff \nu[X] = 1$

b) $\mu \perp \nu \iff \mu(X = \infty) = 1 \iff \nu[X] = 0$

Remark III.4. It is easy to verify that under the conditions of Lemma III.3, $(\frac{d\mu_n}{d\nu_n})$ is always an $(\mathcal{F}_n)$-martingale under ν and $(\frac{d\nu_n}{d\mu_n})$ an $(\mathcal{F}_n)$-martingale under μ.

Before continuing on the convergence of W_n, we give an important theorem, which allows us to use random walk techniques to study the branching random walk and which follows easily from the measure change introduced above.

Theorem III.5. *Let $n \in \mathbb{N}_0$ and $f : \overline{\mathbb{R}}^{n+1} \to \mathbb{R}_+$ be measurable. Then*

$$\mathbb{B}[\sum_{|u|=n} \mathrm{e}^{-z_u} f(z_v; v \le u)] = \hat{\mathbb{B}}^*[f(z_{\xi_i}; i \le n)] = E[f(S_k; k \le n)],$$

where $(S_k)_{k\in\mathbb{N}_0}$ is a random walk starting at $S_0 = 0$ and

$$E[g(S_1)] = \mathbb{B}[\sum_{i\in\mathbb{N}} \mathrm{e}^{-z_i} g(z_i)] = \int_{\mathbb{R}} g(t)\mathrm{e}^{-t}\mu(dt)$$

for any measurable function g.

Proof. We begin with the first equality:

$$
\begin{aligned}
&\mathbb{B}[\sum_{|u|=n} \mathrm{e}^{-z_u} f(z_v; v \le u)] \\
=&\mathbb{B}^*[\sum_{|u|=n} \mathbb{1}\{\xi_n = u\}\mathrm{e}^{-z_u} f(z_v; v \le u)] && \text{by equation (III.1)} \\
=&\sum_{|u|=n} \mathbb{B}^*[\mathbb{1}\{\xi_n = u\}\mathrm{e}^{-z_{\xi_n}} f(z_{\xi_n}; i \le n)] \\
=&\sum_{|u|=n} \hat{\mathbb{B}}^*[\mathbb{1}\{\xi_n = u\} f(z_{\xi_n}; i \le n)] && \text{by equation (III.2)} \\
=&\hat{\mathbb{B}}^*[f(z_{\xi_i}; i \le n)]
\end{aligned}
$$

This equality gives with $n = 1$: $\hat{\mathbb{B}}^*[g(z_{\xi_1})] = E[g(S_1)]$ and so the second equality follows since the increments $z_{\xi_i} - z_{\xi_{i-1}}$ are independent under $\hat{\mathbb{B}}^*$ by definition. □

We may now state the theorem we were all waiting for:

Theorem III.6. *Let $v(1) = 0$, $v'(1)$ exist with $v'(1) < 0$ and $\mathbb{B}[W_1 \log^+(W_1)] < \infty$. Then W_n converges in L^1.*

Proof. For a labelled tree with trunk (z, ξ) we define for all $n \in \mathbb{N}_0$:

$$
\begin{aligned}
\zeta_{n,i} &= z_{\xi_n i} - z_{\xi_n}, \\
X_n &= \sum_{i \in \mathbb{N}} \mathrm{e}^{-\zeta_{n,i}} \text{ and}
\end{aligned}
$$

$\mathcal{H}$ the σ-field generated by ξ and the $\zeta_{n,i}$, $n \in \mathbb{N}_0$, $i \in \mathbb{N}$. Then

$$
\begin{aligned}
\hat{\mathbb{B}}^*[W_{n+1} \mid \mathcal{H}] &= \hat{\mathbb{B}}^*[\sum_{\substack{|u|=n \\ u \neq \xi_n}} \sum_{i \in \mathbb{N}} \mathrm{e}^{-z_{ui}} + \mathrm{e}^{-z_{\xi_n}} X_n \mid \mathcal{H}] \\
&= \hat{\mathbb{B}}^*[\sum_{\substack{|u|=n \\ u \neq \xi_n}} \mathrm{e}^{-z_u} \mid \mathcal{H}] \cdot \mathrm{e}^{v(1)} + \mathrm{e}^{-z_{\xi_n}} X_n \\
&= \hat{\mathbb{B}}^*[W_n \mid \mathcal{H}] + \mathrm{e}^{-z_{\xi_n}}(X_n - 1).
\end{aligned}
$$

Hence,

$$
\hat{\mathbb{B}}^*[W_n \mid \mathcal{H}] = 1 + \sum_{k=0}^{n-1} \mathrm{e}^{-z_{\xi_k}}(X_k - 1).
$$

Theorem III.5 tells us that z_{ξ_k} is a random walk under $\hat{\mathbb{B}}^*$ with

$$
E[z_{\xi_1}] = \mathbb{B}[\sum_{i \in \mathbb{N}} z_i \mathrm{e}^{-z_i}] = -v'(1) > 0
$$

by assumption. Thus, the terms $\mathrm{e}^{-z_{\xi_k}}$ decay at least exponentially $\hat{\mathbb{B}}^*$-a.s.

To estimate the growth of $(X_k)_k$, we will use the following

Lemma III.7. *On some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $Z_1, Z_2, \ldots$ be non-negative i.i.d. random variables. Then, $\mathbb{P}$-a.s.:*

$$\limsup_{n\to\infty} \frac{1}{n} Z_n = \begin{cases} 0 & \text{if } E[Z_1] < \infty \\ \infty & \text{if } E[Z_1] = \infty \end{cases}$$

Proof. This follows from the Borel-Cantelli lemma and the fact that

$$\frac{1}{a} E[Z_1] \le \sum_{n=0}^{\infty} \mathbb{P}(Z_1 \ge na) \le \frac{1}{a} E[Z_1 + 1]$$

for every $a > 0$. □

We can now apply Lemma III.7 with $Z_i = \log X_i$ since the X_i are i.i.d. under $\hat{\mathbb{B}}^*$, $X_1 = W_1$ and

$$\begin{aligned} \hat{\mathbb{B}}^*[\log W_1] &= \hat{\mathbb{B}}[\log W_1] \\ &= \mathbb{B}[W_1 \log W_1] && \text{by equation (III.3)} \\ &< \infty && \text{by assumption.} \end{aligned}$$

Thus, $\limsup_{n\to\infty} \frac{1}{n} \log X_n = 0$ $\hat{\mathbb{B}}^*$-a.s., or, in other words, X_n grows at most subexponentially $\hat{\mathbb{B}}^*$-a.s. It follows that

$$\liminf_{n\to\infty} \hat{\mathbb{B}}^*[W_n \mid \mathcal{H}] < \infty \quad \hat{\mathbb{B}}^*\text{-a.s.}$$

and so by the conditional Fatou lemma:

$$\hat{\mathbb{B}}^*[\liminf_{n\to\infty} W_n \mid \mathcal{H}] < \infty \quad \hat{\mathbb{B}}^*\text{-a.s.},$$

whence,

$$\liminf_{n\to\infty} W_n < \infty \quad \hat{\mathbb{B}}^*\text{-a.s.}$$

and since W_n does not depend on ξ,

$$\liminf_{n\to\infty} W_n < \infty \quad \hat{\mathbb{B}}\text{-a.s.}$$

But $(\frac{1}{W_n})$ is a martingale under $\hat{\mathbb{B}}$ (see Remark III.4), hence $\lim_{n\to\infty} W_n = W$ exists $\hat{\mathbb{B}}$-a.s. Applying Lemma III.3 now gives $\mathbb{B}[W] = 1$, which concludes the proof. □

Remark III.8. Under the additional condition $v'(1) > -\infty$, [Lyo97] showed further that the conditions $v'(1) < 0$ and $\mathbb{B}[W_1 \log^+ W_1] < \infty$ are actually necessary for W_n to converge in L^1, and, furthermore, that the limit is 0 a.s. if W_n does not converge in L^1. [Iks04] established sufficient and necessary conditions for the remaining cases, i.e. if $v'(1) = -\infty$ or if $v'(1)$ does not exist.

We are now going to drop the condition $v(1) = 0$ with the help of the so-called *stable transformation* coined by [DL83]. We obtain the following

Theorem III.9. *Suppose there exists* $\alpha \in (0, 1]$ *with*

$$v(\alpha) = 0,\ v'(\alpha) < 0 \text{ and } E[(\sum_i A_i^\alpha) \log^+(\sum_i A_i^\alpha)] < \infty.$$

Then $\mathcal{L} \neq \emptyset$.

Proof. Set $\tilde{A}_i := A_i^\alpha$ and define $\tilde{v}, \widetilde{\mathcal{L}}$ in terms of $(\tilde{A}_i)_i$. Then $\tilde{v}(\theta) = v(\alpha\theta)$, so that $\tilde{v}(1) = v(\alpha) = 0$, $\tilde{v}'(1) = \alpha v'(\alpha) < 0$ and $E[(\sum_i \tilde{A}_i) \log^+(\sum_i \tilde{A}_i)] < \infty$. Theorem III.6 and Proposition III.1 now imply $\widetilde{\mathcal{L}} \neq \emptyset$. Let $\psi \in \widetilde{\mathcal{L}}$ and put $\phi(\theta) := \psi(\theta^\alpha)$. Then

$$E[\prod_{i \in \mathbb{N}} \phi(\theta A_i)] = E[\prod_{i \in \mathbb{N}} \psi(\theta^\alpha \tilde{A}_i)] = \psi(\theta^\alpha) = \phi(\theta).$$

For ϕ to be in $\mathcal{L}$, we have to verify that it is indeed a Laplace transform of a probability measure. Let $(X(t))_t$ be the one-sided stable Lévy process with index α. The Laplace transform of $X(t)$ is $\mathrm{e}^{-t\theta^\alpha}$. Let τ be a r.v. with Laplace transform ψ, independent of X. Then

$$E[\mathrm{e}^{-\theta X(\tau)}] = E[\mathrm{e}^{-\theta^\alpha \tau}] = \psi(\theta^\alpha) = \phi(\theta),$$

so that ϕ is the Laplace transform of $X(\tau)$, hence $\phi \in \mathcal{L}$. □

Remark III.10. The stable transformation we have seen above works only for $\alpha \leq 1$, for if we set $\phi(\theta) = \psi(\theta^\alpha)$ for some $\alpha > 1$ and some Laplace transform ψ of a r.v. of finite mean, then $\phi'(\theta) = \alpha\theta^{\alpha-1}\psi'(\theta^\alpha)$, hence $\phi'(0) = 0$ and so $\phi \equiv 1$. This argument will be extended later for general ψ and will be used to sharpen the necessary condition for $\mathcal{L} \neq \emptyset$ established in the previous chapter (see Theorem II.12).

III.3 The boundary case

Theorem III.9 is already a good start, but not yet satisfying because of the (in our opinion) strict conditions. Following [Big77b] we could now try to drop the L log L condition by establishing a Seneta-Heyde norming of the martingale (W_n) s.t. the normed process (W_n/c_n) converges in distribution to a non-degenerate r.v. W satisfying the fixed point equation (FPE). In following this approach one has to keep the assumptions $v(1) = 0$, $v'(1) < 0$ and must introduce another one:

$$\text{(N)} \quad N := \sum_{i \in \mathbb{N}} \mathbb{1}\{A_i > 0\} = \sum_{i \in \mathbb{N}} \mathbb{1}\{z_i \in \mathbb{R}\} < \infty \quad \text{a.s.}$$

We will follow the more direct approach of [Liu98] instead, which will still need the assumption (N), but allows for example $v'(1) = 0$. We'll obtain the following

Theorem III.11. *Suppose that* $v(0) > 0$, $\inf_{\theta \in (0,1]} v(\theta) \leq 0$ *and that the assumption (N) is verified. Then* $\mathcal{L} \neq \emptyset$.

Before giving the proof a nice

Proposition III.12. *Assume (N). Let* $\phi \in \mathcal{L}$. *Then* $\phi(\infty)$ *is the extinction probability of the branching random walk z.*

Proof. By dominated convergence, we have

$$\phi(\infty) = \lim_{x\to\infty} \phi(x) = E[\lim_{x\to\infty} \prod_{i\in\mathbb{N}} \phi(xA_i)] = E[\phi(\infty)^N]$$

Hence, $\phi(\infty)$ is a fixed point of the generating function $f(s) = E[s^N]$ of N and thus, since $\phi(\infty) < 1$, it is the extinction probability of the Galton-Watson process $(Z_n(z))$ defined in Remark II.6. □

Proof. For this proof, fix the probability space Ω we are working on and let it support the $(A_i)_{i\in\mathbb{N}}$ and $U_i \sim \mathrm{Unif}(0,1)$ i.i.d. random variables independent of $(A_i)_{i\in\mathbb{N}}$. Also, assume w.l.o.g. that the A_i are a.s. decreasing with i.

For any $M \in \mathbb{R}_+$ define

$$A_i^{(M)} := \begin{cases} (A_i \wedge M)\cdot \mathbb{1}\{U_i \geq \mathrm{e}^{-M}\} & \text{if } i \leq M \\ 0 & \text{otherwise} \end{cases}$$

Let v_M, $\mathcal{L}_M$ and N_M be defined in terms of $(A_i^{(M)})$. Then v_M and $\mathcal{L}_M$ have the following properties:

1. $v_M(\theta) < \infty\ \forall\theta \geq 0\ \forall M \in \mathbb{R}_+$
2. $v_M(\theta) < v(\theta)\ \forall\theta \geq 0$ and therefore
3. $\min_{\theta\in[0,1]} v_M(\theta) < 0$
4. $\forall\theta \in \mathbb{R} : v_M(\theta) \uparrow v(\theta)$ as $M \to \infty$ by monotone convergence

We deduce from 2. and 4. that there exists $M_0 \in \mathbb{R}_+$, s.t. $\forall M \geq M_0 : v_M(0) > 0$. Together with 1. and 3. this gives:

$$\forall M \geq M_0\ \exists\alpha_M \in (0,1] : v_M(\alpha) = 0 \text{ and } v_M'(\alpha) < 0.$$

In addition, $E[\left(\sum_i (A_i^{(M)})^{\alpha_M}\right)\log^+\left(\sum_i (A_i^{(M)})^{\alpha_M}\right)] \leq M\cdot M^{\alpha_M}\cdot \log^+(M\cdot M^{\alpha_M}) < \infty$. Hence, $\mathcal{L}_M \neq \emptyset$ for all $M \geq M_0$ by Theorem III.9.

For $M \geq M_0$ choose $\eta_M \in \mathcal{L}_M$ and set $q_M = \eta_M(\infty) < 1$. Proposition III.12 then shows that q_M is the extinction probability of the Galton-Watson process with offspring distribution $\mathscr{L}(N_M) = \mathscr{L}(\sum_{i\in\mathbb{N}} \mathbb{1}\{A_i^{(M)} > 0\})$. Since the $(A_i^{(M)})$ increase with M, it follows that q_M decreases with M, s.t. $\bar{q} := \lim_{M\to\infty} q_M$ exists and $\bar{q} < 1$. If q denotes the extinction probability of the Galton-Watson process with offspring distribution $\mathscr{L}(N)$, then $q \leq q_M\ \forall M$, hence $q \leq \bar{q}$.

Now choose c_M such that $\eta_M(c_M) = \frac{1}{2}(q_M + 1)$ and set $\phi_M(x) = \eta_M(c_M x)$. Evidently, $\phi_M \in \mathcal{L}_M$ and $\phi_M(1) = \frac{1}{2}(q_M + 1)$. By Helly's selection theorem for distributions and Lévy's continuity theorem for Laplace transforms (see e.g. [Fel71], Theorems VIII.6.1 and XIII.1.2) there exists a sequence $M_n \to \infty$ s.t. $\phi_{M_n} \to \phi$ pointwise as $n \to \infty$, where ϕ is the Laplace transform of a possibly defective probability measure. Evidently, $\phi(1) = \lim \phi_{M_n}(1) = \frac{1}{2}(\bar{q} + 1)$.

Since for almost each $\omega \in \Omega$ and every $i \in \mathbb{N}$ there exists an $n_i \in \mathbb{N}$ s.t. $A_i^{(M_n)} = A_i \ \forall n \geq n_i$, it follows

$$\phi_{M_n}(x) = E[\prod_{i=1}^{N} \phi_{M_n}(xA_i^{(M_n)}) \overset{n\to\infty}{\longrightarrow} \prod_{i=1}^{N} \phi(xA_i) \text{ a.s.}$$

It follows by the dominated convergence theorem that ϕ satisfies (FE).

Letting $x \downarrow 0$ we see that $\phi(0+) = f(\phi(0+))$, where $f(s) = E[s^N]$. Thus, $\phi(0+) \in \{q, 1\}$. But since $\phi(0+) \geq \phi(1) = \frac{1}{2}(\bar{q}+1) > q$, $\phi(0+)$ must be 1. Thus ϕ is the Laplace transform of a proper probability distribution and so $\phi \in \mathcal{L}$. □

We summarise the results of this section in the following

Theorem III.13. *Suppose* $v(0) > 0$ *and that at least one of the two following conditions hold:*

(H1) There exists an $\alpha \in (0,1]$ *with* $v(\alpha) = 0,\ v'(\alpha) < 0$ *and* $E[(\sum_i A_i^{\alpha}) \log^+(\sum_i A_i^{\alpha})] < \infty$

(H2) $\inf_{\theta\in(0,1]} v(\theta) \leq 0$ *and* $\sum_{i\in\mathbb{N}} \mathbb{1}\{A_i > 0\} < \infty$ *a.s.*

Then $\mathcal{L} \neq \emptyset$.

Remark III.14. Condition (H2) contains the case where $\inf_{\theta\in(0,1]} v(\theta) < 0$ but there is no $\alpha \in (0,1]$ with $v(\alpha) = 0$. This is possible for example if $v(\theta) < 0$ on an interval $[a,b]$ and $v(\theta) = +\infty$ outside of this interval.

Chapter IV

The general framework

It is now time to present the general framework due to [BK04], of which the boundary case is a special case. We will further introduce optional lines, the analogue to stopping times for Markov processes indexed by a subset of the real numbers. We will use this to prove uniqueness of the solutions to (FE).

IV.1 Branching processes with Markovian reproduction

Let $U = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ be the space of individuals, or universe, as before. Instead of assigning to each individual a position on the real line, i.e. elements of $\overline{\mathbb{R}}$, we use an arbitrary measurable space $\mathcal{S}$, called the *label space* and define the *space of labelled trees* $\mathbb{T} = \mathcal{S}^U$. Instead of a reproduction Q on $\mathcal{S}^{\mathbb{N}}$, we how have a general *reproduction kernel* $P_s(d\vec{x})$, $s \in \mathcal{S}$, $\vec{x} \in \mathcal{S}^{\mathbb{N}}$ that defines now an individual with label s reproduces. The filtration $(\mathcal{F}_n)$ is defined as before. In analogue to the BRW we present the

Definition IV.1. Let $\mathcal{S}$ be a measurable space, $s_0 \in \mathcal{S}$, $P_s(d\vec{x})$ a reproduction kernel, $\mathbb{P}$ a probability measure on $\mathbb{T} = \mathcal{S}^U$ and z the canonical process. $\mathbb{P}$ is called the *law of a branching process with reproduction kernel $P_s(d\vec{x})$ starting at s_0* iff

1. $z_\varnothing = s_0$ $\mathbb{P}$-a.s.
2. for all measurable functions $f_u : \mathcal{S}^{\mathbb{N}} \to \mathbb{R}_+$ we have

$$\mathbb{P}[\prod_{|u|=n} f_u((z_{ui})_{i\in\mathbb{N}}) \mid \mathcal{F}_n] = \prod_{|u|=n} \mathbb{P}_{z_u}[f_u].$$

As before, uniqueness is evident and existence follows from Ionescu Tulcea's theorem, so that we can define $\mathbb{A}_{s_0} := \mathbb{P}$. If the starting label is not specified (and possibly a random variable), or if it is implicitly known, then we will also simply write $\mathbb{A}$.

As before, we have the *branching property*:

Proposition IV.2. *For each $s \in \mathcal{S}$, let $\mathbb{A}_s$ be the law of a branching process with reproduction kernel P starting at s. Then for all $s_0 \in \mathcal{S}$ and all measurable functions $f_u : \mathbb{T} \to \mathbb{R}_+$ we have*

$$\mathbb{A}[\prod_{|u|=n} f_u(z_{u\bullet}) \mid \mathcal{F}_n] = \prod_{|u|=n} \mathbb{A}_{z_u}[f_u].$$

One of the primary objects of study in the BRW was the martingale (W_n), so we would like to generalise it to our new model. The right notion for this is the following

Definition IV.3. Let $h : \mathcal{S} \to \mathbb{R}_+$ be a measurable function. h is called *mean-harmonic* iff

1. $h \not\equiv 0$ and
2. $P_s[\sum_{i\in\mathbb{N}} h(x_i)] = h(s)\ \forall s \in \mathcal{S}$.

If (z_u) is a branching process with reproduction kernel $P_s(d\vec{x})$, then we define the *additive martingale*

$$W_n(z) := \sum_{|u|=n} h(z_u),$$

which is an $(\mathcal{F}_n)$-martingale, since

$$\mathbb{A}[W_{n+1} \mid \mathcal{F}_n] = \mathbb{A}[\sum_{|u|=n}\sum_{i\in\mathbb{N}} h(z_{ui}) \mid \mathcal{F}_n] = \sum_{|u|=n} P_{z_u}[\sum_{i\in\mathbb{N}} h(x_i)] = W_n.$$

To be sure that we are on the right track, we verify that in setting

$$\begin{aligned} \mathcal{S} &= \overline{\mathbb{R}}, \\ P_s &= Q_s \text{ and} \\ h(s) &= \mathrm{e}^{-s} \end{aligned}$$

we are indeed in the setting of chapter II.

The measure change of [Lyo97] readily transfers to our new model. Again, we define a *trunk* $\xi = (\xi_i)_{i\in\mathbb{N}}$ as a sequence of individuals with $\xi_0 = \varnothing$ and $\xi_i = m(\xi_{i+1})$, $i = 0, 1, 2, \ldots$ and identify it with $\mathbb{N}^{\mathbb{N}}$. Then we set $\mathbb{T}^* = \mathbb{T} \times \mathbb{N}^{\mathbb{N}}$ the space of labelled trees with trunk and define $\mathcal{F}_n^*$, $\mathcal{G}_n^*$ as before.

Again, we note that every $\mathcal{F}_n^*$-measurable function $f : \mathbb{T}^* \to \mathbb{R}$ can be written as

$$f(z, \xi) = \sum_{|u|=n} f_u(z)\mathbb{1}\{\xi_n = u\},$$

where the f_u are $\mathcal{F}_n$-measurable.

Define $\mathbb{A}_n$ the restriction of $\mathbb{A}$ to $(\mathcal{F}_n)$ and call $\mathbb{A}_n^*$ the measure over $\mathbb{T}^*$ which satisfies

$$\int f(z, \xi) d\mathbb{A}_n^* = \int \sum_{|u|=n} f_u(z) d\mathbb{A}_n$$

for every $\mathcal{F}_n^*$-measurable function f. Define the reproduction kernel $\hat{P}_s(d\vec{x})$ by

$$\frac{d\hat{P}_s}{dP_s}(\vec{x}) := \begin{cases} \frac{\sum_{i\in\mathbb{N}} h(x_i)}{h(s)} & \text{if } h(s) > 0 \\ 1 & \text{if } h(s) = 0 \end{cases}$$

The mean-harmonicity of h assures that $\hat{P}_s$ is indeed a probability measure for every $s \in \mathcal{S}$.

Now assume $\mathbb{B} = \mathbb{B}_{s_0}$ with $s_0 \in \mathcal{S}^h := \{s \in \mathcal{S} : h(s) > 0\}$. We define a branching process with trunk as follows: Start with $z_\varnothing = s_0$, $\xi_0 = \varnothing$. Given $\mathcal{F}_n^*$, generate the next generation of the tree as for the BRW except the children of ξ_n: $(z_{\xi_n i})_{i\in\mathbb{N}}$ follows the law $\hat{P}_{z_{\xi_n}}$ instead. In the second step, conditionally on $\mathcal{G}_n^*$ choose $\xi_{n+1} = u$ with probability $h(z_{\xi_n i})/\sum_{i\in\mathbb{N}} h(z_{\xi_n i})$. The law of the branching process thus defined is denoted by $\hat{\mathbb{A}}^*$

If $\hat{\mathbb{A}}_n^*$ denotes the restriction of $\hat{\mathbb{A}}^*$ to $\mathcal{F}_n^*$, then we see by the above definition, that we can also define $\hat{\mathbb{A}}_n^*$ recursively by

$$\begin{aligned}\frac{d\hat{\mathbb{A}}_{n+1}^*}{d\mathbb{A}_{n+1}^*}(z,\xi) &= \frac{d\hat{\mathbb{A}}_n^*}{d\mathbb{A}_n^*}(z,\xi)\cdot\frac{d\hat{P}_{\xi_n}}{dP_{\xi_n}}((z_{\xi_n i})_{i\in\mathbb{N}})\cdot\frac{h(\xi_{n+1})}{\sum_{i\in\mathbb{N}} h(\xi_n i)}\\ &= \frac{d\hat{\mathbb{A}}_n^*}{d\mathbb{A}_n^*}(z,\xi)\cdot\frac{h(\xi_{n+1})}{h(\xi_n)}\end{aligned}$$

Then we have the simple relation

$$\frac{d\hat{\mathbb{A}}_n^*}{d\mathbb{A}_n^*}(z,\xi) = \frac{h(z_{\xi_n})}{h(z_\varnothing)}, \tag{IV.1}$$

which entails for the projection $\hat{\mathbb{A}}$ of $\hat{\mathbb{A}}^*$ to $\mathbb{T}$ that

$$\frac{d\hat{\mathbb{A}}_n}{d\mathbb{A}_n}(z) = \frac{W_n(z)}{h(z_\varnothing)} = \frac{1}{h(z_\varnothing)}\cdot\sum_{|u|=n} h(z_u).$$

On the other hand, taking into account the recursive construction of the process, we have a result similar to Theorem III.5:

$$\begin{aligned}\hat{\mathbb{A}}^*[f(z_{\xi_{n+1}}) \mid \mathcal{F}_n^*] &= \hat{\mathbb{A}}^*[\sum_{i\in\mathbb{N}}\frac{h(z_{\xi_n i})}{\sum_{j\in\mathbb{N}} h(z_{\xi_n j})}\cdot f(z_{\xi_n i}) \mid \mathcal{F}_n^*]\\ &= \hat{P}_{z_{\xi_n}}[\sum_{i\in\mathbb{N}}\frac{h(x_i)}{\sum_{j\in\mathbb{N}} h(x_i)}\cdot f(x_i)]\\ &= \frac{1}{h(z_{\xi_n})}P_{z_{\xi_n}}[\sum_{i\in\mathbb{N}} h(x_i)f(x_i)]\end{aligned}$$

This gives us

Proposition IV.4. *$(\zeta_n) := (z_{\xi_n})$ is an $(\mathcal{F}_n)$-Markov chain under $\hat{\mathbb{A}}^*$ living on $\mathcal{S}^h$ and with transition kernel*

$$\Pi_s[f] := \frac{1}{h(s)}P_s[\sum_{i\in\mathbb{N}} h(x_i)f(x_i)]\ \forall s\in\mathcal{S}^h.$$

Remark IV.5. In the case of the BRW, this becomes for $s \in \mathbb{R}$:

$$\Pi_s[f] = \mathrm{e}^s\cdot Q[\sum_{i\in\mathbb{N}}\mathrm{e}^{-(s+z_i)}f(s+z_i)] = Q[\sum_{i\in\mathbb{N}}\mathrm{e}^{-z_i}f(s+z_i)],$$

so that (ζ_n) is the random walk defined in Theorem III.5, as expected.

IV.2 Convergence of the additive martingale

In this section we develop criteria for the martingale (W_n) to converge in mean. The material, taken entirely from [BK04], is fairly advanced and can be skipped by impatient readers. It will only serve us in section VI.2.

Throughout this section, let $\mathbb{A}$ be the law of a branching process z with reproduction kernel $P_s(d\vec{x})$, let h be some mean-harmonic function and suppose that $z_\varnothing$ is constant and that $z_\varnothing \in \mathcal{S}^h$. As in the proof of Theorem III.6 we set

$$X_n := \sum_{i\in\mathbb{N}} \frac{h(z_{\xi_n i})}{h(z_{\xi_n})}$$

and define $\mathcal{H}$ to be the σ-field generated by ξ and the $z_{\xi_n i}$ for all $n \in \mathbb{N}_0$, $i \in \mathbb{N}$. We get again

$$\hat{\mathbb{A}}^*[W_n \mid \mathcal{H}] = 1 + \sum_{k=0}^{n-1} h(z_{\xi_k})(X_k - 1)$$

and conclude as before that if

$$\sum_{k=0}^{\infty} h(z_{\xi_k})X_k < \infty \quad \hat{\mathbb{A}}^*\text{-a.s.}$$

then $W < \infty$ $\hat{\mathbb{A}}$-a.s.

On the other hand, since $W_n \geq \sum_{i\in\mathbb{N}} h(z_{\xi_{n-1}i}) = h(x_{\xi_{n-1}})X_{n-1}$, we have $W = \infty$ $\hat{\mathbb{A}}$-a.s. if $\limsup_{n\to\infty} h(z_{\xi_n})X_n = \infty$ $\hat{\mathbb{A}}^*$-a.s. An application of Lemma III.3 then gives

Theorem IV.6. *a) If*

$$\sum_{n=1}^{\infty} h(z_{\xi_n})X_n < \infty \quad \hat{\mathbb{A}}^*\text{-a.s.},$$

then $\mathbb{A}[W] = W_0$ *and so* (W_n) *converges in* L^1 *to* W *under* $\mathbb{A}$.

b) If

$$\limsup_{n\to\infty} h(z_{\xi_n})X_n = \infty \quad \hat{\mathbb{A}}^*\text{-a.s.},$$

then $\mathbb{A}[W] = 0$.

In the proof of Theorem III.6 we used the fact that z_{ξ_n} was a random walk with positive drift and that the X_n were i.i.d. with $E[\log^+ X_n] < \infty$ to conclude that the sum converged a.s. In general we don't have there nice conditions, so that we have to work harder to obtain convergence.

First of all, we will exploit the Markovian character of $\hat{\mathbb{A}}^*$ with the help of Lévy's *conditional Borel-Cantelli lemma* (see for example [Che78] for a short proof):

Lemma IV.7. *Let* (X_n) *be a sequence of nonnegative r.v.'s defined on some probability space* $(\Omega, \mathcal{F}, P)$, *adapted to a filtration* $(\mathcal{F}_n)$ *and uniformly bounded by a constant. Then*

$$\sum_{n=1}^{\infty} X_n < \infty \text{ a.s.} \iff \sum_{n=1}^{\infty} E[X_n \mid \mathcal{F}_{n-1}] \text{ a.s.}$$

We get the following

Theorem IV.8. *Set* $X := X_0 = (W_1/W_0)$ *and let* ζ *be the Markov chain defined in Proposition IV.4.*

a) If

$$\sum_{n=1}^{\infty} \mathbb{A}_{\zeta_n}[X \cdot (W_0 X \wedge 1)] < \infty \quad a.s.,$$

then $\mathbb{A}[W] = W_0$.

b) If, for all $y > 0$,

$$\sum_{n=1}^{\infty} \mathbb{A}_{\zeta_n}[X \cdot \mathbb{1}\{W_0 X \geq y\}] = \infty \quad a.s.,$$

then $\mathbb{A}[W] = 0$.

Proof. Part a). We have $\hat{\mathbb{A}}^*$-a.s.

$$\begin{aligned}\sum_{n=1}^{\infty} h(z_{\xi_n})X_n < \infty &\iff \sum_{n=1}^{\infty}(h(z_{\xi_n})X_n \wedge 1) < \infty \\ &\iff \sum_{n=1}^{\infty} \hat{\mathbb{A}}^*[h(z_{\xi_n})X_n \wedge 1 \mid \mathcal{F}_n^*] < \infty \qquad \text{by Lemma IV.7}\end{aligned}$$

But, $\hat{\mathbb{A}}^*$-a.s.:

$$\hat{\mathbb{A}}^*[h(z_{\xi_n})X_n \wedge 1 \mid \mathcal{F}_n^*] = \hat{\mathbb{A}}^*_{z_{\xi_n}}[W_0 X \wedge 1] = \mathbb{A}_{z_{\xi_n}}[X \cdot (W_0 X \wedge 1)].$$

Since (z_{ξ_n}) under $\hat{\mathbb{A}}^*$ follows the same law as (ζ_n), we can apply Theorem IV.6a to prove part a).

Part b). As in the proof of a) we get for $y > 0$:

$$\begin{aligned}&\sum_{n=1}^{\infty} \mathbb{A}_{\zeta_n}[X \cdot \mathbb{1}\{W_0 X \geq y\}] = \infty \quad \text{a.s.} \\ \iff &\sum_{n=1}^{\infty} \hat{\mathbb{A}}^*[\mathbb{1}\{h(z_{\xi_n})X_n \geq y\} \mid \mathcal{F}_n^*] < \infty \quad \hat{\mathbb{A}}^*\text{-a.s.} \\ \iff &\sum_{n=1}^{\infty} \mathbb{1}\{h(z_{\xi_n})X_n \geq y\} < \infty \quad \hat{\mathbb{A}}^*\text{-a.s.} \qquad \text{by Lemma IV.7}\end{aligned}$$

But this is true iff

$$\limsup_{n\to\infty} h(z_{\xi_n})X_n = \infty \quad \hat{\mathbb{A}}^*\text{-a.s.}$$

and so an application of Theorem IV.6b proves part b). □

Theorem IV.8 readily applies if we know exactly the laws of (ζ_n) and of X under $\mathbb{A}_s$ for all $s \in \mathcal{S}^h$. In the example we are going to study in section VI.2, the second point is the tricky one and we will be glad to have some upper and lower bounds on $\mathbb{A}_s(X > x)$, which have some uniformity in s. We will then apply the following

Theorem IV.9. *a) Let $I \in \mathbb{N}$ fixed. For $i = 1, \dots, I$ let X_i^* be a random variable, g_i a positive function on $\mathcal{S}$ and let $F \subset \mathcal{S}$, s.t. ζ is eventually in F almost surely. Suppose that*

$$\forall s \in F : \mathbb{A}_s(X > x) \le P(\sum_i g_i(s) X_i^* > x)$$

Let

$$A_i(x) := \sum_{n=1}^{\infty} g_i(\zeta_n) \mathbb{1}\{x \ge (g_i(\zeta_n) h(\zeta_n))^{-1}\}.$$

Suppose there are positive increasing functions $L_i(x)$ slowly varying at infinity, such that a.s.

$$\max_i \sup_{x>0} \frac{A_i(x)}{L_i(x)} < \infty \quad \text{and} \quad \max_i E[X_i^* L_i(X_i^*)] < \infty.$$

Then $\mathbb{A}[W] = W_0$.

b) Suppose there is a r.v. X_*, a positive function g on S and $F \subset \mathcal{S}$, s.t.

$$\forall s \in F : \mathbb{A}_s(X > x) \ge P(g(s) X_* > x)$$

Let

$$A(x) := \sum_{n=1}^{\infty} g(\zeta_n) \mathbb{1}\{x \ge (g(\zeta_n) h(\zeta_n))^{-1}\} \mathbb{1}\{\zeta_n \in F\}.$$

Suppose there is a positive increasing function $L(x)$ slowly varying at infinity, such that, for some y, a.s.

$$\inf_{x>y} \frac{A(x)}{L(x)} > 0 \quad \text{and} \quad E[X_* L(X_*)] = \infty.$$

Then $\mathbb{A}[W] = 0$.

Proof. Let us first prove the (easier) part b). In view of Theorem IV.8b, we have to show that

$$\sum_{n=1}^{\infty} \mathbb{A}_{\zeta_n}[X \cdot \mathbb{1}\{W_0 X \ge y\}] = \infty \quad \text{a.s.},$$

Since the function $(x \mapsto x\mathbb{1}\{wx \ge y\})$ is increasing for every $w, y > 0$, we have for every $s \in F$:

$$\mathbb{A}_s[X \cdot \mathbb{1}\{W_0 X \ge y\}] \ge E[g(s) X_* \mathbb{1}\{h(s) g(s) X_* \ge y]$$

and therefore (E^* denotes expectation w.r.t. X_* only)

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{A}_{\zeta_n}[X \cdot \mathbb{1}\{W_0 X \ge y\}] &\ge \sum_{n=1}^{\infty} E^*[g(\zeta_n) X_* \mathbb{1}\{h(\zeta_n) g(\zeta_n) X_* \ge y)] \mathbb{1}\{\zeta_n \in F\} \\ &= E^*[X_* A(X_*/y)],\end{aligned}$$

which equals ∞ almost surely, since $E[X_* L(X_*)] = \infty$ implies $E[X_* L(X_*/y)] = \infty$ because L is slowly varying. This proves b).

For part a), note that for any positive w and $x_i,\ i = 1, \dots, I$:

$$\left(\sum_i x_i\right)\left(\left(w \sum_i x_i\right) \wedge 1\right) \le \left(\sum_i x_i\right)\left(\sum_i (wx_i \wedge 1)\right) \le I^2 \sum_i x_i(wx_i \wedge 1),$$

so that for $s \in F$, since the function $(x \mapsto x(wx \wedge 1))$ is increasing, we have

$$\begin{aligned} \mathbb{A}_s[X \cdot (W_0 X \wedge 1)] &\le E\left[\left(\sum_i g_i(s)X_i^*\right)\left(\left(h(s)\sum_i g_i(s)X_j^*\right) \wedge 1\right)\right] \\ &\le I^2 \sum_i E\left[g_i(s)X_i^*\left((h(s)g_i(s)X_i^*) \wedge 1\right)\right]. \end{aligned}$$

Hence, in order to apply Theorem IV.8a, we need to show that for every i,

$$\sum_{n=1}^{\infty} E^*\left[g_i(\zeta_n)X_i^*\left((h(\zeta_n)g_i(\zeta_n)X_i^*) \wedge 1\right)\right] < \infty \text{ a.s.},$$

(where again, E^* denotes expectation w.r.t X^* only). We will need the following lemma, the proof of which can be found in [BK04].

Lemma IV.10. *Suppose A is an increasing function with $A(0) = 0$ and $x \ge 0$. Then*

$$\int_0^\infty \frac{x}{y} \wedge 1 \, dA(y) = \int_1^\infty \frac{A(yx)}{y^2} dy.$$

Now fix i and set $g := g_i$, $X^* := X_i^*$ and $A := A_i$. Set $C := \sup_{x>0}(A(x)/L(x))$, which is finite a.s. by assumption. Then

$$\int_1^\infty \frac{A(yx)}{y^2} dy \le CL(x) \int_1^\infty \frac{L(yx)}{L(x)y^2} dy$$

and the integral on the right side is uniformly bounded for large x (this follows from the representation theorem for slowly varying functions (see e.g. [Fel71], VIII.9). Hence,

$$\int_1^\infty \frac{E^*[X^* A(yX^*)]}{y^2} < \infty \text{ a.s.},$$

since $E[X^* L(X^*)] < \infty$ by assumption. This concludes the proof, because

$$\begin{aligned} &\sum_{n=1}^{\infty} E^*\left[g_i(\zeta_n)X_i^*\left((h(\zeta_n)g_i(\zeta_n)X_i^*) \wedge 1\right)\right] \\ &= E^* \int_0^\infty X^*((X^*/y) \wedge 1) dA(y) && \text{by definition of } A \\ &= \int_0^\infty E[X^*((X^*/y) \wedge 1)] dA(y) && \text{by Fubini's theorem} \\ &= \int_1^\infty \frac{E^*[X^* A(yX^*)]}{y^2} && \text{by Lemma IV.10} \end{aligned}$$

□

IV.3 Optional lines

In the paper [Cha88] treating branching Brownian motion, the author had the brilliant idea to build sums over sets of individuals other that the n-th generation, say. [Jag89] generalised this concept to general branching Markov processes, similar to the general model we have just introduced. Further contributions have been made, particularly by [Kyp00] and [BK04], the results of whom will be useful to us.

The basic object is a *line*, i.e. a set of individuals $l \subset U$ with the property

$$u, v \in l \text{ implies } u \nless v$$

i.e. there is at most one $u \in l$ on every line of descent. In this sense, lines cut through the genealogical tree, "perpendicularly" to the lines of descent. Note however that the definition does not imply that every individual $u \in U$ is either a descendent of or has a descendent in l. If this is the case, we call l a *covering line*.

For a line l and $u \in U$, we write $u \geq l$ if $\exists v \in l : u \geq v$ and say that *u stems from l*. Furthermore, we write $l_1 \leq l_2$ for two lines if $l_1 \leq u$ for all $u \in l_2$. This renders the set of lines a partially ordered set and it is indeed a semi-lattice, i.e. for all lines l_1, l_2 there is a line $l := l_1 \wedge l_2$ with: $l \leq l_1$, $l \leq l_2$ and $l' \leq l$ for all lines l' with $l' \leq l_1$ and $l' \leq l_2$. To see this, one simply has to set

$$l := \{u \in l_1 : \exists v \in l_2 : u \leq v\} \cup \{u \in l_2 : \exists v \in l_1 : u \leq v\}$$

and check that this set has the above properties. The special case $l \wedge \mathbb{N}^n$ will be shorthanded by $l \wedge n$. We can thus define (with $\min \emptyset = \infty$)

$$\begin{aligned} \overline{g}(l) &:= \min\{n : l \wedge n = l\} \text{ and} \\ \underline{g}(l) &:= \min\{|u| : u \in l\} \end{aligned}$$

the *last* and *first generation of* l, respectively. We always have $\underline{g}(l) \leq \overline{g}(l)$. Also note that $\overline{g}(l) < \infty$ implies that l is covering.

The real power of lines is unleashed if on considers *random lines*, i.e. lines that are random variables, analogously to random times for stochastic processes indexed by a subset of the reals. In the one-dimensional case, *stopping times*, i.e. random times T with $\{T \leq t\} \in \mathcal{F}_t$ for all t are of particular interest, specifically for the strong Markov property and the optional sampling theorem. We will consider an analogue for random lines:

Put $\mathcal{F}_l := \{\pi_u; u \ngeq l\}$, where $\pi_u : \mathbb{T} \to \mathcal{S}$ denotes the projection $z \mapsto z_u$. Thus, $\mathcal{F}_l$ contains the information about all individuals except of those that are a child or grand-child or grand-grand-child, etc., of an individual in l. If l is covering, then these are exactly those individuals u with $u \leq v$ for some $v \in l$, so that $\mathcal{F}_l$ captures the information about the process until l. Note that the notation is slightly different from Jagers', since in his model, every mother stores the information about the position (i.e. birth times) and types of her children, whereas in our model this information is kept by the children themselves. The definition of $\mathcal{F}_l$ in [Kyp00] and [BK04] is therefore erroneous.

We can now define optional lines in an analogous way to stopping times: A random line L is called an *optional line* iff for every line l we have $\{L \leq l\} \in \mathcal{F}_l$. In this case we define

$$\mathcal{F}_L := \{A \in \mathcal{F} : \forall \text{ line } l : A \cap \{L \leq l\} \in \mathcal{F}_l\}.$$

[Jag89] has good news for us:

Theorem IV.11. *A branching process* $\mathbb{A}$ *with state space* $\mathcal{S}$ *and reproduction kernel* $P_s(d\vec{x})$ *possesses the* strong branching property*: If* L *is an optional line, then*

$$\mathbb{A}[\prod_{u \in L} f_u(z_{u\bullet}) \mid \mathcal{F}_L] = \prod_{u \in L} P_{z_u}[f_u]$$

We omit the proof.

Setting $W_L(z) := \sum_{u \in L} h(z_u)$, we extend the additive martingale to sums over (optional) lines. Here the question is if an equivalent of the optional sampling theorem holds, i.e. under which conditions do we have $\mathbb{A}[W_L] = \mathbb{A}[W_0] = \mathbb{A}[h(z_\varnothing)]$? An answer is given by the following

Proposition IV.12. *Let* L *be an optional line with* $\overline{g}(L) < \infty$ *a.s. and let* (W_n) *be uniformly integrable. Then*

$$\mathbb{A}[W \mid \mathcal{F}_L] = W_L$$

and in particular

$$\mathbb{A}[W_L] = \mathbb{A}[W] = \mathbb{A}[W_0].$$

Proof. We first show the following lemma from [BK04]:

Lemma IV.13. *For any optional line* L*,* $\mathbb{A}[W_n \mid \mathcal{F}_L] = W_{L \wedge n}$*.*

Proof. Set $L_n := L \wedge n$. If we view L_n as a (random) function from U to $\{0, 1\}$, we get

$$W_n = \sum_{|u|=n} L_n(u) W_{n-|u|}(z_{u\bullet}).$$

Now, if $L_n(u) = 1$ in the above sum then either

- $|u| = n$, so that $W_{n-|u|}(z_{u\bullet}) = h(z_u)$, or
- $|u| < n$, so that $u \in L$. In this case, the strong branching property (Theorem IV.11 gives us $\mathbb{A}[W_{n-|u|}(z_{u\bullet}) \mid \mathcal{F}_L] = h(z_u)$.

Hence, since $L_n(u)$ is $\mathcal{F}_L$-measurable for each u,

$$\mathbb{A}[W_n \mid \mathcal{F}_L] = \sum_{|u| \le n} L_n(u) \mathbb{A}[W_{n-|u|}(z_{u\bullet}) \mid \mathcal{F}_L] = \sum_{|u| \le n} L_n(u) h(z_u) = W_{L_n}.$$

□

Now, since (W_n) is uniformly integrable, $\mathbb{A}[W_n \mid \mathcal{F}_L] \overset{n\to\infty}{\longrightarrow} \mathbb{A}[W \mid \mathcal{F}_L]$ and since $\overline{g}(L) < \infty$, $W_{L \wedge n} \overset{n\to\infty}{\longrightarrow} W_L$. Thus, the proposition follows. □

Remark IV.14. In general, if L is an optional line, $L \wedge n$ need not be optional, as the example

$$L = \begin{cases} \{11, 12, 13, \ldots, 2, 3, 4, \ldots\} & \text{if } h(11) > 17 \\ \mathbb{N}^2 & \text{if } h(11) \le 17 \end{cases}$$

shows ($L \wedge 1$ is not $\mathcal{F}_1$-measurable). For simple lines, which we will define next, this is true however.

Definition IV.15. An optional line L is called *simple*, if for all $u \in U$ the function $L(u)$ is measurable w.r.t. $\mathcal{F}_{|u|}$.

For simple optional lines, [BK04] found an optimal optional sampling theorem:

Theorem IV.16. *Let $\mathbb{A}$ be the law of a branching process with reproduction kernel, L a simple optional line, $f : \mathcal{S} \to \mathbb{R}_+$ measurable and τ defined by $\xi_\tau \in L$ if there exists an n, s.t. $\xi_n \in L$ and $\tau = \infty$ otherwise. Then*

$$\mathbb{A}[\sum_{u\in L} f(z_u)\frac{h(z_u)}{h(z_\varnothing)}] = \hat{\mathbb{A}}^*[\mathbb{1}\{\tau < \infty\} f(z_{\xi_\tau})]$$

Proof. We have

$$\begin{aligned}
&\mathbb{A}[\sum_{u\in L} f(z_u)\frac{h(z_u)}{h(z_\varnothing)}] \\
=&\sum_{n\geq 0} \mathbb{A}[\sum_{|u|=n} L(u) f(z_u)\frac{h(z_u)}{h(z_\varnothing)}] \\
=&\sum_{n\geq 0} \mathbb{A}^*_n[\sum_{|u|=n} L(u) f(z_u)\frac{h(z_u)}{h(z_\varnothing)}\mathbb{1}\{\xi_n = u\}] && \text{since } L \text{ is simple} \\
=&\sum_{n\geq 0} \hat{\mathbb{A}}^*_n[L(\xi_n) f(z_{\xi_n})] && \text{by equation (IV.1)} \\
=&\hat{\mathbb{A}}^*[\sum_{n\geq 0} L(\xi_n) f(z_{\xi_n})] \\
=&\hat{\mathbb{A}}^*[\mathbb{1}\{\tau < \infty\} f(z_{\xi_\tau})]
\end{aligned}$$

□

Corollary IV.17. *Let L be a simple optional line and τ be defined as in Theorem IV.16, and let $z_\varnothing$ be a.s. constant. Then*

$$\mathbb{A}[W_L] = W_0 \iff \tau < \infty \ \hat{\mathbb{A}}^*\text{-a.s.}$$

Proof. Set $f \equiv 1$ in Theorem IV.16 □

Later we will often obtain results about the convergence of the martingale (W_n) by studying the convergence of the process $(W_{L_t})_{t\geq 0}$, where L_t are simple optional lines increasing with t, i.e. $L_s \leq L_t \ \forall s \leq t$. The key for this will be the combination of Theorem IV.16 and the next lemma, whose proof is rather technical and will be omitted; the reader should instead consult [BK04].

Lemma IV.18. *Let $\{L_t; t \geq 0\}$ be simple optional lines that are increasing with t and satisfy $\mathbb{A}[W_{L_t}] = h(z_\varnothing)$ for every t. Then $(W_{L_t})_{t\geq 0}$ is a positive $(\mathcal{F}_{L_t})$-martingale. If, for each n, $W_{L_t \wedge n} \overset{t\to\infty}{\longrightarrow} W_n$ $\mathbb{A}$-a.s., then $W_{L_t} \overset{t\to\infty}{\longrightarrow} W$ $\mathbb{A}$-a.s.*

Chapter V

Slow variation and consequences

The results from the previous chapter, in particular the theory of optional lines, will enable us to examine the behaviour at the origin of solutions to the functional equation and to derive some interesting limit theorems. As a by-product, we will obtain a criterium for uniqueness of the solutions (FE).

V.1 Slow variation

We return back to the branching random walk and the functional equation (FE) and study the behaviour of $\phi \in \mathcal{L}$ near the origin under the assumption

$$\text{(H)} \quad v(0) > 0,\ v(1) = 0,\ v'(1) \leq 0 \text{ and } \sum_{i \in \mathbb{N}} \mathbb{1}\{A_i > 0\} < \infty \text{ a.s.}$$

Hereby, we are going to use optional lines to reduce the case $v'(1) = 0$ to the easier one $v'(1) < 0$. We define for $t \in \mathbb{R}$:

$$C_t := \{u \in U : z_u > t \text{ and } z_v \leq t\ \forall v < u\}$$

and set $C := C_0$. The following theorem due to [BK97] gives us results similar to Lemma IV.18 but for the multiplicative martingale instead:

Theorem V.1. *Let $\phi \in \mathcal{L}$ and $\{L_t; t \in \mathbb{R}\}$ be an increasing sequence of optional lines with*

$$\overline{g}(L_t) < \infty\ \forall t \in \mathbb{R} \quad a.s.$$

Then, with $x \in \mathbb{R}_+$ fixed,

$$M_{L_t}(x) = \prod_{u \in L_t} \phi(x e^{-z_u})$$

is a (bounded) $(\mathcal{F}_{L_t})$-martingale. Moreover, if

$$\underline{g}(L_t) \overset{t\to\infty}{\longrightarrow} \infty \ a.s.,$$

then $M_{L_t} \overset{t\to\infty}{\longrightarrow} M$ a.s. and in L^1.

Proof (sketched). We fix $x \in \mathbb{R}_+$ and discard it from our notation. Let M be the a.s. and L^1-limit of the martingale (M_n). In a similar way to Proposition IV.12, i.e. in using the strong branching property, one can show that $\mathbb{A}[M \mid \mathcal{F}_{L_t}] = M_{L_t}\ \forall t \in \mathbb{R}$. This gives for $s < t$:

$$\mathbb{A}[M_{L_t} \mid \mathcal{F}_{L_s}] = \mathbb{A}[\mathbb{A}[M \mid \mathcal{F}_{L_t}] \mid \mathcal{F}_{L_s}] = \mathbb{A}[M \mid \mathcal{F}_{L_s}] = M_{L_s}$$

For the second part, it suffices to see that $\underline{g}(L_t) \to \infty$ implies $\mathbb{A}[M \mid \mathcal{F}_{L_t}] \to M$ a.s. and in L^1. □

In order to apply this result to the lines $(C_t)_{t\in\mathbb{R}}$, we have to prove that $\overline{g}(C_t) < \infty\ \forall t \geq 0$, which will follow from the following

Theorem V.2. *Assume (H). Let L_n be the left-most individual in the n-th generation of the BRW. Then we have $L_n \overset{n\to\infty}{\longrightarrow} \infty$ a.s.*

Proof. We follow [Big98], Theorem 3. Let $Y := \limsup_{n\to\infty} \mathrm{e}^{-L_n}$. Then $Y \leq \limsup_{n\to\infty} W_n = W$ and so, by Fatou's lemma,

$$E[Y] \leq E[W] \leq \liminf_{n\to\infty} E[W_n] = 1$$

Splitting the BRW at the first generation gives

$$L_{n+1} = \min_{i\in\mathbb{N}}\{z_i + \min_{|u|=n}(z_{iu} - z_i)\} \overset{(\mathrm{d})}{=} \min_{i\in\mathbb{N}}\{z_i + L_n^{(i)}\},$$

where the $L_n^{(i)}$ are i.i.d. copies of L_n, independent of $(z_i)_{i\in\mathbb{N}}$. Thus, letting $n \to \infty$ gives

$$Y \overset{(\mathrm{d})}{=} \max_{i\in\mathbb{N}}\{\mathrm{e}^{-z_i} Y_i\},$$

with Y_i i.i.d. copies of Y independent of $(z_i)_{i\in\mathbb{N}}$.

Assume $E[Y] > 0$. Since $v(0) > 0$, there are with positive probability at least two individuals in the first generation, so that

$$E[Y] = E[\max_{i\in\mathbb{N}}\{\mathrm{e}^{-z_i} Y_i\}] < E[\sum_{i\in\mathbb{N}} \mathrm{e}^{-z_i} Y_i] = \mathrm{e}^{v(1)} E[Y] = E[Y],$$

which is a contradiction. Hence, $E[Y] = 0$ and thus $L_n \to \infty$ a.s. □

Remark V.3. Much more is known about the behaviour of the left-most individual, see for example [McD95] or [HS09] and the references therein.

With Theorem V.2 we conclude that $\overline{g}(C_t) < \infty\ \forall t \in \mathbb{R}$. Moreover, since the assumption (H) implies that every generation is finite a.s., we have $\underline{g}(C_t) \to \infty$ and so we can apply both parts of Theorem V.1 to obtain

Corollary V.4. *(M_{C_t}) is a (bounded) $(\mathcal{F}_{C_t})$-martingale converging a.s. and in L^1 to M, the limit of the martingale (M_n). In particular,*

$$\phi(x) = \mathbb{A}[\prod_{u\in C} \phi(x\mathrm{e}^{-z_u})].$$

In other words: If we enumerate the individuals in C arbitrarily and set $(A_i^s; i \in \mathbb{N}) := (\mathrm{e}^{-u_1}, \mathrm{e}^{-u_2}, \ldots; u_i \in C)$, then

$$\text{(FE*)} \quad \phi(x) = \mathbb{E}[\prod_{i\in\mathbb{N}} \phi(xA_i^*)],$$

i.e. ϕ is a solution of a functional equation with weights $(A_i^*)_{i\in\mathbb{N}}$. The good thing about (FE*) is that the A_i^* behave quite nicely:

Theorem V.5. *Let v^*, N^*, μ^* be defined in terms of (A_i^*) as v, N, μ are in terms of (A_i). Assume (H). Then:*

a) $N^* < \infty$ *a.s.*

b) $\max_{i\in\mathbb{N}} A_i^* < 1$ *a.s.*

c) (H) holds for (A_i^*). Furthermore, v^* is strictly decreasing on $\{\theta : v^*(\theta) < \infty\}$ and in particular $v^{*\prime}(1) < 0$

d) If the assumption

$$(A) \quad \exists \theta < 1 : v(\theta) < \infty$$

holds for (A_i), then it holds for (A_i^*) with the same θ

Proof. a) follows from the fact that $\overline{g}(C) < \infty$ a.s. and $N < \infty$ a.s., b) from a) and the definition of C.

To prove c), let τ be defined as in Theorem IV.16. Theorem III.5 tells us that $(z_{\xi_n})_{n\in\mathbb{N}_0}$ is a random walk under $\hat{\mathbb{B}}^*$ with a mean increment of $-v'(1) \geq 0$. Therefore $\tau = \inf\{n \in \mathbb{N} : z_{\xi_n} > 0\} < \infty$ $\hat{\mathbb{B}}^*$-a.s. Corollary IV.17 tells us therefore that

$$v^*(1) = \log \mathbb{B}[W_C] = \log W_0 = 0 \text{ and}$$
$$\mathbb{B}[\#C] > \mathbb{B}[\sum_{u\in C} \mathrm{e}^{-z_u}] = \mathbb{B}[W_C] = 1, \text{ thus } v^*(0) > 0.$$

That v^* is strictly decreasing on $\{\theta : v^*(\theta) < \infty\}$ follows immediately from b). Thus, c) is proved.

Now assume (A). Using Theorem IV.16 we get

$$\mathbb{B}[\sum_{u\in C} \mathrm{e}^{-\theta z_u}] = \mathbb{B}[\sum_{u\in C} \mathrm{e}^{(1-\theta)z_u} \cdot \mathrm{e}^{-z_u}] = \hat{\mathbb{B}}^*[\mathrm{e}^{(1-\theta)z_{\xi_\tau}}] = E[\mathrm{e}^{(1-\theta)S_\tau}],$$

where S denotes the random walk of Theorem III.5 and $\tau = \inf\{n : S_n > 0\}$ in this case. In random walk theory, S_τ is called the *ladder height* of the random walk and it is known that $E[\mathrm{e}^{(1-\theta)S_1}] < \infty$ implies $E[\mathrm{e}^{(1-\theta)S_\tau}] < \infty$ (see for example XII (3.6a) in [Fel71]). But this follows from

$$E[\mathrm{e}^{(1-\theta)S_1}] = \int \mathrm{e}^{(1-\theta)x}\mathrm{e}^{-x}\mu(dx) = \int \mathrm{e}^{-\theta x}\mu(dx) = \mathrm{e}^{v(\theta)} < \infty.$$

□

Our first application of Theorem V.5 will be

Theorem V.6. *Assume (H). Let $\phi \in \mathcal{L}$ and define*

$$L(x) := \frac{1-\phi(x)}{x}.$$

Then L is slowly varying at the origin.

Proof. Thanks to Theorem V.5 we can restrict ourselves to the case $v'(1) < 0$. We follow [BK97], Theorem 1.4. Fix $n \in \mathbb{N}$ and order the set $\{|u| = n\}$ arbitrarily by $\prec$. Then we have

$$L(x) = \mathbb{B}[\frac{1-\prod_{|u|=n}\phi(x\mathrm{e}^{-z_u})}{x}] = \mathbb{B}[\sum_{|u|=n} \frac{1-\phi(x\mathrm{e}^{-z_u})}{x} \cdot \prod_{v\prec u} \phi(x\mathrm{e}^{-z_u})],$$

the second equality arising from a telescoping sum. This gives

$$1 = \mathbb{B}[\sum_{|u|=n} \mathrm{e}^{-z_u} \frac{L(x\mathrm{e}^{-z_u})}{L(x)} \cdot \prod_{v\prec u} \phi(x\mathrm{e}^{-z_u})] \tag{V.1}$$

Since ϕ is a Laplace transform of a probability measure, L is the Laplace transform of a positive measure (see [Fel71], XIII.2) and thus monotone decreasing with x. If $L(x)$ is not slowly varying, then there exists a constant $\beta < 1$ and a sequence (x_k) with $x_k \downarrow 0$ s.t. $\frac{L(x_k\beta)}{L(x_k)} \stackrel{k\to\infty}{\longrightarrow} \eta > 1$. By monotonicity this implies

$$\liminf_{k\to\infty} \frac{L(x_k y)}{L(x_k)} \geq \eta \;\forall y \leq \beta.$$

Letting $k \to \infty$, equality (V.1) and Fatou's lemma give

$$1 \geq \mathbb{B}[\sum_{\substack{|u|=n \\ z_u \geq -\log\beta}} \mathrm{e}^{-z_u}\eta] + \mathbb{B}[\sum_{\substack{|u|=n \\ 0\leq z_u < -\log\beta}} \mathrm{e}^{-z_u}]$$

and since $\mathbb{B}[\sum_{|u|=n} \mathrm{e}^{-z_u}] = 1$, this gives

$$(\eta-1)\mathbb{B}[\sum_{\substack{|u|=n \\ z_u \geq -\log\beta}} \mathrm{e}^{-z_u}] \leq \mathbb{B}[\sum_{\substack{|u|=n \\ z_u \leq 0}} \mathrm{e}^{-z_u}]$$

and with the notation of Theorem III.5:

$$(\eta-1)P(S_n \geq -\log\beta) \leq P(S_n \leq 0).$$

Since $E[S_1] = -v'(1) > 0$, the left and right sides tend to one and zero, respectively, which is a contradiction to $\eta > 1$. □

Remark V.7. It is possible to prove Theorem V.6 with $v'(1) = 0$ directly without too much pain, as [Kyp98] shows.

Remark V.8. The situation $v(\alpha) = 0$, $v'(\alpha) \leq 0$ for $\alpha \in (0,1)$ is more complex than the case $\alpha = 1$. In light of the stable transformation (see Theorem III.9) one could imagine that $\frac{1-\phi(x)}{x^\alpha}$ is slowly varying, but this need not be the case when the points $\{z_i; i \in \mathbb{N}\}$ are a.s. situated on a lattice. In that case periodicities occur, as shown in [DL83] and [Liu98].

With Theorem V.6 and the stable transformation we can finally finish the necessary condition for $\mathcal{L} \neq \emptyset$.

Theorem V.9. *Assume $\mathcal{L} \neq \emptyset$. Then $v(0) > 0$ and there exists a $\theta \in (0, \infty)$ with $v(\theta) \leq 0$. If condition (N) holds and there is a θ, s.t. $v(\theta) = 0$ and $v'(\theta) \leq 0$, then $\theta \in (0, 1]$.*

Proof. Theorem II.12 gives us the first part. For the second part, assume (N) and that there exists θ, s.t. $v(\theta) = 0$ and $v'(\theta) \leq 0$. Let $\phi \in \mathcal{L}$. Define $\psi(x) := \phi(x^{1/\theta})$. Then ψ is a Laplace transform of a probability measure (see the proof of Theorem III.9) and

$$\psi(x) = \phi(x^{1/\theta}) = E[\prod_{i \in \mathbb{N}} \phi(x^{1/\theta} A_i)] = E[\prod_{i \in \mathbb{N}} \psi(x A_i^\theta)],$$

so that $\psi \in \mathcal{L}_\theta$. Furthermore, $v_\theta(1) = 0$, $v'_\theta(1) \leq 0$ and $N_\theta < \infty$. Thus, according to Theorem V.6, $\frac{1-\psi(x)}{x}$ is slowly varying at the origin, and therefore

$$\frac{1 - \phi(x^{1/\theta})}{x^{1/\theta}} = x^{1-1/\theta} \frac{1 - \psi(x)}{x} \xrightarrow{x \to 0} 0.$$

Hence, $\psi'(0) = 0$ and thus $\psi \equiv 1$, contradiction. □

V.2 Some limit theorems

Knowing the slow variation of L, we are able to establish some limit theorems that involve the limits of the additive and the multiplicative martingale. We start with the multiplicative one:

Lemma V.10. *Assume (H). Let $\{L_t; t \geq 0\}$ be an increasing sequence of optional lines with a.s.:*

$$\overline{g}(L_t) < \infty \ \forall t \text{ and } \underline{g}(L_t) \xrightarrow{t \to \infty} \infty.$$

Then a.s.

$$\forall x \in \mathbb{R}_+ : \lim_{t \to \infty} \sum_{u \in L_t} x \mathrm{e}^{-z_u} L(x \mathrm{e}^{-z_u}) = -\log M(x)$$

Proof. First note that since $\underline{g}(L_i) \xrightarrow{t \to \infty} \infty$, we have by Theorem V.2 $\min_{u \in L_t} z_u \xrightarrow{t \to \infty} \infty$ a.s. and therefore

$$\max_{u \in L_t} \mathrm{e}^{-z_u} \xrightarrow{t \to \infty} 0.$$

Fix $\varepsilon > 0$. Then for n large enough, a.s. for all $x \in \mathbb{R}_+$:

$$\begin{aligned} -\log M_{L_t}(x) &= -\sum_{u \in L_t} \log \phi(x \mathrm{e}^{-z_u}) \geq -\sum_{u \in L_t} (1 - \phi(x \mathrm{e}^{-z_u})) \\ &\geq -(1 - \varepsilon) \sum_{u \in L_t} \log \phi(x \mathrm{e}^{-z_u}) = -(1 - \varepsilon) \log M_{L_t}(x) \end{aligned}$$

Since $1 - \phi(x \mathrm{e}^{-z_u}) = x \mathrm{e}^{-z_u} L(x \mathrm{e}^{-z_u})$, by passing to the limit as $t \to \infty$, we have a.s. for all $x \in \mathbb{R}_+$:

$$-(1 - \varepsilon) \log M_{L_t}(x) \leq \liminf_{n \to \infty} \sum_{u \in L_t} x \mathrm{e}^{-z_u} L(x \mathrm{e}^{-z_u}) \leq \limsup_{n \to \infty} \sum_{u \in L_t} x \mathrm{e}^{-z_u} L(x \mathrm{e}^{-z_u}) \leq -\log M_{L_t}(x).$$

Letting $\varepsilon \to 0$ completes the proof. □

Lemma V.10 entails

Theorem V.11. *Assume (H) and let $\phi \in \mathcal{L}$. The limit M of the multiplicative martingale associated to ϕ satisfies:*

a) $M(x) \equiv M(1)^x$ *a.s.*

b) $-\log M(1)$ *has Laplace transform* ϕ

c) $P(M(x) = 0) = 0$

d) $\{M(x) < 1\}$ *is the survival set, a.s.*

Proof. Since L is slowly varying and $\max_{|u|=n} \mathrm{e}^{-z_u} \to 0$ by Theorem V.2, we have

$$\sup_{|u|=n} \frac{L(x\mathrm{e}^{-z_u})}{L(\mathrm{e}^{-z_u})} \xrightarrow{n\to\infty} 1 \quad \text{a.s.}$$

Hence, $-\log M(x) = \lim_{t\to\infty} \sum_{|u|=n} x\mathrm{e}^{-z_u} L(\mathrm{e}^{-z_u})$ by Lemma V.10. Thus, a) follows. Since $\phi(x) = E[M(x)] = E[M(1)^x]$, b) follows. This implies c), since ϕ is the Laplace transform of a proper probability distribution and thus $P(M(1) = 0) = 0$. For part d), note that extinction implies $M(x) = 1$ and that $P(M(x) = 1) = \phi(\infty)$. But by Proposition III.12, $\phi(\infty)$ is the extinction probability and so d) follows. □

Using Lemma V.10 with the optional lines $\{C_t; t \geq 0\}$ one could expect $L(\mathrm{e}^{-t}) \sum_{u\in C_t} \mathrm{e}^{-z_u}$ to converge to $-\log M(1)$, since z_u is approximately equal to t for $u \in C_t$. In order to estimate the error we are committing there we are going to use results of the theory of *Crump-Mode-Jagers processes*, of which we give a quick overview based on Chapter X in [AH83].

A *Crump-Mode-Jagers (CMJ) process* (also called *general branching process*) for historical reasons) is a system of individuals (indexed for example by the Ulam-Harris labelling) where to each individual u there is associated

- a point process ξ_u (the *reproduction*) on $(0, \infty)$ specifying the age of u at the times when she gave birth to her children
- a r.v. $\tau_u \in [0, \infty]$, the *life length* of u
- a collection of $[0, \infty)$-valued stochastic processes $(\chi_u(t))_{t\geq 0}$, $(\rho_u(t))_{t\geq 0}, \ldots$ called *random characteristics*.

We assume that the vectors $(\xi_u, \tau_u, \chi_u, \rho_u, \ldots)$ for $u \in J$ are independent copies of some vector $(\xi, \tau, \chi, \rho, \ldots)$, where J is the set of individuals ever born. No assumptions are made about the dependence of $\xi, \tau, \chi, \rho, \ldots$. Note further that in the cases we consider here, τ is always ∞ a.s.

If we set $z_\varnothing = 0$ and for $u \in U$ and $i \in \mathbb{N} : z_{ui} = z_u + \inf\{t : \xi_u[0, t] = i\}$ (remember ∂ is to be treated like ∞), then $(z_u)_{u\in U}$ evidently constitutes a branching random walk, whose reproduction Q satisfies $Q(x_i > 0\ \forall i \in \mathbb{N}) = 1$. In this sense, CMJ-processes can be studied through techniques of branching random walks. However, the questions raised in the field of CMJ processes are different from those concerning the BRW. The main objects of study are of the form

$$Z^\chi(t) := \sum_{\substack{u \in J \\ z_u \le t}} \chi_u(t - z_u),$$

and one usually tries to prove limit theorems for $Z^\chi(t)$ and related quantities as $t \to \infty$. Basic examples are $\chi(t) \equiv 1$ and $\rho(t) = \mathbb{1}[0, a \wedge \tau]$, then Z^χ counts the individuals born at or before t and Z^ρ counts those alive at time t and of age less than or equal to a.

First results can be achieved for the mean of $Z^\chi(t)$ using renewal theory. Let μ be the intensity measure of the point process ξ and let $m^\chi(t) := E[Z^\chi(t)]$. Then, since the processes initiated by the children of the ancestor are shifted copies of the original process, we obtain the renewal equation

$$m^\chi(t) = E[\chi(t)] + \int_0^t m^\chi(t-u)\mu(du) \tag{V.2}$$

If we want this equation to have a solution m with $m(\infty) < \infty$, we have to normalise μ. As usual in the theory of branching processes, this is done by multiplication with an exponential function. We therefore define $\mu_\alpha(dt) := \mathrm{e}^{-\alpha t}\mu(dt)$ and $Z^\chi_\alpha(t) := \mathrm{e}^{-\alpha t} Z^\chi(t)$. Then we call μ *Malthusian* if the *Malthusian parameter* α, i.e. the (unique) solution to $\mu_\alpha(\infty) = 1$, exists and if the mean of μ_α, $\int_{(0,\infty)} t\mu_\alpha(dt)$, is finite. Note that this corresponds to the case $v(\alpha) = 0$ and $|v'(\alpha)| < \infty$ in the setting of the branching random walk. Multiplying (V.2) with $\mathrm{e}^{-\alpha t}$ gives

$$m^\chi_\alpha(t) = \mathrm{e}^{-\alpha t} E[\chi(t)] + \int_0^t m^\chi_\alpha(t-u)\mu_\alpha(du)$$

and thus, if $\mathrm{e}^{-\alpha t}E[\chi(t)]$ is directly Riemann integrable (see [Fel71], XI.1 for this notion) and μ non-lattice, the key renewal theorem then gives us

$$m^\chi_\alpha(t) \overset{t\to\infty}{\longrightarrow} m^\chi_\alpha(\infty) := \frac{\int_0^\infty \mathrm{e}^{-\alpha t} E[\chi(t)] dt}{\int_0^\infty t\mu_\alpha(dt)}$$

It is a natural question to ask whether one can make $Z^\chi_\alpha(t)$ converge, too. This was solved by [Ner81] (Theorem 5.4), who showed that under some mild conditions, $Z^\chi_\alpha(t)$ converges a.s. to $m^\chi_\alpha(\infty) \cdot W_\infty$, where W is the limit of the *Nerman martingale* $W_{C_t} = \sum_{u \in C_t} \mathrm{e}^{-\alpha z_u}$, the C_t being defined as at the beginning of section V.1. In the context of CMJ processes, C_t is called the *coming generation at time t*. With the tools established in chapter IV we can easily verify that this is indeed a martingale and that W is nothing more that the limit of the martingale (W_n) (assume $\alpha = 1$ in order to stay in our setting). In light of Lemma IV.18 it is enough to show that $\mathbb{E}[W_{C_t}] = 1\ \forall t \ge 0$ and that $W_{C_t \wedge n} \overset{t\to\infty}{\longrightarrow} W_n$ a.s. for each n. The first follows follows from Corollary IV.17 and the fact that ζ_n is a RW with $E[\zeta_1] > 0$. The second follows readily from the fact that $\underline{g}(L_t) \overset{t\to\infty}{\longrightarrow} \infty$ a.s.

Since W is simply the limit of (W_n), we know that we need an L log L condition if we want W to be non-degenerate. Therefore the above result about the convergence of

$Z^{\chi}_{\alpha}(t)$ is not very useful if this condition is not fulfilled. In this case the following result about the ratio $Z^{\chi}_{\alpha}(t)/Z^{\rho}_{\alpha}(t)$ for two characteristics χ and ρ is very handy, since it is valid also without the L log L condition. It is due to [Ner81] (Theorem 6.3) and can also be found in [AH83] (Theorem X.5.1). The lattice case is proven in [Gat00], Theorem 5.2.

Theorem V.12. *Suppose there is a $\beta < \alpha$ s.t.*

- $\int_0^\infty \mathrm{e}^{-\beta t}\mu(dt) = E[\int_0^\infty \mathrm{e}^{-\beta x}\xi(dx)] < \infty$ *and*
- $E[\sup_{i\in\mathbb{R}}\{\mathrm{e}^{-\beta t}\chi(t)\}]$ *and* $E[\sup_{i\in\mathbb{R}}\{\mathrm{e}^{-\beta t}\rho(t)\}]$ *are both finite and have cadlag paths.*

Then, on the survival set of the process,

- *if μ is non-lattice:*
$$\frac{Z^{\chi}_{\alpha}(t)}{Z^{\rho}_{\alpha}(t)} \xrightarrow{t\to\infty} \frac{\int_0^\infty \mathrm{e}^{-\alpha t}E[\chi(t)]dt}{\int_0^\infty \mathrm{e}^{-\alpha t}E[\rho(t)]dt} \quad a.s.$$
- *if μ is lattice with span λ, $\forall s,t$:*
$$\frac{Z^{\chi}_{\alpha}(s+n\lambda)}{Z^{\rho}_{\alpha}(t+n\lambda)} \xrightarrow{n\to\infty} \frac{\sum_{k=-\infty}^{\infty}\mathrm{e}^{-\alpha k\lambda}E[\chi(s+k\lambda)]}{\sum_{k=-\infty}^{\infty}\mathrm{e}^{-\alpha k\lambda}E[\chi(s+k\lambda)]} \quad a.s.$$

Let us turn back to the branching random walk and see how we can make use of Theorem V.12 in this context. Using the optional lines $(C_t)_{t\in\mathbb{R}}$ we define a CMJ process embedded in the BRW z. Set $J := \{u \in U : \exists t \in \mathbb{R} : u \in C_t\}$ and for every $u \in J$ we keep z_u as her birth time and set
$$\xi_u := \sum_{\substack{v\in C_{z_u}\\ v\geq u}} \delta_{(z_v - z_u)},$$
so that her children are exactly the individuals in $\{v \in C_{z_u} : v \geq u\}$. The strong branching property implies that this defines a CMJ process with reproduction
$$\xi \overset{(d)}{=} \sum_{u\in C} \delta_{z_u}.$$
Moreover, it is not difficult to show that for every $t \in \mathbb{R}$ the line C_t is indeed the coming generation at time t of this embedded process, i.e. if C_t was defined in terms of the individuals of the embedded process the definitions would match. We now have the tools we need to prove

Theorem V.13. *Assume (H) and (A). Then*
$$L(\mathrm{e}^{-t})\sum_{u\in C_t}\mathrm{e}^{-z_u} \xrightarrow{t\to\infty} -\log M(1) \quad a.s.$$

Proof. We follow [BK97], Theorem 8.6 and [BK05], Lemma 19. By Theorem V.11, we have
$$\sum_{u\in C_t}\mathrm{e}^{-z_u}L(\mathrm{e}^{-z_u}) \xrightarrow{t\to\infty} -\log M(1) \quad \text{a.s.}$$

We decompose C_t into those individuals with $z_u > t + c$ for some $c > 0$ and the remainder. We thus obtain, a.s. on the survival set,

$$\begin{aligned} 1 &\leq \frac{\sum_{u \in C_t} \mathrm{e}^{-z_u} L(\mathrm{e}^{-z_u})}{L(\mathrm{e}^{-t}) \sum_{u \in C_t} \mathrm{e}^{-z_u}} \\ &\leq \frac{L(\mathrm{e}^{-(t+c)}) \sum_{u \in C_t} \mathrm{e}^{-z_u} \cdot \mathbb{1}\{z_u \leq t + c\}}{L(\mathrm{e}^{-t}) \sum_{u \in C_t} \mathrm{e}^{-z_u}} + \frac{\sum_{u \in C_t} \mathrm{e}^{-z_u} L(\mathrm{e}^{-z_u}) \cdot \mathbb{1}\{z_u > t + c\}}{L(\mathrm{e}^{-t}) \sum_{u \in C_t} \mathrm{e}^{-z_u}} \\ &\leq \frac{L(\mathrm{e}^{-(t+c)})}{L(\mathrm{e}^{-t})} + \frac{\sum_{u \in C_t} \mathrm{e}^{t - z_u} (L(\mathrm{e}^{-z_u})/L(\mathrm{e}^{-t})) \cdot \mathbb{1}\{z_u > t + c\}}{L(\mathrm{e}^{-t}) \sum_{u \in C_t} \mathrm{e}^{t - z_u}} \end{aligned} \tag{V.3}$$

Because L is slowly varying (see Theorem V.6), the first term goes to 1 as $t \to \infty$. In order to estimate the second term we use the fact that for a slowly varying function L and for any $\varepsilon, \varepsilon_1 > 0$ there is a $\delta > 0$ s.t. for all $y < 1$:

$$\sup_{x < \delta} \frac{L(yx)}{L(x)} \leq (1 + \varepsilon_1) y^{-\varepsilon}$$

(this follows for example from the integral representation of L, see e.g. [Fel71], VIII.9). Thus, if $\mathrm{e}^{-t} < \delta$ and $z_u > t$, we have

$$\mathrm{e}^{t - z_u} \frac{L(\mathrm{e}^{-z_u})}{L(\mathrm{e}^{-t})} = \mathrm{e}^{t - z_u} \frac{L(\mathrm{e}^{t - z_u} \mathrm{e}^{-t})}{L(\mathrm{e}^{-t})} \leq (1 + \varepsilon_1) \mathrm{e}^{(1-\varepsilon)(t - z_u)}.$$

Thus, if we show that for some $\varepsilon > 0$,

$$\lim_{c \to \infty} \lim_{t \to \infty} \frac{\sum_{u \in C_t} \mathrm{e}^{(1-\varepsilon)(t - z_u)} \cdot \mathbb{1}\{z_u \geq t + c\}}{\sum_{u \in C_t} \mathrm{e}^{t - z_u}} = 0, \tag{V.4}$$

then it follows from (V.3) that

$$\lim_{t \to \infty} L(\mathrm{e}^{-t}) \sum_{u \in C_t} \mathrm{e}^{-z_u} = \lim_{t \to \infty} \sum_{u \in C_t} \mathrm{e}^{-z_u} L(\mathrm{e}^{-z_u}) = -\log M(1) \quad \text{a.s.}$$

To prove (V.4), we express numerator and denominator as functions Z^χ, Z^ρ of the embedded CMJ process with the characteristics

$$\begin{aligned} \chi(a) &= \mathbb{1}\{a > 0\} \int_{t+c}^{\infty} \mathrm{e}^{-(1-\varepsilon)(u-a)} \xi(du) \text{ and} \\ \rho(a) &= \mathbb{1}\{a > 0\} \int_{t}^{\infty} \mathrm{e}^{-(u-a)} \xi(du) \end{aligned}$$

(draw a picture to convince yourself). Now, with the notations of Theorem V.5, μ^* is the intensity measure of the embedded CMJ process. Part c) and d) of that theorem now tell us that assumptions (H) and (A) hold for μ^*, since they hold for μ by assumption. Thus, the Malthusian parameter of the embedded CMJ process is 1 and there exists $\beta < 1$, s.t.

$$\int_0^\infty \mathrm{e}^{-\beta t} \mu^*(dt) < \infty. \tag{V.5}$$

Moreover, if we set $\varepsilon = 1-\beta$, we have for $t > 0$:

$$\mathrm{e}^{-\beta t}\chi(t) = \int_{t+c}^{\infty} \mathrm{e}^{-\beta u}\xi(du) \leq \int_0^{\infty} \mathrm{e}^{-\beta u}\xi(du)$$

and

$$\mathrm{e}^{-\beta t}\rho(t) = \mathrm{e}^{-\beta t}\int_t^{\infty} \mathrm{e}^{t-u}\xi(du) \leq \mathrm{e}^{-\beta t}\int_t^{\infty} \mathrm{e}^{-\beta(u-t)}\xi(du) \leq \int_0^{\infty} \mathrm{e}^{-\beta u}\xi(du),$$

thus $\sup_{t\geq 0} \mathrm{e}^{-\beta t}\chi(t)$ and $\sup_{t\geq 0} \mathrm{e}^{-\beta t}\rho(t)$ are bounded in mean by (V.5). Thus, we can apply Theorem V.12 to conclude that

$$\lim_{t\to\infty} \frac{\sum_{u\in C_t} \mathrm{e}^{(1-\varepsilon)(t-z_u)} \cdot \mathbb{1}\{z_u \geq t+c\}}{\sum_{u\in C_t} \mathrm{e}^{t-z_u}} = \frac{\int_0^{\infty} \mathrm{e}^{-\varepsilon a}\left(\int_{a+c}^{\infty} \mathrm{e}^{-(1-\varepsilon)u}\mu^*(du)\right)da}{\int_0^{\infty}\int_a^{\infty} \mathrm{e}^{-u}\mu^*(du)da}$$

in the non-lattice case and that

$$\limsup_{t\to\infty} \frac{\sum_{u\in C_t} \mathrm{e}^{(1-\varepsilon)(t-z_u)} \cdot \mathbb{1}\{z_u \geq t+c\}}{\sum_{u\in C_t} \mathrm{e}^{t-z_u}} \leq \mathrm{e}^{2\lambda}\frac{\sum_{k=1}^{\infty} \mathrm{e}^{-\varepsilon k\lambda}\int_{k\lambda+c}^{\infty} \mathrm{e}^{-(1-\varepsilon)u}\mu^*(du)}{\sum_{k=1}^{\infty}\int_{k\lambda}^{\infty} \mathrm{e}^{-u}\mu^*(du)}$$

in the lattice case. By dominated convergence the denominator tends to 0 as $c \to \infty$ in both cases, so that (V.4) holds. This proves the theorem. □

Corollary V.14. *Under (H) and (A), the non-trivial solution to the functional equation is unique up to a multiplicative constant in the argument.*

Proof. We follow [BK97], Theorem 1.5. Let $\phi_1, \phi_2 \in \mathcal{L}$ and M_1, M_2 the corresponding multiplicative martingales. By parts c) and d) of Theorem V.11 we have

$$-\log M_1(1) \in (0,\infty) \text{ and } -\log M_2(1) \in (0,\infty)$$

a.s. on the survival set and thus

$$\frac{-\log M_1(1)}{-\log M_2(1)} = \lim_{t\to\infty} \frac{L_1(\mathrm{e}^{-t})\sum_{u\in C_t} \mathrm{e}^{-z_u}}{L_2(\mathrm{e}^{-t})\sum_{u\in C_t} \mathrm{e}^{-z_u}} = \lim_{t\to\infty}\frac{L_1(\mathrm{e}^{-t})}{L_2(\mathrm{e}^{-t})}$$

by Theorem V.13. Hence, the limit $\mathrm{c} = \lim_{t\to\infty}\frac{L_1(\mathrm{e}^{-t})}{L_2(\mathrm{e}^{-t})}$ exists and satisfies $0 < c < \infty$. Thus, $\log M_1(1) = c\log M_2(1)$ a.s. and an application of Theorem V.11c completes the proof. □

Chapter VI

The boundary case: detailed study

In the last chapter, we will concentrate on the case $v'(1) = 0$ and will study the behaviour of $L(x)$ at the origin. For this, we are going to introduce a new martingale and a new branching process, the branching random walk with absorption.

VI.1 The derivative martingale

In this section, we assume

$$\text{(H')} \quad v(0) > 0,\ v(1) = v'(1) = 0 \text{ and } N := \sum_{i\in\mathbb{N}} \mathbb{1}\{A_i > 0\} < \infty \text{ a.s.}$$

Let z be a branching random walk. Define

$$\partial W_n(z) := \sum_{|u|=n} z_u \mathrm{e}^{-z_u}$$

We then have

$$\begin{aligned}
\mathbb{B}[\partial W_{n+1} \mid \mathcal{F}_n] &= \mathbb{B}[\sum_{|u|=n}\sum_{i\in\mathbb{N}}(z_{ui} - z_u + z_u)\mathrm{e}^{-(z_{ui}-z_u)-z_u} \mid \mathcal{F}_n] \\
&= \sum_{|u|=n} \mathrm{e}^{-z_u}(\mathbb{B}[\sum_{i\in\mathbb{N}}(z_{ui} - z_u)\mathrm{e}^{-(z_{ui}-z_u)}] + z_u\mathbb{B}[\sum_{i\in\mathbb{N}} \mathrm{e}^{-(z_{ui}-z_u)}]) \\
&\qquad \text{by independence of } (z_{ui} - z_u) \text{ and } \mathcal{F}_n \\
&= \sum_{|u|=n} \mathrm{e}^{-z_u}(-v'(1) + z_u \mathrm{e}^{v(1)}) \\
&= \partial W_n \qquad \text{by assumption (H')}
\end{aligned}$$

Therefore, ∂W is a (signed) martingale, which we call the *derivative martingale* because it is obtained by formally deriving the expression $\sum_{|u|=n} \mathrm{e}^{-\theta z_u}$ w.r.t. θ. The origins of the derivative martingale go at least as far as [Nev88], where the analogue for branching Brownian motion was studied. In [Kyp98] and [Liu00], the authors analysed it in the same setting as ours, using results about the solutions to our equation (FE). We are going to take the reverse approach: We will establish results about the convergence of the derivative martingale directly and use them to determine the behaviour at the origin of solutions to (FE). The key will be the following theorem from [BK05]. Remember that $L(x) = (1 - \phi(x))/x$ for $\phi \in \mathcal{L}$ and that $C_t = \{u \in U : z_u > t \text{ and } z_v \le t \ \forall v < u\}$.

Theorem VI.1. *Assume (H') and (A) and let $\phi \in \mathcal{L}$. Suppose that the process $(\partial W_{C_t})_{t \in \mathbb{R}}$ converges a.s. to a finite non-negative limit Δ as $t \to \infty$. Then*

$$P(\Delta > 0) > 0 \iff \frac{L(x)}{-\log x} \xrightarrow{x \downarrow 0} c \in (0, \infty)$$

and

$$P(\Delta = 0) = 1 \iff \frac{L(x)}{-\log x} \xrightarrow{x \downarrow 0} \infty$$

Proof. The proof draws on the proof of Theorem V.13. We will use again the optional lines C_t to fix individuals with positions "only a bit bigger than t" and use Nerman's convergence result about CMJ processes to control the error we are taking there.

Let $c > 0$. On the survival set, we have

$$\begin{aligned} 1 &\le \frac{\sum_{u \in C_t} z_u e^{-z_u}}{\sum_{u \in C_t} t e^{-z_u}} \\ &= \frac{\sum_{u \in C_t} z_u e^{-z_u} \mathbb{1}\{z_u \le t + c\}}{\sum_{u \in C_t} t e^{-z_u}} + \frac{\sum_{u \in C_t} z_u e^{-z_u} \mathbb{1}\{z_u > t + c\}}{\sum_{u \in C_t} t e^{-z_u}} \\ &\le \frac{t + c}{t} + \frac{\sum_{u \in C_t} (z_u/t) e^{t - z_u} \mathbb{1}\{z_u > t + c\}}{\sum_{u \in C_t} e^{t - z_u}} \end{aligned}$$

The first term tends to 1 as t goes to infinity. As for the second term, since for $x \ge 1$ and any $\varepsilon \in (0,1)$, $\varepsilon x \le e^{\varepsilon(x-1)}$, the second term is less than

$$\frac{\sum_{u \in C_t} \frac{1}{\varepsilon} e^{(1 - (\varepsilon/t))(t - z_u)} \cdot \mathbb{1}\{z_u \ge t + c\}}{\sum_{u \in C_t} e^{t - z_u}} \tag{VI.1}$$

For $t \ge 1$, this quantity is less than $\frac{1}{\varepsilon}$ times the quantity in equation (V.4) in the proof of Theorem V.13. Since we have established there that for some $\varepsilon \in (0,1)$ that quantity tends to 0 when first t and then c tend to infinity, the same holds true for (VI.1), so that we can conclude that

$$\lim_{t \to \infty} t \sum_{u \in C_t} e^{-z_u} = \lim_{t \to \infty} \sum_{u \in C_t} z_u e^{-z_u} = \Delta \quad \text{a.s.}$$

by assumption.

The result of Theorem V.13 now gives a.s. on the survival set:

$$\frac{\Delta}{-\log M(1)} = \lim_{t \to \infty} \frac{t \sum_{u \in C_t} e^{-z_u}}{L(e^{-t}) \sum_{u \in C_t} e^{-z_u}} = \lim_{t \to \infty} \frac{t}{L(e^{-t})},$$

which proves the theorem. □

In order to study the convergence of (∂W_n) and (∂W_{C_t}), we are going to establish a link to the *branching random walk with absorption*, which we will analyse in the next section with the help of the tools introduced in section IV.2.

VI.2 Branching random walk with absorption

Let z be a branching random walk with reproduction Q starting at $x_0 > 0$. Assume that condition (H′) holds (see beginning of the previous section). Define a new branching process $(y_u)_{u\in U}$ by

$$y_u := \begin{cases} z_u & \text{if } z_v > 0 \ \forall v \le u \\ \partial & \text{otherwise} \end{cases} \tag{VI.2}$$

We call this process a *branching random walk with absorption*, since the individuals are "absorbed" when crossing the origin (think of lemmings falling down a cliff ending at the point 0). Study of this process goes back to [Kes78], we are going to follow the treatment in [BK04].

From the branching property of the branching random walk it follows that y is a branching process on $(0,\infty) \cup \{\partial\}$ starting at x_0 with reproduction kernel

$$P_s(x_1 \in A_1, \dots, x_n \in A_n) = Q_s(x_1 \in A_1, \dots, x_n \in A_n)$$

for all Borel subsets $A_1, \dots, A_n$ of $(0,\infty)$ (without ∂!). We can thus use the machinery of chapter IV.

Let us first find a suitable mean-harmonic function h. If h satisfies $h(\partial) = 0$, then

$$\begin{aligned} P_s(\sum_{i\in\mathbb{N}} h(x_i)) = Q_s[\sum_{i\in\mathbb{N}} h(x_i)\mathbb{1}\{x_i > 0\}] &= Q[\sum_{i\in\mathbb{N}} h(x_i + s)\mathbb{1}\{x_i + s > 0\}] \\ &= \int_{\mathbb{R}} h(t+s)\mathbb{1}\{t+s>0\}\mu(dt). \end{aligned}$$

If we guess that $h(x)$ is of the form $V(x)\mathrm{e}^{-x}$ for some V growing subexponentially, then the last inequality says that

$$P_s(\sum_{i\in\mathbb{N}} V(x_i)\mathrm{e}^{-x_i}) = \int_{\mathbb{R}} V(t+s)\mathrm{e}^{-t-s}\mathbb{1}\{t+s>0\}\mu(dt) = \mathrm{e}^{-s}E[V(S_1+s)\mathbb{1}\{S_1+s>0\}],$$

where (S_n) is the random walk of Theorem III.5 (we recall that the assumption (H′) implies that $E[S_1] = 0$). Thus, the function V should satisfy

$$V(s) = E[V(S_1+s)\mathbb{1}\{S_1+s>0\}] \quad \forall s \ge 0$$

Lemma 1 in [Tan89] shows that this is true for

$$V(s) := \begin{cases} E[\sum_{i=0}^{\tau} \mathbb{1}\{S_i > -x\}] & x > 0 \\ 1 & x = 0 \end{cases},$$

where $\tau := \inf\{n : S_n > 0\}$. Thus, $V(s)$ is the expected number times S_n visits $(-x, 0]$ before entering $(0,\infty)$.

We arrive at

Proposition VI.2.

$$\overline{W}_n := \sum_{|u|=n} V(y_u)\mathrm{e}^{-y_u}$$

is the additive martingale associated to the mean-harmonic function $h(s) = V(s)\mathrm{e}^{-s}$. It is non-negative and therefore converges a.s.

The following lemma gives a hint why the BRW with absorption will help us in the study of the derivative martingale. It comes from [Don80] and we will not prove it.

Lemma VI.3.

$$\frac{V(x)}{x} \to c \in (0, \infty]$$

and $c < \infty$ *if* $\int_{-\infty}^{0} x^2 \mathrm{e}^{-x} \mu(dx) < \infty$.

Corollary VI.4. *If* $c < \infty$*, then there are* $0 < a < b < \infty$ *such that* $a(x+1) < V(x) < b(x+1)$ *for all* $x \geq 0$.

By Proposition IV.12, the Markov chain $(\zeta_n) = (y_{\xi_n})$ arising from the measure change (see section IV.1) has the transition kernel

$$\Pi_s[f] = \frac{\mathrm{e}^s}{V(s)} P_s[\sum_{i \in \mathbb{N}} f(x_i) V(x_i) \mathrm{e}^{-x_i}] = \int_{\mathbb{R}} f(t+s) \frac{V(t+s)}{V(s)} \mathrm{e}^{-t} \mathbb{1}\{t + s > 0\} \mu(dt).$$

This process will play a crucial role for our study. [Tan89] found that it can be obtained from the random walk S_n by a certain time reversal similar to the one used to derive the 3-dimensional Bessel process from Brownian motion. Since the 3-dim. Bessel process can be regarded as a Brownian motion conditioned to be non-negative, ζ_n is also called the "random walk conditioned to be non-negative", which is further explained in [BD94], for example. For us, it will be important to know its asymptotic behaviour as time tends to infinity. In the case of finite variance [Big03] established a law of the iterated logarithm which tells that this process (and the 3-dimensional Bessel process as well) eventually stays in a zone around $\sqrt{n}$ of height approximately proportional to $\log \log n$. A rigorous formulation is provided by

Theorem VI.5. *Assume that the random walk* S_n *has finite variance, i.e.*

$$(V) \quad \int_{-\infty}^{\infty} x^2 \mathrm{e}^{-x} \mu(dx) < \infty.$$

Let $\varphi(x) = \log \log x$ *for* $x > 3$*. Define* $D(x) := \sum_{n=0}^{\infty} \mathbb{1}\{\zeta_n \leq x\}$*. Then for suitable (non-random) L and U*

$$\limsup_{x \to \infty} \frac{D(x)}{x^2 \varphi(x)} \leq U < \infty \quad \text{and} \quad \liminf_{x \to \infty} \frac{D(x)}{x^2 / \varphi(x)} \geq L > 0 \quad \text{a.s.}$$

Setting $\tilde{D}(x) := \sum_{n=1}^{\infty} V(\zeta_n)^{-1} \mathbb{1}\{\zeta_n \leq x\}$, Lemma VI.3 and Theorem VI.5 entail (modulo some rigour)

$$\begin{aligned}
\tilde{D}(x) &= \int_0^x V(z)^{-1} dD(z) \\
&\leq c_1 \int_0^x \frac{1}{z} d(z^2 \varphi(z)) && \text{by Lemma VI.3 and Theorem VI.5} \\
&\leq c_2 \int_0^x \frac{1}{z} z \varphi(z) dz && \text{because } \varphi \text{ is slowly varying} \\
&= c_2 \int_0^x \varphi(z) dz \\
&\leq c_3 x \varphi(x) && \text{because } \varphi \text{ is slowly varying}
\end{aligned}$$

and analogously for the lower bound. We thus obtain

Lemma VI.6. *Assume (V). For some suitable (non-random) $\tilde{L}$ and $\tilde{U}$*

$$\limsup_{x\to\infty} \frac{\tilde{D}(x)}{x\varphi(x)} \le \tilde{U} < \infty \quad \text{and} \quad \liminf_{x\to\infty} \frac{\tilde{D}(x)}{x/\varphi(x)} \ge \tilde{L} > 0 \quad a.s.$$

We now have the tools we need to prove the following theorem, which almost provides an L log L-condition for the martingale $(\overline{W}_n)$ to converge in mean.

Theorem VI.7. *Assume (V). Let the random vector $\vec{x} = (x_i)_{i\in\mathbb{N}}$ follow the law Q (the reproduction of the BRW). Define the random variables $\tilde{X}_1$, $\tilde{X}_2$ and $\tilde{X}_3(s)$ and the (slowly varying) functions l, L_1, L_2, L_3, L_4 by*

$$\begin{aligned} \tilde{X}_1 &:= \sum_{i\in\mathbb{N}} x_i \mathrm{e}^{-x_i} \mathbb{1}\{x_i > 0\} & L_1(x) &:= (\log x) l(x) \\ \tilde{X}_2 &:= \sum_{i\in\mathbb{N}} \mathrm{e}^{-x_i} & L_2(x) &:= (\log x)^2 l(x) \\ \tilde{X}_3(s) &:= \sum_{i\in\mathbb{N}} \mathrm{e}^{-x_i} \mathbb{1}\{x_i > -s\} & L_3(x) &:= (\log x)/l(x) \\ l(x) &:= \log\log\log x & L_4(x) &:= (\log x)^2/l(x) \end{aligned}$$

a) If both $E[\tilde{X}_1 L_1(\tilde{X}_1)]$ and $E[\tilde{X}_2 L_2 \tilde{X}_2]$ are finite, then $\overline{W}_n$ converges in L^1.

b) If either $E[\tilde{X}_1 L_3(\tilde{X}_1)]$ or $E[\tilde{X}_3(s) L_4(\tilde{X}_3(s))]$ is infinite for some s, then $\overline{W}_n \to 0$ a.s.

Proof. In order to apply Theorem IV.9, our goal is to find suitable bounding variables X^* and X_*. Let us start with the upper bound. Under $\mathbb{A}_s$, $s > 0$, we have

$$\begin{aligned} X = \frac{\overline{W}_1}{\overline{W}_0} &= \frac{\sum_i V(y_i)\mathrm{e}^{-y_i}}{V(s)\mathrm{e}^{-s}} \\ &= \frac{\sum_i V(z_i)\mathrm{e}^{-z_i}\mathbb{1}\{z_i > 0\}}{V(s)\mathrm{e}^{-s}} \\ &\overset{(d)}{=} \frac{\sum_i V(x_i+s)\mathrm{e}^{-x_i-s}\mathbb{1}\{x_i+s > 0\}}{V(s)\mathrm{e}^{-s}} \\ &\le \frac{\sum_i b(x_i+s+1)\mathrm{e}^{-x_i}\mathbb{1}\{x_i+s > 0\}}{V(s)} && \text{by Corollary VI.4} \\ &\le \frac{b}{V(s)} \sum_i x_i \mathrm{e}^{-x_i}\mathbb{1}\{x_i > 0\} + \frac{b}{a}\sum_i \mathrm{e}^{-x_i} && \text{idem} \\ &= \frac{b\tilde{X}_1}{V(s)} + \frac{b}{a}\tilde{X}_2. \end{aligned}$$

Thus, putting $X_1^* = b\tilde{X}_1$, $g_1(s) = (V(s))^{-1}$, $X_2^* = (b/a)\tilde{X}_2$ and $g_2(s) = 1$, we are in the setting of Theorem IV.9a. The corresponding functions A_1 and A_2 are:

$$A_1(x) = \sum_{n=1}^{\infty} (V(\zeta_n))^{-1} \mathbb{1}\{V(\zeta_n)(h(\zeta_n))^{-1} \le x\} \qquad A_2(x) = \sum_{n=1}^{\infty} \mathbb{1}\{(h(\zeta_n))^{-1} \le x\}$$
$$= \sum_{n=1}^{\infty} (V(\zeta_n))^{-1} \mathbb{1}\{e^{\zeta_n} \le x\} \qquad \approx \sum_{n=1}^{\infty} \mathbb{1}\{\zeta_n \le \log x\} \quad \text{for large } x$$
$$= \sum_{n=1}^{\infty} (V(\zeta_n))^{-1} \mathbb{1}\{\zeta_n \le \log x\} \qquad = D(\log x)$$
$$= \tilde{D}(\log x)$$

Theorem VI.5 and Lemma VI.6 now give

$$A_1(x) \le \tilde{U} L_1(x) \quad \text{and} \quad A_2(x) \le (U+1) L_2(x) \quad \text{for large } x.$$

An application of Theorem IV.9 concludes part a). Part b) is treated similarly, the functions A_1 and A_2 are even the same, the details are left to the reader. □

We finish the section with the following

Theorem VI.8. *$(\overline{W}_{C_t})_{t\in\mathbb{R}}$ is an $(\mathcal{F}_{C_t})$-martingale converging to $\overline{W} = \lim \overline{W}_n$.*

Proof. In Theorem IV.16, τ is the first time $z_{\xi_n} = \zeta_n$ exceeds t. Theorem VI.5 says in particular that $\zeta_n \to \infty$ a.s., so that $\tau < \infty$ a.s. Hence, $E[\overline{W}_{C_t}] = \overline{W}_0$ $\forall t \in \mathbb{R}$ by Corollary IV.17. Moreover, since every generation is finite, $C_t \wedge n$ is ultimately equal to $\mathbb{N}^n$, so that $\overline{W}_{C_t \wedge n} \overset{t\to\infty}{\longrightarrow} \overline{W}_n$ for every n. Lemma IV.18 then concludes the proof. □

VI.3 *Boucler la boucle*

Let us come back to the branching random walk z and the derivative martingale $\partial W_n = \sum_{|u|=n} z_u e^{-z_u}$ and prove the following

Theorem VI.9. *Assume (H') and (V). Then the martingale (∂W_n) and the process (∂W_{C_t}) converge a.s. to the same limit Δ, which is finite and non-negative a.s. and satisfies the fixed point equation (FPE). If the conditions of Theorem VI.7a hold, then Δ has infinite mean; if the conditions of Theorem VI.7b hold, then $\Delta = 0$ a.s.*

Proof. We first note that the martingale $W_n(z)$ converges to 0 $\mathbb{A}$-a.s., by Remark III.8, or by Theorem IV.6b, since $h(z_{\xi_n})X_n \ge h(z_{\xi_n}) = e^{-z_{\xi_n}}$ and z_{ξ_n} is a symmetric random walk under $\hat{\mathbb{A}}^*$.

From z, construct a BRW with absorption at $-H$ called y^H, i.e. $y^H + H$ is a BRW with absorption constructed from $z + H$ as in (VI.2). Let the random lines L_t be either $\mathbb{N}^{\lfloor t \rfloor}$ or C_t. Then, by Proposition VI.2 and Theorem VI.8,

$$W^H_{L_t} = \frac{1}{e^{-H} V(H)} \sum_{u \in L_t} V(H + y^H_u) e^{-H - y^H_u} = \frac{1}{V(H)} \sum_{u \in L_t} V(H + y^H_u) e^{-y^H_u}$$

is a positive martingale of mean 1. We denote its a.s. limit by W^H.

Let $A^H := \{\omega : \inf_n L_n(z) \geq -H\}$, where L_n is the left-most individual in the n-th generation. By Theorem V.2, $L_n \to \infty$ a.s., and so $\mathbb{A}(A^H) \overset{H\to\infty}{\longrightarrow} 1$. We have

$$\begin{aligned}
W^H &= \frac{1}{V(H)} \lim_{t\to\infty} \sum_{u\in L_t} V(H + y_u^H)\mathrm{e}^{-y_u^H} \\
&= \frac{c}{V(H)} \lim_{t\to\infty} \sum_{u\in L_t} (H + y_u^H)\mathrm{e}^{-y_u^H} && \text{by Lemma VI.3 and Theorem V.2} \\
&\leq \frac{c}{V(H)} \lim_{t\to\infty} \sum_{u\in L_t} (H + z_u)\mathrm{e}^{-z_u} \mathbb{1}\{H + z_u > 0\} \\
&= \frac{c}{V(H)} \lim_{t\to\infty} W_{L_t} H + \partial W_{L_t} && \text{by Theorem V.2} \\
&= \frac{c}{V(H)} \lim_{t\to\infty} \partial W_{L_t},
\end{aligned}$$

with equality on A^H. Thus, ∂W_{L_t} converges to $\Delta = (V(H)/c)W^H$ on A^H, which is obviously finite and non-negative. Since $\mathbb{A}[A^H] \to 1$ and the convergence takes place on every A^H, we have $W_{L_t} \to \Delta$ a.s.

Now, if the conditions of Theorem IV.6b hold, $W^H = 0$ a.s. for every H, and so $\Delta = 0$ a.s. On the other hand, if the conditions of Theorem IV.6a hold, then $\mathbb{A}[W^H] = 1$ for every H, and so $\mathbb{A}[\Delta] \geq V(H)/c$ for every H because of the above inequalities, hence, $\mathbb{A}[\Delta] = \infty$.

That Δ satisfies (FPE) can be proved directly, but at the end of the proof of Theorem VI.1 we have already shown that $\Delta/(-\log M(1))$ is constant a.s., so that we can conclude by Theorem V.11b. □

To finish this paper, we resume the results of Theorem VI.1 and Theorem VI.9:

Theorem VI.10. *Assume (H'), (A) and (V). Let $\phi \in \mathcal{L}$. Let $\tilde{X}_1$, $\tilde{X}_2$ and $\tilde{X}_3(s)$ and the functions l, L_1, L_2, L_3, L_4 be defined as in Theorem VI.7. Then*

a) If both $E[\tilde{X}_1 L_1(\tilde{X}_1)]$ and $E[\tilde{X}_2 L_2 \tilde{X}_2]$ are finite, then $\frac{L(x)}{-\log x} \overset{x\downarrow 0}{\longrightarrow} c \in (0,\infty)$.

b) If either $E[\tilde{X}_1 L_3(\tilde{X}_1)]$ or $E[\tilde{X}_3(s) L_4(\tilde{X}_3(s))]$ is infinite for some s, then $\frac{L(x)}{-\log x} \overset{x\downarrow 0}{\longrightarrow} \infty$.

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