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par

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Marches aléatoires branchantes *et al.*

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Publications

Articles publiés ou acceptés

[M1] R. Görke, P. Maillard, C. Staudt, D. Wagner (2010). Modularity-driven clustering of dynamic graphs. In P. Festa, editor, *Experimental Algorithms*, volume 6049 of *Lecture Notes in Computer Science*, 436–448. Springer Berlin / Heidelberg

[M2] R. Görke, P. Maillard, A. Schumm, C. Staudt, D. Wagner (2013). Dynamic graph clustering combining modularity and smoothness. *J. Exp. Algorithmics*, **18**, 1, 1.5:1.1–1.5:1.29

[M3] P. Maillard (2013). The number of absorbed individuals in branching Brownian motion with a barrier. *Annales de l'Institut Henri Poincaré Probabilités et Statistiques*, **49**, 2, 428–455

[M4] P. Maillard (2013). A note on stable point processes occurring in branching Brownian motion. *Electronic Communications in Probability*, **18**, no. 5, 1–9

[M5] J. Bérard, P. Maillard (2014). The limiting process of N-particle branching random walk with polynomial tails. *Electronic Journal of Probability*, **19**, no. 22, 1–17

[M6] P. Maillard, O. Zeitouni (2014). Performance of the Metropolis algorithm on a disordered tree: the Einstein relation. *Ann. Appl. Probab.*, **24**, 5, 2070–2090

[M7] I. Benjamini, P. Maillard (2014). Point-to-point distance in first passage percolation on (Tree)×Z. In B. Klartag, E. Milman, editors, *Geometric Aspects of Functional Analysis, Israel Seminar (GAFA) 2011-2013*, Lecture Notes in Mathematics, Vol. 2116, 47–51. Springer

[M8] P. Maillard, O. Zeitouni (2016). Slowdown in branching Brownian motion with inhomogeneous variance. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, **52**, 3, 1144–1160

[M9] P. Maillard (2016). Speed and fluctuations of N-particle branching Brownian motion with spatial selection. *Probability Theory and Related Fields*, **166**, 3, 1061–1173

[M10] P. Maillard, E. Paquette (2016). Choices and intervals. *Israel Journal of Mathematics*, **212**, 1, 337–384

[M11] O. Hénard, P. Maillard (2016). On trees invariant under edge contraction. *Journal de l'Ecole Polytechnique*, **3**, 365–400

[M12] P. Maillard, R. Rhodes, V. Vargas, O. Zeitouni (2016). Liouville heat kernel: regularity and bounds. *Ann. Inst. H. Poincaré Probab. Statist.*, **52**, 3, 1281–1320

[M13] P. Maillard (2016). The maximum of a tree-indexed random walk in the big jump domain. *ALEA, Lat. Am. J. Probab. Math. Stat.*, **13**, 2, 545–561

[M14] P. Maillard (2018). The λ-invariant measures of subcritical Bienaymé–Galton–Watson processes. *Bernoulli*, **24**, 1, 297–315

Articles soumis

[M15] L. Chen, N. Curien, P. Maillard. The perimeter cascade in critical Boltzmann quadrangulations decorated by an $O(n)$ loop model. *arXiv:1702.06916*

[M16] P. Maillard, M. Pain. 1-stable fluctuations in branching Brownian motion at critical temperature I: the derivative martingale. *arXiv:1806.05152*

Thèse de Doctorat

[Thèse] P. Maillard (2012). Mouvement brownien branchant avec sélection. Thèse de doctorat sous la direction de Zhan Shi, Université Pierre et Marie Curie, disponible sur *arXiv:1210.3500*

Ma thèse **[Thèse]** comporte les articles **[M3]**,**[M9]**, **[M4]**.

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Introduction

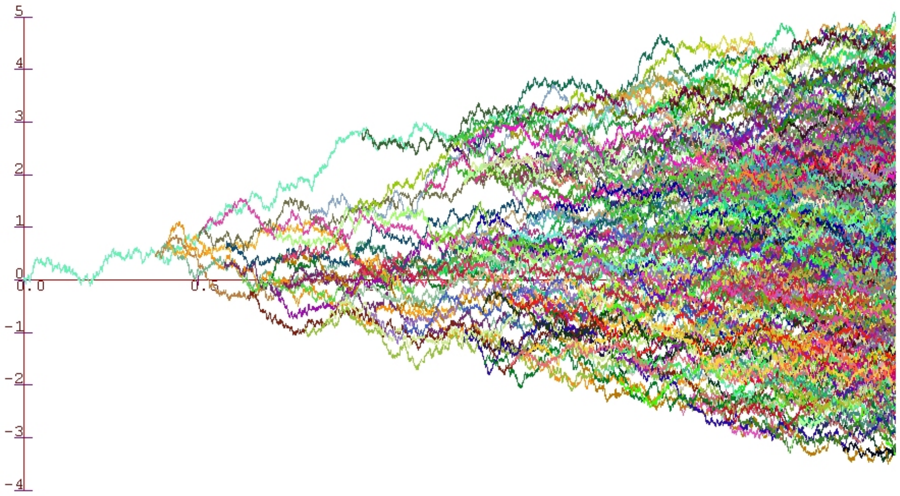


FIGURE 1.1 – Une simulation d'un mouvement brownien branchant avec branchement dyadique. Abscisse = temps, ordonnée = espace. Chaque particule correspond à une couleur. Image par Matt Roberts.

Ce mémoire d'habilitation à diriger des recherches présente les résultats de mes recherches effectuées depuis l'obtention de mon doctorat ainsi que, brièvement, ceux issus de ma thèse **[Thèse]**. Les sujets principaux sont la *marche aléatoire branchante (MAB)* et son analogue continu, le *mouvement brownien branchant (MBB)*. Il me paraissait donc judicieux d'utiliser le cadre fourni par ce mémoire pour dresser un panorama de la MAB, de ces propriétés mathématiques, l'historique de son développement, les interactions avec d'autres disciplines et son rôle dans un contexte scientifique général. Il me semble qu'un tel panorama n'existe pas dans la littérature ; de manière générale et de manière assez surprenante, compte tenu le nombre de travaux sur la MAB/le MBB, il existe peu de livres, notes de cours ou autres notes de synthèse (ceux connus de l'auteur sont [Big10, Shi11, Ber14, Zei16, Shi15, Bov17]), et ceux-ci sont plutôt récents et généralement orientés vers le comportement des particules extrêmes. On pourrait espérer avoir plus de chance dans la littérature sur les *cascades multiplicatives* ou *cascades de Mandelbrot*, dont la communauté de chercheurs est assez disjointe de la communauté des processus de branchement, même si une cascade multiplicative n'est formellement rien d'autre que l'exponentielle d'une marche aléatoire branchante. Cependant, dans cette communauté, l'attention est surtout tournée vers des modèles continus pour lesquelles la cascade multiplicative est considérée plutôt comme un « modèle jouet », voir par exemple [BM04].

J'espère alors que le panorama que je propose dans les sections 1.1 et 1.2 de cette introduction pourra être utile pour une lectrice désirant mieux comprendre l'intérêt que porte un grand nombre de chercheurs [1] à la marche aléatoire branchante et pour avoir un aperçu de la grande diversité des applications de ce modèle. Forcément, chaque présentation est biaisée et je risque d'oublier des contributions importantes, que l'on m'en excuse ! Mes propres travaux sur le sujet seront résumés dans la section 1.3. Les chapitres restants contiennent des présentations détaillées de mes travaux, des synthèses des développements ultérieurs et des questions ouvertes.

1.1 Présentation des acteurs

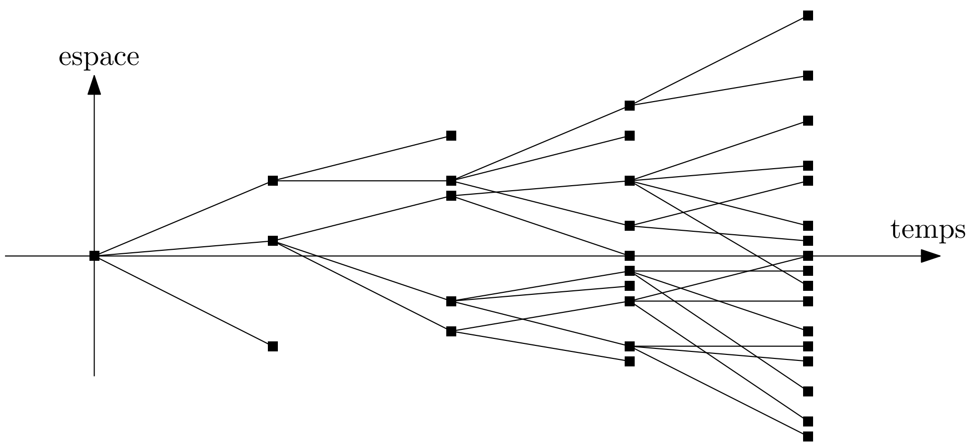


FIGURE 1.2 – Une illustration de la marche aléatoire branchante. Les particules sont indiqués par des carrés, les lignes correspondent aux relations ancestrales.

Commençons par un petit aperçu de l'objet mathématique qu'est la marche aléatoire branchante et des défis qu'elle soulève. Une façon de la voir est en tant que système de particules. Les particules de la marche aléatoire branchante vivent sur la droite réelle et évoluent selon la dynamique suivante : si l'on note $x_1, x_2, \ldots$ les positions des particules à l'étape n, alors à l'étape $n+1$, le système est composée de particules en les positions $x_i + \xi_j^i$, $i, j = 1, 2, \ldots$, où $\xi^i = (\xi_1^i, \xi_2^i, \ldots)$, $i = 1, 2, \ldots$ sont des copies indépendantes d'un vecteur ξ, de longueur finie ou infinie. Ce système est illustré dans la figure 1.2. On précise que plusieurs particules peuvent se trouver au même endroit, les particules n'interagissent alors

1. La section A dans l'annexe contient des statistiques concernant le nombre de publications sur la marche aléatoire branchante et le mouvement brownien branchant.

pas, ou seulement à travers les positions de leurs « ancêtres ». Le vocabulaire que l'on utilise ici est celui des processus de branchements, ainsi on appelle les particules de l'étape $n+1$ les *enfants* des particules de l'étape n et l'on définit de manière évidente les *descendants, parents, ancêtres, sœurs*, etc. La loi de ξ est alors appelée la *loi de reproduction*.

Le mouvement brownien branchant est une version continue en temps et en espace de ce processus. Ici, chaque particule évolue indépendamment selon deux mécanismes : 1) diffusion sur la droite réelle selon un mouvement brownien standard, 2) reproduction à taux constant $\lambda > 0$, i.e. remplacement d'une particule par L particules au même endroit, avec L une variable aléatoire. Après un événement de reproduction, les enfants continuent indépendamment ce processus. Le paramètre λ est appelé *taux de branchement* et la loi de L la *loi de reproduction*. Voir figure 1.1 pour une simulation de ce processus (pour $L = 2$ constante).

La marche aléatoire branchante est un concept plus général (d'un sens) que le mouvement brownien branchant. En effet, le *squelette* en temps discret du MBB, c'est-à-dire, la dynamique des particules le long d'une sous-suite linéaire $t_n = n\delta$ du temps, n'est rien d'autre qu'une MAB. Ceci permet de déduire un très grand nombre de résultats sur le mouvement brownien branchant à travers ceux de la marche aléatoire branchante (avec, parfois, des arguments supplémentaires de continuité). Néanmoins, le mouvement brownien branchant a son intérêt car il est souvent plus facile de montrer des résultats avancés sur le MBB que sur la MAB, voici quelques raisons :

1. La continuité des trajectoires simplifie souvent des arguments, en particulier lors de décompositions trajectorielles.
2. La présence du mouvement brownien permet des calculs explicites qui ne sont valables qu'approximativement pour des marches aléatoires.
3. Le mouvement brownien branchant possède des liens de dualité avec certaines EDPs et EDOs, cf section 1.2.2.
4. Le MBB peut être vu comme un processus gaussien indexé par un arbre (aléatoire), ceci permet d'utiliser des outils de cette théorie telle que l'interpolation gaussienne et/ou des théorèmes de comparaison. Pour ce point-de-vue, une excellente référence est le livre récent de Bovier [Bov17].

Revenons-en à la marche aléatoire branchante. Dans cette introduction (et de manière générale dans ce mémoire d'habilitation), nous nous intéressons principalement à son comportement *asymptotique*, c'est-à-dire quand le temps n tend vers l'infini. La première question concerne alors l'évolution du nombre de particules au cours du temps. Ce nombre évolue comme un processus de (Bienaymé–)Galton-Watson ou processus de branchement, en particulier, le processus survit avec probabilité non nulle si et seulement si la moyenne m du nombre d'enfants d'une particule est strictement supérieure à 1, c'est-à-dire si le branchement est *surcritique*. Il est important de souligner le fait suivant : *dans la grande majorité de la littérature sur les MAB ou le MBB, on travaille sous cette hypothèse de branchement surcritique*. Cela ne veut pas dire que les MAB avec branchement critique ne sont pas étudiés, mais certainement moins et par des communautés différentes, plus en lien avec les superprocessus ou les arbres aléatoires.

Dans la suite de cette section, nous donnons un aperçu de quelques résultats sur la MAB et des questions typiques. Pour simplifier la présentation, nous supposons à partir de maintenant que chaque particule donne naissance à 2 enfants et que leurs déplacements sont indépendants et de même loi. Autrement dit, $\xi = (\xi_1, \xi_2)$, avec ξ_1 et ξ_2 iid. On peut alors voir la MAB comme une marche aléatoire indexée par l'arbre binaire : on associe des v.a. iid aux branches de l'arbre et l'on définit pour chaque sommet v de l'arbre une v.a.

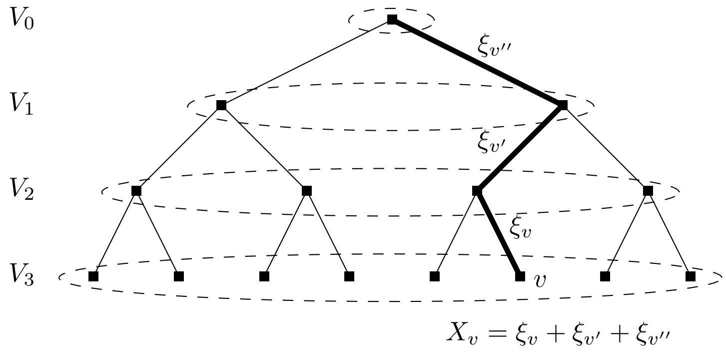


FIGURE 1.3 – Une illustration de la marche aléatoire indexée par l'arbre binaire (voir le texte pour la notation).

X_v comme étant la somme des v.a. sur les branches du chemin reliant ce sommet v à la racine, *cf.* figure 1.3. Pour chaque chemin descendant dans l'arbre, les valeurs X_u le long des sommets de ce chemin forment alors une marche aléatoire, i.e. une somme de variables aléatoires iid.

Pour $n \in \mathbf{N}$, notons V_n l'ensemble des sommets à hauteur n dans l'arbre binaire, si bien que $\#V_n = 2^n$. Nous avons alors 2^n marches aléatoires de valeurs terminales X_v, $v \in V_n$. Que peut-on dire sur la répartition des valeurs X_v ? Par exemple, quelle est la loi ou l'ordre de grandeur de $\max_{v \in V_n} X_v$, ou, de manière générale, quelle est la loi (asymptotique) du nombre de particules dans une certaine région ?

1.1.1 Principe des grandes déviations

Entre la théorie des *grandes déviations* [2]. Car comme nous avons un nombre exponentiel de particules, la particule maximale aura suivi une trajectoire très atypique, de probabilité exponentiellement petite pour la marche aléatoire. Les objets de base de cette théorie pour les marches aléatoires sont la *(log-)transformée de Cramér* et sa *transformée de Fenchel–Legendre.* Pour la marche aléatoire branchante, l'analogue de ces fonctions sont :

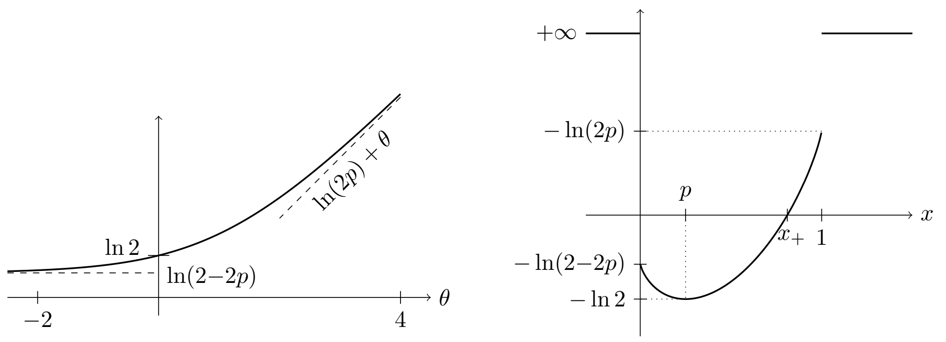
$$\varphi(\theta) = \ln \mathbf{E}[\sum_i e^{\theta \xi_i}] = \ln 2 + E[e^{\theta \xi_1}], \quad \theta \in \mathbf{R} \tag{1.1}$$

$$\varphi^*(x) = \sup_{\theta \in \mathbf{R}} \{\theta x - \varphi(\theta)\}, \quad x \in \mathbf{R}. \tag{1.2}$$

Ces fonctions sont convexes et peuvent valoir $+\infty$. Typiquement, on s'intéressera au cas ou il existe $\theta \neq 0$ tel que $\varphi(\theta) < \infty$. Dans le cas contraire [3], la MAB se comporte de manière très différente et le contenu des résultats ci-dessous devient essentiellement trivial.

Un résultat célébré de Biggins [Big77a] dit alors la chose suivante [4] :

2. Enfin, une petite partie de cette théorie.
3. Voir chapitre 3 pour ce cas.
4. On ne précise pas la signification exacte du symbole « $\approx$ ». La bonne formulation de ce résultat utilise le langage des *principes de grandes déviations.*



$\varphi(\theta) = \ln(1 + p(e^\theta - 1)) + \ln 2 \qquad \varphi^*(x) = (1-x)\ln\frac{1-x}{1-p} + x\ln\frac{x}{p} - \ln 2$

FIGURE 1.4 – Tracé des fonctions φ et φ^* (en gras) de (1.1) et (1.2) quand ξ_1 suit la loi de Bernoulli de paramètre $p = 1/4$. Gauche : la fonction $\varphi(\theta)$ et ses deux asymptotes de pentes respectives 0 et 1 correspondants aux minimum et maximum du support de ξ_1. Droite : la fonction φ^* avec le point $x_+ \approx 0.8107$ correspondant à la vitesse de la particule maximale (la vitesse minimale de la particule minimale est $x_- = 1$). Les propriétés qualitatives du tracé sont les mêmes pour tout $p \in (0, 1/2)$.

Théorème 1.1 (Biggins [Big77a]). *Presque sûrement quand $n \to \infty$,*

$$\frac{\max_{v\in V_n} X_v}{n} \to x_+ := \sup\{x \in \mathbf{R} : \varphi^*(x) < 0\} = \inf_{\theta>0} \frac{\varphi(\theta)}{\theta}$$

$$\textit{and} \quad \frac{\min_{v\in V_n} X_v}{n} \to x_- := \inf\{x \in \mathbf{R} : \varphi^*(x) < 0\} = \sup_{\theta<0} \frac{\varphi(\theta)}{\theta}.$$

De plus, pour tout $x \in \mathbf{R}$, si $\varphi^* < 0$ dans un voisinage de x, alors presque sûrement,

$$\frac{1}{n}\ln \#\{v \in V_n : \frac{X_v}{n} \approx x\} \approx -\varphi^*(x).$$

La figure 1.4 montre l'exemple de la loi de Bernoulli de paramètre $p = 1/4$. Ici, $x_+ \in (0, 1)$ et $\varphi^*(x_+) = 0$. En revanche, x_- est égal à 0, le minimum du support de la loi, et $\varphi^*(x_-)$ est strictement négative. Ces deux situations sont très différentes. Par exemple, un nombre d'ordre 1 de particules se trouvent proches de la particule maximale (voir Théorème 1.3 ci-dessous), mais un nombre exponentiellement grand de particules se trouvent proches de la particule minimale.

Comment pourrait-on démontrer le théorème de Biggins ? La borne supérieure est facile : il suffit de borner *l'espérance* du nombre de particules[5]. On obtient alors pour un $v_0 \in V_n$ arbitraire,

$$\mathbf{E}\left[\#\left\{v \in V_n : \frac{X_v}{n} \approx x\right\}\right] = 2^n \mathbf{P}\left(\frac{X_{v_0}}{n} \approx x\right) = e^{-(\varphi^*(x)+o(1))n},$$

5. Il s'avère que l'espérance est du bon ordre de grandeur. Nous allons introduire plus tard une variation de la MAB, le *CREM* pour lequel ce n'est plus forcément le cas, voir sections 1.2.1 et 2.6.

par un théorème classique des grandes déviations d'une marche aléatoire, le *théorème de Cramér*. En particulier, si $\varphi^*(x) > 0$, cette quantité est exponentiellement petite en n et une application de l'inégalité de Markov montre alors que cette quantité est nulle à partir d'un certain rang [6].

Pour la borne inférieure, les choses sont moins simples. Une approche naturelle est de calculer le *second moment* du nombre de particules, donc la quantité

$$(*) := \mathbf{E}\left[\left(\#\left\{v \in V_n : \frac{X_v}{n} \approx x\right\}\right)^2\right]$$

Si celle-ci est du même ordre que le carré de l'espérance, alors cela montre qu'avec probabilité positive (et bornée inférieurement), le nombre de particules est du même ordre que son espérance. En appliquant cela à un grand nombre de particules (par exemple à celles à hauteur $K \gg 1$ dans l'arbre), cela donne la borne inférieure souhaitée.

En poursuivant cette approche on tombe sur un problème : le second moment du nombre de particules est du même ordre que le carré de l'espérance *seulement pour x dans un certain intervalle $I \subset \{\varphi^* < 0\}$, l'inclusion étant stricte dans la plupart des cas.* Voilà ce qui se passe pour x en dehors de cette intervalle : écrivons le second moment un peu différemment :

$$(*) = \mathbf{E}\left[\#\left\{(u,v) \in (V_n)^2 : \frac{X_u}{n} \approx \frac{X_v}{n} \approx x\right\}\right].$$

Pour que cette quantité soit du même ordre que le carré du premier moment, il faudrait que la contribution principale à cette somme provienne des couples (u,v) tels que leur plus récent ancêtre commun est proche de la racine. En effet, c'est dans cette situation que $(*)$ se comporte comme si les X_v, $v \in V_n$ étaient indépendantes (et dans ce cas il est facile de montrer que le second moment est en effet du même ordre que le carré du premier moment). Ceci est en effet le cas quand $x \in I$. En revanche, quand $x \notin I$ (où plutôt, $x \notin \overline{I}$), la contribution principale provient de couples (u,v) tels que leur plus récent ancêtre commun est à hauteur environ $r \times n$, pour un certain $r = r(x) \in (0,1)$. Ces deux marches aléatoires partagent donc une partie macroscopique de leur trajectoire avant de se séparer ! Notons que le nombre de ces couples est d'ordre $2^{(2-r)n}$ et non $2^{2n} = \#(V_n)^2$, l'espérance $(*)$ est alors dominée par une proportion exponentiellement petite de couples. En termes physiques, on assiste au phénomène « énergie *versus* entropie » : ici l'énergie que l'on gagne en partageant sa trajectoire l'emporte sur la perte en entropie, c'est-à-dire à la diminution (exponentielle) du nombre de couples.

Une analyse plus fine révèle la trajectoire optimale de ces deux marches aléatoires corrélées. Supposons $x > \sup I$. Pendant la première portion de la trajectoire, lorsque les marches sont couplées, la trajectoire monte alors avec une vitesse x' plus grande que la vitesse maximale x_+, puis après la séparation, les marches continuent à une vitesse réduite, voir figure 1.5. Cette analyse montre comment on peut résoudre le problème du calcul du second moment : en ajoutant une *troncature*. Plus précisément, dans l'ensemble $\left\{v \in V_n : \frac{X_v}{n} \approx x\right\}$, on ne considère que le sous-ensemble des v tels que *pendant toute la trajectoire*, la particule reste en-dessous une droite de pente x_+. On applique alors la méthode du premier et second moment à cette quantité-là (et ça marche !).

Remarquons que cette preuve n'est pas la preuve originale de Biggins, qui emploie un argument de renormalisation pour minorer le nombre de particules suivant approximativement une droite de pente x par un processus de Galton–Watson. Cette preuve

6. On utilise ici le fait (trivial en apparence) que les particules sont indivisibles et donc le nombre de particules est forcément zéro ou au moins égal à 1.

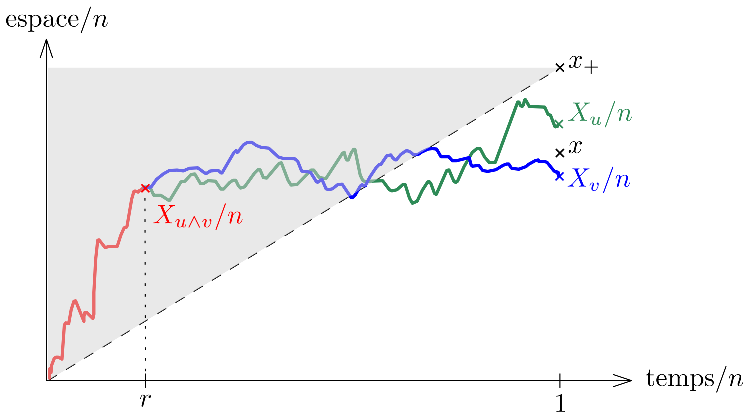


FIGURE 1.5 – Les trajectoires des couples (u, v) qui font échouer la méthode du second moment. En rouge la trajectoire de leur plus récent ancêtre commun $u \wedge v$, en bleu et vert la trajectoire des deux descendants. Typiquement, aucune particule ne rentre dans la zone grise ; l'échec de la méthode provient donc d'événements atypiques qui biaisent le calcul du second moment.

nécessite moins d'hypothèses et est d'un sens conceptuellement plus simple, mais repose sur la propriété de branchement et est donc moins « universelle » que la preuve esquissée ci-dessus. Celle-ci se généralise facilement à d'autres modèles tels que les champs gaussiens log-corrélés, voir par exemple [Ber17] pour une belle application à la construction de l'exponentielle de ces champs, appelée *chaos multiplicatif gaussien*, cf. section 1.2.4.

1.1.2 Martingales additives

On peut grandement raffiner le théorème de Biggins (Théorème 1.1). Pour cela, on introduit une famille de martingales dites *additives* et définies comme suit :

$$W_n(\theta) = \sum_{v \in V_n} e^{\theta X_v - \varphi(\theta) n}, \quad \theta \in \mathbf{R},\ \varphi(\theta) < \infty.$$

On peut voir $W_n(\theta)$ comme la transformée de Cramér renormalisée de la mesure empirique des particules au temps n. Comme dans le théorème de Cramér classique, on s'attend alors à ce que la valeur de $W_n(\theta)$ soit dominée par les particules proches de nx pour $x = \varphi'(\theta)$, ou, de manière équivalente, $\theta = (\varphi^*)'(x)$. Bien sûr, il faut que ces particules existent, c'est-à-dire que $\varphi^*(x) < 0$. Ceci mène au théorème suivant, également dû à Biggins :

Théorème 1.2 (Biggins [Big77b, Big79])**.** *Soit $x \in \mathbf{R}$ tel que φ^* est finie et C^1 dans un voisinage de x. Posons $\theta = (\varphi^*)'(x)$. Alors*

1. *La martingale $(W_n(\theta))_{n\geqslant 0}$ est uniformément intégrable si et seulement si $\varphi^*(x) < 0$. Si elle ne l'est pas, sa limite est 0 p.s.*
2. *Supposons $\varphi^*(x) < 0$. Notons $W_\infty(\theta)$ la limite p.s. et dans L^1 de la martingale*

$(W_n(\theta))_{n\geqslant 0}$. *On a :*

$$\frac{\#\left\{v\in V_n : \frac{X_v}{n}\approx x\right\}}{\mathbf{E}\left[\#\left\{v\in V_n : \frac{X_v}{n}\approx x\right\}\right]}\longrightarrow W_\infty(\theta),\quad \textit{p.s. quand } n\to\infty.$$

Ce théorème répond de manière très précise et satisfaisante à la question de la répartition asymptotique des particules au temps n : la mesure empirique des particules converge (dans un certain sens) vers son espérance, multipliée par un facteur aléatoire qui dépend de la position *macroscopique* (i.e. à l'échelle n). On peut bien sûr raffiner encore grandement ce résultat, en voici quelques résultats :

— La question de la loi de $W_\infty(\theta)$, ou plus particulièrement de l'asymptotique de sa queue, a fait couler beaucoup d'encre. Sous des hypothèses raisonnables, la queue est asymptotiquement polynomiale avec un exposant très explicite et une constante multiplicative qui l'est moins. Une approche passe par une transformation de biais par la taille, puis une application du théorème de renouvellement implicite de Goldie [Gol91] (ou son prédecesseur, le théorème de Kesten–Grincevićius [Kes73, Gri75]). Voir par exemple [Gui90, Liu00, JOC12] ou le livre [BDM16]).

— En ce qui concerne la loi jointe des $(W_\infty(\theta))$, Biggins [Big92] a montré que $W_\infty(\theta)$ est p.s. analytique en θ dans le domaine où la martingale est uniformément intégrable (voir aussi Barral [Bar99] pour une approche « analyse réelle » à la question de continuité). En particulier, ses dérivées de toute ordre existent. Ceci a été utilisé par Grübel et Kabluchko [GK17] pour donner des développements asymptotiques en $1/\sqrt{n}$ de la mesure empirique en termes de ces dérivées.

1.1.3 Transition de phase

Rappelons que x_+, défini en haut, est la vitesse des particules aux positions maximales (si x_+ existe). Une grande partie des recherches sur la MAB depuis les années 2000 a été consacrée aux particules proches du maximum, c'est-à-dire proches de nx_+ au temps n. En effet, c'est ici que se produit une *transition de phase* due à l'absence de particules au-delà de ce niveau. Un résultat phare concerne les processus ponctuel formé par les positions des particules maximales. Il peut être résumé ainsi :

Théorème 1.3 ([ABBS13, ABK13, Mad17]). *Supposons qu'il existe x_+ tel que $\varphi^*(x_+) = 0$, $\theta_0 := (\varphi^*)'(x_+)\in(0,\infty)$ et φ est finie dans un voisinage de θ_0. Posons*

$$m_n = x_+n - \frac{3}{2\theta_0}\ln n,$$

et définissons le processus ponctuel

$$\Pi_n = \sum_{v\in V_n}\delta_{X_v-m_n}.$$

Alors Π_n converge en loi vers un processus ponctuel Π. De plus, il existe une v.a. $Z>0$ et un processus ponctuel Δ sur $\mathbf{R}_- =]-\infty,0]$ tel que Π peut être construit de façon suivante :

— *Soit Ξ un processus ponctuel de Poisson sur $\mathbf{R}$ de mesure d'intensité (aléatoire) $Ze^{-\theta_0 x}\,dx$ (de manière équivalente, un processus ponctuel de Poisson sur $\mathbf{R}$ de mesure d'intensité $e^{-\theta_0 x}\,dx$ translaté par $\frac{1}{\theta_0}\ln Z$).*

— *On remplace chaque atome δ_x de Ξ par une copie de Δ, translatée de x, toutes les copies étant indépendantes.*

— Π *est égal en loi au processus ponctuel qui en résulte.*

Ce théorème contient un grand nombre d'informations et résulte de décennies de recherche. Sans rentrer dans l'historique du théorème, prenons un peu de temps pour digérer ces informations.

La première chose à constater est le facteur $\frac{3}{2}$ dans la définition de m_n. Si les 2^n marches aléatoires $(X_v)_{v\in V_n}$ étaient indépendantes, alors un calcul relativement facile de grandes déviations montre que le maximum se trouverait proche de $x_+n - \frac{1}{2\theta_0}\ln n$, le facteur $\frac{3}{2}$ est donc remplacé par un facteur $\frac{1}{2}$. De plus, la trajectoire de cette particule ressemblerait à un pont brownien et oscillerait donc autour de la droite de pente x_+ (avec des oscillations d'une amplitude d'ordre $\sqrt{n}$). Dans la marche aléatoire branchante, un phénomène différent se produit : toutes les particules restent en-dessous de cette droite à partir d'un certain temps. La trajectoire de la particule se trouvant à la position maximale au temps n ressemble donc à un pont brownien conditionné à rester en-dessous de cette droite, ce qui donne une excursion brownien à l'envers. C'est effectivement ce qui se produit [Che15]. Notons que la probabilité qu'un mouvement brownien standard reste au-dessus de l'origine jusqu'au temps n et revienne proche de l'origine au temps n est de l'ordre $n^{-3/2}$: c'est cet exposant qui correspond au facteur $\frac{3}{2}$, contre le facteur $\frac{1}{2}$ dans le cas de marches indépendantes qui provient simplement d'un théorème central limite local.

La deuxième chose à constater est la présence d'une variable aléatoire Z qui induit un décalage de $\frac{1}{\theta_0}\ln Z$ du processus des particules extrémales. Cette v.a. est en fait (à constante multiplicative près) la limite de la *martingale dérivée* définie par

$$Z_n = -\frac{d}{d\theta}W_n(\theta)\Big|_{\theta=\theta_0} = \sum_{v\in V_n}(x_+n - X_v)e^{\theta X_v - \varphi(\theta)n}.$$

Il se trouve que cette martingale (qui n'est plus une martingale positive) n'est pas uniformément intégrable mais converge quand même p.s. (vers une limite strictement positive).

Finalement, la structure du processus ponctuel s'explique heuristiquement ainsi : Chaque point de Ξ correspond à une particule, et pour chaque couple parmis ces particules, leur plus récent ancêtre commun est à la génération $o(n)$. Le processus ponctuel Δ correspond alors à la « famille » d'une de ces particules, donc aux particules telles que le plus récent ancêtre commun avec cette particule est à la génération $(1 - o(1))n$. Dans les articles [ABBS13, ABK13, Mad17] mentionnés ci-dessus, ce processus Δ est plutôt implicite, mais a été récemment rendu beaucoup plus explicite [CHL17].

1.1.4 Sélection

Nous venons de voir que la trajectoire de la particule qui se trouve au maximum au temps n descend typiquement en-dessous de la droite de pente x_+ et s'en éloigne à une distance d'ordre $\sqrt{n}$. On peut se demander quel est le plus petit exposant α tel qu'il existe des trajectoires qui ne descendent jamais à distance d'ordre plus grand que t^α en-dessous de cette droite. Le bon exposant s'avère être $\alpha = 1/3$. En fait, on peut être encore plus précis :

Théorème 1.4 ([FZ10, FHS12]). *Il existe une constante explicite $C_0 > 0$ telle que pour tout $C > C_0$ ($C < C_0$), la probabilité qu'il existe une trajectoire qui reste à distance $Cn^{1/3}$ de la droite de pente x_+ jusqu'au temps n tend vers 1 (vers 0).*

Expliquons d'où vient l'exposant 1/3. Le nombre de particules qui terminent leur trajectoire en $x_+n - O(n^\alpha)$ est $\exp(O(n^\alpha))$, en effet, la densité des particules croît exponentiellement quand on descend en-dessous de x_+n (ceci peut-être deviné par exemple à partir du Théorème 1.1). Que cela coûte-t-il de contraindre la trajectoire d'une telle particule à rester au-dessus de $x_+t - O(n^\alpha)$ pour tout $t \leqslant n$ (et en-dessous de la droite de pente x_+) ? C'est un événement dit de *petites déviations* d'une marche aléatoire et il est bien connu que la probabilité de cet événement est $\exp(-O(n^{1-2\alpha}))$ (pour $\alpha \leqslant 1/2$). Par conséquent, l'espérance du nombre de trajectoires avec ce comportement est

$$\exp(O(n^\alpha)) \times \exp(-O(n^{1-2\alpha}).$$

Pour que cette espérance soit d'ordre 1 ou plus, il faut alors que $\alpha \geqslant 1-2\alpha$, donc $\alpha \geqslant 1/3$.

Cet exemple peut être formulée d'une manière un peu différente : Supposons que l'on *tue* les particules dès qu'elles descendent de plus de $O(n^\alpha)$ en-dessous de la droite de pente x_+. Quelle est la probabilité de *survie* du système de particules ? Le fait de tuer des particules est généralement appelée *sélection* en référence à la sélection génétique en biologie. Un grand nombre de mécanismes de sélection peuvent être considérées : tuer des particules en-dessous d'une courbe autre qu'une droite, tuer des particules quand leur nombre devient trop grand, tuer des particules en fonction du nombre de particules proches etc. Ce sujet comprend une partie importante de mon travail de recherche et nous allons y revenir dans la section 1.3 ainsi que dans le chapitre 2.

1.2 Contexte scientifique général

Une particularité de la marche aléatoire branchante est le nombre d'interactions qu'elle possède avec d'autres domaines en probabilités, en mathématiques et dans d'autres sciences. Il s'agit d'un va-et-vient : la MAB a fourni des outils pour étudier d'autres modèles mais elle a aussi beaucoup été nourrie par des questions soulevées dans d'autres contextes, notamment en mécanique statistique et en biologie théorique. Nous donnons dans cette section un panorama de ces liens.

1.2.1 Mécanique statistique et verres de spin

La mécanique statistique a grandement influencé l'étude de la marche aléatoire branchante de ces dernières décennies. On voit cela par le langage employé : les termes *transition de phase*, *fonction de partition*, *énergie libre*, *température inverse* ou encore *mesure de Gibbs* sont couramment utilisés. Plus précisément, la marche aléatoire branchante peut être vue comme un modèle de *verre de spin*. Ce terme, qui décrivait au départ certains matériaux avec des impuretés en forme d'atomes ayant des propriétés magnétiques [SN13], englobe aujourd'hui un grand nombre de modèles de mécanique statistique dits *désordonnés*, c'est-à-dire avec des interactions aléatoires entre les sites. Un modèle emblématique est le modèle *Sherrington–Kirkpatrick (SK)* : c'est un modèle en *champ moyen* (c'est-à-dire sans géométrie) sur n sites ou *spins*, où chaque spin peut être dans deux états, notés -1 et 1. L'espace des *configurations* est donc $\Sigma_n = \{-1,1\}^n$ et l'*hamiltonien* (c'est-à-dire la fonction qui associe une énergie à une configuration) s'écrit [7]

$$H_{SK,n}(\sigma) = \frac{1}{\sqrt{n}} \sum_{1\leqslant i<j\leqslant n} J_{ij}\sigma_i\sigma_j,$$

avec J_{ij} des v.a. iid gaussiennes standard. Notons que $\mathrm{Var}(H_{SK,n}(\sigma)) = O(n)$, et puisque le nombre de configurations est exponentielle en n, on s'attend alors à ce que $H_{SK,n}$ varie à l'échelle n, comme c'est le cas pour des énergies indépendantes (ou pour la MAB). La

7. Sans champ magnétique externe.

première question est alors celle de la répartition asymptotique des énergies, autrement dit, l'analogue du Théorème 1.1 pour la marche aléatoire branchante. Cette question déjà est d'une difficulté formidable. Les années 80 ont vu des avancées spectaculaires par Parisi et autres, aboutissant à différentes expressions de la répartition des énergies (une de celle porte le nom de la *formule de Parisi*), mais dont les méthodes étaient hautement non-rigoureuses. Le livre [MPV87] décrit l'état de l'art à cette époque, le livre récent de Mézard et Montanari [MM09] donne une introduction plus digeste, surtout pour des mathématiciens. Des travaux rigoureux ont été menés dans les années 2000 avec comme points d'orgue la démonstration de la formule de Parisi par Talagrand [Tal06] et la preuve de la conjecture d'ultramétricité par Panchenko [Pan13a]. Voir les livres [Tal11a, Tal11b, Pan13b] pour des traitements auto-contenus.

On voit dans la description ci-dessus que la marche aléatoire branchante peut être vue comme un verre de spin avec espace de configuration $V_n \simeq \Sigma_n$ et Hamiltonien $H_n(v) = X_v$ pour tout $v \in V_n$. Plus précisément, c'est un cas particulier du *continuous random energy model (CREM)* de Bovier et Kurkova [BK04], une généralisation du *generalized random energy model (GREM)* de Derrida [Der85]. Ces modèles ont été introduits car ils sont plus simples que les modèles de champ moyen mais peuvent reproduire une partie de leurs propriétés. Le CREM est une marche aléatoire branchante avec branchement binaire et à incréments gaussiens centrés, mais avec une variance qui dépend du temps. Plus précisément, on considère pour chaque n la marche aléatoire branchante où les incréments au temps k sont de variance $\sigma^2(k/n)$, pour une fonction $\sigma^2 : [0,1] \to \mathbf{R}_+$ donnée. Le cas σ^2 constant correspond alors à la marche aléatoire branchante habituelle. Il est important que ce cas est d'un sens un cas critique : par exemple, le cas σ^2 strictement décroissante est très différent du cas σ^2 croissante. Nous reviendrons sur ce fait en plus de détails dans la section 2.6.

L'étude récente de la marche aléatoire branchante (et du mouvement brownien branchant) a ainsi bénéficié de questions inspirées au moins partiellement par des résultats (prouvés ou conjecturés) sur les verres de spins. Cela inclut par exemple :

- La structure des particules extrémales [BD09, BD11, ABBS13, ABK13, Mad17].
- La généalogie de la marche aléatoire branchante avec sélection et le lien avec le coalescent de Bolthausen–Sznitman et les cascades de probabilité de Ruelle [BDMM06b, BBS13].
- L'étude des martingales additives à paramètre complexe [KK14, MRV15, HK15, HK17, KM17]

On voit dans ces exemples le rôle important joué par Derrida et ses coauteurs dans la dissémination d'idées à travers les disciplines.

Pour un point de vue « verre de spin » au mouvement brownien branchant, je recommande le livre [Bov17].

1.2.2 Equation FKPP et propagation de fronts

L'équation *Fisher–Kolmogorov-Petrovskii-Piskounov (FKPP)* est l'équation à dérivées partielles semi-linéaire parabolique suivante :

$$u_t = \frac{1}{2}u_{xx} + F(u), \tag{1.3}$$

dont les solutions sont supposées être à valeurs dans $[0,1]$ et la non-linéarité $F(u)$ satisfait aux conditions KPP :

- $F(0) = F(1) = 0$

— $F(u) > 0$, $0 < u < 1$

— $F'(0) > 0$, $F'(u) < F'(0)$, $0 < u \leqslant 1$.

Cette équation a été introduite indépendemment par Fisher [8] [Fis37] et Kolmogorov, Petrovskii et Piskounov [KPP37] comme modèle pour la propagation d'un gène avantageux dans une population vivant dans un environnement unidimensionnel (disons, un littoral). Cette équation est devenue le prototype des équations dites de réaction-diffusion, le terme « réaction » faisant référence au terme $F(u)$. Comme beaucoup de ces équations, elle possède des *fronts* ou *ondes progressives* (en anglais *travelling waves*), c'est-à-dire des solutions de la forme $u(x,t) = \psi(x - ct)$ pour une constante c.

L'importance de l'équation FKPP pour nous vient du fait qu'elle est liée au MBB par une *dualité produit*, pour une certaine classe de non-linéarités F [Sko64, INW]. Notons $f(s)$ la fonction génératrice de la loi de reproduction du MBB, β le taux de branchement, $\mathcal{N}_t$ l'ensemble des particules au temps t et $X_u(t)$ la position de la particule u au temps t. Soit $u_0 : \mathbf{R} \to [0,1]$ mesurable. Alors la fonction

$$u(x,t) = 1 - \mathbf{E}_x\left[\prod_{u\in\mathcal{N}_t}(1 - u_0(X_u(t)))\right],$$

est solution de l'équation FKPP (1.3) avec condition initiale $u(x,0) = u_0(x)$ et $F(u) = \beta(1 - u - f(1-u))$. Si $f(0) = 0$, alors $F(u)$ satisfait aux conditions KPP avec $F'(0) = \beta(f'(1) - 1) > 0$, car $f'(1)$ équivaut la moyenne de la loi de reproduction qui est supposée plus grande que 1. En particulier, si $u_0(x) = \mathbf{1}_{x<0}$, en utilisant la symétrie et l'invariance par translation du mouvement brownien, on obtient

$$u(x,t) = \mathbf{P}(\max_{u\in\mathcal{N}_t} X_u(t) > x).$$

Cette dualité et ses généralisations (conditions de bord, inhomogénéité en temps) a été et continue d'être une très riche source d'interactions entre ces deux domaines, incorporant les communautés probabilistes, edépistes et physiques. Une bibliographie extensive est contenue dans [Bra83, HNRR13, Ber14]. Du coté probabiliste, cette dualité apporte un outil puissant pour l'étude du mouvement brownien branchant, notamment pour les particules extrémales. Cependant, sa généralisation à la marche aléatoire branchante n'est pas aisée. Pour cette raison, la tendance de la dernière décennie a plutôt été de revenir à des arguments purement probabilistes. Voir par exemple [Shi15] pour un aperçu de résultats récents sur les particules extrémales de la marche aléatoire branchante.

Mentionnons également qu'il y a un autre lieu entre l'équation FKPP et le mouvement brownien branchant : cette première apparait comme limite hydrodynamique du MBB avec sélection locale.

1.2.3 Cascades multiplicatives et analyse multifractale

Les *cascades multiplicatives* ont été introduites par Mandelbrot [Man74] pour clarifier certains aspects du modèle statistique de turbulence de Kolmogorov. Formellement, la cascade multiplicative n'est autre que l'exponentielle d'une marche aléatoire branchante : au lieu de particules qui se reproduisent et se déplacent sur la droite réelle selon des marches

8. Même si l'article de Fisher est en apparence dépourvu de l'idéologie eugéniste, je me sens à chaque fois mal à l'aise de citer cet article paru dans les *Annals of Eugenics*. J'invite le lecteur à s'informer sur cette idéologie, en particulier sur son influence sur l'idéologie Nazi (`en.wikipedia.org/wiki/Eugenics_in_the_United_States`, version du 27 mars 2018) et sur l'implication de Fisher (`en.wikipedia.org/wiki/Ronald_Fisher`, version du 27 mars 2018). Je ferme cette parenthèse un peu gênante.

aléatoires, on s'imagine plutôt une hiérarchie de poids aléatoires où les poids du niveau $n+1$ sont obtenus en multipliant les poids du niveau n par des contributions aléatoires. Sous certaines conditions, cela permet de construire une *mesure aléatoire* sur la frontière de l'arbre ; l'intérêt se porte alors sur l'étude des propriétés fractales de cette mesure. La construction de cette mesure et la détermination de la dimension de son « support » ont été achevés dans un célèbre article par Kahane et Peyrière [KP76].

L'étude complète des propriétés fractales de cette mesure requiert la notion de mesure *multifractale*. Ce terme, introduit par Frisch et Parisi [FP85] (selon Barral et Mandelbrot [BM04]) signifie que la régularité de la mesure est une notion locale et non globale. Plus précisément, si μ est une mesure de probabilité sur $[0,1]$ (disons), alors on dit que μ est multifractale si la fonction $f(\alpha)$ suivante est non-dégénérée :

$$f(\alpha) = \dim \left\{ x \in [0,1] : \lim_{r\to 0} \frac{\ln \mu(B_r(x))}{\ln r} = \alpha \right\}.$$

Ici, « dim » est une notion adéquate de dimension (par ex. Hausdorff ou packing) qu'on suppose bien définie [9]. Cette fonction, introduite indépendemment [10] dans les articles [FP85, HJK$^+$86b] encode un grand nombre de propriétés fractales sur la mesure. Elle est souvent appelée le *spectre multifractal* [11] de la mesure μ. Une autre quantité importante est la fonction $\tau(q)$ suivante :

$$\tau(q) = \lim_{r\to\infty} \frac{1}{\ln r} \ln \sum_{n=1}^{\lfloor 1/r \rfloor} \mu([(n-1)r, nr])^q$$

La fonction $\tau(q)$ est parfois appelée *l'exposant de Rényi*. Notons que la quantité $D_q = (q-1)^{-1}\tau(q)$ peut être interprétée comme dimension généralisée [Rén70, HP83] et contient un grand nombre de dimensions qui ont été considérées auparavant pour des mesures multifractales. Par exemple, la quantité $\tau'(1) = \lim_{q\to 1} D_q$ est la *dimension de Rényi* [Rén59] (appelée également *entropie de Rényi* ou encore *information dimension* en anglais) qui peut être interprétée comme la dimension du « support » de μ [BM04].

Le *formalisme multifractal* est une relation entre la fonction $f(\alpha)$ et la fonction $\tau(q)$ valable en grande généralité. Il stipule que ces fonctions, ou plutôt, leurs opposés $-f$ et $-\tau$, sont *duales dans le sens de la dualité de Legendre–Fenchel*, c'est-à-dire que $-f = (-\tau)^*$ et $(-\tau) = (-f)^*$. Entre autre, cela implique qu'une mesure est multifractale si et seulement si τ est non-linéaire. Cette relation de dualité a été démontrée dans un grand nombre de cas, entre autre pour la cascade multiplicative, voir les références dans [BM04]. Dans le cadre de la cascade multiplicative, on remarque le lien avec les fonctions φ^* et φ de la section 1.1.1.

D'autres travaux sur la cascade multiplicative traitent par exemple la continuité du spectre multifractal [Bar00] ou la construction de la cascade multiplicative *critique*, c'est-à-dire dont la dimension du support est zéro [BKN$^+$13].

En résumé, même si la cascade multiplicative n'est formellement « que » l'exponentielle d'une marche aléatoire branchante, ces deux domaines se distinguent par les questions posées : tandis que dans le domaine des cascades multiplicatives il est question de construire et d'étudier des propriétes fractales d'une certaine mesure limite munie d'une structure de

9. De même, on suppose ici et après que toutes les limites sont bien définies.

10. Selon [HJK$^+$86a].

11. Contrairement à l'usage habituel en mathématiques, où un spectre est généralement une plage de valeurs numériques (par exemple le spectre d'un opérateur), un *spectre* en physique peut être une fonction qui donne une densité ou intensité en fonction d'un paramètre (exemple : le spectre des fréquences en acoustique qui décrit l'intensité des fréquences d'un signal sonore).

géométrie héritée de l'arbre, dans le domaine de la marche aléatoire branchante il est plutôt question d'étudier les trajectoires des particules à temps fini (mais tendant vers l'infini), sans considération particulière d'une structure géométrique.

1.2.4 Chaos multiplicatif gaussien et champs gaussiens log-corrélés

Les mesures limites des cascades multiplicatives de la section précédente possèdent une forme d'auto-similarité *discrète* de par leur construction récursive à l'aide d'un arbre. Le *chaos multiplicatif gaussien* de Kahane [Kah85] et ses généralisations [BM03, RV10] possèdent une auto-similarité analogue. Pour une introduction détaillée au chaos multiplicatif gaussien, nous renvoyons le lecteur à l'excellente note de synthèse [RV14a]. Ici, nous nous bornons à mentionner que la chaos multiplicatif gaussien est une mesure aléatoire sur $[0,1]^d$ (disons), qui est formellement l'exponentielle d'un champ logarithmiquement corrélé :

$$« M_\gamma(dx) = e^{\gamma X_x}\, dx »,$$

où X est un champ gaussien centré sur $[0,1]^d$ satisfaisant à $\mathbf{E}[X_x X_y] = \ln_+ \frac{1}{|x-y|} + g(x,y)$, avec g une fonction continue bornée, et $\gamma \geqslant 0$ est un paramètre. Le noyau de covariance du champ X étant singulier sur la diagonale, ce champ doit être défini [12] comme distribution aléatoire (dans $H^{-\epsilon}$, par exemple) et la mesure M_γ par renormalisation. Plusieurs façons existent de le faire, de manière générale on choisit une famille X^ε, $\varepsilon > 0$, de champs gaussiens qui approchent le champ X quand $\varepsilon \to 0$ et on définit :

$$M_\gamma(dx) = \lim_{\varepsilon \to 0} \exp\left(\gamma X_x^\varepsilon - \frac{\gamma^2}{2}\mathbf{E}[(X_x^\varepsilon)^2]\right)\, dx.$$

On peut ainsi montrer l'existence et la non-dégénérescence de cette limite si $\gamma < 2d$ et la dégénérescence dans le cas contraire $\gamma \geqslant 2d$.

Le chaos multiplicatif gaussien possède des propriétés multifractales similaires à la cascade multiplicative et a été utilisé comme brique de base dans de nombreux modèles mathématiques dans des domaines variés tels que la turbulence ou les mathématiques financières [RV14a, section 5], [CGRV17]. Il intervient également dans la description des propriétés statistiques de la fonction ζ sur la droite critique [SW16, ABB$^+$16] ou du polynôme caractéristique de matrices aléatoires unitaires [ABB17, CNM16, PZ16]. Le chaos multiplicatif gaussien est également un élément de base pour la construction de la *gravité de Liouville quantique*. Ce terme provenant de la physique désigne une certaine théorie conforme des champs, mais a été utilisé dans la littérature mathématique pour différentes constructions inspirés par cette théorie [DS11, DKRV16]. Ces constructions ont en commun entre autre le fait qu'elles reposent sur le chaos multiplicatif gaussien avec X un champ libre gaussien deux-dimensionnel. Ceci leur procure certaines propriétés d'invariance (ou plutôt *covariance*) conforme et donne naissance à une théorie extrèmement riche notamment en lien avec de nombreux objects aléatoires deux-dimensionnels tels que les *évolutions de Schramm-Loewner* ou la *carte brownienne*, voir par exemple [MS15] et ses références. L'autre courant de recherche, plus proche de la physique, a notamment débouché récemment sur la preuve rigoureuse d'une formule fondamentale de la théorie, la formule DOZZ [KRV17].

De manière plus directement liée à la cascade multiplicative ou la marche aléatoire branchante, le champ libre gaussien deux-dimensionnel *discret* a également reçu beaucoup

12. On peut éviter de définir le champ X pour définir M_γ, c'est ce qui est d'ailleurs fait dans la plupart des cas.

d'attention ces dernières années, notamment en ce qui concerne son maximum. Plusieurs résultats sur les particules extrémales de la marche aléatoire branchante ainsi que les concepts qui entraient dans ces résultat (notamment la *martingale dérivée*) ont ainsi été transporté dans ce cadre, voir notamment [BDZ16, BL16a]. De manière générale, les avancées sur la marche aléatoire branchante ont inspiré de nombreux travaux sur les extrêmes de champs gaussiens logarithmiquement corrélés, en plus des articles cités ci-dessus, voir par exemple [Mad15]. Finalement, mentionnons que la cascade multiplicative intervient de manière exacte dans l'étude du champ libre gaussien (continu) en deux dimensions [APS17].

1.2.5 *Smoothing transform*, équation $X = AX + B$

La limite $W = W_\infty(\theta)$ de la martingale additive de la marche aléatoire branchante (voir section 1.1.2) satisfait à l'équation suivante :

$$W \overset{\text{loi}}{=} \sum_i A_i W^{(i)}, \tag{1.4}$$

avec $A_i = e^{\theta X_i - \varphi(\theta)}$ et $W^{(i)}$, $i = 1, 2, \ldots$ des copies iid de la variable W, indépendantes du vecteur $(A_i)_{i\geqslant 1}$. On peut voir le côté droit de cette équation comme une transformation de la loi de W en la loi de $\sum_i A_i W^{(i)}$. Cette transformation porte le nom de *smoothing transform* en référence à [HL81, DL83]. Elle intervient dans de nombreuses applications, voir par exemple les références dans [Liu98] ou [BDM16, section 5.2]. L'étude des solutions de cette équation (existence, unicité, propriétés) a suscité beaucoup d'intérêt dans le passé et peut être considérée comme bien comprise, même si des questions restent ouvertes. Un outil majeur pour étudier cette équation consiste à se ramener à l'équation plus simple

$$X \overset{\text{loi}}{=} AX + B, \tag{1.5}$$

où X, A, B sont des variables aléatoires avec X indépendante du couple (A, B). Cette réduction est possible grâce à la transformation de *biais par la taille*, c'est-à-dire on considère la variable aléatoire $\widetilde{W}$ de loi satisfaisant à $E[f(\widetilde{W})] = E[Wf(W)]/E[W]$ pour toute fonction f mesurable bornée. En effet, on montre aisément que $\widetilde{W}$ est solution de 1.5 avec un certain (A, B) [Dur83, Gui90, Liu00]. L'équation (1.5), qui porte le nom *d'équation de perpétuités*, est omniprésente en probabilités. Un livre y est même dédiée [BDM16]. Les solutions de cette équation on généralement des queues polynomiales, ce qu'on peut montrer à l'aide du théorème de Kesten–Grincevićius–Goldie [Kes73, Gri75, Gol91]. Ceci est donc aussi le cas pour les solutions de (1.4) [13].

Notons finalement que des extensions multidimensionnelles de (1.4) ont également suscité de l'intérêt récemment, notamment pour leur lien avec les marches aléatoires branchantes à valeurs complexes, voir par exemple [MM17] et les références dans cet article.

1.2.6 Et encore...

Voici quelques autres modèles ayant des liens forts avec la MAB ; j'invite la lectrice à consulter les références citées.

- Marches aléatoires en milieu aléatoire sur un arbre : voir section 4.2
- Processus de fragmentation (compensés) [Ber06, BBCK17, SW17, BM17]
- Graphes d'attachement préférentiel [EM14, EMO17]
- Arbres binaires de recherche [CKMR05]

13. Voir [Liu00] pour plus de détails. Pour une preuve alternative qui adapte l'approche de [Gol91] à (1.4), voir [JOC12].

1.3 Synthèse de mes travaux scientifiques

La marche aléatoire branchante et le mouvement brownien branchant ont été le point pivot de mes recherches depuis le début de ma thèse de doctorat. Je présente ici un résumé de mes travaux qui suit à peu près l'organisation du corps du texte.

Un volet important de mes recherches constitue l'étude du mouvement brownien branchant avec sélection (*cf.* section 1.1.4) et l'application de la sélection à l'étude du mouvement brownien branchant sans sélection. Ces travaux sont présentés dans le chapitre 2 de ce mémoire. Le MBB avec sélection était le sujet de ma thèse de doctorat **[Thèse]** qui regroupe les articles **[M3]**,**[M9]**,**[M4]**. Ces deux premiers sont présentés brièvement dans les sections 2.1 et 2.3. L'article **[M9]** ainsi que mes articles suivants sur ce sujet développent et utilisent les techniques de Berestycki, Berestycki et Schweinsberg [BBS13] sur le MBB avec absorption en une ligne droite dans un régime presque critique. Ces techniques sont exposés dans la section 1.1.4.

Dans un travail en cours avec J. Berestycki et J. Schweinsberg, nous reprenons ces techniques pour l'étude du MBB avec absorption dans le régime critique (section 2.4). Nous donnons une étude très détaillée de ce processus, avec entre autre des asymptotiques précises sur la probabilité de survie du processus jusqu'à un temps t et une description du processus au temps t conditionellement à cet événement, type « limite de Yaglom ». Un outil technique important que nous développons sont des estimées très précises sur le mouvement brownien branchant et sur le noyau de la chaleur dans certains domaines courbes.

Dans un autre travail récent avec M. Pain (**[M16]**, section 2.5) nous étudions les fluctuations de la martingale dérivée, la martingale additive (au paramètre critique) et autres fonctionnelles du mouvement brownien branchant. Nous obtenons des théorèmes centraux limites non-standards pour ces quantités, avec des limites 1-stables de n'importe quel paramètre d'asymétrie. Les idées de base sont inspirées du MBB avec sélection, mais vont bien au-delà de ça et nécessitent entre autre une étude fine à plusieurs échelles du processus.

La dernière section du chapitre 2 concerne un travail en collaboration avec O. Zeitouni (**[M8]**, section 2.6). Il concerne le MBB avec variance inhomogène (et décroissante) en temps, une version continue du GREM de Derrida (voir section 1.2.1). En tuant les particules à un certaine courbe, nous obtenons des asymptotiques fines sur la position de la particule maximale de ce processus. La preuve de ce résultat utilise un mélange d'idées probabilistes issus des processus de branchement et des champs gaussiens ainsi qu'une analyse précise d'une certaine équation différentielle partielle de type Airy.

Le chapitre 3 contient la présentation de deux travaux sur des marches aléatoires branchantes avec queues lourdes (polynomiales). Celles-ci présentent des propriétés très différentes du MBB ou des MAB présentées dans cette introduction. Le premier article, avec J. Bérard (**[M5]**, section 3.1), concerne un analogue du N-MBB de la section 2.3, la N-MAB où la loi des sauts des particules admet une queue à variation régulière. Nous construisons un processus limite de ce système de particules, ce qui nous permet de donner un équivalent de sa vitesse quand $N \to \infty$. De plus, ce processus limite fournit une très bonne description du comportement global de ce système de particules qui est très différent du N-MBB. L'étude est purement probabiliste et utilise un certain nombre de couplages judicieux.

L'autre article sur ce sujet (**[M13]**, section 3.2) concerne des marches aléatoires à queues lourdes indexées par des arbres généraux. Le focus est ici sur des arbres de « dimension » fini, comme c'est le cas pour des marches aléatoires branchantes à branchement *critique*. Je donne alors un critère suffisant pour que la position maximale coïncide asymptotiquement avec le taille du plus grand saut. Ceci permet d'éclaircir et de généraliser quelques résultats dans la littérature pour des modèles particuliers.

Le chapitre 4 finalement est dédié à mes travaux sur des sujets connexes où la MAB apparaît de manière plus ou moins explicite. Le premier (**[M4]**, section 4.1), issu de ma thèse, traite les processus ponctuels qui décrivent les particules extrémales de la marche aléatoire branchante (*cf.* Théorème 1.3). La Section 4.2 présente un travail en commun avec O. Zeitouni **[M6]** sur une certaine marche aléatoire sur un arbre desordonné où à chaque sommet est associé une valeur aléatoire, l'ensemble des valeurs formant une MAB. Cette marche aléatoire, qui a un biais vers les sommets de grandes valeurs « d'intensité » β, a été introduite par D. Aldous comme algorithme randomisé pour trouver des sommets de grandes valeurs dans l'arbre. Si S_n note la valeur du sommet visité par la marche aléatoire au temps n, nous montrons l'existence d'une valeur $\beta = \beta_0$ critique pour laquelle S_n est un processus réversible ainsi qu'une « relation d'Einstein » quand β est perturbée autour de cette valeur critique.

L'article **[M7]** (section 4.3) avec I. Benjamini traite la percolation de premier passage sur un certain graphe de Cayley à croissance exponentielle : le produit d'un arbre régulier avec $\mathbf{Z}$. Nous montrons que les fluctuations de la distance entre deux points à distance n dans le graphe sont très petites (d'ordre $\log n$ au plus). Nous utilisons pour cela une technique de Dekking et Host développée pour la MAB. Il s'agit à notre connaissance du premier exemple de ce genre.

Les MAB interviennent aussi dans un modèle de carte aléatoire décorée par un modèle de boucles « $O(n)$ ». En effet, avec L. Chen et N. Curien **[M15]** (section 4.4), nous montrons pour un tel modèle que l'arbre représentant les boucles et leurs périmètres admet comme limite d'échelle une certaine cascade multiplicative, donc l'exponentielle d'une MAB. La loi de reproduction de cette MAB est donnée en fonction des sauts d'un certain processus de Lévy et nous arrivons à calculer sa transformée φ (1.1) par un résultat sur les marches aléatoires d'intérêt indépendant. De plus, l'étude une certaine martingale de cette MAB et de l'équation de point fixe (1.4) satisfaite par sa limite permet d'obtenir la loi limite du volume renormalisé de cette carte aléatoire. Finalement, nous exposons les liens avec la *gravité de Liouville quantique* (voir section 1.2.4) et les *conformal loop ensembles (CLE)*.

La gravité de Liouville quantique, ou plus proasaïquement le chaos multiplicatif gaussien, est aussi présent dans un autre travail avec R. Rhodes, V. Vargas et O. Zeitouni **[M12]**, présenté dans la section 4.5. Nous y étudions le noyau de la chaleur du « mouvement brownien Liouville », un mouvement brownien changé de temps par un chaos multiplicatif gaussien. Nous en donnons des bornes inférieures et supérieures qui peuvent s'interpréter comme des bornes sur la dimension de Hausdorff de la gravité de Liouville quantique. Ce travail représentait la première étude quantitative d'un noyau de la chaleur dans un environnement multifractal.

La dernière section 4.6 présente un travail avec E. Paquette **[M10]** ainsi qu'un autre travail en cours sur un certain processus de fragmentation d'intervalles avec interaction. Cette interaction est inspirée par le paradigme « power of choice » provenant de l'informatique théorique et de la théorie des graphes aléatoires. Les processus de fragmentation d'intervalles ont des liens très forts avec les marche aléatoires branchantes et satisfont entre autre à une certaine propriété de branchement. L'interaction que nous introduisons détruit cette propriété et rend l'étude asymptotique plus difficile. Nous déterminons la limite en loi de la longueur d'un intervalle typique et démontrons l'équirépartition asymptotique des points jonction d'intervalles. Nous espérons que ce travail pourra être utile pour d'autres processus incorporant une structure de branchement et de l'interaction.

Pour préserver une cohérence du sujet, les travaux **[M11]** et **[M14]** ne sont pas présentés en détail, car ils sortent du cadre de la marche aléatoire branchante. Le premier des deux, avec O. Hénard, étudie une certaine transformation d'un arbre qui peut être vue comme une sorte de renormalisation aléatoire de celui-ci. Nous caractérisons les arbres (aléatoires) qui sont invariants par cette transformation. Nous montrons qu'ils peuvent s'obtenir à partir d'un arbre aléatoire *réel* muni d'une mesure et invariant par une transformation de renormalisation plus simple. Ces arbres sont de dimension 1 et donc très atypiques, ce qui peut être vu comme une explication pourquoi les tranformations habituelles sur les arbres aléatoires agissent plutôt sur les feuilles que sur les sommets internes.

L'article **[M14]** a été inspiré par **[M11]** et traite des lois quasi-stationnaires de processus de Galton–Watson sous-critiques. En effet, ces lois interviennent dans les lois de certains des arbres étudiés dans **[M11]**. Je caractérise toutes les lois quasi-stationnaires (ou plus généralement, λ-invariantes) de ces processus.

Perspectives. MAB et MBB continuent à fournir des sujets de recherche intéressants, notamment à cause de leurs liens et interactions avec d'autres modèles probabilistes, dont la section 1.2 donne un aperçu. Ces interactions vont dans les deux sens : d'une part elles apportent des nouvelles questions sur les MAB et d'autre part elles permettent des applications des résultats sur la MAB et les techniques utilisés pour les démontrer dans d'autres contextes. De plus, elles justifient une étude de plus en plus fine de la MAB et du MBB. Ceci ouvre la voie à un grand nombre de projets de recherche possibles, sur la MAB « pure » ou sur ses applications.

Dans le corps du texte et à la fin de (presque) chaque section, je présente un certain nombre de questions qui m'intéressent personnellement et qui sont en lien avec mes propres travaux. Pour la plupart, ces questions pourraient être traitées dans le cadre d'une thèse de doctorat.

Chapter 2

Branching Brownian motion and selection

A important part of my past and present research work has been devoted to branching Brownian motion under various forms of selection. In this chapter, I present the articles **[M3]**, **[M9]**, **[M16]** and **[M8]**, as well as some work in progress. The first two articles are part of my PhD thesis **[Thèse]** and their presentation here will be brief and focused on later developments. In this chapter and the following, the theorems I obtained will be written in a different color in order to distinguish them from known facts in the literature.

During the whole chapter, the object of study will be a branching Brownian motion as defined in Section 1.1. We recall its definition: Starting with an initial individual at the origin of the real line, this individual moves according to a standard Brownian motion until an independent exponentially distributed time with rate $\lambda > 0$. At that moment it dies and produces L (identical) offspring, L being a random variable taking values in the non-negative integers with $\mathbf{P}(L = 1) = 0$. Starting from the position at which its parent has died, each child repeats this process, all independently of one another and of their parent. We denote throughout $m = \mathbf{E}[L] - 1$ and assume that $0 < m < \infty$. This means that the branching is supercritical and the process has positive probability of survival. The set of particles at time t is denoted by $\mathcal{N}_t$. The position of a particle $u \in \mathbf{N}_t$ is denoted by $X_u(t)$.

It will be convenient to add a drift $-\mu \in \mathbf{R}$ to the motion of the particles, i.e. a shift of the particles' positions by $-\mu t$ at time t. Also, we will always set the branching rate to $\lambda = 1/2m$, note that one can always reduce to this choice by scaling time and space.

The functions φ and φ^* from Section 1.1.1 have the following counterparts: for any $t > 0$,

$$\varphi(\theta) = \frac{1}{t} \ln \mathbf{E}_0 \left[\sum_{u \in \mathcal{N}_t} e^{\theta X_u(t)} \right] = \frac{1}{2}(1 + \theta^2) - \mu\theta, \quad \theta \in \mathbf{R}$$

$$\varphi^*(a) = \sup_{\theta \in \mathbf{R}} \{\theta a - \varphi(\theta)\} = \frac{1}{2}(a + \mu - 1)(a + \mu + 1), \quad a \in \mathbf{R}.$$

In particular, Biggins' theorem (Theorem 1.1, or rather its analog in continuous time) shows that the number of particles near at at time a grows exponentially if $-1 - \mu < a < 1 - \mu$ and is almost surely 0 for large t if $a > 1 - \mu$ or $a < -1 - \mu$. In particular, the maximum satisfies

$$\frac{\max_{u \in \mathcal{N}_t} X_u}{t} \to 1 - \mu, \quad \text{a.s. as } t \to \infty.$$

We also recall from Section 1.1.2 the family of *additive martingales* defined in our context by

$$W_t(\theta) = \sum_{u \in \mathcal{N}_t} e^{\theta X_u(t) - \varphi(\theta) t}, \quad \theta \in \mathbf{R}.$$

As for the branching random walk, the main contribution to the martingale $W_t(\theta)$ comes from the particles travelling with speed a, i.e. which are near at at time t, for $a = \varphi'(\theta) = \theta - \mu$.

Throughout the section, we will consider various ways of killing particles. Consider for now the simplest one where we kill particles as soon as they hit a point $-x \leqslant 0$ (equivalently, we can start with a particle at $x \geqslant 0$ and kill particles at the origin). Instead of killing particles it will be technically more convenient to *absorb* them—this means that the particles are frozen and do not move nor branch. This particular process has been considered first by Kesten [Kes78], although general branching diffusions in general domains of $\mathbf{R}^d$ and even general Markov processes have been considered before by Sevast'yanov, Skorohod, Watanabe and others, see for example [INW]. In particular, Kesten proved that this process becomes extinct almost surely if and only if $\mu \geqslant 1$; here extinction is meant in the sense that at some finite time all particles have been absorbed. He actually needed $\mathbf{E}[L^2] < \infty$ for the "only if" part, but the statement holds in general. In fact, the case $\mu \neq 1$ can be easily deduced from Biggins' theorem and the case $\mu = 1$ follows from the fact $\max_{u \in \mathcal{N}_t} X_u(t) \to -\infty$ almost surely as $t \to \infty$, which is a consequence of the fact that the positive martingale $(W_t(1))_{t \geqslant 0}$ converges almost surely.

In fact, one can show the following :

Fact 2.1. *Let N_x denote the number of particles absorbed at $-x$ during the whole course of the process, $x > 0$.*

- *If $|\mu| \geqslant 1$, then $\mathbf{E}[N_x] = e^{\theta_- x}$, where θ_- is the smaller solution of $\varphi(\theta) = 0$, i.e. $\theta_- = \mu - \sqrt{\mu^2 - 1}$.*
- *If $|\mu| < 1$, then $N_x = +\infty$ a.s. on the survival event.*

There are several proofs of (parts of) this fact. See for example [Nev88] or **[M3]**, Lemma 2.4. For the first part ($|\mu| \geqslant 1$), probably the most illuminating argument is the following: Consider $\widetilde{W}_t(\theta_-)$ defined as $W_t(\theta_-)$ but summing over the particles of the process with absorption at the origin. Note that since $\varphi(\theta_-) = 0$, the time-dependent exponent vanishes, so the contribution of an absorbed particle is $e^{-\theta_- x}$, no matter when it is absorbed. Also, one can show that $\widetilde{W}_t(\theta_-)$ is a martingale as well. Furthermore, $\widetilde{W}_t(\theta_-)$ is for large t approximately equal to $e^{-\theta_- x} N_x$ plus a remainder term corresponding to the particles not absorbed at the origin. But since the contribution to the martingale $W_t(\theta_-)$ comes from the particles travelling with speed $\theta_- - \mu = -\sqrt{\mu^2 - 1} \leqslant 0$, the contribution from the particles which do not get absorbed can be shown to converge to 0 in L^1, as $t \to \infty$ (it is here that we used the fact that θ_- is the *smaller* root of φ). This shows that $\widetilde{W}_t(\theta_-)$ converges to $e^{-\theta_- x} N$ in L^1 as $t \to \infty$, and since it is a mean-one martingale, this shows that $\mathbf{E}[N_x] = e^{\theta_- x}$.

To summarize, we have three qualitatively different regimes for the value of μ:

- $\mu \geqslant 1$: The process gets extinct almost surely and the number of absorbed particles is finite almost surely, in fact, $\mathbf{E}[N_x] = e^{\theta_- x}$.
- $|\mu| < 1$: The process survives with positive probability and the number of absorbed particles is a.s. infinite on the survival event.
- $\mu \leqslant -1$: The process survives with positive probability and the number of absorbed particles is finite almost surely, in fact, $\mathbf{E}[N_x] = e^{\theta_- x}$.

2.1 The number of absorbed particles in the case $\mu \geqslant 1$ [M3]

In my thesis, I studied the law of N_x more closely in the case $\mu \geqslant 1$, providing fairly detailed estimates on its tail. Other than previous works on the subject [ABB11, Aïd10], I took a more analytic route, inspired by Pemantle [Pem99]. To start with, I exploited an observation by Neveu [Nev88] that in the case of branching Brownian motion, the process $(N_x)_{x\geqslant 0}$ has the law of a Galton–Watson process in continuous time. This allows to relate the law of N_x for any x to the *infinitesimal generating function* of this Galton–Watson process, i.e., the function

$$a(s) = \frac{d}{dx} E[s^{N_x}]\Big|_{x=0}.$$

Neveu had shown that the function $a(s)$ could be expressed in terms of *travelling wave solutions of the FKPP equation* (see Section 1.2.2). Namely, if ψ is such a travelling wave, then $a(s)$ admits the expression

$$a(s) = \psi'(\psi^{-1}(s)). \tag{2.1}$$

I put this expression to fruitful use in two different ways in order to derive asymptotics for $a(s)$ at $s = 1$:

1. By a real analysis approach, based on the asymptotic study of the two-dimensional system of ODE satisfied by (ψ', ψ). This worked out in the critical case and yielded a precise asymptotic for $a(s)$ for real s at $s = 1$.
2. By a complex analysis approch: I discovered that the function $a(s)$ satisfies a *first-order* ODE in the complex plane, which by a certain change of coordinates (taken from Bieberbach's textbook [Bie65]) can be transformed into a classical ODE called the *Briot–Bouquet equation*:

$$zw' = \lambda w + pz + [w, z]_2,$$

where $\lambda, p \in \mathbf{C}$ and the term $[w, z]_2$ is a power series in w and z with all terms of degree two or more. Classical results on this equation then yielded asymptotics for $a(s)$ for *complex* s in a certain domain near $s = 1$, which could be exploited using theorems by Flajolet and Odlyzko [FO90].

Later developments.

— The analytic approach to the number of absorbed particles from my paper has set the ground for later papers which considered the cases $|\mu| < 1$ [Cor16] and $\mu \leqslant -1$ [BBHM17]. My approach also allowed me subsequently to obtain precise asymptotics for the tail of the *derivative martingale* in branching Brownian motion, see Section 2.4 of my thesis **[Thèse]**.

— For branching random walks, the precise tail asymptotics for N_x were obtained in [AHZ13]. This work also contained an expression for an implicit constant in my results. Their approach can be adapted to the branching Brownian motion (with significant simplifications).

— A few years later, I realized that the case $\mu > 1$ could be treated more easily. Indeed, as noted by Neveu [Nev88], one can show that as $x \to \infty$, $e^{-\theta_- N_x}$ converges almost surely to the limit $W_\infty(\theta_-)$ of the martingale $W_t(\theta_-)$. The tail of $W_\infty(\theta_-)$ has been subject of intense studies, and under a certain power-law tail estimate on L it is known that

$$\mathbf{P}(W_\infty(\theta_-) > x) \sim \frac{C}{x^{\theta_+/\theta_-}}, \quad x \to \infty,$$

for some constant $C > 0$ which admits an expression in terms of some fractional powers of L and $W_\infty(\theta_-)$ [Liu00, JOC12]. One can now use the Galton–Watson process structure of $(N_x)_{x\geqslant 0}$ to relate the tail of $W_\infty(\theta_-)$ to the tail of N_x [BD74, De 82] (one can also use (2.1) and the fact that ψ is related to the Laplace transform of $W_\infty(\theta_-)$ [Nev88]). Yet, I feel that my approach still retains its interest, since it can be reinterpreted as an independent proof of the tail estimate of $W_\infty(\theta_-)$, based on the relation between FKPP travelling waves and the Briot–Bouquet equation. It also gives access to complex variables techniques, which as far as I know have not been applied before in the study of FKPP travelling waves.

2.2 The near-critical case $\mu = 1 - \varepsilon$: The BBS approach

The work by Berestycki, Berestycki and Schweinsberg on the near-critical case $\mu = 1-\varepsilon$, with ε small, has had an important influence on my work on branching Brownian motion with selection. In the seminal papers [BBS13, BBS11], they performed a very precise study of the long-time behavior and the genealogy of this model, inspired by non-rigorous results from the physics literature [BDMM06a, BDMM07]. I will briefly review these results here.

The first quantity that one might want to study is the probability that BBM with absorption, started from one particle at 1, say, survives when $\mu = 1 - \varepsilon$. A first result in this direction was shown by Gantert, Hu and Shi [GHS11]. Let us reparametrize μ by setting

$$\mu = \mu_a = \sqrt{1 - \frac{\pi^2}{a^2}} = 1 - \frac{\pi^2 + o(1)}{2a^2}, \quad a \to \infty. \tag{2.2}$$

Then Gantert, Hu and Shi [GHS11] proved (for branching random walk in fact), that the probability p_a of survival satisfied

$$\ln p_a \sim -a, \quad a \to \infty.$$

The reason for this behavior is that roughly speaking, in order for the system to survive, there has to be a particle moving approximately up to height a, and this event turns out to have probability roughly e^{-a}, see Figure 2.1 for an illustration.

Berestycki, Berestycki and Schweinsberg [BBS13, BBS11] gave a much more precise form of this result (though for BBM only): Write $p_a(x)$ for the probability that the BBM with absorption survives when started from one particle at $x \geqslant 0$. They showed the following two facts:

1. There exists a function $\psi : \mathbf{R} \to (0,1)$, such that $p_a(a+x) \to \psi(x)$ as $a \to \infty$.
2. If $x = x(a)$ such that $a - x \to \infty$, then $p_a(x) \sim C w_a(x)$ as $a \to \infty$, where

   $$w_a(x) = a \sin(\pi x/a) e^{-\mu_a(a-x)},$$

   and $C > 0$ is some constant.

Let us briefly explain the heuristics behind this result. Suppose we add a second absorbing barrier at a. Denote by $q_a(t,x,y)\,dy$ the *expected* number of particles at time t in an infinitesimal interval of length dy around y when starting with a single particle at x. Then the function q_a is the fundamental solution to the following PDE:

$$\begin{cases} u_t = A_a u := \frac{1}{2}u_{xx} + \mu_a u_x + \frac{1}{2}u \\ u(t,x,0) = u(t,x,a) = 0. \end{cases}$$

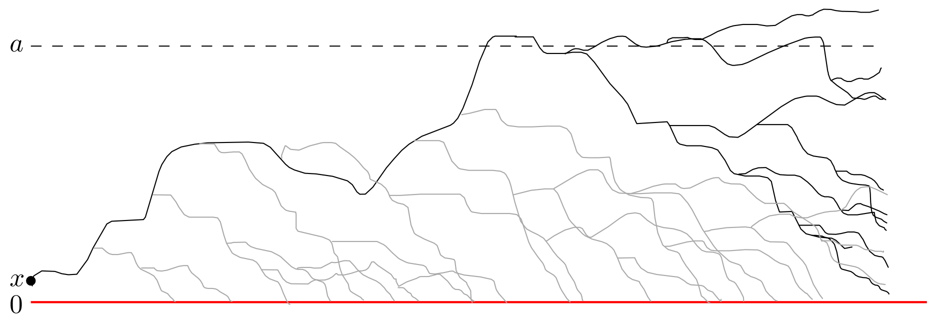


Figure 2.1 – Illustration of branching Brownian motion with absorption at the origin (red line) and drift $-\mu_a = -\sqrt{1-\pi^2/a^2}$, for large a and fixed starting point $x > 0$. In order for the system to survive, a particle has to reach height $(1+o(1))a$, which has a probability of $e^{-(1+o(1))a}$ [GHS11, BBS11]. This particle and its descendants after the time it reaches that height are drawn in black, the other particles in gray.

The three terms on the right-hand side of the equation correspond to diffusion, drift and branching, respectively, and the Dirichlet boundary condition to the fact that particles are killed at 0 and a. The (discrete) spectrum and the eigenfunctions of A are easily calculated:

$$\lambda_k = \frac{1}{2} - \frac{\mu_a^2}{2} - \frac{\pi^2 k^2}{2a^2} = -\frac{\pi^2(k^2-1)}{2a^2}, \quad k = 1, 2, \ldots,$$

and an orthogonal basis of eigenfunctions is given by

$$e_{a,k}(y) = a \sin(\pi k y/a) e^{-\mu_a y}.$$

In particular, with our choice of μ_a, the largest eigenvalue λ_1 equals zero, which means that for every $x \in (0,a)$, $q_a(t,x,y)$ will converge as $t \to \infty$ to a multiple of $e_a := e_{a,1}$. If instead we move up the barrier to height $a' > a$, then $\lambda_1 > 0$, meaning that the expected number of particles will grow exponentially in time. Conversely, if we move down the barrier, then the expected number of particles will decay exponentially in time. This suggests that it is indeed the particles going up to height a which generate enough descendants for the system to survive in the long run and thus explains that $p_a(a)$ stays bounded between 0 and 1.

In order to explain the asymptotic of $p_a(x)$ when $a - x \gg 1$, we note that the function w_a is an eigenfunction of the formal adjoint A^* of A, and so $q_a(t,x,y)$ converges to a constant multiple of $w_a(x)e_a(y)$ as $t \to \infty$, in fact,

$$q_a(t,x,y) \to 2w_a(x)e_a(y), \quad t \to \infty.$$

This explains that $p_a(x) \asymp w_a(x)$ for x such that $a - x \gg 1$.

In order to make a rigorous proof out of these truncated first moment estimates, Berestycki, Berestycki and Schweinsberg proceeded as follows: They lowered the upper barrier by a large constant, i.e. they put a barrier at $a - A$ for $A \gg 1$ but such that first $a \to \infty$, then $A \to \infty$. By first and second moment estimates they then showed that the particles which always stayed inside the interval $(0, a-A)$ evolved almost deterministically over the time scale a^3, which turns out the be the important time scale to consider. As for those particles hitting the upper barrier, they evaluated precisely the law of the number of descendants of these particles. This allowed them to show that *the (renormalized) number of particles in the system, seen as a stochastic process, converges over the time scale a^3 to a certain limit process called* Neveu's continuous-state branching process (CSBP). Finally, they proved that the event of survival of the BBM with absorption corresponds with high probability to the event of survival of this CSBP which is explicitly given in terms of the initial condition.

As a consequence of their methods, they were also able to show that the *genealogy* of the BBM with absorption converges, over the time scale a^3, to the famous *Bolthausen–Sznitman coalescent*, which describes the genealogy of Neveu's CSBP as shown in a seminal work by Bertoin and Le Gall [BL00].

The work by Berestycki, Berestycki and Schweinsberg has opened the door for many other results, as we will see in the next sections.

2.3 N-particle branching Brownian motion with spatial selection [M9]

This work, which represents a large part of my PhD thesis **[Thèse]**, treats a model which I call the N-BBM: a branching Brownian motion in which only the N particles at the largest positions survive at any time, the others being killed, i.e. removed from the system. This system has been introduced by Brunet and Derrida in order to model amongst others a population of fixed size under the influence of reproduction, mutation and selection: the position of a particle is then interpreted as the fitness of an individual and only the N fittest individuals are kept at any time.

The main result in **[M9]** is the following limit theorem:

Theorem 2.2. *Let $X_N(t)$ denote the position of the median (i.e. the $N/2$-th particle from the top). Let*

$$v_N = 1 - \frac{\pi^2}{2\ln^2 N} + \frac{3\pi^2 \ln\ln N}{\ln^3 N}. \tag{2.3}$$

Then under "good" initial conditions, the finite-dimensional distributions of the process

$$\big(X_N\big(t\ln^3 N\big) - v_N t \ln^3 N\big)_{t\geqslant 0}$$

converge as $N\to\infty$ towards those of the Lévy process $(L_t)_{t\geqslant 0}$ with

$$\ln E[e^{i\lambda(L_1-L_0)}] = i\lambda c + \pi^2 \int_0^\infty e^{i\lambda x} - 1 - i\lambda x \mathbf{1}_{(x\leqslant 1)}\,\Lambda(dx),$$

where Λ is the push-forward of the measure $(x^{-2}\mathbf{1}_{(x>0)})dx$ by the map $x\mapsto \ln(1+x)$ and $c\in\mathbb{R}$ is a non-explicit constant.

The proof of this result makes heavy use of the ideas from [BBS13] and [BDMM06a], but adds to it several new ones. The basic idea is to compare the N-BBM with a BBM with absorption at a suitable random barrier. This process is dubbed "B-BBM" and is depicted in Figure 2.2. The actual comparison is done by coupling with two variants of this process.

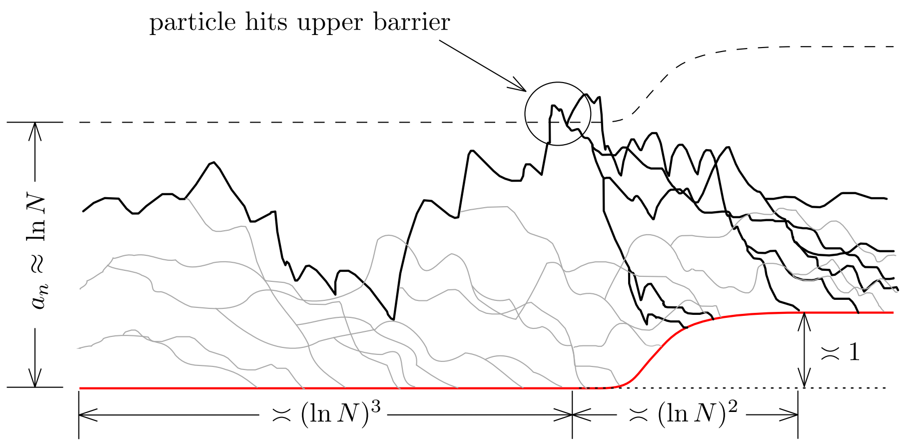


Figure 2.2 – A caricatural graphical description of the B-BBM, which approximates the N-BBM. Particles have drift $-v_N$ and are killed at the lower barrier (in red). Most of the time this barrier stays constant. When a particle reaches the upper barrier (dashed line), which typically happens over the timescale $(\ln N)^3$, this particle will spawn a large number of descendants. In order to keep the population size roughly constant, the lower barrier moves continuously over the timescale $(\ln N)^2$ by an amount of order 1. Then the process starts afresh.

Later developments and perspectives. Many "facts" about the N-BBM still lack rigorous proofs. For example, the usual definition of "speed" is the limit $X_N(t)/t$ as $t \to \infty$. Since we only study the N-BBM over timescales of order $(\log N)^3$, our Theorem 2.2 says nothing on the asymptotic of the speed for large N, although its asymptotic should be similar to v_N. The first order correction term is known [BG10], but apart from that no further progress has been made. For a supposedly related model, namely, the FKPP equation with a certain multiplicative noise, the next order term has indeed be shown to be of order $(\log\log N)/(\log N)^3$ (with our notation) [MMQ10].

The empirical measure seen from the left-most particle of N-BBM is an ergodic measure-valued process and thus has a stationary distribution. Under this stationary distribution, the empirical measure (multiplied by $1/N$) should converge to the deterministic measure $xe^{-x}\,dx$ as $N \to \infty$. This problem is still unsolved. One obstacle to this question has recently been removed, namely, the existence of the hydrodynamic limit of N-BBM, which is a solution of a certain free-boundary partial differential equation [DFPSL17, BBP18, Lee17]. For a certain continuous-time branching random walk, this has been known previously [DR11]. However, the convergence to a stationary limit of the solution to the (deterministic) equation still lacks a proof, even in the case of the branching random walk. Without this, studying the limiting distribution of the empirical measure of N-BBM seems fairly difficult.

Another interesting question is about the genealogy of N-BBM. For the BBM with absorption at near-critical drift, it is known [BBS13] that the genealogy is given, at the timescale $(\log N)^3$, by the celebrated Bolthausen–Sznitman coalescent. Our methods for studying the N-BBM (namely, using upper and lower bounds constructed via coupling), deforms the genealogy and therefore *a priori* does not allow to obtain information on the genealogy. On the other hand, it should not be difficult to show that the genealogy of the B-BBM converges to the Bolthausen–Sznitman coalescent. It might therefore be of reach to prove this for the N-BBM as well, albeit technically challenging.

Finally, we mention that other types of N-BRW or related models have been considered in the literature, see for example [CM16] and the references therein.

2.4 Yaglom-type limit theorems at critical drift $\mu = 1$ (in progress)

In this work in progress with Julien Berestycki and Jason Schweinsberg, we consider branching Brownian motion with absorption at the critical drift $\mu = 1$. As stated above, the probability of survival of this process is zero, but one can consider $p(t, x)$, the probability of survival *until time* t when starting from x. One then hopes to be able to let x and t go to infinity together in such a way as to obtain a non-trivial limit. A related question is a Yaglom-type theorem: conditioned on survival until time t, what does the configuration of particles look like? First results on these two questions have been obtained by Kesten [Kes78]. Define throughout the section

$$c = \left(\frac{3\pi^2}{2}\right)^{1/3}.$$

Kesten showed the following:

1. For sufficiently large t, $p(t, 1) = e^{-ct^{1/3}+O(\ln t)^2}$.
2. Let M_t and N_t denote the maximum position and the number of particles at time t, respectively. Starting from a particle at 1 and conditioned on survival until time t, with high probability, for sufficiently large t,

$$M_t = O(t^{2/9}(\ln t)^{2/3}), \quad N_t = e^{O(t^{2/9}(\ln t)^{2/3})}.$$

We greatly refine these estimates and obtain several precise convergence results. To simplify, we only state two of them. The first one gives the precise estimates for the probability of survival:

Theorem 2.3. *There is a function $\phi : \mathbf{R} \to (0, 1)$ such that for all $x \in \mathbf{R}$,*

$$\lim_{t\to\infty} p(t, ct^{1/3} + x) = \phi(x), \tag{2.4}$$

with $\lim_{x\to-\infty} \phi(x) = 0$ and $\lim_{x\to\infty} \phi(x) = 1 - q$, where q is the extinction probability for the branching Brownian motion without *absorption. Furthermore, if $x = x(t)$ such that $ct^{1/3} - x \to \infty$, then for some constant $C > 0$,*

$$p(t, x) \sim Cw_t(x), \quad w_t(x) = ct^{1/3} \sin(\pi x/(ct^{1/3}))e^{x-ct^{1/3}}.$$

Theorem 2.3 shows that the extinction time is indeed approximately t when starting from a particle near $ct^{1/3}$. It actually easily allows to determine the fluctuation of the extinction time: if ζ denotes the time of extinction, then Theorem 2.3 implies that when starting with one particle at $ct^{1/3}$, $(\zeta - t)/t^{2/3}$ converges in law to an explicit distribution.

We remark that Berestycki, Berestycki and Schweinsberg [BBS14] had already obtained a weaker form of Theorem 2.3, with the same estimates but up to multiplicative constants.

The following theorem concerns the Yaglom limit:

Theorem 2.4. *Starting with one particle at 1, conditioned on survival until time t, the random variables* $\ln N_t/t^{2/9}$ *and* $M_t/t^{2/9}$ *converge in law to the same (non-degenerate) limit, as* $t \to \infty$.

As Kesten remarked, this result is in stark contrast with the case $\mu > 1$, where the configuration of particles at time t, conditioned on survival until time t, converges in law to a limit (without renormalization). This result was claimed by Kesten and recently proven by Louidor and Saglietti [LS17].

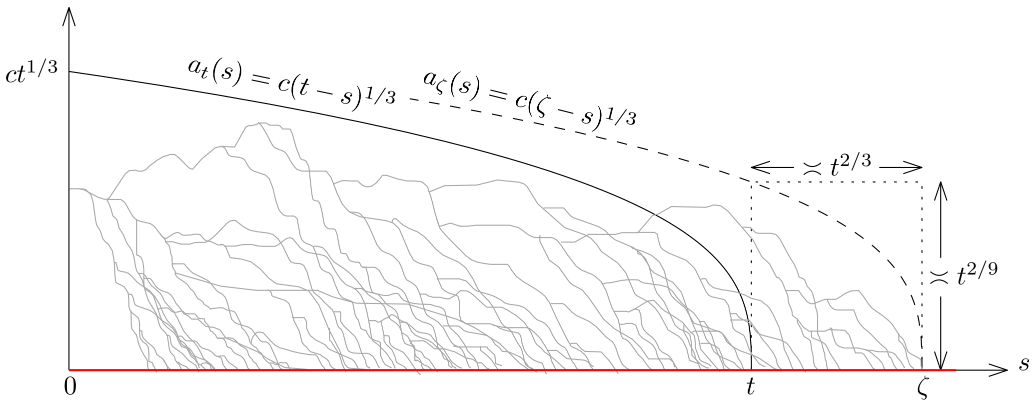


Figure 2.3 – Illustration of the BBM with absorption conditioned on survival until time t. The curve $a_t(s)$ is shown under which particles typically stay for a large time. Between the times t and ζ, the system behaves like a rescaled copy of the original process, staying most of the time under the shifted curve $a_\zeta(s)$.

In order to prove Theorems 2.3 and 2.4, we adapt the techniques from [BBS13, BBS11] to the case of critical drift. That is, we add a second barrier and treat separately the particles that stay below the barrier and those that get absorbed by it. Of course, this barrier cannot be a straight line anymore, as in the near-critical case, since the barrier there moves to infinity as $\mu \to 1$. Instead, the barrier is a *curve*, or rather a family of curves indexed by t:

$$a_t(s) = c(t-s)^{1/3}, \quad s \in [0,t].$$

Note that $a_t(0) = ct^{1/3}$. By arguments similar to those in [BBS13] (but more involved due to the time-inhomogeneity), we can again show that after a certain time-change which blows up near t, the number of particles in the process, suitably renormalized, converges in law to Neveu's CSBP and the probability of survival until time t converges to the probability of *ultimate* survival of this CSBP. This yields Theorem 2.3.

For Theorem 2.4 one has to work more and the proof is based on the fact that the system exhibits a certain *self-similarity*: many features of the system are preserved under the space-time scaling $t = \alpha t'$, $x = \alpha^{1/3}x'$. For example, just before the extinction time ζ, the particles behave as if they were trapped under the curve $a_\zeta(s)$ (see Figure 2.3). In particular, conditioned on $\zeta > t$, the maximum M_t at time t will be roughly at position $a_\zeta(t) = c(\zeta - t)^{1/3}$. The part about M_t in Theorem 2.4 then follows from the fact that conditioned on $\zeta > t$, the renormalized difference $(\zeta - t)/t^{2/3}$ converges in law as $t \to \infty$.

2.5 1-stable fluctuations in BBM [M16]

In this work we consider branching Brownian motion *without* absorption, but, as we will see, the ideas from the previous sections will come into the game. Amongst other things, we are interested in the additive martingale and derivative martingale of BBM with critical parameter $\theta = 1$ (see Section 1.1.2) and more precisely, their fluctuations. To fix the setting, let the parameters be given as in the beginning of the chapter and add drift 1 to the particles, i.e. set $\mu = -1$. Then the derivative martingale (see Section 1.1.2) is defined by

$$Z_t = \sum_{u \in \mathcal{N}_t} X_u(t) e^{-X_u(t)},$$

and it is known that it almost surely converges to a non-degenerate limit Z_∞ if (and only if) $\mathbf{E}[L(\ln L)^2] < \infty$.

More generally, define for each $t \geqslant 0$ the random measure

$$\mu_t = \sum_{u \in \mathcal{N}_t} X_u(t) e^{-X_u(t)} \delta_{X_u(t)/\sqrt{t}},$$

so that $\mu_t(\mathbf{R}) = Z_t$. It is known under slightly stronger assumptions ([Mad16], see also **[M8]** for a simple proof under the hypothesis $\mathbf{E}[L^2] < \infty$) that

$$\mu_t \to \mu_\infty := Z_\infty \sqrt{\frac{2}{\pi}} x^2 e^{-x^2/2}\, dx, \quad \text{a.s., w.r.t. the weak topology.}$$

It is natural to ask about the rate of convergence or, more precisely, whether a central limit theorem holds, i.e., given a function F, is there a deterministic sequence $(b_t)_{t \geqslant 0}$ such that

$$b_t \left(\int F\, d\mu_t - \int F\, d\mu_\infty \right) \text{ converges in law as } t \to \infty?$$

It happens that the answer is not that simple. One has to consider first the derivative martingale, then general functions F. In case of the derivative martingale, we have the following result:

Theorem 2.5 (**[M16]**). *Suppose* $\mathbf{E}[L(\ln L)^3] < \infty$. *We have the following convergence in law w.r.t. finite-dimensional distributions:*

$$(\sqrt{t}(Z_\infty - Z_{at} + \frac{\ln t}{\sqrt{2\pi a t}} Z_\infty))_{a \geqslant 1} \to S_{Z_\infty/\sqrt{a}})_{a \geqslant 1},$$

where $(S_t)_{t \geqslant 0}$ *is a certain spectrally positive 1-stable Lévy process, independent of* Z_∞.

Thus, Theorem 2.5 shows that a non-standard CLT involving a 1-stable limit law holds for Z_t and even gives a functional version. Note the presence of the logarithmic correction term, which is typical for convergences to 1-stable laws.

As for general functions F, since we still have to work out some details, I only state an incomplete result here:

Theorem 2.6 (M., Pain, work in progress). *For every $F : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying $F(x) \leqslant \frac{C}{x}e^{Cx}$ for some $C > 0$ (plus some smoothness assumptions), there exist $\rho_F \in \mathbf{R}$ and a 1-stable Lévy process $(S_t)_{t\geqslant 0}$, independent of Z_∞ and in general not spectrally positive, such that*

$$\sqrt{t}\left(\int F\,d\mu_t - \int F\,d\mu_\infty - \rho_F\frac{\ln t}{\sqrt{t}}Z_\infty\right) \to S_{Z_\infty}, \quad \text{in law as } t \to \infty.$$

Both ρ_F and the law of $(S_t)_{t\geqslant 0}$ are explicit. In particular, if $F(x) = \frac{1}{x}$, then $\rho_F = 0$ and $(S_t)_{t\geqslant 0}$ is a Cauchy process (i.e., the coefficient of asymmetry is 0).

This theorem is surprising in several aspects. First of all, the limit laws are in general not asymmetric which is intriguing given the previous theorem. Secondly, in the case $F(x) = 1/x$ the logarithmic correction term vanishes and the fluctuations are therefore really of order $1/\sqrt{t}$. This last case is particularly important, since $\int x^{-1}\,d\mu_t = \sqrt{t}W_t$, where W_t is the usual additive martingale at critical temperature,

$$W_t = \sum_{u\in\mathcal{N}_t} e^{-X_u(t)}.$$

Note that it is known [AS14] that $\sqrt{t}W_t \to \sqrt{2/\pi}Z_\infty$ in probability as $t \to \infty$, Theorem 2.6 refines this by showing that

$$\sqrt{t}\left(\sqrt{t}W_t - \sqrt{2/\pi}Z_\infty\right) \tag{2.5}$$

converges to a mixture of Cauchy distributions. We will comment below on other possible consequences of Theorem 2.6.

Before giving some details of the proofs, I will describe the approach by the physicists Mueller and Munier [MM14] (it is this article that led us to investigating these questions). In this article, the authors focus on the martingale $(W_t)_{t\geqslant 0}$ and provide arguments for the convergence of the term in (2.5) to a certain random variable (it is not clear to us at the moment whether it is the same as our limit). Their method relies on a heuristic decomposition of the branching Brownian motion into a deterministic and a stochastic part, in the spirit of [BDMM06a] (see Section 2.3), but with additional ideas from [EvS00]. Specifically, the deterministic part consists of the particles above the typical position of the minimal particle, namely, $\frac{3}{2}\log t$ (see Theorem 1.3). The authors of [MM14] approximate the density of the particles in this area by a deterministic function and, assuming the function admits a certain expansion in terms of powers of $t^{-1/2}$, explicitly calculate the first few terms in this expansion in terms of hypergeometric functions. They then model the particles crossing the line as certain stochastic fluctuations, the effect of which they calculate again by deterministic approximations. The computations are fairly involved and, as said above, rely on several unverified assumptions.

Our methods are partly inspired by [MM14] in the sense that we also add absorbing barriers and treat seperately the particles crossing these barriers and those that stay above them. However, the choice of the barriers is fairly different, as well as the way we treat the particles crossing these barriers and is closer in spirit to the methods from [BBS13] (see Section 2.2). It turns out that in order to obtain Theorem 2.6, one first has to obtain Theorem 2.5, i.e. the non-standard CLT for the derivative martingale, for reasons which will soon be clear. The heart of the proof of Theorem 2.5 is the addition of an absorbing barrier *from time t on* at a *fixed point* γ_t (instead of a curved barrier as in [MM14]). There is a lot of freedom in choosing γ_t, it must only satisfy

$$\gamma_t - \frac{1}{2}\log t \to \infty, \quad \text{and} \quad \gamma_t = o(t^{1/3}), \quad t \to \infty.$$

For example, we may think of it as $\gamma_t = \frac{1}{2}\log t + K$ and let first $t \to \infty$ and then $K \to \infty$. We then look at the translated derivative martingale

$$\widetilde{Z}_s = \sum_{u \in \mathcal{N}_s} (X_u(s) - \gamma_t) e^{-X_u(t)} = Z_s - \frac{\log t}{2} W_s - K W_s,$$

which is also a martingale when particle are killed when going below γ_t. With the barrier, we then show by first- and second moment estimates that this martingale is very concentrated around its limit $\widetilde{Z}_\infty = Z_\infty$ and therefore one has

$$Z_t - \frac{\log t}{2} W_t - K W_t \simeq \widetilde{Z}_\infty + o\left(\frac{1}{\sqrt{t}}\right) \tag{2.6}$$

for K large. On the other hand, $Z_\infty - \widetilde{Z}_\infty$ is the sum of the contributions of the killed particles, which are i.i.d. copies of $e^{-\gamma_t} Z_\infty = e^{-K} Z_\infty / \sqrt{t}$. Moreover, the number of killed particles can be shown to be approximately equal to $\sqrt{2/\pi} Z_\infty e^K$. We now use the fact that Z_∞ is in the domain of attraction of a totally asymmetric 1-stable random variable to deduce that for large K, the sum of the contributions of the killed particles is, after a proper compensation, roughly distributed as $S_{Z_\infty}/\sqrt{t}$, with the notations of Theorem 2.5. It turns out that this compensation term is exactly the term $-KW_t$ appearing in (2.6). This gives a proof of Theorem 2.5 in case of one-dimensional martingales which can be extended to yield functional convergence. The proof also shows that the fluctuations of the derivative martingale are due to the particles that come down exceptionally low: around $\frac{1}{2}\log t + O(1)$. The fact that there are particles coming that low has been known previously [HS09, AS14].

The proof of Theorem 2.5 can not be extended as-is to prove Theorem 2.6. Roughly speaking, the reason why the proof of Theorem 2.5 works is that we can compare Z_t and $Z_{t'}$ for $t' > t$ because the expected value of a certain variant of $Z_{t'}$, conditioned on the process at time t, can be compared to Z_t. In other words, we were able to approximate Z_t by a certain martingale. This is not at all the case for general functionals, as can be seen by the fact that all functionals we consider in Theorem 2.6 converge to a multiple of Z_∞. We therefore have to resort to a certain bootstrap procedure, where some a rough polynomial estimates on the rate of convergence are successively transformed into finer and finer estimates. Luckily, this procedure terminates after a finite number of steps and allows to obtain Theorem 2.6.

Perspectives. We believe that our results will help in providing a probabilistic proof of the famous Ebert–van Saarloos asymptotic for the FKPP equation. In the abovementioned article [EvS00], Ebert and van Saarloos predicted a rate of convergence of $1/\sqrt{t}$ of the solution to the FKPP equation to the travelling wave, which has recently been proven rigorously [NRR16]. Thanks to the duality between FKPP and BBM (see Section 1.2.2), this asymptotic is equivalent to showing that the median of the law of the minimal particle, recentered by $\frac{3}{2}\log n$, converges at rate $1/\sqrt{t}$ (with an explicit multiplicative constant). It is well-known that there is a relation between the Gibbs measure at critical temperature and the position of the minimum particle at a later time. It is therefore probable that our results can aid in proving the Ebert–van Saarloos prediction and possibly even more, such as a convergence result in the mod-ϕ sense [DKN15].

It would be also of interest to extend our results to branching random walks. Our methods have to be refined in this case, since they make use of the continuity of the particle's trajectories. However, we believe the adaptation to be possible, in the spirit of [AHZ13].

2.6 Branching Brownian motion with variance decreasing in time [M8]

In this paper, we consider a variant of branching Brownian motion, where the diffusion constant or variance of the particle motion changes over time. More precisely, let $\sigma : [0,1] \to \mathbf{R}_+$. We then consider the generalization of branching Brownian motion where the infinitesimal variance of the particle motion at time $t \in [0,T]$ is given by $\sigma^2(t/T)$. So formally, we are considering a family of branching processes indexed by T.

The discrete-time analogue of this process is also known as the *continuous random energy model (CREM)* and has been introduced and studied by Bovier and Kurkova [BK04, BK07]. Instead of Bovier and Kurkova, who were interested in the free energy and the overlap distribution of the CREM, we are interested in fine asymptotics for the maximum position M_T at time T. These depend very much on the function σ. To make things simple, we consider here only $\sigma \in C^2([0,1])$ which are strictly decreasing and with $\sigma(1) > 0$ and $\inf_{t\in[0,1]} |\sigma'(t)| > 0$. The most important of these assumptions is the fact that σ is strictly decreasing.

To state our results, introduce the functions $v, w : [0,1] \to \mathbf{R}_+$ by

$$v(t) = \int_0^t \sigma(s)\,\mathrm{d}s\,, \tag{2.7}$$

and

$$w(t) = 2^{-1/3}\alpha_1 \int_0^t \sigma(s)^{1/3}|\sigma'(s)|^{2/3}\,\mathrm{d}s\,, \tag{2.8}$$

where $-\alpha_1 = -2.33811...$ is the largest zero of the *Airy function of the first kind*

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt\,, \tag{2.9}$$

see [AS64, Section 10.4] for definitions; note that Ai satisfies the Airy differential equation $\mathrm{Ai}''(x) - x\mathrm{Ai}(x) = 0$. Note also that $v_\sigma = v(1)$. Set

$$m_T = v(1)T - w(1)T^{1/3} - \sigma(1)\ln T.$$

Our main result is the following.

Theorem 2.7. *The family of random variables $(M_T - m_T)_{T\geqslant 0}$ is tight. Further, there exists a solution $\phi(x)$ to (1.3) and a function m'_T with $C_\sigma = \limsup_{T\geqslant 0} |m'_T - m_T| < \infty$, such that for all $x \in \mathbf{R}$,*

$$\lim_{T\to\infty} \mathbf{P}(M_T \leqslant m'_T + x) = \phi(x/\sigma(0)).$$

Let us comment on this result. For the uninitiated, the appearance of a $T^{1/3}$ correction term is probably most surprising. However, this term is of the same kind as the $t^{1/3}$ term from Section 1.1.4. Indeed, because of the decreasing variance, the maximum particle at time T stays close to the maximum *at all time* $t \in [0,T]$ with high probability, as noted already by Fang and Zeitouni [FZ12]. It is well-known that such constraint forces the particle to move in a band of width of order $T^{1/3}$, see for example [Kes78, FHS12, FZ10, Jaf12, Rob15, ABP17] for related results.

The core of the proof of Theorem 2.7 is based on a constrained first and second moment analysis of the number of particles that reach a target value but remain below a barrier for the duration of their lifetime. Due to the time-inhomogeneity of $\sigma(\cdot)$, the choice of barrier is not straight-forward, and in particular it is not a straight line; "rectifying" it introduces a killing potential. The analysis of the survival of Brownian motion in this potential eventually leads to a time-inhomogeneous Airy-type differential equation which we study by analytic means, exploiting the anti-symmetry of the differential operator. (As pointed out to us by Dima Ioffe, a similar phenomenon with related $T^{1/3}$ scaling was already observed in [Gro89, FS05].) These methods together lead to estimates of the right tail of M_T which are sharp up to a multiplicative factor. By a bootstrapping procedure that may be of independent interest, these estimates are then turned into convergence in law by using a convergence result for the derivative Gibbs measure of (time-homogeneous) branching Brownian motion.

Parallel to our work, and an inspiration to it, was the study [NRR14], by PDE techniques, of a class of time-inhomogeneous Fisher–Kolmogorov-Petrovskii-Piskunov equations. As in the time-homogeneous case (see Section 1.2.2), there is a strong link with our model. Indeed, let $F(x,T) = P(M_T \leqslant x)$ be the cumulative distribution function of M_T and assume $L = 2$ for simplicity. Then $F(x,t)$ is the solution of the

$$\begin{cases} F_t(x,t) = \frac{1}{2}\sigma^2(1-t/T)F_{xx}(x,t) - \frac{1}{2}F(x,t)(1-F(x,t)), & t \in [0,T],\ x \in \mathbf{R} \\ F(x,0) = \mathbf{1}_{x\geqslant 0}. \end{cases}$$

Compared with [NRR14], we deal with a slightly restricted class of equations, but are able to obtain finer (up to order 1) asymptotics and convergence to a travelling wave.

Mallein [Mal15] has also simultanuously published results similar to ours which are less precise but hold for a rather general class of (not necessarily Gaussian) time-inhomogeneous branching random walks.

Perspectives An obvious next step is to prove convergence in distribution of the family $(M_T - m_T)_{T\geqslant 0}$ (instead of $(M_T - m'_T)_{T\geqslant 0}$). The major difficulty here is that unlike in the time-homogeneous case, extremal particles at time T will, with positive probability, be extremal at some random intermediate time between ϵT and $(1-\epsilon)T$ (this is related to the *full replica symmetry breaking* of the model in spin glass jargon). However, I believe that this can be handled as in BBM with absorption (see the previous sections), namely by carefully reintegrating the particles which hit the barrier, and showing convergence of a suitable statistic to Neveu's CSBP. This approach would then also allow to show convergence of the *extremal process*, in the spirit of Theorem 1.3. It would also provide a way of studying the genealogy of the maximal particles. Its law is known for the CREM [BK07], but the proof there relies very much on the Gaussian integration by parts formula and therefore does not carry over to non-Gaussian settings. The approach using extremal particles, on the other hand, although technically more involved, could probably be extended to non-Gaussian branching random walks. See e.g. [Mal16] for the time-homogeneous case.

Another interesting question is large deviation behavior of the maximum, i.e., what is the probability that the maximum is a constant times larger than its typical value and what does the process look like conditioned on this event? For the time-homogeneous case, this has been treated long ago by Chauvin and Rouault [CR88]. It is possible that their techniques (martingale change of measures) can be adapted to the time-inhomogeneous case.

Chapter 3

Heavy-tailed branching random walk

Branching random walk with heavy-tailed displacement behaves very differently than BRW with light-tailed displacement: in some ways, the behavior is simpler. In this chapter, we will see two examples of this phenomenon. The first concerns the heavy-tailed analogue of the N-BBM from Section 2.3, the second one is (among other things) about branching random walk with heavy tailed jumps and *critical* branching.

3.1 N-particle branching random walk with polynomial tails [M5]

We consider a system of N particles on the real line that evolves through iteration of the following steps: 1) every particle splits into two, 2) each particle jumps according to a prescribed displacement distribution supported on the positive reals and 3) only the N right-most particles are retained, the others being removed from the system. We call this system the N-BRW; it is the discrete-time analogue of the N-BBM from Section 2.3. As we have seen there, its behavior for large N is now well understood – both from a physical and mathematical viewpoint – in the case where the displacement distribution has light tails, say it admits all exponential moments. Here, we consider the case of displacements with regularly varying tails, where the relevant space and time scales are markedly different.

We denote by $X \geqslant 0$ a random variable following the law of a displacement step of a particle. Define $h(x)$ by

$$\mathbf{P}(X > x) = 1/h(x),\ x \geqslant 0. \tag{3.1}$$

We assume that the function $h(x)$ is regularly varying at $+\infty$ with index $\alpha > 0$.

We now decribe the global heuristic picture of the N-BRW. One can expect that the picture will be very different from the one for N-BBM. Indeed, in both cases, in order for a particle to survive it has to move faster than usual, which can be described by a large deviation event for a random walk. If the Cramér condition is verified, i.e. if the law of the displacement admits sufficiently high exponential moments, then on such an event, the random walk behaves like a Brownian motion with drift. When the Cramér condition is not met, the picture is quite different: the large deviation typically comes from a single jump which occurs at a random time (this is also called the "single big jump principle"). Of course, with many interacting particles as in the N-BRW one has to be more careful, but it happens that in the case of regularly varying tails, the basic picture is quite simple.

As it turns out, the relevant scales are $\log_2 N$ for the time and $c_N = h^{-1}(2N \log_2 N)$ for the space, where h^{-1} is the generalized inverse of h. As we will show, at a typical time and viewed on the space scale c_N, the particles are divided into one big "tribe" located near the left-most particle and containing all but $o(N)$ particles, the remaining particles to the right being split into a $O(1)$ number of smaller tribes. At each time step, the number of particles in every small tribe is multiplied by two, which eventually leads to extinction of the big tribe and another tribe taking over. Furthermore, new tribes are formed by particles performing (rightward) jumps of magnitude c_N out of the big tribe. The value of c_N has been chosen so that these events occur on the time scale $\log_2 N$, which is precisely the time it takes for a new tribe to grow to size N.

The stairs process. We now define the *stairs process*, a real-valued stochastic process which captures the heuristic dynamics state above and which will be shown to approximate the N-BRW when N is large. Let μ be the measure on $(0,\infty)$ defined by $\mu([x,\infty)) = x^{-\alpha}$. The α-stairs process then is the real-valued stochastic process $(\mathcal{R}(t))_{t\geqslant 0}$ defined inductively as follows: For $t \leqslant 0$, $\mathcal{R}(t) = 0$. For integer $n = 0,1,2,\ldots$, suppose $\mathcal{R}(t)$ is defined for $t \leqslant n \in \mathbf{N}$. Now generate points in $(n, n+1] \times \mathbf{R}_+$ according to the Poisson point process with intensity $\mathrm{d}t \otimes \mu$ and translate every atom (t,x) by $\mathcal{R}(t-1)$ in the x-direction. Then define $(\mathcal{R}(t))_{t\in(n,n+1]}$ to be the record process of these points. See Figure 3.1 for a graphical representation.

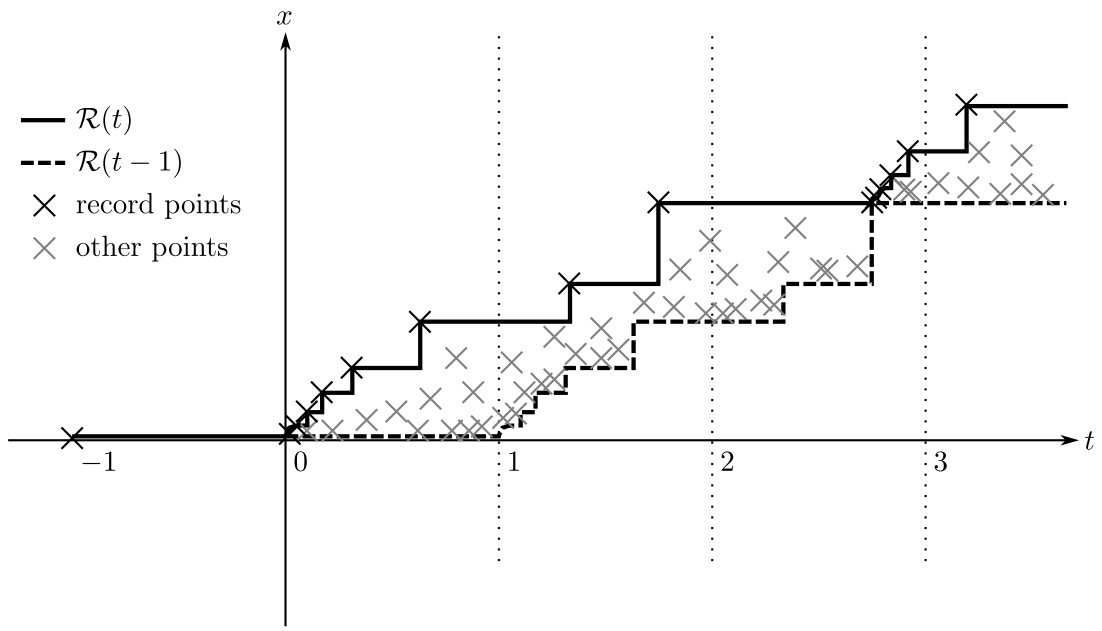


Figure 3.1 – Graphical representation of the stairs process.

Remark. The stairs process can be interpreted as a self-interacting version of a process known in the literature as a *Poisson paced record process* [BG01]. Its long-term behaviour is quite different: While a Poisson paced record process usually grows logarithmically in t, the stairs process grows like a random walk, due to the existence of regeneration times.

Statements of the results Denote the positions of the particles at time n of the N-BRW by

$$\mathcal{X}(n) = \{\mathcal{X}_1(n) \leqslant \cdots \leqslant \mathcal{X}_N(n)\}.$$

In particular, $\mathcal{X}_1(n)$ denotes the position of the minimum and $\mathcal{X}_N(n)$ the position of the maximum at time n. Our first theorem gives convergence in law of the maximum and the minimum of the N-BRW to the α-stairs process, after rescaling of space and time.

Theorem 3.1. *Let $(\mathcal{R}(t))_{t\geqslant 0}$ be an α-stairs process. We have the following convergences in law w.r.t. finite-dimensional distributions, as $N \to \infty$:*

$$(c_N^{-1}\mathcal{X}_N(\lfloor t\log_2 N\rfloor, c_N^{-1}\mathcal{X}_1(\lfloor t\log_2 N\rfloor))_{t\geqslant 0} \Longrightarrow (\mathcal{R}(t), \mathcal{R}(t-1))_{t\geqslant 0}.$$

In fact, we show that the convergence holds as well w.r.t. the so-called SM_1-topology, see [Whi02, Chapter 12].

Theorem 3.1, although very satisfying due to its conciseness, leaves open the question about the long-time behavior of N-BRW. It turns out that it is the same as the α-stairs process in general only if $\alpha > 1$ and that one has to distinguish four cases:

Theorem 3.2. *We distinguish between the following cases:*

— $\alpha > 1$*: The limit* $v_N = \lim_{n\to\infty}\mathcal{X}_N(n)/n = \lim_{n\to\infty}\mathcal{X}_1(n)/n$ *exists almost surely and in* L^1 *and satisfies*

$$v_N \sim \rho_\alpha \frac{c_N}{\log_2 N}.$$

Here, $\rho_\alpha = \lim_{t\to\infty}\mathcal{R}(t)/t$ *for the α-stairs process* $(\mathcal{R}(t))_{t\geqslant 0}$*, where the limit holds almost surely and in* L^1 *and is a positive and finite constant.*

— $\alpha = 1$, $\mathbf{E}[X] < \infty$*: The limit* $v_N = \lim_{n\to\infty}\mathcal{X}_N(n)/n = \lim_{n\to\infty}\mathcal{X}_1(n)/n$ *exists almost surely and in* L^1 *and satisfies*

$$v_N \sim \frac{c_N}{\log_2 N}\int_1^\infty h(c_N)/h(c_N x)\,\mathrm{d}x.$$

— $\alpha = 1$, $\mathbf{E}[X] = \infty$*: Set* $b_n^N = \int_1^{h^{-1}(n)} h(c_N)/h(c_N x)\,\mathrm{d}x$*. Then, for* $i \in \{1, N\}$,

$$\frac{\log_2 N}{c_N}\frac{\mathcal{X}_i(n)}{nb_n^N} \to 1 \quad \textit{in law, as } n\to\infty, \textit{ then } N\to\infty.$$

— $0 < \alpha < 1$*: There is an α-stable random variable* W_α*, such that for* $i \in \{1, N\}$,

$$(2N)^{-1/\alpha}\frac{\mathcal{X}_i(n)}{h^{-1}(n)} \to W_\alpha \quad \textit{in law, as } n\to\infty, \textit{ then } N\to\infty.$$

The proof of Theorem 3.2 is fairly intricate. It relies on a clever coupling with a variant of the stairs process which evolves in discrete time and which features a Poisson process with an intensity measure more closely related to the law of X. In fact, two different couplings are used in order to get upper and lower bounds for the positions of the right- and left-most particles. It turns out that one can bound the difference between the N-BRW and its approximation in L^p for $p < 2\alpha$, which is enough to deduce Theorem 3.2 from an analog result for the generalized stairs process.

Perspectives A natural question and possibility for future research is to investigate what happens in between the two scenarios of light-tailed displacement distributions (satisfying an exponential moment assumption) and the polynomial tails considered in the present article. This seems to be a delicate and interesting question. Even the heuristic picture is not clear. For example, it is not clear to us whether for all subexponential displacement distributions the advance of the cloud of particles arises from jumps of single particles as is the case for the regularly varying tails considered here. As mentioned in the beginning of the section, one would expect this behavior from the large deviation behavior of general random walks with subexponential displacement distributions, but the interaction between the particles might cause other effects.

This question is related to the genealogy of the model. In the regularly varying case considered here, we expect the genealogy to be described by the star-shaped coalescent (i.e. the coalescent where all particles coagulate to a single block). Does this coalescent describe the genealogy of N-BRW for every subexponential displacement distribution?

3.2 Tree-indexed random walks in the big jump domain [M13]

Let T be a tree with root ρ. T can be finite or infinite, deterministic or random. Denote by $\mathcal{V}$ the set of vertices and set $\mathcal{V}^* = \mathcal{V}\backslash\{\rho\}$. A *random walk indexed by the tree* T is defined as follows: let $(X_v)_{v\in\mathcal{V}^*}$ be iid random variables, independent of T if T is random. Set

$$\forall v \in \mathcal{V} : S_v = \sum_{\rho\neq u\leqslant v} X_u,$$

where $u \leqslant v$ means that u lies on the path from the root to v (including v itself). The collection $(S_v)_{v\in\mathcal{V}}$ is then called the *random walk indexed by the tree* T.

We are interested here in tree-indexed random walks whose jumps have regularly vaying tails with index $-\alpha$, for some $\alpha > 0$. For simplicity, we will also assume that the law of the jumps is symmetric; this can be much weakened, see **[M13]** for details. We then ask under which conditions the maximum of the walk on the tree is asymptotically equal to the *maximal jump*. If this is the case, we say that the tree-indexed random walk is in the *big jump domain*. As opposed to the previous sections, where we considered branching random walks with supercritical branching, i.e. random walks indexed by exponentially growing trees, our emphasis here is on *polynomially growing trees*, with the particular example of branching random walk with *critical* branching.

Let us start with the simplest example: let $S_n = X_1 + \cdots + X_n$ be a usual random walk. It is well-known and easy to show that $M_n^X = \max(X_1, \ldots, X_n) = n^{1/\alpha+o(1)}$ as $n \to \infty$. On the other hand, standard random walk theory (see, e.g. [Fel71]) gives that $S_n = n^{1/(2\wedge\alpha)+o(1)}$ as $n \to \infty$. In other words, S_n and M_n^X are (roughly) of the same order if and only if $\alpha \leqslant 2$.

Now consider branching random walk with critical offspring distribution of finite variance. In other words, this is a random walk indexed by a Galton–Watson tree with offspring distribution of mean 1 and finite variance. Condition the tree on having n non-root vertices and let M_n^X and M_n^S be the maximum over all X_v and S_v, respectively. Of course, $M_n^X = n^{1/\alpha+o(1)}$ as above. As for M_n^S, Kesten [Kes95] proved that if $\alpha > 4$, then $M_n^S n^{1/4+o(1)}$. Subsequently, Janson and Marckert [JM05] showed that in general, $M_n^S = n^{1/(4\wedge\alpha)+o(1)}$, so that M_n^S and M_n^X are (roughly) of the same order if and only if $\alpha \leqslant 4$. Lalley and Shao [LS16] consider a continuous analog, namely symmetric stable branching Lévy processes of index $\alpha \in (0, 2)$ with critical binary offspring distribution.

They show through analytic methods that for all $\alpha \in (0, 2)$, the tail of the maximal displacement and the tail of the maximal jump are asymptotically equivalent. On motivation of my work was to find a probabilitic proof of their result.

It is well-known that a critical, finite variance Galton–Watson tree conditioned on having n vertices converges after rescaling to Aldous' *continuum random tree* [Ald93], a random metric space of Hausdorff dimension 2. Looking at the above results, one question immediately comes to mind: for a symmetric random walk with regularly varying tails indexed by a large "D-dimensional" random tree, a notion to be made precise, is it true that M^S and M^X are of the same order if and only if $\alpha \leqslant 2D$, with $M^S = \max_v S_v$ and $M^X = \max_v X_v$? In **[M13]**, I provide a partial response to this question.

We first have to define precisely what we mean by "dimension". It turns out that we only need a very crude notion: we say that a tree is of dimension (at least) $D > 1$ if its height is at most of order $V^{1/D}$, where V is the number of its vertices. We make this definition precise in two ways, yielding two different settings:

(S1) through a growing sequence of (possibly random) trees of height H_n and number of vertices V_n, satisfying for each $\varepsilon > 0$, $H_n \leqslant V_n^{1/D+\varepsilon}$ with high probability, and

(S2) through a single random tree of height H and number of vertices V satisfying for each $\varepsilon > 0$, $\mathbf{P}(H \leqslant V^{1/D+\varepsilon}, V \geqslant n) \leqslant C_\varepsilon n^{-\kappa}$ for some large enough κ.

In setting 1, we obtain the following result:

Theorem 3.3. *Assume (S1) and $\alpha < 2D$. Denote by M_n^S and M_n^X being respectively the maximal displacement and the size of the maximal jump in the n-th process. Then*

$$\frac{M_n^S}{M_n^X} \to 1 \text{ in probability as } n \to \infty.$$

Under the assumption (S2) and a further assumption on the tail of V, we obtain a similar theorem, stating now that the tails of M^S and M^X are asymptotically equivalent. See **[M13]** for details. We then show that the results can be applied to branching random walks with critical offspring distribution in the domain of attraction of a β-stable law, $\beta \in (1, 2]$. Roughly speaking, we conclude that the maximal displacement and the maximal jump of such branching random walks behave the same as long as $\alpha < 2D_\beta$, with $D_\beta = \beta/(\beta - 1)$. This is indeed the Hausdorff (or packing) dimension of the β-stable tree [DL04], which is the scaling limit of the Galton–Watson tree underlying this branching random walk [Duq03].

Proof idea. It is natural to try to apply classical large deviation results for random walks with regularly varying tails [1]. As recalled in the previous section, a large deviation event for a random walk with regularly vaying tails consists with high probability of a single big jump happening at a (uniformly) random time, the other jumps being negligible. In order to apply this, one could use the following strategy (recall that V and H denote the number of vertices and the height of the tree, respectively, and assume for simplicity $H = V^{1/D}$),

1. Show that the size M^X of the largest jump is of order $V^{1/\alpha+o(1)}$ (easy calculation).

1. Classical references are [Lin61a, Lin61b, Hey67, Nag69a, Nag69b], for more see [EKM97, Section 8.6], [Dur79, CH98, MN98, DDS08]

2. Let us suppose we want to show that $M^S = \max_v S_v \leqslant (1+\varepsilon)M^X$ with high probability, for every $\varepsilon > 0$. By a union bound, for every $x \geqslant 0$,

$$\mathbf{P}(\max_v S_v \geqslant x,\, M^X \leqslant (1-\varepsilon)x) \leqslant \sum_{v\in\mathcal{V}} \mathbf{P}(S_v \geqslant x,\, \max_{u\leqslant v} X_u \leqslant (1-\varepsilon)x).$$

One could then bound the term under the sum by classical large deviation estimates for random walks, for x of order $V^{1/\alpha+o(1)}$, using for example the results from [Dur79]. Unfortunately, in doing so, one only obtains a vanishing probability for $\varepsilon \geqslant \varepsilon_0$, for some $\varepsilon_0 > 0$, unless H grows subpolynomially in V (formally, $D = \infty$), which is the case for example for branching random walks with *supercritical* branching, where this strategy works [Dur83].

It turns out that there is an easy way of circumventing this problem, which is the approach in **[M13]**. One has to seperate the small from the big jumps more globally, instead of separately on each branch (this approach was in fact implicit in [JM05]). In particular, we show the following two facts:

1. No two large jumps (meaning, not much smaller than the order of the maximum) occur on the same branch of the tree with high probability.
2. The contribution of the small jumps are asymptotically negligible.

The first fact follows from a simple union bound on the jumps. The second then follows by a union bound on the vertices, similar to the above bound, together with a quite general bound on suitably truncated random walks from [DDS08].

Chapter 4

Branching random walks in disguise

As explained in the introduction, branching random walks appear in many situations and have a variety of applications. In this chapter I present my work on some of its interactions with other models in probability.

4.1 Stable point processes occurring in branching Brownian motion [M4]

This work was part of my thesis **[Thèse]**. It was motivated by a question asked by Brunet and Derrida about the point processes arising in the study of the extremes of branching Brownian motion, see Theorem 1.3. These point processes are of the following form:

$$\Pi = \sum_{i=1}^{\infty} T_{\xi_i} D_i, \tag{4.1}$$

where

- $(D_i)_{i\geqslant 1}$ are independent copies of a point process D on $\mathbf{R}$ (the letter "D" stands for "decoration")
- $(\xi_i)_{i\geqslant 1}$ are the atoms of a Poisson process of intensity $e^{-x}\,\mathrm{d}x$ on $\mathbf{R}$ and independent of $(D_i)_{i\geqslant 1}$
- T_x is translation by x.

It is easy to see that for every $\alpha, \beta \in \mathbf{R}$ with $e^{\alpha} + e^{\beta} = 1$, Π is equal in law to $T_\alpha \Pi + T_\beta \Pi'$, where Π' is an independent copy of Π. We call this property *exp-1-stability* or *exponential 1-stability*. In the case of the point processes describing the extremes of branching Brownian motion, this property has a fundamental interpretation in terms of the superposition of two branching Brownian motions. Brunet and Derrida asked the following question: Does every exp-1-stable point process Π admit the decomposition (4.1)? The article **[M4]** gives an affirmative answer to that question and in fact to the more general question where Π is a random measure on $\mathbf{R}$. The main result can be stated as follows:

Theorem 4.1. *Let Z be an exp-1-stable point process. Then there exists a point process D on $\mathbf{R}$, such that Z admits the representation* (4.1)*. Moreover, if $P(Z = 0) > 0$, then D can be chosen such that for some $z \in \mathbf{R}$, $\max D = z$ a.s., and as such, D and z are unique.*

This gives a definite answer to the question. This question actually was not only academic: Theorem 4.1 was subsequently used by Madaule [Mad11] in order to prove Theorem 1.3 for the branching random walk, avoiding the explicit description of the process D as in the works on branching Brownian motion [ABBS13, ABK13]. Let us briefly explain how the exp-1-stability property comes into play in branching random walk. It is based on the following observation from [BD11] that Madaule rigorously proves:

— There is a point process Π, independent of the branching random walk, such that for every initial condition of the branching random walk, the limiting process of the extremal particles is equal in law to $T_{\ln Z}\Pi$, where Z is the limit of the derivative martingale of the branching random walk.

Now observe that when starting with two particles at time 0, the branching random walk can be written as a superposition of two branching random walks starting with one particle. This shows that

$$T_{\ln(Z+Z')}\Pi \overset{\text{law}}{=} T_{\ln Z}\Pi + T_{\ln Z'}\Pi',$$

where Z, Z', Π and Π' are all independent. This idea (essentially) allows to prove exp-1-stability of the process Π.

Perspectives. Exponentially 1-stable point processes or random measures appear in the description of the extremes of many random processes, for example the 2D Gaussian Free Field [BL16a, BL16b]. More generally, the Poisson point process with intensity measure $e^{-x}\,dx$ is a well-known limiting process in extreme value theory (see e.g. [Res87, Corollary 4.19] or [KSdH09]). The property of exponential stability played an important role in the characterization of the extremal process in branching random walk [Mad17], can it be exploited in other models?

Note that other characterizations of the Poisson point process with intensity measure $e^{-x}\,dx$ and variants have been derived and used in the literature, see the references in [SZ17, p.779]. Here we mention only one of them, essentially due to Biskup and Louidor [BL16a], but we present the simpler version contained in [SZ17] (which seems to be a special case of [RA05]). It has been successfully applied in the first two papers to certain Gaussian models, namely, the 2D Gaussian Free Field and the spherical p-spin glass, respectively. The characterization is based on a theorem by Liggett [Lig78] and ultimately, the Choquet–Deny theorem. It says the following: Let Π be a point process such that its law is invariant under the transformation which shifts every atom indepently by a Gaussian r.v. with mean $t/2$ and variance t, for any $t \geqslant 0$ (this transformation is called "Dysonization" in [BL16a]). Then Π is a Poisson point process with random intensity measure $(Ze^{-x} + Z')\,dx$ for some pair (Z, Z') of random variables (this is also called a "Cox process").

A generalizations of this "Dysonization" transformation has been studied by Kabluchko [Kab12] who characterizes a certain class of point processes which are invariant under replacements of the atoms by independent *clusters* as randomly shifted exp-1-stable processes (relying, in fact, on Theorem 4.1). The replacement by clusters is motivated by the branching random walk: the extremal particles at time $n+1$ are in fact obtained by replacing the extremal particles at time n independently by clusters with the law of the first generation of the branching random walk. However, the results in [Kab12] require a certain finite intensity assumption which usually does not hold for the point process of the extremal particles in various systems such as the branching random walk. It is an interesting question whether the results can be generalized to these cases. This would give another proof of the characterization of the point process of extremes in branching random walk and would probably also allow to characterize the point process D (the "decoration") more explicitly in terms of the law of the original branching random walk.

Finally, we note that Subag and Zeitouni [SZ15] give another characterization of randomly shifted exp-1-stable processes in terms of their Laplace functional. In this characterization, they rely on several "technical" assumptions; it is an interesting question whether these can be further weakened.

4.2 Performance of the Metropolis algorithm on a disordered tree [M6]

Random walks in random environments on trees, sometimes also called biased random walks on trees is a huge subject, see for example [Shi15, Chapter 7]. We study in **[M6]** a particular model introduced by Aldous [Ald98], and slightly different from the usual model. To fix notation, attach to each edge e of a d-regular rooted tree a random variable $X(e)$, such that the variables are independent copies of some random variable X. For a vertex v in the tree, denote by $S(v)$ the sum of the variables $X(e)$ over all edges e on the path from the root to v. Then the collection $(S(v))_v$ is a random walk indexed by the d-regular tree, or, in other words, a $(d-1)$-ary branching random walk with the exception that the root has d children instead of $d-1$. We assume that the speed of the maximum of the branching random walk is positive, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \max_{|v|=n} S(v) > 0.$$

Aldous asked the following question: *What are efficient algorithms for finding a vertex v in the tree, so that $S(v)$ is larger than some prescribed value?* He proposed two algorithms:

— The *greedy algorithm* [Ald92]: Suppose we have examined vertices $\rho = v_0, v_1, ..., v_n$. Among the subset of these vertices which still have unexamined neighbors, choose a vertex v for which $S(v)$ is maximal and then choose some unexamined neighbor of v to be the next vertex v_{n+1} to be examined.

— The *Metropolis algorithm* [Ald98]: Fix a parameter $\beta > 0$. Consider a Markov chain $(V_n)_{n\geqslant 0}$ on the tree whose invariant measure is $\mu_\beta(v) = e^{\beta S(v)}$; such a chain can be constructed by the Metropolis–Hastings recipe. Sample the Markov chain until $S(V_n)$ reaches the prescribed value.

Note that the greedy algorithm can be chosen to be deterministic given the realisation of the branching random walk (choose any deterministic prescription to break ties), whereas the Metropolis algorithm is inherently probabilistic and falls into the class of MCMC (Markov chain Monte Carlo) algorithms.

Both algorithms can be run forever, in that case we call the quantity $\lim_{n\to\infty} S(v_n)/n$ or $\lim_{n\to\infty} S(V_n)/n$ (if it exists) the *speed* of the algorithm.

The analysis of the greedy algorithm proved to be mathematically very pleasant: it turns out that its speed can be expressed in terms of a certain branching random walk with absorption. This allowed Aldous to give estimates for its speed in the regime where the speed of the maximum of the branching random walk approaches zero and Pemantle [Pem09] showed that the algorithm is in a certain sense optimal. Note that this latter paper inspired Bérard and Gouéré [BG11] and Gantert, Hu and Shi [GHS11] to study branching random walk killed at a linear barrier close to criticality, thus Aldous' greedy algorithm had a big impact on the research on branching random walks.

In **[M6]**, we continued the study of the other algorithm that Aldous proposed, namely, the Metropolis algorithm. For this algorithm, Aldous showed (in the case where X is concentrated on $\{-1, 1\}$) that there exists $\beta_0 > 0$ such that the speed of the algorithm is zero and asked whether there exists $\beta > 0$ such that the speed is positive. He conjectured yes and also conjectured an expression for the derivative of the speed w.r.t. β at β_0. Proving this is the content of **[M6]**. We consider general bounded X satisfying the following symmetry assumption:

(XR) There exists $\beta_0 > 0$ such that $E[e^{(\beta_0/2+\beta)X}] = E[e^{(\beta_0/2-\beta)X}]$ for all $\beta \in \mathbf{R}$.

This assumption is in particular satisfied if X takes on two values only.

Our main theorem can be stated as follows:

Theorem 4.2. *Set $S_n = S(V_n)$ and write $\mathbb{P}_\beta$ for the law of the Metropolis algorithm with parameter β.*

1. *The limit $\sigma^2 = \lim_{n\to\infty} S_n^2/n$ exists $\mathbb{P}_{\beta_0}$-almost surely and is a strictly positive and finite constant.*
2. *For each $\beta \in \mathbf{R}$, the deterministic limit $v_\beta = \lim_{n\to\infty} S_n/n$ exists $\mathbb{P}_\beta$-almost surely and satisfies*

$$\lim_{\beta\to\beta_0} \frac{v_\beta}{\beta - \beta_0} = \frac{\sigma^2}{2}. \tag{4.2}$$

Theorem 4.2 hence in particular shows that there exists $\beta > \beta_0$, such that $v_\beta > 0$, answering positively Aldous' question. Of course, it says nothing about the *maximal* speed $\max_\beta v_\beta$ that one can attain. Aldous also has a conjecture about that on which we will comment below.

Returning to Theorem 4.2 and in particular to (4.2), we remark that results of this type are known as *Einstein relations* or *fluctuation-dissipation theorems* both in the mathematics and physics literature. This relation says that the perturbation of a stationary and reversible dynamics entails a shift proportionally to the strength of the perturbation and to the variance of this dynamics under stationarity. It is named in honor of Albert Einstein who discovered such a relation in his seminal article on Brownian motion [Ein05] and used it to devise an experiment for estimating the Avogadro number (and, in fine, to validate the hypothesis that heat is due to molecular motion).

Mathematically, the Einstein relation (ER) links the asymptotic variance of additive functionals of (reversible) Markov chains in equilibrium to the chains' response to small perturbations. The work by Lebowitz and Rost [LR94] was very influential and forms the basis of our proof of Theorem 4.2, as well as of other recent advances in the analysis of the Einstein relation for disordered systems (Aldous also refers speculatively to [LR94] as relevant to the analysis near $\beta = 0$). Indeed, Lebowitz and Rost [LR94] provide a general recipe (based on the Kipnis-Varadhan theory, see [KLO12] for a comprehensive account) for the validity of a weak form of the ER, which holds on a time-scale which is inversely proportional to the square of the perturbation.

Significant progress to obtain a *full ER* (which holds as time goes to infinity) was achieved in [GMP12], where the Lebowitz–Rost approach was combined with good *uniform in the environment* estimates on certain regeneration times in the transient regime. Such uniform estimates are typically not available for random walks on (random) trees, and a completely different approach, based on explicit recursions, was taken in [BHOZ11], where (biased) random walks on Galton–Watson trees were analyzed. Yet, our situation is more closely related to [GMP12].

For the proof of Theorem 4.2, we rely on the crucial observation (initially due to Aldous, although he only had a quite complicated argument) that the *environment seen from the* *particle* is a reversible process under $\mathbb{P}_{\beta_0}$, using crucially assumption (XR) on X. This fact allows to apply the Lebowitz–Rost theory to yield the above-mentioned weak form of (4.2); the factor 1/2 there comes from an neat algebraic trick exploiting the reversibility. In order to obtain the full ER, we work with *level* regeneration times, in the spirit of [PZ08]. These involve the random walk location $\{V_n\}$, not its value in the branching random walk $\{S_n\}$. We can then exploit the fact that the random walk location V_n has a ballistic speed, even at $\beta = \beta_0$: it is the process of values S_n which is recurrent under $\mathbb{P}_{\beta_0}$, not V_n. This is shown using an argument by Aidékon [Aïd08]. Together with a structure lemma of Grimmett and Kesten, this allows to obtain uniform (in β) annealed stretched-exponential bounds for the regeneration times, which in turn allow to transform the weak ER to a full ER.

Perspectives. Suppose $X(e)$ is supported on $\{-1, 1\}$ with $P(X(e) = -1) = p$. Then there exists a critical parameter p_c, such that the speed of the maximal particle in the branching random walk is zero. Based on numerical experiments, Aldous [Ald98] formulated a conjecture for the asymptotics of $\max_\beta v_\beta$, i.e. the maximal speed for the Metropolis algorithm in the regime where $p \downarrow p_c$. I have convincing arguments for this conjecture which rely on recent results on BRW or BBM with selection at near-critical drift (section 2.2). However, a rigorous proof seems technically challenging. A helpful tool one should develop in order to attack this conjecture (and which would be of independent interest), is to determine the invariant measure of the environment seen from the particle for any β. This law has been indentified in the related setting of biased random walks on Galton–Watson trees by [Aïd14] but extending it to other random walks in random environments on trees has not yet been successful. It is possible that the case of the Metropolis algorithm is, at least for the distribution mentioned above, solvable as well.

In another direction of research, it is interesting to study other disordered tree models for their algorithmic complexity, i.e. for the complexity to find vertices of large value. With Louigi Addario-Berry, we are currently investigating this issue for the *continuous random energy model (CREM)*, see Sections 1.2.1 and 2.6. We show that there is in general a threshold level x^*, such that it typically needs exponential time to find vertices at height n in the tree of value $S_n > x^*n$. We think this question is of importance to the study of general disordered models, such as spin glasses.

4.3 Point-to-point distance in first passage percolation on (Tree) $\times$ **Z** [M7]

First passage percolation (FPP) on a graph is the following model: Attach to each edge of a graph a copy of a random variable X; the value of this random variable is interpreted as the *length* of this edge. The length of a *path* is defined to be the sum over the lengths of the edges along that path. The most basic question is then: what it the length of the *shortest* path between two vertices (or, in general, between two sets of vertices)?

First passage percolation is a model of random perturbation of a given geometry. It has been studied intensively in Euclidean space and lattices or subsets thereof, see e.g. [How04, GK12, ADH15] for reviews on old and new results. On the two-dimensional lattice, there are fascinating conjectured relations with the Kardar–Parisi–Zhang equation, see for instance [Cha13]. The simplest incarnation of FPP is arguably on trees, where it is nothing but the branching random walk, which of course has been studied in detail. Other setups considered include the complete graph [Jan99, BvdH12], the Erdös-Rényi graph [BVH11], random planar maps [CG15] and a class of graphs admitting a certain recursive structure [BZ12] (see more on that below).

If the graph is exponentially growing and fairly homogeneous, say a Cayley graph of a group, then one would expect the FPP distance to be very much concentrated, since it is morally a very well-behaved function of an exponential number of independent variables. However, results in this direction are sparse. In order to provide a prototypical example where one can get fairly good estimates, we study in **[M7]** first passage percolation on $\mathbb{T}_d \times \mathbf{Z}$, where $\mathbb{T}_d$ is a d-regular tree, $d \geqslant 3$. We consider the FPP distance along the $\mathbf{Z}$-direction, i.e. the distance $D(n)$ between two points $(\rho, 0)$ and (ρ, n), where ρ is the root of the tree and $n \in \mathbf{N}$. We show the following theorem:

Theorem 4.3. *Under some mild assumptions on the law of X,*

$$D(n) - \mathbf{E}[D(n)] = O(\ln n) \qquad \textit{in probability.}$$

To our knowledge, this is the first example where the fluctuation of the point-to-point distance in FPP on the Cayley graph of a finitely generated group is shown to be small.

The proof of Theorem 4.3 is based on the following innocent-looking observation by Dekking and Host [DH91]: Let Z be a random variable and suppose there exist Z', Z'' and Y such that Z' and Z'' are independent and of the same law as Z and

$$Z \leqslant \min(Z', Z'') + Y.$$

Then

$$\mathbf{E}[|Z - \mathbf{E}[Z]|] \leqslant 2\mathbf{E}[Y].$$

The proof of this inequality is simple: write $\min(a, b) = \frac{a+b}{2} - \frac{|a-b|}{2}$ and take expectations to get

$$\mathbf{E}[Z' - Z''] \leqslant 2\mathbf{E}[Y].$$

Then use the fact that $\mathbf{E}[|Z - \mathbf{E}[Z]|] \leqslant \mathbf{E}[|Z - Z'|]$ for random variable Z, by Jensen's inequality. Dekking and Host [DH91] used this argument to show tightness of the recentered maximum of branching random walk, and Benjamini and Zeitouni [BZ12] generalized this to the point-to-sphere FPP distance on rooted graphs G containing two vertex-disjoint rooted subgraphs G_1 and G_2 which are isomorphic to G.

Our proof of Theorem 4.3 proceeds by two steps: First, we construct an auxiliary graph. This graph is the product $\mathbb{T}' \times \mathbf{Z}$, where $\mathbb{T}'$ is the tree obtained from $\mathbb{T}_d$ by choosing an arbitrary vertex at distance $K \ln n$ from the root, K large, and removing it together with its subtree. This auxiliary graph has the property mentioned above and we can adapt the Dekking-Host argument to the point-to-point distance on this graph. Second, we show that the FPP distance on $\mathbb{T}_d \times \mathbf{Z}$ can be efficiently compared to the FPP distance on $\mathbb{T}' \times \mathbf{Z}$. This step uses crucially the symmetry of $\mathbb{T}_d$ in order to bound the probability that the two distances differ.

Perspectives. As said above, our article is to our knowledge the first example where the fluctuation of the point-to-point distance in FPP on the Cayley graph of a finitely generated group is shown to be small. First passage percolation on groups of growth larger than polynomial growth is a subject in its infancy and much can be done in this area. The method used in this article may be of help for other graphs, but it is quite rigid and requires a certain type of structure of the graph. It is of interest to check whether more classical techniques such as those based on hypercontractivity as in [BKS03] can give good results. These techniques allow to give an upper bound on the variance of a function of N independent random variables by a bound which usually has an improvement over "trivial" bounds by a factor of $1/\log N$. In case of FPP on groups with exponential or intermediate growth, the FPP distance is a function of a very large (more than polynomially large) number of independent random variables, so that hypercontractivity could yield very good results.

4.4 The perimeter cascade in $O(n)$ loop-decorated random planar maps [M15]

A planar map is a planar graph together with an embedding into the two-dimensional sphere modulo homeomorphisms of the sphere. A random planar map with n vertices is a planar map chosen uniformly among all planar maps of n vertices. It is a combinatorial object which has an important place in the combinatorics literature since the works by Tutte in the 1970s. It also plays an important role in physics as it appears not only as a Feynman diagram in certain quantum field theories, but also as a canonical model for a random discrete surface. The last years have seen the appearence of spectacular results on random planar maps in the probability literature, notably the convergence as metric spaces of certain classes of random planar maps with n vertices, as $n \to \infty$, to a certain limiting metric space called the *Brownian map*, by Le Gall and Miermont. In a different vein, it has been shown recently by Miller and Sheffield that the Brownian map can be contructed from a certain measure on the sphere, called the *Liouville measure*, thus building a bridge between *Liouville theory* (a certain conformal field theory) and planar maps.

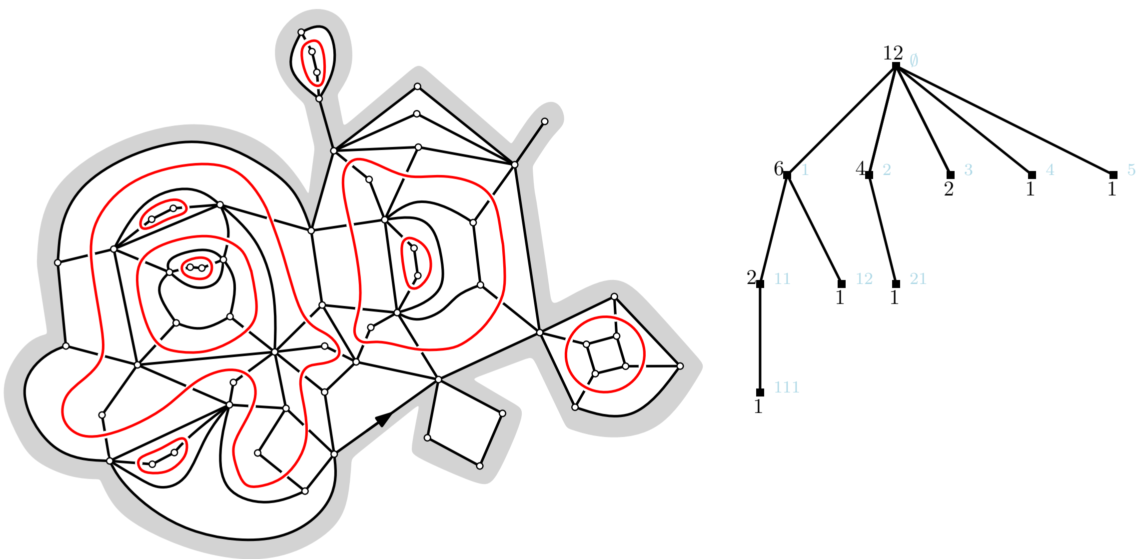


Figure 4.1 – A quadrangulation of perimeter 24 decorated with loops and the tree of half-perimeters of the loops.

In our article **[M15]** we study a certain class of random planar maps decorated by an

$O(n)$ loop model, $n \in (0,2)$. More precisely, we consider quadrangulations with a boundary of length $2p$ and containing non-crossing loops on its planar dual, see Figure 4.1. We endow such a configuration with the following Boltzmann–Gibbs weight:

$$g^{\#\text{faces not crossed by a loop}} \times h^{\#\text{faces crossed by a loop}} \times n^{\#\text{loops}}.$$

For this model, Borot–Bouttier–Guitter [BBG12] have determined the critical manifold in the space of parameters, i.e. for which the object has fractal properties, and also the sub-manifold where the random quadrangulation behaves very differently from a uniform random quadrangulation, this is called the "non-generic critical case". We are interested in this case and study the "genealogical" tree of the half-perimeters of the loops, where a loop is considered the descendant of another loop if it is contained in its interior. We index these loops by the Ulam tree $U = \bigcup_{n\geqslant 0}(\mathbf{N}^*)^n$ and denote by $\chi_u^{(p)}$ the half-perimeter of the loop indexed by u. We prove the following theorem:

Theorem 4.4 (Convergence of the perimeter cascade)**.** *We have the following convergence in distribution*

$$(\chi_u^{(p)}/p)_{u\in U} \xrightarrow[p\to\infty]{(d)} (Z_u)_{u\in U},$$

in $\ell^\infty(U)$, where $Z = (Z_u)_{u\in U}$ is a certain multiplicative cascade, i.e. $(\ln Z_u)_{u\in U}$ is a branching random walk.

We also explicitly determine the law of the multiplicative cascade Z. For this, we only have to describe its offspring distribution. We introduce another parameter $\alpha \in (1,2)$, which is related to n by

$$\alpha = \frac{3}{2} \pm \frac{1}{\pi}\arccos(n/2),$$

with $+$ or $-$ according to whether we are in of two possible subcases called respectively "dilute" and "dense" [BBG12]. Now let $(\zeta_t)_{t\geqslant 0}$ be a spectrally positive and centered α-stable Lévy process and denote by τ the hitting time of -1 of this process. We write $(\boldsymbol{\Delta\zeta})^{\downarrow}_{\tau}$ for the infinite vector consisting of the sizes of the jumps of ζ before time τ, ranked in decreasing order. We define a probability distribution ν_α on $(\mathbf{R}_+)^{\mathbf{N}^*}$ by

$$\int d\nu_\alpha(\mathbf{x})F(\mathbf{x}) = \frac{\mathbf{E}\left[\frac{1}{\tau}F((\boldsymbol{\Delta\zeta})^{\downarrow}_{\tau})\right]}{\mathbf{E}\left[\frac{1}{\tau}\right]}.$$

Then ν_α is the offspring distribution of the multiplicative cascade Z from Theorem 4.4, i.e. $(Z_i)_{i\in\mathbf{N}^*}$ follows the law ν_α.

The proof of Theorem 4.4 consists of two parts. First, we determine the limit law of the first generation by exploiting certain bijections between planar maps and certain two-type trees (called *mobiles*) and between mobiles and one-type trees (specifically, the BDG and the Janson–Stefansson bijections, respectively). Second, to get from this the limit law of the whole cascade, one has to make sure that microscopic loops do not generate macroscopic loops in later generations. This is done by a combination of L^p-estimates, a Scheffé-type lemma and the use of an explicit supermartingale for the discrete model from [Bud].

In a second part of the article, we calculate several quantities associated to this multiplicative cascade. In particular, we show that the function φ from (1.1) associated to this multiplicative cascade equals

$$\varphi(\theta) = \ln \mathbf{E}[\sum_{i\in\mathbf{N}^*} Z_i^\theta] = \ln \frac{\sin(\pi(2-\alpha))}{\sin(\pi(\theta-\alpha))}, \quad \theta \in (\alpha, \alpha+1).$$

For the calculation of φ, we find in fact a general lemma on expectations of certain quantities related to random walks, which might be of independent interest.

The *Malthusian martingale* associated to the multiplicative cascade is the additive martingale corresponding to $\theta = \theta_0$, where θ_0 is the smaller root of φ. By solving the fixed point equation (1.4) of the limit of this martingale, we are able to determine the law of the limit which is equal (in the dilute phase) or related to (in the dense phase) to the inverse Gamma distribution of parameter $\alpha - 1/2$. The proof of this fact relies on a certain identity for its Laplace transform, which follows from an identity on Whittaker functions and, to our knowledge, has not been observed before. We further conjecture that this martingale limit has the same law as the limit law of the renormalized volume of the planar map, which may be related to results on continuous models [HRV15].

The other additive martingales are also of interest. In particular, they allow to guess the large deviation rate function for the number of loops surrounding a fixed vertex. Under a conjectural relation with Liouville quantum gravity decorated by a conformal loop ensemble (for a certain value of its parameter depending on α), one can recover heuristically a result about the number of loops in this conformal loop ensemble surrounding a small Euclidean ball.

Perspectives. An obvious continuation of our work is to extend it to $n = 2$. It is well-known that the model degenerates in this case and needs a different renormalization. This is apparent in our work by the fact that the multiplicative cascade is critical: the two roots of φ coagulate and the Malthusian martingale converges to 0 almost surely. Studying the model therefore should benefit from the machinery developed for the extremal particles of the branching random walk (derivative martingale and Seneta–Heyde norming).

An interesting aspect of our work which may go beyond planar maps is the identity we discovered on the law of the volume of the map (equation (1.4) specialized to the Malthusian martingale). I am not aware of other examples of non-trivial branching random walk martingales for which the limit law can be exactly calculated. As an analogy, calculating the law of the volume of Gaussian Multiplicative Chaos measures is a hard problem as well and still largely misunderstood [Ost18]. It would be interesting to understand the mechanism behind this fact, i.e. vaguely stated, *for which class of branching random walk martingales can the law of the limit be explicitly calculated?* Also, in a different vein, *does our this identity, or variations of it, appear in other contexts, for example where a (fractal) object is decomposed into a countable number of rescaled copies of itself? Can it maybe hint at the possibility of such decompositions?*

4.5 Liouville heat kernel: regularity and bounds [M12]

In Section 1.2.4, I introduced Gaussian multiplicative chaos, a random measure on $[0,1]^d$, say, which has the property of *multifractality*, i.e. a measure whose regularity (measured in terms of the Hölder exponent) varies locally. We will in this section consider the planar case $d = 2$. To fix notation, denote the measure by $M_\gamma(dx)$, where $\gamma \in (0,2)$ is a parameter representing the "intensity" of the singularities; in particular, M_γ is almost surely concentrated on a set of Hausdorff dimension D_γ satisfying $D_\gamma \to 0$ as $\gamma \to 2$. Recall that M_γ is formally the exponential of a logarithmically correlated centered Gaussian field $(X_x)_{x\in[0,1]^2}$:

$$\text{“}M_\gamma(dx) = e^{\gamma X_x}\, dx\text{”},$$

which must be properly renormalized in order for it to be defined.

There has been considerable interest in defining a Riemannian metric on $[0,1]^2$ formally given by

$$\text{“}e^{\gamma X_x} dx^2\text{”},$$

i.e. a metric conformally equivalent to the Euclidean metric with conformal factor “$e^{\gamma X_x}$”. Defining such a metric, i.e., a distance function $d_\gamma(\cdot,\cdots)$, is a very hard problem, about which we say more later. However, even without a proper definition of the metric, Garban, Rhodes and Vargas [GRV16] and independently, Berestycki [Ber15] were able to introduce what would be the canonical diffusion in this metric. Indeed, in two dimensions, a conformal change in the metric simply translates into a time-change by the inverse of the conformal factor, so the infinitesimal generator of this diffusion should formally be

$$\text{“}\Delta_X = e^{-\gamma X_x}\tfrac{1}{2}\Delta\text{”},$$

where Δ is the usual (negative) Laplacian on the plane. In the above-mentioned papers, the authors were able to properly define this diffusion as the time-change of Brownian motion by a positive continuous additive functional (PCAF) defined through renormalization. The Gaussian multiplicative chaos measure M_γ then arises as the so-called *Revuz measure* of this PCAF. The authors of [GRV16] furthermore showed that this diffusion, dubbed the *Liouville Brownian motion (LBM)*, has the Feller property and that the measure M_γ is an invariant measure for the LBM.

Since LBM is a time-change of usual two-dimensional Brownian motion, interesting questions are not about its trace, but rather about its heat kernel. Indeed, in monofractal geometries with a distance function $d(\cdot,\cdot)$, one usually observes that the heat kernel $p_t(x,y)$ of the diffusion satisfies the following asymptotic for small t:

$$p_t(x,y) \asymp t^{-d_H/b} \exp\left(-\frac{d(x,y)^{1/b}}{t^{1/(b-1)}}\right), \tag{4.3}$$

for some $b > 0$ and with d_H denoting the *Hausdorff dimension* of the space, i.e. the exponent describing the growth of the volume of balls in this geometry. The ratio $2d_H/b$ is known as the *spectral dimension* and describes the asymptotic of the heat kernel *on the diagonal* $x = y$. Thus, knowing the spectral dimension and *off diagonal* small-time asymptotics of the heat kernel gives access to the Hausdorff dimension of the space. In the case of the *multifractal* geometry of Liouville Brownian motion, it is of course not at all clear whether the above asymptotic should hold, nor whether the hypothetic metric space should have a well-defined Hausdorff dimension. This last fact though has been conjectured in the physics literature by Watabiki [Wat93] who conjectured that the Liouville metric is locally monofractal and heuristically derived a formula for its Hausdorff dimension. Thus, supposing that the asymptotic (4.3) of the heat kernel holds for Liouville Brownian motion, knowing its spectral dimension and off-diagonal small time asymptotics would allow to test the validity of Watabiki's formula.

In order to implement this program, Rhodes and Vargas [RV14b] proved that the spectral dimension of Liouville Brownian motion was shown to be equal to 2 for all $\gamma \in (0,2)$. In **[M12]**, we attacked the off-diagonal behavior of the Liouville heat kernel $p_t^\gamma(x,y)$.

To our knowledge, this was the first work to investigate the quantitative behavior of a heat kernel in a multifractal setting. Apart from regularity estimates and certain (fairly weak) uniform upper bounds, our main result was the following lower bound on the heat kernel:

Theorem 4.5. *For fixed $x \neq y$ and $\eta > 0$, there is a random time $T > 0$ such that for all $t \in (0, T)$,*

$$p_t^\gamma(x, y) \geqslant \exp\left(-t^{1/(1+\gamma^2/4-\eta)}\right).$$

The outcome of our results was that our upper and lower bounds did not contradict Watabiki's formula, in the sense that the bounds on the Hausdorff dimension obtained through the *Ansatz* (4.3) were compatible with the expression obtained by Watabiki. Still, again assuming the validity of (4.3), Theorem 4.5 suggests the lower bound $d_H \geqslant 2 + \gamma^2/4$ on the Hausdorff dimension which is non-trivial in the sense that it is strictly bigger than the dimension of the ambient Euclidean space (which is 2).

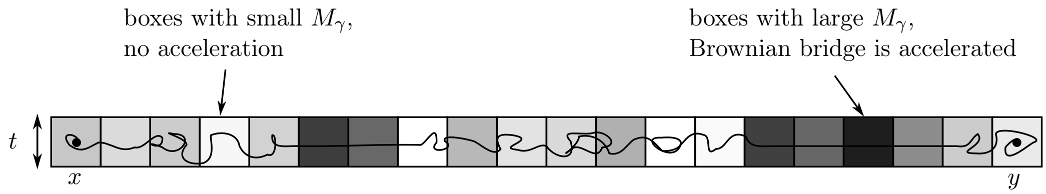


Figure 4.2 – Proof of Theorem 4.5: the "strategy" of the Brownian bridge for minimizing the exponent in (4.4).

The basic idea of the proof of Theorem 4.5 is fairly simple and exploits little of the two-dimensional geometry. The first step is to write the *resolvent* of the Liouville heat kernel in terms of the expectation of a certain functional involving M_γ and an independent Brownian bridge from x to y. This reduces the problem to finding a lower bound on the following expectation:

$$\mathbf{E}_x\left[\exp\left(-\lambda \int_0^t e^{\gamma X_{B_r}}\, dr\right)\right], \tag{4.4}$$

where $(B_r)_{r\in[0,t]}$ is a Brownian bridge from x to y of length t (small), $\lambda > 0$ is a large parameter (which will be chosen as a polynomial in t), and the integral has to be defined again through renormalization. In order to get a lower bound for this expectation, on therefore only has to find a good "strategy" for the Brownian bridge which minimizes the integral in the exponent and is not too costly. The strategy we choose is depicted in Figure 4.2. Basically, we ask the Brownian bridge to follow a straight line, but accelerate it when it crosses areas where the field X is large. Although this strategy sounds simple, writing it down requires some care.

Later developments and perspectives. Shortly after our article **[M12]** was published, Andres and Kajino [AK16] published an article whose results overlapped with ours. In particular, they proved a better upper bound on the heat kernel (but no lower bound). This upper bound is again compatible with Watabiki's formula. However, more recently, and partly inspired by **[M12]**, Ding and co-authors have published results that shed doubt on the validity of Watabiki's prediction or at the very least, show that one has to be extremely careful in interpreting Watabiki's formula. Among their many articles we mention two:

— In [DZZ17] the authors construct for every $\varepsilon > 0$ and every $\gamma \in (0, 1/2)$ a log-correlated Gaussian field whose covariance is a bounded perturbation of the Gaussian Free Field and such that the Liouville heat kernel w.r.t. the Gaussian multiplicative chaos with parameter γ defined by this field satisfies for fixed $x \neq y$,

$$\exp\left(-t^{-1/(1+\gamma^2/2)+\varepsilon}\right) \leqslant p_t^\gamma(x, y) \leqslant \exp\left(-t^{-1/(1+\gamma^2/2)-\varepsilon}\right). \tag{4.5}$$

This shows a certain non-universality of the diffusive exponent of Liouville Brownian motion and, in particular, shows that Watabiki's formula, if true, may only hold for a very special field.

— In [DG16], the authors study a related model, namely, the first passage percolation metric with respect to weights given by a discretization of the exponential of (γ times) a Gaussian Free Field. They show that the asymptotic behavior for small γ is very different from what one would reasonably guess from Watabiki's formula. Again, this shows that Watabiki's formula, if true, may have a very special meaning for a very particular model and might not generalize to seemingly similar settings.

The articles by Ding *et al.* are very intriguing and open up interesting new questions, for example:

— To which extent can universality be expected for the heat kernel? In particular, is it possible to find a reasonably large class of Gaussian fields including the Gaussian Free Field, for which bounds can be obtained which are different than the ones in (4.5)?

— Can one build a direct link between the first passage percolation metric from [DG16] and the Liouville heat kernel?

— Can a refinement of the techniques in our paper be used to yield a lower bound as in [DG16], contradicting (our interpretation of) Watabiki's formula?

Finally, we remark that the construction of a metric for the special parameter $\gamma = \sqrt{8/3}$ together with its identification as the so-called *Brownian map* has been achieved in groundbreaking work by Miller and Sheffield [MS15, MS16, MS17]. The Brownian map has Hausdorff dimension 4 [Le 07], in agreement with Watabiki's formula.

4.6 An interval fragmentation process with interaction [M10]

Throw n points one by one independently and uniformly at random in the unit interval, thus dividing the interval into $n + 1$ segments. One can generalize this process (let us call it the *uniform* process) as follows: at each step, first throw two points at random, then reject the one that falls into the smaller segment – the larger segment is thus split into two. We call this process the *max-2* process. Interchanging the role of the smaller and larger interval yields the *min-2* process. We introduced these processes (in fact, a much more general class including for example the min-k and max-k processes for all $k \geqslant 2$) in **[M10]** and tried to answer the question: do these rules fundamentally change the behavior of the process?

This question is inspired by the general paradigm known as the "power of choices," which has seen considerable attention in the computer science and random graph literature [ABKU99, ADS09, RW12]. A very simple but already non-trivial example is the following balls-and-bins model from [ABKU99]: Suppose one throws n balls into n bins, each uniformly at random. It is a simple exercise to see the maximum load (i.e. the number of balls in the fullest bin) is about $\ln n / \ln \ln n$. What happens if one modifies process such that for each ball, two bins are selected uniformly at random and the ball is placed in the bin with more balls (max-2 case) or fewer balls (min-2 case)? In the max-2 case, it is easy to show that the maximum load is still about $\ln n / \ln \ln n$. In the min-2 case however, it was shown in [ABKU99] that the maximal load is $O(\ln \ln n)$, a considerable decrease from the same model without the two choices, reflecting the power of such a simple strategy. This power-of-choices paradigm has since been applied in several other settings, for example for the growth of random graphs [ADS09, RW12, MP14] or random permutations [Tra15].

In **[M10]**, motivated by a question by Itai Benjamini, we proposed to apply the power-of-choices paradigm to the uniform process described above, which is the most basic example of an *interval fragmentation process*. These processes have received a great deal of attention in the last decades, see [Ber06] for a comprehensive account. An interval fragmentation process in this sense is usually defined in continuous time: intervals split independently at a rate depending on their length. For example, in the seminal work by Brennan and Durrett [BD87], an interval of length L is subdivided with rate L^α, $\alpha \in \mathbf{R}$ a parameter. For $\alpha = 1$, this is exactly the uniform process and for $\alpha = 0$, the process is called *homogeneous* and can be identified with a multiplicative cascade. For $\alpha = \infty$, the process can be formally identified with the so-called *Kakutani process*, a discrete-time process where at each step the *largest* interval splits randomly into two and which has received some attention around the year 1980 [Loo77, vZ78, Slu78, Pyk80].

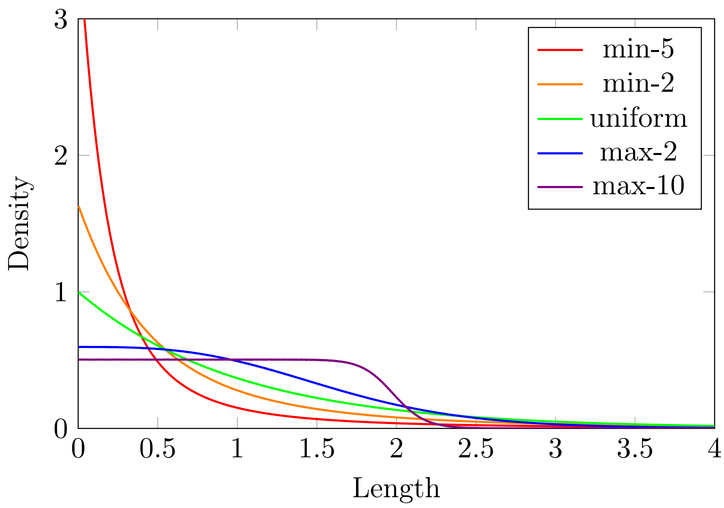


Figure 4.3 – Empirical density of interval lengths in simulation of max-10, max-2, uniform, min-2 and min-5 processes with 10^9 points. For the plot, the x-axis has been discretized into 1024 equally sized bins.

The study of (interval) fragmentation processes relies very much on the branching property, that is on the mutual independence of the future evolution of the intervals, conditioned on the past. In fact, many concepts from branching random walks such as the additive martingales carry over to fragmentation processes, see e.g. the references cited in section 1.2.6. The branching property is however lost with the addition of a power of choices rule and makes the study of the process more challenging. In **[M10]**, we were able to obtain the following result:

Theorem 4.6. *In the max-2 or min-2 process, denote by $I_1^{(n)}, \dots, I_n^{(n)}$ the lengths of the n intervals at time $n-1$. Then there exists a probability measure μ on $\mathbf{R}_+$, such that*

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{nI_i^{(n)}} \to \mu, \quad \textit{almost surely w.r.t. the weak topology.}$$

Furthermore, μ admits the following tail asymptotic:

- *max-2 process: $\mu([x,\infty)) \sim Ce^{-2x}$ for some $C > 0$ as $x \to \infty$*
- *min-2 process: $\mu([x,\infty)) \sim (3/4)x^{-2}$ as $x \to \infty$.*

It is striking to see the analogy with the balls-and-bins model: the max-2 process behaves quite similarly as the original process (in which a typical interval is exponentially distributed in the limit), whereas the min-2 process behaves drastically different. The reason is that in both models, the evolution of the large objects (the bins with high load/the large intervals) is simply accelerated by a factor of 2 in the max-version, whereas it is substantially slowed down in the min-version. To wit, in the uniform splitting model, the size of the largest interval is $\approx \ln n/n$ [Dar53, Whi97]. In the max-2 process, the tail of the interval distribution is of order e^{-2x}, which suggests that the size of the largest interval is $\approx \frac{1}{2}\ln n/n$. In the min-2 process on the other hand, the size of the largest interval is $n^{-1/2+o(1)}$ and thus on a completely different scale, mirroring what occurs in the balls-and-bins model.

We now describe the inner workings of Theorem 4.6. As mentioned before, we actually work with a much larger class of processes, which we call the Ψ-*processes* and defined as follows: Given the subdivision of the unit interval into intervals of lengths $I_1^{(n)}, \dots, I_n^{(n)}$, we choose the next interval to split as follows:

- denote by $\widetilde{D}_n(x) = \sum_{i=1}^n I_i^{(n)} \mathbf{1}_{I_i^{(n)} \leqslant x}$ the size-biased empirical distribution function of interval lengths and by $\widetilde{D}_n^{-1}$ its generalized inverse
- the next interval to split is the interval of length $\widetilde{D}_n^{-1}(U)$, where U is sampled from a law on $(0,1]$ whose distribution function we denote by Ψ.

This randomly chosen interval is now subdivided into two pieces at a point chosen uniformly inside the interval. This produces a new sequence of interval lengths $I_1^{(n+1)}, \dots, I_{n+1}^{(n+1)}$ and the process is repeated. We call the resulting process the Ψ-process. Note that the max-2, uniform, min-2 and Kakutani processes are Ψ-processes with $\Psi(u) = u^2$, u, $1-(1-u)^2$ and $\mathbf{1}_{u \geqslant 1}$, respectively. Our methods are able to handle all Ψ processes under some mild assumptions on Ψ including continuity, in particular the max-2, uniform and min-2 processes (but not the Kakutani process).

The main observable that we study is the function $D_n(x) = \widetilde{D}_n(x/n)$. We begin by embedding the discrete-time process $D_n(x)$ into a continuous time process $A_t(x)$ in such a way that $n \approx e^t$. This continuous time process A_t essentially evolves according to a stochastic evolution equation

$$\partial_t A_t(x) = -x\partial_x A_t(x) + x^2 \int_x^\infty \frac{1}{y} d\Psi(A_t(y)) + \dot{M}_t(x) \tag{4.6}$$

for some centered noise $M_t(x)$. We first treat the deterministic part of the equation. The key to this is the following carefully selected norm,

$$\|f\|_{x^{-2}} = \int_0^\infty x^{-2}|f(x)|\, dx,$$

with respect to which the evolution operator associated to the deterministic part of (4.6) quite surprisingly turns out to be a contraction. This assures that there is a unique distribution F^Ψ so that for any starting distribution, the large-time limit of the deterministic evolution is F^Ψ.

Finally, in order to show that A_t converges to F^Ψ despite the presence of noise, we show that the family $\{A_t\}_{t\geqslant 0}$ is almost surely precompact in a suitable topology. For this, tightness of A_t is needed which is obtained through entropy estimates. We then show that the limit points of this sequence are solutions of the deterministic evolution, from which we can conclude that the unique limit is the stationary evolution $\boldsymbol{F}^* \equiv F^\Psi$. This yields almost sure convergence of the stochastic evolution A_t. This compactness-continuity method is also called the Kushner–Clark method [KC78, Section 2.1] or the method of asymptotic pseudotrajectories [BH96, Ben99]. This method is usually applied in finite-dimensional settings and we adapt it here to a functional setting (see [BLR02] for another example in an infinite-dimensional setting).

As a consequence of the method, we obtain a characterization of the limiting fonction F^Ψ as the unique solution of the following integro-differential equation:

$$F'(x) = x \int_x^\infty \frac{1}{z}\, d\Psi(F(z)),$$

which allows us to derive tail estimates for many choices of Ψ.

In ongoing work, we are interested in the empirical measure of the endpoints of the intervals. We obtain the following theorem:

Theorem 4.7. *Under mild conditions on Ψ, the empirical measure of the endpoints of the intervals, renormalized to mass 1, converges almost surely to the Lebesgue measure on $[0, 1]$.*

This theorem has been obtained for a fairly restrictive class of processes including the max-2 but excluding the min-2 process by Matthew Junge [Jun14]. His idea was to seperate the unit interval at a point $\alpha \in (0, 1)$ and to consider the joint evolution of the empirical measure of the lengths of the intervals in $[0, \alpha]$ and those in $[\alpha, 1]$. This yields a pair of processes $(A_t^\alpha, A_t^{\alpha'})$ whose joint evolution has less pleasing algebraic properties as the one for A_t, but in certain cases might still be shown to be contractive in the limit. We build up on this idea but with the important change that we make use of the fact that we know already that A_t converges to a limit F^Ψ as $t \to \infty$. Supposing we can replace A_t by its limit, the deterministic part of the evolution of A_t^α becomes a *linear* equation, which can be interpreted as the equation satisfied by (α times) the distribution function of the law of a certain piecewise deterministic Markov process, namely, the tagged

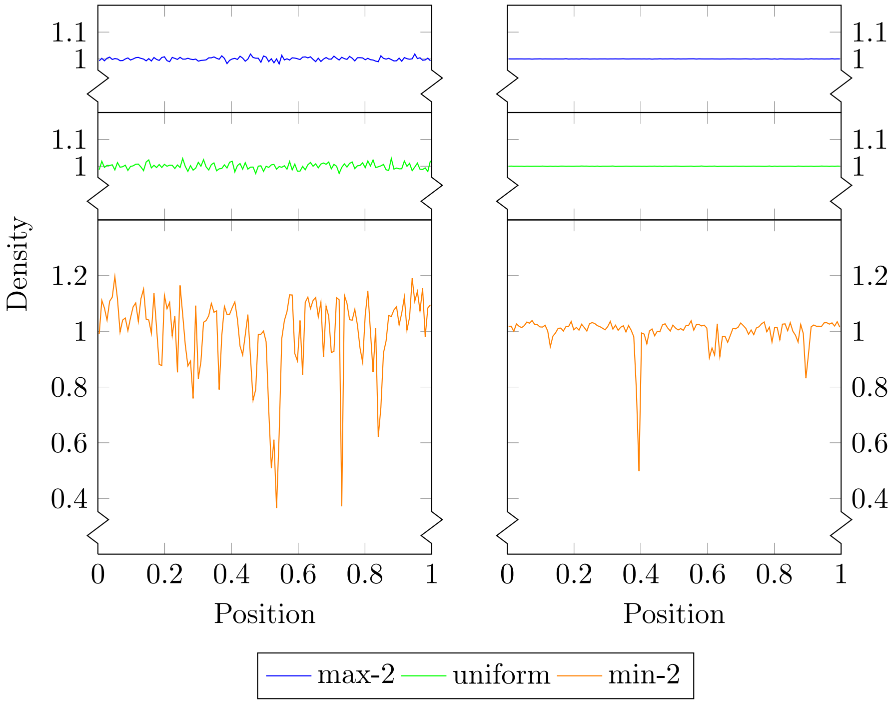


Figure 4.4 – Empirical density of points in the unit interval in simulation of max-2, uniform and min-2 processes with 10^6 points (left) and 10^9 points (right). For the plot, the x-axis has been discretized into 128 equally sized bins.

fragment in a certain (growth-)fragmentation process. Furthermore, the function F^Ψ turns out to be the distribution function of a stationary distribution of this process. Proving these facts rigorously needs some work and combines probabilistic arguments together with functional analysis (in particular, semigroups on L^p-spaces). Using regeneration times, it is then shown that this Markov process is ergodic and thus converges to its stationary distribution. In the end, we show that for every $\alpha \in (0,1)$, A_t^α converges almost surely to $\alpha \times F^\Psi$ from which one can easily deduce Theorem 4.7 as in [Jun14].

Perspectives. Our work sets the stone for the study of more general interval fragmentation processes with interaction. One obvious generalization would be to consider more general splitting laws instead of the mere splitting at a uniform point inside an interval. Interestingly, the algebra becomes more complicated in this case. In particular, the norm $\|f\|_{x^{-2}}$ needs to be replaced by a norm with a different weight tailored to the splitting distribution (and might not always be possible to find). Any advance on this subject would be interesting.

In general, variations of the Ψ-process might be difficult to study for the lack of observables with nice algebraic properties. For example, consider the process where at each step, 2 points are randomly dropped in the unit interval and the point whose distance to the previous points is larger (resp., smaller) is retained, the other one is discarded. Show that the empirical measure of points converges to the Lebesgue measure (this was the original question by I. Benjamini). This problem seems to be difficult and as far as we know, no progress has been made since.

Returning to the Ψ-process, another open problem is the rate of convergence of the empirical distribution of the interval lengths. Our method, relying on a compactness-continuity argument, does not give any results on this. The most serious difficulty arises from the fact that the norm $\|\cdot\|_{x^{-2}}$, which is our only tool to study convergence of the (deterministic) evolution, is very sensitive to perturbations, due to the absolute value appearing inside the integral. As a consequence, we are not able to directly control the stochastic evolution A_t or the noise M_t in terms of this norm. Simulations indicate though that this convergence is quite fast, possibly polynomial in n (see Figure 4.3, in which the noise is completely invisible despite the high resolution of the data).

Finally, I would like to mention the paper [DFGGR16] in which the authors study an interesting, much more powerful power of choice method: throw $2n$ points in the interval and select n of them (with a low-complexity algorithm) such that the distance (in some sense) between the empirical measure and Lebesgue measure is small.

Annexe A

Une étude bibliométrique

Le domaine de la marche aléatoire branchante et du mouvement brownien branchant est en pleine effervescence avec un nombre d'articles sur le sujet qui est en croissance constante depuis quelques années. Pour mesurer cette croissance, j'ai conduit le 8 mars 2018 une recherche bibliométrique sur MathSciNet (`mathscinet.ams.org`). J'ai mesuré le nombre de résultats par année pour une recherche avec un champ "Title" ou "Anywhere" donné. Les termes recherchés étaient :

1. "branching random walk" OR "branching random walks"
2. "branching Brownian motion" OR "branching Brownian motions"
3. "multiplicative cascade" OR "multiplicative cascades" OR "Mandelbrot cascades" OR "Mandelbrot cascade"
4. "branching diffusion" OR "branching diffusions" OR "branching Markov process" OR "branching Markov processes"

Les résultats de ces recherches sont illustrés dans la figure A.1. Les données montrent une forte croissance du nombre de résultats pour les deux premières recherches (marche aléatoire branchante et mouvement brownien branchant) depuis le milieu des années 2000. A noter qu'à cause du délai de publication, des articles très récents ne figurent pas dans ces données, il est donc probable que la croissance soit en réalité encore plus forte pour les dernières années. En comparaison, ce nombre reste à peu près stable pour les deux autres (cascades multiplicatives et diffusions ou processus de Markov branchants). Ces résultats sont à prendre avec précaution, car le termes recherchés ne sont pas utilisés par tous les auteurs. Les recherches risquent donc de sous-estimer le nombre d'articles sur un sujet donné et ce de manière inégale selon le sujet ainsi que selon l'année, la nomenclature pouvant changer au fil du temps. Il serait intéressant de raffiner la recherche pour prendre en compte ce fait. Néanmoins, il me semble que les données sont déjà suffisamment significatifs pour affirmer que l'engouement pour la MAB et le MBB ne cesse de croître. De plus, cette croissance ne semble pas être portée par un intérêt croissant pour les processus de branchement généraux (diffusions ou processus de Markov branchants), mais semble être propre à la MAB et au MBB. Une raison probable est donnée par le grand nombre d'interactions de la MAB avec d'autres domaines dont la section 1.2 donne un aperçu.

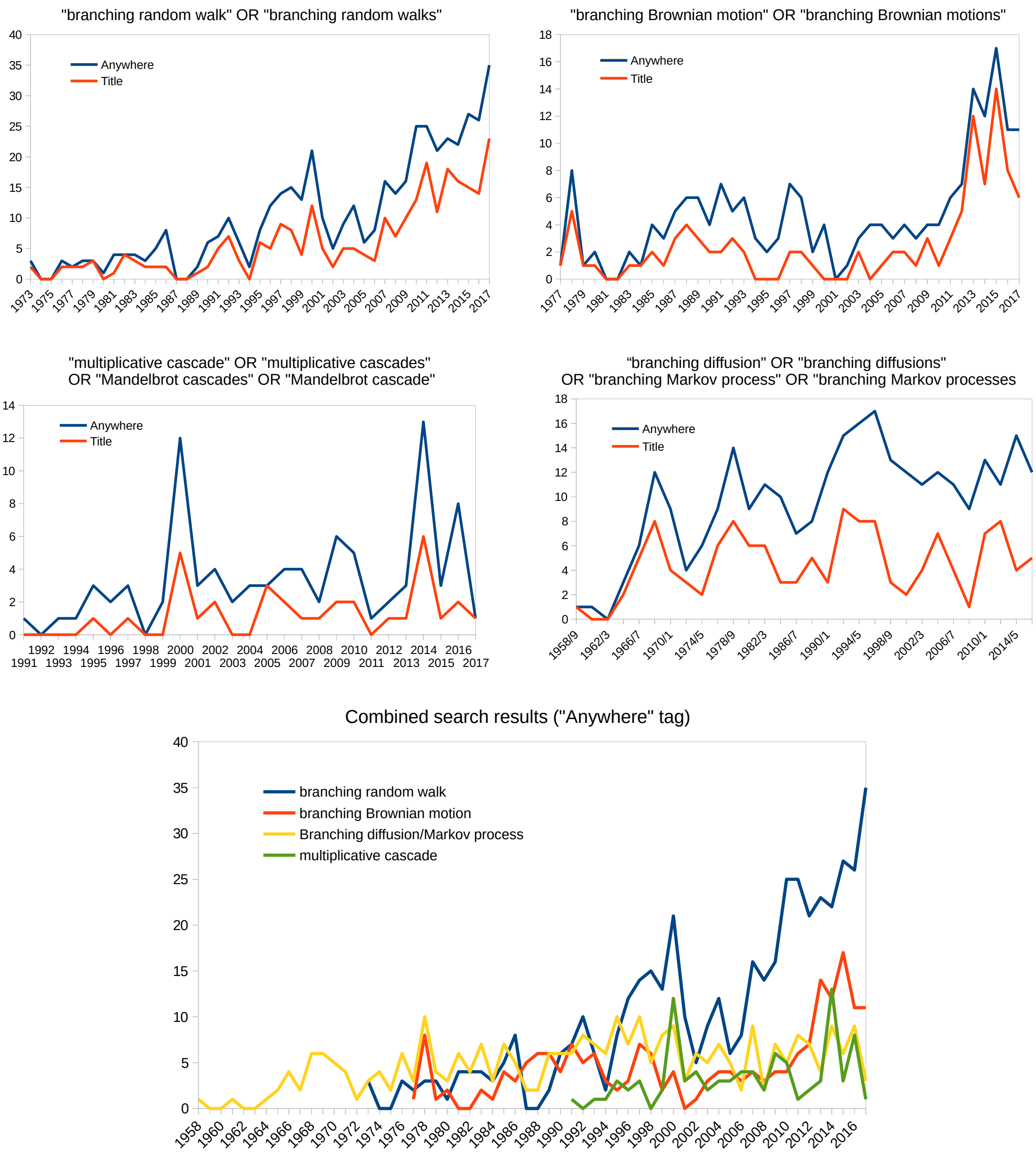


FIGURE A.1 – Résultats d'une recherche bibliométrique effectuée le 8 mars 2018 sur MathSciNet (`mathscinet.ams.org`). Données : nombre de résultats par année pour une recherche avec un champ "Title" ou "Anywhere" donné.

Bibliographie exogène

[ABB11] Louigi Addario-Berry and Nicolas Broutin. Total progeny in killed branching random walk. *Probability Theory and Related Fields*, 151(1-2):265–295, may 2011.

[ABB+16] Louis-Pierre Arguin, David Belius, Paul Bourgade, Maksym Radziwiłł, and Kannan Soundararajan. Maximum of the Riemann zeta function on a short interval of the critical line. *arXiv:1612.08575*, 2016.

[ABB17] Louis Pierre Arguin, David Belius, and Paul Bourgade. Maximum of the Characteristic Polynomial of Random Unitary Matrices. *Communications in Mathematical Physics*, 349(2):703–751, 2017.

[ABBS13] E. Aïdékon, J. Berestycki, É Brunet, and Z. Shi. Branching Brownian motion seen from its tip. *Probability Theory and Related Fields*, 157(1-2):405–451, 2013.

[ABK13] Louis-Pierre Arguin, Anton Bovier, and Nicola Kistler. The extremal process of branching Brownian motion. *Probability Theory and Related Fields*, 157(3-4):535–574, dec 2013.

[ABKU99] Yossi Azar, Andrei Z. Broder, Anna R. Karlin, and Eli Upfal. Balanced Allocations. *SIAM Journal on Computing*, 29(1):180–200, 1999.

[ABP17] Louigi Addario-Berry and Sarah Penington. The front location in branching Brownian motion with decay of mass. *Annals of Probability*, 45(6):3752–3794, 2017.

[ADH15] Antonio Auffinger, Michael Damron, and Jack Hanson. 50 Years of First Passage Percolation. *arXiv:1511.03262*, 2015.

[ADS09] D. Achlioptas, R. M. D'Souza, and J. Spencer. Explosive Percolation in Random Networks. *Science*, 323(5920):1453–1455, mar 2009.

[AHZ13] Elie Aïdékon, Yueyun Hu, and Olivier Zindy. The precise tail behavior of the total progeny of a killed branching random walk. *Annals of Probability*, 41(6):3786–3878, feb 2013.

[Aïd08] Elie Aïdékon. Transient random walks in random environment on a Galton–Watson tree. *Probability Theory and Related Fields*, 142(3-4):525–559, nov 2008.

[Aïd10] Elie Aïdékon. Tail asymptotics for the total progeny of the critical killed branching random walk. *Electronic Communications in Probability*, 15:522–533, 2010.

[Aïd14] Elie Aïdékon. Speed of the biased random walk on a Galton–Watson tree. *Probability Theory and Related Fields*, 159(3-4):597–617, 2014.

[AK16] Sebastian Andres and Naotaka Kajino. Continuity and estimates of the Liouville heat kernel with applications to spectral dimensions. *Probability Theory and Related Fields*, 166(3-4):713–752, 2016.

[Ald92] David Aldous. Greedy Search on the Binary Tree with Random Edge-Weights. *Combinatorics, Probability and Computing*, 1(04):281–293, dec 1992.

[Ald93] David Aldous. The Continuum Random Tree III. *The Annals of Probability*, 21(1):248–289, jan 1993.

[Ald98] David Aldous. A Metropolis-type optimization algorithm on the infinite tree. *Algorithmica*, 22(4):388–412, dec 1998.

[APS17] Juhan Aru, Ellen Powell, and Avelio Sepúlveda. Liouville measure as a multiplicative cascade via level sets of the Gaussian free field. *arXiv:1701.05872*, 2017.

[AS64] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office, Washington, D.C., 1964.

[AS14] Elie Aïdékon and Zhan Shi. The Seneta-Heyde scaling for the branching random walk. *Annals of Probability*, 42(3):959–993, feb 2014.

[Bar99] Julien Barral. Moments, continuité, et analyse multifractale des martingales de Mandelbrot. *Probability Theory and Related Fields*, 113(4):535–569, 1999.

[Bar00] Julien Barral. Continuity of the Multifractal Spectrum of a Random Statistically Self-Similar Measure. *Journal of Theoretical Probability*, 13(4):1027–1060, 2000.

[BBCK17] Jean Bertoin, Timothy Budd, Nicolas Curien, and Igor Kortchemski. Martingales in self-similar growth-fragmentations and their connections with random planar maps. *Probability Theory and Related Fields*, pages 1–62, dec 2017.

[BBG12] G. Borot, J. Bouttier, and E. Guitter. A recursive approach to the $O(n)$ model on random maps via nested loops. *Journal of Physics A: Mathematical and Theoretical*, 45(4):045002, feb 2012.

[BBHM17] Julien Berestycki, Éric Brunet, Simon C. Harris, and Piotr Miłoś. Branching Brownian motion with absorption and the all-time minimum of branching Brownian motion with drift. *Journal of Functional Analysis*, 273(6):2107–2143, 2017.

[BBP18] Julien Berestycki, Éric Brunet, and Sarah Penington. Global existence for a free boundary problem of Fisher-KPP type. *arXiv:1805.03702*, 2018.

[BBS11] Julien Berestycki, Nathanaël Berestycki, and Jason Schweinsberg. Survival of near-critical branching Brownian motion. *Journal of Statistical Physics*, 143(5):833–854, may 2011.

[BBS13] Julien Berestycki, Nathanaël Berestycki, and Jason Schweinsberg. The genealogy of branching Brownian motion with absorption. *The Annals of Probability*, 41(2):527–618, mar 2013.

[BBS14] Julien Berestycki, Nathanaël Berestycki, and Jason Schweinsberg. Critical branching Brownian motion with absorption: survival probability. *Probability Theory and Related Fields*, 160(3-4):489–520, 2014.

[BD74] N. H. Bingham and Ronald A Doney. Asymptotic properties of supercritical branching processes I: The Galton-Watson process. *Advances in Applied Probability*, 6(4):711–731, jan 1974.

[BD87] Michael D. Brennan and Richard Durrett. Splitting intervals II: Limit laws for lengths. *Probability Theory and Related Fields*, 75(1):109–127, may 1987.

[BD09] É Brunet and B. Derrida. Statistics at the tip of a branching random walk and the delay of traveling waves. *EPL (Europhysics Letters)*, 87(6):60010, sep 2009.

[BD11] Éric Brunet and Bernard Derrida. A branching random walk seen from the tip. *Journal of Statistical Physics*, 143(3):420–446, apr 2011.

[BDM16] Dariusz Buraczewski, Ewa Damek, and Thomas Mikosch. *Stochastic models with power-law tails.* Springer Series in Operations Research and Financial Engineering. Springer, [Cham], 2016.

[BDMM06a] Éric Brunet, Bernard Derrida, A. Mueller, and S. Munier. Phenomenological theory giving the full statistics of the position of fluctuating pulled fronts. *Physical Review E*, 73(5):056126, may 2006.

[BDMM06b] Éric Brunet, Bernard Derrida, A. H Mueller, and S Munier. Noisy traveling waves: Effect of selection on genealogies. *Europhysics Letters (EPL)*, 76(1):1–7, oct 2006.

[BDMM07] Éric Brunet, Bernard Derrida, A Mueller, and S. Munier. Effect of selection on ancestry: An exactly soluble case and its phenomenological generalization. *Physical Review E*, 76(4):1–20, oct 2007.

[BDZ16] Maury Bramson, Jian Ding, and Ofer Zeitouni. Convergence in Law of the Maximum of the Two-Dimensional Discrete Gaussian Free Field. *Communications on Pure and Applied Mathematics*, 69(1):62–123, 2016.

[Ben99] Michel Benaïm. Dynamics of Stochastic Approximation Algorithms. In Jacques Azéma, M. Emery, M. Ledoux, and M. Yor, editors, *Séminaire de Probabilités XXXIII*, volume 1709 of *Lect. Notes Math.*, pages 1–68. Springer Berlin, 1999.

[Ber06] Jean Bertoin. *Random fragmentation and coagulation processes*, volume 102 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[Ber14] Julien Berestycki. Topics on branching Brownian motion. Lecture notes, available at www.stats.ox.ac.uk/∼berestyc/articles.html, 2014.

[Ber15] Nathanaël Berestycki. Diffusion in planar Liouville quantum gravity. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 51(3):947–964, 2015.

[Ber17] Nathanaël Berestycki. An elementary approach to gaussian multiplicative chaos. *Electronic Communications in Probability*, 22(27):1–12, 2017.

[BG01] John Bunge and Charles M. Goldie. Record sequences and their applications. In D.N. Shanbhag and C.R. Rao, editor, *Stochastic Processes: Theory and Methods*, volume Volume 19 of *Handbook of Statistics, volume 19*, pages 277–308. Elsevier, 2001.

[BG10] Jean Bérard and Jean-Baptiste Gouéré. Brunet-Derrida behavior of branching-selection particle systems on the line. *Communications in Mathematical Physics*, 298(2):323–342, jun 2010.

[BG11] Jean Bérard and Jean-Baptiste Gouéré. Survival probability of the branching random walk killed below a linear boundary. *Electronic Journal of Probability*, 16(14):396–418, 2011.

[BH96] Michel Benaïm and Morris W. Hirsch. Asymptotic pseudotrajectories and chain recurrent flows, with applications. *Journal of Dynamics and Differential Equations*, 8(1):141–176, jan 1996.

[BHOZ11] Gérard Ben Arous, Yueyun Hu, Stefano Olla, and Ofer Zeitouni. Einstein relation for biased random walk on Galton–Watson trees. *arXiv:1106.4387*, jun 2011.

[Bie65] Ludwig Bieberbach. *Theorie der gewöhnlichen Differentialgleichungen auf funktionentheoretischer Grundlage dargestellt*, volume 66 of *Die Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition, 1965.

[Big77a] John D. Biggins. Chernoff's theorem in the branching random walk. *Journal of Applied Probability*, 14(3):630, sep 1977.

[Big77b] John D. Biggins. Martingale Convergence in the Branching Random Walk. *Journal of Applied Probability*, 14(1):25, mar 1977.

[Big79] John D. Biggins. Growth rates in the branching random walk. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 48(1):17–34, 1979.

[Big92] John D. Biggins. Uniform Convergence of Martingales in the Branching Random Walk. *The Annals of Probability*, 20(1):137–151, dec 1992.

[Big10] John D. Biggins. Branching out. In N. H. Bingham and C. M. Goldie, editors, *Probability and Mathematical Genetics: Papers in Honour of Sir John Kingman*, London Mathematical Society Lecture Note Series, pages 112–133. Cambridge University Press, 2010.

[BK04] Anton Bovier and Irina Kurkova. Derrida's Generalized Random Energy models 2: models with continuous hierarchies. *Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques*, 40(4):481–495, aug 2004.

[BK07] Anton Bovier and Irina Kurkova. Much Ado about Derrida's GREM. In Erwin Bolthausen and Anton Bovier, editors, *Spin Glasses*, Lecture Notes in Mathematics, pages 81–115. Springer Berlin Heidelberg, jan 2007.

[BKN+13] Julien Barral, Antti Kupiainen, Miika Nikula, Eero Saksman, and Christian Webb. Critical Mandelbrot Cascades. *Communications in Mathematical Physics*, pages 1–27, nov 2013.

[BKS03] Itai Benjamini, Gil Kalai, and Oded Schramm. First passage percolation has sublinear distance variance. *Annals of Probability*, 31(4):1970–1978, 2003.

[BL00] Jean Bertoin and Jean-François Le Gall. The Bolthausen-Sznitman coalescent and the genealogy of continuous-state branching processes. *Probability Theory and Related Fields*, 117(2):249–266, jun 2000.

[BL16a] Marek Biskup and Oren Louidor. Extreme Local Extrema of Two-Dimensional Discrete Gaussian Free Field. *Communications in Mathematical Physics*, 345(1):271–304, 2016.

[BL16b] Marek Biskup and Oren Louidor. Full extremal process, cluster law and freezing for two-dimensional discrete Gaussian Free Field. *arXiv:1606.00510*, 2016.

[BLR02] Michel Benaïm, Michel Ledoux, and Olivier Raimond. Self-interacting diffusions. *Probability Theory and Related Fields*, 122(1):1–41, jan 2002.

[BM03] E. Bacry and J. F. Muzy. Log-infinitely divisible multifractal processes. *Communications in Mathematical Physics*, 236(3):449–475, 2003.

[BM04] Julien Barral and Benoît B Mandelbrot. Introduction to infinite products of independent random functions (Random multiplicative multifractal measures. I-III). In *Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 2*, volume 72 of *Proc. Sympos. Pure Math.*, pages 3–90. Amer. Math. Soc., Providence, RI, 2004.

[BM17] Jean Bertoin and Bastien Mallein. Biggins' Martingale Convergence for Branching Lévy Processes. *arXiv:1712.04769*, 2017.

[Bov17] Anton Bovier. *Gaussian processes on trees*, volume 163 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017.

[Bra83] Maury D. Bramson. Convergence of solutions of the Kolmogorov equation to travelling waves. *Memoirs of the American Mathematical Society*, 44(285), 1983.

[Bud] Timothy Budd. The peeling process on random planar maps coupled to an $O(n)$ loop model. with an appendix by L. Chen. In preparation.

[BvdH12] Shankar Bhamidi and Remco van der Hofstad. Weak disorder asymptotics in the stochastic mean-field model of distance. *The Annals of Applied Probability*, 22(1):29–69, feb 2012.

[BVH11] Shankar Bhamidi, Remco Van Der Hofstad, and Gerard Hooghiemstra. First Passage Percolation on the Erd$\mathscr{H}$os–Rényi Random Graph. *Combinatorics, Probability and Computing*, 20(05):683–707, 2011.

[BZ12] Itai Benjamini and Ofer Zeitouni. Tightness of Fluctuations of First Passage Percolation on Some Large Graphs. In Bo'az Klartag, Shahar Mendelson, and Vitali D. Milman, editors, *Geometric aspects of functional analysis: Papers from the Israel Seminar (GAFA) held 2006–2010.*, volume 2050 of *Lecture Notes in Math.*, pages 127–132. Springer, Heidelberg, oct 2012.

[CG15] Nicolas Curien and Jean-François Le Gall. First-passage percolation and local modifications of distances in random triangulations. *arXiv:1511.04264*, 2015.

[CGRV17] Laurent Chevillard, Christophe Garban, Rémi Rhodes, and Vincent Vargas. On a skewed and multifractal uni-dimensional random field, as a probabilistic representation of Kolmogorov's views on turbulence. *arXiv:1712.00332*, 2017.

[CH98] Darren B. H. Cline and Tailen Hsing. Large deviation probabilities for sums of random variables with heavy or subexponential tails. Technical report, Statistics Department, Texas A&M University, available online at http://www.stat.tamu.edu/∼dcline/Papers/large5.pdf, 1998.

[Cha13] Sourav Chatterjee. The universal relation between scaling exponents in first-passage percolation. *Annals of Mathematics. Second Series*, 177(2):663–697, may 2013.

[Che15] Xinxin Chen. Scaling Limit of the Path Leading to the Leftmost Particle in a Branching Random Walk. *Theory of Probability & Its Applications*, 59(4):567–589, 2015.

[CHL17] Aser Cortines, Lisa Hartung, and Oren Louidor. The Structure of Extreme Level Sets in Branching Brownian Motion. *arXiv:1703.06529*, 2017.

[CKMR05] B. Chauvin, T. Klein, J. F. Marckert, and A. Rouault. Martingales and profile of binary search trees. *Electronic Journal of Probability*, 10:420–435, 2005.

[CM16] Aser Cortines and Bastien Mallein. A N-branching random walk with random selection. *arXiv:1605.03401*, 2016.

[CNM16] Reda Chhaibi, Joseph Najnudel, and Thomas Madaule. On the maximum of the CβE field. *arXiv:1607.00243*, pages 1–73, 2016.

[Cor16] Pierre-Antoine Corre. Number of particles absorbed in a BBM on the extinction event. *arXiv:1611.01833*, 2016.

[CR88] Brigitte Chauvin and Alain Rouault. KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. *Probability Theory and Related Fields*, 80(2):299–314, dec 1988.

[Dar53] D. A. Darling. On a Class of Problems Related to the Random Division of an Interval. *The Annals of Mathematical Statistics*, 24(2):239–253, jun 1953.

[DDS08] D. Denisov, A. B. Dieker, and V. Shneer. Large deviations for random walks under subexponentiality: The big-jump domain. *The Annals of Probability*, 36(5):1946–1991, 2008.

[De 82] A. De Meyer. On a theorem of Bingham and Doney. *Journal of Applied Probability*, 19(01):217–220, mar 1982.

[Der85] Bernard Derrida. A generalization of the random energy model which includes correlations between energies. *Journal de Physique Lettres*, 46(9):401–407, 1985.

[DFGGR16] Raaz Dwivedi, Ohad N. Feldheim, Ori Gurel-Gurevich, and Aaditya Ramdas. The power of online thinning in reducing discrepancy. *arXiv:1608.02895*, 2016.

[DFPSL17] Anna De Masi, Pablo A. Ferrari, Errico Presutti, and Nahuel Soprano-Loto. Hydrodynamics of the N-BBM process. *arXiv:1707.00799*, 2017.

[DG16] Jian Ding and Subhajit Goswami. Upper bounds on Liouville first passage percolation and Watabiki's prediction. *arXiv:1610.09998*, 2016.

[DH91] F M Dekking and B Host. Limit distributions for minimal displacement of branching random walks. *Probability Theory and Related Fields*, 90(3):403–426, 1991.

[DKN15] Freddy Delbaen, Emmanuel Kowalski, and Ashkan Nikeghbali. Mod-ϕ convergence. *International Mathematics Research Notices*, 2015(11):3445–3485, 2015.

[DKRV16] François David, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Liouville Quantum Gravity on the Riemann Sphere. *Communications in Mathematical Physics*, 342(3):869–907, oct 2016.

[DL83] Richard Durrett and Thomas M Liggett. Fixed points of the smoothing transformation. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 64:275–301, 1983.

[DL04] Thomas Duquesne and Jean-François Le Gall. Probabilistic and fractal aspects of Lévy trees. *Probability Theory and Related Fields*, 131(4):553–603, nov 2004.

[DR11] Rick Durrett and Daniel Remenik. Brunet–Derrida particle systems, free boundary problems and Wiener–Hopf equations. *The Annals of Probability*, 39(6):2043–2078, nov 2011.

[DS11] Bertrand Duplantier and Scott Sheffield. Liouville quantum gravity and KPZ. *Inventiones mathematicae*, 185(2):333–393, aug 2011.

[Duq03] Thomas Duquesne. A limit theorem for the contour process of conditioned Galton-Watson trees. *The Annals of Probability*, 31(2):996–1027, apr 2003.

[Dur79] Richard Durrett. Maxima of branching random walks vs. independent random walks. *Stochastic Processes and their Applications*, 9(2):117–135, nov 1979.

[Dur83] Richard Durrett. Maxima of branching random walks. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 62(2):165–170, jun 1983.

[DZZ17] Jian Ding, Ofer Zeitouni, and Fuxi Zhang. On the Liouville heat kernel for k-coarse MBRW and nonuniversality. *arXiv:1701.01201*, 2017.

[Ein05] Albert Einstein. Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Annalen der Physik*, 322(8):549–560, 1905.

[EKM97] Paul Embrechts, Claudia Klüppelberg, and Thomas Mikosch. *Modelling extremal events*, volume 33 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1997.

[EM14] Maren Eckhoff and Peter Mörters. Vulnerability of robust preferential attachment networks. *Electronic Journal of Probability*, 19:1–38, 2014.

[EMO17] Maren Eckhoff, Peter Morters, and Marcel Ortgiese. Near critical preferential attachment networks have small giant components. *arXiv:1712.00400*, 2017.

[EvS00] Ute Ebert and Wim van Saarloos. Front propagation into unstable states: universal algebraic convergence towards uniformly translating pulled fronts. *Physica D: Nonlinear Phenomena*, 146:1–99, nov 2000.

[Fel71] William Feller. *An introduction to probability theory and its applications. Vol II.* John Wiley & Sons Inc., New York, second edition, 1971.

[FHS12] Gabriel Faraud, Yueyun Hu, and Zhan Shi. Almost sure convergence for stochastically biased random walks on trees. *Probability Theory and Related Fields*, 154(3-4):621–660, dec 2012.

[Fis37] R. A. Fisher. The wave of advance of advantageous genes. *Annals of Eugenics*, 7(4):355–369, jun 1937.

[FO90] Philippe Flajolet and Andrew Odlyzko. Singularity analysis of generating functions. *SIAM Journal on Discrete Mathematics*, 3(2):216–240, 1990.

[FP85] U Frisch and G Parisi. On the singularity structure of fully developed turbulence. In *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, Proceedings of the International School of Physics "Enrico Fermi", Course LXXXVIII., pages 84 – 88. North-Holland, New York, 1985.

[FS05] Patrik L. Ferrari and Herbert Spohn. Constrained Brownian motion: Fluctuations away from circular and parabolic barriers. *Annals of Probability*, 33(4):1302–1325, 2005.

[FZ10] Ming Fang and Ofer Zeitouni. Consistent Minimal Displacement of Branching Random Walks. *Electronic Communications in Probability*, 15:no. 11, 106–118, mar 2010.

[FZ12] Ming Fang and Ofer Zeitouni. Slowdown for time inhomogeneous branching Brownian motion. *Journal of Statistical Physics*, 149(1):1–9, oct 2012.

[GHS11] Nina Gantert, Yueyun Hu, and Zhan Shi. Asymptotics for the survival probability in a killed branching random walk. *Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques*, 47(1):111–129, feb 2011.

[GK12] Gr Grimmett and H Kesten. Percolation since Saint-Flour. *arXiv preprint arXiv:1207.0373*, pages 1–19, 2012.

[GK17] Rudolf Grübel and Zakhar Kabluchko. Edgeworth expansions for profiles of lattice branching random walks. *Annales de l'institut Henri Poincare (B) Probability and Statistics*, 53(4):2103–2134, 2017.

[GMP12] Nina Gantert, Pierre Mathieu, and Andrey Piatnitski. Einstein relation for reversible diffusions in a random environment. *Communications on Pure and Applied Mathematics*, 65(2):187–228, 2012.

[Gol91] Charles M. Goldie. Implicit Renewal Theory and Tails of Solutions of Random Equations. *The Annals of Applied Probability*, 1(1):126–166, 1991.

[Gri75] A. K. Grincevićius. One limit distribution for a random walk on the line. *Lithuanian Mathematical Journal*, 15(4):580–589, 1975.

[Gro89] Piet Groeneboom. Brownian motion with a parabolic drift and airy functions. *Probability Theory and Related Fields*, 81(1):79–109, 1989.

[GRV16] Christophe Garban, Rémi Rhodes, and Vincent Vargas. Liouville Brownian motion. *Annals of Probability*, 44(4):3076–3110, jan 2016.

[Gui90] Yves Guivarc'h. Sur une extension de la notion de loi semi-stable. *Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques*, 26(2):261–285, 1990.

[Hey67] C. C. Heyde. A contribution to the theory of large deviations for sums of independent random variables. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 7(5):303–308, jan 1967.

[HJK+86a] Thomas C. Halsey, Mogens H. Jensen, Leo P. Kadanoff, Itamar Procaccia, and Boris I. Shraiman. Erratum: Fractal measures and their singularities: The characterization of strange sets [Phys. Rev. A 33 , 1141 (1986)]. *Physical Review A*, 34(2):1601–1601, aug 1986.

[HJK+86b] Thomas C. Halsey, Mogens H. Jensen, Leo P. Kadanoff, Itamar Procaccia, and Boris I. Shraiman. Fractal measures and their singularities: The characterization of strange sets. *Physical Review A*, 33(2):1141–1151, feb 1986.

[HK15] Lisa Hartung and Anton Klimovsky. The glassy phase of the complex branching Brownian motion energy model. *Electronic Communications in Probability*, 20(78):1–15, 2015.

[HK17] Lisa Hartung and Anton Klimovsky. The phase diagram of the complex branching Brownian motion energy model. *arXiv:1704.05402*, pages 1–22, 2017.

[HL81] Richard Holley and Thomas M Liggett. Generalized potlatch and smoothing processes. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 55:165–195, 1981.

[HNRR13] François Hamel, James Nolen, Jean-Michel Roquejoffre, and Lenya Ryzhik. A short proof of the logarithmic Bramson correction in Fisher-KPP equations. *Networks and Heterogeneous Media*, 8(1):275–289, apr 2013.

[How04] C. Douglas Howard. Models of first-passage percolation. In Harry Kesten, editor, *Probability on discrete structures*, volume 110 of *Encyclopaedia Math. Sci.*, pages 125–173. Springer, Berlin, 2004.

[HP83] H. G.E. Hentschel and Itamar Procaccia. The infinite number of generalized dimensions of fractals and strange attractors. *Physica D: Nonlinear Phenomena*, 8(3):435–444, 1983.

[HRV15] Yichao Huang, Rémi Rhodes, and Vincent Vargas. Liouville Quantum Gravity on the unit disk. *arXiv:1502.04343*, 2015.

[HS09] Yueyun Hu and Zhan Shi. Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. *The Annals of Probability*, 37(2):742–789, mar 2009.

[INW] Nobuyuki Ikeda, Masao Nagasawa, and Shinzo Watanabe. Branching Markov processes. I–III. *Journal of Mathematics of Kyoto University.*

[Jaf12] Bruno Jaffuel. The critical barrier for the survival of branching random walk with absorption. *Annales de l'institut Henri Poincare (B) Probability and Statistics*, 48(4):989–1009, nov 2012.

[Jan99] Svante Janson. One, two and three times $\log n/n$ for paths in a complete graph with random weights. *Combinatorics, Probability and Computing*, 8(04):347–361, 1999.

[JM05] Svante Janson and Jean-François Marckert. Convergence of Discrete Snakes. *Journal of Theoretical Probability*, 18(3):615–645, jul 2005.

[JOC12] Predrag R. Jelenković and Mariana Olvera-Cravioto. Implicit Renewal Theory and Power Tails on Trees. *Advances in Applied Probability*, 44(2):528–561, jun 2012.

[Jun14] Matthew Junge. Choices, intervals and equidistribution. *arXiv:1410.6537*, 2014.

[Kab12] Zakhar Kabluchko. Persistence and equilibria of branching populations with exponential intensity. *Journal of Applied Probability*, 49(1):226–244, 2012.

[Kah85] Jean-Pierre Kahane. Sur le chaos multiplicatif. *Ann. Sci. Math. Québec*, 9(2):105–150, 1985.

[KC78] Harold J. Kushner and Dean S. Clark. *Stochastic approximation methods for constrained and unconstrained systems*, volume 26 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1978.

[Kes73] Harry Kesten. Random difference equations and Renewal theory for products of random matrices. *Acta Mathematica*, 131(1):207–248, 1973.

[Kes78] Harry Kesten. Branching Brownian motion with absorption. *Stochastic Processes and their Applications*, 7(1):9–47, mar 1978.

[Kes95] Harry Kesten. Branching random walk with a critical branching part. *Journal of Theoretical Probability*, 8(4):921–962, oct 1995.

[KK14] Zakhar Kabluchko and Anton Klimovsky. Complex random energy model: Zeros and fluctuations. *Probability Theory and Related Fields*, 158(1-2):159–196, 2014.

[KLO12] Tomasz Komorowski, Claudio Landim, and Stefano Olla. *Fluctuations in Markov processes*, volume 345 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2012.

[KM17] Konrad Kolesko and Matthias Meiners. Convergence of complex martingales in the branching random walk: The boundary. *Electronic Communications in Probability*, 22(18):1–14, 2017.

[KP76] J.-P. Kahane and J Peyrière. Sur certaines martingales de Benoit Mandelbrot. *Adv. Math.*, 22:131–145, 1976.

[KPP37] Andrei N Kolmogorov, I.G. Petrovskii, and N.S. Piskunov. Étude de l'équation de la diffusion avec croissance de la quantite de matière et son application à un problème biologique. *Bulletin de l'Université d'État à Moscou : Série internationale. Section A, Mathématiques et mécanique*, 1(6):1–25, 1937.

[KRV17] Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Integrability of Liouville theory: proof of the DOZZ Formula. *arXiv:1707.08785*, 2017.

[KSdH09] Zakhar Kabluchko, Martin Schlather, and Laurens de Haan. Stationary max-stable fields associated to negative definite functions. *The Annals of Probability*, 37(5):2042–2065, sep 2009.

[Le 07] Jean François Le Gall. The topological structure of scaling limits of large planar maps. *Inventiones Mathematicae*, 169(3):621–670, 2007.

[Lee17] Jimyeong Lee. A free boundary problem in biological selection models. *arXiv:1707.01232*, 2017.

[Lig78] Thomas M. Liggett. Random invariant measures for Markov chains, and independent particle systems. *Zeitschrift f??r Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 45(4):297–313, 1978.

[Lin61a] Yu. V. Linnik. Limit theorems for sums of independent variables taking into account large deviations. I. *Theory of Probability and its Applications*, 6(2):131–148, 1961.

[Lin61b] Yu. V. Linnik. Limit theorems for sums of independent variables, taking large deviations into account. II. *Theory of Probability and its Applications*, VI(4):345–360, 1961.

[Liu98] Quansheng Liu. Fixed points of a generalized smoothing transformation and applications to the branching random walk. *Advances in Applied Probability*, 30(1):85–112, mar 1998.

[Liu00] Quansheng Liu. On generalized multiplicative cascades. *Stochastic Processes and their Applications*, 86(2):263–286, apr 2000.

[Loo77] J. C. Lootgieter. Sur la répartition des suites de Kakutani. I. *Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques*, 13(4):385–410, 1977.

[LR94] Joel L. Lebowitz and Hermann Rost. The Einstein relation for the displacement of a test particle in a random environment. *Stochastic Processes and their Applications*, 54(2):183–196, dec 1994.

[LS16] Steven P. Lalley and Yuan Shao. Maximal displacement of critical branching symmetric stable processes. *Annales de l'institut Henri Poincare (B) Probability and Statistics*, 52(3):1161–1177, jul 2016.

[LS17] Oren Louidor and Santiago Saglietti. A Strong Law of Large Numbers for Super-critical Branching Brownian Motion with Absorption. *arXiv:1708.08440*, 2017.

[Mad11] Thomas Madaule. Convergence in law for the branching random walk seen from its tip. arXiv:1107.2543, jul 2011.

[Mad15] Thomas Madaule. Maximum of a log-correlated Gaussian field. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 51(4):1369–1431, 2015.

[Mad16] Thomas Madaule. First order transition for the branching random walk at the critical parameter. *Stochastic Processes and their Applications*, 126(2):470–502, feb 2016.

[Mad17] Thomas Madaule. Convergence in Law for the Branching Random Walk Seen from Its Tip. *Journal of Theoretical Probability*, 30(1):27–63, mar 2017.

[Mal15] Bastien Mallein. Maximal displacement of a branching random walk in time-inhomogeneous environment. *Stochastic Processes and their Applications*, 125(10):3958–4019, 2015.

[Mal16] Bastien Mallein. Genealogy of the extremal process of the branching random walk. *arXiv:1606.01748*, pages 1–20, 2016.

[Man74] Benoit B. Mandelbrot. Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier. *Journal of Fluid Mechanics*, 62(02):331–358, 1974.

[MM09] Marc Mézard and Andrea Montanari. *Information, physics, and computation.* Oxford Graduate Texts. Oxford University Press, Oxford, 2009.

[MM14] A. H. Mueller and S. Munier. Phenomenological picture of fluctuations in branching random walks. *Physical Review E*, 90(4):042143, oct 2014.

[MM17] Matthias Meiners and Sebastian Mentemeier. Solutions to complex smoothing equations. *Probability Theory and Related Fields*, 168(1-2):199–268, 2017.

[MMQ10] Carl Mueller, Leonid Mytnik, and Jeremy Quastel. Effect of noise on front propagation in reaction-diffusion equations of KPP type. *Inventiones mathematicae*, 184(2):405–453, nov 2010.

[MN98] T. Mikosch and A. V. Nagaev. Large Deviations of Heavy-Tailed Sums with Applications in Insurance. *Extremes*, 1(1):81–110, jan 1998.

[MP14] Yury Malyshkin and Elliot Paquette. The power of choice combined with preferential attachment. *Electronic Communications in Probability*, 19(44):1–13, 2014.

[MPV87] Marc Mézard, Giorgio Parisi, and Miguel Angel Virasoro. *Spin glass theory and beyond*, volume 9 of *World Scientific Lecture Notes in Physics*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1987.

[MRV15] Thomas Madaule, Rémi Rhodes, and Vincent Vargas. The Glassy Phase of Complex Branching Brownian Motion. *Communications in Mathematical Physics*, 334(3):1157–1187, 2015.

[MS15] Jason Miller and Scott Sheffield. Liouville quantum gravity and the Brownian map I: The QLE(8/3,0) metric. *arXiv:1507.00719*, 2015.

[MS16] Jason Miller and Scott Sheffield. Liouville quantum gravity and the Brownian map II: geodesics and continuity of the embedding. *arXiv:1605.03563*, 2016.

[MS17] Jason Miller and Scott Sheffield. Liouville quantum gravity and the Brownian map III: the conformal structure is determined. *arXiv:1608.05391*, 2017.

[Nag69a] AV Nagaev. Integral limit theorems taking large deviations into account when Cramer's condition does not hold. I. *Theory of Probability and its Applications*, XIV(1):51–64, 1969.

[Nag69b] AV Nagaev. Integral limit theorems taking large deviations into account when Cramer's condition does not hold. II. *Theory of Probability and its Applications*, XIV(2):193–208, 1969.

[Nev88] Jacques Neveu. Multiplicative martingales for spatial branching processes. In *Seminar on Stochastic Processes, 1987 (Princeton, NJ, 1987)*, volume 15 of *Progress in Probability and Statistics*, pages 223–242. Birkhäuser Boston, Boston, MA, 1988.

[NRR14] James Nolen, Jean-Michel Roquejoffre, and Lenya Ryzhik. Power-Like Delay in Time Inhomogeneous Fisher-KPP Equations. *Communications in Partial Differential Equations*, 40(3):475–505, dec 2014.

[NRR16] James Nolen, Jean-Michel Roquejoffre, and Lenya Ryzhik. Refined long time asymptotics for Fisher-KPP fronts. *arXiv:1607.08802*, 2016.

[Ost18] Dmitry Ostrovsky. A Review of Conjectured Laws of Total Mass of Bacry-Muzy GMC Measures on the Interval and Circle and Their Applications. *arXiv:1803.06677*, 2018.

[Pan13a] Dmitry Panchenko. The Parisi ultrametricity conjecture. *Annals of Mathematics*, 177(1):383–393, 2013.

[Pan13b] Dmitry Panchenko. *The Sherrington-Kirkpatrick model*. Springer Monographs in Mathematics. Springer, New York, 2013.

[Pem99] Robin Pemantle. Critical killed branching process tail probabilities. Manuscript, 1999.

[Pem09] Robin Pemantle. Search cost for a nearly optimal path in a binary tree. *The Annals of Applied Probability*, 19(4):1273–1291, aug 2009.

[Pyk80] Ronald Pyke. The Asymptotic Behavior of Spacings Under Kakutani's Model for Interval Subdivision. *The Annals of Probability*, 8(1):157–163, feb 1980.

[PZ08] Yuval Peres and Ofer Zeitouni. A central limit theorem for biased random walks on Galton–Watson trees. *Probability Theory and Related Fields*, 140(3-4):595–629, mar 2008.

[PZ16] Elliot Paquette and Ofer Zeitouni. The maximum of the CUE field. *arXiv:1602.08875*, 2016.

[RA05] Anastasia Ruzmaikina and Michael Aizenman. Characterization of invariant measures at the leading edge for competing particle systems. *The Annals of Probability*, 33(1):82–113, jan 2005.

[Rén59] A. Rényi. On the dimension and entropy of probability distributions. *Acta Mathematica Academiae Scientiarum Hungaricae*, 10(1-2):193–215, 1959.

[Rén70] Alfréd Rényi. *Probability Theory*. Elsevier Science Publishing Co Inc.,U.S., revised edition, 1970.

[Res87] Sidney I. Resnick. *Extreme values, regular variation, and point processes*, volume 4 of *Applied Probability. A Series of the Applied Probability Trust*. Springer-Verlag, New York, 1987.

[Rob15] Matthew I. Roberts. Fine asymptotics for the consistent maximal displacement of branching brownian motion. *Electronic Journal of Probability*, 20:1–26, sep 2015.

[RV10] Raoul Robert and Vincent Vargas. Gaussian multiplicative chaos revisited. *Annals of Probability*, 38(2):605–631, 2010.

[RV14a] Rémi Rhodes and Vincent Vargas. Gaussian multiplicative chaos and applications: A review. *Probability Surveys*, 11(0):315–392, may 2014.

[RV14b] Rémi Rhodes and Vincent Vargas. Spectral Dimension of Liouville Quantum Gravity. *Annales Henri Poincaré*, 15(12):2281–2298, dec 2014.

[RW12] Oliver Riordan and Lutz Warnke. Achlioptas process phase transitions are continuous. *Annals of Applied Probability*, 22(4):1450–1464, 2012.

[Shi11] Zhan Shi. Random Walks and Trees. *ESAIM: Proceedings*, 31:1–39, mar 2011.

[Shi15] Zhan Shi. *Branching random walks*, volume 2151 of *Lecture Notes in Mathematics*. Springer, Cham, 2015.

[Sko64] A V Skorohod. Branching diffusion processes. *Theory Probab. Appl.*, 9:445–449, 1964.

[Slu78] Eric Slud. Entropy and maximal spacings for random partitions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 41(4):341–352, dec 1978.

[SN13] Daniel L Stein and Charles M Newman. *Spin Glasses and Complexity*. Princeton University Press, 2013.

[SW16] Eero Saksman and Christian Webb. The Riemann zeta function and Gaussian multiplicative chaos: statistics on the critical line. *arXiv:1609.00027*, 2016.

[SW17] Quan Shi and Alexander R. Watson. Probability tilting of compensated fragmentations. *arXiv:1707.00732*, 2017.

[SZ15] Eliran Subag and Ofer Zeitouni. Freezing and Decorated Poisson Point Processes. *Communications in Mathematical Physics*, 337(1):55–92, 2015.

[SZ17] Eliran Subag and Ofer Zeitouni. The extremal process of critical points of the pure p-spin spherical spin glass model. *Probability Theory and Related Fields*, 168(3-4):773–820, 2017.

[Tal06] Michel Talagrand. The Parisi formula. *Annals of Mathematics*, 163(1):221–263, jan 2006.

[Tal11a] Michel Talagrand. *Mean field models for spin glasses. Volume I*, volume 54 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2011.

[Tal11b] Michel Talagrand. *Mean field models for spin glasses. Volume II*, volume 55 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Heidelberg, 2011.

[Tra15] Nicholas Travers. Inversions and longest increasing subsequence for k-card-minimum random permutations. *Electronic Journal of Probability*, 20(no. 11):1–28, 2015.

[vZ78] W. R. van Zwet. A Proof of Kakutani's Conjecture on Random Subdivision of Longest Intervals. *The Annals of Probability*, 6(1):133–137, feb 1978.

[Wat93] Yoshiyuki Watabiki. Analytic Study of Fractal Structure of Quantized Surface in Two-Dimensional Quantum Gravity. *Progress of Theoretical Physics Supplement*, 114(0):1–17, 1993.

[Whi97] W. A. Whitworth. *Choice and Chance.* Cambridge University Press, 1897.

[Whi02] Ward Whitt. *Stochastic-process limits.* Springer Series in Operations Research. Springer-Verlag, New York, 2002.

[Zei16] Ofer Zeitouni. Branching random walks and Gaussian fields. In *Probability and statistical physics in St. Petersburg*, volume 91 of *Proc. Sympos. Pure Math.*, pages 437–471. Amer. Math. Soc., Providence, RI, 2016.