Symmetric and non-symmetric variants of Nitsche's method for contact problems in elasticity: theory and numerical experiments

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Abstract

A general Nitsche method, which encompasses symmetric and non-symmetric variants, is proposed for frictionless unilateral contact problems in elasticity. The optimal convergence of the method is established both for two and three-dimensional problems and Lagrange affine and quadratic finite element methods. Two and three-dimensional numerical experiments illustrate the theory.

Key words: unilateral contact, finite elements, Nitsche's method.
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1 Introduction

In solid mechanics, the numerical implementation of contact and impact problems generally uses the Finite Element Method (FEM) (see [28, 18, 21, 19, 31, 40, 38]). The non-linear contact conditions involved on a part of the boundary lead naturally to a variational inequality (see, e.g., [15]).

We consider in this paper a special FEM inspired from Nitsche's method [33] (see also [3] for an early extension to the Discontinuous Galerkin framework) in which the coercivity condition for the domain bilinear form $A_{\theta\gamma}$ (see (8)) is the same as in the standard formulation (see, e.g., [4]). This method aims at treating the boundary or interface conditions in a weak sense, thanks to a consistent penalty term. So it differs from standard penalization techniques which are typically non-consistent [28]. Moreover, unlike mixed methods (see, e.g., [21]), no additional unknown (Lagrange multiplier) is needed. Nitsche's method has been widely applied during these last years to problems involving linear conditions on the boundary of a domain or in the interface between sub-domains: see, e.g,. [36] for the Dirichlet problem or [4] for domain decomposition with non-matching meshes. More recently, in [20, 22] it has been adapted for bilateral (persistent) contact, which still involves linear boundary conditions on the contact zone (note that an algorithm for unilateral contact which involves Nitsche's method in its original form is given and implemented in [20]). An extension to large strain bilateral contact has been performed in [41].

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In a previous work a symmetric Nitsche-based method has been proposed for (non-linear) unilateral contact conditions (see [9]). We established its optimal convergence in the $H^1(\Omega)$-norm of order $O(h^{\frac{1}{2}+\nu})$ for a solution of regularity $H^{\frac{3}{2}+\nu}(\Omega)$ $(0 < \nu \leq \frac{1}{2})$. Furthermore, no additional assumption on the contact zone is needed for the proof, such as an increased regularity of the contact stress or a finite number of transitions between contact and non-contact. Besides, the standard FEM for contact consists in a direct conforming approximation of the variational inequality, with the elastic displacement as the only unknown. For this standard FEM and also for all the other approaches such as mixed/hybrid methods (e.g., [23, 6, 30]), stabilized mixed methods (e.g., [25]), penalty methods (e.g., [10]), no such proof of optimal convergence has been established to the best of our knowledge in the case the solution $\mathbf{u}$ is in $H^{\frac{3}{2}+\nu}(\Omega)$ $(0 < \nu \leq \frac{1}{2})$: see e.g. [27, 26] for recent reviews on a priori error estimates.

In the present paper we introduce an extension of the symmetric method of [9], which allows nonsymmetric variants, depending upon a new parameter called θ. The symmetric case is recovered when $\theta = 1$. In this new method the advantage of the symmetric formulation consisting in the positivity of the Nitsche penalty term is generally lost. Nevertheless this extension presents some other advantages, mostly from the numerical viewpoint. In particular, one of its variants $(\theta = 0)$ involves a reduced quantity of terms, which makes it easier to implement and to extend to contact problems involving non-linear elasticity. Also, for $\theta = -1$, the well-posedness of the discrete formulation and the optimal convergence are preserved irrespectively of the value of the Nitsche parameter. For all the values of θ, the optimal convergence rates can still be obtained. We prove this optimal convergence property and illustrate it on several numerical experiments. Note that the behavior of the generalized Newton algorithm when applied to this method has already been studied in [35], in the frictionless and frictional cases and for two values of θ (0 and 1). This study illustrated the better numerical performances of the nonsymmetric variant $\theta = 0$, which requires less Newton iterations to converge, for a wider range of the Nitsche parameter. Values of θ different from 0 and 1 have not been studied previously.

Our paper is outlined as follows. In Section 2, we recall the continuous (strong and weak) formulations for unilateral contact problems and introduce our Nitsche-based FEM. In Section 3, we carry out the numerical analysis of this method: we prove its consistency, the existence and uniqueness of solutions, and at last its optimal convergence. Numerical experiments in 2D and 3D are described in Section 4, which illustrate the convergence properties for different values of the numerical parameters, and in particular θ. In Section 5 conclusions are drawn and some perspectives are given.

Let us introduce some useful notations. In what follows, bold letters like $\mathbf{u}, \mathbf{v}$, indicate vector or tensor valued quantities, while the capital ones (e.g., $\mathbf{V}, \mathbf{K} \ldots$) represent functional sets involving vector fields. As usual, we denote by $(H^s(.))^d$, $s \in \mathbb{R}, d = 1, 2, 3$, the Sobolev spaces in one, two or three space dimensions (see [1]). The usual norm of $(H^s(D))^d$ is denoted by $\| \cdot \|_{s,D}$ and we keep the same notation when $d = 1$ or $d > 1$. The letter C stands for a generic constant, independent of the discretization parameters.

2 Setting

2.1 The unilateral contact problem

We consider an elastic body whose reference configuration is represented by the domain Ω in $\mathbb{R}^d$ with $d = 2$ or $d = 3$. Small strain assumptions are made, as well as plane strain when $d = 2$. The boundary $\partial\Omega$ of Ω is polygonal or polyhedral and we suppose that $\partial\Omega$ consists

in three nonoverlapping parts Γ_D, Γ_N and the contact boundary Γ_C, with $\text{meas}(\Gamma_D) > 0$ and $\text{meas}(\Gamma_C) > 0$. The contact boundary is supposed to be a straight line segment when $d = 2$ or a polygon when $d = 3$ to simplify. The normal unit outward vector on $\partial\Omega$ is denoted $\mathbf{n}$. In its initial stage, the body is in contact on Γ_C with a rigid foundation (the extension to two elastic bodies in contact can be easily made, at least for small strain models) and we suppose that the unknown final contact zone after deformation will be included into Γ_C. The body is clamped on Γ_D for the sake of simplicity. It is subjected to volume forces $\mathbf{f} \in (L^2(\Omega))^d$ and to surface loads $\mathbf{g} \in (L^2(\Gamma_N))^d$.
The unilateral contact problem in linear elasticity consists in finding the displacement field $\mathbf{u} : \Omega \to \mathbb{R}^d$ verifying the equations and conditions (1)–(2):

$$\begin{aligned} \mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} & \qquad \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{A}\,\boldsymbol{\varepsilon}(\mathbf{u}) & \qquad \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \qquad \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{g} & \qquad \text{on } \Gamma_N, \end{aligned} \tag{1}$$

where $\boldsymbol{\sigma} = (\sigma_{ij})$, $1 \le i, j \le d$, stands for the stress tensor field and $\mathbf{div}$ denotes the divergence operator of tensor valued functions. The notation $\boldsymbol{\varepsilon}(\mathbf{v}) = (\boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\mathbf{v}^T)/2$ represents the linearized strain tensor field and $\mathbf{A}$ is the fourth order symmetric elasticity tensor having the usual uniform ellipticity and boundedness property. For any displacement field $\mathbf{v}$ and for any density of surface forces $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$ defined on $\partial\Omega$ we adopt the following notation

$$\mathbf{v} = v_n\mathbf{n} + v_{\mathbf{t}} \quad \text{ and } \quad \boldsymbol{\sigma}(\mathbf{v})\mathbf{n} = \sigma_n(\mathbf{v})\mathbf{n} + \sigma_{\mathbf{t}}(\mathbf{v}),$$

where $v_{\mathbf{t}}$ (resp. $\sigma_{\mathbf{t}}(\mathbf{v})$) are the tangential components of $\mathbf{v}$ (resp. $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$). The conditions describing unilateral contact without friction on Γ_C are:

$$\begin{aligned} u_n \le\ & 0, \quad (i) \\ \sigma_n(\mathbf{u}) \le\ & 0, \quad (ii) \\ \sigma_n(\mathbf{u})\, u_n =\ & 0, \quad (iii) \\ \sigma_{\mathbf{t}}(\mathbf{u}) =\ & 0. \quad (iv) \end{aligned} \tag{2}$$

We introduce the Hilbert space $\mathbf{V}$ and the convex cone $\mathbf{K}$ of admissible displacements which satisfy the noninterpenetration on the contact zone Γ_C:

$$\mathbf{V} := \left\{ \mathbf{v} \in \left(H^1(\Omega)\right)^d \ : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \right\}, \quad \mathbf{K} := \{ \mathbf{v} \in \mathbf{V} \ : \ v_n = \mathbf{v}\cdot\mathbf{n} \le 0 \text{ on } \Gamma_C \}.$$

Define

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\, d\Omega, \qquad L(\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\Omega + \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v}\, d\Gamma,$$

for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}$. From the previous assumptions, we deduce that $a(\cdot,\cdot)$ is bilinear, symmetric, $\mathbf{V}$-elliptic and continuous on $\mathbf{V}\times\mathbf{V}$. We see also that $L(\cdot)$ is a continuous linear form on $\mathbf{V}$. The weak formulation of Problem (1)-(2), as a variational inequality (see [15, 21, 28]), reads as:

$$\left\{ \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{K} \text{ such that:} \\ a(\mathbf{u}, \mathbf{v}-\mathbf{u}) \ge L(\mathbf{v}-\mathbf{u}), \qquad \forall\, \mathbf{v} \in \mathbf{K}. \end{array} \right. \tag{3}$$

Stampacchia's Theorem ensures that Problem (3) admits a unique solution.

2.2 A general Nitsche-based finite element method

Let $\mathbf{V}^h \subset \mathbf{V}$ be a family of finite dimensional vector spaces (see [11, 14, 7]) indexed by h coming from a family $\mathcal{T}^h$ of triangulations of the domain Ω ($h = \max_{T\in\mathcal{T}^h} h_T$ where h_T is the diameter of T). The family of triangulations is supposed regular (i.e., there exists $\sigma > 0$ such that $\forall T \in \mathcal{T}^h, h_T/\rho_T \leq \sigma$ where ρ_T denotes the radius of the inscribed ball in T) and conformal to the subdivision of the boundary into Γ_D, Γ_N and Γ_C (i.e., a face of an element $T \in \mathcal{T}^h$ is not allowed to have simultaneous non-empty intersection with more than one part of the subdivision). To fix ideas, we choose a standard Lagrange finite element method of degree k with $k = 1$ or $k = 2$, i.e.:

$$\mathbf{V}^h := \left\{ \mathbf{v}^h \in (\mathscr{C}^0(\overline{\Omega}))^d : \mathbf{v}^h{}_{|T} \in (P_k(T))^d, \forall T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D \right\}. \tag{4}$$

However, the analysis would be similar for any $\mathscr{C}^0$-conforming finite element method.
Let us introduce the notation $[\cdot]_+$ for the positive part of a scalar quantity $a \in \mathbb{R}$:

$$[a]_+ := \begin{cases} a & \text{if } a > 0, \\ 0 & \text{otherwise.} \end{cases}$$

In the following, we will use the properties:

$$a \leq [a]_+, \quad a[a]_+ = [a]_+^2, \quad \forall a \in \mathbb{R}. \tag{5}$$

Note that conditions (5) can be straightforwardly extended to real valued functions. The derivation of a Nitsche-based method comes from a classical reformulation (see for instance [2]) of the contact conditions (2), for a given $\gamma > 0$:

$$\sigma_n(\mathbf{u}) = -\frac{1}{\gamma}[u_n - \gamma\sigma_n(\mathbf{u})]_+. \tag{6}$$

Remark 2.1. *Note that condition (6) is still equivalent to* (2) *(i)-(iii) on Γ_C when γ is a positive function defined on Γ_C instead of a positive constant.*

Let now $\theta \in \mathbb{R}$ be a fixed parameter, and let $\mathbf{u}$ be the solution of the unilateral contact problem in its strong form (1)–(2). We assume that $\mathbf{u}$ is sufficiently regular so that all the following calculations make sense. From the Green formula and equations (1) and (2)-(iv), we get for every $\mathbf{v} \in \mathbf{V}$:

$$a(\mathbf{u}, \mathbf{v}) - \int_{\Gamma_C} \sigma_n(\mathbf{u})\, v_n \, d\Gamma = L(\mathbf{v}).$$

Note that $v_n = (v_n - \theta\gamma\sigma_n(\mathbf{v})) + \theta\gamma\sigma_n(\mathbf{v})$. So:

$$a(\mathbf{u}, \mathbf{v}) - \int_{\Gamma_C} \theta\gamma\, \sigma_n(\mathbf{u})\, \sigma_n(\mathbf{v}) \, d\Gamma - \int_{\Gamma_C} \sigma_n(\mathbf{u})\, (v_n - \theta\gamma\sigma_n(\mathbf{v}))\, d\Gamma = L(\mathbf{v}).$$

Finally, using condition (6), we obtain:

$$a(\mathbf{u}, \mathbf{v}) - \int_{\Gamma_C} \theta\gamma\, \sigma_n(\mathbf{u})\, \sigma_n(\mathbf{v}) \, d\Gamma + \int_{\Gamma_C} \frac{1}{\gamma}[u_n - \gamma\sigma_n(\mathbf{u})]_+ (v_n - \theta\gamma\sigma_n(\mathbf{v}))\, d\Gamma = L(\mathbf{v}). \tag{7}$$

Formula (7) is the starting point of our Nitsche-based formulation. We remark that it may have no sense at the continuous level if $\mathbf{u}$ lacks of regularity (the only assumption $\mathbf{u} \in \mathbf{V}$ is not sufficient to justify the above calculations). Nevertheless, and as in the stabilized Lagrange multiplier method

[25], we consider in what follows that γ is a positive piecewise constant function on the contact interface Γ_C : for any $x \in \Gamma_C$, let T be an element such that $x \in T$ and set

$$\gamma(x) = \gamma_0 h_T,$$

where γ_0 is a positive constant. This allows to define a discrete counterpart of (7). Let us introduce for this purpose the discrete linear operator

$$P_\gamma : \begin{array}{ccc} \mathbf{V}^h & \to & L^2(\Gamma_C) \\ \mathbf{v}^h & \mapsto & v_n^h - \gamma\,\sigma_n(\mathbf{v}^h) \end{array},$$

and also the bilinear form:

$$A_{\theta\gamma}(\mathbf{u}^h, \mathbf{v}^h) := a(\mathbf{u}^h, \mathbf{v}^h) - \int_{\Gamma_C} \theta\gamma\,\sigma_n(\mathbf{u}^h)\sigma_n(\mathbf{v}^h)\,d\Gamma. \tag{8}$$

Remark 2.2. *With the previous notations, we see that problem (3) could be formally written as follows :*

$$\begin{cases} \text{Find a sufficiently regular } \mathbf{u} \in \mathbf{V} \text{ such that:} \\ A_{\theta\gamma}(\mathbf{u}, \mathbf{v}) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma}\,[P_\gamma(\mathbf{u})]_+ P_{\theta\gamma}(\mathbf{v})\,d\Gamma = L(\mathbf{v}), \text{ for all sufficiently regular } \mathbf{v} \in \mathbf{V}, \end{cases}$$

with $P_{\theta\gamma}(\mathbf{v}) := v_n - \theta\gamma\,\sigma_n(\mathbf{v})$.

Our generalized Nitsche-based method then reads:

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that:} \\ A_{\theta\gamma}(\mathbf{u}^h, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma}\,[P_\gamma(\mathbf{u}^h)]_+ P_{\theta\gamma}(\mathbf{v}^h)\,d\Gamma = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases} \tag{9}$$

Remark 2.3. *The introduction of an additional numerical parameter θ allows us to introduce some new interesting variants acting on the symmetry / skew-symmetry / non-symmetry of the discrete formulation. Moreover, a unified analysis of all these variants can be performed. Note that some values of θ may be of special interest:*

- *for $\theta = 1$ we recover the symmetric method proposed and analyzed in [9],*
- *for $\theta = 0$ we recover a very simple non-symmetric version close to the classical penalty method,*
- *for $\theta = -1$ we obtain a skew-symmetric version which has the remarkable property to be well-posed and convergent irrespectively of the value of $\gamma_0 > 0$ (see Theorem 3.4 and Theorem 3.8).*

Remark 2.4. *As in the symmetric case [9], we can rewrite the formulation* (9) *in a mixed form as follows:*

$$\begin{cases} \text{Find } (\mathbf{u}^h, \lambda^h) \in \mathbf{V}^h \times L^2_-(\Gamma_C) \text{ such that:} \\ a(\mathbf{u}^h, \mathbf{v}^h) - \displaystyle\int_{\Gamma_C} \lambda^h v_n^h\,d\Gamma + \int_{\Gamma_C} \theta\gamma(\lambda^h - \sigma_n(\mathbf{u}^h))\,\sigma_n(\mathbf{v}^h)\,d\Gamma = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ \displaystyle\int_{\Gamma_C} (\mu - \lambda^h) u_n^h\,d\Gamma + \int_{\Gamma_C} \gamma(\mu - \lambda^h)(\lambda^h - \sigma_n(\mathbf{u}^h))\,d\Gamma \geq 0, \quad \forall\, \mu \in L^2_-(\Gamma_C), \end{cases}$$

where $L^2_-(\Gamma_C) := \{\mu \in L^2(\Gamma_C) \mid \mu \leq 0$ a.e. on $\Gamma_C\}$ and with $\lambda^h := -\frac{1}{\gamma}[P_\gamma(\mathbf{u}^h)]_+$. Note that formally, in the case $\theta \neq 1$, this mixed form is somehow different from the stabilized method applied to unilateral contact (see [25]). In particular the stabilization term in the first equation vanishes when $\theta = 0$.

3 Analysis of the Nitsche-based method

In this section, we carry out the mathematical analysis of the method (9). A difference between Nitsche's method and classical penalty methods [28, 10] is the property of consistency, which we first show in §3.1. The proof of the well-posedness of the (non-linear) discrete problem (9) is carried out in §3.2. The error analysis is finally detailed in §3.3. We show that the method converges in an optimal way when the mesh size h vanishes.

3.1 Consistency

Like Nitsche's method for second order elliptic problems with Dirichlet boundary conditions or domain decomposition [4], our Nitsche-based formulation (9) for unilateral contact is consistent:

Lemma 3.1. *The Nitsche-based method for contact is consistent: suppose that the solution* $\mathbf{u}$ *of* (1)–(2) *lies in* $(H^{\frac{3}{2}+\nu}(\Omega))^d$ *with* $\nu > 0$ *and* $d = 2, 3$. *Then* $\mathbf{u}$ *is also solution of*

$$A_{\theta\gamma}(\mathbf{u}, \mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\gamma} [P_\gamma(\mathbf{u})]_+ P_{\theta\gamma}(\mathbf{v}^h)\, d\Gamma = L(\mathbf{v}^h), \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h.$$

Proof: Let $\mathbf{u}$ be the solution of (1)–(2) and set $\mathbf{v}^h \in \mathbf{V}^h$. Since $\mathbf{u} \in (H^{\frac{3}{2}+\nu}(\Omega))^d$ and $\nu > 0$, we have $\sigma_n(\mathbf{u}) \in H^\nu(\Gamma_C) \subset L^2(\Gamma_C)$. As a result, $A_{\theta\gamma}(\mathbf{u}, \mathbf{v}^h)$ makes sense and $P_\gamma(\mathbf{u}) \in L^2(\Gamma_C)$. On the one hand, we use the definition of P_γ, of $A_{\theta\gamma}(\cdot, \cdot)$ and the reformulation (6) of the contact conditions to obtain:

$$\begin{aligned}
&A_{\theta\gamma}(\mathbf{u}, \mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\gamma} [P_\gamma(\mathbf{u})]_+ P_{\theta\gamma}(\mathbf{v}^h)\, d\Gamma \\
&= a(\mathbf{u}, \mathbf{v}^h) - \int_{\Gamma_C} \theta\gamma\, \sigma_n(\mathbf{u})\sigma_n(\mathbf{v}^h)\, d\Gamma + \int_{\Gamma_C} \frac{1}{\gamma} (-\gamma\sigma_n(\mathbf{u}))(v_n^h - \theta\gamma\, \sigma_n(\mathbf{v}^h))\, d\Gamma \\
&= a(\mathbf{u}, \mathbf{v}^h) - \int_{\Gamma_C} \sigma_n(\mathbf{u}) v_n^h\, d\Gamma.
\end{aligned}$$

On the other hand, with equations (1)–(2) and an integration by parts, we get:

$$a(\mathbf{u}, \mathbf{v}^h) - \int_{\Gamma_C} \sigma_n(\mathbf{u}) v_n^h\, d\Gamma = L(\mathbf{v}^h),$$

which ends the proof. □

3.2 Well-posedness

To prove well-posedness of our Nitsche-based formulation, we first need the following classical property.

Lemma 3.2. *There exists $C > 0$, independent of the parameter γ_0 and of the mesh size h, such that:*

$$\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h)\|^2_{0,\Gamma_C} \leq C\gamma_0\|\mathbf{v}^h\|^2_{1,\Omega}, \tag{10}$$

for all $\mathbf{v}^h \in \mathbf{V}^h$.

Proof: It follows from the definition of $\sigma_n(\mathbf{v}^h)$ and the boundedness of $\mathbf{A}$ that:

$$\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h)\|^2_{0,\Gamma_C} \leq \gamma_0 h\|\sigma_n(\mathbf{v}^h)\|^2_{0,\Gamma_C} \leq C\gamma_0 h\|\nabla\mathbf{v}^h\|^2_{0,\Gamma_C}.$$

Then estimation (10) is obtained using a scaling argument: see [37, Lemma 2.1, p.24] for a detailed proof in the general case (for an arbitrary degree k and any dimension d). □

Remark 3.3. *Note that contrary to the penalty case or to the symmetric case of Nitsche's method (i.e., $\theta = 1$), the integral term on Γ_C:*

$$\int_{\Gamma_C} \frac{1}{\gamma}\,[P_\gamma(\mathbf{u}^h)]_+ P_{\theta\gamma}(\mathbf{v}^h)$$

is not necessarily non-negative when $\mathbf{v}^h = \mathbf{u}^h$ and $\theta \neq 1$.

We then show that Problem (9) is well-posed using an argument by Brezis for M-type and pseudo-monotone operators [8] (see also [32] and [29]).

Theorem 3.4. *Suppose that one of the two following assumptions hold:*

1. *$\theta \neq -1$ and $\gamma_0 > 0$ is sufficiently small,*
2. *$\theta = -1$ and $\gamma_0 > 0$.*

Then Problem (9) admits one unique solution $\mathbf{u}^h$ in $\mathbf{V}^h$.

Proof: Using the Riesz representation theorem, we define a (non-linear) operator $\mathbf{B}^h : \mathbf{V}^h \to \mathbf{V}^h$, by means of the formula:

$$(\mathbf{B}^h\mathbf{v}^h, \mathbf{w}^h)_{1,\Omega} := A_{\theta\gamma}(\mathbf{v}^h, \mathbf{w}^h) + \int_{\Gamma_C} \frac{1}{\gamma}\,[P_\gamma(\mathbf{v}^h)]_+ P_{\theta\gamma}(\mathbf{w}^h)\, d\Gamma,$$

for all $\mathbf{v}^h, \mathbf{w}^h \in \mathbf{V}^h$, and where $(\cdot,\cdot)_{1,\Omega}$ stands for the scalar product in $(H^1(\Omega))^d$. Note that Problem (9) is well-posed if and only if $\mathbf{B}^h$ is a one-to-one operator.
Let $\mathbf{v}^h, \mathbf{w}^h \in \mathbf{V}^h$. We have:

$$\begin{aligned}
(\mathbf{B}^h\mathbf{v}^h - \mathbf{B}^h\mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h)_{1,\Omega} &= a(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) - \theta\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h - \mathbf{w}^h)\|^2_{0,\Gamma_C} \\
&\quad + \int_{\Gamma_C} \frac{1}{\gamma}\,([P_\gamma(\mathbf{v}^h)]_+ - [P_\gamma(\mathbf{w}^h)]_+)(v^h_n - w^h_n - \theta\gamma\sigma_n(\mathbf{v}^h - \mathbf{w}^h))\, d\Gamma \\
&= a(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) - \theta\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h - \mathbf{w}^h)\|^2_{0,\Gamma_C} \\
&\quad + \int_{\Gamma_C} \frac{1}{\gamma}\,([P_\gamma(\mathbf{v}^h)]_+ - [P_\gamma(\mathbf{w}^h)]_+)P_\gamma(\mathbf{v}^h - \mathbf{w}^h)\, d\Gamma \\
&\quad + (1-\theta)\int_{\Gamma_C} \frac{1}{\gamma}\,([P_\gamma(\mathbf{v}^h)]_+ - [P_\gamma(\mathbf{w}^h)]_+)\gamma\sigma_n(\mathbf{v}^h - \mathbf{w}^h)\, d\Gamma.
\end{aligned} \tag{11}$$

Due to the properties (5), we observe that, for all $a, b \in \mathbb{R}$:

$$\begin{aligned}([a]_+ - [b]_+)(a-b) &= a[a]_+ + b[b]_+ - b[a]_+ - a[b]_+ \\ &\geq [a]_+^2 + [b]_+^2 - 2[a]_+[b]_+ \\ &= ([a]_+ - [b]_+)^2 \geq 0.\end{aligned} \tag{12}$$

Using the latter inequality in (11) and Cauchy-Schwarz inequality, we get

$$\begin{aligned}&(\mathbf{B}^h\mathbf{v}^h - \mathbf{B}^h\mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h)_{1,\Omega} \\ &\geq a(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) - \theta\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h - \mathbf{w}^h)\|_{0,\Gamma_C}^2 \\ &\quad + \|\gamma^{-\frac{1}{2}}([P_\gamma(\mathbf{v}^h)]_+ - [P_\gamma(\mathbf{w}^h)]_+)\|_{0,\Gamma_C}^2 \\ &\quad - |1-\theta|\ \|\gamma^{-\frac{1}{2}}([P_\gamma(\mathbf{v}^h)]_+ - [P_\gamma(\mathbf{w}^h)]_+)\|_{0,\Gamma_C}\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h - \mathbf{w}^h)\|_{0,\Gamma_C}.\end{aligned} \tag{13}$$

If $\theta = 1$, we use the coercivity of $a(\cdot,\cdot)$ and the property (10) in the previous expression (13). Therefore:

$$\begin{aligned}(\mathbf{B}^h\mathbf{v}^h - \mathbf{B}^h\mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h)_{1,\Omega} &\geq a(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) - \|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h - \mathbf{w}^h)\|_{0,\Gamma_C}^2 \\ &\geq C\|\mathbf{v}^h - \mathbf{w}^h\|_{1,\Omega}^2,\end{aligned} \tag{14}$$

when γ_0 is chosen sufficiently small. This corresponds to the symmetric version already studied in [9].
We now suppose that $\theta \neq 1$. Let $\beta > 0$. Applying Young inequality in (13) yields:

$$\begin{aligned}&(\mathbf{B}^h\mathbf{v}^h - \mathbf{B}^h\mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h)_{1,\Omega} \\ &\geq a(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) - \theta\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h - \mathbf{w}^h)\|_{0,\Gamma_C}^2 \\ &\quad + \|\gamma^{-\frac{1}{2}}([P_\gamma(\mathbf{v}^h)]_+ - [P_\gamma(\mathbf{w}^h)]_+)\|_{0,\Gamma_C}^2 \\ &\quad - \frac{|1-\theta|}{2\beta}\|\gamma^{-\frac{1}{2}}([P_\gamma(\mathbf{v}^h)]_+ - [P_\gamma(\mathbf{w}^h)]_+)\|_{0,\Gamma_C}^2 - \frac{|1-\theta|\beta}{2}\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h - \mathbf{w}^h)\|_{0,\Gamma_C}^2 \\ &= a(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) - \left(\theta + \frac{|1-\theta|\beta}{2}\right)\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h - \mathbf{w}^h)\|_{0,\Gamma_C}^2 \\ &\quad + \left(1 - \frac{|1-\theta|}{2\beta}\right)\|\gamma^{-\frac{1}{2}}([P_\gamma(\mathbf{v}^h)]_+ - [P_\gamma(\mathbf{w}^h)]_+)\|_{0,\Gamma_C}^2.\end{aligned} \tag{15}$$

Choosing $\beta = |1-\theta|/2$ in (15), we get:

$$\begin{aligned}(\mathbf{B}^h\mathbf{v}^h - \mathbf{B}^h\mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h)_{1,\Omega} &\geq a(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) - \frac{1}{4}(1+\theta)^2\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h - \mathbf{w}^h)\|_{0,\Gamma_C}^2 \\ &\geq C\|\mathbf{v}^h - \mathbf{w}^h\|_{1,\Omega}^2,\end{aligned} \tag{16}$$

when $\theta \neq -1$ and γ_0 sufficiently small, or when $\theta = -1$. Note that in the latter case we do not need the smallness assumption on γ_0.
Next, let us show that $\mathbf{B}^h$ is also hemicontinuous. Since $\mathbf{V}^h$ is a vector space, it is sufficient to show that

$$[0,1] \ni t \mapsto \varphi(t) := (\mathbf{B}^h(\mathbf{v}^h - t\mathbf{w}^h), \mathbf{w}^h)_{1,\Omega} \in \mathbb{R}$$

is a continuous real function, for all $\mathbf{v}^h, \mathbf{w}^h \in \mathbf{V}^h$. Let $s, t \in [0,1]$, we have:

$$
\begin{aligned}
& |\varphi(t) - \varphi(s)| \\
= \; & |(\mathbf{B}^h(\mathbf{v}^h - t\mathbf{w}^h) - \mathbf{B}^h(\mathbf{v}^h - s\mathbf{w}^h), \mathbf{w}^h)_{1,\Omega}| \\
= \; & \left| A_{\theta\gamma}((s-t)\mathbf{w}^h, \mathbf{w}^h) + \int_{\Gamma_C} \frac{1}{\gamma} \Big([P_\gamma(\mathbf{v}^h - t\mathbf{w}^h)]_+ - [P_\gamma(\mathbf{v}^h - s\mathbf{w}^h)]_+ \Big) P_{\theta\gamma}(\mathbf{w}^h) \, d\Gamma \right| \\
\leq \; & |s-t| \, A_{\theta\gamma}(\mathbf{w}^h, \mathbf{w}^h) + \int_{\Gamma_C} \frac{1}{\gamma} \Big| [P_\gamma(\mathbf{v}^h - t\mathbf{w}^h)]_+ - [P_\gamma(\mathbf{v}^h - s\mathbf{w}^h)]_+ \Big| \, |P_{\theta\gamma}(\mathbf{w}^h)| \, d\Gamma.
\end{aligned}
$$

With help of the bound $|[a]_+ - [b]_+| \leq |a-b|$, for all $a, b \in \mathbb{R}$, and using the linearity of P_γ, we deduce that:

$$
\begin{aligned}
& \int_{\Gamma_C} \frac{1}{\gamma} \Big| [P_\gamma(\mathbf{v}^h - t\mathbf{w}^h)]_+ - [P_\gamma(\mathbf{v}^h - s\mathbf{w}^h)]_+ \Big| \, |P_{\theta\gamma}(\mathbf{w}^h)| \, d\Gamma \\
\leq \; & \int_{\Gamma_C} \frac{1}{\gamma} \Big| P_\gamma(\mathbf{v}^h - t\mathbf{w}^h) - P_\gamma(\mathbf{v}^h - s\mathbf{w}^h) \Big| \, |P_{\theta\gamma}(\mathbf{w}^h)| \, d\Gamma \\
= \; & \int_{\Gamma_C} \frac{1}{\gamma} \, |(s-t) P_\gamma(\mathbf{w}^h)| |P_{\theta\gamma}(\mathbf{w}^h)| \, d\Gamma.
\end{aligned}
$$

It results that:

$$
|\varphi(t) - \varphi(s)| \leq |s-t| \left(A_{\theta\gamma}(\mathbf{w}^h, \mathbf{w}^h) + \int_{\Gamma_C} \frac{1}{\gamma} \, |P_\gamma(\mathbf{w}^h)| |P_{\theta\gamma}(\mathbf{w}^h)| \, d\Gamma \right),
$$

which means that φ is Lipschitz, so that $\mathbf{B}^h$ is hemicontinuous. Since properties (14), (16) also hold, we finally apply the Corollary 15 (p.126) of [8] to conclude that $\mathbf{B}^h$ is a one-to-one operator. This ends the proof. □

Remark 3.5. *When γ_0 is large and $\theta \neq -1$ we can neither conclude to uniqueness, nor to existence of a solution. In Appendix B, we show some simple explicit examples of nonexistence and nonuniqueness of solutions.*

3.3 A priori error analysis

Our Nitsche-based method (9) converges in a optimal way as the mesh parameter h vanishes. This is proved in the following theorems. First, we establish an abstract error estimate.

Theorem 3.6. *Suppose that the solution $\mathbf{u}$ of Problem (3) belongs to $(H^{\frac{3}{2}+\nu}(\Omega))^d$ with $\nu > 0$ and $d = 2$ or $d = 3$.*
1. Let $\theta \in \mathbb{R}$. Suppose that the parameter $\gamma_0 > 0$ is sufficiently small. Then, the solution $\mathbf{u}^h$ of Problem (9) satisfies the following abstract error estimate:

$$
\begin{aligned}
& \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + \|\gamma^{\frac{1}{2}} (\sigma_n(\mathbf{u}) + \frac{1}{\gamma} [P_\gamma(\mathbf{u}^h)]_+)\|_{0,\Gamma_C} \\
\leq \; & C \inf_{\mathbf{v}^h \in \mathbf{V}^h} \Big(\|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega} + \|\gamma^{-\frac{1}{2}} (u_n - v_n^h)\|_{0,\Gamma_C} + \|\gamma^{\frac{1}{2}} \sigma_n(\mathbf{u} - \mathbf{v}^h)\|_{0,\Gamma_C} \Big),
\end{aligned}
\tag{17}
$$

where C is a positive constant, independent of h, $\mathbf{u}$ and γ_0.
2. Set $\theta = -1$. Then for all values of $\gamma_0 > 0$, the solution $\mathbf{u}^h$ of Problem (9) satisfies the abstract error estimate (17) where C is a positive constant, dependent of γ_0 but independent of h and $\mathbf{u}$.

Proof: Let $\mathbf{v}^h \in \mathbf{V}^h$. We first use the $\mathbf{V}$-ellipticity and the continuity of $a(\cdot,\cdot)$, as well as Young's inequality, to obtain:

$$\begin{aligned}\alpha\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}^2 &\le a(\mathbf{u}-\mathbf{u}^h,\mathbf{u}-\mathbf{u}^h)\\ &= a(\mathbf{u}-\mathbf{u}^h,(\mathbf{u}-\mathbf{v}^h)+(\mathbf{v}^h-\mathbf{u}^h))\\ &\le C\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega}+a(\mathbf{u}-\mathbf{u}^h,\mathbf{v}^h-\mathbf{u}^h)\\ &\le \frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}^2+\frac{C^2}{2\alpha}\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega}^2+a(\mathbf{u},\mathbf{v}^h-\mathbf{u}^h)-a(\mathbf{u}^h,\mathbf{v}^h-\mathbf{u}^h),\end{aligned}$$

where $\alpha > 0$ is the ellipticity constant of $a(.,.)$. We can transform the term $a(\mathbf{u},\mathbf{v}^h-\mathbf{u}^h) - a(\mathbf{u}^h,\mathbf{v}^h-\mathbf{u}^h)$ since $\mathbf{u}$ solves (3), $\mathbf{u}^h$ solves (9) and using Lemma 3.1, which yields:

$$\begin{aligned}\frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}^2 \le& \frac{C^2}{2\alpha}\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega}^2-\theta\int_{\Gamma_C}\gamma\,\sigma_n(\mathbf{u}^h-\mathbf{u})\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\,d\Gamma\\ &+\int_{\Gamma_C}\frac{1}{\gamma}\Big([P_\gamma(\mathbf{u}^h)]_+-[P_\gamma(\mathbf{u})]_+\Big)P_{\theta\gamma}(\mathbf{v}^h-\mathbf{u}^h)\,d\Gamma.\end{aligned}\tag{18}$$

The first integral term in (18) is bounded as follows:

$$\begin{aligned}&-\theta\int_{\Gamma_C}\gamma\,\sigma_n(\mathbf{u}^h-\mathbf{u})\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\,d\Gamma\\ &=\theta\int_{\Gamma_C}\gamma\,\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\,d\Gamma-\theta\int_{\Gamma_C}\gamma\,\sigma_n(\mathbf{v}^h-\mathbf{u})\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\,d\Gamma\\ &\le\theta\|\gamma^{\frac12}\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\|_{0,\Gamma_C}^2+|\theta|\|\gamma^{\frac12}\sigma_n(\mathbf{v}^h-\mathbf{u})\|_{0,\Gamma_C}\|\gamma^{\frac12}\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\|_{0,\Gamma_C}\\ &\le\frac{\beta_1\theta^2}{2}\|\gamma^{\frac12}\sigma_n(\mathbf{v}^h-\mathbf{u})\|_{0,\Gamma_C}^2+\left(\frac{1}{2\beta_1}+\theta\right)\|\gamma^{\frac12}\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\|_{0,\Gamma_C}^2,\end{aligned}\tag{19}$$

with $\beta_1 > 0$. The second integral term in (18) is considered next:

$$\begin{aligned}&\int_{\Gamma_C}\frac{1}{\gamma}\Big([P_\gamma(\mathbf{u}^h)]_+-[P_\gamma(\mathbf{u})]_+\Big)P_{\theta\gamma}(\mathbf{v}^h-\mathbf{u}^h)\,d\Gamma\\ =&\int_{\Gamma_C}\frac{1}{\gamma}\Big([P_\gamma(\mathbf{u}^h)]_+-[P_\gamma(\mathbf{u})]_+\Big)P_\gamma(\mathbf{v}^h-\mathbf{u})\,d\Gamma+\int_{\Gamma_C}\frac{1}{\gamma}\Big([P_\gamma(\mathbf{u}^h)]_+-[P_\gamma(\mathbf{u})]_+\Big)P_\gamma(\mathbf{u}-\mathbf{u}^h)\,d\Gamma\\ &+\int_{\Gamma_C}\frac{1}{\gamma}\Big([P_\gamma(\mathbf{u}^h)]_+-[P_\gamma(\mathbf{u})]_+\Big)\gamma(1-\theta)\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\,d\Gamma.\end{aligned}\tag{20}$$

Using the bound $([a]_+-[b]_+)(b-a)\le-([a]_+-[b]_+)^2$ (see (12)) and two times Cauchy-Schwarz and Young's inequalities in (20) we obtain

$$\begin{aligned}&\int_{\Gamma_C}\frac{1}{\gamma}\Big([P_\gamma(\mathbf{u}^h)]_+-[P_\gamma(\mathbf{u})]_+\Big)P_{\theta\gamma}(\mathbf{v}^h-\mathbf{u}^h)\,d\Gamma\\ \le&\frac{1}{2\beta_2}\|\gamma^{\frac12}(\sigma_n(\mathbf{u})+\frac{1}{\gamma}[P_\gamma(\mathbf{u}^h)]_+)\|_{0,\Gamma_C}^2+\frac{\beta_2}{2}\|\gamma^{-\frac12}P_\gamma(\mathbf{v}^h-\mathbf{u})\|_{0,\Gamma_C}^2-\|\gamma^{\frac12}(\sigma_n(\mathbf{u})+\frac{1}{\gamma}[P_\gamma(\mathbf{u}^h)]_+)\|_{0,\Gamma_C}^2\\ &+\frac{|1-\theta|}{2\beta_3}\|\gamma^{\frac12}(\sigma_n(\mathbf{u})+\frac{1}{\gamma}[P_\gamma(\mathbf{u}^h)]_+)\|_{0,\Gamma_C}^2+\frac{|1-\theta|\beta_3}{2}\|\gamma^{\frac12}\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\|_{0,\Gamma_C}^2,\end{aligned}\tag{21}$$

with $\beta_2 > 0$ and $\beta_3 > 0$. Noting that

$$\|\gamma^{-\frac12}P_\gamma(\mathbf{v}^h-\mathbf{u})\|_{0,\Gamma_C}^2\le2\|\gamma^{-\frac12}(u_n-v_n^h)\|_{0,\Gamma_C}^2+2\|\gamma^{\frac12}\sigma_n(\mathbf{u}-\mathbf{v}^h)\|_{0,\Gamma_C}^2$$

and putting together estimates (19) and (21) in (18) gives

$$\begin{aligned}\frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}^2 \le \frac{C^2}{2\alpha}\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega}^2 + \left(\frac{\beta_1\theta^2}{2}+\beta_2\right)\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u}-\mathbf{v}^h)\|_{0,\Gamma_C}^2 + \beta_2\|\gamma^{-\frac{1}{2}}(u_n-v_n^h)\|_{0,\Gamma_C}^2 \\ +\left(-1+\frac{1}{2\beta_2}+\frac{|1-\theta|}{2\beta_3}\right)\|\gamma^{\frac{1}{2}}(\sigma_n(\mathbf{u})+\frac{1}{\gamma}[P_\gamma(\mathbf{u}^h)]_+)\|_{0,\Gamma_C}^2 \\ +\left(\frac{1}{2\beta_1}+\theta+\frac{|1-\theta|\beta_3}{2}\right)\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\|_{0,\Gamma_C}^2. \qquad (22)\end{aligned}$$

The norm term on the discrete normal constraints in (22) is bounded as follows by using (10):

$$\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\|_{0,\Gamma_C} \le C\gamma_0^{\frac{1}{2}}\|\mathbf{v}^h-\mathbf{u}^h\|_{1,\Omega} \le C\gamma_0^{\frac{1}{2}}(\|\mathbf{v}^h-\mathbf{u}\|_{1,\Omega}+\|\mathbf{u}^h-\mathbf{u}\|_{1,\Omega}). \qquad (23)$$

For a fixed $\theta\in\mathbb{R}$ we then choose β_2 and β_3 large enough such that

$$-1+\frac{1}{2\beta_2}+\frac{|1-\theta|}{2\beta_3} < -\frac{1}{2}.$$

Choosing then γ_0 small enough in (23) and putting the estimate in (22) establishes the first statement of theorem.
We now consider separately the case $\theta=-1$ for which (22) becomes:

$$\begin{aligned}\frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}^2 \le \frac{C^2}{2\alpha}\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega}^2 + \left(\frac{\beta_1}{2}+\beta_2\right)\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u}-\mathbf{v}^h)\|_{0,\Gamma_C}^2 + \beta_2\|\gamma^{-\frac{1}{2}}(u_n-v_n^h)\|_{0,\Gamma_C}^2 \\ +\left(-1+\frac{1}{2\beta_2}+\frac{1}{\beta_3}\right)\|\gamma^{\frac{1}{2}}(\sigma_n(\mathbf{u})+\frac{1}{\gamma}[P_\gamma(\mathbf{u}^h)]_+)\|_{0,\Gamma_C}^2 \\ +\left(\frac{1}{2\beta_1}-1+\beta_3\right)\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\|_{0,\Gamma_C}^2. \qquad (24)\end{aligned}$$

Let be given $\eta>0$. Set $\beta_1=1/(2\eta)$, $\beta_2=1+(1/\eta)$ and $\beta_3=1+\eta$. Therefore (24) becomes:

$$\begin{aligned}\frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega}^2 \le \frac{C^2}{2\alpha}\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega}^2 + \left(\frac{5}{4\eta}+1\right)\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u}-\mathbf{v}^h)\|_{0,\Gamma_C}^2 + \frac{1+\eta}{\eta}\|\gamma^{-\frac{1}{2}}(u_n-v_n^h)\|_{0,\Gamma_C}^2 \\ -\frac{\eta}{2(1+\eta)}\|\gamma^{\frac{1}{2}}(\sigma_n(\mathbf{u})+\frac{1}{\gamma}[P_\gamma(\mathbf{u}^h)]_+)\|_{0,\Gamma_C}^2 + 2\eta\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{v}^h-\mathbf{u}^h)\|_{0,\Gamma_C}^2.\end{aligned}$$

Let $\gamma_0>0$. Set $\eta=\alpha/(16C^2\gamma_0)$ where C is the constant in (23) to conclude the proof of the theorem. □

Remark 3.7. *In the case $\theta=-1$, note that the convergence result holds for any value of $\gamma_0>0$. However, in that case, the abstract estimate takes the form*

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega} + \left(\frac{C}{\gamma_0+C}\right)^{\frac{1}{2}}\|\gamma^{\frac{1}{2}}(\sigma_n(\mathbf{u})+\frac{1}{\gamma}[P_\gamma(\mathbf{u}^h)]_+)\|_{0,\Gamma_C} \\ \le\ & C(1+\gamma_0)^{\frac{1}{2}}\inf_{\mathbf{v}^h\in\mathbf{V}^h}\left(\|\mathbf{u}-\mathbf{v}^h\|_{1,\Omega}+\|\gamma^{-\frac{1}{2}}(u_n-v_n^h)\|_{0,\Gamma_C}+\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u}-\mathbf{v}^h)\|_{0,\Gamma_C}\right),\end{aligned}$$

where C is a positive constant, independent of h, $\mathbf{u}$ and γ_0. As a result, the estimation is deteriorated when γ_0 increases, in particular the estimation of the contact stress on Γ_C.

The optimal convergence of the method is stated below.

Theorem 3.8. *Suppose that the solution* $\mathbf{u}$ *to Problem* (3) *belongs to* $(H^{\frac{3}{2}+\nu}(\Omega))^d$ *with* $0 < \nu \leq k-\frac{1}{2}$ *(*$k = 1, 2$ *is the degree of the finite element method, given in (4)) and* $d = 2, 3$*. When* $\theta \neq -1$*, suppose in addition that the parameter* γ_0 *is sufficiently small. The solution* $\mathbf{u}^h$ *of Problem* (9) *satisfies the following error estimate:*

$$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + \|\gamma^{\frac{1}{2}}(\sigma_n(\mathbf{u}) + \frac{1}{\gamma}[P_\gamma(\mathbf{u}^h)]_+)\|_{0,\Gamma_C} \leq Ch^{\frac{1}{2}+\nu}\|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega}, \tag{25}$$

where C *is a positive constant, independent of* h *and* $\mathbf{u}$*.*

Proof: We need to bound the right terms in estimate (17) and we choose $\mathbf{v}^h = \mathcal{I}^h\mathbf{u}$ where $\mathcal{I}^h$ stands for the Lagrange interpolation operator mapping onto $\mathbf{V}^h$. The estimation of the Lagrange interpolation error in the H^1-norm on a domain Ω is classical (see, e.g., [7, 13, 14]):

$$\|\mathbf{u} - \mathcal{I}^h\mathbf{u}\|_{1,\Omega} \leq Ch^{\frac{1}{2}+\nu}\|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega}, \tag{26}$$

for $-\frac{1}{2} < \nu \leq k - \frac{1}{2}$.
The estimation of the term $\|\gamma^{-\frac{1}{2}}(u_n - (\mathcal{I}^h\mathbf{u})_n)\|_{0,\Gamma_C}$ can be done in a very similar manner to [25]. Indeed, let E be an edge (resp. a face) of a triangle (resp. tetrahedron) $T \in \mathcal{T}^h$ on Γ_C:

$$\|\gamma^{-\frac{1}{2}}(u_n - (\mathcal{I}^h\mathbf{u})_n)\|_{0,E} \leq Ch_T^{-\frac{1}{2}}h_T^{1+\nu}\|u_n\|_{1+\nu,E} = Ch^{\frac{1}{2}+\nu}\|u_n\|_{1+\nu,E},$$

(see [13] for instance). By summation on all the edges (resp. faces) and the trace theorem, it results:

$$\|\gamma^{-\frac{1}{2}}(u_n - (\mathcal{I}^h\mathbf{u})_n)\|_{0,\Gamma_C} \leq Ch^{\frac{1}{2}+\nu}\|u_n\|_{1+\nu,\Gamma_C} \leq Ch^{\frac{1}{2}+\nu}\|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega}. \tag{27}$$

From Lemma A.1 in appendix, (see also [16, 17]), the following estimate also holds:

$$\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u} - \mathcal{I}^h\mathbf{u})\|_{0,\Gamma_C} \leq Ch^{\frac{1}{2}+\nu}\|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega}. \tag{28}$$

We conclude by inserting the three estimates (26)–(28) into (17). □

Remark 3.9. *Note that for the theoretical estimation of Theorem 3.8, in the case* $\theta \neq -1$*, and for* γ_0 *small, the constant* C *in the estimate* (25) *behaves in* $O(\gamma_0^{-\frac{1}{2}})$ *(due to the term* $\|\gamma^{-\frac{1}{2}}(u_n - v_n^h)\|_{0,\Gamma_C}$*), so that taking* γ_0 *too small theoretically deteriorates the convergence. In the case* $\theta = -1$*, the same observation can be made, with the additional fact that the convergence is also deteriorated in* $O(\gamma_0)$ *if* γ_0 *is too large (see Remark 3.7). However, numerical experiments of Section 4 reveal that in practice the method is quite robust relatively to the value of* γ_0*, provided it is sufficiently small when* $\theta \neq -1$*.*

Actually we are not able to obtain estimates for the displacements in the L^2-norm ($\|\mathbf{u} - \mathbf{u}^h\|_{0,\Omega}$ and also $\|\mathbf{u} - \mathbf{u}^h\|_{0,\Gamma_C}$) by using the Aubin-Nitsche argument as it is achieved in the linear case (see [17]). Note also that the $L^2(\Omega)$-norm estimates for contact problems are not easy to prove and there are to our knowledge only few estimates (see [12]). Nevertheless, we can easily obtain the following error estimate on the weighted $L^2(\Gamma_C)$-norm on the normal constraint $\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u} - \mathbf{u}^h)\|_{0,\Gamma_C}$ (note that $\sigma_n(\mathbf{u}^h) \neq -\frac{1}{\gamma}[P_\gamma(\mathbf{u}^h)]_+$ on Γ_C contrary to the continuous case).

Corollary 3.10. *Suppose that the solution* $\mathbf{u}$ *to Problem* (3) *belongs to* $(H^{\frac{3}{2}+\nu}(\Omega))^d$ *with* $0 < \nu \leq k - \frac{1}{2}$ *and* $d = 2, 3$. *When* $\theta \neq -1$, *suppose in addition that the parameter* γ_0 *is sufficiently small. The solution* $\mathbf{u}^h$ *of Problem* (9) *satisfies the following error estimate:*

$$\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u} - \mathbf{u}^h)\|_{0,\Gamma_C} \leq Ch^{\frac{1}{2}+\nu}\|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega},$$

where C *is a positive constant, independent of* h *and* $\mathbf{u}$.

Proof: We use (28), (10), (26) and (25) to establish the bound:

$$\begin{aligned}\|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u} - \mathbf{u}^h)\|_{0,\Gamma_C} &\leq \|\gamma^{\frac{1}{2}}\sigma_n(\mathbf{u} - \mathcal{I}^h\mathbf{u})\|_{0,\Gamma_C} + \|\gamma^{\frac{1}{2}}\sigma_n(\mathcal{I}^h\mathbf{u} - \mathbf{u}^h)\|_{0,\Gamma_C} \\ &\leq Ch^{\frac{1}{2}+\nu}\|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega} + C\gamma_0^{\frac{1}{2}}(\|\mathcal{I}^h\mathbf{u} - \mathbf{u}\|_{1,\Omega} + \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}) \\ &\leq Ch^{\frac{1}{2}+\nu}\|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega}.\end{aligned}$$

□

Remark 3.11. *Although contact problems are known to be limited in regularity (the regularity* $H^{5/2}(\Omega)$ *can not generally be passed beyond for such inequality problems), the use of quadratic finite element methods can be of interest in particular for the most regular solutions lying in* $H^s(\Omega)$, $2 < s < 5/2$ *(as it is considered in e.g., [5, 24, 34, 39]).*

4 Numerical experiments

In this section, the numerical results of two and three-dimensional Hertz's contact problems of a disk/sphere with a plane rigid foundation are presented. This slightly exceeds the scope defined in Section 2 since a non-zero initial gap between the elastic solid and the rigid foundation is considered in the computations. Moreover, the tests are performed with P_1 and isoparametric P_2 Lagrange finite elements on meshes which are approximations of the real domain.
The finite element library Getfem++[1] is used. The discrete contact problem is solved by using a generalized Newton method, which means that Problem (9) is derived with respect to $\mathbf{u}^h$ to obtain the tangent system. The term "generalized Newton's method" comes from the fact that the positive part $[x]_+$ is non-differentiable at $x = 0$. However, no special treatment is considered. If a point of non-differentiability is encountered, the tangent system corresponding to one of the two alternatives ($x < 0$ or $x > 0$) is chosen arbitrarily. Integrals of the non-linear term on Γ_C are computed with standard quadrature formulas. Note that the situation where the solution is non-differentiable at an integration point is very rare and corresponds to what is called a "grazing contact" (both $u_n = 0$ and $\sigma_n = 0$). Further details on generalized Newton's method applied to contact problems can be found for instance in [35] and the references therein.

4.1 Two-dimensional numerical tests

The numerical situation is represented in Fig. 1. A disc of radius 20cm is considered with a contact boundary Γ_C which is restricted to the lower part ($y < 20$cm) of the boundary. A homogeneous Neumann condition is applied on the remaining part of the boundary. Since no Dirichlet condition is considered, the problem is not fully coercive. To overcome the non-definiteness coming from

[1] see `http://download.gna.org/getfem/html/homepage/`

the free rigid motions, the horizontal displacement is prescribed to be zero on the two points of coordinates (0cm, 10cm) and (0cm, 30cm) which blocks the horizontal translation and the rigid rotation. Homogeneous isotropic linear elasticity in plane strain approximation is considered with a Young modulus fixed at $E = 25MPa$ and a Poisson ratio $P = 0.25$. A vertical density of volume forces of $20MN/m^3$ is applied.

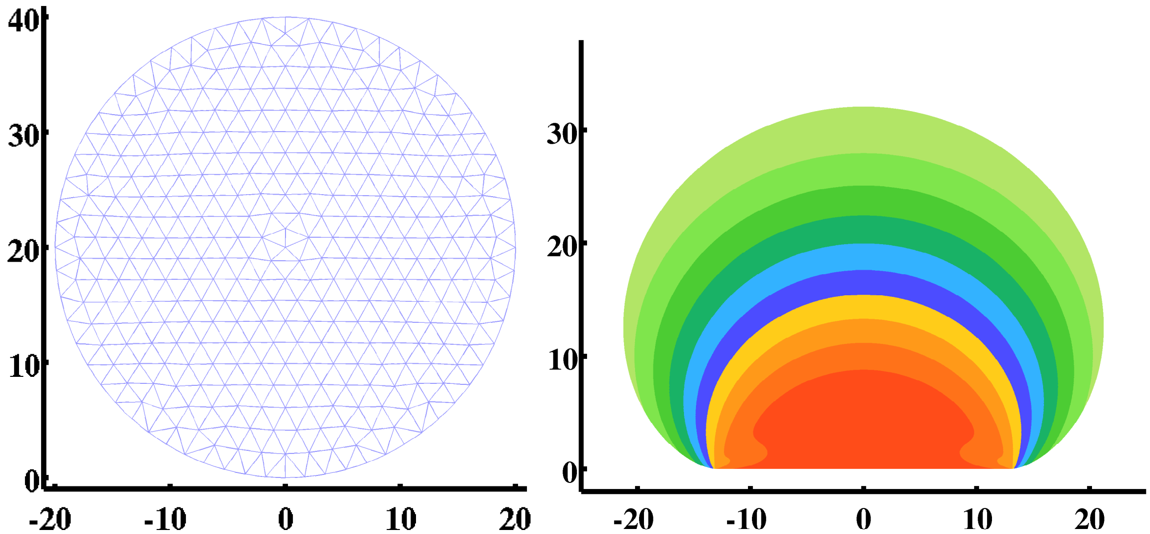

Figure 1: *Example of two-dimensional mesh and reference solution (with color plot of the Von-Mises stress).*

The convergence curves when using a P_1 Lagrange finite element method are shown in Figs. 2, 3 and 4. The solution for mesh sizes $h = 0.5\text{cm}, 1\text{cm}, 3\text{cm}, 4.5\text{cm}$ and $h = 10\text{cm}$ are compared with a reference solution on a very fine mesh ($h = 0.15\text{cm}$) using quadratic isoparametric finite elements. Moreover, the reference solution is computed with a different discretization of the contact problem (Lagrange multipliers and Alart-Curnier augmented Lagrangian, see [35]). On all the figures, the graph on the left represents the relative $H^1(\Omega)$-norm of the error between the computed solution and the reference one and the graph on the right represents the following relative $L^2(\Gamma_C)$-norm:

$$\frac{\|\gamma^{\frac{1}{2}}(\gamma^{-1}[P_\gamma(\mathbf{u}^h)]_+ - \gamma^{-1}[P_\gamma(\mathbf{u}^h_{ref})]_+)\|_{0,\Gamma_C}}{\|\gamma^{-1}[P_\gamma(\mathbf{u}^h_{ref})]_+\|_{0,\Gamma_C}},$$

where $\mathbf{u}^h$ is the discrete solution and $\mathbf{u}^h_{ref}$ the reference solution. Note that $-\frac{1}{\gamma}[P_\gamma(\mathbf{u}^h)]_+$ is an approximation of the contact stress with a convergence of order 1 (see Theorem 3.8).
Let us recall that the formulation is symmetric if and only if $\theta = 1$. The results corresponding to this symmetric case are shown in Fig. 2. Optimal convergence is obtained for both $H^1(\Omega)$ and weighted $L^2(\Gamma_C)$-norms of the error, but only for the smallest value of the parameter γ_0 ($\gamma_0 = 1/100E$). For higher values of γ_0, at least the convergence of the contact stress is non-optimal. This corroborates the theoretical result of Theorem 3.8 for which the optimal rate of convergence is obtained for a sufficiently small γ_0. We also noted that for the largest value of γ_0 (i.e., $\gamma_0 = 100/E$) the convergence of Newton's method is not always achieved (in this case, Newton's method is stopped after 100 iterations). In some experiments, the residual of Newton's

method diminishes to a value which is greatly higher than the one we considered for the tests achieving the convergence. Our interpretation is that for large values of γ_0, when coercivity is lost, there might be no solution of the discrete problem.
Now, when $\theta = 0$, the convergence curves are plotted in Fig. 3. For this version, which is in a sense simpler to implement than the other ones (since the number of additional terms is lower), Newton's method always converges even for the largest value of γ_0. For this latter value of γ_0 the convergence remains sub-optimal. However, for the intermediate value of γ_0 ($\gamma_0 = 1/E$) the optimal convergence is reached.
Concerning the version with $\theta = -1$, which corresponds to an unconditionally coercive problem, one can see in Fig. 4 that optimal convergence is reached for all values of γ_0. Moreover, the smallest error on the contact stress corresponds to the intermediate value of γ_0.
Globally, it is remarkable that the convergence curves for the smallest value of γ_0 are rather the same for the three values of θ. A strategy to guarantee an optimal convergence is of course to consider a sufficiently small γ_0. However, the price to pay is an ill-conditioned discrete problem. The study presented in [35] for the versions $\theta = 1$ and $\theta = 0$ shows that Newton's method has important difficulties to converge when γ_0 is small. When symmetry is not required, a better strategy seems to consider the version with $\theta = -1$ or an intermediate value of $\theta = 0$ which ensure both a optimal convergence rate and few iterations of Newton's method to converge.
We have no interpretation of the slight super-convergence noted on most of the convergence curves (theoretically, the convergence rates should be close to 1).

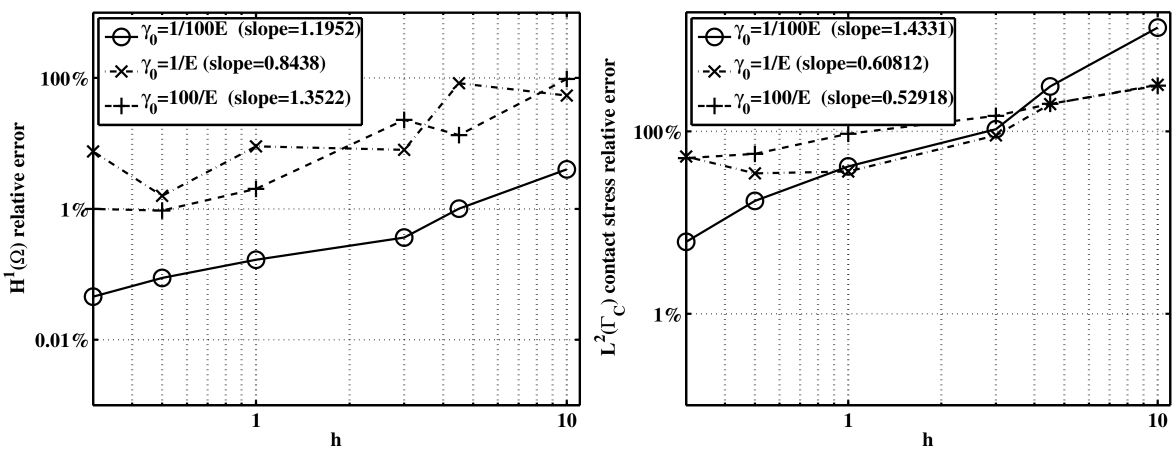

Figure 2: Convergence in the 2D case with P_1 elements and $\theta = 1$. Left : relative $H^1(\Omega)$-norm on the displacements. Right : relative weighted $L^2(\Gamma_C)$-norm on the contact pressure.

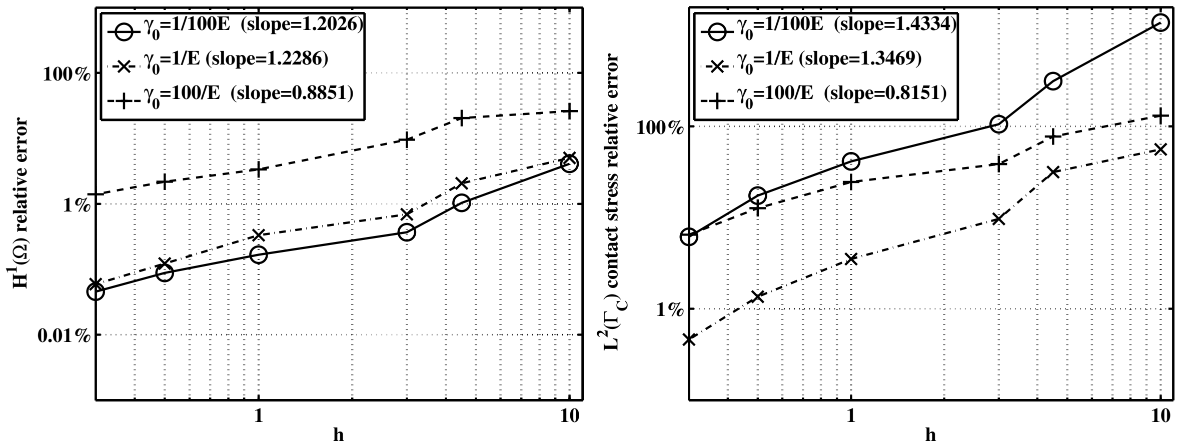

Figure 3: Convergence in the 2D case with P_1 elements and $\theta = 0$. Left : relative $H^1(\Omega)$-norm on the displacements. Right : relative weighted $L^2(\Gamma_C)$-norm on the contact pressure.

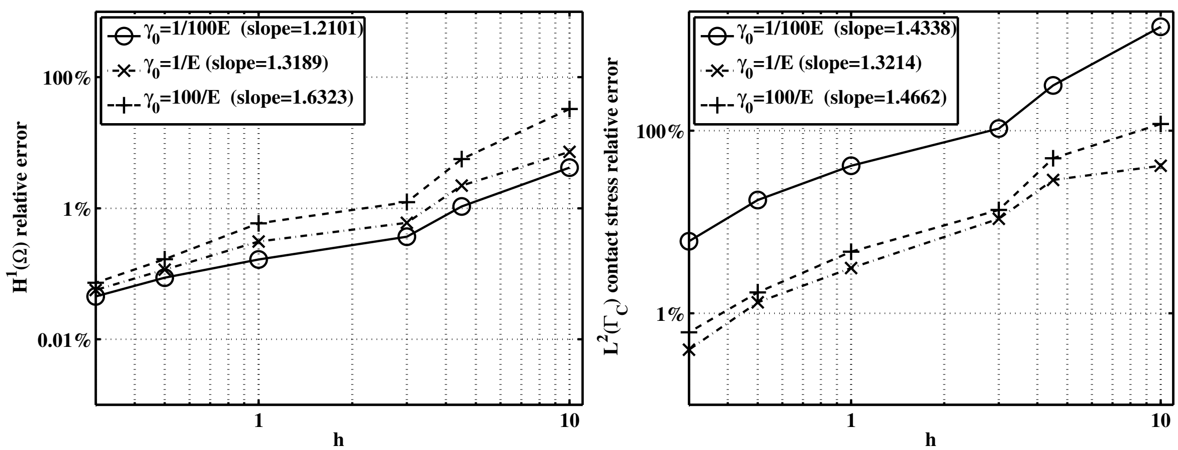

Figure 4: Convergence in the 2D case with P_1 elements and $\theta = -1$. Left : relative $H^1(\Omega)$-norm on the displacements. Right : relative weighted $L^2(\Gamma_C)$-norm on the contact pressure.

The same numerical experiment has been extended to the P_2 Lagrange isoparametric finite element method. The corresponding convergence curves are presented in Figs. 5, 6 and 7. The results are quite similar compared with the P_1 Lagrange method. The convergence is even poorer for large values of γ_0 and $\theta = 1$. The error levels are smaller compared with the P_1 method. However, the convergence rates are only slightly better. This comes probably from the fact that there is a transition between effective contact and non-contact. Due to this transition, the solution is expected to be at most in $(H^s(\Omega))^d$ with $s < 5/2$ which limits the convergence rate of quadratic finite elements.

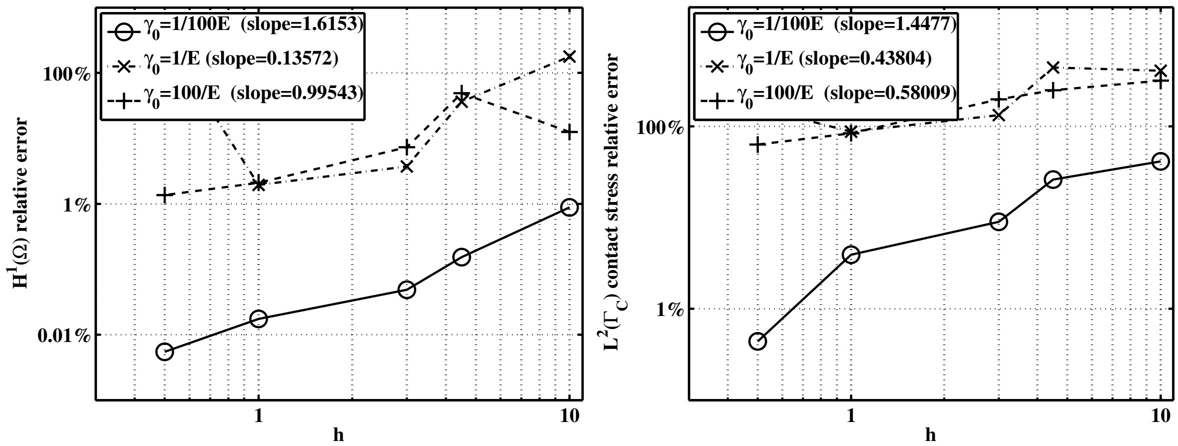

Figure 5: Convergence in the 2D case with P_2 elements and $\theta = 1$. Left : relative $H^1(\Omega)$-norm on the displacements. Right : relative weighted $L^2(\Gamma_C)$-norm on the contact pressure.

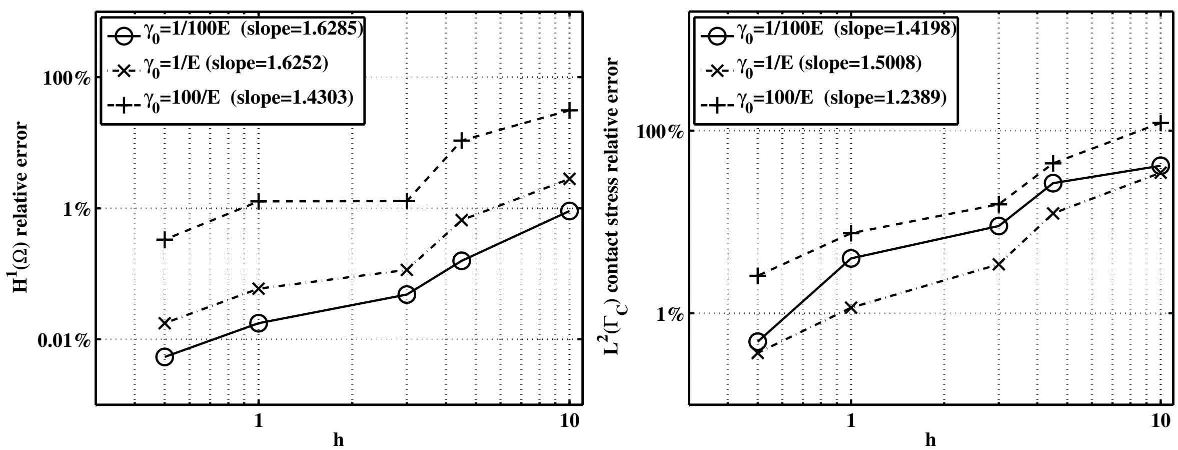

Figure 6: Convergence in the 2D case with P_2 elements and $\theta = 0$. Left : relative $H^1(\Omega)$-norm on the displacements. Right : relative weighted $L^2(\Gamma_C)$-norm on the contact pressure.

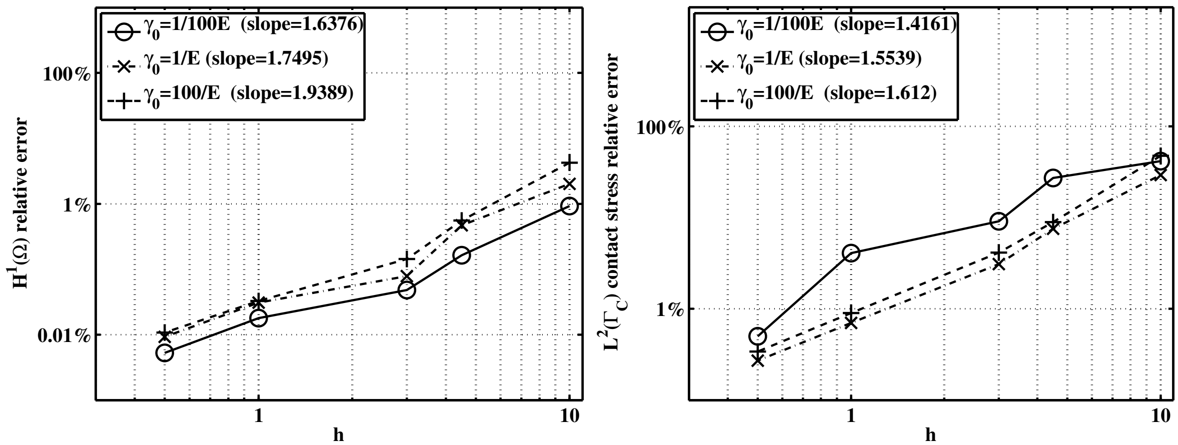

Figure 7: Convergence in the 2D case with P_2 elements and $\theta = -1$. Left : relative $H^1(\Omega)$-norm on the displacements. Right : relative weighted $L^2(\Gamma_C)$-norm on the contact pressure.

4.2 Three-dimensional numerical tests

The three-dimensional tests are similar to the two-dimensional ones. We consider a sphere of radius 20cm with mesh sizes $h = 3.6\text{cm}, 6\text{cm}, 11\text{cm}, 23\text{cm}$ and a P_1 Lagrange finite element method (an example of a mesh and a reference solution are presented in Fig. 8). Homogeneous isotropic linear elasticity is considered with still a Young modulus $E = 25MPa$ and a Poisson ratio $P = 0.25$. A vertical density of volume forces of $20MN/m^3$ is also still considered. Similarly to the two-dimensional case, the horizontal rigid motions and the rotations are blocked by prescribing the displacement on specific chosen points. The reference solution is still computed with quadratic isoparametric finite elements on a fine mesh ($h = 1\text{cm}$) using Lagrange multipliers. The convergence curves are presented in Fig. 9, 10 and 11. Very similar conclusions can be drawn compared with the two-dimensional case. In particular, the method for $\theta = -1$ allows us to obtain an optimal convergence rate for any value of γ_0. However, note that the convergence for large values of γ_0 for $\theta = 1$ and $\theta = 0$ is even worse than in the two-dimensional case.

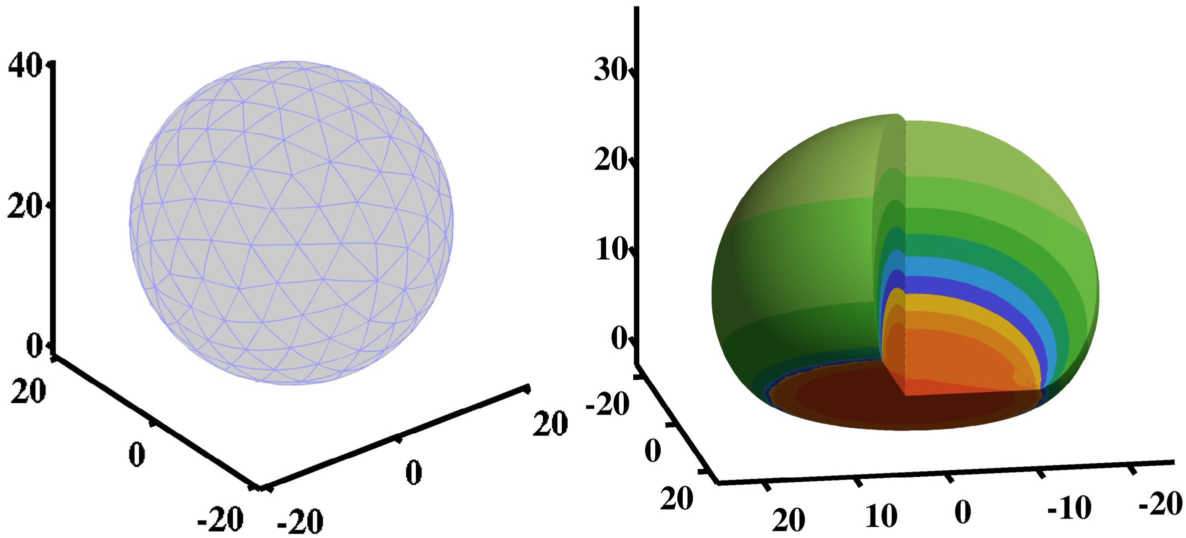

Figure 8: *Three-dimensional mesh example and reference solution (sectional view, with color plot of the Von-Mises stress).*

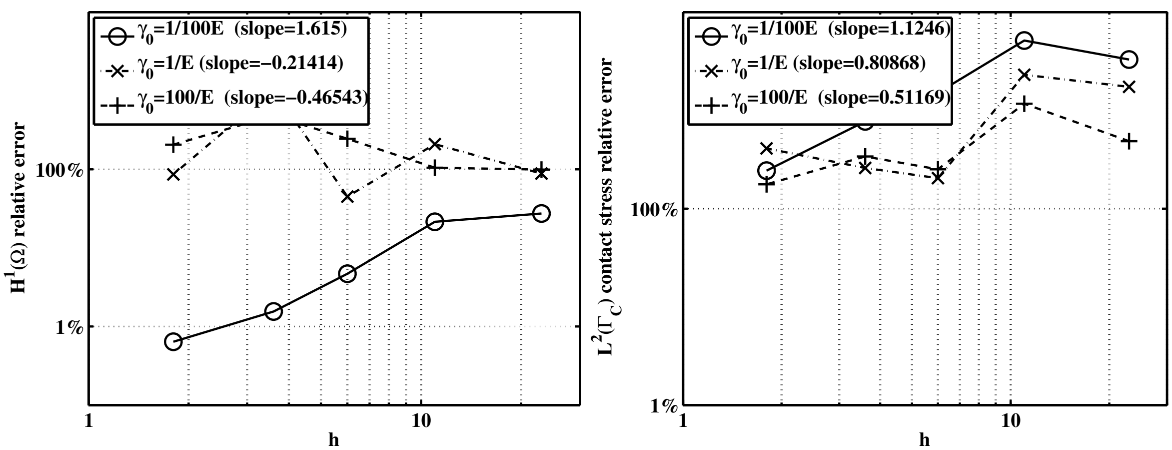

Figure 9: Convergence in the 3D case with P_1 elements and $\theta = 1$. Left : relative $H^1(\Omega)$-norm on the displacements. Right : relative weighted $L^2(\Gamma_C)$-norm on the contact pressure.

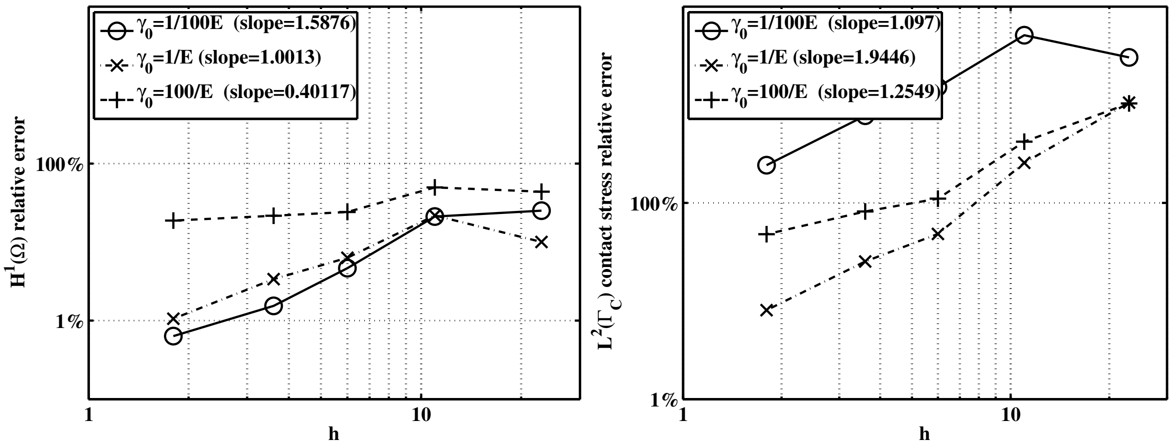

Figure 10: Convergence in the 3D case with P_1 elements and $\theta = 0$. Left : relative $H^1(\Omega)$-norm on the displacements. Right : relative weighted $L^2(\Gamma_C)$-norm on the contact pressure.

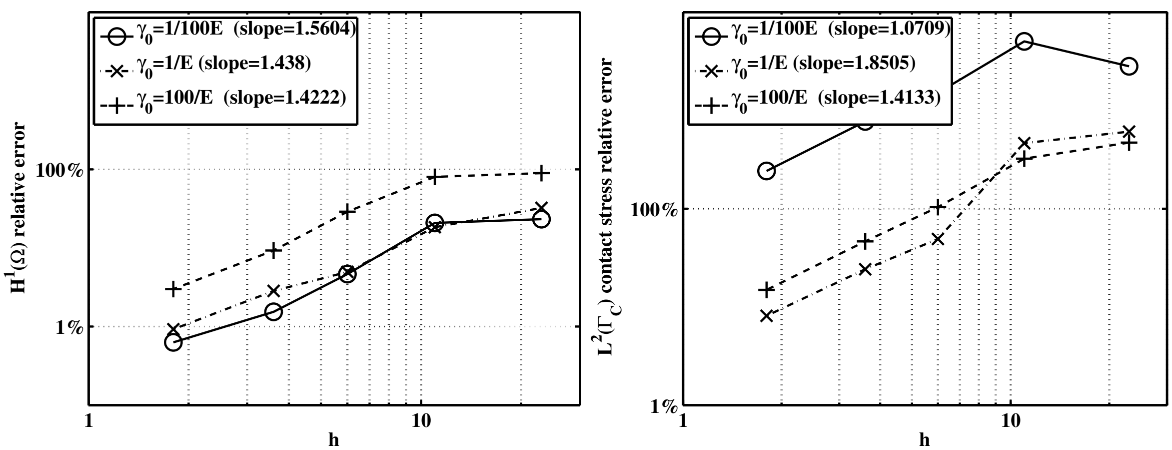

Figure 11: Convergence in the 3D case with P_1 elements and $\theta = -1$. Left : relative $H^1(\Omega)$-norm on the displacements. Right : relative weighted $L^2(\Gamma_C)$-norm on the contact pressure.

5 Conclusion and perspectives

In comparison with the other existing methods for unilateral contact, Nitsche's method has the great advantage that no other unknown than the displacement field is introduced. Indeed, most of the methods are based on a mixed formulation which introduces Lagrange multipliers: see e.g. [38] for a recent review and [35] for a recent numerical comparison of existing methods. From this point of view, Nitsche's method is close to the well-known penalty method in its simplicity, with the advantage of remaining consistent. Moreover, it admits some variants that have interesting properties from the theoretical and/or computational point of view: the variant $\theta = 1$ preserves symmetry and has a positive penalty term, the variant $\theta = 0$ involves less terms and the variant $\theta = -1$ preserves well-posedness and the convergence irrespectively of the parameter γ_0. For all the variants, i.e., all the values of θ, the discrete problem is well-posed and optimal convergence is achieved. Numerical experiments point out the good convergence properties of Nitsche's formulation when solved with a generalized Newton's method. In particular, the variant $\theta = -1$

presents very good convergence properties, for both small and large values of γ_0.
Among future directions of work is extension of the method for friction problems, in particular Tresca's friction and then Coulomb's friction.

Appendix A: interpolation error estimate for the gradient on the boundary

Lemma A.1. *Let $u \in H^{\frac{3}{2}+\nu}(\Omega)$ where $\Omega \subset \mathbb{R}^d$ and $0 < \nu \leq k - \frac{1}{2}$ ($k = 1, 2$ is the degree of the finite element method, given in (4)). Set $\Gamma := \partial\Omega$ and consider an element $T \in \mathcal{T}^h$ such that $\Gamma \cap T$ is a face of T. Then there exists a positive constant C independent of T and h such that:*

$$\|\nabla(u - \mathcal{I}^h u)\|_{0,\Gamma\cap T} \leq C h_T^{\nu} |u|_{\frac{3}{2}+\nu,T},$$

where h_T is the diameter of T and $|u|_{\frac{3}{2}+\nu,T}$ is the usual $H^{\frac{3}{2}+\nu}(T)$ seminorm of u.

Proof: We will use a classical scaling argument (see for instance [14, Theorem 1.103]) . Let us consider the reference element $\hat{T}$ (which is independent of T and h_T) and the jacobian matrix J_T of the linear geometric transformation from $\hat{T}$ to T. Then, due to the regularity of the family of meshes $\mathcal{T}^h$, we have

$$|\det(J_T)| = \frac{|T|}{|\hat{T}|}, \quad \|J_T\| \leq \frac{h_T}{\rho_{\hat{T}}}, \quad \|J_T^{-1}\| \leq \frac{h_{\hat{T}}}{\rho_T},$$

where $\|J_T\| := \sup_{\hat{x}\neq 0} (\|J_T\hat{x}\|/\|\hat{x}\|)$ is the matrix norm associated to the usual euclidean norm in $\mathbb{R}^d$ and $|T|$, $|\hat{T}|$ stand for the areas of T, $\hat{T}$. Using this and the regularity of the mesh, we deduce from a basic calculus

$$\|\nabla(u - \mathcal{I}^h u)\|_{0,\Gamma\cap T} \leq C h_T^{\frac{d-3}{2}} \|\hat{\nabla}(\hat{u} - \hat{\mathcal{I}}^h \hat{u})\|_{0,\hat{\Gamma}},$$

where $\hat{\Gamma}$ is the corresponding face on the reference element, $\hat{\nabla}$, $\hat{\mathcal{I}}^h$ are the gradient and the Lagrange interpolation operator in the reference coordinates, respectively, and $\hat{u}(\hat{x}) = u(J_T(\hat{x}))$. Now, we consider the map

$$\begin{array}{rcl} \mathscr{F} : H^{\frac{3}{2}+\nu}(\hat{T}) & \to & (L^2(\hat{\Gamma}))^d, \\ \hat{u} & \mapsto & \hat{\nabla}(\hat{u} - \hat{\mathcal{I}}^h \hat{u}). \end{array}$$

From standard trace theorems (see [1]), we deduce that $\mathscr{F}$ is continuous. Using the property $\mathscr{F}(\hat{p}) = 0$ for all $\hat{p} \in P_k(\hat{T})$, we can write

$$\begin{array}{rcl} \|\hat{\nabla}(\hat{u} - \hat{\mathcal{I}}^h \hat{u})\|_{0,\hat{\Gamma}} & = & \|\mathscr{F}(\hat{u} + \hat{p})\|_{0,\hat{\Gamma}} \quad \forall \hat{p} \in P_k(\hat{T}), \\ & \leq & \|\mathscr{F}\|_{\mathscr{L}(H^{\frac{3}{2}+\nu}(\hat{T}),(L^2(\hat{\Gamma}))^d)} \|\hat{u} + \hat{p}\|_{\frac{3}{2}+\nu,\hat{T}} \quad \forall \hat{p} \in P_k(\hat{T}), \\ & \leq & C \inf_{\hat{p}\in P_k(\hat{T})} \|\hat{u} + \hat{p}\|_{\frac{3}{2}+\nu,\hat{T}} \\ & \leq & C|\hat{u}|_{\frac{3}{2}+\nu,\hat{T}}. \end{array}$$

The last estimate is the application of the extension to fractional order spaces of Deny-Lions lemma. Such an extension can be found for instance in [13] (Theorem 6.1). Now, proceeding to the reverse change of variable, and still using the regularity of the mesh and the expression of $|\hat{u}|_{\frac{3}{2}+\nu,\hat{T}}$ given also for instance in [13] it comes

$$\begin{aligned}\|\nabla(u-\mathcal{I}^h u)\|_{0,\Gamma\cap T} &\leq Ch_T^{\frac{d-3}{2}}|\hat{u}|_{\frac{3}{2}+\nu,\hat{T}}\\ &\leq Ch_T^{\frac{d-3}{2}}h_T^{-\frac{d-3-2\nu}{2}}|u|_{\frac{3}{2}+\nu,T}\\ &= Ch_T^{\nu}|u|_{\frac{3}{2}+\nu,T}.\end{aligned}$$

□

This result can be straightforwardly extended to the vectorial case. The global interpolation estimate on the whole Γ_C can be obtained by summation on all the faces of elements lying on Γ_C.

Appendix B: a simple example of nonexistence and nonuniqueness of solutions

We consider the triangle Ω of vertexes $A = (0,0)$, $B = (\ell,0)$ and $C = (0,\ell)$. We define $\Gamma_D = [B,C]$, $\Gamma_N = [A,C]$, $\Gamma_C = [A,B]$ and $\{X_1, X_2\}$ denotes the canonical orthonormal basis (see Figure 12). We suppose that the volume forces $\mathbf{f}$ are absent and that the surface forces denoted $\mathbf{g} = g_1X_1 + g_2X_2$ are such that g_1 and g_2 are constant on Γ_N.

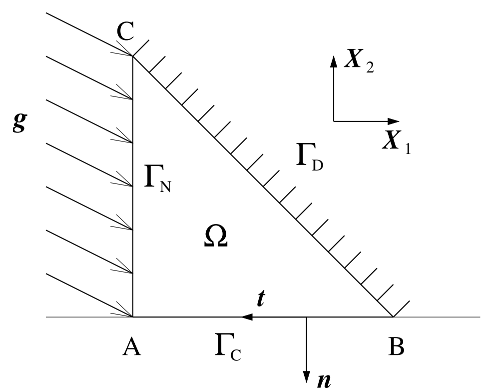

Figure 12: Setting of the problem.

We suppose that Ω is discretized with a single finite element of degree one. Consequently, the finite element space becomes:

$$\mathbf{V}^h := \Big\{\mathbf{v}^h = (v_1^h, v_2^h) \in (P_1(\Omega))^2, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D\Big\}.$$

Clearly, $\mathbf{V}^h$ is of dimension two. For $\mathbf{v}^h \in \mathbf{V}^h$ (resp. $\mathbf{u}^h \in \mathbf{V}^h$), we denote by (V_T, V_N) (resp. (U_T, U_N)) the value of $\mathbf{v}^h(A)$ corresponding to the tangential and the normal displacements at point A respectively (in our example, we have $V_T = -v_1^h(A)$ and $V_N = -v_2^h(A)$). Then, for any $\mathbf{v}^h \in \mathbf{V}^h$, one obtains

$$\boldsymbol{\varepsilon}(\mathbf{v}^h) = \frac{1}{2\ell}\begin{pmatrix} 2V_T & V_T + V_N \\ V_T + V_N & 2V_N \end{pmatrix}.$$

Young's modulus and Poisson's ratio are denoted by E and P, respectively, or equivalently $\lambda = (EP)/((1-2P)(1+P))$ and $\mu = E/(2(1+P))$ are the corresponding Lamé coefficients. So we get

$$\boldsymbol{\sigma}(\mathbf{v}^h) = \frac{1}{\ell}\begin{pmatrix} (\lambda+2\mu)V_T + \lambda V_N & \mu(V_T+V_N) \\ \mu(V_T+V_N) & (\lambda+2\mu)V_N + \lambda V_T \end{pmatrix}.$$

We next consider a very simple case:

$$E = 1, P = 0 \text{ (i.e., } \lambda = 0, \mu = 1/2), \theta = 1, \ell = 1, \gamma = 1 \text{ (i.e., } \gamma_0 = 1/\sqrt{2}).$$

Therefore

$$A_1(\mathbf{u}^h, \mathbf{v}^h) = \frac{3}{4}(U_T V_T + U_N V_N) + \frac{1}{4}(U_T V_N + U_N V_T) - U_N V_N.$$

Denoting by ϕ_N the (normal component) basis function of $\mathbf{V}^h$ ($\phi_N(A) = 1, \phi_N(B) = 0$), we get

$$P_1(\mathbf{v}^h) = V_N \phi_N - V_N \text{ and } [P_1(\mathbf{u}^h)]_+ = (1-\phi_N)(-U_N)_+,$$

so

$$\int_{\Gamma_C} \frac{1}{\gamma}[P_1(\mathbf{u}^h)]_+ P_1(\mathbf{v}^h)\, d\Gamma = -\frac{1}{3}(-U_N)_+ V_N.$$

Besides

$$L(\mathbf{v}^h) = -\frac{1}{2}(g_1 V_T + g_2 V_N).$$

The discrete problem (9) consists then of finding $(U_T, U_N) \in \mathbb{R}^2$ such that:

$$\begin{cases} \frac{3}{2}U_T + \frac{1}{2}U_N = -g_1, \\ \frac{1}{2}U_T - \frac{1}{2}U_N - \frac{2}{3}(-U_N)_+ = -g_2. \end{cases} \tag{29}$$

Clearly, a solution of (29) satisfies either $U_N \geq 0$ or $U_N < 0$. We now show that for certain values of g_1 and g_2, system (29) admits an infinity of solutions or no solution.

• Suppose that $g_1 = 3g_2$, then there exists an infinity of solutions $(U_T, U_N) = (-2g_1/3 - x/3, x)$ for any $x \leq 0$.

• Suppose that $g_1 > 3g_2$, then (29) admits no solution.

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