Multiple solutions of stick and separation type in the Signorini model with Coulomb friction

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Abstract

This paper proves the existence of multiple solutions to the Coulomb friction problem with Signorini contact conditions in continuum linear elasticity. We consider a body lying on a rigid foundation and we propose a method in order to exhibit two solutions to the frictional contact problem when the friction coefficient is large enough: one solution which separates from the foundation and another one which remains stuck on the foundation. We apply the method to the simple class of problems with triangular bodies and linear displacement fields and we describe the cases in which such multiple solutions exist. Denoting by μ the friction coefficient, we come to the conclusion that such nonuniqueness cases may appear when $\mu > 1$.

Keywords : Coulomb friction, unilateral contact, linear elasticity, multiplicity of solutions.

2000 Mathematics Subject Classification : 74G35, 74M10, 74M15

1. Introduction

In continuum mechanics of solids, the Coulomb law of friction [1] is the most common model used when describing slipping or sticking bodies on a contact surface. This law is very often considered together with the unilateral contact model (i.e., the Signorini law introduced in [10]) in order to take into account the possible separation of the body from the surface. In the case of elastostatics, the variational formulation of the unilateral contact problem with Coulomb friction was obtained in [2] (see also [3]) and followed in [9] by the existence proof in the case of an infinitely long strip with small friction. These results were extended to more general geometries and greater bounds ensuring existence of solutions were obtained in [8] and more recently in [4]. Nevertheless the understanding of this frictional contact problem is not complete so that there does not exist, to our knowledge, neither uniqueness results nor nonexistence examples. Concerning nonuniqueness of solutions in the continuum case, the approach introduced in [6] consists of searching sufficient conditions leading to an

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infinity of solutions with slip, located on a continuous branch for precise (eigen)values of the friction coefficient. It has been recently shown in [7] that such non-unique slipping solutions exist.

This paper is also concerned with nonuniqueness of solutions to the unilateral contact problem with Coulomb friction in two-dimensional static linear elasticity. The approach chosen in this paper is different from the one in [6, 7] and it does not deal with slipping solutions but only with solutions involving separation and stick. We introduce a simple setting in order to obtain at least two solutions for the problem with assumptions which require in particular that the friction coefficient μ is large enough. The behaviour of the two solutions on the contact surface is quite different: the first one represents separation of the body from the rigid foundation whereas the second one corresponds to stick on the contact area.

After defining the problem in Section 2, we introduce the setting in Section 3 and we prove that this method allows to exhibit at least two solutions when appropriate hypotheses are satisfied. Section 4 is concerned with the application of the results to a simple case: triangular bodies and linear displacement fields. We study in detail this class of problems and we show that some of them fulfill the assumptions of the theoretical setting.

The main result obtained in this paper is issued from the discussion in Section 4 and can be summarized as follows: the unilateral contact problem with Coulomb friction in continuum elastostatics does not admit unique solutions in the general case when $\mu > 1$.

2. Problem statement

Let us consider the deformation of an elastic body occupying, in the initial unconstrained configuration a domain Ω in $\mathbb{R}^2$. The boundary $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_C}$ of Ω consists of three non-overlapping domains Γ_D, Γ_N and Γ_C where the measures of Γ_D and Γ_C are positive. The body Ω is submitted to given displacements $\mathbf{U}$ on Γ_D, it is subjected to surface traction forces $\mathbf{F}$ on Γ_N and the body forces are denoted by $\boldsymbol{f}$. In the initial configuration, the part Γ_C is considered as the candidate contact surface on a rigid foundation which means that the contact zone cannot enlarge during the deformation process. The contact is assumed to be frictional and the stick, slip and separation zones on Γ_C are not known in advance. We denote by $\mu \geq 0$ the given friction coefficient on Γ_C. The unit outward normal and tangent vectors on $\partial\Omega$ are $\boldsymbol{n} = (n_x, n_y)$ and $\boldsymbol{t} = (-n_y, n_x)$ respectively.

The unilateral contact problem with the Coulomb friction law consists of finding the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ satisfying (2.1)–(2.6):

$$\mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \mathbf{0} \quad \text{in } \Omega, \tag{2.1}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbf{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{u} = \mathbf{U} \quad \text{on } \Gamma_D, \tag{2.3}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \mathbf{F} \quad \text{on } \Gamma_N. \tag{2.4}$$

The notation $\boldsymbol{\sigma}(\boldsymbol{u}) : \Omega \to S_2$ represents the stress tensor field lying in S_2, the space of second order symmetric tensors on $\mathbb{R}^2$. The linearized strain tensor field is $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\boldsymbol{\nabla u} + \boldsymbol{\nabla}^T \boldsymbol{u})/2$ and $\mathbf{C}$ is the fourth order symmetric and elliptic tensor of linear elasticity.

Afterwards we choose the following notation for any displacement field $\boldsymbol{u}$ and for any density of surface forces $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}$ defined on $\partial\Omega$:

$$\boldsymbol{u} = u_n \boldsymbol{n} + u_t \boldsymbol{t} \quad \text{and} \quad \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \sigma_n(\boldsymbol{u})\boldsymbol{n} + \sigma_t(\boldsymbol{u})\boldsymbol{t}.$$

On Γ_C, the three conditions representing unilateral contact are as follows

$$u_n \leq 0, \qquad \sigma_n(\boldsymbol{u}) \leq 0, \qquad \sigma_n(\boldsymbol{u})\, u_n = 0, \tag{2.5}$$

and the Coulomb friction law on Γ_C is described by the following conditions:

$$\begin{cases} u_t = 0 \implies |\sigma_t(\boldsymbol{u})| \leq \mu |\sigma_n(\boldsymbol{u})|, \\ u_t \neq 0 \implies \sigma_t(\boldsymbol{u}) = -\mu |\sigma_n(\boldsymbol{u})| \dfrac{u_t}{|u_t|}. \end{cases} \tag{2.6}$$

As far as we know there only exist existence results in the case of small friction coefficients (see in particular [9, 8, 4]) and some nonuniqueness examples involving slipping solutions (see [7]) for problem (2.1)–(2.6). There are neither uniqueness results (unless the loads $\boldsymbol{f}$,$\mathbf{F}$ and $\mathbf{U}$ are equal to zero) nor nonexistence examples available. Let us mention that the frictionless case which corresponds to $\mu = 0$ or equivalently $\sigma_t(\boldsymbol{u}) = 0$ in (2.6) admits a unique solution according to [5].

3. Sufficient conditions of existence of at least two solutions for large friction coefficients

First we consider a solution $\boldsymbol{u}$ of the unilateral contact problem without friction (i.e., when $\mu = 0$ in (2.1)–(2.6)) which separates the body from the rigid foundation almost everywhere on the contact zone Γ_C. Therefore $\boldsymbol{u}$ solves the following system of equations:

$$\begin{cases} \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \mathbf{0} \quad \text{in } \Omega, \\ \boldsymbol{\sigma}(\boldsymbol{u}) = \mathbf{C}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \\ \boldsymbol{u} = \mathbf{U} \quad \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \mathbf{F} \quad \text{on } \Gamma_N, \\ u_n < 0 \quad \text{on } \Gamma_C, \\ \sigma_n(\boldsymbol{u}) = \sigma_t(\boldsymbol{u}) = 0 \quad \text{on } \Gamma_C. \end{cases} \tag{3.1}$$

It is easy to check that a displacement field $\boldsymbol{u}$ verifying (3.1) is also a solution of the frictional unilateral contact problem (2.1)–(2.6) for any $\mu > 0$. Having at our disposal the field $\boldsymbol{u}$ solving (3.1), we consider the following elasticity problem with Dirichlet conditions on $\Gamma_D \cup \Gamma_C$ and Neumann conditions on Γ_N.

$$\begin{cases} \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{\Phi}) = \mathbf{0} \quad \text{in } \Omega, \\ \boldsymbol{\sigma}(\boldsymbol{\Phi}) = \mathbf{C}\, \boldsymbol{\varepsilon}(\boldsymbol{\Phi}) \quad \text{in } \Omega, \\ \boldsymbol{\Phi} = \mathbf{0} \quad \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(\boldsymbol{\Phi})\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_N, \\ \Phi_n = -u_n \quad \text{on } \Gamma_C, \\ \Phi_t = -u_t \quad \text{on } \Gamma_C. \end{cases} \tag{3.2}$$

When the compatibility conditions at the interfaces of the boundary parts Γ_D and Γ_C are satisfied which we assume for the sake of simplicity, problem (3.2) admits a unique solution $\boldsymbol{\Phi} \neq \mathbf{0}$ according to the Lax-Milgram theorem.

The following Proposition establishes sufficient conditions for the nonuniqueness of the equilibrium solution $\boldsymbol{u}$ to problem (2.1)–(2.6) under assumptions which require that the friction coefficient μ is large enough in a sense which is detailed hereafter. Let us mention that the framework we propose in this study deals only with ”regular” solutions. As a consequence, the normal and tangential stresses we consider on the contact zone are at least defined almost everywhere.

Proposition 3.1 *Let $\boldsymbol{u}$ be a displacement field satisfying the conditions* (3.1) *and let $\boldsymbol{\Phi}$ be the solution of problem* (3.2). *If $\mu > 0$ and $-\mu\sigma_n(\boldsymbol{\Phi}) \geq |\sigma_t(\boldsymbol{\Phi})|$ on Γ_C, then $\boldsymbol{u}$ and $\boldsymbol{u} + \boldsymbol{\Phi}$ are two distinct solutions of Coulomb's frictional contact problem* (2.1)–(2.6).

Proof. As already mentioned, the displacement field $\boldsymbol{u}$ satisfies (2.1)–(2.6) for any nonnegative μ. Let us check that $\boldsymbol{u} + \boldsymbol{\Phi}$ also satisfies these conditions when $\mu > 0$ and $-\mu\sigma_n(\boldsymbol{\Phi}) \geq |\sigma_t(\boldsymbol{\Phi})|$ on Γ_C. It is straightforward that $\mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u} + \boldsymbol{\Phi}) + \boldsymbol{f} = \mathbf{0}$ in Ω, $\boldsymbol{\sigma}(\boldsymbol{u} + \boldsymbol{\Phi}) = \mathbf{C}\, \boldsymbol{\varepsilon}(\boldsymbol{u} + \boldsymbol{\Phi})$ in Ω, $\boldsymbol{u} + \boldsymbol{\Phi} = \mathbf{U}$ on Γ_D and $\boldsymbol{\sigma}(\boldsymbol{u} + \boldsymbol{\Phi})\boldsymbol{n} = \mathbf{F}$ on Γ_N. It remains to show the fulfillment of the conditions (2.5)–(2.6) on the contact zone Γ_C. One gets on Γ_C:

$$u_n + \Phi_n = 0 \quad \text{and} \quad \sigma_n(\boldsymbol{u} + \boldsymbol{\Phi}) \leq 0$$

so that (2.5) holds. The conditions (2.6) on Γ_C are verified since

$$u_t + \Phi_t = 0 \quad \text{and} \quad |\sigma_t(\boldsymbol{u} + \boldsymbol{\Phi})| \leq \mu|\sigma_n(\boldsymbol{u} + \boldsymbol{\Phi})|.$$

Hence $\boldsymbol{u} + \boldsymbol{\Phi}$ solves (2.1)–(2.6). □

Remark 3.2 *The above result is a sufficient condition for the existence of at least two solutions of* (2.1)–(2.6). *It suffices first to determine loads $\mathbf{F}, \boldsymbol{f}, \mathbf{U}$ such that separation occurs almost everywhere on Γ_C for $\boldsymbol{u}$ ($\boldsymbol{u}$ is the unique solution of the frictionless unilateral contact problem). Then one solves an elasticity problem involving $\boldsymbol{u}$, admitting $\boldsymbol{\Phi}$ as solution, and yielding $\sigma_n(\boldsymbol{\Phi})$ (which must be nonpositive) and $\sigma_t(\boldsymbol{\Phi})$. Besides note that there does not always exist a positive number μ such that*

$-\mu\sigma_n(\boldsymbol{\Phi}) \geq |\sigma_t(\boldsymbol{\Phi})|$ *on* Γ_C. *Clearly* $\boldsymbol{u}+\boldsymbol{\Phi}$ *is a displacement field with stick everywhere on the contact zone. Moreover if there exists a positive number* μ *such that* $-\mu\sigma_n(\boldsymbol{\Phi}) \geq |\sigma_t(\boldsymbol{\Phi})|$ *then* $\boldsymbol{u}$ *and* $\boldsymbol{u}+\boldsymbol{\Phi}$ *are solutions of the frictional contact problem* (2.1)–(2.6) *for any* $\overline{\mu}$ *verifying* $\overline{\mu} \geq \mu$.

Remark 3.3 *Let* $\boldsymbol{u}$ *and* $\boldsymbol{\Phi}$ *verify* (3.1) *and* (3.2) *respectively. Then*

$$\min_{\boldsymbol{v}=\mathbf{U}\ on\ \Gamma_D, v_n\leq 0\ on\ \Gamma_C} J(\boldsymbol{v}) = J(\boldsymbol{u}) < J(\boldsymbol{u}+\boldsymbol{\Phi}) = \min_{\boldsymbol{v}=\mathbf{U}\ on\ \Gamma_D, \boldsymbol{v}=\mathbf{0}\ on\ \Gamma_C} J(\boldsymbol{v}), \tag{3.3}$$

where J *denotes the energy functional defined by*

$$J(\boldsymbol{v}) = \frac{1}{2}a(\boldsymbol{v},\boldsymbol{v}) - L(\boldsymbol{v})$$

and

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega (\mathbf{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, d\Omega, \qquad L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}.\boldsymbol{v}\, d\Omega + \int_{\Gamma_N} \mathbf{F}.\boldsymbol{v}\, d\Gamma,$$

for any $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *in the Sobolev space* $(H^1(\Omega))^2$. *In these definitions the notations* $\cdot$ *and* : *represent the canonical inner products in* $\mathbb{R}^2$ *and* S_2 *respectively. This means that* $\boldsymbol{u}$ *and* $\boldsymbol{u}+\boldsymbol{\Phi}$ *are solutions of an unilateral contact problem without friction and an elasticity problem respectively and that the corresponding energy functional* J *admits a value which is lower for* $\boldsymbol{u}$ *than for* $\boldsymbol{u}+\boldsymbol{\Phi}$.

As a matter of fact the two equalities in (3.3) *follow from the equivalence between the formulations* (3.1)–(3.2) *and the corresponding minimization problems. Moreover both minimizers* $\boldsymbol{u}$ *and* $\boldsymbol{u}+\boldsymbol{\Phi}$ *in* (3.3) *are unique. Finally*

$$\begin{aligned} J(\boldsymbol{u}+\boldsymbol{\Phi}) - J(\boldsymbol{u}) &= a(\boldsymbol{u},\boldsymbol{\Phi}) - L(\boldsymbol{\Phi}) + \frac{1}{2}a(\boldsymbol{\Phi},\boldsymbol{\Phi}) \\ &= \int_\Omega (-\mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) - \boldsymbol{f}).\boldsymbol{\Phi}\, d\Omega + \int_{\Gamma_N} (\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} - \mathbf{F}).\boldsymbol{\Phi}\, d\Gamma \\ &\quad + \int_{\Gamma_D} \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}.\boldsymbol{\Phi}\, d\Gamma \int_{\Gamma_C} \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}.\boldsymbol{\Phi}\, d\Gamma + \frac{1}{2}a(\boldsymbol{\Phi},\boldsymbol{\Phi}) \\ &= \frac{1}{2}a(\boldsymbol{\Phi},\boldsymbol{\Phi}) > 0 \end{aligned}$$

where the second equality follows from the Green formula and the positiveness of $a(\boldsymbol{\Phi},\boldsymbol{\Phi})$ *results from the Korn inequality.*

Remark 3.4 *If* $\boldsymbol{u}$ *and* $\boldsymbol{\Phi}$ *satisfy* (3.1) *and* (3.2) *then* $\boldsymbol{\Phi}$ *and* $\boldsymbol{u}+\boldsymbol{\Phi}$ *are not rigid body displacements.*

As a matter of fact, assume that $\boldsymbol{\Phi}$ *is a rigid body displacement. One has* $\boldsymbol{\Phi} = \mathbf{0}$ *on* Γ_D *which is of positive measure. Therefore* $\boldsymbol{\Phi} = \mathbf{0}$ *on* Ω *which contradicts* $\Phi_n \neq 0$ *on* Γ_C. *Suppose now that* $\boldsymbol{u}+\boldsymbol{\Phi}$ *is a rigid body displacement. From the definition of* $\boldsymbol{u}$ *and* $\boldsymbol{\Phi}$, *we get* $\boldsymbol{u}+\boldsymbol{\Phi} = \mathbf{0}$ *on* Γ_C *which is of positive measure; so* $\boldsymbol{u}+\boldsymbol{\Phi} = \mathbf{0}$ *on* Ω. *Since* $\mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}+\boldsymbol{\Phi}) = -\boldsymbol{f}$ *on* Ω, $\boldsymbol{\sigma}(\boldsymbol{u}+\boldsymbol{\Phi})\boldsymbol{n} = \mathbf{F}$ *on* Γ_N *and* $\boldsymbol{u}+\boldsymbol{\Phi} = \mathbf{U}$ *on* Γ_D, *we deduce that* $\boldsymbol{f} = \mathbf{F} = \mathbf{U} = \mathbf{0}$. *This together with conditions* $\sigma_n(\boldsymbol{u}) = \sigma_t(\boldsymbol{u}) = 0$ *on* Γ_C *implies that* $\boldsymbol{u} = \mathbf{0}$ *on* Ω *which contradicts* $u_n < 0$ *in* (3.1).

4. Study of the linear case

In this section we show that the theory can be illustrated in the case when Ω is a triangle (in which the edges represent Γ_D, Γ_N and Γ_C) and the displacement fields $\boldsymbol{u}$ and $\boldsymbol{\Phi}$ are linear. Afterwards we look after fields $\boldsymbol{u}$ and $\boldsymbol{\Phi}$ satisfying the assumptions of Proposition 3.1 in order to exhibit some examples of non-unique solutions to the frictional contact problem.

So we consider the triangle Ω of vertexes $A = (0,0)$, $B = (1,0)$ and $C = (x_c, y_c)$ with $y_c > 0$ and we define $\Gamma_D =]B,C[$, $\Gamma_N =]A,C[$, $\Gamma_C =]A,B[$. The body Ω lies on a rigid foundation, the half-space delimited by the straight line (A,B) as suggested in Figure 1.

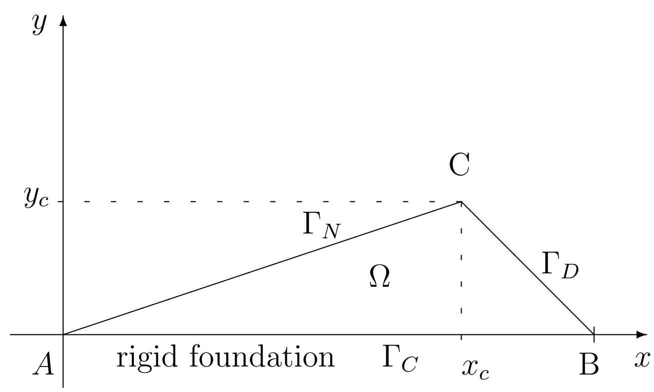

Figure 1: The geometry of the body Ω

We suppose that the body Ω is governed by Hooke's law concerning homogeneous isotropic materials so that (2.2) becomes

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \frac{E\nu}{(1-2\nu)(1+\nu)} tr(\boldsymbol{\varepsilon}(\boldsymbol{u}))\mathbf{I} + \frac{E}{1+\nu}\boldsymbol{\varepsilon}(\boldsymbol{u}) \tag{4.1}$$

where $\mathbf{I}$ represents the identity matrix, tr is the trace operator, E and ν denote Young's modulus and Poisson ratio, respectively with $E > 0$ and $0 \leq \nu < 1/2$. Let $(x = (1,0), y = (0,1))$ stand for the canonical basis of $\mathbb{R}^2$. We suppose that the volume forces $\boldsymbol{f} = (f_x, f_y) = (0,0)$ are absent in Ω and that the surface forces on Γ_N are denoted by $\mathbf{F} = (F_x, F_y)$. Let $\mathbf{U} = (U_x, U_y)$ represent the given displacements on Γ_D.

We begin with the determination of $\boldsymbol{\Phi} = (\Phi_x, \Phi_y)$ in (3.2). Since $\boldsymbol{\Phi} = \mathbf{0}$ on $\Gamma_D =]B,C[$ and $\boldsymbol{\Phi}$ is linear, we get

$$\Phi_x = \alpha\Big(y_c x + (1-x_c)y - y_c\Big), \tag{4.2}$$

$$\Phi_y = \beta\Big(y_c x + (1-x_c)y - y_c\Big), \tag{4.3}$$

where α and β are real numbers.

Let us now focus on the field $\boldsymbol{u} = (u_x, u_y)$ solving problem (3.1). Since $\boldsymbol{u}$ is linear and $u_x + \Phi_x = u_y + \Phi_y = 0$ on Γ_C, it can be written

$$u_x = -\alpha y_c x + ay + \alpha y_c,$$
$$u_y = -\beta y_c x + by + \beta y_c,$$

with a, b in $\mathbb{R}$. Inserting the previous expression of $\boldsymbol{u}$ in the constitutive law (4.1) and writing $\sigma_n(\boldsymbol{u}) = \sigma_t(\boldsymbol{u}) = 0$ on Γ_C (where $\boldsymbol{n} = (0, -1)$) yields the conditions $E(\nu\alpha y_c + \nu b - b)/((1+\nu)(-1+2\nu)) = E(-a + \beta y_c)/(1+\nu) = 0$.
Therefore $a = \beta y_c, b = \alpha\nu y_c/(1-\nu)$ and the field $\boldsymbol{u}$ becomes

$$u_x = -\alpha y_c x + \beta y_c y + \alpha y_c, \tag{4.4}$$
$$u_y = -\beta y_c x + \frac{\alpha\nu y_c}{1-\nu} y + \beta y_c. \tag{4.5}$$

Obviously $\mathbf{div}\,(\boldsymbol{\sigma}(\boldsymbol{u})) = \mathbf{0}$ and $u_n|_{\Gamma_C} = \beta y_c(x-1)$ which according to (3.1) requires

$$\beta > 0. \tag{4.6}$$

Inserting now the expressions (4.2)–(4.3) of $\boldsymbol{\Phi}$ in the constitutive law (4.1) yields

$$\boldsymbol{\sigma}(\boldsymbol{\Phi}) = \begin{pmatrix} \dfrac{E(\alpha y_c(\nu-1) + \nu\beta(x_c-1))}{(1+\nu)(-1+2\nu)} & \dfrac{E(\beta y_c + \alpha(1-x_c))}{2(1+\nu)} \\ \dfrac{E(\beta y_c + \alpha(1-x_c))}{2(1+\nu)} & \dfrac{E(\nu\alpha y_c + \beta(1-x_c)(1-\nu))}{(1+\nu)(1-2\nu)} \end{pmatrix}, \tag{4.7}$$

and $\mathbf{div}\,(\boldsymbol{\sigma}(\boldsymbol{\Phi})) = \mathbf{0}$. Then we consider the Neumann conditions: $\boldsymbol{\sigma}(\boldsymbol{\Phi})\boldsymbol{n} = \mathbf{0}$ on Γ_N. Since the unit outward normal vector on Γ_N is $\boldsymbol{n} = (-y_c/\sqrt{x_c^2+y_c^2}, x_c/\sqrt{x_c^2+y_c^2})$, the stress vector on Γ_N becomes

$$\boldsymbol{\sigma}(\boldsymbol{\Phi})\boldsymbol{n} = \begin{pmatrix} \dfrac{E(\alpha(2\nu y_c^2 - 2y_c^2 - x_c^2 + 2x_c^2\nu + x_c - 2x_c\nu) + \beta(-2y_c\nu + x_c y_c))}{2(1-2\nu)(1+\nu)\sqrt{x_c^2+y_c^2}} \\ \dfrac{E(\alpha(y_c x_c - y_c + 2\nu y_c) + \beta(-y_c^2 + 2y_c^2\nu + 2\nu x_c^2 - 2\nu x_c - 2x_c^2 + 2x_c))}{2(1-2\nu)(1+\nu)\sqrt{x_c^2+y_c^2}} \end{pmatrix}.$$

So the Neumann condition is equivalent to the linear system

$$M \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $(\alpha, \beta) \neq (0, 0)$, we deduce that $\det(M) = 0$. After some calculation (and keeping in mind that $y_c > 0$ and $0 \leq \nu < 1/2$) this leads to the expression of the Poisson ratio:

$$\nu = \frac{(y_c^2 - x_c + x_c^2)^2}{((x_c-1)^2 + y_c^2)(x_c^2+y_c^2)}. \tag{4.8}$$

Inserting the expression of ν in the linear system reduces the condition $\boldsymbol{\sigma}(\boldsymbol{\Phi})\boldsymbol{n} = \boldsymbol{0}$ to one of the two following cases (4.9) or (4.10):

$$\alpha = 0,\ y_c = \sqrt{x_c(2-x_c)},\ \left(\nu = \frac{x_c}{2}\right) \tag{4.9}$$

$$\beta = -\frac{x_c^3 - x_c^2 + y_c^2 x_c + y_c^2}{(x_c^2 - 2x_c + y_c^2)y_c}\alpha. \tag{4.10}$$

Let us show that case (4.9) does not fit with the assumptions of the Proposition 3.1. If condition (4.9) holds, it follows that on Γ_C, $\sigma_n(\boldsymbol{\Phi}) = \beta E(2-x_c)/(2+x_c)$. According to (4.6) we deduce that $\sigma_n(\boldsymbol{\Phi}) > 0$. As a consequence, such a field $\boldsymbol{\Phi}$ does not satisfy the assumptions of the Proposition.

Therefore we consider condition (4.10). In that case the normal constraint on Γ_C given by (4.7) becomes

$$\sigma_n(\boldsymbol{\Phi}) = \frac{E\beta y_c^2(y_c^2 + (x_c-1)^2)^2(x_c^2+y_c^2)}{(x_c^3 - x_c^2 + x_c y_c^2 + y_c^2)(2(y_c^2 - x_c + x_c^2)^2 + y_c^2)}$$

and the assumptions of Proposition 3.1 require that

$$x_c^3 - x_c^2 + x_c y_c^2 + y_c^2 < 0. \tag{4.11}$$

Obviously, the Poisson ratio ν is nonnegative in (4.8). The condition $\nu < 1/2$ is equivalent to

$$(x_c^2 - x_c + y_c^2 + y_c)(x_c^2 - x_c + y_c^2 - y_c) < 0 \tag{4.12}$$

which means that (x_c, y_c) belongs to the open disk centered at (1/2,1/2) of radius $1/\sqrt{2}$ and not to the closed disk centered at (1/2,-1/2) of radius $1/\sqrt{2}$. Putting together conditions (4.11) and (4.12) yields

$$x_c \in]0,1[, \quad \sqrt{\frac{1}{4} + x_c - x_c^2} - \frac{1}{2} < y_c < x_c\sqrt{\frac{1-x_c}{1+x_c}}. \tag{4.13}$$

The admissible domain Σ in which are located the pairs (x_c, y_c) satisfying (4.13) is depicted in Figure 2.

Note that $(x_c, y_c) \in \Sigma$ and $\beta > 0$ implies $\alpha < 0$ according to (4.10). Besides, we obtain on Γ_C:

$$\left|\frac{\sigma_t(\boldsymbol{\Phi})}{\sigma_n(\boldsymbol{\Phi})}\right| = \frac{x_c}{y_c}.$$

According to (4.4)–(4.5), (4.8) and (4.10) the field $\boldsymbol{u}$ is

$$u_x = \beta y_c\left(\frac{(x_c^2 - 2x_c + y_c^2)y_c}{x_c^3 - x_c^2 + y_c^2 x_c + y_c^2}(x-1) + y\right), \tag{4.14}$$

$$u_y = -\beta y_c\left((x-1) + \left(\frac{(y_c^2 - x_c + x_c^2)^2(x_c^2 - 2x_c + y_c^2)}{y_c(x_c^3 - x_c^2 + x_c y_c^2 + y_c^2)}\right)y\right), \tag{4.15}$$

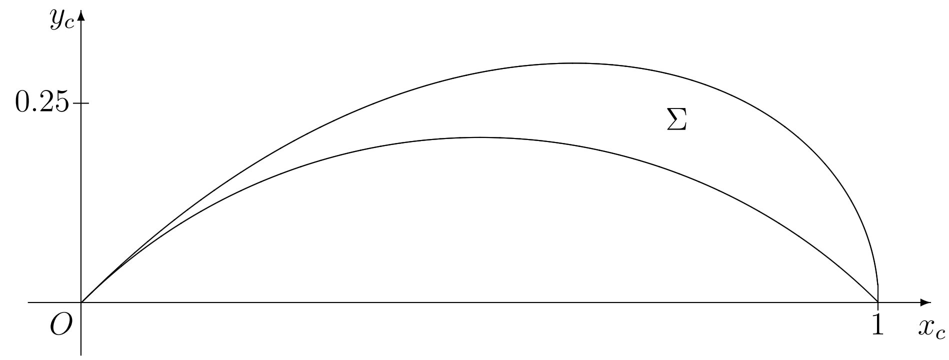

Figure 2: The open admissible region Σ for point $C = (x_c, y_c)$.

and we set $(U_x, U_y) = (u_x, u_y)$ on Γ_D. The densities of surface forces $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = (F_x, F_y)$ on Γ_N are then

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \left(\begin{array}{c} \dfrac{-E\beta y_c(x_c^2 - 2x_c + y_c^2)(y_c^2 + (x_c-1)^2)^2(x_c^2+y_c^2)^{3/2}}{(2(x_c^2 - x_c + y_c^2)^2 + y_c^2)(x_c^3 - x_c^2 + x_c y_c^2 + y_c^2)} \\ 0 \end{array} \right). \qquad (4.16)$$

Finally, combining (4.14)–(4.15) with (4.2)–(4.3) and (4.10) leads to the expression of $\boldsymbol{u} + \boldsymbol{\Phi}$:

$$(u+\Phi)_x = \frac{2\beta x_c y_c((x_c-1)^2 + y_c^2)}{x_c^3 - x_c^2 + x_c y_c^2 + y_c^2} y, \qquad (4.17)$$

$$(u+\Phi)_y = \frac{-\beta((x_c-1)^2 + y_c^2)(y_c^2 + y_c - x_c + x_c^2)(y_c^2 - y_c - x_c + x_c^2)}{x_c^3 - x_c^2 + x_c y_c^2 + y_c^2} y. \qquad (4.18)$$

Let us remark that the displacement field $\boldsymbol{u}$ moves points A and C to the new positions

$$A' = \left(-\beta \frac{(x_c^2 - 2x_c + y_c^2) y_c^2}{x_c^3 - x_c^2 + y_c^2 x_c + y_c^2}, \beta y_c \right) \qquad (4.19)$$

and

$$C' = \Bigg(x_c + \beta \frac{2y_c^2 x_c((x_c-1)^2 + y_c^2)}{x_c^3 - x_c^2 + x_c y_c^2 + y_c^2}, \\ y_c - \beta \frac{y_c((x_c-1)^2 + y_c^2)(y_c^2 + y_c - x_c + x_c^2)(y_c^2 - y_c - x_c + x_c^2)}{x_c^3 - x_c^2 + x_c y_c^2 + y_c^2} \Bigg), \qquad (4.20)$$

respectively whereas position of point B remains unchanged.

When considering the field $\boldsymbol{u} + \boldsymbol{\Phi}$, the points A and B are stuck on the rigid foundation and point C admits after deformation the new coordinates given by C' in (4.20).

Note that we can add an additional (and facultative) smallness assumption on β which is not linked to the equations (2.1)–(2.6) but to the small strain hypothesis. Clearly the point C' should remain over the straight line (A', B) after deformation for both linear displacement fields $\boldsymbol{u}$ and $\boldsymbol{u}+\boldsymbol{\Phi}$ to avoid some turning over of the triangle Ω. In fact it suffices to check this property for the displacement field $\boldsymbol{u}$. Denoting to simplify $A' = (-\beta\delta_1, \beta\delta_2)$ and $C' = (x_c - \beta\delta_3, y_c - \beta\delta_4)$ where $(\delta_i)_{1\le i\le 4}$ are positive constants depending on x_c and y_c (obtained from (4.19) and (4.20)), the point C' remains over the straight line (A', B) if and only if

$$y_c - \beta\delta_4 > -\frac{\beta\delta_2}{\beta\delta_1 + 1}(x_c - \beta\delta_3 - 1),$$

which is satisfied for small positive β. In other words β should be chosen small enough so that $0 < \beta < \overline{\beta}(x_c, y_c)$.

The latter discussion and the statement of Proposition 3.1 prove the next result of nonuniqueness when the friction coefficient is large enough.

Proposition 4.1 *Let be given the triangle Ω of vertexes $A = (0,0)$, $B = (1,0)$ and $C = (x_c, y_c)$ with $y_c > 0$. Set $\Gamma_D =]B, C[$, $\Gamma_N =]A, C[$, $\Gamma_C =]A, B[$ and let $E > 0$. Assume that the pair (x_c, y_c) satisfies (4.13) and that $\beta > 0$. Suppose that ν and α are given by (4.8) and (4.10) respectively and that $\boldsymbol{f} = \boldsymbol{0}$. Let $\mathbf{F}$ and $\mathbf{U}$ given by (4.16) and (4.14)–(4.15) respectively.*

For any $\mu \ge x_c/y_c$ there exist at least two solutions (given by (4.14)–(4.15) and (4.17)–(4.18)) of the Coulomb frictional contact problem (2.1)–(2.6).

Remark 4.2 *We mentioned in Section 3 that $\boldsymbol{\Phi}$ and $\boldsymbol{u}+\boldsymbol{\Phi}$ are never rigid body displacements in the general case. We remark that in the particular case of the functions involved in Proposition 4.1, $\boldsymbol{u}$ is never a rigid body displacement. This results from the definition of $\boldsymbol{u}$ in (4.14)–(4.15) since for any $(x_c, y_c) \in \Sigma$, e.g., $\partial u_x/\partial x \ne 0$.*

A consequence of the previous study is the following result.

Corollary 4.3 *The Coulomb friction problem (2.1)–(2.6) does not admit a unique solution in the general case when $\mu > 1$. More precisely, for any $0 < \nu < 1/2$, the problem (2.1)–(2.6) does generally not admit a unique solution when*

$$\mu > \sqrt{\frac{1-\nu}{\nu}}.$$

Proof. Let us consider the domain Σ depicted in Figure 2:

$$\Sigma = \left\{ (x_c, y_c) \in \mathbb{R}^2,\ x_c \in]0, 1[,\ \sqrt{\frac{1}{4} + x_c - x_c^2} - \frac{1}{2} < y_c < x_c \sqrt{\frac{1 - x_c}{1 + x_c}} \right\}$$

and introduce the function f defined on Σ as follows: $f(x_c, y_c) = x_c/y_c$. Set $g(x) = \sqrt{1/4 + x - x^2} - 1/2$ and $h(x) = x\sqrt{(1-x)/(1+x)}$. From the definition of Σ and since the right derivatives of g and h at 0 are equal to 1 we deduce that

$$f(\Sigma) =]1, +\infty[.$$

Combining this with Proposition 4.1 proves the existence of non-unique solutions when $\mu > 1$.

Next, we consider the second part of the Corollary and we show that the non-unique solutions of the Proposition 4.1 occur when the Poisson ratio is greater than $1/(1+\mu^2)$. It suffices to study the function $\nu(x_c, y_c)$ introduced in (4.8) when $(x_c, y_c) \in \Sigma$ and $y_c = x_c/\mu$ with $\mu \in]1, +\infty[$. This consists of studying the function ν on the straight line segment $]D, E[$ where $D = ((\mu^2 - \mu)/(\mu^2 + 1), (\mu - 1)/(\mu^2 + 1))$ and $E = ((\mu^2 - 1)/(\mu^2 + 1), (\mu - 1/\mu)/(\mu^2 + 1))$. On the interval $]D, E[$, $y_c = x_c/\mu$ and the function $\nu(x_c)$ is decreasing from $1/2$ to $1/(1 + \mu^2)$ since

$$\nu'(x_c) = 2\frac{\mu^2(x_c - \mu^2 + x_c\mu^2)}{(x_c^2 + \mu^2(x_c - 1)^2)^2(1 + \mu^2)} < 0, \quad \forall x_c \in \left]\frac{\mu^2 - \mu}{\mu^2 + 1}, \frac{\mu^2 - 1}{\mu^2 + 1}\right[.$$

Hence the problem does generally not admit unique solutions when $\mu > 1$ and $\nu > 1/(1 + \mu^2)$. That ends the proof. □

5. Concluding remarks

The setting introduced in this work allows to prove that the unilateral contact problem with Coulomb friction in continuum elastostatics does not admit unique solutions for large friction coefficients in the general case. Nevertheless numerous problems concerning nonuniqueness seem to remain open. Let us briefly enumerate some of them.

The existence of an example with multiple solutions to (2.1)–(2.6) for arbitrary small friction coefficients is an open problem. The weaker problem which consists of finding for any small friction coefficient an example of nonuniqueness remains open. Another question concerns the existence of multiple solutions for some problems in the pure compressible case when the Poisson ratio vanishes.

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