A posteriori error control of finite element approximations for Coulomb's frictional contact

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This paper is concerned with the frictional unilateral contact problem governed by Coulomb's law. We define an a posteriori error estimator based on the concept of error in the constitutive relation to quantify the accuracy of a finite element approximation of the problem. We propose and study different mixed finite element approaches and discuss their properties in order to compute the estimator. The information given by the error estimates is then coupled with a mesh adaptivity technique which provides the user with the desired quality and minimizes the computation costs. The numerical implementation of the error estimator as well as optimized computations are performed.

Keywords : Coulomb's friction law, a posteriori error estimates, finite elements, error in the constitutive relation, optimized computations.

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1. Introduction and problem set-up

The finite element method is currently used in the numerical realization of frictional contact problems occurring in many engineering applications (see [13]). An important task consists of evaluating numerically the quality of the finite element computations by using a posteriori error estimators. In elasticity, several different approaches leading to various error estimators have been developed, in particular the error estimators introduced in [2] based on the residual of the equilibrium equations, the estimators linked to the smoothing of finite element stresses (see [22]) and the estimators based on the errors in the constitutive relation (see [14, 17]). A review of different a posteriori error estimators can be found in [21].

For frictionless unilateral contact problems, the residual based method was considered and studied in [3] (see also the references quoted therein) using a penalized approach and the study of error in the constitutive relation was performed in [5].

In the present paper, we are interested in the more general and currently used Coulomb's frictional contact model and we choose the estimators in the constitutive relation to quantify the accuracy of the finite element approximations. As far as we know, there is no literature concerning a posteriori error estimators for Coulomb's frictional unilateral contact model. The latter is recalled hereafter.

Let be given an elastic body occupying a bounded domain Ω in $\mathbb{R}^2$ whose generic point is denoted $\boldsymbol{x} = (x_1, x_2)$. The boundary Γ of Ω is Lipschitz and divided as follows: $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_C}$ where Γ_D, Γ_N and Γ_C are three open disjoint parts. We suppose that the displacement field is given on Γ_D (to simplify, we assume afterwards that the body is clamped on Γ_D). On the boundary part Γ_N, a density of forces denoted $\boldsymbol{F} \in (L^2(\Gamma_N))^2$ is applied. The third part is the segment Γ_C, in frictional contact with a rigid foundation (see Figure 1). The body Ω is submitted to a given density of volume forces $\boldsymbol{f} \in (L^2(\Omega))^2$. Let the notation $\boldsymbol{n} = (n_1, n_2)$ represent the unit outward normal vector on Γ and define the unit tangent vector $\boldsymbol{t} = (-n_2, n_1)$. Let us denote by $\mu > 0$ the friction coefficient on Γ_C.

The problem consists of finding the displacement field $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^2$ and the stress tensor field $\boldsymbol{\sigma} : \Omega \longrightarrow \mathscr{S}_2$ satisfying (1.1)-(1.10)

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{1.1}$$

$$\boldsymbol{div}\ \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{F} \quad \text{on } \Gamma_N, \tag{1.3}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_D. \tag{1.4}$$

where $\mathscr{S}_2$ stands for the space of second order symmetric tensors on $\mathbb{R}^2$, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u})$ denotes the linearized strain tensor field, $\mathcal{C}$ is a fourth order symmetric and elliptic tensor of linear elasticity and $\boldsymbol{div}$ represents the divergence operator of tensor valued functions.

In order to introduce the equations on Γ_C, let us adopt the following notation: $\boldsymbol{u} = u_n\boldsymbol{n} + u_t\boldsymbol{t}$ and $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \sigma_n(\boldsymbol{u})\boldsymbol{n} + \sigma_t(\boldsymbol{u})\boldsymbol{t}$. The equations modelling unilateral contact with Coulomb friction are as follows on Γ_C:

$$u_n \leq 0, \tag{1.5}$$

$$\sigma_n(\boldsymbol{u}) \leq 0, \tag{1.6}$$

$$\sigma_n(\boldsymbol{u})\, u_n = 0, \tag{1.7}$$

$$|\sigma_t(\boldsymbol{u})| \leq \mu|\sigma_n(\boldsymbol{u})|, \tag{1.8}$$

$$|\sigma_t(\boldsymbol{u})| < \mu|\sigma_n(\boldsymbol{u})| \Longrightarrow u_t = 0, \tag{1.9}$$

$$|\sigma_t(\boldsymbol{u})| = \mu|\sigma_n(\boldsymbol{u})| \Longrightarrow \exists\lambda \geq 0 \ \text{ such that } \ u_t = -\lambda\sigma_t(\boldsymbol{u}). \tag{1.10}$$

The variational formulation of problem (1.1)-(1.10) has been obtained by Duvaut and Lions in [8]. It consists of finding $\boldsymbol{u}$ such that

$$\boldsymbol{u} \in \boldsymbol{K}_{ad}, \qquad a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geq L(\boldsymbol{v} - \boldsymbol{u}), \quad \forall \boldsymbol{v} \in \boldsymbol{K}_{ad}, \tag{1.11}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \, d\Omega, \qquad j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_C} \mu |\sigma_n(\boldsymbol{u})| |v_t| \, d\Gamma,$$

$$L(\boldsymbol{v}) = \int_{\Omega} \boldsymbol{f}.\boldsymbol{v} \, d\Omega + \int_{\Gamma_N} \boldsymbol{F}.\boldsymbol{v} \, d\Gamma,$$

are defined for any $\boldsymbol{u}$ and $\boldsymbol{v}$ in

$$\boldsymbol{V} = \left\{ \boldsymbol{v} \in (H^1(\Omega))^2; \ \boldsymbol{v} = 0 \text{ on } \Gamma_D \right\}.$$

The notation $H^1(\Omega)$ represents the standard Sobolev space, $\cdot$ and $:$ stand for the inner product in $\mathbb{R}^2$ and $\mathscr{S}_2$ respectively. In (1.11), $\boldsymbol{K}_{ad}$ denotes the closed convex cone of admissible displacement fields satisfying the non-penetration condition

$$\boldsymbol{K}_{ad} = \left\{ \boldsymbol{v} \in \boldsymbol{V}; \ v_n \leq 0 \text{ on } \Gamma_C \right\}.$$

The first existence result for problem (1.11) has been obtained in [20] when Ω is an infinitely long strip and if the friction coefficient of compact support in Γ_C is sufficiently small. The extension of these results to domains with smooth boundaries can be found in [12]. A recent improvement in [9] states existence when the friction coefficient μ is lower than $\frac{\sqrt{3-4\nu}}{2-2\nu}$, ν denoting Poisson's ratio in Ω $(0 < \nu < \frac{1}{2})$. When the loads $\boldsymbol{f}$ and $\boldsymbol{F}$ are not equal to zero, there is to our knowledge neither uniqueness result nor non-uniqueness example for problem (1.11). Let us mention that there exists several laws "mollifying" Coulomb's frictional contact model (see, e.g. [13, 19] and the references quoted therein) and that such regularizations lead to more existence and uniqueness properties.

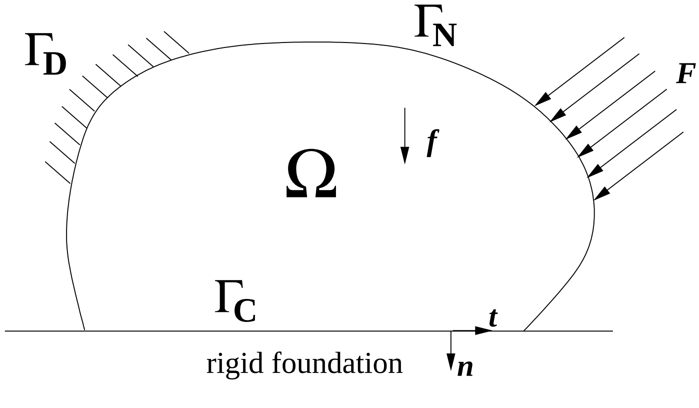

Figure 1: Setting of the problem

Our paper is outlined as follows. In section 2, we first recall the convenient setting which consists of separating the kinematic conditions, the equilibrium equations and the constitutive relations in order to define the error estimator and to study its properties. In section 3, we propose two mixed finite element methods for Coulomb's frictional unilateral contact problem. We prove the existence of solutions and we study the discrete frictional contact properties satisfied by such solutions. Section 4 is concerned with the practical construction of such an estimator. In section 5, several numerical studies in which we compute and couple the estimator with a mesh adaptivity procedure are performed.

2. The error estimator for Coulomb's frictional contact problem

The aim of this section is to introduce the concept of error in the constitutive relation for the frictional unilateral contact problem. Before defining the estimator, let us begin with some useful setting and notation.

2.1. *The appropriate setting for error in the constitutive relation*

To define the error in the constitutive relation, the contact part Γ_C is considered like in [15, 5] as an interface on which two unknowns $\boldsymbol{w}$ (displacement field) and $\boldsymbol{r}$ (density of surfacic forces due to the frictional contact with the rigid foundation) are to be found. If $\boldsymbol{n} = (n_1, n_2)$ and $\boldsymbol{t} = (-n_2, n_1)$ stand for the unit outward normal and tangent on Γ, we adopt afterwards the notation $\boldsymbol{z} = z_n\boldsymbol{n} + z_t\boldsymbol{t}$ for any vector $\boldsymbol{z}$.

The unilateral contact problem with Coulomb's friction law (1.1)-(1.10) is reformulated by using these quantities and it consists of finding the displacement field $\boldsymbol{u}$ on Ω, the stress tensor field $\boldsymbol{\sigma}$ on Ω and $\boldsymbol{w}$, $\boldsymbol{r}$ on Γ_C satisfying the following equations (2.1)–(2.9).

- The displacement fields $\boldsymbol{u}$ and $\boldsymbol{w}$ verify the kinematic conditions:

$$\boldsymbol{u} = 0 \text{ on } \Gamma_D \quad \text{and} \quad \boldsymbol{w} = \boldsymbol{u} \text{ on } \Gamma_C. \tag{2.1}$$

- The fields $\boldsymbol{\sigma}$ and $\boldsymbol{r}$ satisfy the equilibrium equation:

$$-\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v})\, d\Omega + \int_\Omega \boldsymbol{f}.\boldsymbol{v}\, d\Omega + \int_{\Gamma_N} \boldsymbol{F}.\boldsymbol{v}\, d\Gamma + \int_{\Gamma_C} \boldsymbol{r}.\boldsymbol{v}\, d\Gamma = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{2.2}$$

- The fields $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ are linked by the constitutive law of linear elasticity:

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}). \tag{2.3}$$

- The displacement field $\boldsymbol{w} = w_n\boldsymbol{n} + w_t\boldsymbol{t}$ and the density of forces $\boldsymbol{r} = r_n\boldsymbol{n} + r_t\boldsymbol{t}$

satisfy the unilateral contact conditions with Coulomb's friction law along Γ_C:

$$w_n \leq 0, \tag{2.4}$$
$$r_n \leq 0, \tag{2.5}$$
$$r_n w_n = 0, \tag{2.6}$$
$$|r_t| \leq \mu |r_n|, \tag{2.7}$$
$$|r_t| < \mu |r_n| \quad \Longrightarrow \quad w_t = 0, \tag{2.8}$$
$$|r_t| = \mu |r_n| \quad \Longrightarrow \quad \exists \lambda \geq 0 \text{ such that } w_t = -\lambda r_t. \tag{2.9}$$

Let us define the convex cones

$$K = \Big\{ \boldsymbol{z}; \boldsymbol{z} = z_n \boldsymbol{n} + z_t \boldsymbol{t} \text{ such that } z_n \leq 0 \Big\},$$

and

$$C_\mu = \Big\{ \boldsymbol{s}; \boldsymbol{s} = s_n \boldsymbol{n} + s_t \boldsymbol{t} \text{ such that } s_n \leq 0 \text{ and } |s_t| \leq \mu |s_n| \Big\}.$$

Denoting by I_A the indicator function of the set A (i.e. $I_A(\boldsymbol{z}) = 0$ if $\boldsymbol{z} \in A$ and $I_A(\boldsymbol{z}) = +\infty$ if $\boldsymbol{z} \notin A$), it can be easily checked that the frictional contact conditions (2.4)-(2.9) can be also written in a more compact form

$$I_K(\boldsymbol{w}) + I_{C_\mu}(\boldsymbol{r}) + \mu |r_n||w_t| + r_t w_t + r_n w_n = 0, \qquad \text{on } \Gamma_C. \tag{2.10}$$

2.2. *Definitions*

We begin with recalling the definition of an admissible pair:

Definition 2.1 *A pair $\hat{s} = ((\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}), (\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{r}}))$ is admissible if the kinematic conditions* (2.1) *and the equilibrium equations* (2.2) *are fulfilled.*

We are now in a position to define the estimator based on the error in the constitutive relation:

Definition 2.2 *Let $\hat{s} = ((\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}), (\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{r}}))$ be admissible. The error estimator $e(\hat{s})$ is as follows:*

$$e(\hat{s}) = \Bigg(\|\hat{\boldsymbol{\sigma}} - \mathcal{C}\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}})\|^2_{\sigma,\Omega} + 2 \int_{\Gamma_C} \Big(I_K(\hat{\boldsymbol{w}}) + I_{C_\mu}(\hat{\boldsymbol{r}}) + \mu |\hat{r}_n||\hat{w}_t| + \hat{r}_t \hat{w}_t + \hat{r}_n \hat{w}_n \Big) \, d\Gamma \Bigg)^{\frac{1}{2}}, \tag{2.11}$$

where the norm $\|.\|_{\sigma,\Omega}$ on the stress tensor fields is defined by

$$\|\boldsymbol{\sigma}\|_{\sigma,\Omega} = \Big(\int_\Omega (\mathcal{C}^{-1}\boldsymbol{\sigma}) : \boldsymbol{\sigma} \, d\Omega \Big)^{\frac{1}{2}}.$$

Let us notice that the function in the integral term of (2.11) is always nonnegative at $\boldsymbol{x} \in \Gamma_C$: it is equal to $+\infty$ if $\hat{\boldsymbol{w}}(\boldsymbol{x}) \notin K$ or $\hat{\boldsymbol{r}}(\boldsymbol{x}) \notin C_\mu$; otherwise it is nonnegative owing to $(\mu|\hat{r}_n||\hat{w}_t|+\hat{r}_t\hat{w}_t)(\boldsymbol{x}) \geq 0$ and $(\hat{r}_n\hat{w}_n)(\boldsymbol{x}) \geq 0$. To avoid more notation, we will skip over the regularity aspects of the functions defined on Γ_C which are beyond the scope of this paper and we write afterwards integral terms instead of duality pairings. The first natural property arising directly from the definition of $e(\hat{s})$ becomes:

Property 2.3 *Let $\hat{s}$ be admissible. Then $e(\hat{s}) = 0$ if and only if $\hat{s} = ((\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}), (\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{r}}))$ is solution to the reference problem* (2.1)-(2.9).

Let us define some quantities useful for the forthcoming study:

Definition 2.4 *Let $\hat{s}$ be admissible. The relative error $\epsilon(\hat{s})$ is as follows:*

$$\epsilon(\hat{s}) = \frac{e(\hat{s})}{\|\hat{\boldsymbol{\sigma}} + \mathcal{C}\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}})\|_{\sigma,\Omega}}. \tag{2.12}$$

Given a part E of Ω, the local error contribution $\epsilon_E(\hat{s})$ is defined as

$$\epsilon_E(\hat{s}) = \tag{2.13}$$
$$\frac{\left(\|\hat{\boldsymbol{\sigma}} - \mathcal{C}\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}})\|^2_{\sigma,E} + 2\int_{\Gamma_C\cap E}\Big(I_K(\hat{\boldsymbol{w}}) + I_{C_\mu}(\hat{\boldsymbol{r}}) + \mu|\hat{r}_n||\hat{w}_t| + \hat{r}_t\hat{w}_t + \hat{r}_n\hat{w}_n\Big)\, d\Gamma\right)^{\frac{1}{2}}}{\|\hat{\boldsymbol{\sigma}} + \mathcal{C}\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}})\|_{\sigma,\Omega}}$$

where $\|\boldsymbol{\sigma}\|_{\sigma,E} = \left(\int_E (\mathcal{C}^{-1}\boldsymbol{\sigma}) : \boldsymbol{\sigma}\, d\Omega\right)^{\frac{1}{2}}$.

For the sake of simplicity of notations, we will write ϵ and ϵ_E instead of $\epsilon(\hat{s})$ and $\epsilon_E(\hat{s})$ in the following studies. It is straightforward that

$$\bigcup_{E_i\cap E_j=\emptyset,\ i\neq j} E_i = \Omega \quad \Longrightarrow \quad \epsilon^2 = \sum_i \epsilon^2_{E_i}.$$

2.3. *Link between the estimator and the other errors*

This part is concerned with the relation between the error in the constitutive law and the other errors. We suppose that a solution to the exact problem (2.1)-(2.9) exists which is satisfied when μ is small enough (see [9]). The next proposition generalizes former results (see [5]) obtained in the frictionless case (corresponding to $\mu = 0$).

Proposition 2.5 *Let $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\sigma}, \boldsymbol{r})$ be solution to Coulomb's frictional contact problem* (2.1)-(2.9)*. Let $\hat{s} = ((\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}), (\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{r}}))$ be admissible. Then*

$$\|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}\|^2_{\sigma,\Omega} + \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|^2_{u,\Omega} + 2\mu\int_{\Gamma_C}(r_n - \hat{r}_n)(|\hat{w}_t| - |w_t|)\, d\Gamma \leq e^2(\hat{s}), \tag{2.14}$$

where the norm $\|.\|_{u,\Omega}$ *on the displacement fields is defined by*

$$\|\boldsymbol{u}\|_{u,\Omega} = \Big(\int_\Omega (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{u})\; d\Omega\Big)^{\frac{1}{2}} = \big(a(\boldsymbol{u},\boldsymbol{u})\big)^{\frac{1}{2}}.$$

Consequently

$$\|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}\|^2_{\sigma,\Omega} + 2\mu \int_{\Gamma_C} (r_n - \hat{r}_n)(|\hat{w}_t| - |w_t|)\; d\Gamma \le e^2(\hat{s}), \tag{2.15}$$

and

$$\|\boldsymbol{u} - \hat{\boldsymbol{u}}\|^2_{u,\Omega} + 2\mu \int_{\Gamma_C} (r_n - \hat{r}_n)(|\hat{w}_t| - |w_t|)\; d\Gamma \le e^2(\hat{s}). \tag{2.16}$$

Proof. We begin with noticing that the property obviously holds when $\hat{\boldsymbol{w}} \notin K$ or $\hat{\boldsymbol{r}} \notin C_\mu$ on a set of positive measure. In such a case the error estimator is equal to infinity. Next, we then suppose that $\hat{\boldsymbol{w}} \in K$ and $\hat{\boldsymbol{r}} \in C_\mu$ almost everywhere. One immediately gets

$$\begin{aligned}\|\hat{\boldsymbol{\sigma}} - \mathcal{C}\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}})\|^2_{\sigma,\Omega} &= \|\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma} + \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u} - \hat{\boldsymbol{u}})\|^2_{\sigma,\Omega} \\ &= \|\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\|^2_{\sigma,\Omega} + \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|^2_{u,\Omega} + 2\int_\Omega (\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}) : \boldsymbol{\varepsilon}(\boldsymbol{u} - \hat{\boldsymbol{u}})\; d\Omega.\end{aligned}$$

The stress fields $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$ satisfy the equilibrium equation (2.2) and the displacement fields $\boldsymbol{u}$ and $\hat{\boldsymbol{u}}$ verify the kinematic conditions (2.1). Hence

$$\|\hat{\boldsymbol{\sigma}} - \mathcal{C}\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}})\|^2_{\sigma,\Omega} = \|\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\|^2_{\sigma,\Omega} + \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|^2_{u,\Omega} + 2\int_{\Gamma_C} (\hat{\boldsymbol{r}} - \boldsymbol{r}).(\boldsymbol{w} - \hat{\boldsymbol{w}})\; d\Gamma. \tag{2.17}$$

Developing the integral term yields

$$\begin{aligned}&\int_{\Gamma_C} (\hat{\boldsymbol{r}} - \boldsymbol{r}).(\boldsymbol{w} - \hat{\boldsymbol{w}})\; d\Gamma = \\ &\qquad \int_{\Gamma_C} \hat{r}_n w_n\; d\Gamma + \int_{\Gamma_C} \hat{r}_t w_t\; d\Gamma + \int_{\Gamma_C} r_n \hat{w}_n\; d\Gamma + \int_{\Gamma_C} r_t \hat{w}_t\; d\Gamma \\ &\qquad - \int_{\Gamma_C} r_n w_n\; d\Gamma - \int_{\Gamma_C} r_t w_t\; d\Gamma - \int_{\Gamma_C} \hat{r}_n \hat{w}_n\; d\Gamma - \int_{\Gamma_C} \hat{r}_t \hat{w}_t\; d\Gamma.\end{aligned} \tag{2.18}$$

Putting together (2.17) and (2.18) in the definition (2.11) of the estimator leads to

$$\begin{aligned}e^2(\hat{s}) = {}& \|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}\|^2_{\sigma,\Omega} + \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|^2_{u,\Omega} \\ &+2\int_{\Gamma_C} \hat{r}_n w_n\; d\Gamma + 2\int_{\Gamma_C} \hat{r}_t w_t\; d\Gamma + 2\int_{\Gamma_C} r_n \hat{w}_n\; d\Gamma + 2\int_{\Gamma_C} r_t \hat{w}_t\; d\Gamma \\ &-2\int_{\Gamma_C} r_n w_n\; d\Gamma - 2\int_{\Gamma_C} r_t w_t\; d\Gamma + 2\int_{\Gamma_C} \mu|\hat{r}_n||\hat{w}_t|\; d\Gamma.\end{aligned}$$

Noting that $r_n\hat{w}_n \geq 0$, $\hat{r}_n w_n \geq 0$ and $r_n w_n = 0$ on Γ_C, we get

$$\begin{aligned} e^2(\hat{s}) \geq & \|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}\|^2_{\sigma,\Omega} + \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|^2_{u,\Omega} \\ & +2\int_{\Gamma_C} \hat{r}_t w_t \, d\Gamma + 2\int_{\Gamma_C} r_t \hat{w}_t \, d\Gamma - 2\int_{\Gamma_C} r_t w_t \, d\Gamma + 2\int_{\Gamma_C} \mu|\hat{r}_n||\hat{w}_t| \, d\Gamma. \end{aligned}$$

According to (2.7)-(2.9), the equality

$$-r_t w_t = \mu|r_n||w_t|$$

holds on Γ_C. Moreover $\boldsymbol{r} \in C_\mu$ and $\hat{\boldsymbol{r}} \in C_\mu$ lead to the bounds

$$r_t\hat{w}_t \geq -\mu|r_n||\hat{w}_t| \qquad \text{and} \qquad \hat{r}_t w_t \geq -\mu|\hat{r}_n||w_t|.$$

Consequently

$$e^2(\hat{s}) \geq \|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}\|^2_{\sigma,\Omega} + \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|^2_{u,\Omega} + 2\mu\int_{\Gamma_C} (|r_n| - |\hat{r}_n|)(|w_t| - |\hat{w}_t|) \, d\Gamma.$$

The bound (2.14) is obtained thanks to $r_n \leq 0$ and $\hat{r}_n \leq 0$. Both bounds (2.15) and (2.16) are an obvious consequence. □

It is easy to check that no information on the sign of the integral term in (2.14) is available. This is not at all surprising because the evaluation of such a term corresponds also to the study of the uniqueness for the (quasi-)variational inequality (1.11) with classical arguments (i.e. by choosing and subtracting two solutions) which does not lead to a successful conclusion. Nevertheless, the following remark shows that the integral term can be bounded at least in a particular case.

Remark 2.6 *If the exact solution and the admissible solution satisfy $w_t \geq 0$ and $\hat{w}_t \geq 0$ on Γ_C (or $w_t \leq 0$ and $\hat{w}_t \leq 0$ on Γ_C), and if the measure of Γ_D is positive, then inequality* (2.14) *becomes more relevant since the integral term in* (2.14) *can be estimated as follows:*

$$\begin{aligned} \left| \int_{\Gamma_C} (r_n - \hat{r}_n)(|\hat{w}_t| - |w_t|) \, d\Gamma \right| &= \left| \int_{\Gamma_C} (r_n - \hat{r}_n)(\hat{w}_t - w_t) \, d\Gamma \right| \\ &\leq \|r_n - \hat{r}_n\|_{H^{-\frac{1}{2}}(\Gamma_C)} \|w_t - \hat{w}_t\|_{H^{\frac{1}{2}}(\Gamma_C)} \\ &\leq C\|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}\|_{(L^2(\Omega))^4} \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|_{(H^1(\Omega))^2} \\ &\leq C'\|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}\|_{\sigma,\Omega} \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|_{u,\Omega} \\ &\leq C''\Big(\|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}\|^2_{\sigma,\Omega} + \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|^2_{u,\Omega}\Big), \end{aligned}$$

where $H^{\frac{1}{2}}(\Gamma_C)$ stands for a fractionally Sobolev space (see [1]*) and $H^{-\frac{1}{2}}(\Gamma_C)$ is its dual space. The bounds of $\|r_n - \hat{r}_n\|_{H^{-\frac{1}{2}}(\Gamma_C)}$ and $\|w_t - \hat{w}_t\|_{H^{\frac{1}{2}}(\Gamma_C)}$ are obtained using Green's formula and the trace theorem respectively. Moreover the norms $\|.\|_{(H^1(\Omega))^2}$ and $\|.\|_{u,\Omega}$ are equivalent since meas$(\Gamma_D) > 0$. In such a case, the integral term can* *be removed from* (2.14), (2.15) *and* (2.16) *and we come to the conclusion that there exists a positive constant* C *such that for small friction coefficients,* $\|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}\|_{\sigma,\Omega}$ *and* $\|\boldsymbol{u} - \hat{\boldsymbol{u}}\|_{u,\Omega}$ *can be bounded by* $(1/\sqrt{1-\mu C})e(\hat{s})$. *Concerning the general case, we think that one could reasonably expect that if the exact and admissible solutions are smooth enough and if the friction coefficient is small then the integral term multiplied by* 2μ *is small in comparison with* $\|\boldsymbol{u} - \hat{\boldsymbol{u}}\|^2_{u,\Omega}$ *and* $\|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}\|^2_{\sigma,\Omega}$.

Remark 2.7 *If instead of Coulomb's law* (2.10), *one considers a Tresca's type friction law:*

$$I_K(\boldsymbol{w}) + I_C(\boldsymbol{r}) + k|w_t| + r_t w_t + r_n w_n = 0, \qquad \text{on } \Gamma_C, \tag{2.19}$$

where $k \geq 0$ *and where*

$$C = \Big\{ \boldsymbol{s}; \boldsymbol{s} = s_n \boldsymbol{n} + s_t \boldsymbol{t} \text{ such that } s_n \leq 0 \text{ and } |s_t| \leq k \Big\},$$

then the problem (2.1)-(2.3), (2.19) *admits an unique solution* $(\boldsymbol{u}_k, \boldsymbol{w}_k, \boldsymbol{\sigma}_k, \boldsymbol{r}_k)$ *and the bound*

$$\|\boldsymbol{\sigma}_k - \hat{\boldsymbol{\sigma}}\|^2_{\sigma,\Omega} + \|\boldsymbol{u}_k - \hat{\boldsymbol{u}}\|^2_{u,\Omega} \leq e^2(\hat{s}), \tag{2.20}$$

holds for any admissible $\hat{s} = ((\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}), (\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{r}}))$.

Estimate (2.20) *is obtained following the same points as in the proof of estimate* (2.14). *In particular, if* $k = 0$ *in* (2.19) *or equivalently* $\mu = 0$ *in* (2.10), *we recover the frictionless unilateral contact model.*

3. The discrete Coulomb's frictional contact problem

In this section, we propose and study the properties of two mixed discrete finite element formulations for Coulomb's frictional contact in order to implement the error estimator. Let us mention that a detailed study of several (different) mixed finite element methods for frictionless and frictional contact problems can be found in [10], [11].

3.1. *The mixed finite element formulations*

The body Ω is discretized by using a family of triangulations $(\mathscr{T}_h)_h$ made of finite elements of degree one. For technical purposes, we assume (in Section 3.1 only) that $\overline{\Gamma_D} \cap \overline{\Gamma_C} = \emptyset$ which is generally not restrictive in engineering applications and that the bilinear form $a(.,.)$ is $\boldsymbol{V}-$elliptic. Let us denote by $h > 0$ the discretization parameter representing the greatest diameter of a triangle in $\mathscr{T}_h$. The space approximating $\boldsymbol{V}$ becomes:

$$\boldsymbol{V}_h = \Big\{ \boldsymbol{v}_h;\ \boldsymbol{v}_h \in (\mathscr{C}(\overline{\Omega}))^2,\ \boldsymbol{v}_h|_T \in (P_1(T))^2\ \ \forall T \in \mathscr{T}_h,\ \boldsymbol{v}_h = 0 \text{ on } \Gamma_D \Big\},$$

where $\mathscr{C}(\overline{\Omega})$ stands for the space of continuous functions on $\overline{\Omega}$ and $P_1(T)$ represents the space of polynomial functions of degree one on T. On the boundary of Ω, we still keep the notation $\boldsymbol{v}_h = v_{hn}\boldsymbol{n} + v_{ht}\boldsymbol{t}$ for every $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and we denote by $(T_h)_h$ the family of monodimensional meshes on Γ_C inherited by $(\mathscr{T}_h)_h$.

We next introduce two convex sets of Lagrange multipliers denoted $\boldsymbol{M}'_h(g)$ and $\boldsymbol{M}''_h(g)$. The convex $\boldsymbol{M}'_h(g)$ is defined by $\boldsymbol{M}'_h(g) = M'_{hn} \times M'_{ht}(g)$ where

$$M'_{hn} = \Big\{\nu; \nu \in \mathscr{C}(\overline{\Gamma_C}),\ \nu|_S \in P_1(S),\ \ \forall S \in T_h,\ \nu \le 0 \text{ on } \Gamma_C\Big\},$$

and for $g \in -M'_{hn}$, we define $M'_{ht}(g)$ as follows:

$$M'_{ht}(g) = \Big\{\nu; \nu \in \mathscr{C}(\overline{\Gamma_C}),\ \nu|_S \in P_1(S),\ \ \forall S \in T_h,\ |\nu| \le g \text{ on } \Gamma_C\Big\}.$$

We denote by p the number of nodes of the triangulation on Γ_C and by $\psi_i, 1 \le i \le p$ the monodimensional basis functions on Γ_C (the function ψ_i is continuous on Γ_C, linear on each segment of T_h, equal to 1 at node i and to 0 at the other nodes). The second convex $\boldsymbol{M}''_h(g)$ is given by $\boldsymbol{M}''_h(g) = M''_{hn} \times M''_{ht}(g)$ with

$$M''_{hn} = \Big\{\nu; \nu \in \mathscr{C}(\overline{\Gamma_C}),\ \nu|_S \in P_1(S),\ \ \forall S \in T_h,\ \int_{\Gamma_C} \nu\psi_i \, d\Gamma \le 0,\ \ \forall 1 \le i \le p\Big\}.$$

If $g \in -M''_{hn}$, $M''_{ht}(g)$ is given by:

$$M''_{ht}(g) = \Big\{\nu; \nu \in \mathscr{C}(\overline{\Gamma_C}),\ \nu|_S \in P_1(S),\ \ \forall S \in T_h,\ \Big|\int_{\Gamma_C} \nu\psi_i \, d\Gamma\Big| \le \int_{\Gamma_C} g\psi_i \, d\Gamma,\ \ \forall 1 \le i \le p\Big\}.$$

Next, the notation $\boldsymbol{M}_h(g) = M_{hn} \times M_{ht}(g)$ denotes either $\boldsymbol{M}'_h(g) = M'_{hn} \times M'_{ht}(g)$ or $\boldsymbol{M}''_h(g) = M''_{hn} \times M''_{ht}(g)$.

We then introduce a intermediary problem with a given slip limit $-\mu g_{hn}$ where $g_{hn} \in M_{hn}$. This problem denoted $P(g_{hn})$ consists of finding $\boldsymbol{u}_h \in \boldsymbol{V}_h$ and $(\lambda_{hn}, \lambda_{ht}) \in M_{hn} \times M_{ht}(-\mu g_{hn}) = \boldsymbol{M}_h(-\mu g_{hn})$ such that:

$$(P(g_{hn}))\quad \begin{cases} a(\boldsymbol{u}_h, \boldsymbol{v}_h) - \displaystyle\int_{\Gamma_C} \lambda_{hn} v_{hn} \, d\Gamma - \int_{\Gamma_C} \lambda_{ht} v_{ht} \, d\Gamma = L(\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ \displaystyle\int_{\Gamma_C} (\nu_{hn} - \lambda_{hn}) u_{hn} \, d\Gamma + \int_{\Gamma_C} (\nu_{ht} - \lambda_{ht}) u_{ht} \, d\Gamma \ge 0, \\ \qquad\qquad \forall (\nu_{hn}, \nu_{ht}) \in \boldsymbol{M}_h(-\mu g_{hn}). \end{cases} \tag{3.1}$$

Problem $P(g_{hn})$ is equivalent of finding a saddle-point $(\boldsymbol{u}_h, \lambda_{hn}, \lambda_{ht}) = (\boldsymbol{u}_h, \boldsymbol{\lambda}_h) \in \boldsymbol{V}_h \times \boldsymbol{M}_h(-\mu g_{hn})$ verifying

$$\mathscr{L}(\boldsymbol{u}_h, \boldsymbol{\nu}_h) \le \mathscr{L}(\boldsymbol{u}_h, \boldsymbol{\lambda}_h) \le \mathscr{L}(\boldsymbol{v}_h, \boldsymbol{\lambda}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\ \forall \boldsymbol{\nu}_h \in \boldsymbol{M}_h(-\mu g_{hn}),$$

where

$$\mathscr{L}(\boldsymbol{v}_h, \boldsymbol{\nu}_h) = \frac{1}{2} a(\boldsymbol{v}_h, \boldsymbol{v}_h) - \int_{\Gamma_C} \nu_{hn} v_{hn} \, d\Gamma - \int_{\Gamma_C} \nu_{ht} v_{ht} \, d\Gamma - L(\boldsymbol{v}_h).$$

By using classical arguments on saddle-point problems as Haslinger, Hlaváček and Nečas (1996, p.338), we deduce that there exists such a saddle-point. The strict convexity of $a(.,.)$ implies that the first argument $\boldsymbol{u}_h$ is unique. Besides, the assumption $\overline{\Gamma_D} \cap \overline{\Gamma_C} = \emptyset$ allows us to write

$$\int_{\Gamma_C} \nu_{hn} v_{hn} \, d\Gamma - \int_{\Gamma_C} \nu_{ht} v_{ht} \, d\Gamma = 0, \ \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \qquad \Longrightarrow \qquad \nu_{hn} = 0, \nu_{ht} = 0.$$

Consequently, the second argument $\boldsymbol{\lambda}_h$ is unique and $P(g_{hn})$ admits a unique solution.

It becomes then possible to define a map Φ_h as follows

$$\begin{aligned} \Phi_h : M_{hn} &\longrightarrow M_{hn} \\ g_{hn} &\longmapsto \lambda_{hn}, \end{aligned}$$

where $(\boldsymbol{u}_h, \lambda_{hn}, \lambda_{ht})$ is the solution of $P(g_{hn})$. The introduction of this map allows the definition of a discrete solution of Coulomb's frictional contact problem.

Definition 3.1 *Let $\boldsymbol{M}_h(g) = \boldsymbol{M}'_h(g)$ or $\boldsymbol{M}_h(g) = \boldsymbol{M}''_h(g)$. A solution of Coulomb's discrete frictional contact problem is the solution of $P(\lambda_{hn})$ where $\lambda_{hn} \in M_{hn}$ is a fixed point of Φ_h.*

Proposition 3.2 *Let $\boldsymbol{M}_h(g) = \boldsymbol{M}'_h(g)$ or $\boldsymbol{M}_h(g) = \boldsymbol{M}''_h(g)$. Then for any μ, there exists a solution to Coulomb's discrete frictional contact problem.*

Proof. To establish existence, we use Brouwer's fixed point theorem.
Step 1. We prove that the mapping Φ_h is continuous. Set

$$\tilde{\boldsymbol{V}}_h = \Big\{ \boldsymbol{v}_h \in \boldsymbol{V}_h; \ v_{ht} = 0 \text{ on } \Gamma_C \Big\}, \qquad W_h = \Big\{ \nu; \nu \in \mathscr{C}(\overline{\Gamma_C}), \ \nu|_S \in P_1(S), \ \ \forall S \in T_h \Big\}.$$

Since $\overline{\Gamma_D} \cap \overline{\Gamma_C} = \emptyset$, it is easy to check that the definition of $\|.\|_{-\frac{1}{2},h}$ given by

$$\|\nu\|_{-\frac{1}{2},h} = \sup_{\boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h} \frac{\displaystyle\int_{\Gamma_C} \nu v_{hn} \, d\Gamma}{\|\boldsymbol{v}_h\|_1},$$

is a norm on W_h. The notation $\|.\|_1$ represents the $(H^1(\Omega))^2$-norm.

Let $(\boldsymbol{u}_h, \lambda_{hn}, \lambda_{ht})$ and $(\overline{\boldsymbol{u}_h}, \overline{\lambda_{hn}}, \overline{\lambda_{ht}})$ be the solutions of $(P(g_{hn}))$ and $(P(\overline{g_{hn}}))$ respectively. On the one hand, we get

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) - \int_{\Gamma_C} \lambda_{hn} v_{hn} \, d\Gamma = L(\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h,$$

and

$$a(\overline{\boldsymbol{u}_h}, \boldsymbol{v}_h) - \int_{\Gamma_C} \overline{\lambda_{hn}} v_{hn} \, d\Gamma = L(\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h.$$

Subtracting the previous equalities and using the continuity of the bilinear form $a(.,.)$ gives

$$\int_{\Gamma_C} (\lambda_{hn} - \overline{\lambda_{hn}}) v_{hn} \, d\Gamma = a(\boldsymbol{u}_h - \overline{\boldsymbol{u}_h}, \boldsymbol{v}_h) \leq M \|\boldsymbol{u}_h - \overline{\boldsymbol{u}_h}\|_1 \|\boldsymbol{v}_h\|_1 \qquad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h.$$

Hence, we get a first estimate

$$\|\lambda_{hn} - \overline{\lambda_{hn}}\|_{-\frac{1}{2},h} \leq M \|\boldsymbol{u}_h - \overline{\boldsymbol{u}_h}\|_1. \tag{3.2}$$

On the other hand, we have from (3.1)

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) - \int_{\Gamma_C} \lambda_{hn} v_{hn} \, d\Gamma - \int_{\Gamma_C} \lambda_{ht} v_{ht} \, d\Gamma = L(\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.3}$$

and

$$a(\overline{\boldsymbol{u}_h}, \boldsymbol{v}_h) - \int_{\Gamma_C} \overline{\lambda_{hn}} v_{hn} \, d\Gamma - \int_{\Gamma_C} \overline{\lambda_{ht}} v_{ht} \, d\Gamma = L(\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.4}$$

Choosing $\boldsymbol{v}_h = \overline{\boldsymbol{u}_h} - \boldsymbol{u}_h$ in (3.3) and $\boldsymbol{v}_h = \boldsymbol{u}_h - \overline{\boldsymbol{u}_h}$ in (3.4) implies by addition:

$$a(\boldsymbol{u}_h - \overline{\boldsymbol{u}_h}, \boldsymbol{u}_h - \overline{\boldsymbol{u}_h}) = \int_{\Gamma_C} (\lambda_{hn} - \overline{\lambda_{hn}})(u_{hn} - \overline{u_{hn}}) \, d\Gamma + \int_{\Gamma_C} (\lambda_{ht} - \overline{\lambda_{ht}})(u_{ht} - \overline{u_{ht}}) \, d\Gamma. \tag{3.5}$$

Let us notice that the inequality in (3.1) is obviously equivalent to the two following conditions:

$$\int_{\Gamma_C} (\nu_{hn} - \lambda_{hn}) u_{hn} \, d\Gamma \geq 0, \qquad \forall \nu_{hn} \in M_{hn}, \tag{3.6}$$

$$\int_{\Gamma_C} (\nu_{ht} - \lambda_{ht}) u_{ht} \, d\Gamma \geq 0, \qquad \forall \nu_{ht} \in M_{ht}(-\mu g_{hn}). \tag{3.7}$$

According to the definitions of M'_{hn} and M''_{hn}, we can choose $\nu_{hn} = 0$ and $\nu_{hn} = 2\lambda_{hn}$ in (3.6) which gives

$$\int_{\Gamma_C} \lambda_{hn} u_{hn} \, d\Gamma = 0 \quad \text{and} \quad \int_{\Gamma_C} \nu_{hn} u_{hn} \, d\Gamma \geq 0, \qquad \forall \nu_{hn} \in M_{hn},$$

from which we deduce that

$$\int_{\Gamma_C} (\lambda_{hn} - \overline{\lambda_{hn}})(u_{hn} - \overline{u_{hn}}) \, d\Gamma \leq 0.$$

Denoting by α the ellipticity constant of the bilinear form $a(.,.)$, (3.5) becomes

$$\alpha \|\boldsymbol{u}_h - \overline{\boldsymbol{u}_h}\|_1^2 \leq \int_{\Gamma_C} (\lambda_{ht} - \overline{\lambda_{ht}})(u_{ht} - \overline{u_{ht}}) \, d\Gamma. \tag{3.8}$$

To evaluate the latter integral term, let us first introduce the p-by-p mass matrix $\mathcal{M} = (m_{ij})_{1\le i,j\le p}$ on Γ_C as

$$m_{ij} = \int_{\Gamma_C} \psi_i \psi_j \, d\Gamma, \qquad 1 \le i, j \le p, \tag{3.9}$$

and let U_T, $\overline{U_T}$, G_N, $\overline{G_N}$, denote the vectors of components the nodal values of u_{ht}, $\overline{u_{ht}}$, g_{hn} and $\overline{g_{hn}}$ respectively.

• We begin with considering the mixed method where $\boldsymbol{M}_h(g) = \boldsymbol{M}'_h(g)$. From (3.7), we get

$$\int_{\Gamma_C} \lambda_{ht} u_{ht} \, d\Gamma \le \int_{\Gamma_C} \nu_{ht} u_{ht} \, d\Gamma, \quad \forall \nu_{ht} \in M_{ht}(-\mu g_{hn}),$$

or equivalently

$$\int_{\Gamma_C} \lambda_{ht} u_{ht} \, d\Gamma \le \sum_{i=1}^{p} M_i (\mathcal{M} U_T)_i, \quad \forall M \in \mathbb{R}^p \text{ such that } |M_i| \le -\mu (G_N)_i, \ 1 \le i \le p.$$

It is easy to construct a vector M minimizing the sum and yielding the following bound:

$$\int_{\Gamma_C} \lambda_{ht} u_{ht} \, d\Gamma \le \mu \sum_{i=1}^{p} (G_N)_i |(\mathcal{M} U_T)_i|.$$

A similar expression can be obtained when integrating the term $\overline{\lambda_{ht}} \overline{u_{ht}}$. The two remaining terms of the integral in (3.8) are roughly bounded as follows:

$$-\int_{\Gamma_C} \lambda_{ht} \overline{u_{ht}} \, d\Gamma \le -\mu \sum_{i=1}^{p} (G_N)_i |(\mathcal{M} \overline{U_T})_i| \ ; \ -\int_{\Gamma_C} \overline{\lambda_{ht}} u_{ht} \, d\Gamma \le -\mu \sum_{i=1}^{p} (\overline{G_N})_i |(\mathcal{M} U_T)_i|$$

Finally, (3.8) becomes

$$\begin{aligned} \alpha \|\boldsymbol{u}_h - \overline{\boldsymbol{u}_h}\|_1^2 &\le \mu \sum_{i=1}^{p} (G_N - \overline{G_N})_i \Big(|(\mathcal{M} U_T)_i| - |(\mathcal{M} \overline{U_T})_i| \Big) \\ &\le \mu \left(\sum_{i=1}^{p} (G_N - \overline{G_N})_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{p} (\mathcal{M}(U_T - \overline{U_T}))_i^2 \right)^{\frac{1}{2}} \\ &= \mu \|G_N - \overline{G_N}\|_{\mathbb{R}^p} \, \|U_T - \overline{U_T}\|_{\mathbb{R}^p, \mathcal{M}}, \end{aligned} \tag{3.10}$$

where $\big||x| - |y|\big| \le |x - y|$ and Hölder inequality have been used. The notations $\|.\|_{\mathbb{R}^p}$ and $\|.\|_{\mathbb{R}^p,\mathcal{M}}$ whose definitions are straightforward represent norms on $\mathbb{R}^p$ (the mass matrix $\mathcal{M}$ is nonsingular). As a consequence, there exists constants $C_1(h)$ and $C_2(h)$ depending on h (or equivalently on p) such that

$$\|G_N - \overline{G_N}\|_{\mathbb{R}^p} \le C_1(h) \|g_{hn} - \overline{g_{hn}}\|_{-\frac{1}{2},h} \tag{3.11}$$

and

$$\|U_T - \overline{U_T}\|_{\mathbb{R}^p,\mathcal{M}} \leq C_2(h)\|u_{ht} - \overline{u_{ht}}\|_{L^2(\Gamma_C)} \leq C_3(h)\|\boldsymbol{u}_h - \overline{\boldsymbol{u}_h}\|_1, \tag{3.12}$$

where the trace theorem has been used. Combining (3.10), (3.11), (3.12) and (3.2) implies that there exists a constant $C(h)$ such that

$$\|\lambda_{hn} - \overline{\lambda_{hn}}\|_{-\frac{1}{2},h} \leq \mu C(h)\|g_{hn} - \overline{g_{hn}}\|_{-\frac{1}{2},h}. \tag{3.13}$$

Hence Φ_h is continuous.

- In the case where $\boldsymbol{M}_h(g) = \boldsymbol{M}_h''(g)$, the proof is analogous: first (3.7) implies

$$\int_{\Gamma_C} \lambda_{ht} u_{ht} \, d\Gamma \leq \sum_{i=1}^{p} (\mathcal{M}M)_i (U_T)_i, \;\; \forall M \in \mathbb{R}^p \text{ s.t. } |(\mathcal{M}M)_i| \leq -\mu(\mathcal{M}G_N)_i, \; 1 \leq i \leq p.$$

Therefore

$$\int_{\Gamma_C} \lambda_{ht} u_{ht} \, d\Gamma \leq \mu \sum_{i=1}^{p} (\mathcal{M}G_N)_i |(U_T)_i|.$$

Expression (3.8) leads then to

$$\begin{aligned} \alpha\|\boldsymbol{u}_h - \overline{\boldsymbol{u}_h}\|_1^2 &\leq \mu \sum_{i=1}^{p} (\mathcal{M}(G_N - \overline{G_N}))_i \Big(|(U_T)_i| - |(\overline{U_T})_i| \Big) \\ &\leq \mu \left(\sum_{i=1}^{p} (\mathcal{M}(G_N - \overline{G_N}))_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{p} (U_T - \overline{U_T})_i^2 \right)^{\frac{1}{2}} \\ &= \mu \|G_N - \overline{G_N}\|_{\mathbb{R}^p,\mathcal{M}} \, \|U_T - \overline{U_T}\|_{\mathbb{R}^p}, \end{aligned}$$

and the continuity of Φ_h is proved following the same arguments as in the first case. Step 2. Let $(\boldsymbol{u}_h, \lambda_{hn}, \lambda_{ht})$ be the solution of $(P(g_{hn}))$. Taking $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (3.1) gives

$$a(\boldsymbol{u}_h, \boldsymbol{u}_h) - \int_{\Gamma_C} \lambda_{hn} u_{hn} \, d\Gamma - \int_{\Gamma_C} \lambda_{ht} u_{ht} \, d\Gamma = L(\boldsymbol{u}_h). \tag{3.14}$$

According to

$$\int_{\Gamma_C} \lambda_{hn} u_{hn} \, d\Gamma = 0 \quad \text{ and } \quad \int_{\Gamma_C} \lambda_{ht} u_{ht} \, d\Gamma \leq 0,$$

we deduce from (3.14), the $\boldsymbol{V}-$ellipticity of $a(.,.)$ and the continuity of $L(.)$:

$$\alpha\|\boldsymbol{u}_h\|_1^2 \leq a(\boldsymbol{u}_h, \boldsymbol{u}_h) \leq L(\boldsymbol{u}_h) \leq C\|\boldsymbol{u}_h\|_1,$$

where the constant C depends on the loads $\boldsymbol{f}$ and $\boldsymbol{F}$. So, we get

$$\|\boldsymbol{u}_h\|_1 \leq \frac{C}{\alpha}.$$

In other respects

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) - \int_{\Gamma_C} \lambda_{hn} v_{hn} \, d\Gamma = L(\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h,$$

leads to

$$\int_{\Gamma_C} \lambda_{hn} v_{hn} \, d\Gamma \leq M \|\boldsymbol{u}_h\|_1 \|\boldsymbol{v}_h\|_1 + C \|\boldsymbol{v}_h\|_1, \qquad \forall \boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h.$$

That implies

$$\|\lambda_{hn}\|_{-\frac{1}{2},h} \leq M \|\boldsymbol{u}_h\|_1 + C \leq \Big(\frac{M}{\alpha} + 1\Big) C.$$

Finally

$$\|\Phi_h(g_{hn})\|_{-\frac{1}{2},h} \leq C', \qquad \forall g_{hn} \in M_{hn},$$

where C' depends only on the applied loads $\boldsymbol{f}, \boldsymbol{F}$ and on the continuity and ellipticity constant of $a(.,.)$. This together with the continuity of Φ_h proves that there exists at least a solution of Coulomb's discrete frictional contact problem according to Brouwer's fixed point theorem. □

Remark 3.3 *From* (3.13) *when* $\boldsymbol{M}_h(g) = \boldsymbol{M}'_h(g)$ *or from the equivalent bound which is obtained when* $\boldsymbol{M}_h(g) = \boldsymbol{M}''_h(g)$, *we get a (quite weak) uniqueness result when* $\mu \ C(h) \leq 1$. *That means that uniqueness holds when* μ *is small enough where the denomination "small" depends on the discretization parameter. A more detailed study would show that we are not able to prove that* $C(h)$ *remains bounded as* h *tends towards* 0. *Using another mixed finite element formulation (with a single multiplier instead of two which is not adapted to our a posteriori error estimator) leads to similar existence and uniqueness results* (*see* [10],[11]).

3.2. *The matrix formulation of the frictional contact conditions*

Let us consider a solution $(\boldsymbol{u}_h, \lambda_{hn}, \lambda_{ht}) \in \boldsymbol{V}_h \times M_{hn} \times M_{ht}(-\mu\lambda_{hn})$ of Coulomb's discrete frictional contact problem. We are interested in the matrix translation of the frictional contact conditions:

$$\int_{\Gamma_C} (\nu_{hn} - \lambda_{hn}) u_{hn} \, d\Gamma \geq 0, \qquad \forall \nu_{hn} \in M_{hn}, \tag{3.15}$$

$$\int_{\Gamma_C} (\nu_{ht} - \lambda_{ht}) u_{ht} \, d\Gamma \geq 0, \qquad \forall \nu_{ht} \in M_{ht}(-\mu\lambda_{hn}). \tag{3.16}$$

As previously, Γ_C contains p nodes of the triangulation and $\psi_i, \ 1 \leq i \leq p$ denote the scalar monodimensional basis functions on Γ_C. The p-by-p mass matrix $\mathcal{M} = (m_{ij})_{1\leq i,j\leq p}$ on Γ_C is given by (3.9).

Let U_N and U_T denote the vectors of components the nodal values of u_{hn} and u_{ht} respectively and let L_N and L_T denote the vectors of components the nodal values of λ_{hn} and λ_{ht} respectively. We begin with considering the mixed method where $\boldsymbol{M}_h(g) = \boldsymbol{M}'_h(g)$.

Proposition 3.4 *Let $\boldsymbol{M}_h(g) = \boldsymbol{M}'_h(g)$. The vectors U_N, U_T, L_N, L_T associated with a solution of Coulomb's discrete frictional contact problem satisfy for any $1 \le i \le p$:*

$$
\begin{aligned}
(L_N)_i &\le 0, && (3.17)\\
(\mathcal{M}U_N)_i &\le 0, && (3.18)\\
(L_N)_i(\mathcal{M}U_N)_i &= 0, && (3.19)\\
|(L_T)_i| &\le -\mu(L_N)_i, && (3.20)\\
|(L_T)_i| &< -\mu(L_N)_i \implies (\mathcal{M}U_T)_i = 0, && (3.21)\\
(L_T)_i(\mathcal{M}U_T)_i &\le 0. && (3.22)
\end{aligned}
$$

Proof. From $\lambda_{hn} \in M'_{hn}$, we immediately get (3.17). Condition (3.15) is equivalent to

$$
\int_{\Gamma_C} \nu_{hn} u_{hn} \, d\Gamma \ge 0, \quad \forall \nu_{hn} \in M'_{hn} \qquad \text{and} \qquad \int_{\Gamma_C} \lambda_{hn} u_{hn} \, d\Gamma = 0. \tag{3.23}
$$

Choosing in the inequality of (3.23), $\nu_{hn} = -\psi_i$, $1 \le i \le p$, and writing $u_{hn} = \sum_{j=1}^{p} (U_N)_j \ \psi_j$ gives (3.18). Putting $\lambda_{hn} = \sum_{i=1}^{p} (L_N)_i \ \psi_i$ and $u_{hn} = \sum_{j=1}^{p} (U_N)_j \ \psi_j$ in the equality of (3.23) yields

$$
\sum_{i=1}^{p} (L_N)_i (\mathcal{M}U_N)_i = 0.
$$

The latter estimate together with (3.17) and (3.18) implies (3.19).

Inequality (3.20) follows directly from $\lambda_{ht} \in M'_{ht}(-\mu\lambda_{hn})$. For any $1 \le i \le p$, choose ν_{ht} in (3.16) as follows: $\nu_{ht} = \mu\lambda_{hn}$ at node i and $\nu_{ht} = \lambda_{ht}$ at the $p-1$ other nodes. We obtain

$$
\begin{aligned}
\int_{\Gamma_C} (\nu_{ht} - \lambda_{ht}) u_{ht} \, d\Gamma &= (\mu L_N - L_T)_i \int_{\Gamma_C} \psi_i u_{ht} \, d\Gamma \\
&= (\mu L_N - L_T)_i (\mathcal{M}U_T)_i \ge 0. && (3.24)
\end{aligned}
$$

Similarly, take $\nu_{ht} = -\mu\lambda_{hn}$ at node i and $\nu_{ht} = \lambda_{ht}$ at the $p-1$ other nodes. We get

$$
\int_{\Gamma_C} (\nu_{ht} - \lambda_{ht}) u_{ht} \, d\Gamma = (-\mu L_N - L_T)_i (\mathcal{M}U_T)_i \ge 0. \tag{3.25}
$$

Putting together estimates (3.24) and (3.25) implies (3.21).

It remains to prove (3.22). Define ν_{ht} in (3.16) as follows: $\nu_{ht} = \frac{1}{2}\lambda_{ht}$ at node i and $\nu_{ht} = \lambda_{ht}$ at the $p-1$ other nodes. Therefore

$$
\begin{aligned}
\int_{\Gamma_C} (\nu_{ht} - \lambda_{ht}) u_{ht} \, d\Gamma &= -\frac{1}{2}(L_T)_i \int_{\Gamma_C} \psi_i u_{ht} \, d\Gamma \\
&= -\frac{1}{2}(L_T)_i (\mathcal{M}U_T)_i \ge 0.
\end{aligned}
$$

Hence inequality (3.22). □

Proceeding in a similar way when $\boldsymbol{M}_h(g) = \boldsymbol{M}''_h(g)$, we obtain the following proposition.

Proposition 3.5 *Let $\boldsymbol{M}_h(g) = \boldsymbol{M}''_h(g)$. The vectors U_N, U_T, L_N, L_T associated with a solution of Coulomb's discrete frictional contact problem satisfy for any $1 \leq i \leq p$:*

$$\begin{aligned}
(\mathcal{M}L_N)_i &\leq 0, \\
(U_N)_i &\leq 0, \\
(\mathcal{M}L_N)_i(U_N)_i &= 0, \\
|(\mathcal{M}L_T)_i| &\leq -\mu(\mathcal{M}L_N)_i, \\
|(\mathcal{M}L_T)_i| &< -\mu(\mathcal{M}L_N)_i \implies (U_T)_i = 0, \\
(\mathcal{M}L_T)_i(U_T)_i &\leq 0.
\end{aligned}$$

Remark 3.6 *We show in the next section that the choice of the method using $\boldsymbol{M}'_h(g)$ is quite appropriate and easier than $\boldsymbol{M}''_h(g)$ to compute the estimator. But most of the finite element codes solving contact problems (with or without friction) make use of nodal displacements U_N, U_T and of nodal forces F_N, F_T as dual unknowns (and not pressures like L_N, L_T) on the contact part Γ_C. This means that the frictional contact conditions are generally: $(F_N)_i \leq 0$, $(U_N)_i \leq 0$, $(F_N)_i(U_N)_i = 0$, $|(F_T)_i| \leq -\mu(F_N)_i$, and so on. This is precisely the choice of $\boldsymbol{M}''_h(g)$ when supposing that pressures and forces are linked by $F_N = \mathcal{M}L_N$ and $F_T = \mathcal{M}L_T$. As a consequence, we must also be able to propose a practical computation of the estimator for this widespread case.*

4. Construction of admissible fields

The purpose of this section is to describe the building of admissible fields $\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{r}}$ satisfying the kinematic conditions (2.1) and the equilibrium equations (2.2) to compute the error estimator (2.11). Moreover, in order to obtain a finite value of the error estimator, the displacement fields $\hat{\boldsymbol{w}}$ on the contact part Γ_C must satisfy the non-penetration conditions and the densities of forces $\hat{\boldsymbol{r}}$ should belong to Coulomb's friction cone C_μ on Γ_C.

To perform such a construction, we will obviously make use of the finite element solution of Coulomb's frictional contact problem $(\boldsymbol{u}_h, \lambda_{hn}, \lambda_{ht}) \in \boldsymbol{V}_h \times M_{hn} \times M_{ht}(-\mu\lambda_{hn}) = \boldsymbol{V}_h \times \boldsymbol{M}_h(-\mu\lambda_{hn})$ which satisfies:

$$\begin{cases}
a(\boldsymbol{u}_h, \boldsymbol{v}_h) - \displaystyle\int_{\Gamma_C} \lambda_{hn} v_{hn}\, d\Gamma - \int_{\Gamma_C} \lambda_{ht} v_{ht}\, d\Gamma = L(\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\
\displaystyle\int_{\Gamma_C} (\nu_{hn} - \lambda_{hn}) u_{hn}\, d\Gamma + \int_{\Gamma_C} (\nu_{ht} - \lambda_{ht}) u_{ht}\, d\Gamma \geq 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad \forall(\nu_{hn}, \nu_{ht}) \in \boldsymbol{M}_h(-\mu\lambda_{hn}),
\end{cases}$$

where $\boldsymbol{M}_h(-\mu\lambda_{hn}) = M_{hn} \times M_{ht}(-\mu\lambda_{hn})$ denotes either $\boldsymbol{M}'_h(-\mu\lambda_{hn}) = M'_{hn} \times M'_{ht}(-\mu\lambda_{hn})$ or $\boldsymbol{M}''_h(-\mu\lambda_{hn}) = M''_{hn} \times M''_{ht}(-\mu\lambda_{hn})$.

We begin with building the displacements fields $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{w}}$ satisfying the kinematic conditions (2.1) and the non-penetration conditions.

4.1. *Construction of the displacement fields*

For both finite element approaches (i.e. $\boldsymbol{M}_h(g) = \boldsymbol{M}'_h(g)$ or $\boldsymbol{M}_h(g) = \boldsymbol{M}''_h(g)$) the finite element displacement field $\boldsymbol{u}_h$ verifies the embedding conditions in (2.1).

When $\boldsymbol{M}_h(g) = \boldsymbol{M}'_h(g)$, the finite element displacement field may not fulfill the non-penetration conditions according to Proposition 3.4. At the nodes $\boldsymbol{x}_j$ which are not located on Γ_C, we set $\hat{\boldsymbol{u}}(\boldsymbol{x}_j) = \boldsymbol{u}_h(\boldsymbol{x}_j)$. At the nodes $\boldsymbol{x}_i$ lying on Γ_C, we set $\hat{u}_t(\boldsymbol{x}_j) = u_{ht}(\boldsymbol{x}_j)$ and $\hat{u}_n(\boldsymbol{x}_j) = \min(u_{hn}(\boldsymbol{x}_j), 0)$. Using these nodal values, the displacement field $\hat{\boldsymbol{u}}$ is then built in $\boldsymbol{V}_h$ (and $\hat{\boldsymbol{w}} = \hat{\boldsymbol{u}}$ on Γ_C).

When $\boldsymbol{M}_h(g) = \boldsymbol{M}''_h(g)$, the finite element displacement field satisfies also the non-penetration conditions according to Proposition 3.5. In that case, we simply take $\hat{\boldsymbol{u}} = \boldsymbol{u}_h$ in Ω (and $\hat{\boldsymbol{w}} = \hat{\boldsymbol{u}}$ on Γ_C).

4.2. *Construction of the stress fields*

Let us describe the building of the stress fields $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{r}}$ verifying the equilibrium equations and $\hat{\boldsymbol{r}} \in C_\mu$ on Γ_C.

4.2.1. *Building of $\hat{\boldsymbol{r}} \in C_\mu$ on Γ_C*

When $\boldsymbol{M}_h(g) = \boldsymbol{M}'_h(g)$, the building is straightforward. According to Proposition 3.4, we can choose directly $\hat{\boldsymbol{r}} = \boldsymbol{\lambda}_h$ (i.e. $\hat{r}_n = \lambda_{hn}$ and $\hat{r}_t = \lambda_{ht}$).

When $\boldsymbol{M}_h(g) = \boldsymbol{M}''_h(g)$, the situation is more complicated. However, from Proposition 3.5, we are not assured that the multipliers $\boldsymbol{\lambda}_h = (\lambda_{hn}, \lambda_{ht})$ belong always to C_μ on Γ_C (i.e. satisfy $\lambda_{hn} \le 0$ and $|\lambda_{ht}| \le -\mu\lambda_{hn}$) as in the previous case. Nevertheless, there is in the construction a freedom on the choice of the tangential components $\hat{r}_t$. Indeed, they can be modified edge by edge by adding a density with null resultant and moment. More precisely, when the computed multipliers satisfy $|\lambda_{ht}| > -\mu\lambda_{hn}$ at the node i, it is possible to compute a new density $\tilde{\lambda}_{ht}$ on the edge $[i, j]$ (j is one of the neighboring nodes) as follows:

$$\begin{aligned} \tilde{\lambda}_{ht} &= \lambda_{ht} - d, \text{ at node } i, \\ \tilde{\lambda}_{ht} &= \lambda_{ht} + d, \text{ at node } j. \end{aligned} \tag{4.1}$$

If $d \in \mathbb{R}$ is chosen such that $|\tilde{\lambda}_{ht}| \le -\mu\lambda_{hn}$ at nodes i and j, then the modification is satisfying. In such a case, the modified tangential pressure $\tilde{\lambda}_{ht}$ is piecewise linear on each mesh and possibly discontinuous on Γ_C. We finally choose $\hat{r}_n = \lambda_{hn}$ and $\hat{r}_t = \tilde{\lambda}_{ht}$.

4.2.2. *Building of $\hat{\boldsymbol{\sigma}}$ verifying the equilibrium equations*

Next, having at our disposal $\hat{\boldsymbol{r}}$, the stress fields $\hat{\boldsymbol{\sigma}}$ satisfying (2.2) are to be constructed. It is straightforward that the stress field obtained from the finite element displacement field $\boldsymbol{u}_h$ with the constitutive relation: $\boldsymbol{\sigma}_h = \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}_h)$ does not satisfy the equilibrium equation (2.2). If we want to compute the error estimator, a stress field $\hat{\boldsymbol{\sigma}}$ that strictly satisfies the equilibrium equations must be obtained. The construction of $\hat{\boldsymbol{\sigma}}$ is performed in two steps which can be summarized as follows:

• the first step consists of building densities of forces $\hat{\boldsymbol{F}}$ on each edge of the mesh satisfying equilibrium with the body forces $\boldsymbol{f}$:

$$\int_E \boldsymbol{f}.\boldsymbol{v}\, d\Omega + \int_{\partial E} \eta_E \hat{\boldsymbol{F}}.\boldsymbol{v}\, d\Gamma = 0, \qquad \text{for all } \boldsymbol{v} \text{ such that } \boldsymbol{\varepsilon}(\boldsymbol{v}) = 0,$$

where η_E is a function defined on the boundary of the triangular element E, constant on each edge of E, equal to 1 or equal to -1 and satisfying $\eta_E + \eta_{E'} = 0$ on the common edge to adjacent elements E and E'. Note that the construction of $\hat{\boldsymbol{F}}$ is always possible and it is generally not unique. In the numerical experiments, the non-uniqueness of $\hat{\boldsymbol{F}}$ is handled in minimizing (locally, on each patch of elements connected to a node) the difference (least squares) with the finite element solution. The choice of such a technique leads very often to satisfactory effectivity indexes (between 1 and 1.5, see 5.2.2 hereafter). The details of these techniques can be found in [18].

• the second step is devoted to the construction of $\hat{\boldsymbol{\sigma}}$ locally on each element E by solving:

$$\begin{cases} \boldsymbol{div}\ \hat{\boldsymbol{\sigma}} + \boldsymbol{f} = 0 & \text{in } E, \\ \hat{\boldsymbol{\sigma}}\boldsymbol{n} = \eta_E \hat{\boldsymbol{F}} & \text{on } \partial E, \end{cases}$$

where $\boldsymbol{n}$ stands for the unit outward normal on ∂E.

There are two techniques to compute locally the stress admissible field from the densities:

− analytical construction; it is easy to check that there does not exist an $\hat{\boldsymbol{\sigma}}$ linear on E due to the stress symmetry requirement. The chosen technique for determining $\hat{\boldsymbol{\sigma}}$ on each triangle E is then to divide E into three subtriangles and to search $\hat{\boldsymbol{\sigma}}$ which is linear on each subtriangle. The details of this construction can be found in [18].

− numerical construction by using higher-degree polynomials (see [4]).

5. Numerical studies

5.1. *Mesh adaption*

The aim of adaptive procedures is to offer the user a level of accuracy denoted ϵ_0 with a minimal computational cost. We use the h-version which is the most widespread procedure of adaptivity currently in use: the size and the topology of the elements are modified but the same kind of basis functions for the different meshes are retained. A mesh T^* is said to be optimal with respect to a measure of the error ϵ^* if [16]:

$$\begin{cases} \epsilon^* = \epsilon_0 \\ N^*\text{minimal } (N^*\text{: number of elements of } T^*) \end{cases} \tag{5.1}$$

To solve problem (5.1), the following procedure is applied:

1. an initial analysis is performed on a relatively uniform and coarse mesh T,

2. the corresponding global error ϵ in (2.12) and the local contributions ϵ_E in (2.13) are computed,

3. the characteristics of the optimal mesh T^* are determined in order to minimize the computational costs in respect of the global error,

4. a second finite element analysis is performed on the mesh T^*.

The optimal mesh T^* is determined by the computation of a size modification coefficient r_E on each element E of the mesh T:

$$r_E = \frac{h_E^*}{h_E},$$

where h_E denotes the size of E and h_E^* represents the size that must be imposed to the elements of T^* in the region of E in order to ensure optimality. The computation of the coefficients r_E uses the rate of convergence of the error which depends on the used element but also on the regularity of the solution [6]. So, to compute the coefficients r_E, we use a technique detailed in [7] that automatically takes into account the steep gradient regions. The mesh T^* is generated by an automatic mesher able to respect accurately a map of sizes. Practically, the previous procedure allows to divide in two or three the error ϵ. If the user wishes more accuracy, then the procedure is repeated as far as a precision close to ϵ_0 is reached (see [6]).

5.2. *Examples*

We consider two-dimensional plane strain problems where no body forces are applied. As a constitutive relation in Ω, we choose Hooke's law of homogeneous isotropic elastic materials:

$$\boldsymbol{\sigma}_{ij} = \frac{E\nu}{(1-2\nu)(1+\nu)}\delta_{ij}\boldsymbol{\varepsilon}_{kk}(\boldsymbol{u}) + \frac{E}{1+\nu}\boldsymbol{\varepsilon}_{ij}(\boldsymbol{u}),$$

where E and ν denote Young's modulus and Poisson's ratio respectively and the notation δ_{ij} stands for Kronecker's symbol. The implementation is achieved using CASTEM 2000 developed at the CEA and an HP-C3000 computer has been used.

Three examples are studied with various friction coefficients. The computations have been carried out by using the mixed finite element method with $\boldsymbol{M}_h(g) = \boldsymbol{M}_h''(g)$ (see Remark 3.6 for comments).

5.2.1. *First example*

We consider the problem depicted in Figure 2. The dimensions of the rectangular body are 40mm $\times$160mm and computations are performed on the left half of the structure due to symmetry. The material characteristics are $E = 13$ Gpa, $\nu = 0.2$ and $\mu = 0.5$ is the friction coefficient. The load on the left side is 10 N.mm^{-2} and the upper side is clamped.

The initial mesh comprises 1088 three-node elements and 627 nodes for an accuracy ϵ of 9.74% (Fig. 3). We show the deformed body in which separation occurs in Figure 4 and the contributions to the error ϵ_E in Figure 5. The contact pressures are reported on Figure 6. We show the normal pressure $-\lambda_{hn}$, the friction cone (between $-|\lambda_{hn}|$ and $|\lambda_{hn}|$), the tangential pressure $-\lambda_{ht}$ and the modified tangential pressure $-\tilde{\lambda}_{ht}$

Due to the separation on the left part of Γ_C and the choice of the finite element method $\boldsymbol{M}_h(g) = \boldsymbol{M}''_h(g)$, we see that the normal pressure does not always satisfy the convenient sign property. Moreover, at the last mesh on the right part, the tangential pressure is outside the friction cone. By using the modification of the densities (4.1), we are able to compute a modified tangential pressure inside the cone which allows the computing of the statically admissible field. Concerning the normal pressure, the difficulty can not solved by the densities modification. The proposed mixed finite element method $\boldsymbol{M}_h(g) = \boldsymbol{M}'_h(g)$ will allow us to solve this difficulty.

The prescribed accuracy ϵ_0 is 5%. The optimized mesh is obtained in one step and comprises 1148 three-node elements, 647 nodes for an accuracy ϵ of 4.28% (Fig. 7).

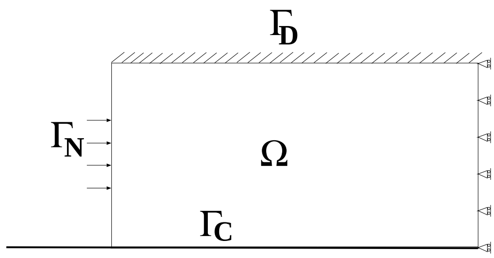

Figure 2: Setting of the problem

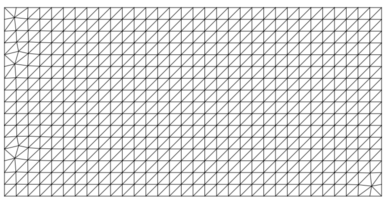

Figure 3: Initial mesh: 1088 three-node elements, 627 nodes, ϵ=9.74%

Figure 4: Deformed configuration

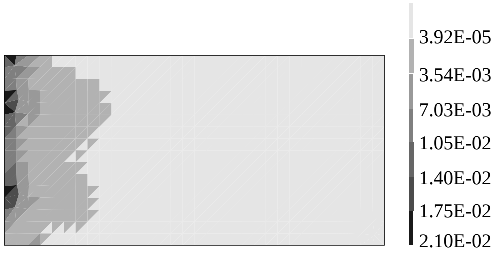

Figure 5: Local contributions ϵ_E

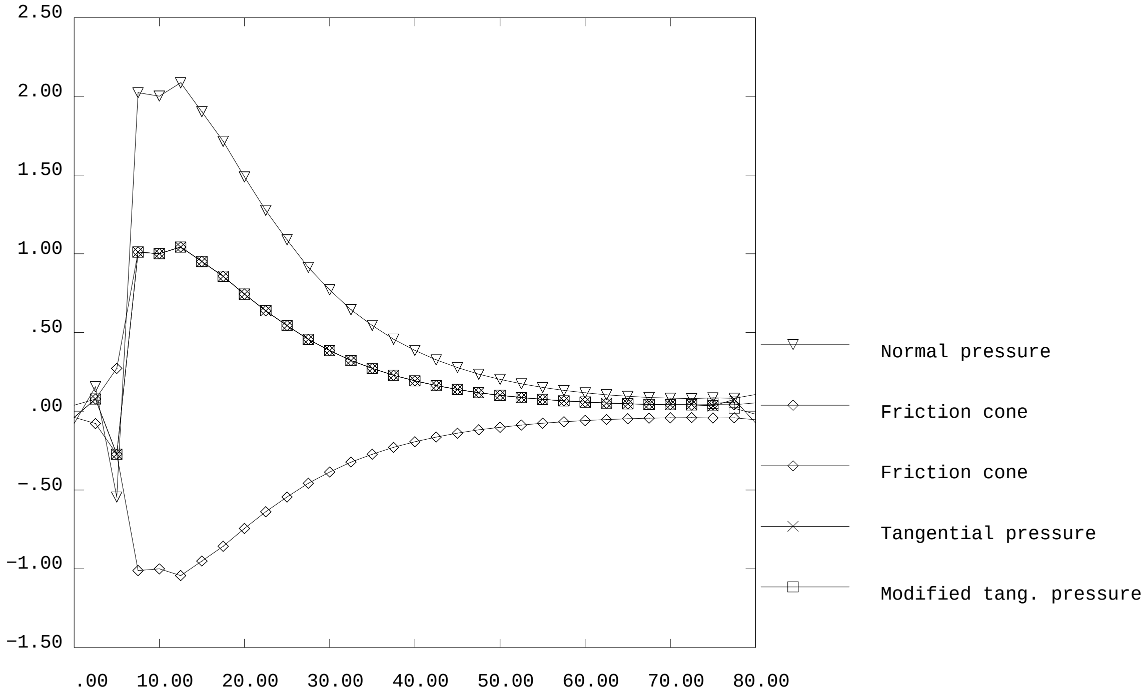

Figure 6: Contact pressures

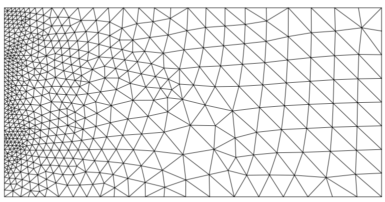

Figure 7: Optimized mesh: 1148 three-node elements, 647 nodes, ϵ=4.28%

5.2.2. *Second example*

Next, we consider the structure depicted in Figure 8. The dimensions of the rectangle are 40mm ×80mm, and symmetry conditions are adopted. We choose

$E = 13$Gpa, $\nu = 0.2$ and a friction coefficient of 0.3. The load on the upper side is 5 N.mm^{-2} and no embedding conditions are applied. In such a case, the bilinear form $a(.,.)$ is no longer $\boldsymbol{V}-$elliptic but satisfies some semi-coercivity property (see [11], Theorem 6.3).

The initial mesh is made of 544 three-node elements and 323 nodes corresponding to an accuracy ϵ of 1.46% (Fig. 9). The deformed configuration and the map of local contributions ϵ_E are shown in Figures 10 and 11 respectively. On Figure 12, the normal contact pressure, the friction cone and the tangential pressure are reported. We can notice than on the left node, the tangential pressure is outside the cone. By using the modification of the densities (4.1), we compute a modified tangential pressure which gives a new pressure inside the cone. Notice that in this example, we have stick on the contact zone whereas the body was slipping in the previous example (see also Proposition 3.5 for some corroboration).

The prescribed accuracy ϵ_0 is 0.5%. The optimized mesh (obtained in one step) comprising 1178 three-node elements and 666 nodes for an accuracy ϵ of 0.57% is represented on Figure 13.

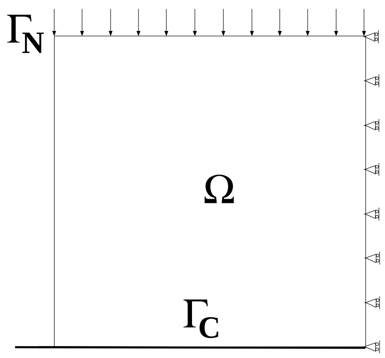

Figure 8: Setting of the problem

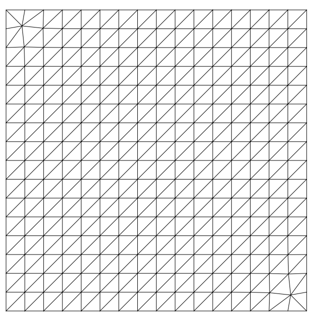

Figure 9: Initial mesh: 544 three-node elements, 323 nodes, ϵ=1.46%

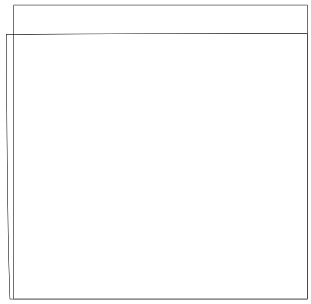

Figure 10: Deformed configuration

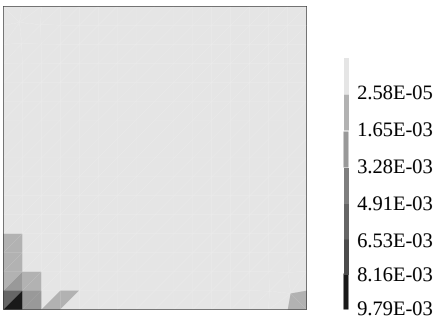

Figure 11: Local contributions ϵ_E

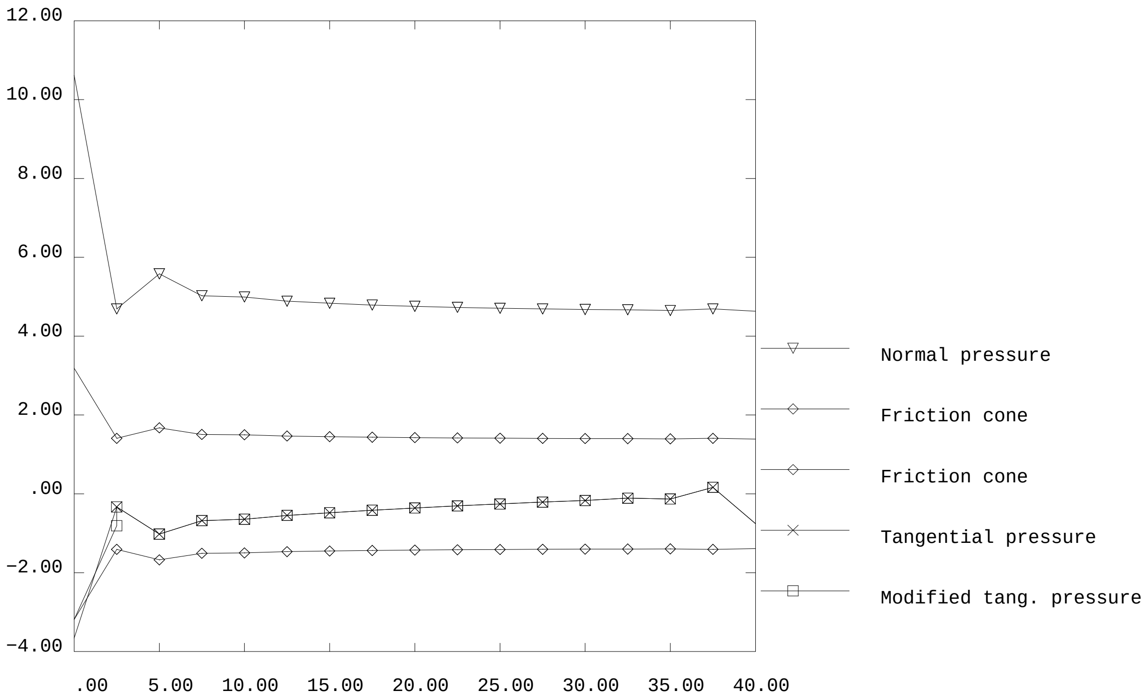

Figure 12: Contact pressures

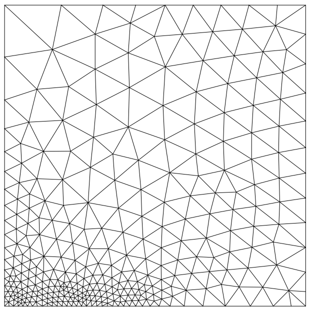

Figure 13: Optimized mesh: 1178 three-node elements, 666 nodes, ϵ=0.57%

Next, we consider again the initial mesh in Figure 9 and we compute the effectivity indexes as a function of the friction coefficient μ. Since no analytical solution is available, we use a reference solution denoted $\boldsymbol{u}_{ref}$ corresponding to a very refined mesh. The exact error denoted ϵ_{ex} and the effectivity index γ can be defined as follows:

$$\epsilon_{ex} = \frac{\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_{u,\Omega}}{\|\boldsymbol{u}_{ref} + \boldsymbol{u}_h\|_{u,\Omega}}, \qquad \gamma = \frac{e}{\|\boldsymbol{u}_{ref} - \boldsymbol{u}_h\|_{u,\Omega}},$$

where e is the error estimator defined in (2.11). The results are reported in Table 1. Note that the frictionless case is not interesting because it corresponds to a pure compression case and the error is negligible. The effectivity indexes close to 1 show the accuracy of the error estimator.

μ	ϵ(in%)	ϵ_{ex}(in%)	effectivity γ
0.2	1 , 26	0 , 93	1 , 36
0.4	1 , 50	1 , 02	1 , 48
0.6	1 , 50	1 , 02	1 , 48
0.8	1 , 50	1 , 02	1 , 48

Table 1: effectivity indexes

5.2.3. *Third example: numerical extension to two bodies in frictional contact*

We consider the problem of two elastic bodies initially in contact (Fig. 14). The upper body is submitted to an uniform load of 10 N.mm^{-2}. We have adopted symmetry conditions on the lower side of Ω^2 in order to avoid a greater number of singularities and the lower body is fixed on the left node of its lower side. The two materials are identical (E = 200GPa, $\nu = 0.25$), the dimensions are 100mm ×100mm and 200mm ×200mm and the friction coefficient μ is 0.3.

The initial mesh with 640 three-node elements, 370 nodes and an accuracy ϵ of 8.51% is shown on Figure 15. We show the deformed bodies (Fig. 16) and the contributions to the error ϵ_E (Fig. 17). The contact pressures are drawn on Figure 18. As in the previous example the computed tangential pressure is outside the friction cone (on both extreme meshes) and as previously, it can be successfully modified to compute the estimator.

An accuracy ϵ_0 of 5% is prescribed. The optimized mesh, obtained in one step, made of 381 three-node elements and 230 nodes for an accuracy ϵ of 5.46% is shown in Figure 19.

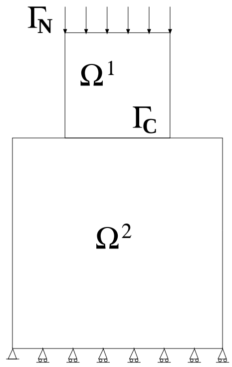

Figure 14: Setting of the problem

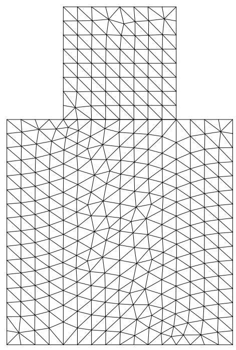

Figure 15: Initial mesh: 640 three-node elements, 370 nodes, ϵ=8.51%

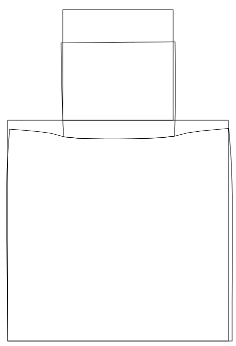

Figure 16: Deformed configuration

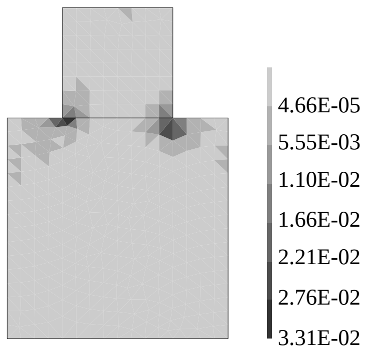

Figure 17: Local contributions ϵ_E

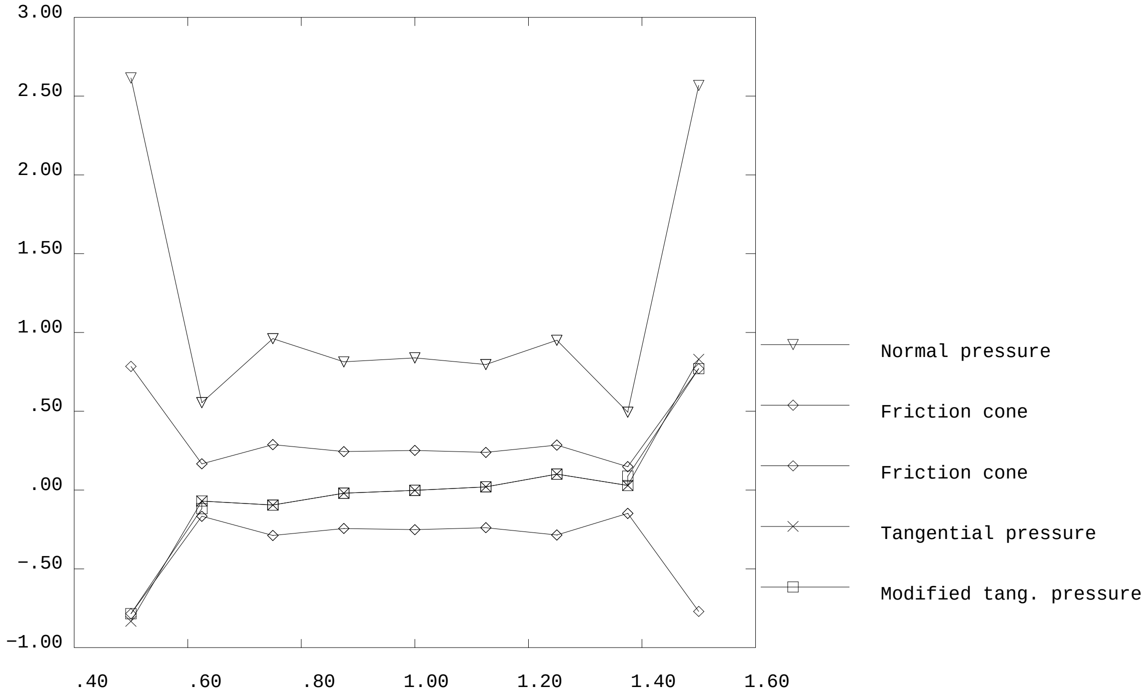

Figure 18: Contact pressures

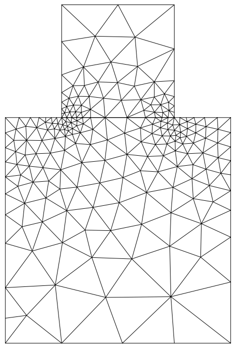

Figure 19: Optimized mesh: 381 three-node elements, 230 nodes, ϵ=5.46%

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