Numerical Approximation of the Elastic-Viscoplastic Contact Problem with Non-matching Meshes

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Abstract

This work deals with the approximation of a time dependent variational inequality modelling the unilateral contact problem of elastic-viscoplastic bodies in a bidimensional context. The problem is approximated in the space variable with nonconforming finite element methods which allow the handling of non-matching meshes on the contact zone. Several error estimates are established and the corresponding numerical experiments are achieved.

Keywords : Elastic-viscoplastic materials, unilateral contact, nonconforming finite element methods, nonmatching meshes

1 Introduction and problem set-up

In computational structural mechanics, the contact phenomena between deformable solids are generally simulated using finite element methods (see [13, 14, 18, 23]). The contact problems in which we are interested in this paper involve the (large) class of elastic-viscoplastic materials (see [7, 17, 19, 20]). Considering the contact problem between two bodies, we focus on the finite element approximations which involve nonmatching meshes on the contact part. Such a configuration in which the nodes inherited from the discretizations of the bodies may not coincide arises in computational situations when the different bodies are independently meshed and/or there is an initial distance between the bodies and/or an evolution process is considered. Our first aim is to study the convergence of finite element methods for such models in generalizing the techniques developed in [2] for linear elastic bodies with nonmatching meshes and the discussion in [12] concerning the elastic-viscoplastic case with matching meshes. Our second aim is to carry out the numerical experiments associated with the theoretical studies.

The paper is outlined as follows. We first begin with introducing the equations modelling the problem and the associated weak formulation is exhibited. In the second section, we present a fully discrete approximation based on nonconforming finite element methods in space and an implicit scheme in time. In these approximations, nonmatching meshes are allowed on the contact part and two different ways for defining the discrete non penetration conditions are proposed: a local approach of node-on-segment type and a global approach corresponding to a generalization of the mortar finite element method (see [3]) to the variational inequality from the unilateral contact model. The third section is dedicated to the convergence analysis of the approximations. We provide an extension of Falk's lemma to our problem which leads us to establish the error estimates. In the fourth section, we describe the algorithm which is adopted in our finite element computations. Finally, the fifth section shows the results of some numerical experiments in which both local and global finite element approaches are implemented.

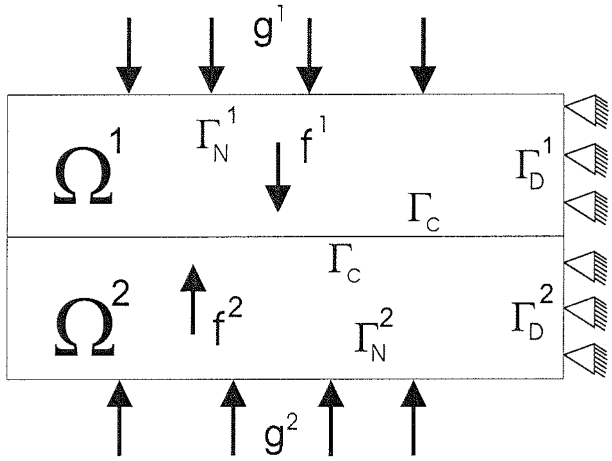

Figure 1: Contact problem between two viscoplastic bodies.

We consider two elastic-viscoplastic bodies which occupy two bounded domains Ω^1 and Ω^2 of $\mathbb{R}^2$ and we denote by $[0,T]$, $T>0$, the time interval during which the evolution of the bodies will be investigated (see Figure 1). For $\ell=1,2$, the boundary

Γ^ℓ of Ω^ℓ is assumed to be "smooth" and is the union of three nonoverlapping portions Γ_D^ℓ, Γ_N^ℓ and Γ_C^ℓ. Assume that both bodies are submitted to prescribed displacements (supposed equal to zero for the sake of simplicity although other admissible situations could also be considered) on $\Gamma_D^\ell \times (0,T)$. The normal unit outward vector on Γ^ℓ is denoted $\boldsymbol{\nu}^\ell = (\nu_1^\ell, \nu_2^\ell)$. Both bodies are subjected to densities of volume forces $\boldsymbol{f}^\ell(t) = (f_1^\ell(t), f_2^\ell(t))$ on $\Omega^\ell \times (0,T)$ and densities of surface forces $\boldsymbol{g}^\ell(t) = (g_1^\ell(t), g_2^\ell(t))$ are assumed on $\Gamma_N^\ell \times (0,T)$. In the time interval $[0,T]$, the bodies can be in contact on their common boundary part $\Gamma_C^1 = \Gamma_C^2$ which we denote by Γ_C, and frictionless unilateral contact conditions without initial gap are assumed on $\Gamma_C \times (0,T)$. We assume that the contact process is quasistatic.

This frictionless unilateral contact problem consists of finding in the time interval $[0,T]$ the displacement fields $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2)$ (where the notation $\boldsymbol{u}^\ell$ stands for $\boldsymbol{u}|_{\Omega^\ell}$) with $\boldsymbol{u}^\ell = (u_i^\ell)$, $1 \le i \le 2$ and the stress tensor field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)$ (where the notation $\boldsymbol{\sigma}^\ell$ stands for $\boldsymbol{\sigma}|_{\Omega^\ell}$) with $\boldsymbol{\sigma}^\ell = (\sigma_{ij}^\ell)$, $1 \le i,j \le 2$ which satisfy equations and conditions (1.1)-(1.10) for $\ell = 1, 2$.

The following equation

$$\frac{\partial \sigma_{ij}^\ell}{\partial x_j} + f_i^\ell = 0 \qquad \text{in } \Omega^\ell \times (0,T), \quad 1 \le i \le 2, \tag{1.1}$$

is the equilibrium equation where the summation convention of repeated indices is adopted and where the stress tensor field $\boldsymbol{\sigma}^\ell$ is linked to the displacement field by the constitutive relation of rate-type elastic-viscoplastic models

$$\dot{\boldsymbol{\sigma}}^\ell = \mathcal{E}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + G^\ell(\boldsymbol{\sigma}^\ell, \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell)) \qquad \text{in } \Omega^\ell \times (0,T). \tag{1.2}$$

In (1.2) and in all what follows the upper dot indicates the time derivative, the notation $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$, $1 \le i,j \le 2$ stands for the linearized strain tensor field:

$$\varepsilon_{ij}(\boldsymbol{u}^\ell) = \frac{1}{2}\Big(\frac{\partial u_i^\ell}{\partial x_j} + \frac{\partial u_j^\ell}{\partial x_i}\Big),$$

and $\mathcal{E}^\ell$ and G^ℓ are constitutive functions (see, for example, [7]).

The equations on the boundary parts submitted to prescribed displacements and prescribed loads are:

$$\boldsymbol{u}^\ell = 0 \qquad \text{on } \Gamma_D^\ell \times (0,T), \tag{1.3}$$

$$\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell = \boldsymbol{g}^\ell \qquad \text{on } \Gamma_N^\ell \times (0,T). \tag{1.4}$$

The conditions on the boundary part $\Gamma_C \times (0,T)$ submitted to frictionless unilateral contact conditions incorporate the Signorini type conditions:

$$\boldsymbol{\nu}^1.(\boldsymbol{\sigma}^1\boldsymbol{\nu}^1) = \boldsymbol{\nu}^2.(\boldsymbol{\sigma}^2\boldsymbol{\nu}^2) = \sigma_\nu, \tag{1.5}$$

$$\boldsymbol{u}^1.\boldsymbol{\nu}^1 + \boldsymbol{u}^2.\boldsymbol{\nu}^2 \leq 0, \tag{1.6}$$

$$\sigma_\nu \leq 0, \tag{1.7}$$

$$\sigma_\nu(\boldsymbol{u}^1.\boldsymbol{\nu}^1 + \boldsymbol{u}^2.\boldsymbol{\nu}^2) = 0, \tag{1.8}$$

where the notation $\cdot$ represents the inner product in $\mathbb{R}^2$. The conditions stating the absence of friction are

$$\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell = \sigma_\nu\boldsymbol{\nu}^\ell \qquad \text{on } \Gamma_C \times (0,T), \tag{1.9}$$

which means that for $\ell = 1, 2$ the density of surface forces $\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell$ on the contact zone are parallel to the normal. The initial conditions at $t = 0$ and $\ell = 1, 2$ are as follows

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}^{0,\ell} \quad \text{and} \quad \boldsymbol{\sigma}^\ell(0) = \boldsymbol{\sigma}^{0,\ell} \qquad \text{in } \Omega^\ell. \tag{1.10}$$

Let us begin by introducing some useful notation and several functional spaces.

Let Ω be an open bounded subset of $\mathbb{R}^2$ whose generic point is denoted $\boldsymbol{x} = (x_1, x_2)$ and denote by $L^2(\Omega)$ the Hilbert space of square integrable functions endowed with the inner product

$$(\phi, \psi) = \int_\Omega \phi(\boldsymbol{x})\psi(\boldsymbol{x})\, d\Omega.$$

Given $m \in \mathbb{N}$, introduce the Sobolev space

$$H^m(\Omega) = \Big\{\psi \in L^2(\Omega), D^\alpha\psi \in L^2(\Omega), |\alpha| \leq m\Big\},$$

where $\alpha = (\alpha_1, \alpha_2)$ represents a multi–index in $\mathbb{N}^2$ and $|\alpha| = \alpha_1 + \alpha_2$. The notation D^α denotes a partial derivative and the convention $H^0(\Omega) = L^2(\Omega)$ is adopted. The spaces $H^m(\Omega)$ are equipped with the norm

$$||\psi||_{H^m(\Omega)} = \Big(\sum_{|\alpha|\leq m} ||D^\alpha\psi||^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}.$$

In order to give the variational formulation associated with problem (1.1)-(1.10), we introduce the spaces $\mathbf{V}(\Omega^\ell)$ and $\mathcal{H}(\Omega^\ell)$ for $\ell = 1$ and $\ell = 2$ as follows:

$$\mathbf{V}(\Omega^\ell) = \Big\{\boldsymbol{v}^\ell = (v_i^\ell), \quad v_i^\ell \in H^1(\Omega^\ell), \quad v_i^\ell = 0 \text{ on } \Gamma_D^\ell,\ 1 \leq i \leq 2\Big\},$$

$$\mathcal{H}(\Omega^\ell) = \Big\{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell), \quad \tau_{ij}^\ell \in L^2(\Omega^\ell),\ 1 \leq i, j \leq 2\Big\}.$$

The Hilbert spaces $\mathbf{V}(\Omega^\ell)$ and $\mathcal{H}(\Omega^\ell)$ are endowed with their canonical inner products denoted $(.,.)_{\mathbf{V}(\Omega^\ell)}$ and $(.,.)_{\mathcal{H}(\Omega^\ell)}$. Setting $\mathbf{V} = \mathbf{V}(\Omega^1) \times \mathbf{V}(\Omega^2)$ and $\mathcal{H} = \mathcal{H}(\Omega^1) \times \mathcal{H}(\Omega^2)$

these product spaces are equipped with the inner products $(.,.)_{\mathbf{V}}$ and $(.,.)_{\mathcal{H}}$; the associated norms are denoted $\|.\|_{\mathbf{V}}$ and $\|.\|_{\mathcal{H}}$ respectively. Let $\mathcal{S}_2$ be the space of second order symmetric tensors in $\mathbb{R}^2$ and let the notations $\cdot$ and $|.|$ denote the inner product and the norm on these spaces.

Next, we specify the properties satisfied by the functions $\mathcal{E}^\ell$ and G^ℓ ($\ell = 1, 2$) incorporated in the constitutive law (1.2).

Function $\mathcal{E}^\ell : \Omega^\ell \times \mathcal{S}_2 \to \mathcal{S}_2$ is a bounded symmetric positive definite fourth order tensor

$$\begin{cases} \mathcal{E}^\ell_{ijkl} \in L^\infty(\Omega^\ell),\ 1 \leq i, j, k, l \leq 2, \\ \mathcal{E}^\ell \boldsymbol{\sigma}.\boldsymbol{\tau} = \boldsymbol{\sigma}.\ \mathcal{E}^\ell \boldsymbol{\tau},\ \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{S}_2, \\ \exists\ \alpha^\ell > 0\ s.t.\ \mathcal{E}^\ell \boldsymbol{\tau}.\boldsymbol{\tau} \geq \alpha^\ell |\boldsymbol{\tau}|^2,\ \forall \boldsymbol{\tau} \in \mathcal{S}_2. \end{cases} \tag{1.11}$$

Function $G^\ell : \Omega^\ell \times \mathcal{S}_2 \times \mathcal{S}_2 \to \mathcal{S}_2$ satisfies

$$\begin{cases} \exists\ L^\ell > 0\ s.t.\ |G^\ell(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\epsilon}) - G^\ell(\boldsymbol{x}, \tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\epsilon}})| \leq L^\ell(|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}| + |\boldsymbol{\epsilon} - \tilde{\boldsymbol{\epsilon}}|), \\ \forall \boldsymbol{\sigma}, \tilde{\boldsymbol{\sigma}}, \boldsymbol{\epsilon}, \tilde{\boldsymbol{\epsilon}} \in \mathcal{S}_2, \quad a.e.\ in\ \Omega^\ell, \\ G^\ell(\cdot, \boldsymbol{\sigma}, \boldsymbol{\epsilon}) : \boldsymbol{x} \mapsto G^\ell(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\epsilon}) \in \mathcal{S}_2 \text{ is measurable for any } \boldsymbol{\sigma}, \boldsymbol{\epsilon} \in \mathcal{S}_2, \\ G^\ell(\cdot, 0, 0) \in \mathcal{H}(\Omega^\ell) \text{ and } \dfrac{\partial G^\ell_{ij}(\cdot, 0, 0)}{\partial x_j} \in L^2(\Omega^\ell),\ 1 \leq i, j \leq 2. \end{cases} \tag{1.12}$$

The given densities of forces verify

$$\boldsymbol{f}^\ell \in W^{1,\infty}(0, T; (L^2(\Omega^\ell))^2) \quad \text{and} \quad \boldsymbol{g}^\ell \in W^{1,\infty}(0, T; (L^2(\Gamma^\ell_N))^2).$$

For the sake of simplicity, we shall adopt the following notations:
$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\boldsymbol{\varepsilon}(\boldsymbol{v}^1), \boldsymbol{\varepsilon}(\boldsymbol{v}^2))$, $\forall \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \mathbf{V}, \mathcal{E}\boldsymbol{\varepsilon} = (\mathcal{E}^1\boldsymbol{\varepsilon}^1, \mathcal{E}^2\boldsymbol{\varepsilon}^2)$, $\forall \boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}^1, \boldsymbol{\varepsilon}^2) \in \mathcal{H}$ and $G(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) = (G^1(\boldsymbol{\sigma}^1, \boldsymbol{\varepsilon}^1), G^2(\boldsymbol{\sigma}^2, \boldsymbol{\varepsilon}^2))$, $\forall \boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}^1, \boldsymbol{\varepsilon}^2) \in \mathcal{H}$, $\forall \boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \in \mathcal{H}$.

Let $\mathbf{V}'$ be the strong dual of $\mathbf{V}$ and let $\langle .,. \rangle$ denote the duality pairing between $\mathbf{V}$ and $\mathbf{V}'$. For all $t \in [0, T]$, let $\mathbf{F}(t)$ denote the element of $\mathbf{V}'$ given by

$$\langle \mathbf{F}(t), \boldsymbol{v} \rangle = \sum_{\ell=1}^{2} (\boldsymbol{f}^\ell(t), \boldsymbol{v}^\ell)_{(L^2(\Omega^\ell))^2} + \sum_{\ell=1}^{2} (\boldsymbol{g}^\ell(t), \boldsymbol{v}^\ell)_{(L^2(\Gamma^\ell_N))^2},\ \forall \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \mathbf{V}.$$

Using the regularity assumptions on $\boldsymbol{f}^\ell$ and $\boldsymbol{g}^\ell$, we deduce that

$$\mathbf{F} \in W^{1,\infty}(0, T; \mathbf{V}').$$

The convex set of admissible displacements is defined as follows

$$\mathbf{K} = \{\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \mathbf{V}, \qquad [\boldsymbol{v}.\boldsymbol{\nu}] \leq 0 \text{ on } \Gamma_C\}, \tag{1.13}$$

where $[\boldsymbol{v}.\boldsymbol{\nu}] = \boldsymbol{v}^1.\boldsymbol{\nu}^1 + \boldsymbol{v}^2.\boldsymbol{\nu}^2$ denotes the jump of the relative normal displacement across interface Γ_C.

Let us suppose that

$$\boldsymbol{u}^0 = (\boldsymbol{u}^{0,1}, \boldsymbol{u}^{0,2}) \in \mathbf{K} \quad \text{and} \quad (\boldsymbol{\sigma}^0, \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}^0))_{\mathcal{H}} \geq \langle \mathbf{F}(0), \boldsymbol{v} - \boldsymbol{u}^0 \rangle \quad \forall \boldsymbol{v} \in \mathbf{K}. \quad (1.14)$$

The variational formulation of the elastic-viscoplastic unilateral contact problem (see [19]) consists then of finding the displacement fields $\boldsymbol{u} : [0,T] \to \mathbf{K}$ and the stress fields $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}_1$ such that:

$$\begin{cases} \boldsymbol{u}(0) = \boldsymbol{u}^0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}^0, \\ \dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + G(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \quad a.e. \quad t \in (0,T), \\ (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \geq \langle \mathbf{F}(t), \boldsymbol{v} - \boldsymbol{u}(t) \rangle, \forall \boldsymbol{v} \in \mathbf{K}, \, t \in [0,T]. \end{cases} \quad (1.15)$$

The existence and uniqueness statement for this problem has been established (see [19]). We recall this result in the following proposition.

Proposition 1.1. *Let the assumptions* (1.11), (1.12) *and* (1.14) *hold. Then there exists a unique solution* $(\boldsymbol{u}, \boldsymbol{\sigma})$ *of problem* (1.15) *having the regularity*

$$(\boldsymbol{u}, \boldsymbol{\sigma}) \in W^{1,\infty}(0,T; \mathbf{K} \times \mathcal{H}).$$

2 Discretization

2.1 The nonconforming finite element approximations

We suppose that each subdomain Ω^ℓ, $\ell = 1, 2$, is a polygon. With each subdomain Ω^ℓ, we then associate a regular family of discretizations $\mathcal{T}_h^\ell$ made of triangular elements denoted κ such that $\overline{\Omega^\ell} = \bigcup_{\kappa \in \mathcal{T}_h^\ell} \overline{\kappa}$. The discretization parameter h_ℓ on Ω^ℓ is given by $h_\ell = \max_{\kappa \in \mathcal{T}_h^\ell} h_\kappa$ where h_κ denotes the diameter of the triangle κ. We suppose that the end points $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ of the contact zone Γ_C are common nodes of the triangulations $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ and that the monodimensional traces of triangulations of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ on Γ_C denoted θ_h^1 and θ_h^2 are uniformly regular. For any integer $q \geq 1$, the notation $\mathcal{P}_q(\kappa)$ represents the space of the polynomials with the global degree $\leq q$ on κ.

The space of constant symmetrical tensors on each element of the mesh is chosen for the approximation of the stress fields:

$$Q_h(\Omega^\ell) = \left\{ \boldsymbol{\tau}_h^\ell \in (L^2(\Omega^\ell))^4_{Sym}, \quad \boldsymbol{\tau}_h^\ell|_\kappa \in (\mathcal{P}_0(\kappa))^4_{Sym}, \quad \forall \kappa \in \mathcal{T}_h^\ell \right\},$$

and we set $Q_h = Q_h(\Omega^1) \times Q_h(\Omega^2)$.

The finite element space used in Ω^ℓ for the displacement fields is defined by (see [6])

$$\mathbf{V}_h(\Omega^\ell) = \left\{ \boldsymbol{v}_h^\ell \in (\mathcal{C}(\overline{\Omega^\ell}))^2,\ \boldsymbol{v}_h^\ell|_\kappa \in (\mathcal{P}_1(\kappa))^2,\ \forall \kappa \in \mathcal{T}_h^\ell,\quad \boldsymbol{v}_h^\ell|_{\Gamma_D^\ell} = 0 \right\}.$$

Denoting $\mathbf{V}_h = \mathbf{V}_h(\Omega^1) \times \mathbf{V}_h(\Omega^2)$, it is straightforward that $\boldsymbol{\varepsilon}(\mathbf{V}_h) \subset Q_h$. Moreover, we assume that $G(Q_h, Q_h) \subset Q_h$ and $\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{V}_h) \subset Q_h$.

To translate in the finite element context the noninterpenetration conditions contained in (1.13), define

$$\begin{aligned} W_h^\ell(\Gamma_C) = \{\psi_h : \overline{\Gamma}_C \to \mathbb{R}\ ;\quad & \psi_{h|_\tau} \in P_1(\tau),\quad \forall \tau \in \theta_h^\ell, \\ & \psi_h \text{ continuous on each straight line segment of } \Gamma_C\}. \end{aligned}$$

We also consider the subspace

$$M_h^\ell(\Gamma_C) = \left\{ \psi_h \in W_h^\ell(\Gamma_C),\ \forall \tau \in \theta_h^\ell,\ \psi_h|_\tau \in \mathcal{P}_1(\tau) \text{ and } \psi_h|_\tau \in \mathcal{P}_0(\tau) \text{ if } \boldsymbol{c}_1 \text{ or } \boldsymbol{c}_2 \in \tau \right\}.$$

This allows to introduce the following operator π_h^ℓ on $W_h^\ell(\Gamma_C)$ defined for any function $\phi : \overline{\Gamma}_C \to \mathbb{R}$, piecewise continuous on $\overline{\Gamma}_C$, by:

$$\begin{aligned} \pi_h^\ell \phi &\in W_h^\ell(\Gamma_C), \\ (\pi_h^\ell \phi)(\boldsymbol{c}_i) &= \phi(\boldsymbol{c}_i), \qquad \text{for } i = 1 \text{ and } 2, \\ \int_{\Gamma_C} (\phi - \pi_h^\ell \phi)\psi_h \, d\Gamma &= 0, \qquad \forall \psi_h \in M_h^\ell(\Gamma_C). \end{aligned}$$

Remark 2.1. *If Γ_C is a straight line segment, operator π_h^ℓ coincides with the mortar projection operator whose stability and approximation properties can be found in [3, 1]. Moreover, let us note that operator π_h^ℓ is very "close" to the orthogonal projection operator $\tilde{\pi}_h^\ell$ onto $W_h^\ell(\Gamma_C)$, defined for any function $\phi \in L^2(\Gamma_C)$ by $\tilde{\pi}_h^\ell \phi \in W_h^\ell(\Gamma_C)$ and $\int_{\Gamma_C} (\phi - \tilde{\pi}_h^\ell \phi)\psi_h \, d\Gamma = 0$, $\forall \psi_h \in W_h^\ell(\Gamma_C)$. The latter operator can be used in the numerical experiments (without loss of quality of computations) in our simple case involving only two bodies in contact without domain decomposition.*

Another approach to discretize the contact condition is based on the linear piecewise interpolation operator on the mesh θ_h^ℓ that we denote by $\mathcal{I}_h^\ell$. It is also defined for any function $\phi : \overline{\Gamma}_C \to \mathbb{R}$, piecewise continuous on $\overline{\Gamma}_C$, by:

$$\mathcal{I}_h^\ell \phi \in W_h^\ell(\Gamma_C), \quad (\mathcal{I}_h^\ell \phi)_{|_\tau}(\boldsymbol{y}_i) = \phi_{|_\tau}(\boldsymbol{y}_i),\ i = 1, 2,\ \forall \tau \in \theta_h^\ell,$$

where $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ are the nodes of $\tau \in \theta_h^\ell$.

We must notice that if Γ_C is a straight line segment, $\mathcal{I}_h^\ell$ coincides with the usual Lagrange interpolation operator of order one in θ_h^ℓ.

The next step consists of defining the discrete convex cone (approximating $\mathbf{K}$) involving the latter projection operator

$$\mathbf{K}_h^\pi = \left\{ \boldsymbol{v}_h = (\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) \in \mathbf{V}_h(\Omega^1) \times \mathbf{V}_h(\Omega^2), \quad \boldsymbol{v}_h^1.\boldsymbol{\nu}^1 + \pi_h^1(\boldsymbol{v}_h^2.\boldsymbol{\nu}^2) \le 0 \text{ on } \Gamma_C \right\},$$

and another approximation convex cone using the Lagrange interpolation operator

$$\mathbf{K}_h^{\mathcal{I}} = \left\{ \boldsymbol{v}_h = (\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) \in \mathbf{V}_h(\Omega^1) \times \mathbf{V}_h(\Omega^2), \quad \boldsymbol{v}_h^1.\boldsymbol{\nu}^1 + \mathcal{I}_h^1(\boldsymbol{v}_h^2.\boldsymbol{\nu}^2) \le 0 \text{ on } \Gamma_C \right\}.$$

Note that both approximations are different and nonconforming so that generally $\mathbf{K}_h^\pi \neq \mathbf{K}_h^{\mathcal{I}}$, $\mathbf{K}_h^\pi \not\subset \mathbf{K}$ and $\mathbf{K}_h^{\mathcal{I}} \not\subset \mathbf{K}$. This means that (slight) interpenetrations are allowed in these approximations. It is easy to see that if all the nodes of the mesh of Ω^2 on Γ_C are also nodes of the mesh of Ω^1 then $\mathbf{K}_h^\pi = \mathbf{K}_h^{\mathcal{I}} \subset \mathbf{K}$. In particular, the case in which matching meshes are considered was studied by Han and Sofonea (see [12]) and the comparison between $\mathbf{K}_h^\pi$ and $\mathbf{K}_h^{\mathcal{I}}$ was performed in the static linear elastic case (see [15]).

Moreover, it is straightforward that the choice of $\mathbf{K}_h^{\mathcal{I}}$ corresponds to the classical local node-on-segment contact conditions and that the choice of $\mathbf{K}_h^\pi$ leads to more global conditions due to the global character of the projection operator. Finally, notice also that the symmetrical definitions of $\mathbf{K}_h^\pi$ and $\mathbf{K}_h^{\mathcal{I}}$ using π_h^2 and $\mathcal{I}_h^2$ could also be chosen. Talking of that, let us mention that we are not able to exhibit a discrete approximation of $\mathbf{K}$ in which both bodies play the same role and which leads to reasonable a priori error estimates.

2.2 Fully discrete approximation: implicit scheme

We consider a fully discrete approximation of problem (1.15). Given a partition of time interval $[0,T]$: $0 = t_0 < t_1 < \ldots < t_N = T$, we denote by k_n the step size $t_n - t_{n-1}$ for $n = 1, 2, \ldots, N$. Let $k = \max_n k_n$ be the time discretization parameter. Given a sequence $\{w^n\}_{n=0}^N$, we define $\delta w^n = (w^n - w^{n-1})/k_n$. Finally, we denote $\mathbf{F}^n = \mathbf{F}(t_n)$, $\boldsymbol{u}^n = \boldsymbol{u}(t_n)$, $\boldsymbol{u}^{1,n} = \boldsymbol{u}^1(t_n)$, $\boldsymbol{u}^{2,n} = \boldsymbol{u}^2(t_n)$, $\boldsymbol{\sigma}^n = \boldsymbol{\sigma}(t_n)$, $\boldsymbol{\sigma}_\nu^n = \boldsymbol{\sigma}_\nu(t_n)$ for $n = 0, 1, \ldots, N$.

The discretized problem derived from (1.15) uses the above-mentioned nonconforming finite element approaches and a backward Euler scheme. It consists then of finding the displacement fields $\boldsymbol{u}_{hk} = \{\boldsymbol{u}_{hk}^n\}_{n=0}^N \subset \mathbf{K}_h$ and the stress fields $\boldsymbol{\sigma}_{hk} = \{\boldsymbol{\sigma}_{hk}^n\}_{n=0}^N \subset Q_h$

such that:

$$\begin{cases} \boldsymbol{u}_{hk}^0 \in \mathbf{K}_h, \quad \boldsymbol{\sigma}_{hk}^0 \in Q_h, \\ \delta\boldsymbol{\sigma}_{hk}^n = \mathcal{E}\delta\boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^n) + G(\boldsymbol{\sigma}_{hk}^n, \boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^n)), \qquad n = 1, 2, \dots, N, \\ (\boldsymbol{\sigma}_{hk}^n, \boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}_{hk}^n))_{\mathcal{H}} \geq \langle \mathbf{F}^n, \boldsymbol{v}_h - \boldsymbol{u}_{hk}^n \rangle, \qquad \forall \boldsymbol{v}_h \in \mathbf{K}_h,\ n = 1, 2, \dots, N, \end{cases} \tag{2.1}$$

where $\mathbf{K}_h = \mathbf{K}_h^\pi$ or $\mathbf{K}_h = \mathbf{K}_h^\mathcal{I}$.

The following existence and uniqueness statement holds for problem (2.1).

Proposition 2.1. *Let the assumptions of Proposition 1.1 still hold. Let $\mathbf{K}_h = \mathbf{K}_h^\pi$ or $\mathbf{K}_h = \mathbf{K}_h^\mathcal{I}$. There exists a constant $k_0 > 0$, such that if $k \leq k_0$ then problem (2.1) admits a unique solution $(\boldsymbol{u}_{hk}, \boldsymbol{\sigma}_{hk}) \in \mathbf{K}_h \times Q_h$.*

Proof. Since $\mathbf{K}_h^\pi$ and $\mathbf{K}_h^\mathcal{I}$ are nonempty closed convex sets (cones), it is easy to check that the proof of the conforming case (with matching meshes) obtained in [12] which uses a fixed point technique is still valid in our context. □

3 Error analysis

The purpose of this section is to prove the convergence of the discrete solution $(\boldsymbol{u}_{hk}, \boldsymbol{\sigma}_{hk})$ towards $(\boldsymbol{u}, \boldsymbol{\sigma})$ for both local and global approaches introduced in the previous section. The key tool is an extension of Falk's lemma (see [9]) to a fully discrete approximation problem. This result is as follows.

Proposition 3.1. *Let the assumptions of Proposition 1.1 still hold. Let $(\boldsymbol{u}, \boldsymbol{\sigma})$ be the solution of (1.15). Suppose that $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ are such that $\boldsymbol{u}^1 \in L^\infty(0, T; (H^2(\Omega^1))^2)$, $\boldsymbol{u}^2 \in L^\infty(0, T; (H^2(\Omega^2))^2)$ and $\sigma_\nu \in L^\infty(0, T; L^2(\Gamma_C))$. Let $(\boldsymbol{u}_{hk}, \boldsymbol{\sigma}_{hk})$ be the solution of (2.1). Then*

$$\begin{aligned} \max_{1 \leq n \leq N} \Big(\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_{hk}^n\|_{\mathcal{H}} + \|\boldsymbol{u}^n - \boldsymbol{u}_{hk}^n\|_{\mathrm{V}} \Big) &\leq C \Big(\|\boldsymbol{\sigma}^0 - \boldsymbol{\sigma}_{hk}^0\|_{\mathcal{H}} + \|\boldsymbol{u}^0 - \boldsymbol{u}_{hk}^0\|_{\mathrm{V}} \Big) \\ +C \max_{1 \leq n \leq N} \Bigg\{ &\inf_{\boldsymbol{v}_h \in \mathbf{K}_h} \Big(\|\boldsymbol{v}_h - \boldsymbol{u}^n\|_{\mathrm{V}} + \Big| \int_{\Gamma_C} \sigma_\nu^n [(\boldsymbol{v}_h - \boldsymbol{u}^n).\boldsymbol{\nu}]\, d\Gamma \Big|^{\frac{1}{2}} \Big) \\ &+ \inf_{\boldsymbol{v} \in \mathbf{K}} \Big(\Big| \int_{\Gamma_C} \sigma_\nu^n [(\boldsymbol{v} - \boldsymbol{u}_{hk}^n).\boldsymbol{\nu}]\, d\Gamma \Big|^{\frac{1}{2}} \Big) \\ &+ \Big| \int_0^{t_n} G(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds - \sum_{i=1}^n k_i G(\boldsymbol{\sigma}^i, \boldsymbol{\varepsilon}(\boldsymbol{u}^i)) \Big| \Bigg\}, \end{aligned}$$

where $\mathbf{K}_h = \mathbf{K}_h^{\pi}$ *or* $\mathbf{K}_h = \mathbf{K}_h^{\mathcal{I}}$ *and constant* C *is independent of* h_1, h_2 *and* k.

Proof. Let $t = t_n$ with $n = 1, \ldots, N$ and $\boldsymbol{v} \in \mathbf{K}, \boldsymbol{v}_h \in \mathbf{K}_h$. We have

$$\begin{aligned}&(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_{hk}^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n) - \boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^n))_{\mathcal{H}} = \\ &(\boldsymbol{\sigma}^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n))_{\mathcal{H}} + (\boldsymbol{\sigma}_{hk}^n, \boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^n))_{\mathcal{H}} - (\boldsymbol{\sigma}^n, \boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^n))_{\mathcal{H}} - (\boldsymbol{\sigma}_{hk}^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n))_{\mathcal{H}}.\end{aligned}$$

The variational inequalities in (1.15) and (2.1) can be written

$$\begin{aligned}(\boldsymbol{\sigma}^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n))_{\mathcal{H}} &\leq (\boldsymbol{\sigma}^n, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} - \langle \mathbf{F}^n, \boldsymbol{v} - \boldsymbol{u}^n \rangle, \\ (\boldsymbol{\sigma}_{hk}^n, \boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^n))_{\mathcal{H}} &\leq (\boldsymbol{\sigma}_{hk}^n, \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_{\mathcal{H}} - \langle \mathbf{F}^n, \boldsymbol{v}_h - \boldsymbol{u}_{hk}^n \rangle.\end{aligned}$$

Hence

$$\begin{aligned}(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_{hk}^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n - \boldsymbol{u}_{hk}^n))_{\mathcal{H}} \leq\ & (\boldsymbol{\sigma}_{hk}^n - \boldsymbol{\sigma}^n, \boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}^n))_{\mathcal{H}} \\ & + (\boldsymbol{\sigma}^n, \boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}^n))_{\mathcal{H}} - \langle \mathbf{F}^n, \boldsymbol{v}_h - \boldsymbol{u}^n \rangle \\ & + (\boldsymbol{\sigma}^n, \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}_{hk}^n))_{\mathcal{H}} - \langle \mathbf{F}^n, \boldsymbol{v} - \boldsymbol{u}_{hk}^n \rangle. \qquad (3.1)\end{aligned}$$

From (2.1), we get

$$\boldsymbol{\sigma}_{hk}^n = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^n) + k_n G(\boldsymbol{\sigma}_{hk}^n, \boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^n)) + \boldsymbol{\sigma}_{hk}^{n-1} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^{n-1}),$$

and we deduce

$$\boldsymbol{\sigma}_{hk}^n = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^n) + \sum_{i=1}^{n} k_i G(\boldsymbol{\sigma}_{hk}^i, \boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^i)) + \boldsymbol{\sigma}_{hk}^0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^0).$$

Integrating the constitutive law in (1.15) from $t = 0$ to $t = t_n$ yields

$$\boldsymbol{\sigma}^n = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^n) + \int_0^{t_n} G(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}^0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^0).$$

Subtracting both previous equalities gives

$$\begin{aligned}\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^n - \boldsymbol{u}_{hk}^n) = \ & \boldsymbol{\sigma}^n - \boldsymbol{\sigma}_{hk}^n + \sum_{i=1}^{n} k_i \Big(G(\boldsymbol{\sigma}_{hk}^i, \boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^i)) - G(\boldsymbol{\sigma}^i, \boldsymbol{\varepsilon}(\boldsymbol{u}^i)) \Big) \\ & - \int_0^{t_n} G(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \sum_{i=1}^{n} k_i G(\boldsymbol{\sigma}^i, \boldsymbol{\varepsilon}(\boldsymbol{u}^i)) \\ & + \boldsymbol{\sigma}_{hk}^0 - \boldsymbol{\sigma}^0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^0 - \boldsymbol{u}^0). \qquad (3.2)\end{aligned}$$

Denote for $n = 1, \ldots, N$

$$I_n = \Big| \int_0^{t_n} G(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds - \sum_{i=1}^{n} k_i G(\boldsymbol{\sigma}^i, \boldsymbol{\varepsilon}(\boldsymbol{u}^i)) \Big|$$

and for $n = 0, ..., N$

$$e_n = \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^n_{hk}\|_{\mathcal{H}} + \|\boldsymbol{u}^n - \boldsymbol{u}^n_{hk}\|_{\mathbf{V}}.$$

From (3.2), properties (1.11) and (1.12) and Korn's inequality, we obtain:

$$\|\boldsymbol{u}^n - \boldsymbol{u}^n_{hk}\|_{\mathbf{V}} \leq C\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^n_{hk}\|_{\mathcal{H}} + C\sum_{i=1}^{n} k_i\Big(\|\boldsymbol{\sigma}^i_{hk} - \boldsymbol{\sigma}^i\|_{\mathcal{H}} + \|\boldsymbol{u}^i_{hk} - \boldsymbol{u}^i\|_{\mathbf{V}}\Big)$$
$$+CI_n + Ce_0. \tag{3.3}$$

Putting together (3.1) and (3.2) leads to

$$\begin{aligned}(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^n_{hk}, \mathcal{E}^{-1}(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^n_{hk}))_{\mathcal{H}} &\leq (\boldsymbol{\sigma}^n_{hk} - \boldsymbol{\sigma}^n, \boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}^n))_{\mathcal{H}} \\ &+(\boldsymbol{\sigma}^n, \boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}^n))_{\mathcal{H}} - \langle \mathbf{F}^n, \boldsymbol{v}_h - \boldsymbol{u}^n\rangle \\ &+(\boldsymbol{\sigma}^n, \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}^n_{hk}))_{\mathcal{H}} - \langle \mathbf{F}^n, \boldsymbol{v} - \boldsymbol{u}^n_{hk}\rangle\end{aligned}$$
$$+\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^n_{hk}\|_{\mathcal{H}}\Big(C\sum_{i=1}^{n} k_i(\|\boldsymbol{\sigma}^i_{hk} - \boldsymbol{\sigma}^i\|_{\mathcal{H}} + \|\boldsymbol{u}^i_{hk} - \boldsymbol{u}^i\|_{\mathbf{V}}) + CI_n + Ce_0\Big).$$

Using the definition of $\mathbf{F}$ and Green's formula, we get

$$\begin{aligned}\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^n_{hk}\|_{\mathcal{H}} \leq\; & C\|\boldsymbol{v}_h - \boldsymbol{u}^n\|_{\mathbf{V}} \\ &+C\Big|\int_{\Gamma_C} \sigma^n_\nu[(\boldsymbol{v}_h - \boldsymbol{u}^n).\boldsymbol{\nu}]\, d\Gamma\Big|^{\frac{1}{2}} \\ &+C\Big|\int_{\Gamma_C} \sigma^n_\nu[(\boldsymbol{v} - \boldsymbol{u}^n_{hk}).\boldsymbol{\nu}]\, d\Gamma\Big|^{\frac{1}{2}} \\ &+C\sum_{i=1}^{n} k_i(\|\boldsymbol{\sigma}^i_{hk} - \boldsymbol{\sigma}^i\|_{\mathcal{H}} + \|\boldsymbol{u}^i_{hk} - \boldsymbol{u}^i\|_{\mathbf{V}}) + CI_n + Ce_0. \end{aligned}\tag{3.4}$$

Combining (3.3) and (3.4), we obtain for any $\boldsymbol{v}_h \in \mathbf{K}_h$, and any $\boldsymbol{v} \in \mathbf{K}$

$$\begin{aligned}\|\boldsymbol{u}^n - \boldsymbol{u}^n_{hk}\|_{\mathbf{V}} + \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^n_{hk}\|_{\mathcal{H}} \leq\; & C\|\boldsymbol{v}_h - \boldsymbol{u}^n\|_{\mathbf{V}} \\ &+C\Big|\int_{\Gamma_C} \sigma^n_\nu[(\boldsymbol{v}_h - \boldsymbol{u}^n).\boldsymbol{\nu}]\, d\Gamma\Big|^{\frac{1}{2}} \\ &+C\Big|\int_{\Gamma_C} \sigma^n_\nu[(\boldsymbol{v} - \boldsymbol{u}^n_{hk}).\boldsymbol{\nu}]\, d\Gamma\Big|^{\frac{1}{2}} + CI_n + Ce_0 \\ &+C\sum_{i=1}^{n} k_i(\|\boldsymbol{\sigma}^i_{hk} - \boldsymbol{\sigma}^i\|_{\mathcal{H}} + \|\boldsymbol{u}^i_{hk} - \boldsymbol{u}^i\|_{\mathbf{V}}). \end{aligned}\tag{3.5}$$

Set for $n = 1, \dots, N$

$$g_n = \|\boldsymbol{v}_h - \boldsymbol{u}^n\|_{\mathbf{V}} + \Big|\int_{\Gamma_C} \sigma^n_\nu[(\boldsymbol{v}_h - \boldsymbol{u}^n).\boldsymbol{\nu}]\, d\Gamma\Big|^{\frac{1}{2}} + \Big|\int_{\Gamma_C} \sigma^n_\nu[(\boldsymbol{v} - \boldsymbol{u}^n_{hk}).\boldsymbol{\nu}]\, d\Gamma\Big|^{\frac{1}{2}} + I_n + e_0.$$

From (3.5), we get

$$e_n \leq Cg_n + C\sum_{i=1}^{n} k_i e_i, \qquad n = 1, \dots, N.$$

Using Gronwall's discrete inequality, one obtains under the assumption that $k = \max_n k_n$ is small enough:

$$e_n \le C g_n + C \sum_{i=1}^{n} k_i g_i, \qquad n = 1, \dots, N.$$

Consequently

$$\max_{1 \le n \le N} \Big(\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^n_{hk}\|_{\mathcal{H}} + \|\boldsymbol{u}^n - \boldsymbol{u}^n_{hk}\|_{\mathbf{V}} \Big) \le C \max_{1 \le n \le N} g_n.$$

That ends the proof. □

Clearly, the previous proposition divides the error committed by the approximation into four different quantities: the error on the initial conditions, the approximation error comprising a norm and an integral term, the consistency error coming from the nonconformity of the finite element methods and an integration error term on G. We restrict us to the case where Γ_C is a straight line segment for many reasons: because operator π_h^ℓ could be decomposed on each straight line segment of Γ_C into different projection operators, to use the already existing results of [2] established in the case of a straight line segment and to avoid a lenghty supplementary convergence study in this paper. Therefore, in the following lemma, we give the estimates for the approximation and the consistency error terms in the case $\mathbf{K}_h = \mathbf{K}_h^\pi$ and $\mathbf{K}_h = \mathbf{K}_h^{\mathcal{I}}$.

Lemma 3.1. *Let the hypothesis of Proposition 3.1 still hold. Assume the following additional conditions:*

a) Γ_C *is a straight line segment.* (3.6)

b) If $\bar{\Gamma}_D^1 \cap \bar{\Gamma}_C \ne \emptyset$, *then for all* $\boldsymbol{a} \in \bar{\Gamma}_D^1 \cap \bar{\Gamma}_C$ *we have:*

i) the line segments of $\bar{\Gamma}_D^1$ *and* $\bar{\Gamma}_C$ *which contain* $\boldsymbol{a}$ *form an angle different of* π. (3.7)

ii) $[u_\nu(t)](\boldsymbol{a}) = u_{2\nu}(t)(\boldsymbol{a}) = 0 \quad \forall t \in [0,T]$. (3.8)

c) $\sigma_\nu \in L^\infty(0,T; H^{\frac{1}{2}}(\Gamma_C))$. (3.9)

(i) Let $\mathbf{K}_h = \mathbf{K}_h^\pi$ *and let* $n = 0, 1, \dots, N$. *There exists* $\boldsymbol{v}_h \in \mathbf{K}_h^\pi$ *that satisfies the estimates*

$$\|\boldsymbol{v}_h - \boldsymbol{u}^n\|_{\mathbf{V}} + \Big| \int_{\Gamma_C} \sigma_\nu^n [(\boldsymbol{v}_h - \boldsymbol{u}^n).\boldsymbol{\nu}] \, d\Gamma \Big|^{\frac{1}{2}} \le C(\boldsymbol{u}^n)(h_1^{\frac{3}{4}} + h_2),$$

where constant $C(\boldsymbol{u}^n)$ *depends linearly on* $\|\boldsymbol{u}^{1,n}\|_{(H^2(\Omega^1))^2}$ *and* $\|\boldsymbol{u}^{2,n}\|_{(H^2(\Omega^2))^2}$.

Also, there exists $\boldsymbol{v} \in \mathbf{K}$ *that satisfies the estimates*

$$\Big| \int_{\Gamma_C} \sigma_\nu^n [(\boldsymbol{v} - \boldsymbol{u}^n_{hk}).\boldsymbol{\nu}] \, d\Gamma \Big|^{\frac{1}{2}} \le \Big(C(\boldsymbol{u}^n) h_1 \|\boldsymbol{u}^n - \boldsymbol{u}^n_{hk}\|_{\mathbf{V}} \Big)^{\frac{1}{2}} + C(\boldsymbol{u}^n)(h_1 + h_2),$$

where constant $C(\boldsymbol{u}^n)$ *depends linearly on* $\|\boldsymbol{u}^{1,n}\|_{(H^2(\Omega^1))^2}$ *and* $\|\boldsymbol{u}^{2,n}\|_{(H^2(\Omega^2))^2}$.

(ii) Let $\mathbf{K}_h = \mathbf{K}_h^{\mathcal{I}}$ *and let* $n = 0, 1, \ldots, N$. *There exists* $\boldsymbol{v}_h \in \mathbf{K}_h^{\mathcal{I}}$ *that satisfies the estimates*

$$\|\boldsymbol{v}_h - \boldsymbol{u}^n\|_{\mathrm{V}} + \left| \int_{\Gamma_C} \sigma_\nu^n [(\boldsymbol{v}_h - \boldsymbol{u}^n).\boldsymbol{\nu}] \, d\Gamma \right|^{\frac{1}{2}} \leq C(\boldsymbol{u}^n)(h_1^{\frac{3}{4}} + h_2),$$

where constant $C(\boldsymbol{u}^n)$ *depends linearly on* $\|\boldsymbol{u}^{1,n}\|_{(H^2(\Omega^1))^2}$ *and* $\|\boldsymbol{u}^{2,n}\|_{(H^2(\Omega^2))^2}$.

Also, there exists $\boldsymbol{v} \in \mathbf{K}$ *that satisfies the estimates*

$$\left| \int_{\Gamma_C} \sigma_\nu^n [(\boldsymbol{v} - \boldsymbol{u}_{hk}^n).\boldsymbol{\nu}] \, d\Gamma \right|^{\frac{1}{2}} \leq \left(C(\boldsymbol{u}^n) h_1^{\frac{1}{2}} \|\boldsymbol{u}^n - \boldsymbol{u}_{hk}^n\|_{\mathrm{V}} \right)^{\frac{1}{2}} + C(\boldsymbol{u}^n)(h_1^{\frac{3}{4}} + h_2^{\frac{3}{4}}),$$

where constant $C(\boldsymbol{u}^n)$ *depends linearly on* $\|\boldsymbol{u}^{1,n}\|_{(H^2(\Omega^1))^2}$ *and* $\|\boldsymbol{u}^{2,n}\|_{(H^2(\Omega^2))^2}$.

Proof. The two bounds in (i) have been proved in the (technical) Lemma 4.2 and Lemma 4.4 of [2]. Replacing projection operator π_h^1 with interpolation operator $\mathcal{I}_h^1$ in these lemmas yields the less satisfactory bounds of (ii). The loss of convergence rate in the consistency term of (ii) comes from the poor approximation properties of the Lagrange interpolation operator in dual Sobolev spaces (see [16]). □

Remark 3.1. *If we do not assume hypothesis (3.7), then we obtain an error estimate of order* $r - \epsilon/2$ *where* $r = \frac{3}{4}$ *if* $\mathbf{K}_h = \mathbf{K}_h^{\pi}$ *and* $r = \frac{1}{2}$ *if* $\mathbf{K}_h = \mathbf{K}_h^{\mathcal{I}}$ *(see [10]).*

We finally obtain the convergence result of the fully discrete approximation.

Theorem 3.1. *Assume the hypothesis of Lemma 3.1 still hold. Let* $(\boldsymbol{u}, \boldsymbol{\sigma}) \in W^{1,\infty}(0, T; \mathbf{K} \times \mathcal{H}_1)$ *be the solution of* (1.15). *Set* $\mathbf{K}_h = \mathbf{K}_h^{\pi}$ *or* $\mathbf{K}_h = \mathbf{K}_h^{\mathcal{I}}$ *and let* $\{\boldsymbol{u}_{hk}^n\}_{n=0}^N$, $\{\boldsymbol{\sigma}_{hk}^n\}_{n=0}^N$ *be the solution of* (2.1). *Let* $h = \max(h_1, h_2)$. *We have the following error estimates:*

$$\max_{1 \leq n \leq N} \left(\|\boldsymbol{u}^n - \boldsymbol{u}_{hk}^n\|_{\mathrm{V}} + \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_{hk}^n\|_{\mathcal{H}} \right) \leq C \left(\|\boldsymbol{u}^0 - \boldsymbol{u}_{hk}^0\|_{\mathrm{V}} + \|\boldsymbol{\sigma}^0 - \boldsymbol{\sigma}_{hk}^0\|_{\mathcal{H}} \right) + C(h^r + k),$$

where $r = \frac{3}{4}$ *if* $\mathbf{K}_h = \mathbf{K}_h^{\pi}$ *and* $r = \frac{1}{2}$ *if* $\mathbf{K}_h = \mathbf{K}_h^{\mathcal{I}}$ *and constant* C *is independent of* h *and* k.

Proof. Let us consider the error terms involved in Proposition 3.1. Using (1.12) the integration error committed on the functional G is bounded as follows:

$$\begin{aligned}
&\Big|\int_0^{t_n} G(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds-\sum_{i=1}^{n}k_iG(\boldsymbol{\sigma}^i,\boldsymbol{\varepsilon}(\boldsymbol{u}^i))\Big|\\
&\leq\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|G(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))-G(\boldsymbol{\sigma}^i,\boldsymbol{\varepsilon}(\boldsymbol{u}^i))|\,ds\\
&\leq C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\|\boldsymbol{\sigma}(s)-\boldsymbol{\sigma}^i\|_{\mathcal{H}}+\|\boldsymbol{\varepsilon}(\boldsymbol{u}(s))-\boldsymbol{\varepsilon}(\boldsymbol{u}^i)\|_{\mathcal{H}}\,ds\\
&\leq CTk\Big(\|\dot{\boldsymbol{\sigma}}\|_{L^\infty(0,T;\mathcal{H})}+\|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\|_{L^\infty(0,T;\mathcal{H})}\Big).
\end{aligned}$$

The announced result is now obtained using the approximation and consistency error bounds of Lemma 3.2 in Proposition 3.1 and writing

$$\Big(C(\boldsymbol{u}^n)h_1^\beta\|\boldsymbol{u}^n-\boldsymbol{u}_{hk}^n\|_{\mathbf{V}}\Big)^{\frac{1}{2}}\leq\gamma\|\boldsymbol{u}^n-\boldsymbol{u}_{hk}^n\|_{\mathbf{V}}+\frac{1}{4\gamma}C(\boldsymbol{u}^n)h_1^\beta,$$

(with $\beta=1$ if $\mathbf{K}_h=\mathbf{K}_h^\pi$ and $\beta=\frac{1}{2}$ if $\mathbf{K}_h=\mathbf{K}_h^{\mathcal{I}}$) for an arbitrary γ small enough. □

4 Solution of the fully discrete problem

Let $\mathbf{V}_h$ and Q_h be the finite element spaces defined in section 2.1 and composed by piecewise linear and constant functions respectively. We first write problem (2.1) in an equivalent form:

Find the displacements fields $\boldsymbol{u}_{hk}=\{\boldsymbol{u}_{hk}^n\}_{n=0}^N\subset\mathbf{K}_h$ and the stress fields $\boldsymbol{\sigma}_{hk}=\{\boldsymbol{\sigma}_{hk}^n\}_{n=0}^N\subset Q_h$ such that:

$$\left\{\begin{array}{l}
\boldsymbol{u}_{hk}^0\in\mathbf{K}_h,\quad\boldsymbol{\sigma}_{hk}^0\in Q_h\quad\text{ and for } n=1,2\ldots,N\\
\boldsymbol{\sigma}_{hk}^n=\boldsymbol{\sigma}_{hk}^{n-1}+\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^n)-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^{n-1})+k_nG(\boldsymbol{\sigma}_{hk}^n,\boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^n)),\\
(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^n),\boldsymbol{\varepsilon}(\boldsymbol{v}_h-\boldsymbol{u}_{hk}^n))_{\mathcal{H}}+(k_nG(\boldsymbol{\sigma}_{hk}^n,\boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^n)),\boldsymbol{\varepsilon}(\boldsymbol{v}_h-\boldsymbol{u}_{hk}^n))_{\mathcal{H}}\geq\\
\langle\mathbf{F}^n,\boldsymbol{v}_h-\boldsymbol{u}_{hk}^n\rangle+(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{hk}^{n-1})-\boldsymbol{\sigma}_{hk}^{n-1},\boldsymbol{\varepsilon}(\boldsymbol{v}_h-\boldsymbol{u}_{hk}^n))_{\mathcal{H}},\quad\forall\boldsymbol{v}_h\in\mathbf{K}_h,
\end{array}\right.\tag{4.1}$$

where $\mathbf{K}_h=\mathbf{K}_h^\pi$ or $\mathbf{K}_h=\mathbf{K}_h^{\mathcal{I}}$.

In order to solve problem (4.1) we apply the same fixed point algorithm as in the proof of Proposition 1.1 and Proposition 2.1 (see [12, 19] for details). This algorithm is formulated as follows:

For some n between 1 and N, let us assume $\boldsymbol{u}_{hk}^{n-1}$ and $\boldsymbol{\sigma}_{hk}^{n-1}$ are known, and let $\boldsymbol{\eta}_{hk}^0\in Q_h$ be given. For $s=0,1,2,\ldots,$ let $\{(\boldsymbol{u}\boldsymbol{\eta}_{hk}^s,\boldsymbol{\sigma}\boldsymbol{\eta}_{hk}^s,\boldsymbol{\eta}_{hk}^{s+1})\}_{s\geq0}$ be the sequence *obtained when solving the following problems* (4.2) *and* (4.3):

$$\begin{cases} \boldsymbol{u}\boldsymbol{\eta}^s_{hk} \in \mathbf{K}_h, \\ (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}\boldsymbol{\eta}^s_{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}\boldsymbol{\eta}^s_{hk}))_{\mathcal{H}} \geq \langle \mathbf{F}^n, \boldsymbol{v}_h - \boldsymbol{u}\boldsymbol{\eta}^s_{hk} \rangle \\ +(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^{n-1}_{hk}) - \boldsymbol{\sigma}^{n-1}_{hk} - k_n \boldsymbol{\eta}^s_{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}_h - \boldsymbol{u}\boldsymbol{\eta}^s_{hk}))_{\mathcal{H}}, \qquad \forall \boldsymbol{v}_h \in \mathbf{K}_h, \end{cases} \tag{4.2}$$

where $\mathbf{K}_h = \mathbf{K}^{\pi}_h$ or $\mathbf{K}_h = \mathbf{K}^{\mathcal{I}}_h$ and

$$\begin{cases} \boldsymbol{\sigma}\boldsymbol{\eta}^s_{hk} = \boldsymbol{\sigma}^{n-1}_{hk} + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}\boldsymbol{\eta}^s_{hk}) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^{n-1}_{hk}) + k_n \boldsymbol{\eta}^s_{hk}, \\ \boldsymbol{\eta}^{s+1}_{hk} = G(\boldsymbol{\sigma}\boldsymbol{\eta}^s_{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}\boldsymbol{\eta}^s_{hk})). \end{cases} \tag{4.3}$$

It can be shown (see [12], Theorem 4.1) that if k is sufficiently small then

$$\lim_{s\to\infty} \boldsymbol{u}\boldsymbol{\eta}^s_{hk} = \boldsymbol{u}^n_{hk}.$$

In order to determine the unique solution $\boldsymbol{u}\boldsymbol{\eta}^s_{hk}$ of the variational inequality (4.2), we use a penalty-duality algorithm (see, e.g., [11, 5, 21, 22]) which is described below.

Given $q^0_{hk} \in E_h = \{q_h \in L^2(\Gamma_C),\ \forall \tau \in \theta^1_h,\ q_h|_\tau \in \mathcal{P}_1(\tau)\}$, $\omega > 0$ *an arbitrary real number and* $\rho \in (0, 1]$, *let* $\{(\boldsymbol{u}^r_{hk}, q^r_{hk})\}_{r\geq 0} \subset \mathbf{V}_h \times E_h$ *be the sequence constructed when solving the following problems* (4.4) *and* (4.5) :

$$\begin{cases} \boldsymbol{u}^r_{hk} \in \mathbf{V}_h, \\ (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^r_{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_{\mathcal{H}} \\ \quad +\omega \displaystyle\int_{\Gamma_C} (\boldsymbol{u}^{1,r}_{hk} \cdot \boldsymbol{\nu}^1 + P^1(\boldsymbol{u}^{2,r}_{hk} \cdot \boldsymbol{\nu}^2))(\boldsymbol{v}^1_h \cdot \boldsymbol{\nu}^1 + P^1(\boldsymbol{v}^2_h \cdot \boldsymbol{\nu}^2))\, d\Gamma = \\ \langle \mathbf{F}^n, \boldsymbol{v}_h \rangle + (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^{n-1}_{hk}) - \boldsymbol{\sigma}^{n-1}_{hk} - k_n \boldsymbol{\eta}^s_{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_{\mathcal{H}} \\ \quad - \displaystyle\int_{\Gamma_C} q^r_{hk}(\boldsymbol{v}^1_h \cdot \boldsymbol{\nu}^1 + P^1(\boldsymbol{v}^2_h \cdot \boldsymbol{\nu}^2))\, d\Gamma, \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h, \end{cases} \tag{4.4}$$

$$\begin{cases} \tilde{q}^r_{hk} = 2(\boldsymbol{u}^{1,r}_{hk} \cdot \boldsymbol{\nu}^1 + P^1(\boldsymbol{u}^{2,r}_{hk} \cdot \boldsymbol{\nu}^2)) + \dfrac{1}{\omega} q^r_{hk}, \\ q^{r+\frac{1}{2}}_{hk} = \omega(\tilde{q}^r_{hk} - 2P_{U_h}(\tilde{q}^r_{hk})), \\ q^{r+1}_{hk} = \rho q^{r+\frac{1}{2}}_{hk} + (1-\rho) q^r_{hk}, \end{cases} \tag{4.5}$$

where P_{U_h} *is the* $L^2(\Gamma_C)$*-projection operator over the closed convex set* U_h *defined by:*

$$U_h = \{q_h \in E_h\, ;\, q_h \leq 0 \ \text{on}\ \Gamma_C\},$$

and P^1 *is either* π^1_h *or* $\mathcal{I}^1_h$. From [5], we get:

$$\lim_{r\to\infty} \boldsymbol{u}^r_{hk} = \boldsymbol{u}\boldsymbol{\eta}^s_{hk}.$$

Remark 4.2. *Problem (4.4) is equivalent to a linear system having the following form:*

$$(A + \omega\, {}^t BB)\overline{\boldsymbol{u}}_r = b - {}^t B\overline{q}_r,$$

where A represents the stiffness matrix, b is the loading vector, $\overline{\boldsymbol{u}}_r$ is the nodal displacements field vector, $\overline{q}_r$ is the multipliers vector and B is a rectangular matrix related to the operator

$$\boldsymbol{v} \to \boldsymbol{v}^1 \cdot \boldsymbol{\nu}^1 + P^1(\boldsymbol{v}^2 \cdot \boldsymbol{\nu}^2).$$

If P^1 is the interpolation operator $\mathcal{I}_h^1$, the matrix formulation of Problem (4.4) is very similar to that of the compatible meshes case and the ideas of [5] have been followed. If P^1 is the projection operator π_h^1, some technical difficulties must be solved in order to calculate the matrix B. Some ideas of [4] are used (see [10] for details).

5 Numerical experiments

In order to verify the accuracy and the performance of the numerical methods described in the previous sections, some experiments have been performed in test problems. In this section we show some results for constitutive models (1.2) when G^ℓ is Perzyna's viscoplastic function (see [8, 17]):

$$G^\ell(\boldsymbol{\sigma}^\ell, \boldsymbol{\varepsilon}^\ell) = -\frac{1}{2\lambda^\ell}\mathcal{E}^\ell(\boldsymbol{\sigma}^\ell - \mathcal{P}_{B^\ell}(\boldsymbol{\sigma}^\ell)),$$

where λ^ℓ is a viscosity constant and $\mathcal{P}_{B^\ell}$ is the projection operator (with respect to the norm $\|\boldsymbol{\tau}^\ell\|^2 = (\boldsymbol{\tau}^\ell, \mathcal{E}^\ell\boldsymbol{\tau}^\ell)_{\mathcal{H}(\Omega^\ell)}$) on the plasticity convex set $B^\ell \subset \mathcal{S}_2$ defined by:

$$B^\ell = \{\boldsymbol{\tau} \in \mathcal{S}_2\,;\ |\boldsymbol{\tau}|_{VM} \le \sigma_Y^\ell\},$$

where $|\cdot|_{VM}$ is the Von-Mises norm for stresses:

$$|\boldsymbol{\tau}|_{VM} = (\tau_{11}^2 + \tau_{22}^2 - \tau_{11}\tau_{22} + 3\tau_{12}^2)^{1/2}, \quad \forall \boldsymbol{\tau} \in \mathcal{S}_2,$$

and σ_Y^ℓ represents the uniaxial yield stress. Moreover, both plane stress elasticity tensors $\mathcal{E}^\ell$ ($\ell = 1, 2$) are given by :

$$\mathcal{E}^\ell\boldsymbol{\tau}^\ell = \frac{E^\ell\kappa^\ell}{1-\kappa^\ell}(\tau_{11}^\ell + \tau_{22}^\ell)\delta_{\alpha\beta} + \frac{E^\ell}{1+\kappa^\ell}\tau_{\alpha\beta}^\ell, \quad 1 \le \alpha, \beta \le 2,$$

where the notations E^ℓ and κ^ℓ denote Young's modulus and Poisson's ratio respectively of the material occupying domain Ω^ℓ.

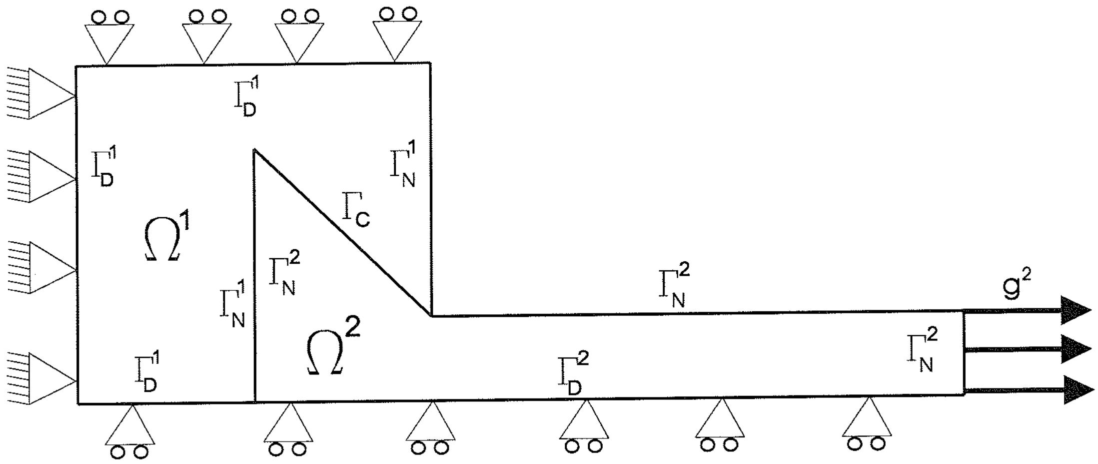

Figure 2: $\mathbf{K}_h^\pi$: Contact boundary is a straight line.

5.1 Test 1: The contact boundary is a straight line

We consider the contact problem between two viscoplastic bodies depicted in Figure 2. The body Ω^2 is submitted to a loading $\boldsymbol{g}^2$ linearly increasing in time. Embedding conditions hold on the left part of the structure and symmetry conditions (normal displacement and tangential component of the stress vector equal to zero) are prescribed on the upper and lower parts of the structure.

The following data have been used for calculations:

$$
\begin{aligned}
&T = 1 \text{ sec.},\\
&\boldsymbol{f}^1 = \mathbf{0}\, N/m^2, \quad \boldsymbol{f}^2 = \mathbf{0}\, N/m^2, \quad \boldsymbol{g}^1 = \mathbf{0}\, N/m,\\
&\boldsymbol{g}^2(x_1, x_2, t) = (10t, 0)\, N/m,\\
&\boldsymbol{\sigma}^0 = \mathbf{0}\, N/m^2, \quad \boldsymbol{u}^0 = \mathbf{0}\, m,\\
&E^\ell = 10^2\, MPa, \quad \kappa^\ell = 0.3, \quad \ell = 1, 2,\\
&\sigma_Y^\ell = \sqrt{10}\, N/m^2, \quad \lambda^\ell = 100\, N \cdot sec/m^2, \quad \ell = 1, 2.
\end{aligned}
$$

The finite element meshes of Ω^1 and Ω^2 are composed by 214 triangles and 358 nodes and 141 triangles and 216 nodes, respectively and they do not match together on the contact zone (see Figure 3).

In Figure 4 we show the deformed configuration and the Von Mises stress norm at final time $T = 1\, sec.$ using the discrete set of admissible displacements $\mathbf{K}_h^\pi$ to approximate $\mathbf{K}$.

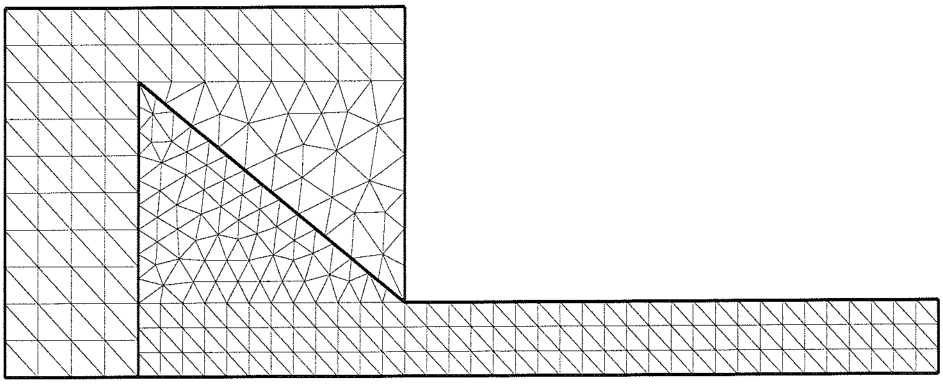

Figure 3: $\mathbf{K}_h^{\pi}$: Non conforming finite element mesh.

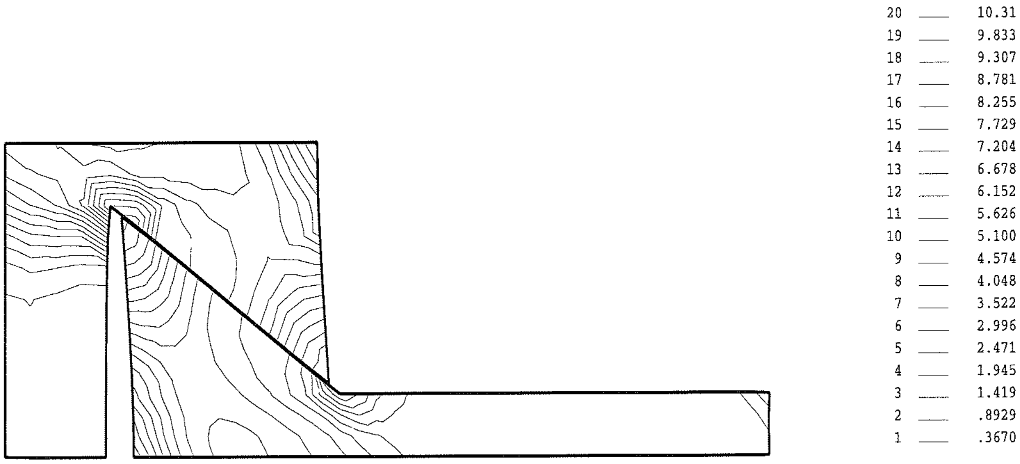

Figure 4: $\mathbf{K}_h^{\pi}$: Contours of the Von-Mises stress norm and deformed configuration after $1sec$.

5.2 Test 2: Comparison of local and global discrete contact conditions

In order to compare results obtained via (local) approximation set $\mathbf{K}_h^{\mathcal{I}}$ and (global) approximation set $\mathbf{K}_h^{\pi}$, we implement both methods for the simple problem shown in Figure 5.

This problem involves two viscoplastic bodies ($E^{\ell} = 3 \times 10^5 \, MPa$, $\kappa^{\ell} = 0.3$, $\lambda^{\ell} = 100 \, N \cdot sec/m^2$, $\sigma_Y^{\ell} = \sqrt{10} \, N/m^2$, $\ell = 1, 2$) whose right boundaries as well as the lower boundary of Ω^2 are submitted to symmetry conditions. A density of forces $\boldsymbol{g}^1(x_1, x_2, t) = (0, -10t) \, N/m$ linearly increasing in time is applied on the upper boundary of Ω^1 whereas body forces are absent. Problem (1.1)-(1.10) is now considered with the following data: $T = 1$ sec., $\boldsymbol{\sigma}^0 = \mathbf{0} \, N/m^2$ and $\boldsymbol{u}^0 = \mathbf{0} \, m$.

Figure 5 shows also the deformed boundaries and the deformed meshes when adopting

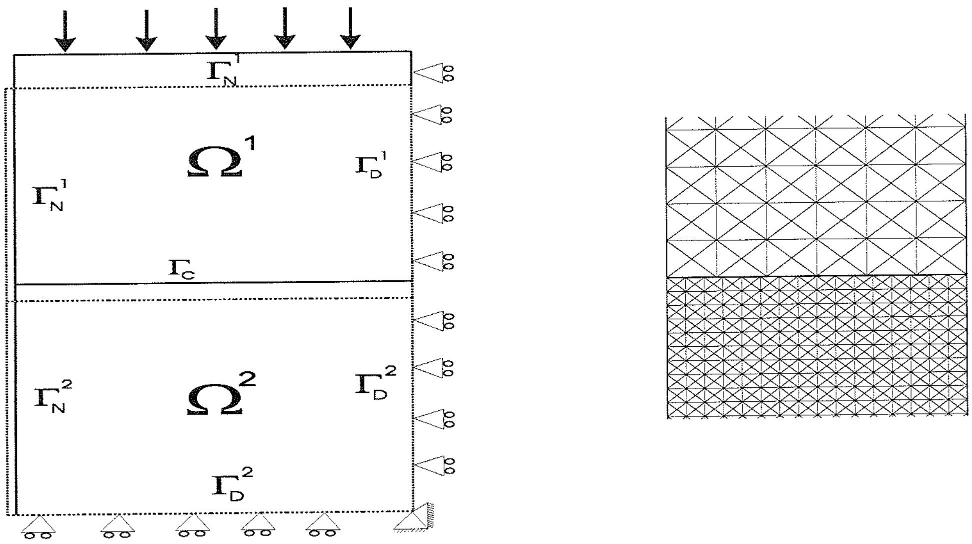

Figure 5: $\mathbf{K}_h^{\pi}$: Initial and deformed boundaries at final time $T = 1\,sec.$ and contact area.

$\mathbf{K}_h^{\pi}$. We remark that no interpenetration has been produced and the obtained stress field is constant ($\sigma_{11} = \sigma_{12} = 0$, $\sigma_{22}(x_1, x_2, t) = 10t$).

In the case where the local approach is chosen, the deformed meshes near the contact area are shown in Figure 6. We now discover a non negligible and non realistic penetration of Ω^2 into Ω^1 as well as artificial stresses, particularly near the contact zone.

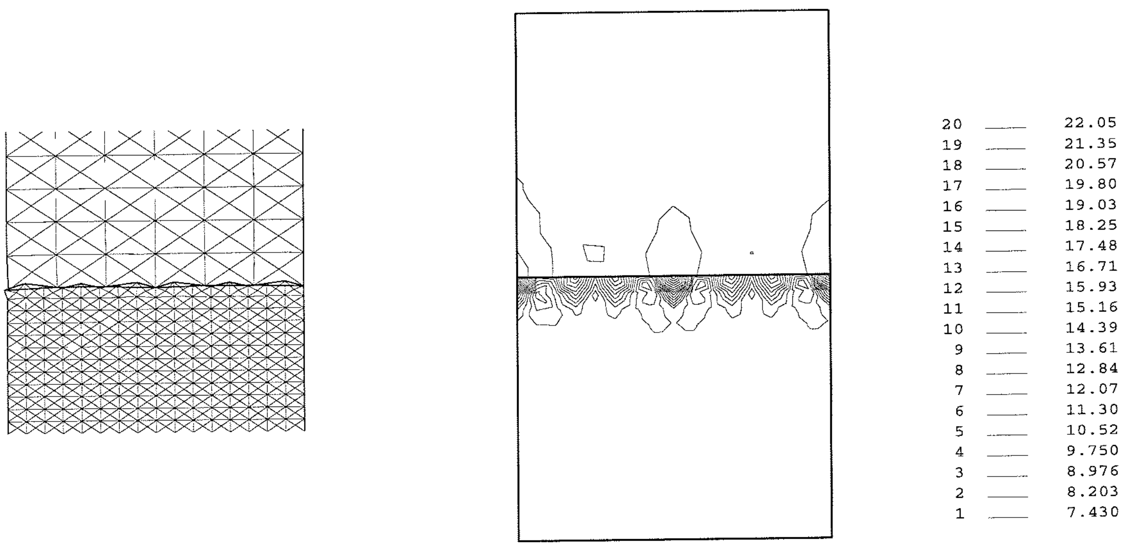

Figure 6: $\mathbf{K}_h^{\mathcal{I}}$: Contact area and stress field at final time $T = 1\,sec.$

5.3 Test 3: The contact boundary is a piecewise straight line

In this last test, we consider a more general contact problem in which the contact boundary is the union of several straight lines (see Figure 7). No body forces are assumed and a boundary force $\boldsymbol{g}^1(x_1, x_2, t) = (0, -10t)\ N/m$ pushing down the upper body to the lower one is applied. The embedding conditions are depicted in Figure 7.

The initial conditions $\boldsymbol{\sigma}^0 = \mathbf{0}\ N/m^2, \boldsymbol{u}^0 = \mathbf{0}\ m$ and a final time of $T = 1$ sec. are chosen. The material characteristics are $E^\ell = 10^2 MPa$, $\kappa^\ell = 0.3$, $\lambda^\ell = 1000 N \cdot sec/m^2$, $\sigma_Y^\ell = \sqrt{10} N/m^2$, $\ell = 1, 2$.

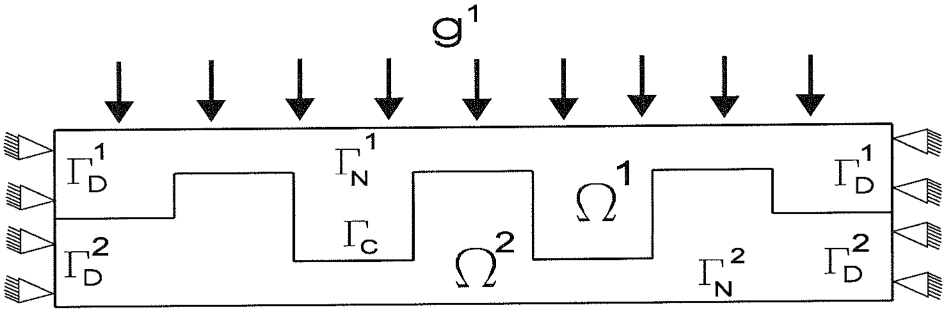

Figure 7: A case with a piecewise straight line as contact boundary.

A first computation using the global contact approach and the nonmatching finite element meshes suggested in Figure 8 is performed.

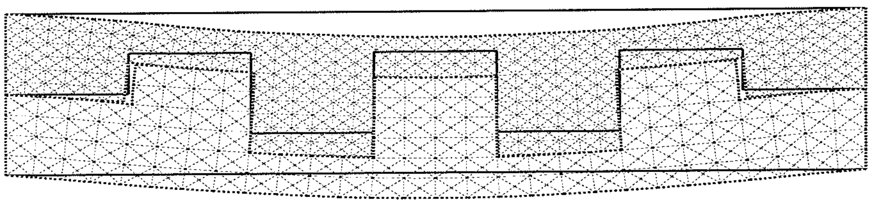

Figure 8: (Non-matching meshes): Initial boundaries and deformed meshes at final time $T = 1\ sec.$ using $\mathbf{K}_h^\pi$

The Von-Mises stress norm at the final time T is plotted (in the deformed configuration) in Figure 9. These results show a good agreement with those obtained when using matching meshes (see Figures 10 and 11).

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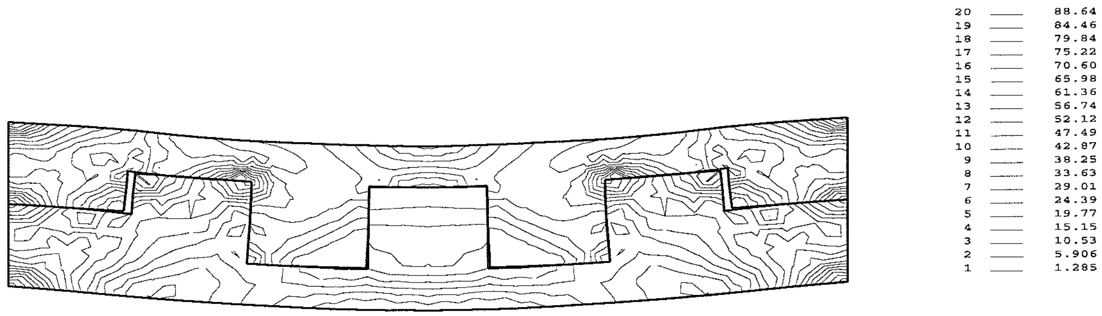

Figure 9: (Non-matching meshes): Von-Mises stress norm at final time $T = 1\,sec.$ using $\mathbf{K}_h^{\pi}$

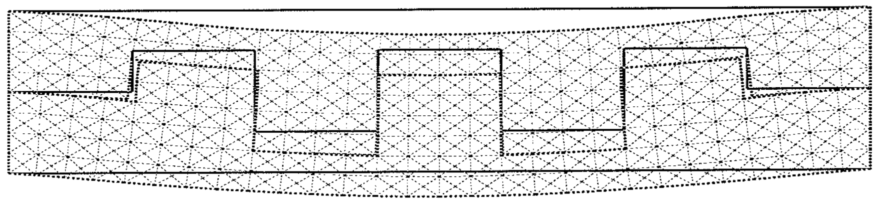

Figure 10: (Matching meshes): Initial boundaries and deformed meshes at final time $T = 1\,sec.$

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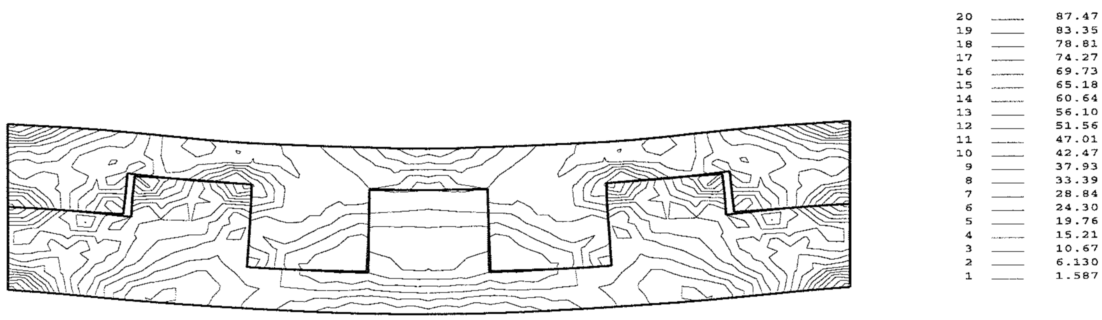

Figure 11: (Matching meshes): Von-Mises stress norm at final time $T = 1\,sec.$

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