The Mortar Finite Element Method for Contact Problems

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Abstract

The purpose of this paper is to describe a domain decomposition technique: *the mortar finite element method* applied to contact problems between two elastic bodies. This approach allows the use of no-matching grids and to glue different discretizations across the contact zone in an optimal way, at least for bilateral contact. We present also an adaptation of this method to unilateral contact problems.

1 Introduction

The *mortar element method* offers a great facility for coupling different variational approximations and therefore using grids that do not match at the interfaces of the subdomains. The beginning of such an approach appeared in [1, 1987], it was reformulated in terms of "mortars" in [2] and took its final shape in [3, 4], which makes it better suited to parallel implementation.

This approach has many advantages regarding both practical and theoretical aspects. Indeed, the mortar process seems to fit naturally to the numerical simulation of contact problems, since this method allows the use of different meshes the size of which will depend on the particularities of each body (elasticity coefficients, geometries, etc ...). In the theoretical area, a substantial number of mathematical proofs have already appeared proving the optimality of the method in spectral and finite element framework for elliptic second order problems [1, 2, 3, 4].

So far, this technique has been applied to partial differential equations expressed in terms of weak formulations. The main aim of our work is to adapt it to variational inequalities arising from unilateral contact.

The paper is outlined as follows. We start with bilateral contact equations without friction, we describe the mortar finite element concept applied to such a problem emphasising the way the approximations are constrainted at the contact zone. Two kinds of couplings studied in [1] are presented yielding to well posed algebraic systems. The first kind is pointwise matching easy to implement and providing satisfactory results for low order finite elements, though not optimal. The second choice turns out to be optimal and consists of enforcing integral matching conditions at the interface. In both situations we extend the error estimate results derived in the above references to the contact problem after some slight modifications.

The main novelty of this work is adressed in the second chapter where we attempt to adapt this nonconforming domain decomposition procedure to unilateral contact problem (without friction). A weak formulation leads to a variational inequality. Again, we propose two types of (pointwise and integral) matching to express the no–interpenetrating conditions at the contact region. We give an abstract error estimate for elliptic inequalities. This basic tool has similar role as that played by the second Strang lemma for variational equations. Then, we prove the strong convergence of the approximated solution towards the exact one and discuss the convergence rate which is not optimal. However we hope to improve these results by using different mathematical tools. Detailed technical proofs and enventual improvements will be given in a forthcoming paper [5].

First, we present the notations we shall use. We are given a bounded domain $\mathcal{O} \subset \mathbb{R}^2$ and a generating point $\boldsymbol{x} = (x_1, x_2)$. In this paper $L^2(\mathcal{O})$

denotes the classical Lebesgue space of square integrable functions, endowed with the inner product:

$$(\varphi, \psi) = \int_{\mathcal{O}} \varphi\psi \, d\boldsymbol{x},$$

for all $\varphi, \psi \in L^2(\mathcal{O})$. We shall, also, make a constant use of the standard Sobolev space $H^m(\mathcal{O})$, $m \geq 1$ provided with the norm:

$$\|\psi\|_{H^m(\mathcal{O})} = \Big(\sum_{0\leq|\alpha|\leq m} \|\partial^\alpha \psi\|^2_{L^2(\mathcal{O})} \Big)^{\frac{1}{2}},$$

where $\alpha = (\alpha_1, \alpha_2)$ denotes a multi-index of $\mathbb{N}^2$ and the symbol ∂^α is the partial derivative $\partial_1^{\alpha_1}\partial_2^{\alpha_2}$. The scale of fractional Sobolev spaces $(H^\tau(\mathcal{O}))_{\tau\in\mathbb{R}\setminus\mathbb{N}}$ are obtained by hilbertian interpolation of $(H^m(\mathcal{O}))_{m\in\mathbb{N}}$. For more details about Sobolev spaces properties we refer the reader to [6]. Bold latin letters like $\boldsymbol{u}, \boldsymbol{v}$, indicate vector quantities while the capital ones (e.g. $\boldsymbol{V}$) are functionnal sets involving vector fields. Afterwards we will consider plane elasticity problems. The symbol σ stands for the stress tensor and ε is the strain tensor; the linearized strain tensor generated by a displacement field $\boldsymbol{v}$ is written :

$$\varepsilon(\boldsymbol{v}) = \frac{1}{2}(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T).$$

2 Setting of the Problem

Let us consider two elastic bodies occupying, in the initial unconstrained configuration, two bounded domains Ω^1 and Ω^2 of the bidimensional space and having a contact zone denoted $\Gamma_c = \Gamma_c^1 = \Gamma_c^2$. For $\ell = 1, 2$ the boundary $\Gamma^\ell = \partial\Omega^\ell$ is assumed to be "smooth" and is the union of three nonoverlapping portions $\Gamma^\ell_{\boldsymbol{u}}$, $\Gamma^\ell_{\boldsymbol{g}}$ and Γ^ℓ_c with the surface measure of $\Gamma^\ell_{\boldsymbol{u}}$ not vanishing and the outward unit normal vector is $\boldsymbol{n}^\ell$. Each of the bodies is subjected to volume forces $\boldsymbol{f}^\ell = (f_1^\ell, f_2^\ell) \in (L^2(\Omega^\ell))^2$ and to surface forces $\boldsymbol{g}^\ell = (g_1^\ell, g_2^\ell) \in (L^2(\Gamma^\ell_{\boldsymbol{g}}))^2$ on $\Gamma^\ell_{\boldsymbol{g}}$.

The bilateral contact problem consists of finding the displacement fields $\boldsymbol{u}^\ell = (u_i^\ell)$, $1 \leq i \leq 2$, and the stress tensors fields $\sigma^\ell = (\sigma^\ell_{ij})$, $1 \leq i, j \leq 2$, satisfying the following equations for $\ell = 1, 2$:

$$\begin{aligned} \mathbf{div}\, \sigma^\ell + \boldsymbol{f}^\ell &= 0 && \text{in } \Omega^\ell, \\ \sigma^\ell \boldsymbol{n}^\ell &= \boldsymbol{g}^\ell && \text{on } \Gamma^\ell_{\boldsymbol{g}}, \\ \boldsymbol{u}^\ell &= 0 && \text{on } \Gamma^\ell_{\boldsymbol{u}}. \end{aligned} \tag{1}$$

The symbol ($\mathbf{div}$) denotes the divergence operator and is defined by

$\mathbf{div}\, \sigma = \left(\frac{\partial \sigma_{ij}}{\partial x_j}\right)_i$ where the summation convention of repeated indices is adopted. The stress tensor is linked to the displacement by the constitutive law

$$\sigma^\ell(\boldsymbol{u}^\ell) = A^\ell\, \varepsilon(\boldsymbol{u}^\ell), \tag{2}$$

where $A^\ell = (a^\ell_{ij,kh})_{1\le i,j,k,h\le 2}$ is a fourth order tensor verifying $a^\ell_{ij,kh} = a^\ell_{ji,kh} = a^\ell_{kh,ij}$ with $a^\ell_{ij,kh} \in L^\infty(\Omega^\ell)$.

The conditions at the contact interface Γ_c are as follows

$$\boldsymbol{u}^1.\boldsymbol{n}^1 + \boldsymbol{u}^2.\boldsymbol{n}^2 = 0, \tag{3}$$

$$\sigma^1(\boldsymbol{u}^1)\boldsymbol{n}^1.\boldsymbol{n}^1 = \sigma^2(\boldsymbol{u}^2)\boldsymbol{n}^2.\boldsymbol{n}^2 = \sigma_N(\boldsymbol{u}), \tag{4}$$

$$\sigma^1_T(\boldsymbol{u}^1) = \sigma^2_T(\boldsymbol{u}^2) = 0, \tag{5}$$

where $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2) = (\boldsymbol{u}|_{\Omega^1}, \boldsymbol{u}|_{\Omega^2})$ and

$$\sigma^\ell_T(\boldsymbol{u}^\ell) = \sigma^\ell(\boldsymbol{u}^\ell)\boldsymbol{n}^\ell - \sigma_N(\boldsymbol{u}^\ell)\boldsymbol{n}^\ell,\ 1 \le \ell \le 2.$$

Condition (3) represents the bilateral contact between the two solids, (4) states the action and the reaction principle and finally (5) represents contact without friction. In this formulation the bodies always remain in contact. The case of possible separation will be considered below. Moreover, there exists positive constants α^ℓ such that

$$a^\ell_{ij,kh}\, \varepsilon_{ij}\, \varepsilon_{kh} \ge \alpha^\ell\, \varepsilon_{ij}\, \varepsilon_{ij}, \qquad \forall \varepsilon_{ij} = \varepsilon_{ji}. \tag{6}$$

In order to study this problem we have to determine an equivalent variational formulation which gives a mathematical sense to the formal equations (1) (see [7, 8]). To this end, we define the spaces $\boldsymbol{V}^\ell$ ($\ell = 1, 2$)

$$\boldsymbol{V}^\ell = \left\{ \boldsymbol{v} \in \left(H^1(\Omega^\ell)\right)^2, \quad \boldsymbol{v} = 0 \text{ on } \Gamma^\ell_{\boldsymbol{u}} \right\},$$

and henceforth the generating vector field of $\boldsymbol{V}^1 \times \boldsymbol{V}^2$ is called $\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2)$. When endowed with the standard inner product

$$(\boldsymbol{u}, \boldsymbol{v})_* = (\boldsymbol{u}^1, \boldsymbol{v}^1)_{(H^1(\Omega^1))^2} + (\boldsymbol{u}^2, \boldsymbol{v}^2)_{(H^1(\Omega^2))^2},$$

$\boldsymbol{V}^1 \times \boldsymbol{V}^2$ is a Hilbert space and the corresponding norm is denoted $\|.\|_*$. Next, we need the following bilinear form :

$$a(\boldsymbol{u}, \boldsymbol{v}) \;=\; \sum_{\ell=1}^{2} \int_{\Omega^\ell} A^\ell \,\varepsilon(\boldsymbol{u}^\ell)\varepsilon(\boldsymbol{v}^\ell)\, d\Omega^\ell,$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}^1 \times \boldsymbol{V}^2$. It is easy to check the continuity, the symmetry and, using (6) together with Korn inequality, the coercivness of $a(.,.)$ on the product space $\boldsymbol{V}^1 \times \boldsymbol{V}^2$. The convenient space $\boldsymbol{V}$ is a subset of $\boldsymbol{V}^1 \times \boldsymbol{V}^2$ and incorporates the contact condition (3)

$$\boldsymbol{V} = \Big\{ \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \boldsymbol{V}^1 \times \boldsymbol{V}^2, \qquad \boldsymbol{v}^1.\boldsymbol{n}^1 + \boldsymbol{v}^2.\boldsymbol{n}^2 = 0 \text{ on } \Gamma_c \Big\}.$$

Now, having the necessary tools, the variational formulation of the bilateral contact problem is: *find* $\boldsymbol{u} \in \boldsymbol{V}$ *such that*:

$$a(\boldsymbol{u}, \boldsymbol{v}) = \sum_{\ell=1}^{2} \Big(\int_{\Omega^\ell} \boldsymbol{f}^\ell.\boldsymbol{v}^\ell \, d\Omega^\ell + \int_{\Gamma^\ell_{\boldsymbol{g}}} \boldsymbol{g}^\ell.\boldsymbol{v}^\ell \, d\Gamma^\ell \Big), \tag{7}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$.
We can check easily the formal equivalence between problems (7) and (1)–(5) (see [7] or [8]). Since the linear form involved on the right side is obviously continuous on $\boldsymbol{V}^1 \times \boldsymbol{V}^2$, using the Lax–Milgram lemma we conclude that problem (7) has only one solution.

3 Finite Element Approximation

The present section is devoted to the construction of a space which will be a good (in a sense that will be precised later) finite element approximation of $\boldsymbol{V}$. First, we describe the discretization used locally within each solid. In order to avoid techniques required for the treatment of curved boundaries, we assume, only for sake of simplicity, that each subdomain Ω^ℓ, $\ell = 1, 2$, is a polygon. Let the approximation parameter $h = (h_1, h_2)$ be given which is a pair of real positive numbers that will decay to 0. With each subdomain Ω^ℓ we then associate a regular triangulation $\mathscr{T}^\ell$, made of elements that are either triangles or rectangles, the diameter of which does not exceed h_ℓ. Nevertheless, we shall focus here only on triangular elements and the extension to the rectangular case is straightforward modulo some slight modifications,

$$\Omega^\ell = \bigcup_{\kappa\in\mathscr{T}^\ell} \overline{\kappa}.$$

When the boundary points $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ of the contact face Γ_c are common nodes of the grids corresponding to the triangulations on both bodies, Γ_c inherits two independent regular meshes, each from one domain (that are all entire edges of an element of the triangulation of Ω^ℓ) and denoted $\mathcal{T}_c^\ell$. We shall assume, in what follows that these (1D) triangulations are uniformely regular so that the inverse inequalities in Sobolev spaces are available (see [9]). For any integer $q \geq 1$, the space $\mathbb{P}_q(\kappa)$ involves the polynomials with the global degree $\leq q$ on κ. With any κ we associate a finite set Ξ_κ of points with barycentric coordinates $(\frac{i}{q}, \frac{j}{q}, \frac{q-i-j}{q})$, $0 \leq i, j \leq q$, in such a way that $(\kappa, \mathbb{P}_q(\kappa), \Xi_\kappa)$ is a finite element of Lagrange type and we set $\Xi^\ell = \cup_{\kappa\in\mathscr{T}^\ell}\Xi_\kappa$. The trace of the two grids on Γ_c provides two sets of nodes denoted ξ^ℓ and because of non conformity $\xi^1 \neq \xi^2$. The finite element space used in Ω^ℓ is then defined by

$$\boldsymbol{V}_h^\ell = \Big\{\boldsymbol{v}_h^\ell \in (\mathscr{C}(\overline{\Omega}^\ell))^2, \qquad \forall\kappa \in \mathscr{T}^\ell, \qquad \boldsymbol{v}_h^\ell|_\kappa \in ((\mathbb{P}_q)(\kappa))^2, \quad \boldsymbol{v}_h^\ell|_{\Gamma_{\boldsymbol{u}}^\ell} = 0\Big\}.$$

In order to express the contact condition (3) on Γ_c in the discrete case we need also to use the spaces

$$M_h^\ell(\Gamma_c) = \Big\{q_h^\ell \in \mathscr{C}(\overline{\Gamma_c}^\ell), \quad \forall T \in \mathcal{T}_c^\ell, \qquad q_h^\ell|_T \in \mathbb{P}_q(T), \\ q_h^\ell|_T \in \mathbb{P}_{q-1}(T) \ \text{ if } \boldsymbol{a}_1 \text{ or } \boldsymbol{a}_2 \in T\Big\}.$$

We are now in a position to construct the approximation space. The first choice corresponds to the pointwise matching, we set $\xi_h = \xi^1$ or ξ^2 and

$$\boldsymbol{V}_h^P = \Big\{\boldsymbol{v}_h = (\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) \in \boldsymbol{V}_h^1 \times \boldsymbol{V}_h^2, \ \boldsymbol{v}_h^1.\boldsymbol{n}^1(\boldsymbol{a}) + \boldsymbol{v}_h^2.\boldsymbol{n}^2(\boldsymbol{a}) = 0, \ \forall \boldsymbol{a} \in \xi_h\Big\}.$$

The second space uses rather integral matching conditions at Γ_c. We set $M_h(\Gamma_c) = M_h^1(\Gamma_c)$ or $M_h^2(\Gamma_c)$ and

$$\boldsymbol{V}_h^I = \Big\{\boldsymbol{v}_h = (\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) \in \boldsymbol{V}_h^1 \times \boldsymbol{V}_h^2, \ \int_{\Gamma_c} (\boldsymbol{v}_h^1.\boldsymbol{n}^1 + \boldsymbol{v}_h^2.\boldsymbol{n}^2) q_h \, d\Gamma = 0, \ \forall q_h \in M_h(\Gamma_c)\Big\},$$

endowed with the norm $\|.\|_*$ both are Hilbert spaces. These spaces are not included in $\boldsymbol{V}$. As a result the approximation is nonconforming in the Hodge sense. Then, the discrete problem is obtained from the exact one (7) by a Galerkin procedure and consists of: *find* $\boldsymbol{u}_h \in \boldsymbol{V}_h^I$ *(resp:* $\boldsymbol{V}_h^P$*) such that:*

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{\ell=1}^{2} \Big(\int_{\Omega^\ell} \boldsymbol{f}^\ell . \boldsymbol{v}_h^\ell \, d\Omega^\ell + \int_{\Gamma_{\boldsymbol{g}}^\ell} \boldsymbol{g}^\ell . \boldsymbol{v}_h^\ell \, d\Gamma^\ell \Big), \tag{8}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h^I$ *(resp:* $\boldsymbol{V}_h^P$*).*
Using, again, the Lax–Milgram lemma shows that this problem has only one solution in $\boldsymbol{V}_h^I$ (resp: $\boldsymbol{V}_h^P$).

Remark: In both cases there are two possibilities for the approximation space corresponding each to different choice of $M_h(\Gamma_c)$ for $\boldsymbol{V}_h^I$ and of ξ_h for $\boldsymbol{V}_h^P$.

4 Error Estimation

We intend in the present section to give an estimate of the error committed on the exact solution by our domain decomposition algorithm. But first, we recall a basic tool, the second Strang lemma ([9]), that allows to obtain such an estimate.

Lemma 4.1 *The solutions* $\boldsymbol{u}$ *and* $\boldsymbol{u}_h$ *of the exact and discrete problems are such that*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_* \leq C \left(\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{u}-\boldsymbol{v}_h\|_* + \sup_{\boldsymbol{w}_h \in \boldsymbol{V}_h} \frac{1}{\|\boldsymbol{w}_h\|_*} \int_{\Gamma_c} \sigma_N(\boldsymbol{u})(\boldsymbol{w}_h^1.\boldsymbol{n}^1 + \boldsymbol{w}_h^2.\boldsymbol{n}^2) \, d\Gamma \right),$$

where $C>0$ *is independent of h.*

Accordingly, the global error results from two contributions. The first infimum is the well known approximation error. The second term is the consistency error caused by the non conformity of the elements.

In the following lemmas and theorems, we will choose $M_h(\Gamma_c) = M_h^1(\Gamma_c)$ and $\xi_h = \xi^1$. Due to the use of inverse inequalities, we suppose, in the case of the pointwise matching only, that $\frac{h_1}{h_2}$ is bounded. This is not restrictive because of the choice of ξ_h.

4.1 Best approximation Error

In this section we deal with the approximation properties of the discrete spaces $\boldsymbol{V}_h^P$ and $\boldsymbol{V}_h^I$. In both cases the expected optimality is obtained.

Lemma 4.2 *When the solution* $\boldsymbol{u}$ *of the exact problem satisfies the following regularity assumptions:* $\boldsymbol{u}|_{\Omega^1} \in (H^{q+1}(\Omega^1))^2$ *and* $\boldsymbol{u}|_{\Omega^2} \in (H^{q+1}(\Omega^2))^2$*, one has*

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h^P} \|\boldsymbol{u} - \boldsymbol{v}_h\|_* \leq C(\boldsymbol{u})(h_1^q + h_2^q),$$

where $C(\boldsymbol{u}) > 0$ *is independent of* h*.*

Lemma 4.3 *Assume that the solution* $\boldsymbol{u}$ *satisifes similar regularity properties as in the previous lemma. Then*

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h^I} \|\boldsymbol{u} - \boldsymbol{v}_h\|_* \leq C(\boldsymbol{u})(h_1^q + h_2^q),$$

where $C(\boldsymbol{u})$ *is independent of* h*.*

4.2 Consistency Error

The consistency error measures the effects of the non conformity. In the pointwise matching case we have the estimate of

Lemma 4.4 *When the solution* $\boldsymbol{u}$ *of the exact problem satisfies the following regularity assumptions:* $\boldsymbol{u}|_{\Omega^1} \in (H^{q+1}(\Omega^1))^2$*, we have*

$$\sup_{\boldsymbol{w}_h \in \boldsymbol{V}_h^P} \frac{1}{\|\boldsymbol{w}_h\|_*} \int_{\Gamma_c} \sigma_N(\boldsymbol{u})(\boldsymbol{w}_h^1.\boldsymbol{n}^1 + \boldsymbol{w}_h^2.\boldsymbol{n}^2)\, d\Gamma \leq C(\boldsymbol{u}) h_1^{\frac{1}{2}},$$

where $C(\boldsymbol{u}) > 0$ *is independent of* h*.*

Unfortunatly, the bound given here is not optimal. This is due to the fact that the interpolation operator doesn't have an optimal truncation error estimate with respect to the negative Sobolev norms. On the contrary, the integral matching provides the optimality. Indeed, we have

Lemma 4.5 *Assume that the solution* $\boldsymbol{u}$ *satisfies similar regularity properties as in the previous lemma. Then*

$$\sup_{\boldsymbol{w}_h \in \boldsymbol{V}_h^I} \frac{1}{\|\boldsymbol{w}_h\|_*} \int_{\Gamma_c} \sigma_N(\boldsymbol{u})(\boldsymbol{w}_h^1.\boldsymbol{n}^1 + \boldsymbol{w}_h^2.\boldsymbol{n}^2)\, d\Gamma \leq C h_1^q \, \|\boldsymbol{u}^1\|_{(H^{q+1}(\Omega^1))^2},$$

where $C(\boldsymbol{u}) > 0$ *is independent of* h*.*

Proof : Due to the integral matching constraints, we have: $\forall q_h \in M_h(\Gamma_c)$

$$\int_{\Gamma_c} \sigma_N(\boldsymbol{u})(\boldsymbol{w}_h^1.\boldsymbol{n}^1 + \boldsymbol{w}_h^2.\boldsymbol{n}^2)\, d\Gamma = \int_{\Gamma_c} (\sigma_N(\boldsymbol{u}) - q_h)(\boldsymbol{w}_h^1.\boldsymbol{n}^1 + \boldsymbol{w}_h^2.\boldsymbol{n}^2)\, d\Gamma.$$

According to the Cauchy-Schwarz inequality, we obtain: $\forall q_h \in M_h(\Gamma_c)$

$$\int_{\Gamma_c} \sigma_N(\boldsymbol{u})(\boldsymbol{w}_h^1.\boldsymbol{n}^1 + \boldsymbol{w}_h^2.\boldsymbol{n}^2)\, d\Gamma \leq \|\sigma_N(\boldsymbol{u}) - q_h\|_{H^{-\frac{1}{2}}(\Gamma_c)} \|\boldsymbol{w}_h^1.\boldsymbol{n}^1 + \boldsymbol{w}_h^2.\boldsymbol{n}^2\|_{H^{\frac{1}{2}}(\Gamma_c)}.$$

Taking the infimum over $M_h(\Gamma_c)$ leads to

$$\int_{\Gamma_c} \sigma_N(\boldsymbol{u})(\boldsymbol{w}_h^1.\boldsymbol{n}^1 + \boldsymbol{w}_h^2.\boldsymbol{n}^2)\, d\Gamma \leq \inf_{q_h \in M_h(\Gamma_c)} \|\sigma_N(\boldsymbol{u}) - q_h\|_{H^{-\frac{1}{2}}(\Gamma_c)} \|\boldsymbol{w}_h^1.\boldsymbol{n}^1 + \boldsymbol{w}_h^2.\boldsymbol{n}^2\|_{H^{\frac{1}{2}}(\Gamma_c)}.$$

This can be estimated as follows (see [2])

$$\int_{\Gamma_c} \sigma_N(\boldsymbol{u})(\boldsymbol{w}_h^1.\boldsymbol{n}^1 + \boldsymbol{w}_h^2.\boldsymbol{n}^2)\, d\Gamma \leq C h_1^q \|\sigma_N(\boldsymbol{u})\|_{H^{q-\frac{1}{2}}(\Gamma_c)} \|\boldsymbol{w}_h^1.\boldsymbol{n}^1 + \boldsymbol{w}_h^2.\boldsymbol{n}^2\|_{H^{\frac{1}{2}}(\Gamma_c)}.$$

Using the trace theorem yields

$$\frac{1}{\|\boldsymbol{w}_h\|_*} \int_{\Gamma_c} \sigma_N(\boldsymbol{u})(\boldsymbol{w}_h^1.\boldsymbol{n}^1 + \boldsymbol{w}_h^2.\boldsymbol{n}^2)\, d\Gamma \leq C h_1^q \, \|\boldsymbol{u}^1\|_{(H^{q+1}(\Omega^1))^2}.$$

4.3 Final Results

The following theorems give the global error on the exact solution for the two matching cases considered. In the pointwise matching case we have the following estimate :

Theorem 4.1 *Assume that the solution* $\boldsymbol{u} \in \boldsymbol{V}$ *of the exact problem is such that:* $\boldsymbol{u}|_{\Omega^1} \in (H^{q+1}(\Omega^1))^2$ *and* $\boldsymbol{u}|_{\Omega^2} \in (H^{q+1}(\Omega^2))^2$. *If* $\boldsymbol{u}_h \in \boldsymbol{V}_h^P$ *is the solution of the discrete problem then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_* \leq C(\boldsymbol{u})(h_1^{\frac{1}{2}} + h_2^q),$$

where $C(\boldsymbol{u}) > 0$ *is independent of* h.

In the integral matching case we have the following estimate :

Theorem 4.2 *Assume that the solution* $\boldsymbol{u}$ *satisifes similar regularity properties as in the previous theorem and let* $\boldsymbol{u}_h \in \boldsymbol{V}_h^I$ *be the solution of the discrete problem. Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_* \leq C(h_1^q \|\boldsymbol{u}^1\|_{(H^{q+1}(\Omega^1))^2} + h_2^q \|\boldsymbol{u}^2\|_{(H^{q+1}(\Omega^2))^2}),$$

where $C > 0$ *is independent of* h.

5 Unilateral contact model

The unilateral contact problem consists in finding for $\ell = 1, 2$ the displacements fields $\boldsymbol{u}^\ell = (u_i^\ell),\ 1 \leq i \leq 2$, and stress tensors fields $\sigma^\ell = (\sigma_{ij}^\ell),\ 1 \leq i, j \leq 2$, satisfying equations (1),(2) and conditions (4) and (5). Moreover, on the initial contact zone Γ_c condition (3) is replaced by

$$\begin{aligned} \boldsymbol{u}^1.\boldsymbol{n}^1 + \boldsymbol{u}^2.\boldsymbol{n}^2 &\leq 0, && (9)\\ \sigma_N(\boldsymbol{u}) &\leq 0, && (10)\\ \sigma_N(\boldsymbol{u})(\boldsymbol{u}^1.\boldsymbol{n}^1 + \boldsymbol{u}^2.\boldsymbol{n}^2) &= 0. && (11) \end{aligned}$$

The two bodies are allowed to leave each other on a portion of the contact zone Γ_c. This formulation is intensively used (see for instance [10] or [11]). The variational principle applied to such a problem leads to a variational inequality. Before presenting it we need to introduce the convex $\boldsymbol{K}$ which takes condition (9) into account explicitly,

$$\boldsymbol{K} = \Big\{\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \boldsymbol{V}^1 \times \boldsymbol{V}^2, \qquad \boldsymbol{v}^1.\boldsymbol{n}^1 + \boldsymbol{v}^2.\boldsymbol{n}^2 \leq 0 \text{ on } \Gamma_c\Big\}.$$

The weak formulation amounts then to *find* $\boldsymbol{u} \in \boldsymbol{K}$ *such that:*

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) \geq \sum_{\ell=1}^{2} \Big(\int_{\Omega^\ell} \boldsymbol{f}^\ell.(\boldsymbol{v}^\ell - \boldsymbol{u}^\ell) \, d\Omega^\ell + \int_{\Gamma^\ell_{\boldsymbol{g}}} \boldsymbol{g}^\ell.(\boldsymbol{v}^\ell - \boldsymbol{u}^\ell) \, d\Gamma^\ell \Big), \qquad (12)$$

for any $\boldsymbol{v} \in \boldsymbol{K}$.
All the assumptions required to use Stampacchia's theorem are fulfilled and the problem (12) has only one solution in $\boldsymbol{K}$.

5.1 Discrete problem

In order to determine a finite element approximation of problem (12) using the mortar technique we define two closed convex sets, each of them corresponding to a different matching relation on Γ_c. They are respectively denoted $\boldsymbol{K}_h^P$ (pointwise matching) and $\boldsymbol{K}_h^I$ (integral matching). Assume the finite element tools of the previous section are still available and we restrict ourselves to the case of elements of degree $q = 1$ (althought conceptually there are no particular problems in defining a mortar space for higher degrees). Then, let us denote $M_{h+}^\ell(\Gamma_c)$ the subset of nonnegative functions of $M_h^\ell(\Gamma_c)$

$$M_{h+}^\ell(\Gamma_c) = \{q_h \in M_h^\ell(\Gamma_c), \quad q_h \geq 0\}.$$

We are, now, in position to determine precisely the approximation spaces. The first one is related to the pointwise matching and, after setting $\xi_h = \xi^1$ or ξ^2, it is

$$\boldsymbol{K}_h^P = \Big\{ \boldsymbol{v}_h = (\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) \in \boldsymbol{V}_h^1 \times \boldsymbol{V}_h^2, \;\; \boldsymbol{v}_h^1.\boldsymbol{n}^1(\boldsymbol{a}) + \boldsymbol{v}_h^2.\boldsymbol{n}^2(\boldsymbol{a}) \leq 0, \quad \forall \boldsymbol{a} \in \xi_h \Big\}.$$

The second choice corresponds to the integral matching conditions and, after setting $M_{h+} = M_{h+}^1$ or M_{h+}^2, it is given by

$$\boldsymbol{K}_h^I = \Big\{ \boldsymbol{v}_h = (\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) \in \boldsymbol{V}_h^1 \times \boldsymbol{V}_h^2, \; \int_{\Gamma_c} (\boldsymbol{v}_h^1.\boldsymbol{n}^1 + \boldsymbol{v}_h^2.\boldsymbol{n}^2) q_h \, d\Gamma \leq 0, \; \forall q_h \in M_{h+}(\Gamma_c) \Big\}.$$

It is clear that, as for the bilateral contact problem, there are two possible choices for the discrete sets.

Next, the discrete problem reads as follows: *find* $\boldsymbol{u}_h \in \boldsymbol{K}_h^I$ (*resp:* $\boldsymbol{K}_h^P$) *such that:*

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) \geq \sum_{\ell=1}^{2} \int_{\Omega^\ell} \boldsymbol{f}^\ell.(\boldsymbol{v}_h^\ell - \boldsymbol{u}_h^\ell) \, d\Omega^\ell + \int_{\Gamma^\ell_{\boldsymbol{g}}} \boldsymbol{g}^\ell.(\boldsymbol{v}_h^\ell - \boldsymbol{u}_h^\ell) \, d\Gamma^\ell, \qquad (13)$$

for all $\boldsymbol{v}_h \in \boldsymbol{K}_h^I$ *(resp:* $\boldsymbol{K}_h^P$*).*
Another use of the Stamppachia theorem asserts the well posedness of the problem and consequently it has a unique solution in $\boldsymbol{K}_h^I$ (resp: $\boldsymbol{K}_h^P$).

5.2 Error Estimation

The second Strang lemma is not useful anymore for the numerical analysis of the mortar approach in the variational inequalities context. Actually a different result is needed and is nothing else but an adaptation of Falk's lemma (see [9], theorem 5.1.1). First, let $\boldsymbol{K}_h$ coincide with $\boldsymbol{K}_h^P$ or $\boldsymbol{K}_h^I$.

Lemma 5.1 *The solutions* $\boldsymbol{u}$ *and* $\boldsymbol{u}_h$ *of the exact and discrete problems are such that*

$$\begin{aligned} \|\boldsymbol{u}-\boldsymbol{u}_h\|_*^2 &\leq C\Big\{\inf_{\boldsymbol{v}_h\in\boldsymbol{K}_h}\Big(\|\boldsymbol{u}-\boldsymbol{v}_h\|_*^2+\int_{\Gamma_c}\sigma_N(\boldsymbol{u})(\boldsymbol{v}_h^1.\boldsymbol{n}^1+\boldsymbol{v}_h^2.\boldsymbol{n}^2)\,d\Gamma\Big) \\ &\quad + \inf_{\boldsymbol{v}\in\boldsymbol{K}}\int_{\Gamma_c}\sigma_N(\boldsymbol{u})((\boldsymbol{v}^1-\boldsymbol{u}_h^1).\boldsymbol{n}^1+(\boldsymbol{v}^2-\boldsymbol{u}_h^2).\boldsymbol{n}^2)\,d\Gamma\Big\}. \quad (14)\end{aligned}$$

Proof : Let us set $\alpha=\min(\alpha^1,\alpha^2)$. Thanks to the ellipticity of the bilinear form $a(.,.)$ on $\boldsymbol{V}^1\times\boldsymbol{V}^2$, we have

$$\alpha\|\boldsymbol{u}-\boldsymbol{u}_h\|_*^2 \leq a(\boldsymbol{u},\boldsymbol{u})-a(\boldsymbol{u},\boldsymbol{u}_h)-a(\boldsymbol{u}_h,\boldsymbol{u})+a(\boldsymbol{u}_h,\boldsymbol{u}_h).$$

Then, noticing that: $\forall\boldsymbol{v}\in\boldsymbol{K},\forall\boldsymbol{v}_h\in\boldsymbol{K}_h$,

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{u}) &\leq a(\boldsymbol{u},\boldsymbol{v})-\sum_{\ell=1}^{2}\Big(\int_{\Omega^\ell}\boldsymbol{f}^\ell.(\boldsymbol{v}^\ell-\boldsymbol{u}^\ell)\,d\Omega^\ell-\int_{\Gamma_{\boldsymbol{g}}^\ell}\boldsymbol{g}^\ell.(\boldsymbol{v}^\ell-\boldsymbol{u}^\ell)\,d\Gamma^\ell\Big), \\ a(\boldsymbol{u}_h,\boldsymbol{u}_h) &\leq a(\boldsymbol{u}_h,\boldsymbol{v}_h)-\sum_{\ell=1}^{2}\Big(\int_{\Omega^\ell}\boldsymbol{f}^\ell.(\boldsymbol{v}_h^\ell-\boldsymbol{u}_h^\ell)\,d\Omega^\ell-\int_{\Gamma_{\boldsymbol{g}}^\ell}\boldsymbol{g}^\ell.(\boldsymbol{v}_h^\ell-\boldsymbol{u}_h^\ell)\,d\Gamma^\ell\Big),\end{aligned}$$

we deduce the following inequality

$$\begin{aligned} &\alpha\|\boldsymbol{u}-\boldsymbol{u}_h\|_*^2 \leq \\ &\leq a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}_h)-\sum_{\ell=1}^{2}\Big(\int_{\Omega^\ell}\boldsymbol{f}^\ell.(\boldsymbol{v}^\ell-\boldsymbol{u}_h^\ell)\,d\Omega^\ell-\int_{\Gamma_{\boldsymbol{g}}^\ell}\boldsymbol{g}^\ell.(\boldsymbol{v}^\ell-\boldsymbol{u}_h^\ell)\,d\Gamma^\ell\Big) \\ &+ a(\boldsymbol{u},\boldsymbol{v}_h-\boldsymbol{u})-\sum_{\ell=1}^{2}\Big(\int_{\Omega^\ell}\boldsymbol{f}^\ell.(\boldsymbol{v}_h^\ell-\boldsymbol{u}^\ell)\,d\Omega^\ell-\int_{\Gamma_{\boldsymbol{g}}^\ell}\boldsymbol{g}^\ell.(\boldsymbol{v}_h^\ell-\boldsymbol{u}^\ell)\,d\Gamma^\ell\Big) \\ &+ a(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{v}_h-\boldsymbol{u}). \quad (15)\end{aligned}$$

Applying Green's formula gives

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}_h) \quad &- \quad \sum_{\ell=1}^{2} \Big(\int_{\Omega^\ell} \boldsymbol{f}^\ell . (\boldsymbol{v}^\ell - \boldsymbol{u}_h^\ell) \; d\Omega^\ell \\ &- \quad \int_{\Gamma_{\boldsymbol{g}}^\ell} \boldsymbol{g}^\ell . (\boldsymbol{v}^\ell - \boldsymbol{u}_h^\ell) \; d\Gamma^\ell \Big) = \sum_{\ell=1}^{2} \int_{\Gamma_c} \sigma_N(\boldsymbol{u}) (\boldsymbol{v}^\ell . \boldsymbol{n}^\ell - \boldsymbol{u}_h^\ell . \boldsymbol{n}^\ell) \; d\Gamma^\ell, \end{aligned}$$

and on the other side

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}) \quad &- \quad \sum_{\ell=1}^{2} \Big(\int_{\Omega^\ell} \boldsymbol{f}^\ell . (\boldsymbol{v}_h^\ell - \boldsymbol{u}^\ell) \; d\Omega^\ell \\ &- \quad \int_{\Gamma_{\boldsymbol{g}}^\ell} \boldsymbol{g}^\ell . (\boldsymbol{v}_h^\ell - \boldsymbol{u}^\ell) \; d\Gamma^\ell \Big) = \sum_{\ell=1}^{2} \int_{\Gamma_c} \sigma_N(\boldsymbol{u}) (\boldsymbol{v}_h^\ell . \boldsymbol{n}^\ell - \boldsymbol{u}^\ell . \boldsymbol{n}^\ell) \; d\Gamma^\ell. \end{aligned}$$

Observing that when M is the norm of $a(.,.)$ we have

$$\begin{aligned} a(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}) \quad &\leq \quad M \|\boldsymbol{u}_h - \boldsymbol{u}\|_* \|\boldsymbol{v}_h - \boldsymbol{u}\|_* \\ &\leq \quad M \Big(\frac{\alpha}{2M} \|\boldsymbol{u}_h - \boldsymbol{u}\|_*^2 + \frac{M}{2\alpha} \|\boldsymbol{v}_h - \boldsymbol{u}\|_*^2 \Big). \end{aligned}$$

The proof is achieved by replacing this inequality in (15) and using condition (11).
We recognize here the approximation error

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{K}_h} \int_{\Gamma_c} \sigma_N(\boldsymbol{u}) [\boldsymbol{v}_h . \boldsymbol{n}] \; d\Gamma + \|\boldsymbol{u} - \boldsymbol{v}_h\|_*^2,$$

where the symbol $[\boldsymbol{v}.\boldsymbol{n}] = (\boldsymbol{v}^1.\boldsymbol{n}^1 + \boldsymbol{v}^2.\boldsymbol{n}^2)$ is the jump of $\boldsymbol{v}.\boldsymbol{n}$ across Γ_c. The consistency error is represented by

$$\inf_{\boldsymbol{v} \in \boldsymbol{K}} \int_{\Gamma_c} \sigma_N(\boldsymbol{u}) [(\boldsymbol{v} - \boldsymbol{u}_h).\boldsymbol{n}] \; d\Gamma,$$

indeed this latter expression is generated by the non conformity. Otherwise we have $\boldsymbol{K}_h \subset \boldsymbol{K}$ and the consistency error desappears.

5.3 Global error estimates

In the beginning we attempted to use the mathematical tools developed in [2] to evaluate the different errors and we have reached a convergence result for both matching types. But, the convergence rate which is of order $(h^{\frac{1}{4}})$ does not seem satisfactory to us and we still hope to prove that the global error decays like h (actually like $h_1 + h_2$). Again, the main difficulty is the evaluation of the consistency error and also the estimate of the integral term on Γ_c involved in the approximation error. Our current investigation is oriented towards finding or developing different techniques that would lead to the optimality or that would improve the error estimates at least for the integral matching case. Here, we shall skip the proofs, which are long and technical, and we give only the final results. In the following theorem, we have choosen $M_h(\Gamma_c) = M_h^1(\Gamma_c)$ and $\xi_h = \xi^1$. Due to the use of inverse inequalities, we have supposed that $\frac{h_1}{h_2}$ is bounded.

Theorem 5.1 *When the solution $\boldsymbol{u} \in \boldsymbol{K}$ of the exact problem satisfies the following regularity assumptions: $\boldsymbol{u}|_{\Omega^1} \in (H^2(\Omega^1))^2$ and $\boldsymbol{u}|_{\Omega^2} \in (H^2(\Omega^2))^2$, then for both matchings the solution of the discrete problem $\boldsymbol{u}_h$ is such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_* \leq C(\boldsymbol{u})(h_1^{\frac{1}{4}} + h_2),$$

where $C(\boldsymbol{u})$ >0 is independent of h.

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